REPRESENTATION THEORY OF C*-ALGEBRAS

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1. Introduction

When talking about Banach algebras one often has in mind the common example of bounded operators on a Hilbert space. However these operator algebras have more structure than just that of a Banach algebra - for example the adjoint operation. $C^*$-algebras are Banach algebras that in addition have such an adjoint operation. The major part of this work will be concerned with showing that the Banach algebra structure together with this adjoint operation characterizes operator algebras. In other words we will show that every $C^*$-algebra is isomorphic to an algebra of operators (that is closed under adjoints) on a Hilbert space.

Section 2 reviews basic results and definitions concerning operators on Hilbert spaces, Banach algebras and $C^*$-algebras. Most important the representation theory for commutative unital $C^*$-algebras will be discussed. The results of Section 2 will be given without proofs. For proofs and further results the reader is referred to [2, 3].

Section 3 starts with the unitization construction. Using this we will extend the result of section 2 to non unital commutative $C^*$-algebras. We then go on with introducing positive elements and establishing some of their most important properties. The section ends with a proof for the existence of approximate units.

Section 4 starts with a general treatment of $^*$-representations. We will work out different characterizations of irreducibility using von Neumann's bicommutant theorem. Afterwards we will introduce the notion of positive functionals in order to obtain representations via the Gelfand-Naimark-Segal construction (GNS construction). Ultimately, considering direct sums of GNS-representations, we prove the Gelfand-Naimark theorem which ensures the existence of an irreducible isometric representation for every $C^*$-algebra.

Most of the results and sketches of proofs are taken from [1]. Furthermore some results and proofs are taken from [4].
2. Basics

In this first part we will review important definitions and facts. For proofs cf. [2].


**Definition 1.** Given a Hilbert space \( H \), we define \( B(H) \) as the space of bounded linear operators on \( H \).

Later on, we will have to deal with direct sums of Hilbert spaces and operators on them.

**Definition 2.** Let \((H_i)_{i \in \Omega}\) be a family of Hilbert spaces. The direct sum \( \bigoplus_{i \in \Omega} H_i \) is the space of tuples \((\xi_i)_{i \in \Omega}\), where \( \sum_{i \in \Omega} \|\xi_i\|^2 < \infty \). With the inner product given by

\[
((\xi_i)_{i \in \Omega}, (\eta_i)_{i \in \Omega}) = \sum_{i \in \Omega} (\xi_i, \eta_i)
\]

this space becomes a Hilbert space.

Given operators \( T_i \in B(H_i) \), \( i \in \Omega \) such that \( \sup_{i \in \Omega} \|T_i\| < \infty \), one can consider their direct sum \( \bigoplus_{i \in \Omega} T_i \) defined by

\[
(\bigoplus_{i \in \Omega} T_i)(\xi_i)_{i \in \Omega} = (T_i \xi_i)_{i \in \Omega}.
\]

\( \bigoplus_{i \in \Omega} T_i \) is bounded with

\[
\left\| \bigoplus_{i \in \Omega} T_i \right\| = \sup_{i \in \Omega} \|T_i\|.
\]

If all \( H_i \) are the same \( H \), the direct sum is called the amplification of \( H \) by \( \text{card}(\Omega) \). In that case, the amplification \( T^\sim \in B(\bigoplus_{i \in \Omega} H) \) of a \( T \in B(H) \) is defined by

\[
T^\sim = \bigoplus_{i \in \Omega} T_i.
\]

Recall some of the most important topologies on \( B(H) \). For their definition we remember, that a topology is characterized by its convergent nets.

**Definition 3.** Let \( H \) be a Hilbert space

- The weak topology is defined by \( T_i \to T \iff (T_i \xi, \eta) \to (T \xi, \eta), \ \xi, \eta \in H \).
- The strong topology is given by \( T_i \to T \iff \|T - T_i\| \to 0, \ \xi \in H \).

**Remark 4.** The strong topology is induced by the family of semi-norms \( p_x : T \mapsto \|Tx\| \). Let \( X \) be a vector space with a topology induced by a family of semi-norms \((p_i)_{i \in I}\). We recall, that for every \( x \in X \) a neighborhood basis is given by

\[
\left\{ \{ y \in X : p_i(x - y) < \epsilon, i \in J \} : J \subseteq I \text{ finite}, \epsilon > 0 \right\}.
\]

We will also need the Spectral Theorem for self-adjoint operators. For the proof and the definition of spectral measures cf. Section 7 in [2].

**Theorem 5.** Let \( H \) be a Hilbert space and \( A \in B(H) \) be a self-adjoint operator. Let \( A \) be the \( \sigma \)-algebra of Borel sets on \( \sigma(A) \). Then there exists a unique spectral measure \( E \) on \((\sigma(A), A, H)\), such that

\[
A = \int t \, dE(t).
\]
Moreover for \( T \in B(H) \)
\[
TA = AT \iff TE(\Delta) = E(\Delta)T, \ \Delta \in A.
\]
The following definition will be useful.

**Definition 6.** Let \( B \subseteq B(H) \) and \( X \subseteq H \). Set
\[
BX := \{ Tx : T \in B, x \in X \}
\]

Finally, we will state a standard argument as a lemma.

**Lemma 7.** Let \( H \) be a Hilbert space, \( S, T \in B(H) \) and \( D \subseteq H \) dense. If
\[
(\xi, \eta) = (T\xi, \eta) \text{ for all } \xi, \eta \in D,
\]
then \( S = T \).

**Proof.** We have \( ((S - T)\xi, \eta) = 0 \) for all \( \xi, \eta \in D \). By the density of \( D \) and continuity of the inner product, this holds for \( \eta \in H \). Thus \( (S - T)\xi \in H^\perp = \{0\} \) for all \( \xi \in D \). Again the density of \( D \) yields \( S = T \). \( \square \)

2.2. **Banach algebras.**

**Definition 8.** A Banach algebra is a Banach space \((A, \|\|)\) together with an associative and distributive multiplication \( \cdot : A \times A \to A \), such that
\[
\|x \cdot y\| \leq \|x\|\|y\| \text{ for all } x, y \in A.
\]

It is called unital if there exists a multiplicative unit. This unit is unique and will be denoted by 1. The set of invertible elements is denoted by
\[
Inv(A) = \{ x \in A : \exists x^{-1} \in A : xx^{-1} = x^{-1}x = 1 \}.
\]

Furthermore the spectrum of an element \( x \in A \) is given by
\[
\sigma(x) = \{ \lambda \in \mathbb{C} : x - \lambda 1 \notin Inv(A) \}.
\]

The spectral radius is given by
\[
r(x) = \max \{ |\lambda| : \lambda \in \sigma(x) \}.
\]

**Remark 9.** Note, that for \( x \in A \), the multiplicative inverse \( x^{-1} \) is unique.

The following properties concerning the spectrum hold true.

**Theorem 10.** Let \( A \) be a unital Banach algebra and \( x, y \in A \). Then

1. \( \sigma_A(x) \) is nonempty and compact.
2. \( r(x) = \lim_{n \to \infty} \|x^n\|^{1/n} \). In particular \( r(x) \leq \|x\| \).
3. \( \sigma_A(xy) \cup \{0\} = \sigma_A(yx) \cup \{0\} \)
4. If \( B \) is a Banach sub-algebra of \( A \) containing 0, then \( \sigma_A(x) \subseteq \sigma_B(x) \) and \( \delta(\sigma_B(x)) \subseteq \delta(\sigma_A(x)) \) where \( \delta \) is the topological boundary. Equivalently, \( \rho_B(x) \) is open and closed as subset of \( \rho_A(x) \).
5. \( \sigma_A(x^{-1}) = \{ \lambda^{-1} : \lambda \in \sigma_A(x) \} \)

Many proofs that technically assume the Banach algebra to be unital, can be extended to the non unital case by the following construction.

**Lemma 11.** Let \( A \) be a non unital Banach algebra. Then \( A_1 := A \oplus \mathbb{C} \) with coordinate wise addition, multiplication defined by \( (x; \lambda)(y; \mu) = (xy + \mu x + \lambda y; \lambda \mu) \) and \( \|(x; \lambda)\|_1 := \|x\| + |\lambda| \) is a unital Banach algebra.
Elements of a Banach algebra $A$ can also be viewed as elements of $B(A)$, if we identify them with their respective multiplication operators.

**Definition 12.** Let $A$ be a Banach algebra. For $x \in A$, we define the *multiplication operator* $M_x \in B(A)$ by

$$y \mapsto xy.$$ 

**Remark 13.** The map $x \mapsto M_x$ from $A$ to $B(A)$ is a homomorphism. Moreover, since $\|xy\| \leq \|x\|\|y\|$, we have $\|M_x\| \leq \|x\|$.

### 2.3. C*-Algebras.

**Definition 14.** A Banach*-algebra is a Banach algebra $A$ with a conjugate-linear and isometric involution $*: A \to A$ such that

$$(xy)^* = y^*x^*$$

for all $x, y \in A$.

If additionally the so called C*-axiom

$$\|x^*x\| = \|x\|^2$$

holds, $A$ is called a C*-algebra.

Let $A, B$ be Banach*-algebras. An algebra-homomorphism $\phi: A \to B$ that satisfies $\phi(x^*) = \phi(x)^*$ for all $x \in A$ is called a *-homomorphism.

A sub-algebra $S \subseteq A$, that is *-closed, i.e. $S^* \subseteq S$, is called a *-sub-algebra.

**Example 15.** The following examples are fundamental in the sense that all C*-algebras, as we will see, can be represented as a *-sub-algebra of one of them.

1. (Unital commutative C*-algebras) Let $C(X)$ be the set of all continuous complex-valued functions on a compact topological space $X$. Equipped with point-wise operations, the point-wise complex-conjugate and the norm given by $\|f\| = \sup \{f(x) : x \in X\}$ it is a C*-algebra.
2. (Non unital commutative C*-algebras) We consider the space $C_0(X) = \{f \in C(X) : \forall \epsilon > 0 \exists K \text{ compact} : f(X \setminus K) \subseteq U_{\epsilon}(0)\}$ for a locally compact space $X$. With the same structure as above one again obtains a C*-algebra.
3. (Non commutative C*-algebras) Consider the space $B(H)$ of bounded linear operators on a Hilbert space $H$. With the operation which maps an operator to its adjoint, the usual algebraic operations - multiplication being the composition of operators - and the operator-norm, $B(H)$ is a C*-algebra.

In analogy to operators we can define normal, self-adjoint and unitary elements for C*-algebras.

**Definition 16.** For a *-algebra $A$, $x \in A$ is called

- **normal** if $x^*x = xx^*$
- **self-adjoint** if $x^* = x$
- **unitary** if $x^* = x^{-1}$ for unital $A$.

**Remark 17.** The set of self-adjoint elements is denoted by $A_{sa}$.

We recall the following facts.

**Theorem 18.** Let $A$ be a unital C*-algebra and $x \in A$.

1. If $x$ is unitary, then $\sigma_A(x) \subseteq \mathbb{T}$ where $\mathbb{T}$ denotes the unit circle.
(2) If $x$ is self-adjoint then $\sigma_A(x) \subseteq \mathbb{R}$.

**Lemma 19.** Let $B$ be a unital $C^*$-sub-algebra of a unital $C^*$-algebra $A$ (with the same unit), then $\sigma_B(x) = \sigma_A(x)$ for all $x \in B$.

A main characteristic of $C^*$-algebras is, that algebraic (spectrum) properties are strongly linked to topological (norm) properties. This connection roots in the innocuous looking C*-axiom.

**Lemma 20.** For a unital $C^*$-algebra, let $x \in A$ be normal. Then $\|x\| = r(x)$ where $r(x)$ is the spectral radius. Furthermore, $\|x\|^2 = r(x^*x)$ for arbitrary $x \in A$.

This also results in additional structure for $*$-homomorphisms.

**Lemma 21.** Let $\varphi : A \to B$ be a $*$-homomorphism between unital $C^*$-algebras $A, B$, such that $\varphi(1_A) = 1_B$. Then $\varphi$ is automatically bounded with $\|\varphi\| \leq 1$. Moreover, $\sigma(\varphi(x)) \subseteq \sigma(x)$ for all $x \in A$.

**Remark 22.** The previous lemma also holds true for the weaker assumption of $A$ being a Banach $*$-algebra.

### 2.4. Unital Commutative $C^*$-algebras

For the proofs in this section, the reader is referred to Section 1 in [3].

**Definition 23.** Let $A$ be a $C^*$-algebra. A multiplicative functional $\phi : A \to \mathbb{C}$ is a linear functional additionally satisfying $\phi(ab) = \phi(a)\phi(b)$. The Gelfand space or spectrum of $A$ is defined by

$$\hat{A} := \{ \phi : A \to \mathbb{C} | \phi \text{ is multiplicative; } \phi \neq 0 \} .$$

**Lemma 24.** Let $\phi \neq 0$ be a multiplicative functional on a $C^*$-algebra $A$. Then $\|\phi\| = 1$. If $A$ is unital, then $\phi(1) = 1$.

We recall that the $w^*$-topology on any subset $Y \subseteq A^*$ of the topological dual of a Banach space $A$, is the initial topology with respect to the maps

$$\iota(a) : Y \to \mathbb{C}; \phi \mapsto \phi(a),$$

where $a \in A$. We denote it by $\sigma(Y, \iota(A))$. The following Banach-Alaoglu type theorem for unital $C^*$-algebras holds true.

**Theorem 25.** Let $A$ be a unital $C^*$-algebra. Its Gelfand space $\hat{A}$ is a compact subspace of the topological dual space $A^*$ with respect to the $w^*$-topology.

**Remark 26.** From now on, if not otherwise stated explicitly, we assume $\hat{A}$ to be endowed with the $w^*$-topology.

The main result of this section is the following representation theorem.

**Theorem 27.** Let $A$ be a commutative unital $C^*$-algebra. The map $\iota : A \to C(\hat{A})$ defined by $\iota(a) = (\phi \mapsto \phi(a))$ is an isometric isomorphism.

**Definition 28.** Let $A$ be a unital $C^*$-algebra. For a normal $x \in A$ define

$$C^*(x) := \text{c.l.s.} \{ x^j (x^*)^k : j,k \in \mathbb{N}_0 \}$$

where $x^0 := 1$.

More generally for commuting $x_1, x_2, \ldots, x_n$, each of which is normal, we define $C^*(x_1, x_2, \ldots, x_n)$ to be the closed linear span of the products of the powers of the $x_k$ and $x_k^*$. 
Remark 29. It is easy to see, that \( C^*(x) \) is a commutative \( C^* \)-sub-algebra of \( A \). Indeed, it is the smallest one containing \( x \) and 1.

As a corollary of Theorem (27) we obtain the continuous functional calculus for normal elements of unital \( C^* \)-algebras.

**Corollary 30.** Let \( A \) be a unital \( C^* \)-algebra. For any normal element \( x \in A \), \( C^*(x) \) is isometrically isomorphic to \( C(\sigma(x)) \). The isomorphism \( \Phi \) maps \( x \) to the identity id : \( \sigma(x) \to \mathbb{C} ; \ t \mapsto t \). For \( f \in C(\sigma(x)) \) we define \( f(x) := \Phi^{-1}(f) \). The following properties hold

1. \( \sigma(f(x)) = f(\sigma(x)) \).
2. If \( f \in C(\sigma(x)), g \in C(\sigma(f(x))) \), then \( g \circ f \in C(\sigma(x)) \) and \( g \circ f(x) = g(f(x)) \).

**Proof.** The proofs found in Section 1 in [3] may be applied to the present situation.

Remark 31. This functional calculus may be used to prove the spectral theorem for normal operators.

**Lemma 32.** Let \( A, B \) be unital \( C^* \)-algebras, \( \varphi : A \to B \) a *-homomorphism between them, such that \( \varphi(1_A) = 1_B \) and \( x \in A \) normal. Then \( \varphi(f(x)) = f(\varphi(x)) \) for any \( f \in C(\sigma(x)) \).

**Proof.** We consider the unital commutative sub-algebras \( C^*(x) \subseteq A, C^*(\varphi(x)) \subseteq B \) and the induced homomorphism between them. If \( f(z) = \sum_{j,k=0}^n \lambda_{j,k} z^j \overline{z}^k \) is a polynomial, the claim is obvious. By Stone-Weierstrass these polynomials are dense in \( C(\sigma(x)) \). From Lemma (21) we know that \( \sigma(\varphi(x)) \subseteq \sigma(x) \). Denote by \( |\varphi(x)| : C(\sigma(x)) \to C(\sigma(\varphi(x))) \) the restriction mapping and by \( \Phi_x(\varphi(x)) \) the mapping from Corollary (30) applied to \( x(\varphi(x)) \). We just showed that \( \varphi \circ \Phi_x^{-1}(f) = \Phi_{\varphi(x)}^{-1} \circ |\varphi(x)| (f) \) for any \( f \in \mathbb{C}[z, \overline{z}] \) viewed as an element of \( C(\sigma(x)) \). According to the continuity of all involved mappings and due to the density of \( \mathbb{C}[z, \overline{z}] \) we see that the above equation holds true for all \( f \in C(\sigma(x)) \), i.e. \( \varphi(f(x)) = f(\varphi(x)) \).

As a first application of the functional calculus, we will strengthen the statement from Lemma (21).

**Lemma 33.** Let \( \phi : A \to B \) be a *-homomorphism such that \( \phi(1_A) = 1_B \) between unital \( C^* \)-algebras \( A, B \). If \( \phi \) is injective it is an isometry.

**Proof.** We already know, that \( \| \phi(x) \| \leq \| x \| \) for all \( x \in A \), cf. Lemma (21). Let us suppose there is an \( x \in A \), such that this inequality is strict. Then also \( r := \| \phi(x^*x) \| < \| x^*x \| =: s \).

We now consider the \( C^* \)-algebra \( C^*(x^*x) \) (note, that \( x^*x \) is normal) and the induced injective *-homomorphism \( \phi : C^*(x^*x) \to B \). It is indeed injective since for \( x \in A \cap C^*(x^*x) \) we have \( \phi(x + \lambda 1) = \phi(x) + \lambda 1 \) where \( \phi(x) \) and 1 are obviously linearly independent. Consider the function \( f := g_{|\sigma(x^*x)} \), where \( g \in C([-s, s]) \) vanishing on \([-s, r] \) with \( g(s) = 1 \). By the functional calculus we have \( 0 = f(\phi(x^*x)) = \phi(f(x^*x)) \).

Since \( f \) does not vanish on \( \sigma(x^*x) \), the equality above contradicts the injectivity of \( \phi \). 

\( \square \)
3. Further Structure

In the following $A$, $B$, ... will always denote a $C^*$-algebra unless otherwise stated. Moreover, the word functional always means linear functional.

3.1. Unitization. A $C^*$-algebra need not be unital. However, there is a canonical way to construct an enveloping unital $C^*$-algebra.

Remark 34. If $A$ is a $C^*$-algebra, the map $x \mapsto M_x$ (cf. Definition (12)) is an isometric homomorphism. Only $\|M_x\| \geq \|x\|$ is left to prove. Indeed

$$\|M_x\| \geq \|M_x \frac{x^*}{\|x^*\|}\| = \frac{1}{\|x^*\|} \|xx^*\| = \|x\|.$$  

Theorem 35. Let $A$ be a non unital $C^*$-algebra. Then $A_1 := A \oplus \mathbb{C}$ with coordinate wise addition, $(x;\lambda)(y;\mu) = (xy + \mu x + \lambda y; \lambda \mu)$, $\|(x;\lambda)\|_1 = \sup \|xy + \lambda y\|$ and $(x;\lambda)^* = (x^*; \overline{\lambda})$ is a unital $C^*$-algebra. The unit is given by $(0; 1)$.

Proof. $\|\|\|_1$ is the operator norm on $A_1$ if its elements are viewed as sums of left multiplication operators and multiples of the identity on $A$, i.e. $(x;\lambda) \simeq M_x + \lambda I$. Thus $\|\|_1$ actually is a norm and it also satisfies $\|xy\|_1 \leq \|\|_1 \|y\|_1$.

To show completeness we first notice that equipped with the norm $\|(x;\lambda)\|_2 := \|x\| + |\lambda|$ and the algebraic rules given above, $A_1$ is a Banach algebra; (cf Lemma (11)).

The next step is to show, that $\|\|_1$ and $\|\|_2$ are equivalent. Since $\|xy + \lambda y\| \leq \|x\||\|y\| + |\lambda||\|y\| = \|(x;\lambda)\|_2$ for every $\|y\| = 1$,

$$\|(x;\lambda)\|_1 \leq \|(x;\lambda)\|_2.$$

Suppose there is no $n \in \mathbb{N}$ with $\|(x;\lambda)\|_2 \leq n\|(x;\lambda)\|_1$ for all $(x;\lambda) \in A_1$. Then there are sequences $x_n, \lambda_n$ with $\|(x_n;\lambda_n)\|_2 \geq n\|x_ny + \lambda_n y\|$ for all $\|y\| = 1$. Take $y = x_n/\|x_n\|$ to see, that $\lambda_n \neq 0$ for $n > 1$. Setting $z_n := x_n/\lambda_n$ one gets

$$\|z_n\| + 1 \geq n\|z_ny + y\| \text{ for all } \|y\| = 1.$$  

$(z_n)_{n \in \mathbb{N}}$ cannot be unbounded since for every $n \in \mathbb{N}$ we may take $y := z_n/\|z_n\|$, to obtain

$$\|z_n\| + 1 \geq n\|z_ny + y\| \geq n\|z_n\| - n.$$

Hence

$$\frac{n + 1}{n - 1} \geq \|z_n\|, \quad n > 1.$$  

So by (3.1) for all $n, m \geq n(\epsilon)$ and $\|y\| = 1$ we get $\|z_ny + y\| \leq \epsilon/2$ and therefore

$$\|(z_n - z_m)y\| \leq \|z_ny + y\| + \|z_my + y\| \leq \epsilon.$$  

By the previous remark $(z_n)$ is a Cauchy sequence. Thus it converges to a limit $z \in A$ satisfying $zy = -y$ for all $y$. But this is impossible because $A$ is non unital. So $\|\|_1$ is a norm equivalent to $\|\|_2$ showing that $(A_1; \|\|_1)$ is complete.

Next we prove that the *-operation is an isometry and that the $C^*$-axiom holds true. In fact

$$\|(x;\lambda)\|_1^2 = \sup_{\|y\| = 1} \|xy + \lambda y\|^2 = \sup_{\|y\| = 1} \|(xy + \lambda y)^*(xy + \lambda y)\|$$

$$= \sup_{\|y\| = 1} \|y^*x^*xy + \lambda y^*x^*y + \lambda^*xy + |\lambda|^2y^*y\| \leq$$
Theorem (35). Moreover, for a normal we set

\[ A \triangleq \{ x \in A : \| x \| = 1 \} \]

Remark 36. By Remark (34) we have that \( \| . \|_A = \| . \| \) for a non unital \( A \). Therefore, \( A \) can be viewed as the subset \( A \oplus \{ 0 \} \) of \( A_1 \) by the isometric embedding \( a \mapsto (a; 0) \). As such it is a closed ideal with \( A_1/A \cong \mathbb{C} \).

Definition 37. Let \( A \) be a unital C*-algebra. If \( A \) is unital, then we set. Otherwise we set \( A_1 := A \oplus \mathbb{C} \) and provide this space with the C*-algebra structure from Theorem (35). Moreover, for a normal \( x \in A \) we define

\[ C^*(x) := \text{c.l.s. } \{ x^j (a^k)^* : j, k \in \mathbb{N}_0 \} \subseteq A_1 \]

where \( x^0 := 1 \). More generally for commuting \( x_1, x_2, \ldots, x_n \), each of which is normal, we define \( C^*(x_1, x_2, \ldots, x_n) \) to be the closed linear span of the products of the powers of the \( x_k \) and \( x_k^* \).

Lemma 38. Let \( X \) be a non compact but locally compact Hausdorff space. If \( ac(X) \) denotes the one-point (Alexandroff) compactification of \( X \), then \( C_0(X)_1 \cong C(ac(X)) \) i.e. there exists an isometric *-isomorphism between those C*-algebras.

Proof. We show, that the map \( \Phi : C_0(X)_1 \to C(ac(X)) \) defined by

\[ \Phi((f; \lambda))(x) = \begin{cases} f(x) + \lambda & x \in X \\ \lambda & x = \infty \end{cases} \]

is an isometric *-isomorphism.

It is easy to see, that the map is well-defined and a *-isomorphism. By Lemma (33) it is also isometric.

Alternatively we can show isometry directly. In order to do so, note that \( f \in C_0(X) \) takes its maximum at a point \( y \in X \). By definition we get \( \| (f; \lambda) \|_1 = \sup \{ \| f + \lambda g \| : g \in C_0(X), \| g \| = 1 \} = f(y) + \lambda \). Since, \( \| \Phi((f; \lambda)) \| = f(y) + \lambda, \Phi \) is an isometry.

Remark 39. Let \( A, B \) be C*-algebras and \( \varphi : A \to B \) a *-homomorphism. If \( A \) is unital, then by the continuity of the product \( \varphi(1_A) \) is a unit of the *-subalgebra \( \varphi(A) \). Nevertheless it can happen, that \( B \) is not unital or that \( 1_B \neq \varphi(1_A) \).

An example for \( 1_B \neq \varphi(1_A) \) is easily constructed:

Let \( \mathbb{M}(n) \) denote the C*-algebra of \( n \)-dimensional matrices (cf. Example (15.3)). Then the embedding

\[ \iota : \mathbb{M}(m) \to \mathbb{M}(n), \quad m < n \]

given by

\[ A \mapsto \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} \]

is a *-homomorphism between this C*-algebras. Obviously, we have \( \iota(E_n) \neq E_m \).
Remark 40. Let $A, B$ be C*-algebras and $\varphi: A \to B$ a *-homomorphism. If $A$ is non unital, then it is easy to check, that $(x; \lambda) \mapsto \phi(x) + \lambda 1$ is a *-homomorphism from $A_1$ to $B_1$ extending $\varphi$. This extension will also be denoted by $\varphi$.

If $\varphi(A)$ is dense in $B$, then $\varphi: A_1 \to B_1$ is the unique extension of $\varphi: A \to B$ to a homomorphism. Assume, that $\psi: A_1 \to B_1$ is another extension of $\varphi: A \to B$. If $B = B_1$, i.e. $B$ is unital, then it follows from the previous remark, that $1_B = \psi(1_{A_1})$ and by linearity $\psi = \varphi$ on $A_1$. If $B$ is not unital, then $\psi(1_{A_1}) \not\in B$, since otherwise $\psi(1_{A_1})$ would be a unit in $B$, which contradicts our assumption. Consequently, $\psi(A_1)$ is dense in $B_1$. As before, we derive $1_{B_1} = \psi(1_{A_1})$ and in turn $\psi = \varphi$ on $A_1$.

Lemma 41. Any *-homomorphism $\varphi: A \to B$ between C*-algebras $A, B$ is a contraction. If $\varphi$ is injective, then it is isometric.

Proof. Replacing $B$ by the closure of $\varphi(A)$, we can assume $\varphi$ to have dense range. We saw in the previous remark, that $\varphi$ then admits a unique extension $\varphi: A_1 \to B_1$ which according to Lemma (21) is a contraction.

If $\varphi$ is injective and $A$ is not unital, then $1_{B_1} \not\in \varphi(A)$, since otherwise injectivity would give the existence of a unit in $A$. Thus, also $\varphi: A_1 \to B_1$ is injective. According to Lemma (33) $\varphi$ is isometric. \qed

Corollary 42. If $A$ is a non-unital C*-algebra and $B$ a unital C*-Algebra containing $A$, then $A + c.l.s.\{1_B\}$ is isometrically isomorphic to $A_1$.

Proof. According to Remark (40) the inclusion mapping $A \to B$ extends to an injective *-homomorphism. By the previous lemma this extension is an isometric isomorphism. \qed

Remark 43. For a C*-algebra $A$ and $x \in A$, let $\sigma(x) = \sigma_A(x)$ be the spectrum of $x$, when $x$ is considered as an element of $A_1$. By the previous corollary and Lemma (19) we have $\sigma_A(x) = \sigma_B(x)$ for any C*-algebra $B \subseteq A$.

3.2. Commutative C*-algebras. In this section we will expand the results of Section 2.3. to the non unital case. We start with examining the Gelfand space of a unitization of a non unital C*-algebra $A$.

Remark 44. By Lemma (24) we have $\phi(1) = 1$ for all $\phi \in \hat{A}_1$ (cf. Def (23)). Thus by Remark (40), we can uniquely extend any $\phi \in \hat{A}$ to a multiplicative functional on $A_1$ by setting $\phi(1) = 1$. By default we will identify $\phi$ with its unique extension.

Moreover, since it also holds, that $\phi|_A$ is a multiplicative functional on $A$ for any $\phi \in \hat{A}_1$, we get $\hat{A}_1 = \hat{A} \cup \{\phi_1\}$ where $\phi_1|_A = 0$ and $\phi_1(1) = 1$.

Indeed an even stronger proposition is true.

Lemma 45. Let $A$ be a non unital C*-algebra. $\hat{A}_1$ is homeomorphic to the one point compactification $ac(\hat{A})$ of $\hat{A}$.

Proof. We use the notation introduced previous to Theorem (25). By the remark above, $(\hat{A}, \sigma_\hat{A}(\iota(\hat{A})))$ is homeomorphic to $(\hat{A}_1 \setminus \{\phi_1\}, \sigma_\hat{A}(\hat{A}_1 \setminus \{\phi_1\}, \iota(A_1)))$. To see this let $\Psi: \hat{A} \to \hat{A}_1 \setminus \{\phi_1\}$ map $\phi \in \hat{A}$ to its unique extension. Obviously this is a bijection. For all $a \in \hat{A}$ one has $\iota(a;\lambda) \circ \Psi = \iota_a + \lambda$ and $\iota_a \circ \Psi^{-1} = \iota(a;0)$. The functions on the right side of the equation are continuous. Thus by the definition of the respective initial topologies, $\Psi$ is bi-continuous.
By Theorem (25) $\hat{A}$ is compact. Furthermore $A_1 \setminus \{\phi_1\}$ is open and dense in $(\hat{A}_1, \sigma(\hat{A}_1, \iota(A_1)))$. So using the uniqueness of the one point compactification we get, that $A_1$ is homeomorphic to ac($A$).

\textbf{Corollary 47.} Let a self-adjoint $A$.

\textbf{Proof.} We recall, that a space is locally compact if its one-point compactification is Hausdorff. Indeed, by the above lemma, the one point compactification is homeomorphic to the Hausdorff space $A_1$.

We now have everything in order, to extend Theorem (27) to the non unital case.

\textbf{Corollary 48.} Let $A$ be a commutative C*-algebra. Then $A$ is isometrically isomorphic to $C_0(\hat{A})$.

\textbf{Proof.} We already know, that $A_1$ is isometrically isomorphic to $C(\hat{A}_1)$, so we can assume, that $A$ is non-unital. By Lemma (45) $\hat{A}$ is homeomorphic to ac($A$). Let $\Phi : \hat{A} \rightarrow ac(A)$ be the corresponding homeomorphism. Then the map $\Upsilon(f) = f \circ \Phi$ from $C(\hat{A}_1)$ to $C(ac(A))$ is an isometric isomorphism. We now use Lemma (45) to see, that $C(ac(A))$ is also isometrically isomorphic to $C_0(\hat{A})$.

Putting it all together one gets an isometric isomorphism $\Omega : A_1 \rightarrow C_0(\hat{A})$. Taking a closer look on this isomorphism one sees, that $\Omega(A) = C_0(\hat{A})$, we consider $C_0(A)$ as a subset of $C_0(\hat{A})_1$ by Remark (36).

3.3. \textbf{positive Elements.} Let $A$ be a C*-algebra for the remainder of this section.

\textbf{Definition 49.} A self-adjoint $x$ is called \textbf{positive} if $\sigma(x) \subseteq [0, \infty)$. The set of all positive elements is denoted $A_+$. We also write $x \geq 0$ to indicate, that $x$ is positive.

\textbf{Remark 49.} Since $\sigma(f) = f(X)$ for $f \in C(X)$ and a compact space $X$, the positive elements of $C(X)$ are exactly the non negative functions. By Lemma (45) the same is true for $C_0(X)$.

It is useful to know, that every element can be written as a linear combination of four positive elements.

\textbf{Lemma 50.} Let $y \in A_{sa}$ and $x \in A$ then

1. $x = Re(x) + iIm(x)$ where $Re(x) = \frac{x + x^*}{2}$ and $Im(x) = \frac{x - x^*}{2i}$ are self-adjoint.

2. For $y \in A_{sa}$ one has the decomposition $y = y_+ - y_-$ where $y_+, y_- \in A_+$ are unique satisfying $y_+ y_- = 0$. We also have $\sigma(y_-) \cap (\neg \infty, 0) = \sigma(y) \cap (\neg \infty, 0)$ and $\sigma(y_+) \cap (0, \infty) = \sigma(y) \cap (0, \infty)$.

\textbf{Proof.} The first assertion is immediately seen by a straightforward calculation.

For (2) consider the representation of the (commutative) C*-algebra $C^*(y)$. It is easy to see, that $y_+, y_-$ represented by $f(t) := \max(t, 0)$, $g(t) := -\min(t, 0)$ respectively, satisfy the conditions. The claim about the spectra follows from the previous theorem.

From the functional calculus in the previous section one derives some basic facts about positive elements.

\textbf{Lemma 51.} For $x \in A$, the following assertions hold true:
If $x$ is normal then $x^*x \geq 0$.
(2) If $x \in A_{sa}$ and $\|x\| \leq 2$, then $x \geq 0 \iff \|1 - x\| \leq 1$ (in $A_1$).
(3) If $x \geq 0$ then there exists a square-root $x^{1/2} \geq 0$.
(4) $(x; \lambda) \geq 0$ in $A_1$ if and only if $x \in A_{sa}$ and $\lambda \geq \|x_-\|$.

Proof. In each case it is possible to consider the $C^*$-algebra $C^*(x)$ instead of $A$. Therefore, $x$ can be represented by some $f \in C(\sigma(x))$. The first three properties then follow immediately using that fact. Concerning (4), note that
$$
\sigma((x_-; \lambda)) = \sigma(x_-) + \lambda.
$$
Then
$$
\sigma((x; \lambda)) \cap (-\infty, 0) = \sigma((x_-; \lambda)) \cap (-\infty, 0) = (\sigma(x_-) + \lambda) \cap (-\infty, 0)
$$
yields the fourth property. \hfill \Box

We continue with some further properties. Note, that (2) is a considerable improvement of (1) in the previous lemma.

Theorem 52. Let $x \in A$.

(1) If $x, y \geq 0$ then $x + y \geq 0$, i.e. $A_+$ is a closed cone.
(2) $x^*x \geq 0$.
(3) $x^*yx = (y^{1/2}x^*y^{1/2}) \geq 0$ for $y \geq 0$.

Proof. (1) By Lemma (51) we have,
$$
A_+ \cap K_1(0) = A_{sa} \cap K_1(0) \cap \{x : \|1 - x\| \leq 1\}
$$
with $K_1(0)$ being the closed unit ball in $A$. Since $A_{sa}$ is closed by the continuity of $^*$, this is a closed and convex set. For $x, y \in A_+$, there is a $C > 0$ such, that
$$
\frac{1}{2C} (x + y) \in A_+ \cap K_1(0).
$$
Since positive multiples of positive elements are positive we have $x + y \in A_+$.

(2) To prove the second property, according to Lemma (50) we decompose $x^*x = c_+ - c_-$ and define $h := xc_-$. A calculation yields
$$
-h^*h = c_-^2 \geq 0.
$$
Using Lemma (51),(1) along with (1) we obtain
$$
hh^* = (h^*h + hh^*) + (-h^*h) = 2 \text{Re}(h)^2 + 2 \text{Im}(h)^2 + (c_-^2) \geq 0.
$$
Since $h^*h \geq 0 \iff hh^* \geq 0$ by Theorem (10),(3), it follows that $c_- = 0$.
(3) immediately follows from (2). \hfill \Box

Remark 53. By property (2) and the existence of square roots we get that $x \in A$ is positive if and only if there exists $y \in A$ such that $x = y^*y$. This characterization does not hold in general Banach $^*$-algebras. The reason for this is that one does not have the functional calculus which is used to obtain the decomposition $x^*x = c_+ - c_-$.  

Definition 54. For $x, y \in A$ one writes $x \geq y$ if $x - y \geq 0$. 

Remark 55. This relation defines a partial order on \( A \) satisfying \( x \leq y \Rightarrow x + z \leq y + z \) for all \( z \in A \). To assert transitivity let \( x \leq y \) and \( y \leq z \). Then

\[
z - x = (z - y) + (y - x) \geq 0
\]

because the sum of positive elements is positive by the previous theorem. We get \( x \leq z \).

Lemma 56. If \( y \leq z \), then \( x^*yx \leq x^*zx \) and if \( y \in A_+ \), then \( 0 \leq x^*yx \leq \|y\|x^*x \).

Proof. By (3) of the previous theorem, we have \( x^*(z - y)x \geq 0 \). Hence we get \( x^*yx \leq x^*zx \). From Theorem (10),(2) we obtain \( y \leq \|y\|1 \). The second inequality immediately follows from this fact and the first part of the present assertion. \( \square \)

The next lemma and its corollary will be used in the next section, to prove the existence of approximate units.

Lemma 57. If \( 0 \leq x \leq y \) and \( x \in \text{Inv}(A) \) then \( y \in \text{Inv}(A) \) and \( 0 \leq y^{-1} \leq x^{-1} \).

Proof. If \( x \in \text{Inv}(A) \cap A_+ \) then \( \varepsilon 1 \leq x \) for some \( \varepsilon > 0 \) as one sees by identifying \( x \) with \( f \in C(\sigma(x)) \). Therefore, also \( \varepsilon 1 \leq y \). Identifying \( y \) with a function \( g \), we see, that \( y \) is invertible. If \( x \) and \( y \) commute, we may restrict our considerations to \( C^*(x,y) \) and use representation theory from Theorem (27) for commutative C*-algebras to get the inequality. Indeed, \( x \) and \( y \) are both represented by functions \( f \) and \( g \), on the Gelfand space of \( C^*(x,y) \), respectively, and we have \( 0 \leq f \leq g \). Thus \( 0 \leq y^{-1} \leq f^{-1} \) and finally \( 0 \leq y^{-1} \leq x^{-1} \). For the general case note, that by Lemma (56)

\[
y^{-1/2}xy^{-1/2} \leq y^{-1/2}yy^{-1/2} = 1.
\]

From the special case we get

\[
1 \leq (y^{-1/2}xy^{-1/2})^{-1} = y^{1/2}x^{-1}y^{1/2}.
\]

Finally,

\[
y^{-1} = y^{-1/2}1y^{-1/2} \leq y^{-1/2}y^{1/2}x^{-1}y^{1/2}y^{-1/2} = x^{-1}.
\]

\( \square \)

Corollary 58. If \( 0 \leq x \leq y \), then \( \|x\| \leq \|y\| \).

Proof. Let \( \lambda > r(y) \), then \( 0 \leq \lambda - y \leq \lambda - x \). By the previous lemma, since \( \lambda - y \in \text{Inv}(A) \), \( \lambda - x \) is also invertible. Therefore, \( \|x\| = r(x) \leq r(y) = \|y\| \). \( \square \)

3.4. Approximate Units.

Definition 59. An approximate unit for a C*-algebra \( A \) is a net \((h_\lambda)_{\lambda \in \Lambda}\) of positive elements in \( A \) with \( \|h_\lambda\| \leq 1 \) and \( h_\lambda x \to x \) for all \( x \in A \).

There always exists an approximate unit for a C*-algebra. Indeed for any dense two-sided ideal \( I \subseteq A \) there is an approximate unit contained in \( I \).

Theorem 60. If \( I \subseteq A \) is a dense two-sided ideal one defines \( \Lambda_I := \{ x \in I_+ : \|x\| < 1 \} \) with \( I_+ := A_+ \cap I \). Then there exists a direction on \( \Lambda_I \), given by the partial order \( \leq \), such that it is an approximate unit.
Proof. Step 1: We start showing, that $\leq$ is a direction on $\Lambda_I$. To do so, we will prove, that the map $\alpha : x \mapsto (1-x)^{-1} - 1$ defines an order-isomorphism from $\Lambda_I$ onto $I_+$, where the order is given by $\leq$.

Indeed, for $x \in \Lambda_I$, $\alpha(x)$ is well defined and belongs to $I_+$, since

$$(1 - x)^{-1} - 1 = \sum_{k=1}^{\infty} x^k = x(\sum_{k=0}^{\infty} x^k) \in I_+$$

by $\|x\| < 1$. By Lemma (57) $\alpha$ is order preserving.

If $y \in I_+$ then, $(1 - (1 + y)^{-1})(1 + y) = y$.

The invertibility of $(1 + y)$ (cf. Lemma (57)) yields

$$1 - (1 + y)^{-1} = y(1 + y)^{-1} \in I_+.$$ 

Let $y$ be represented by the continuous function $g$ on the Gelfand space of $C^*(y)$. Then $y(1 + y)^{-1}$ is represented by $\frac{g}{1+g}$. The compactness of the Gelfand space yields $\|y(1 + y)^{-1}\| < 1$. Therefore,

$$1 - (1 + y)^{-1} \in \Lambda_I.$$ 

Thus the mapping $\beta : x \mapsto 1 - (1 + x)^{-1}$ maps $I_+$ onto $\Lambda_I$ and obviously satisfies $\alpha \circ \beta = \text{id}_{I_+}$, $\beta \circ \alpha = \text{id}_{\Lambda_I}$. Hence, $\alpha$ is a bijection with the inverse mapping $\beta$.

As $I_A$ is a directed set $(x, y \in I_+ \Rightarrow x \leq y)$, the same is true for $\Lambda_I$.

Step 2: In order to prove, that $\Lambda_I$ is an approximate unit, we have to show, that the net $(x^+(1-y)x)_{y \in \Lambda_I}$ converges to 0 for every $x \in \Lambda_A$. Indeed from $(1 - y)^2 = (1 - y)^{\frac{2}{2}}(1-y)(1-y)^{\frac{1}{2}} \leq (1-y)$ we get

$$||(1-y)x||^2 = ||x^+(1-y)^2x|| \leq ||x^+(1-y)x||.$$ 

Hence, from the fact, that by Lemma (50) every $x \in A$ can be written as a linear combination of elements from $\Lambda_A = \{x \in A_+: \|x\| < 1\}$ this suffices.

We first notice that $\Lambda_I$ is dense in $\Lambda_A$. Indeed, let $y \in I$ be a sufficiently good approximation of $x^{\frac{1}{2}}$ for $x \in \Lambda_A$, then by

$$\|x - yy^*\| \leq \|x^{\frac{1}{2}}(x^{\frac{1}{2}} - y^*)\| + \|(x^{\frac{1}{2}} - y)y^*\|$$

As a consequence without loss of generality we may assume that $I = A$. We now fix $x \in \Lambda_A$ and view $x$ as a function on the Gelfandspace of $C^*(x)$. For every $\epsilon > 0$, that is sufficiently small, there exists a point $\mu \in \{\lambda \in C^*(x) : f(\lambda) \geq \epsilon\}$ independent of $\epsilon$, where $f$ obtains its maximum. We see, that for those $\epsilon > 0$ one finds an $n \in \mathbb{N}$ such that

$$\|x(1 - x^\frac{1}{2})x\| \leq \left\| f(1 - f^\frac{1}{2}) \right\| \leq \max \left\{ f(\mu)(1 - \epsilon^\frac{1}{2}) f(\mu), \epsilon^2 \right\} \leq \epsilon.$$ 

By Lemma (56), we see that $x^+(1-y)x$ is decreasing. Therefore, $(x^+(1-y)x)_{y \in \Lambda_I}$ converges to 0 and the proof is completed.

we get that $yy^* \in \Lambda_I$ approximates $x$. \hfill \Box

The next lemma establishes some properties of approximate units.

Lemma 61. If $(h_\lambda)_{\lambda \in \Lambda}$ is an approximate unit in $A$ then

1. $xh_\lambda \to x$ and $h_\lambda xh_\lambda \to x$ for all $x \in A$.
2. $(h_\lambda^n)_{\lambda \in \Lambda}$ is an approximate unit for any $\alpha > 0$. 

Proof. 1) Since \( h_\lambda \) is positive \( xh_\lambda \to x \) follows from \( h_\lambda x^* \to x^* \). By decomposition one can assume that \( x \geq 0 \). Then
\[
h_\lambda xh_\lambda = (h_\lambda x^\frac{1}{2})(x^\frac{1}{2}h_\lambda) \to x^\frac{1}{2}x^\frac{1}{2} = x.
\]
2) One has
\[
\|h_\lambda^2 x - x\| \leq \|h_\lambda^2 x - h_\lambda xh_\lambda\| + \|h_\lambda xh_\lambda - x\| = \|h_\lambda\| \|h_\lambda x - xh_\lambda\| + \|h_\lambda xh_\lambda - x\| \to 0.
\]
By induction
\[
\|h_\lambda^n x - x\| \to 0
\]
for any \( n \in \mathbb{N} \). Hence, there exists a \( \lambda_0 \) such that for all \( \lambda \geq \lambda_0 \), we have
\[
\|x^*h_\lambda^n x - x^*x\| \leq \varepsilon.
\]
If \( \alpha > 0 \), choose \( n \) with \( 2^{n-1} \geq \alpha \). Then
\[
x^*h_\lambda^{2n} x \leq x^*h_\lambda^{2\alpha} x \leq x^*h_\lambda^\alpha x \leq x^* x.
\]
Combining these inequalities, by Corollary (58), we obtain
\[
\|x^*h_\lambda^n x - x^*h_\lambda^{2\alpha} x\| \leq \varepsilon, \quad \|x^*x - x^*h_\lambda^\alpha x\| \leq \varepsilon.
\]
Therefore,
\[
\|x - h_\lambda^n x\|^2 = \|(x^* - x^*h_\lambda^n)(x - h_\lambda^n x)\| = \|x^*x - 2x^*h_\lambda^n x + x^*h_\lambda^{2n} x\|
\leq \|x^*x - x^*h_\lambda^n x\| + \|x^*h_\lambda^n x - x^*h_\lambda^{2n} x\| \leq 2\varepsilon.
\]
\[\square\]
Remark 62. For the approximate unit constructed in Theorem (60) there is an easier way to see (2). Indeed, if we examine the last part of the proof of Theorem (60), we see, that there also exists an \( n \in \mathbb{N} \), such that \( \|x(1 - x^{\alpha/n})x\| \leq \varepsilon.\)
4. Representation Theory of C*-Algebras

4.1. Representations.

Definition 63. A pair \((H, \pi)\) is called a representation of a Banach *-algebra \(A\) if \(\pi: A \to B(H)\) is a *-homomorphism into the space of bounded operators on the Hilbert space \(H\).

Remark 64. Sometimes we will call just \(\pi\) a representation if it is clear what Hilbert space is meant.

We introduce some notions for representations.

Definition 65. Let \((H, \pi)\) be a representation of a Banach *-algebra \(A\).

- It is called faithful if \(\pi\) is injective.
- It is called irreducible if there are no non trivial closed invariant subspaces \(U\), i.e. \(\pi(A)(U) \subseteq U \Rightarrow U = \{0\} \lor U = H\).
- Let \(B \subseteq B(H)\) be a *-sub-algebra (i.e. a sub-algebra satisfying \(B^* \subseteq B\)). Consider the largest subspace \(N \subseteq H\) such that \(B|_N = 0\); i.e. \(N = \bigcap_{a \in B} \ker A\). Its orthogonal complement \(X := N^\perp\) is called the essential subspace of \(B\). If \(X = H\), we say that \(B\) acts nondegenerately on \(H\). The representation \(\pi\) is called nondegenerate if \(\pi(A) \subseteq B(H)\) acts nondegenerately on \(H\).
- \(\xi \in H\) is called a cyclic vector if \(\pi(A)\xi = H\). If there exists a cyclic vector \(\xi\), we call \((H, \pi, \xi)\) a cyclic representation.

Remark 66. If a representation \(\pi\) is irreducible, then it is also nondegenerate.

Irreducibility is formulated in terms of closed invariant subspaces. Since \(SU \subseteq \Sigma U\) for all bounded operators \(S\), we have \(\pi(A)U \subseteq U \Rightarrow \pi(A)\overline{U} \subseteq \overline{U}\). Hence, the only invariant non trivial subspaces of an irreducible representation must be dense in \(H\).

Lemma 67. \(B \subseteq B(H)\) acts nondegenerately if and only if \(\overline{BH} = H\).

Proof. If \(\overline{BH} = X \neq H\), then \(N := X^\perp \neq \{0\}\). Since \(X\) is invariant and \(B\) is a *-sub-algebra, \(N\) also is. To see this take \(x \in N\), then \((Bx, X) = (x, B^* X) = (x, BX)\)

where \(BX \subseteq X\), because \(X\) is invariant. It follows that \((Bx, X) = \{0\}\). Hence \(BN \subseteq N\), as \(BN \subseteq \overline{BH} = X^\perp N\), \(B\) acts degenerately. Now suppose \(B|_N = 0\) on some \(N \neq \{0\}\). Again, because \(N\) is invariant \(X := N^\perp \neq H\) also is. For \(x \in H\) we have \((Bx, N) = (x, B^* N) = \{0\}\). Hence, \(BH \subseteq X\) and further \(\overline{BH} \subseteq X = X\). \(\square\)

Before we give some characterizations of irreducibility we have to prove the von Neumann bicommutant theorem.

Definition 68. For a Hilbert space \(H\) and \(A \subseteq B(H)\) we define the commutant \(A^\prime := \{T \in B(H) : TS = ST\ \text{for all} \ S \in A\}\).

Remark 69. Obviously \(A^\prime \subseteq B(H)\) is a sub-algebra. If \(A\) is * closed, than \(A^\prime\) also is. Indeed \(a^* b = (b^* a)^* = (ab^*)^* = ba^*\) for \(a \in A^\prime\) and all \(b \in A\).

Moreover, we have \(A \subseteq A''\). Therefore, \(A^\prime \subseteq A''\). Since \(A''\) commutes with \(A'\), and hence also with \(A\), we get \(A'' = A''''\) and finally \(A^\prime = A''\).
Lemma 70. Let $A \subseteq B(H)$. Then the commutant $A'$ is closed in the weak topology. In particular, it is strongly closed.

Proof. Let $(T_i)_{i \in I} \in A'$ be a net, with $(T_i \xi, \eta) \rightarrow (T \xi, \eta)$ for all $\xi, \eta \in H$. We need to prove, that $T \in A'$. For an arbitrary $S \in A$, we get

$$(TS \xi, \eta) = \lim_{i \in I}(T_i S \xi, \eta) = \lim_{i \in I}(ST_i \xi, \eta) = \lim_{i \in I}(T_i S^* \eta) = (T \xi, S^* \eta) = (ST \xi, \eta).$$

By Lemma (7) $TS = ST$. □

Theorem 71. (Bicommutant) Let $H$ be a Hilbert space. If $B \subseteq B(H)$ is a *-subalgebra that acts nondegenerately, then $B$ is strongly dense in $B''$.

Proof. Let $T'' \in A''$. We have to show, that every strong neighborhood (cf. Remark (4))

$$N_{\epsilon,x_1,...,x_n}(T'') = \{ T \in B(H) : \| (T'' - T) \xi_j \| < \epsilon, j = 1, \ldots, n \}$$

contains an element of $B$. Fixing $\xi_1, \xi_2, \ldots, \xi_n \in H$ and $\epsilon > 0$ a $T \in B$ with

$$\max \left\{ \| (T'' - T) \xi_j \| : j = 1, \ldots, n \right\} < \epsilon$$

has to be constructed. Therefore, we consider the amplifications (cf. Definition (2))

$$H'' := \bigoplus_{j=1}^{n} H, S'' := \bigoplus_{j=1}^{n} S$$

for $S \in B(H)$. Moreover, we define $B'' = \{ S'' : S \in B \}$. Obviously $B''$ is a *-subalgebra of $B(H'')$ that acts nondegenerately. Now it suffices to show, that there is a $T'' \in B''$ such that

$$\| (T'' - (T'')) \xi_j \| < \epsilon$$

where $\xi = (\xi_1, \ldots, \xi_n)$.

For this consider $X = B'' \xi$. Since both $X$ and $X^\perp$ are invariant under $B''$, we have $P_X \in (B'')'$ for the orthogonal projection $P_X$ onto $X$. Since $(T'')^* \xi$ commutes with $P_X$, we get $(T'')^* X \subseteq X$. If we can show that $\xi \in X$ are we done, for then $(T'')^* \xi \in X$. For all $S \in B''$ we have

$$S[(1 - P_X) \xi] = (1 - P_X)[S \xi] = 0$$

since $S \xi \in X$. Because $B$ acts nondegenerately we may conclude $(1 - P_X) \xi = 0$, i.e. $\xi \in X$. □

We will see, that we can reduce our attention to projections if we want to know whether some $T \in B(H)$ belongs to a certain commutant.

Lemma 72. Let $B \subseteq B(H)$ be a *-subalgebra and $T \in B(H)$. Then $T \in B'$ if and only if the ranges of all spectral projections of $Re(T)$ and $Im(T)$ are $B$-invariant subspaces.

Proof. For $T \in B'$ we have $Re(T), Im(T) \in B'$ as they are linear combinations of $T, T^*$. It follows from Theorem (5), that their spectral projections $E(\Delta)$ are in $B'$. Therefore, their ranges are invariant subspaces. Conversely, if the ranges of all spectral projections are invariant subspaces, then all projections are in $B'$. Again by Theorem (5) we get $Re(T), Im(T) \in B'$. Hence, $T \in B'$. □

Now we have all that we need to prove the following characterizations of irreducible representations.
Theorem 73. Let \((\pi, H)\) be a representation of a \(C^*\)-algebra \(A\). The following assertions are equivalent:

1. \(\pi\) is irreducible.
2. \(\pi(A)' = C1\).
3. \(\pi(A)'' = B(H)\).
4. \(\pi(A)\) is strongly dense in \(B(H)\).
5. If \(\text{dim}(H) \neq 1\), then every \(0 \neq \xi \in H\) is a cyclic vector for \(\pi\).

Proof. Let start with 1) \(\iff\) 2). Suppose \(\pi\) is irreducible. If \(T \in \pi(A)'\), then by the previous lemma and the definition of irreducibility \(R(T)\) and \(I_m(T)\) are multiples of the identity. Thus \(T\) is a multiple of the identity. On the other hand if \(\pi(A)' = C1\), then again by the previous lemma the only invariant subspaces are trivial.

2) \(\implies\) 3) is trivial. For 3) \(\implies\) 2) note, that \(B(H)\) has no nontrivial invariant subspaces. By the previous lemma \(\pi'(A) = \pi(A)^{'''} = B(H)' = C1\).

4) \(\implies\) 4) Since 3) implies 1), this follows by the commutant theorem.

We will now prove 1) \(\iff\) 5). Let \(\pi\) be irreducible. Then for \(\xi \in H\), the closed invariant subspace \(\pi(A)\xi\) must be trivial. If \(\pi(A)\xi = \{0\}\), then \(C\xi\) is a closed invariant subspace and thus the whole space, so \(\text{dim}(H) = 1\).

For the other direction, consider a nonzero closed invariant subspace \(U \subseteq H\). If \(\text{dim}(H) = 1\), then trivially \(U = H\). If on the other hand every \(\xi\) is a cyclic vector, choose \(\xi \in U\). Then \(H = \pi(A)\xi \subseteq U\). □

Definition 74. Let \(A\) be a \(C^*\)-algebra and \((\pi_1, H_1), (\pi_2, H_2)\) representations. We call \(T \in B(H_1, H_2)\) an intertwining operator if

\[ T\pi_1(x) = \pi_2(x)T \]

for every \(x \in A\). The set of all intertwining operators is denoted \(R(\pi_1, \pi_2)\).

If \(U \in R(\pi_1, \pi_2)\) is unitary \((\pi_1, H_1), (\pi_2, H_2)\) are called unitarily equivalent. If in addition, there exist cyclic vectors \(\xi_1, \xi_2\) with \(U\xi_1 = \xi_2\), then \((\pi_1, H_1, \xi_1), (\pi_2, H_2, \xi_2)\) are called unitarily equivalent as cyclic representations.

4.2. Positive Functionals, States. In order to describe commutative \(C^*\)-algebras we used multiplicative functionals. This approach fails in general, since a multiplicative functional cannot distinguish between \(xy\) and \(yx\) for non commuting elements \(x, y \in A\). It turns out, that in order to describe a non-commutative \(C^*\)-algebra one has to introduce the more general notion of states.

Definition 75. Let \(A\) be a \(C^*\)-algebra. A functional \(\omega\) is called positive if \(\omega(x) \geq 0\) for all \(x \in A\). If it is normalized, i.e. \(\|\omega\| = 1\), then it is called a state. The set of states on \(A\) is denoted by \(S(A)\).

Example 76. By Lemma (24) any multiplicative functional on a commutative \(C^*\)-algebra is a state.

Remark 77. For general Banach *-algebras a functional is called positive if \(\omega(x^*x) \geq 0\) for all \(x \in A\). Indeed this defining property is the one that is needed in what follows (see Lemma (81)). However by Remark (53) we see that this definition is equivalent to the one given above in the case of \(C^*\)-algebras. Since we do not have this equivalence for general Banach *-algebras, a lot of the following proofs will not
work in that case. Nevertheless the majority of these results hold in more general settings as well, though the proofs are more elaborate.

Let us start by proving that every positive functional is automatically bounded.

**Lemma 78.** Let \( \omega \) be a positive functional, then there exists \( C > 0 \) such that \( \| \omega \| \leq C \).

It is an important fact, that any positive linear functional gives rise to a pre-inner product.

**Proof.** After decomposition (cf. Lemma (50)) it suffices to show that \( |\omega(x)| \leq C \|x\| \) for all \( x \in A_+ \) and a \( C > 0 \). Suppose there is no such \( C \). Then for every \( n \in \mathbb{N} \) there is an \( x_n \in A_+ \) with \( \|x_n\| = 1 \) and \( \omega(x_n) \geq 4^n \). We now consider \( x := \sum_{n=1}^{\infty} 2^{-n} x_n \). This is a well defined element of \( A \) satisfying \( \omega(x) \geq 2^{-n} \omega(x_n) \geq 2^n \) for any \( n \in \mathbb{N} \), which cannot be true.

**Definition 79.** \( (x, y)_\omega := \omega(y^* x) \) for any positive functional \( \omega \) on \( A \).

**Lemma 80.** Let \( \omega \) be a positive functional. Then \( \omega(x^*) = \overline{\omega(x)} \).

**Proof.** If \( x \in A_{sa} \), then \( x = x_+ - x_- \) with \( x_+, x_- \in A_+ \). We get
\[
\omega(x) = \omega(x_+ - x_-) = \omega(x_+) - \omega(x_-) \in \mathbb{R}.
\]
For \( x \in A \), \( x = a + ib \) with \( a, b \in A_{sa} \), we have
\[
\omega(x^*) = \omega(a - ib) = \omega(a) - i\omega(b) = \overline{\omega(x)}
\]
since \( \omega(a), \omega(b) \in \mathbb{R} \).

**Lemma 81.** \( (.,.)_\omega \) is a pre-inner product on \( A \).

**Proof.** \( (x, y)_\omega = \omega(y^* x) = \overline{\omega(x^* y)} = (y, x)_\omega \) by Lemma (80), and \( (x, x)_\omega = \omega(x^* x) \geq 0 \) by Theorem (52). The sesquilinearity is obvious.

Since we have shown that \( (.,.)_\omega \) is a pre-inner product, one gets the Cauchy-Schwarz inequality.

**Fact 82.** \( |\omega(y^* x)|^2 \leq \omega(x^* x)\omega(y^* y) \) for any \( x, y \in A \).

The next theorem gives a characterization of positive functionals on unital algebras.

**Theorem 83.** Let \( \omega \) be a linear functional on \( A \).

1. If \( \omega \geq 0 \), then \( \|\omega\| = \sup \{\omega(x) : x \geq 0, \|x\| \leq 1\} = \lim \omega(h_\lambda) \) for an approximate unit \( (h_\lambda)_{\lambda \in \Lambda} \). In particular, if \( A \) is unital \( \|\omega\| = \omega(1) = |\omega(1)| \).
2. If \( \omega_1, \omega_2 \geq 0 \), then \( \|\omega_1 + \omega_2\| = \|\omega_1\| + \|\omega_2\| \).
3. If \( A \) is unital, then \( \|\omega\| = \omega(1) \iff \omega \geq 0 \).

**Proof.** 1) We have
\[
\|\omega\| \geq \sup \{\omega(x) : x \geq 0, \|x\| \leq 1\} \geq \limsup \omega(h_\lambda).
\]
Now on the other hand, for every \( \varepsilon > 0 \) there is a \( \|y\| = 1 \) such that,
\[
\|\omega\|^2 - \varepsilon \leq |\omega(y)|^2 = \lim |\omega(h_\lambda^{1/2} y)|^2 \leq \liminf \omega(h_\lambda) \omega(y^* y) \leq \|\omega\| \liminf \omega(h_\lambda).
\]
2) From (1) it follows, that
\[
\|\omega_1 + \omega_2\| = \lim (\omega_1 + \omega_2)(h_\lambda) = \lim \omega_1(h_\lambda) + \lim \omega_2(h_\lambda) = \|\omega_1\| + \|\omega_2\|.
\]
3) It suffices to show, that \( \omega \) is positive on the C*-sub-algebras \( C^*(x) \) where \( x \in A^+ \) if \( \omega(1) = \|\omega\| \). We can use continuous functional calculus for normal elements, to identify \( C^*(x) \cong C(\sigma(x)) \). Therefore we may use the Riesz representation theorem, to identify \( \omega \) with a complex measure \( \mu \). \( \omega(1) = \|\omega\| \) yields \( \|\mu\| = \mu(\sigma(x)) \), which is only possible for \( \mu \geq 0 \). Hence, \( \omega \geq 0 \). □

An important consequence is, that any state on \( A \) uniquely extends to a state on \( A_1 \).

**Fact 84.** Let \( \omega \) be a positive linear functional on the non unital C*-algebra \( A \). Then by \( \omega(1) = \|\omega\| \), it extends to a positive functional on \( A_1 \). In particular, the extension of \( \omega \in S(A) \) belongs to \( S(A_1) \).

The following estimate, will be useful later on.

**Corollary 85.** If \( x \in A \), then \( |\omega(x)|^2 \leq \omega(x^*x) \) for a state \( \omega \).

**Proof.** \( \omega \) extends to \( A_1 \). So one can use Lemma (82) with \( y = 1 \). □

4.3. The GNS Construction. In this section we use the fact, that positive functionals induce pre-inner products to obtain representations. For every positive functional \( \omega \), the so called Gelfand-Naimark-Segal construction, yields a representation \( \pi_\omega \) of the C*-algebra.

**Theorem 86.** Let \( \omega \) be a positive functional on \( A \).

1) \( N_\omega := \{ x \in A : \omega(x^*x) = 0 \} \) is a left ideal in \( A \). The pre-inner product \((.,.)_\omega\) induces a well-defined inner product on \( A/N_\omega \) by \( (x+N_\omega,y+N_\omega)_\omega := (x,y)_\omega \). Its completion \( (H_\omega,(.,.)_\omega) \) is a Hilbert space.

2) Let \( \pi_\omega(x)(y+N_\omega) := xy+N_\omega \) for \( x,y \in A \). Then \( \pi_\omega(x) \) is a bounded operator on \( A/N_\omega \) with \( \|\pi_\omega(x)\| \leq \|x\| \). It follows that it extends to a bounded operator on \( H_\omega \) which we will also call \( \pi_\omega(x) \).

3) The map \( \pi_\omega : A \rightarrow B(H_\omega) \) is a representation, called the GNS representation of \( A \) associated to \( \omega \).

**Proof.** 1) Let us start by showing that \( N_\omega \) is a closed left ideal in \( A \). It is obviously closed since it is the zero-set of a continuous map. For \( y,z \in A \) and \( x \in N_\omega \) using Lemma (82) we obtain

\[
|\omega((yx)^*yz)|^2 = |\omega(x^*y^yxz)|^2 \leq \omega(x^*y^y(x^*y^y)^*)\omega(x^*x) = 0.
\]

From \( |\omega(x^*z)|^2 \leq \omega(z^*z)\omega(x^*x) = 0 \) we infer

\[
\omega((x+z)^*(x+z)) = \omega(x^*x + x^*z + z^*x + z^*z) = \omega(x^*z) + \omega(z^*x) + \omega(z^*z) = \omega(z^*z).
\]

It follows that \( N_\omega \) is a left ideal (for \( N_\omega + N_\omega = N_\omega \) one chooses \( z \in N_\omega \) in the above equation) and by employing the polar formula that the pre-inner product is well defined by acting on the representatives. From

\[
0 = \omega((x+N_\omega)^*(x+N_\omega)) = \omega((x^*+N_\omega)(x+N_\omega)) = \omega(x^*x) \Rightarrow x+N_\omega = 0 + N_\omega
\]

one concludes that \((.,.)_\omega\) is defined on \( A/N_\omega \).

2) Since \( N_\omega \) is a left ideal \( \pi_\omega(x) \) is a well defined operator on \( N/N_\omega \). Applying \( \omega \) to

\[
y^*x^*xy \leq \|x\|^2y^*y
\]

yields

\[
\|\pi_\omega(x)\| \leq \|x\|.
\]
3) $\pi_\omega$ is obviously a homomorphism. Furthermore
\[
(\pi_\omega(x^*)(y + N_\omega), z + N_\omega) = \omega(z^*x^*y) = (y + N_\omega, \pi_\omega(x)(z + N_\omega))_\omega
\]
\[
= (\pi_\omega(x)^*(y + N_\omega), z + N_\omega)_\omega.
\]
Using Lemma (7), we get $\pi_\omega(x^*) = \pi_\omega(x)^*$.

We are going to see that representations constructed in this way are always cyclic and unique in a certain sense.

**Theorem 87.** Let $(H_\omega, \pi_\omega)$ be the GNS-representation of $A$ associated to $\omega$.

1. There is a vector $\xi_\omega \in H_\omega$ satisfying $\pi_\omega(x)\xi_\omega = x + N_\omega$ and $\omega(x) = (\pi_\omega(x)\xi_\omega, \xi_\omega)$. $\xi_\omega$ is cyclic for $\pi_\omega$ with $\|\xi_\omega\|^2 = \|\omega\|$.

2. Let $\pi$ be a representation of $A$ on $B(H)$ with a $\xi$ cyclic for $\pi$ and $\omega(x) = (\pi(x)\xi, \xi)$. Then $(H, \pi, \xi)$ is unitarily equivalent to $(H_\omega, \pi_\omega, \xi_\omega)$.

**Proof.** 1) In the unital case $\xi_\omega$ is simply $1 + N_\omega$. In the general case one considers the extension $\omega_1$ of $\omega$ to $A_1$ with $\|\omega_1\| = \|\omega\|$. By Theorem (83),
\[
\omega_1(1) = \|\omega\| = \lim \omega(h_\lambda) = \lim \omega(h_\lambda^2).
\]
Hence,
\[
\lim ||1 - h_\lambda||^2_{H_{\omega_1}} = \lim \omega_1((1 - h_\lambda)^2) = 0.
\]
It follows that $A/N_\omega$ is dense in $A_1/N_{\omega_1}$ and $H_\omega$ can be identified with $H_{\omega_1}$. Now $\xi_\omega = \xi_{\omega_1}$ satisfies $\omega(x) = (\pi_\omega(x)\xi_\omega, \xi_\omega)_H$.

2) If there was a unitary operator establishing a unitary equivalence, it must certainly send $\pi(x)\xi$ to $\pi_\omega(x)\xi_\omega$. So we define $U : \pi(A)H \to \pi_\omega(A)H_\omega$ by $U\pi(x)\xi = \pi_\omega(x)\xi_\omega$. One calculates
\[
(U\pi(x)\xi, U\pi(y)\xi)_H = (\pi_\omega(x)\xi_\omega, \pi_\omega(y)\xi_\omega)_H = (\pi_\omega(y^*x)\xi_\omega, \xi_\omega)_H =
\]
\[
\omega(y^*x) = (\pi(y^*x)\xi, \xi)_H = (\pi(x)\xi, \pi(y)\xi)_H.
\]
Thus $U$ is an isometry. In particular, it is well defined. Since $\xi, \xi_\omega$ are cyclic, it uniquely extends to a unitary operator also denoted by $U$ from $H$ to $H_\omega$. For all $x, y \in A$ we have
\[
\pi_\omega(x)U\pi(y)\xi = \pi_\omega(xy)\xi_\omega = U\pi(xy)\xi = U\pi(x)y_\omega\xi.
\]
Hence, by a density argument $\pi_\omega(x)U = U\pi(x)$ for all $x \in A$ and $U$ establishes a unitary equivalence.

**Remark 88.** As mentioned in Remarks (53, 77) a lot of the things we have done do not work for Banach $*$-algebras. However, with some effort the results of this section may be obtained for Banach $*$-algebras with a bounded approximate identity. Central in this considerations is the Cohen-Hewitt Factorisation Theorem. For more details see [4].
4.4. The Space of States, Pure States. As mentioned above, so far we did not need all the structure that comes with a C*-algebra. We have seen that for every state there exists a representation but this representation is not necessarily faithful. In order to get a faithful representation we will consider direct sums of GNS representations. If there are sufficiently many states this approach will work. Indeed, this is the point where we need most of the structure of C*-algebras.

We will start with a Banach-Alaoglu like theorem.

Theorem 89. If $A$ is a unital C*-algebra, then $(\mathcal{S}(A), \sigma(S(A), \iota(A)))$ is a compact convex subset of the dual space $A^*$.\[\]

Proof. The proof works similar to Theorem (25) or Banach-Alaoglu respectively. For $x \in A_+$ consider the maps

$$\iota_x : A \to \mathbb{C}; \phi \mapsto \phi(x).$$

Using these maps, we get

$$\mathcal{S}(A) = K_1^{\mathcal{A}}(0) \cap \{\phi \in \mathcal{A}^* : \phi(1) = 1\} \cap \bigcap_{x \in A_+} \iota_x^{-1}([0, \infty)).$$

Because the $\iota_x$ are continuous, this is an intersection of w*-closed sets. Hence, $\mathcal{S}(A)$ is also w*-closed. By Banach-Alaoglu, $K_1^{\mathcal{A}}(0)$ is compact. Hence $\mathcal{S}(A)$ is compact.

To see, that $\mathcal{S}(A)$ is convex, note, that the sum of positive functionals is always a positive functional. Moreover, by Theorem (83)

$$\|\lambda \omega_1 + (1 - \lambda) \omega_2\| = \lambda \|\omega_1\| + (1 - \lambda) \|\omega_2\| = 1$$

if $\omega_1, \omega_2 \in \mathcal{S}(A)$. Therefore, a convex combination of states is always a state.\[\]

The next theorem one could understand as a version of the algebraic Hahn-Banach theorem for states.

Theorem 90. If $B \subseteq A$ is C*-sub-algebra and $\omega$ is a state on $B$, then $\omega$ extends to a state $\omega'$ on $A$.\[\]

Proof. By the Hahn-Banach theorem there is an extension $\omega'$ with norm 1. If $B$ is unital, then by Theorem (83) $\omega'$ is a state since $\omega'(1) = \omega(1) = 1$. If $B$ is not unital, one extends $\omega$ to $B_1 \subseteq A_1$ and then there is an extension $\omega'$ on $A_1$. By restricting to $A$ one gets the wanted extension.\[\]

If the set of positive functionals would fully describe a C*-algebra, obviously the subset of states also would. As it will eventually turn out, there is an even smaller subset sufficient for that task.

Definition 91. A state $\omega$ is called pure if it is an extrem point of $S(A)$, i.e. $\omega = \lambda \omega_1 + (1 - \lambda) \omega_2$ with $0 < \lambda < 1$ and $\omega_1, \omega_2 \in S(A)$, yields $\omega_1 = \omega_2 = \omega$. The set of pure states is denoted $P(A)$.

Lemma 92. A state $\omega$ is pure, if and only if all positive $\omega^-$ with $\omega^- \leq \omega$ are multiples of $\omega$. Here $\omega^- \leq \omega$ mean, that $\omega^-(x) \leq \omega(x)$ for all $x \in A_+$.\[\]

Proof. Let $\omega$ be pure and $\omega^- \leq \omega$. Since $\|\omega - \omega^-\| = \|\omega\| - \|\omega^-\|$ (cf. Theorem (83),(2)), we may assume, that $1 = \|\omega\| > \|\omega^-\|$. We get

$$\omega = \|\omega^-\| \frac{\omega^-}{\|\omega^-\|} + (1 - \|\omega^-\|) \frac{(\omega - \omega^-)}{(1 - \|\omega^-\|)}$$


where by Theorem \((83),(2), \frac{\omega - \omega^*}{\|\omega - \omega^*\|}\) is a state. Thus \(\frac{\omega - \omega^*}{\|\omega - \omega^*\|} = \omega\).

Now let all \(\omega^* \leq \omega\) be multiples of \(\omega\). Consider a convex combination \(\omega = \lambda \omega_1 + (1 - \lambda) \omega_2\) with \(0 < \lambda < 1\). It suffices to show, that \(\omega_1 = \omega\). Obviously \(\lambda \omega_1 = \omega - (1 - \lambda) \omega_2 \leq \omega\). Thus \(\omega_1\) is a multiple of \(\omega\) and since it is a state, \(\omega_1 = \omega\).

**Theorem 93.** Let \(A\) be a C*-algebra and \(B \subseteq A\) a non trivial C*-sub-algebra. If \(\omega\) is a pure state on \(B\), then the set \(\text{ext}(\omega)\) of extensions of \(\omega\) to a state on \(A\) is a compact convex subset of \(S(A)\) and any extremal point of this set is a pure state on \(A\).

**Proof.** Since \(S(A)\) is convex, \(\text{ext}(\omega)\) obviously also is. For compactness, note, that

\[\text{ext}(\omega) = K^*_1(0) \cap \{ \phi \in A^* : \phi|_B = \omega \} \cap \bigcap_{x \in A^*_+} \mathcal{t}^{-1}_x([0, \infty)).\]

Indeed, every functional on the right hand side \(\phi\) satisfies \(\|\phi\| = 1\). This follows from (cf. Theorem \((83)\))

\[\|\phi\| = \sup \{ \omega(x) : x \geq 0, \|x\| \leq 1, x \in A \} \geq \sup \{ \omega(x) : x \geq 0, \|x\| \leq 1, x \in B \} = \|\omega\| = 1.\]

It is easy to see, that \(\{ \phi \in A^* : \phi|_B = \omega \}\) is weakly closed. Hence, \(\text{ext}(\omega)\) is compact.

Let \(\omega'\) be an extremal point of \(\text{ext}(\omega)\). If \(\omega' = \lambda \omega_1 + (1 - \lambda) \omega_2\) for \(\omega_1, \omega_2 \in S(A)\), then \(\omega_1|_B = \omega_2|_B = \omega'\), because \(\omega\) is a pure state, i.e. \(\omega_1, \omega_2 \in \text{ext}(\omega)\). Since \(\omega'\) is an extremal point of \(\text{ext}(\omega)\), we get \(\omega_1 = \omega_2 = \omega'\) on \(A\). Thus \(\omega'\) is a pure state. \(\square\)

**Remark 94.** Since \(\text{ext}(\omega)\) is compact, by the Krein-Milman theorem there always exist extremal points (cf. [2], Section 5).

**Corollary 95.** For a C*-algebra \(A\) and \(x \in A\) there exists a pure state \(\omega\) and a vector \(\xi \in H_\omega\) with \(\|\xi\| = 1\), such that \(\|\pi_\omega(x)\xi\|_{H_\omega} = \|x\|\). In particular, \(\|\pi_\omega(x)\| = \|x\|\).

**Proof.** Step 1: If \(0 \neq x \in A_{\text{sa}}\), then there is a pure state \(\omega\) with \(\|\omega(x)\| = \|x\|\).

One considers the C*-algebra \(C^*(x)\) which is isometrically isomorphic to \(C(\sigma(x))\). Let \(x\) be represented by a real valued \(f\). Then \(f\) takes its maximum for some \(\lambda \in \sigma(x)\). \(t_\lambda : g \to g(\lambda)\) is a pure state on \(C(\sigma(x))\) satisfying \(\|t_\lambda(f)\| = \|f\|\). So there is a pure state on \(C^*(x)\) satisfying the conditions. By Theorem \((93)\) and its subsequent remark, it can be extended to a pure state \(\omega\) on \(A_1\). \(x \in A\) implies, that \(\omega_{|A}\) is a pure state on \(A\) and \(\omega_{|A}\) is even a pure state on \(A\). Indeed, if \(\omega_{|A} = \lambda \omega_1 + (1 - \lambda) \omega_2\), then the unique extensions of \(\omega_1, \omega_2\) to \(A_1\) would give the same convex combination for \(\omega\). Thus \(\omega_1 = \omega_2 = \omega_{|A}\).

Step 2: By Step 1 there exists a pure state \(\omega\) with \(\omega(xx^*) = \|xx^*\| = \|x\|^2\). We get

\[\|\pi_\omega(x)\xi_\omega\|^2_{H_\omega} = (\pi_\omega(xx^*)\xi_\omega, \xi_\omega)_{H_\omega} = \omega(xx^*) = \|x\|^2.\]

\(\square\)
4.5. The Theorem of Gelfand-Naimark. As announced above considering direct sums of GNS-representations, we get a faithful representation.

Theorem 96. For every C*-algebra $A$ there is an isometric representation $\pi$.

Proof. By Corollary (95) and the fact that $\left\| \bigoplus_{\omega \in P(A)} \pi_{\omega}(x) \right\| = \sup_{\omega \in P(A)} \| \pi_{\omega}(x) \|$, we get a faithful representation.

This result is quite remarkable since it states, that every C*-algebra is basically nothing else but an operator algebra. The next few results are concerned with showing, that this representation is a direct sum of irreducible representations. Therefore, we will prove, that a GNS representation is irreducible if and only if the associated state is pure.

At first we make a trivial observation.

Corollary 97. If $\omega$ is a positive functional on $A$, then by $\omega(T) = (T\xi_\omega, \xi_\omega)$ it induces a positive functional with the same norm on $B(H_\omega)$. In particular, a state induces a state.

Proof. We have $\| (T\xi_\omega, \xi_\omega) \| \leq \| T \| \cdot \| \xi_\omega \|^2 = \| T \| \cdot \| \omega \|$. Equality follows, if we set $T = I$.

The next result may be seen as an operator theoretic version of the Radon-Nikodym theorem.

Theorem 98. Given a C*-algebra $A$ let $\omega_1 \leq \omega_2$ (i.e. $\omega_1(x) \leq \omega_2(x)$ for all $x \in A_+$) be two positive functionals on $A$. Then there is an unique operator $T \in \pi_{\omega_2}(A)'$ with $0 \leq T \leq 1$, such that $\omega_1(x) = \omega_2(T\pi_{\omega_2}(x))$ for all $x \in A$.

Proof. We start by defining a pre-inner product that relates $\omega_1$ to $\omega_2$. In fact, $[\pi_{\omega_2}(x)\xi_{\omega_2}, \pi_{\omega_2}(y)\xi_{\omega_2}] := \omega_1(y^*x)$ is a pre-inner product on $\pi_{\omega_2}(A)\xi_{\omega_2}$. To see, that it is well defined, take $x \in A$, such that $x + N_{\omega_2} = \pi_{\omega_2}(x)\xi_{\omega_2} = 0$. Hence, $x \in N_{\omega_2}$, i.e. $\omega_2(x^*x) = 0$. Using $\omega_1 \leq \omega_2$, we obtain

$$\omega_1(x^*x) = 0.$$ 

Hence, $|\omega_1(y^*x)| \leq \omega_1(x^*x) \omega(y^*y)$ $= 0$. Let $z \in A$. Then,

$$|[\pi_{\omega_2}(z)\xi_{\omega_2}, \pi_{\omega_2}(z)\xi_{\omega_2}]|^2 = |\omega_1(z^*z)|^2 \leq |\omega_2(z^*z)|^2 \leq \omega_2(z^*z)\omega_2(z^*z) =$$

$$= (\pi_{\omega_2}(z)\pi_{\omega_2}(z)\xi_{\omega_2}, \xi_{\omega_2})_{H_{\omega_2}} (\pi_{\omega_2}(z)^*\pi_{\omega_2}(z)\xi_{\omega_2}, \xi_{\omega_2})_{H_{\omega_2}} =$$

$$= \| \pi_{\omega_2}(z)\xi_{\omega_2} \|^2 \| \pi_{\omega_2}(z)\xi_{\omega_2} \|^2.$$

Since $[.,.]$ is hermitian, $||[.,.]|| = \sup_{||z|| = 1} ||[z,z]|| \leq 1$. By continuity this pre-inner product extends to $H_{\omega_2}$.

The Lax-Milgram Theorem (cf. Proposition 3.2.6 in [2]), asserts, that there exists a unique operator $T \in B(H_{\omega_2})$ with $[\eta, \xi] = (T\eta, \xi)_{H_{\omega_2}}$ for all $\eta, \xi \in H_{\omega_2}$. By the calculation above we get $0 \leq T \leq 1$. One has $\omega_1(x) = [\pi_{\omega_2}(x)\xi_{\omega_2}, \xi_{\omega_2}] = (T\pi_{\omega_2}(x)\xi_{\omega_2}, \xi_{\omega_2})_{H_{\omega_2}} = \omega_2(T\pi_{\omega_2}(x))$ for all $x \in A$ (cf. Corollary (97)). To show $T \in \pi_{\omega_2}(A)'$ we calculate

$$(T\pi_{\omega_2}(x)[\pi_{\omega_2}(z)\xi_{\omega_2}], \pi_{\omega_2}(y)\xi_{\omega_2})_{H_{\omega_2}} = \omega_1(y^*(xz)) = \omega_1((x^*y)^*z) =$$
Proof. Let \( \pi_\omega \) be irreducible. By the previous theorem every \( \omega' \leq \omega \) can be written as \( \omega'(x) = \omega(T\pi_\omega(x)) \) with \( T \in \pi_\omega(A) \). Irreducibility gives \( T = \text{cf} \) according to Lemma (73). Therefore, \( \omega' \) is a multiple of \( \omega \) and by Lemma (92) \( \omega \) is pure.

Conversely, for a pure \( \omega \), suppose there is a non trivial projection \( P \in \pi_\omega(A)' \). We obtain \( P\xi_\omega \neq 0 \), for otherwise

\[
P[\pi_\omega(x)\xi_\omega] = \pi_\omega(x)P\xi_\omega = 0
\]

for all \( x \in A \). Since \( \xi_\omega \) is a cyclic vector, we get \( P = 0 \). Analogously, \( (1-P)\xi_\omega \neq 0 \). Therefore, we can decompose \( \omega \) into the nonzero functionals (cf. Corollary (97))

\[
\omega_1(x) := (\pi_\omega(x)P\xi_\omega, P\xi_\omega)_{H_\omega} = \omega(P\pi_\omega(x))
\]

\[
\omega_2(x) := (\pi_\omega(x)(1-P)\xi_\omega, (1-P)\xi_\omega)_{H_\omega} = \omega((1-P)\pi_\omega(x)), \ \omega = \omega_1 + \omega_2
\]

These are positive because \( \pi_\omega(x) \) is a positive operator for \( x \in A_+ \). Since \( \omega \) is pure, due to Lemma (92), there exists \( 0 \leq \lambda \leq 1 \) with \( \omega_1 = \lambda\omega \) and \( \omega_2 = (1-\lambda)\omega \).

As \( \omega_1, \omega_2 \neq 0 \) we have \( \lambda \neq 0, 1 \). Using again, that \( \xi_\omega \) is a cyclic vector, for every \( \epsilon > 0 \), we find \( x \in A \) such that

\[
||\pi_\omega(x)\xi_\omega - P\xi_\omega||_{H_\omega} \leq \epsilon \quad \text{and} \quad ||\pi_\omega(x)\xi_\omega||^2_{H_\omega} = ||P\xi_\omega||^2_{H_\omega}.
\]

It follows that

\[
||(1-P)\pi_\omega(x)\xi_\omega||_{H_\omega} = ||(1-P)(\pi_\omega(x)\xi_\omega - P\xi_\omega)||_{H_\omega} \leq \epsilon.
\]

Thus we have

\[
(1-\lambda)||P\xi_\omega||^2_{H_\omega} = (1-\lambda)\omega(x^*x) = \omega_2(x^*x) = ||(1-P)\pi_\omega(x)\xi_\omega||^2_{H_\omega} \leq \epsilon^2.
\]

This is a contradiction to \( \lambda \neq 1 \) and by Lemma (72) we conclude \( \pi_\omega(A)' = C_1 \).

Using Theorem (73) we see that \( \pi_\omega \) is irreducible. \( \square \)

We now get, that the representation in Corollary (96) is a direct sum of irreducible ones.

**Theorem 100.** The representation \( (\oplus_{\omega \in P(A)} H_\omega, \oplus_{\omega \in P(A)} \pi_\omega) \) called the universal representation is isometric and a direct sum of irreducible representations.

To end this section we will look once again at commutative \( C^* \)-algebras. Using the characterization of irreducible GNS representations we can show that for a commutative \( A \) the Gelfand space \( A \) coincides with \( P(A) \).

**Lemma 101.** Let \( A \) be commutative. A functional \( \omega \) is a pure state if and only if it is multiplicative.

**Proof.** As seen in Example (76) any multiplicative functional \( \omega \) is a state. Since \( \omega \) is multiplicative we have \( H_\omega = A \setminus J_\omega = A \setminus \ker \omega \cong C \), as the co-dimension of \( \ker \omega \) is equal to 1. By Theorem (73) \( \pi_\omega \) is irreducible and in turn \( \omega \) is pure.
Now suppose on the other hand, that \( \omega \) is pure. We have \( \pi_\omega(A) \subseteq \pi_\omega(A)' \), since \( A \) is commutative. By Theorem (73), one gets \( \pi_\omega(A)' = CI \) and therefore \( \pi_\omega(A) = CI \). Furthermore,

\[
\omega(x) = (\pi_\omega(x) \xi_\omega, \xi_\omega)_{H_\omega} = (c \xi_\omega, \xi_\omega)_{H_\omega} = c
\]

yields \( \pi_\omega(x) = \omega(x)I \). Hence,

\[
\omega(xy) = (\pi_\omega(x) \pi_\omega(y) \xi_\omega, \xi_\omega)_{H_\omega} = \omega(x)\omega(y)(\xi_\omega, \xi_\omega)_{H_\omega} = \omega(x)\omega(y).
\]

\[\square\]

**Example 102.** It is instructive to derive the representation of commutative C*-algebras by the Gelfand-transform from non commutative representation theory. We now know, that the set of pure states coincides with the Gelfand space. We have \( H_\omega \simeq \mathbb{C} \) and \( \pi_\omega(x)(z) = \omega(x)z \). Therefore, the universal representation acts on \( H = \prod_{\omega \in \hat{A}} \mathbb{C} \) as pointwise multiplication by \( \omega(x) \). Thus it is given by \( f : \hat{A} \to \mathbb{C}, \omega \mapsto \omega(x) \).
References