Bachelorarbeit

De Branges' Theorem on Weighted Polynomial Approximation

Daniel Hainschink

2016S

Contents

Introduction

The subject of this thesis is de Branges' theorem on the classical *Bernstein problem* on weighted polynomial approximation. If, for a lower semicontinuous function $W : \mathbb{R} \to (0, \infty]$, referred to as the *weight function*, we let $\mathrm{C}_0(W)$ be the space of all $f \in \mathrm{C}(\mathbb{R})$ with $f(x)/W(x) \to 0$ as $x \to \pm\infty$, with the norm $\|f\| = \|f/W\|_\infty$, the simplest instance of the Bernstein problem may be stated as:

For which weight functions W are the polynomials $\mathcal{P}$ dense in $\mathrm{C}_0(W)$?

Classical results on the Bernstein problem usually rely on the *Hall majorant*

$$\mathfrak{m}(z) = \sup\{|f(z)| : f \in \mathcal{P}, \|f\|_W \leq 1\}, \tag{I.1}$$

i.e. the norm of the point evaluation at z on $\mathcal{P}$. Characterizations of denseness include (cf. [5, pp. 147,158,164]):

1) $\mathfrak{m}(z) = \infty$ for some (all) $z \in \mathbb{C} \setminus \mathbb{R}$. (Mergelyan)
2) $\int \frac{\log \mathfrak{m}(x)}{1+x^2}\, dx = \infty$. (Akhiezer)
3) $\sup\{\int \frac{\log |f(x)|}{1+x^2}\, dx : f \in \mathcal{P}, \|f\|_W \leq 1\} = \infty$. (Pollard)

The key idea here is that if $\overline{\mathcal{P}} \neq \mathrm{C}_0(W)$, then $\mathfrak{m}$ is everywhere finite, and the elements of $\overline{\mathcal{P}}$ actually extend to entire functions, with the point evaluation on $\overline{\mathcal{P}}$ still being bounded by $\mathfrak{m}$.

Before stating de Branges' theorem, let us remark that only a handful of properties of the class of polynomials are relevant to most of the theory of the Bernstein problem, so that it is natural to work with the following generalization:

I.1 Definition. A linear space $\mathcal{L}$ of entire functions is called an *algebraic de Branges space* if

1) If $F \in \mathcal{L}$, then so is $F^\sharp(z) = \overline{F(\overline{z})}$.
2) If $F \in \mathcal{L}$ and $F(w) = 0$, then $F(z)/(z - w)$ is in $\mathcal{L}$.

This definition includes, in particular, the space $\mathcal{E}_a$ of all entire function of exponential type $\leq a$.

While the above extension property is also relevant to the proof of de Branges' theorem, the motivation behind de Branges' theorem comes from the Hahn-Banach characterization of non-denseness in $\mathrm{C}_0(W)$: If we identify the dual of $\mathrm{C}_0(W)$ with the space $\mathrm{M}(\mathbb{R})$ of complex measures on $\mathbb{R}$ via $(T\mu)f := \int fW^{-1}\, d\mu$ (as is done in Theorem 2.1), then the theorem may be interpreted as a description of the annihilating measures of a given algebraic de Branges space $\mathcal{L} \subset \mathrm{C}_0(W)$ in terms of particular entire functions:

I.2 Definition. The *weighted Krein class* $\mathcal{K}(\mathcal{L}, W)$ consist of all entire functions B such that:

1) $B = B^\sharp$. B has at least one zero, and all zeros of B are real and simple.
2) For $F \in \mathcal{L}$, F/B is of bounded type in the half planes $\mathbb{C}^+, \mathbb{C}^-$, and $yF(iy) = o(B(iy))$ as $y \to \pm\infty$.
3) $\sum_{B(x)=0} \left|\frac{W(x)}{B'(x)}\right| < \infty$.

Given $B \in \mathcal{K}(\mathcal{L}, W)$, condition 3) ensures that

$$\mu = \sum_{B(x)=0} \frac{1}{B'(x)} \delta_x \tag{I.2}$$

defines a complex measure on $\mathbb{R}$ with $\mathcal{L} \subset \mathrm{L}^1(\mu)$. Condition 2) implies that for $F \in \mathcal{L}$, the interpolation formula

$$\frac{(z - x_0)F(z)}{B(z)} = \sum_{B(x)=0} \frac{(x - x_0)F(x)}{B'(x)(z - x)}, \quad x_0 \in \mathbb{R}, B(z) \neq 0, \tag{I.3}$$

holds. To see this, note that both functions are meromorphic in $\mathbb{C}$ with the same simple poles and residues. The left hand side is of bounded type in both half planes and tends to 0 along both imaginary axes by condition 2), and the right hand side has the same properties by Lemma A.6 (being the Cauchy integral of the measure $\sum(x-x_0)F(x)/B'(x)\delta_x$) and dominated convergence. It follows that the difference is an entire function in $\mathrm{N}(\mathbb{C}^\pm)$ tending to 0 along the imaginary axes, which must vanish identically by Lemma A.10. If we choose $x_0 \in \mathbb{R}$ with $B(x_0) \neq 0$ and let $z = x_0$, (I.3) turns into $0 = \int F(x)\, d\mu(x)$, so that $\mu \in \mathcal{L}^\perp$.

We have just shown that $\mathcal{K}(\mathcal{L}, W) \neq \varnothing$ implies the non-denseness of $\mathcal{L}$, which is the simple part of de Branges' theorem:

I.3 De Branges' Theorem. *Let $W : \mathbb{R} \to (0, \infty]$ be lower semicontinuous. An algebraic de Branges space $\mathcal{L} \subset \mathrm{C}_0(W)$ is dense in $\mathrm{C}_0(W)$ if and only if $\mathcal{K}(\mathcal{L}, W) \neq \varnothing$.*

We present two proofs of the necessity of $\mathcal{K}(\mathcal{L}, W) \neq \varnothing$: The original proof of de Branges [3], which constructs the function B from an extremal annihilating measure, using a lemma on extremal measures from [4], and a more recent proof by Sodin and Yuditskii [11], based on a normal family argument, which makes no reference to annihilating measures. Both proofs were orginally formulated for the case of polynomials; the generalizations to algebraic de Branges spaces we present are contained in [1] and [12], respectively.

Before we actually give the proofs in Sections 3 and 4, respectively, we develop some of the theory of approximation by analytic functions in regular function spaces according to Pitt [7] in Section 1, and give Summers' [13] description of the dual space of $\mathrm{C}_0(W)$ as a space of complex measures and de Branges' lemma [4] on extremal annihilating measures in Section 2. For completeness, a few results on entire functions of exponenial and bounded type are given in Appendix A.

Notation

1_Ω	indicator function of Ω, $1_\Omega(x) = 1$ for $x \in \Omega$, 0 otherwise
$\\|f\\|_\Omega$	for $f : X \to \mathbb{C}$ and $\Omega \subset X$, $\\|f\\|_\Omega = \sup_{x\in\Omega} \|f(x)\|$
$\mathrm{C}_0(W)$	weighted C_0-space, page 2
$\mathrm{C}^n(\mathbb{R})$ $[\mathrm{C}^n_c(\mathbb{R})]$	n times differentiable functions [with compact support]
$\mathbb{C}^+, \mathbb{C}^-$	upper and lower half planes, $\mathbb{C}^\pm = \{z \in \mathbb{C} : \pm \operatorname{Im} z > 0\}$
δ_x	Dirac measure $\delta_x(A) = 1$ if $x \in A$, 0 otherwise
$\mathbb{D}, \mathbb{D}(w, r)$	complex unit disc; disc with radius r about w
$\mathscr{D}$	test functions in the sense of distribution theory, cf. e.g. [9, p. 151]
$\mathrm{H}(\Omega)$	holomorphic functions in an open set $\Omega \subset \mathbb{R}$
iff	if and only if
$\mathcal{L}$	algebraic de Branges space, page 2
lsc	lower semicontinuous
$\mathrm{M}(X)$	finite complex Borel measures on X, page 10
$\mathfrak{m}(z)$	Hall majorant in a regular function space, page 8
$\mathrm{N}(\Omega)$	holomorphic functions of bounded type in Ω, page 19
$\mathcal{R}(\Omega)$	holomorphic functions with positive imaginary part in Ω, page 16
R_w	resolvent operator $R_w f(z) = (z-w)^{-1} f(z)$, page 7
$\mathcal{R}_w, \mathcal{R}^1_w$	difference quotient operator $\mathcal{R}_w[F, G](z) = (z-w)^{-1}(F(z)G(w) - F(w)G(z))$ on $\mathcal{L}$; corresponding operator on $\mathrm{L}^1(\mu)$, page 13
W	lower semicontinuous weight function, page 2
$Z(F)$	zero set of an entire function, $Z(F) = \{z \in \mathbb{C} : F(z) = 0\}$

1 Approximation in Regular Function Spaces

We first give results by Pitt [7] implying that for algebraic de Branges subspaces of a large class of function spaces (including $\mathrm{C}_0(W)$, but also $\mathrm{L}^1(\mu)$), non-denseness implies boundedness of the Hall majorant and the extendability of functions in the closure to entire functions, whose quotients are of bounded type in both half-planes.

1.1 Regular Function Spaces

The following concept of function space from [7, Sec. 2] allows us to prove results that will be needed for both $\mathrm{C}_0(W)$ and $\mathrm{L}^1(\mu)$ simultaneously. Let $\mathrm{C}^n(\mathbb{R})$ be endowed with the topology of locally uniform convergence of the first n derivatives.

1.1 Definition. A Banach space B together with a subspace B_0 of $\mathrm{C}^{N_0}(\mathbb{R})$ and a linear map $B_0 \to B$ (in general neither continuous nor injective) is called a *regular function space* if

1) B_0 is dense in B.
2) B_0 is *preclosed* in B, i.e. if $f_n \in B_0$ converge to 0 in $\mathrm{C}^{N_0}(\mathbb{R})$ and f_n form a convergent sequence in B, then they actually converge to 0 in B.
3) The multiplication operators $(T_t f)(x) = e^{itx} f(x), t \in \mathbb{R}$ form a strongly continuous operator group of polynomial growth on B_0, i.e.

$$\|T_t f - f\| \to 0, \quad t \to 0, \quad \text{and} \quad \|T_t\| \le (C + |t|)^{N_1}. \tag{1.1}$$

Notation for the map $B_0 \to B$ will be suppressed throughout. We can interpret the elements of B_0 as representants for equivalence classes of (not necessarily continuous) functions, of which B consists in many important cases. If $f_n \in B_0$ converge to some $f \in \mathrm{C}^{N_0}(\mathbb{R})$ as well as to some element $\widetilde{f}$ of B, then condition 2) says that $\widetilde{f}$ is uniquely determined by f, so that the mapping $B_0 \to B$ and our interpretation extend to the projection B_1 of the closure of B_0 as a subset of $\mathrm{C}^{N_0}(\mathbb{R}) \times B$ to $\mathrm{C}^{N_0}(\mathbb{R})$. The most interesting condition 3) is meant to formalize the requirement for $\|f\|$ to depend only on the first N_1 derivatives of f.

1.2 Proposition. $\mathrm{C}_0(W)$ *for lsc weights* W *and* $\mathrm{L}^p(\mu)$ *for Borel measures* μ *on* $\mathbb{R}$ *are regular function spaces.*

Proof. For $\mathrm{C}_0(W)$, we may let $N_0 = 0$, $B_0 = \mathrm{C}_0(W)$, so that 1) is clear and the embedding is actually injective. To see 2), note that the limit of a sequence in both $\mathrm{C}^{N_0}(\mathbb{R})$ and $\mathrm{C}_0(W)$, if existent, must be equal to the pointwise limit. For 3), choose $M > 0, t > 0$ with $|f/W| < \varepsilon$ outside $[-M, M]$ and $|e^{itx} - 1| < \varepsilon$ on $[-M, M]$, so that

$$\|T_t f - f\|_W = \|(e^{itx} - 1) f/W\|_\infty \le \varepsilon(\|f/W\|_{[-M,M]} + \|e^{itx} - 1\|_{\mathbb{R}\setminus[-M,M]}) \le \varepsilon(\|f\|_W + 2). \tag{1.2}$$

For $\mathrm{L}^p(\mu)$, use $B_0 = \mathrm{C_c}(\mathbb{R})$, the existence of pointwise convergent subsequences of L^p-convergent sequences and dominated convergence. □

1.2 Multipliers

Next, we introduce a space of multipliers for B_1 (which will be used to formalize division of zeros on analytic subspaces of regular function spaces). Write $N = \max\{N_0, N_1\}, N(t) = (C + |t|)^N$, and note $N(t + s) \le N(t)N(s)$.

1.3 Definition. $\mathscr{M}$ is the space of all complex measures μ on $\mathbb{R}$ such that $\int N(t)\, d|\mu|$. $\widehat{\mathscr{M}}$ is the space of all Fourier transforms $\phi = \widehat{\mu}$ of measures in $\mathscr{M}$, with $\|\phi\| = \int N(t)\, d|\mu|$. $\widehat{\mathscr{M}_+}$ is the subspace of all Fourier transforms of measures in $\mathscr{M}$ supported in $(-\infty, 0]$.

1.4 Lemma. *$\widehat{\mathscr{M}}$ forms a Banach algebra under pointwise multiplication. For $\phi \in \widehat{\mathscr{M}}$ and $f \in B_1$, $\phi f \in B_1$ with $\|\phi f\| \le \|\phi\|\|f\|$ and*

$$\phi f(x) = \int T_t f(x)\ d\mu(t), \quad \phi = \widehat{\mu}. \tag{1.3}$$

Note that the right hand side of (1.3) makes sense for all $f \in B$, so that we can use it define multiplication with ϕ for arbitrary $f \in B$.

Proof. To see that $\widehat{\mathscr{M}}$ is a Banach algebra, note that

$$\|\widehat{\mu}\widehat{\nu}\| = \int N(t)\ d|\mu * \nu| \le \int N(t)\ d|\mu| * |\nu| \tag{1.4}$$

$$= \int\int N(t+s)\ d|\mu|\ d|\nu| \le \int\int N(t)N(s)\ d|\mu|\ d|\nu| = \|\widehat{\mu}\|\|\widehat{\nu}\|. \tag{1.5}$$

Since $\|T_t f\| \le N(t)\|f\|$, the B-valued (Riemann) integral in (1.3) has $\|\int T_t f\ d\mu\| \le \int N(t)\|f\|\ d|\mu| = \|\phi\|\|f\|$. It remains to show (1.3). For $\mu = \delta_t$ and $\phi = e^{itx}$, this is trivial. For general μ, get a tight sequence of discrete measures ν_n converging to $N(t)\mu$ weakly[1], and let $\mu_n = N(t)^{-1}\nu_n$. Then $\phi_n = \widehat{\mu}_n \to \phi$ follows immediately from weak convergence, and

$$\left\|\int T_t f\ d\mu_n - \int T_t f\ d\mu\right\| \le \|f\| \int_{|t|\le t_0} N(t)\ d|\mu_n - \mu| + \|f\| \int_{|t|>t_0} (d|\nu_n| + N(t)d|\mu|). \tag{1.6}$$

Now choose $t_0 > 0$ large enough to make the second term $< \varepsilon$ for all n, and note that the first term tends to 0 as $n \to \infty$ if we additionally take t_0 to have $\mu(\{t_0\}) = 0$. Hence the lim sup of the left side is smaller than ε. □

The obvious identity

$$(x-z)^{-1} = i\int_{-\infty}^{0} e^{i(x-z)t}\ dt = i\int_{-\infty}^{0} e^{itx}\ d\mu(t), \quad \operatorname{Im} z > 0 \tag{1.7}$$

where $d\mu = e^{-izt}dt$, allows us to interpret multiplication by $(x-z)^{-1}$ for $z \in \mathbb{C}^+$ as the action of an element of $\widehat{\mathscr{M}_+}$, whose norm we can now bound by a simple analysis of μ.

1.5 Lemma. *If B is a regular function space, $f(x) \in B$ and $z \in \mathbb{C}$ with $y = \operatorname{Im} z \ne 0$, then $(x-z)^{-1}f(x) \in B$ with*

$$\|(x-z)^{-1}f(x)\| \le C\|f\| \left(1/|y| + 1/|y|^{N+1}\right), \tag{1.8}$$

where $C > 0$ does not depend on y or f. □

Proof. With μ as above,

$$\int N(t)\ d|\mu| = \int N(t)e^{-yt}\ dt = y^{-1}\int N(y^{-1}t)e^{-t}\ dt \tag{1.9}$$

$$\le y^{-1}N(y^{-1})\int N(t)e^{-t}\ dt \le Cy^{-1}(1+y^{-N}). \tag{1.10}$$

□

[1] Both concepts are taken to have their probabilistic meaning, although we do share the opinion that weak* convergence would be the more appropriate term. One easily verfies that the measures

$$\nu_n = \sum_{k=-\infty}^{\infty} (N\mu)(n^{-1}(k-1,k])\delta_{n^{-1}k}$$

will do.

1.6 Definition. For $\operatorname{Im} z \neq 0$, $R_z : B \to B$ denotes multiplication by $(x-z)^{-1}$.

R_z is a bounded operator on B by (1.8).

1.7 Lemma. *If A is a linear subspace of B and $R_w A \subset \overline{A}$ for some $w \in \mathbb{C}^+$, then $\widehat{\mathscr{M}_+}\overline{A} \subset \overline{A}$.*

Proof. The continuity of R_w immediately implies $R_w\overline{A} \subset \overline{A}$. R_z satisfies the resolvent equation

$$R_z - R_w = (w-z)R_zR_w. \tag{1.11}$$

For $|z-w| < \|R_w\|^{-1}$, it follows that

$$\sum_{n=0}^{\infty}(z-w)^nR_w^{n+1} = R_z\sum_{n=0}^{\infty}(z-w)^nR_w^n - (z-w)^{n+1}R_w^{n+1} = R_z, \tag{1.12}$$

which gives the invariance of $\overline{A}$ under R_z for z close to w. It also follows that for $f \in B, l \in B^*$, the function $z \to (R_zf, l)$ is analytic on $\mathbb{C}^+$ and vanishes in a neigborhood of w, hence on all of $\mathbb{C}^+$. For arbitrary $\phi = \widehat{\mu} \in \widehat{\mathscr{M}_+}$, $f \in \overline{A}$ and $l \in A^\perp$, we have by (1.3)

$$(\phi f, l) = \int_{-\infty}^{0}(T_tf, l)\; d\mu, \tag{1.13}$$

but we already know that also $l \in (R_z\overline{A})^\perp$, so that the Laplace transform of the integrand vanishes:

$$0 = (R_{iy}f, l) = -i\int_{-\infty}^{0} e^{yt}(T_tf, l)\; dt. \tag{1.14}$$

It follows by the invertibility of the Laplace transform that $(\phi f, l) = 0$, and hence $l \in (\widehat{\mathscr{M}_+}\overline{A})^\perp$. $\square$

The reason why we consider the above lemma is that for A an analytic subspace, its assumptions will follow from infiniteness of the Hall majorant (while having little to do with the division of zeros in algebraic de Branges spaces) and its conclusion will often imply densenss by the criterion formulated below.

1.8 Definition. A subspace A of a regular function space is called *analytic* if its members extend to entire functions.$Z_{\mathbb{R}}(A)$ is the set of common real zeros of A.

Note that according to our definition of algebraic de Branges spaces (which includes division of *real* zeros), $Z_{\mathbb{R}}(A) = \varnothing$ for A an algebraic de Branges space. For this reason, we need only a special case of [7, Prop. 2.4], which also shows denseness in some cases with $Z_{\mathbb{R}}(A) \neq \varnothing$.

1.9 Lemma. *If $A \neq \{0\}$ is an analytic subspace of a regular function space with $\widehat{\mathscr{M}}\overline{A} \subset \overline{A}$ and $Z_{\mathbb{R}}(A) = \varnothing$, then $\overline{A} = B$.*

Proof. Assume $\overline{A} \neq B$, so that $\overline{A} \subsetneq B_0$, and get $l \in A^\perp, f \in B_0$ with $K = \operatorname{supp} f$ compact $\|l\|, \|f\| = 1$ and $(f, l) \neq \varnothing$. Then $u(\phi) = (\phi f, l), \phi \in \mathscr{D}$ defines a nontrivial distribution of order N supported in K.[2] Since $Z_{\mathbb{R}}(A) = \varnothing$ and f has compact support, we may represent f as

$$f = \sum \phi_i a_i f = \sum (\phi_i f)a_i, \quad \phi_i \in \mathscr{D}, a_i \in A, \tag{1.15}$$

where $\phi_i f$ is compactly supported and in C^N, hence $\phi_i f \in \widehat{\mathscr{M}}$ by [9, Thm. 7.15]. But then $f \in A$, so that $l \in A^\perp$ implies $u = 0$. $\square$

[2]As usual, $\mathscr{D}$ is the class of test functions, contained in $\widehat{\mathscr{M}}$ by the invariance of the Schwartz class under Fourier transforms and the Inversion theorem [9, Thm. 7.7]. u being of order N means that u is bounded in the norm of $\mathrm{C}_c^N(K)$. While $1 \notin \mathscr{D}$, it is still clear that u is nontrivial since $\mathscr{D}$ is dense in the norm of $\widehat{\mathscr{M}}$.

1.10 Remark. In the case $Z_{\mathbb{R}}(A) \neq \varnothing$, we could still have applied the construction in (1.15) to $g = \psi f$, where $\psi \in \mathscr{D}$ with $\psi(x) = 1$ for $d(x, Z_{\mathbb{R}}(A) \cap K) < \varepsilon$ and $\phi(x) = 0$ for $d(x, Z_{\mathbb{R}}(A) \cap K) \geq 2\varepsilon$ to obtain that u is supported on $Z_{\mathbb{R}}(A)$. By discreteness of $Z_{\mathbb{R}}(A)$, multiplying f by a function $\psi \in \mathscr{D}$ supported near a single point $x \in Z_{\mathbb{R}}(A)$ reduces u to a distribution with one-point support, which must be of the form $u(\phi) = \sum_{i=0}^{n} c_i \phi^{(i)}(x)$ by [9, Thm. 6.25]. Again multiplying f by $\psi \in \mathscr{D}$ with $\psi^{(i)}(x) = 0, i < n$ and $\psi^{(n)}(x) = c_n^{-1}$, we get a representation of the point evaluation at x on $\widehat{\mathscr{M}}$ in the form $(\phi f, l) = \phi(x)$.

Such points x are said to be in the *point spectrum* $\sigma_{\mathrm{p}}(B)$ of B, and we have just seen that it would suffice to assume $Z_{\mathbb{R}}(A) \cap \sigma_{\mathrm{p}}(B) \neq \varnothing$. In the case $B = \mathrm{C}_0(W)$, this is of no great use since taking $f \in \mathrm{C}_0(W)$ with $f(x) = 1$, and l the point evaluation at x on $\mathrm{C}_0(W)$ shows that $\sigma_{\mathrm{p}}(\mathrm{C}_0(W)) = \mathbb{R}$. For $B = \mathrm{L}^1(\mu)$, $\sigma_{\mathrm{p}}(B)$ is the set of atoms of μ, since for $\mu(\{x\}) = 0$ and $f \in \mathrm{L}^1(\mu)$, there are $\phi \in \widehat{\mathscr{M}}$ of arbitrarily small $\mathrm{L}^1(\mu)$-norm with $\phi(x) = 1$.

1.11 Remark. At this point, the lack of injectivity assumptions on the embedding of A into B will probably have made the reader somewhat suspicious. Therefore, let us remark that the above proof really takes place in B_0, first showing denseness in B_0 and only then using the denseness of B_0 in B. This rules out pathologies such as $A \neq \{0\}$ as a space of functions and $\widehat{\mathscr{M}A} \subset \overline{A}$, but $A = \{0\}$ as a subspace of B, which appear to be possible from the statement of the lemma.

1.3 Hall Majorant

We are now able to give the key statements on finiteness of the Hall majorant and denseness of algebraic de Branges subspaces, as contained in [7, Prop. 3.1,3.2, Thm. 3.1]. Fix an algebraic de Branges subspace H of a regular function space B, and let $H_z = \{h \in H : h(z) = 0\}$.

1.12 Definition. For $f \in B$, $\|f\|_+ := \|R_i f\|$.

Since R_z is $\|\cdot\|$-bounded, $\|\cdot\|_+$ is weaker than $\|\cdot\|$. By the resolvent equation (1.11),

$$\|R_z f\| \leq \|R_w f\| + |w - z| \|R_z R_w f\| \leq (1 + |w - z| \|R_z\|) \|R_w f\|, \quad f \in B, z, w \in \mathbb{C} \tag{1.16}$$

so that the norms $\|R_z f\|$ are all equivalent.

1.13 Definition. For $z \in \mathbb{C}$, the *Hall majorants* of the algebraic de Branges subspace H are defined by

$$\mathfrak{m}(z) = \sup\{|h(z)| : h \in H, \|h\| \leq 1\}, \quad \mathfrak{m}_+(z) = \sup\{|h(z)| : h \in H, \|h\|_+ \leq 1\} \tag{1.17}$$

Clearly, $\mathfrak{m}(z) \leq \|R_i\| \mathfrak{m}_+(z)$. By the symmetry requirement $H^\sharp = H$, $\mathfrak{m}(\overline{z}) = \mathfrak{m}(z), \mathfrak{m}_+(\overline{z}) = \mathfrak{m}_+(z)$.

1.14 Lemma. *If $\mathfrak{m}_+(z) = \infty$ for some $z \in \mathbb{C} \setminus \mathbb{R}$, then $R_z H \subset \overline{H}$. Conversely, if $\overline{R_z H} = \overline{H}$, then $\mathfrak{m}_+(z) = \infty$.*

Proof. $\mathfrak{m}_+(z) = \infty$ occurs iff the point evaluation at z is unbounded on H, which happens iff its kernel H_z is dense in H, ie

$$d_{\|\cdot\|_+}(f, H_z) = 0, f \in H. \tag{1.18}$$

By the equivalence of the norms $\|R_z f\|$, this is the same as

$$0 = d_{\|\cdot\|}(R_z f, R_z H_z) \geq d_{\|\cdot\|}(R_z f, H), \tag{1.19}$$

where the latter inequality is by $R_z H_z \subset H$. This shows that $R_z f \in \overline{H}$.

If $R_z H$ is dense in H, the inequality in (1.19) becomes an equality, so that we can reverse the argument. ☐

Together with Lemma 1.7, Lemma 1.9, the symmetry of algebraic de Branges spaces and $\mathfrak{m}(z) \leq \|R_i\| \mathfrak{m}_+(z)$, we obtain

1.15 Theorem. *If an algebraic de Branges space H is not dense in a regular function space B, then its Hall majorant $\mathfrak{m}(z)$ is finite on $\mathbb{C} \setminus \mathbb{R}$.* □

To obtain analytic extensions for the functions in $\overline{H}$, we actually need local boundedness, which is contained in the next theorem:

1.16 Theorem. *If H is not dense in B, then $\mathfrak{m}(z)$ is continuous and finite on $\mathbb{C}$, and $\log \mathfrak{m}(z)$ is subharmonic and thus locally L^1.*

The statement that $\mathfrak{m}(z)$ is either infinite on all of $\mathbb{C}$ or bounded in the sense of the theorem is commonly referred to as the *Riesz-Mergelyan alternative.* Recall that every subharmonic function $\not\equiv -\infty$ is locally integrable by the sub-mean value property [8, Thm. 2.3,2.4 (ii)], and that $\log|f|$ is subharmonic for any analytic function f [8, Thm. 2.12].

Proof. First, we show local integrability. Note that $\log^- \mathfrak{m}(z)$ is bounded below by each of the subharmonic functions $\log^-|g|, g \in H, \|g\| = 1$, so that it suffices to show $\log^+ \mathfrak{m}(z) \in \mathrm{L}^1_{\mathrm{loc}}$. Since H is not dense, Lemma 1.9 shows that $\overline{H}$ cannot be invariant under all operators R_z. Hence fix $l \in H^\perp$ and $f \in H$ with $(R_z f, l) \neq 0$ for some $z \in \mathbb{C} \setminus \mathbb{R}$. For arbitrary $g \in H$ and z fixed, we have $(w-z)^{-1}(f(z)g(w) - f(w)g(z)) \in H$ as a function of w, so

$$0 = \Big((w-z)^{-1}(f(z)g(w) - f(w)g(z)), l\Big) \tag{1.20}$$

$$= f(z)(R_z g, l) - g(z)(R_z f, l) = f(z)G(z) - g(z)F(z), \tag{1.21}$$

where $G(z) = (R_z g, l)$ and $F(z) = (R_z f, l)$ are entire by (1.12). From (1.8), we obtain that

$$|G(z)| \leq \|g\| C \left(1/|y| + 1/|y|^{N+1}\right), \tag{1.22}$$

where by calculus

$$\int_0^1 \log \left|1/|y| + 1/|y|^{N+1}\right| \leq -C_1 \int_0^1 \log y \; dy = -C_1 \left[y \log y - y\right]_0^1 < \infty. \tag{1.23}$$

Since $\log^+|g| = \log^+|f| + \log^+|G| - \log^+|F|$, the functions $\log^+|g|$ for $\|g\| \leq 1$ can be uniformly estimated by a locally integrable function, so that $\mathfrak{m}(z)$ is indeed locally integrable.
By the area sub-mean value property[3] applied to $\log^+|g|$, it follows that the family $\{g \in H : \|g\| \leq 1\}$ is locally bounded, hence normal. This gives the continuity of $\mathfrak{m}(z)$ and the upper semicontinuity of $\log \mathfrak{m}(z)$. Since

$$\log \mathfrak{m}(z) = \sup_{\|g\| \leq 1} \log|g(z)| \leq \sup_{\|g\| \leq 1} \frac{1}{2\pi} \int_0^{2\pi} \log|g(z + re^{i\theta})| \; d\theta \tag{1.25}$$

$$\leq \frac{1}{2\pi} \int_0^{2\pi} \sup_{\|g\| \leq 1} \log|g(z + re^{i\theta})| \; d\theta = \frac{1}{2\pi} \int \log \mathfrak{m}(z + re^{i\theta}) \; d\theta, \tag{1.26}$$

$\log \mathfrak{m}(z)$ also has the sub-mean value property and is thus subharmonic [8, Thm. 2.3], which completes the proof. □

[3] The *area sub-mean value property* reads

$$u(w) \leq \frac{1}{r^2 \pi} \iint_{|z-w|<r} u(z) \; dxdy, \quad u \text{ subharmonic}, \tag{1.24}$$

and follows from the sub-mean value property by integration over r.

1.4 Analytic Extensions

With the results of the preceding section at hand, we can easily prove the properties of the functions in $\overline{H}$ we need (cf. [7, Prop. 3.3,3.4, Thm. 3.2]).

1.17 Theorem. *If H is not dense in B, then every function $f \in \overline{H}$ has an extension $h \in \mathrm{H}(\mathbb{C})$ satisfying*

$$|h(z)| \leq \|f\| \mathfrak{m}(z). \tag{1.27}$$

Proof. By local boundedness of $\mathfrak{m}(z)$, the unit ball of H is a normal family, so that every $\|\cdot\|$-convergent sequence in H has a locally unformly convergent subsequence. Since each member of the sequence satisfies (1.27), so does the limit. ☐

1.18 Theorem. *If H is not dense in B and $f, g \in \overline{H}$ with $g \neq 0$, then f/g is a meromorphic function of bounded type in $\mathbb{C}^{\pm}$.*

Proof. Since the quotient of funtions of bounded type is again of bounded type, it clearly suffices to prove this for a fixed $g \in \overline{H}$ and all $f \in \overline{H}$, so that we may assume $R_z g \notin \overline{H}$. As in the proof of theorem 1.16, we get a representation

$$f(z)/g(z) = F(z)/G(z), \quad F(z) = (R_z f, l), G(z) = (R_z g, l). \tag{1.28}$$

Estimate (1.22) shows (together with lemma A.5) that f/g is of bounded type in the half planes $\{y < -1\}, \{y > 1\}$. We now prove

$$\int_{-1}^{1} \int_{-\infty}^{\infty} \frac{\log^{+} |f(x+iy)/g(x+iy)|}{1+x^2} \, dxdy < \infty, \tag{1.29}$$

which will imply $f/g \in \mathrm{N}(\mathbb{C}^{\pm})$ by the generalization of Krein's theorem given in [7, Thm. A.1]. This amounts to estimating the same integral with $\log^{+}|F|$ and $\log^{-}|G|$ in place of $\log^{+}|f/g|$. For $\log^{+}|F|$, use (1.23). By subharmonicity of $G(z)$ and the Poisson formula for the half plane [8, Sec. 5.2], we get that

$$\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\log |G(x+ih)|}{1+(x-x_0)^2} \, dx \geq \log |G(x_0 + i + ih)|, \quad h > 0, x_0 \in \mathbb{R}. \tag{1.30}$$

Since $G \not\equiv 0$, the integral of the right hand side over $h \in (-1, 1)$ must be finite for some x_0 by local integrability of $\log|G(z)|$. Since $(1+x^2)/(1+(x-x_0)^2)$ is bounded, this completes the proof. ☐

2 De Branges' Lemma

We proceed to prove Summers' description of the dual space of $\mathrm{C}_0(W)$ from [13], and de Branges' Lemma [4] on extreme points of the annihilator of a subspace of $\mathrm{C}_0(W)$.

2.1 Dual of $\mathrm{C}_0(W)$

In this section, we may as well consider a locally compact space X in place on $\mathbb{R}$, and define $\mathrm{C}_0(X)$ to be the space of all $f \in \mathrm{C}(X)$ such that for every $\varepsilon > 0$, there is $K \subset X$ compact such that $|f|$ is bounded by ε on $X \setminus K$. For $W : X \to (0, \infty]$ lower semicontinuous, $\mathrm{C}_0(W)$ is defined accordingly.

From the Riesz Representation Theorem for $\mathrm{C}_0(X)$, it seems natural to identify the dual $\mathrm{C}_0(W)^*$ of $\mathrm{C}_0(W)$ with the space $\mathrm{M}(X)$ of (finite) complex Radon measures[4] on X via $T\mu(f) := \int f W^{-1} \, d\mu$.

[4] *Radon measures* are locally finite inner regular measures on the Borel sets of X; by σ-compactness, the requirement of inner regularity is void for $X = \mathbb{R}$.

If W is continuous and everywhere finite, then $f \to fW^{-1}$ is an isomorphism between $\mathrm{C}_0(W)$ and $\mathrm{C}_0(X)$, so that the statement indeed reduces to the Riesz Theorem. However, by the following result of Summers [13, Thm. 3.1], lower semicontinuity is already sufficient for the existence of representing measures, although they need not be unique if W is infinite:

2.1 Theorem. *If $W : X \to (0, \infty]$ is a lower semicontinuous function on a locally compact space X, T maps $\mathrm{M}(X)$ onto $\mathrm{C}_0(W)^*$.*

2.2 Lemma. *If $f : X \to [0, \infty]$ is lsc on a Tychonoff space X, then $f = \sup\{g : f \geq g \in \mathrm{C_c}(X)\}$ pointwise.*

Proof. Given $x \in X$ and $c < f(x)$, $\{f > c\}$ is a neighborhood of x, so get $g : X \to [0, 1]$ with $g(x) = 1$, $g|_{\{f \leq c\}} = 0$, and consider cg. ☐

The key to the proof is the following lemma [13, Lemma 3.3]:

2.3 Lemma. *If an inner regular positive measure ν on X satisfies $\int f \, d\nu \leq \|f\|_W, f \in \mathrm{C_c}(X)$, then $\nu = W^{-1}\mu$ for some finite positive measure μ.*

Proof. From inner regularity and $\{W > a\} = \bigcup \{\{f > a\} : W \geq f \in \mathrm{C_c}(X)\}$ it follows easily that

$$\int W \, d\nu = \sup\left\{\int f \, d\nu : W \geq f \in \mathrm{C_c}(X)\right\} \leq 1, \tag{2.1}$$

since $f \leq W$ iff $\|f\|_W \leq 1$, i.e. $W \in \mathrm{L}^1(\nu)$. Hence $\mu = W\nu$ is a finite measure. ☐

2.4 Lemma. *The topology of locally uniform convergence on $\mathrm{C_c}(X)$ is finer than the topology induced by $\mathrm{C}_0(W)$ on $\mathrm{C_c}(X)$.*

Proof. The set $U = \{f \in \mathrm{C_c}(X) : \|f\|_W < 1\}$ is a neighborhood of 0 in $\mathrm{C_c}(X)$, since W is bounded from below by some $c > 0$ on each compact $K \subset X$ and $U \cap \mathrm{C}(K) \supset \{f \in \mathrm{C}(K) : \|f\|_\infty < c/2\}$. ☐

Proof (Theorem 2.1). If $\phi \in \mathrm{C}_0(W)^*$, and $\|\phi\| = 1$, then $\phi|_{\mathrm{C_c}(X)}$ is again continuous, so that by the Riesz Theorem for $\mathrm{C_c}(X)$, there is a measure ν with $\phi(f) = \int f \, d\nu, f \in \mathrm{C_c}(X)$. Applying Lemma 2.3 to the parts of the Jordan decomposition of ν, we obtain $\mu \in \mathrm{M}(X)$ with $\nu = W^{-1}\mu$. Since $\mathrm{C_c}(X)$ is dense in $\mathrm{C}_0(W)$, $\phi = T\mu$ follows by continuity. ☐

2.2 De Branges' Lemma

As a last preparation for de Branges' proof, we need a property of extremal annihilating measures first stated in [4] in a non-weighted context (and used there for a short proof of the Stone-Weierstrass Theorem). We use the version [1, Lemma 2.2], adapted to our specific situation:

2.5 De Branges' Lemma. *Given an lsc weight function W and a non-dense linear subspace $\mathcal{L}$ of $\mathrm{C}_0(W)$, there exists a positive measure μ on $\mathbb{R}$ such that*

a) *$\int W \, d\mu < \infty$ (so that $\mathrm{C}_0(W) \subset \mathrm{L}^1(\mu)$), and*
b) *the annihilator of $\mathcal{L}$ in the duality $(\mathrm{L}^1(\mu), \mathrm{L}^\infty(\mu))$ is spanned by a real function g of constant modulus 1.*

Explicitly, *b*) says that if $\widetilde{g} \in \mathrm{L}^\infty(\mu)$ satisfies $\int f\widetilde{g} \, d\mu = 0$ for all $f \in \mathcal{L}$, then $\widetilde{g}$ is a multiple of g. We denote the annihilators with respect to the dualities $(\mathrm{C}_0(W), \mathrm{M}(\mathbb{R}))$ and $(\mathrm{L}^1(\mu), \mathrm{L}^\infty(\mu))$ by $\perp_0$ and $\perp_1$, respectively.

Proof. The closed convex subset $\Sigma = \{\sigma \in \mathcal{L}^{\perp_0} : \sigma \text{ real}, \|\sigma\| \leq 1\}$ of $\mathrm{M}(\mathbb{R})$ is nonempty, since, $\mathcal{L}$ being a space of real functions, for any nonzero $\sigma \in \mathcal{L}^{\perp_0}$, $\operatorname{Re}\sigma$ and $\operatorname{Im}\sigma$ are in the annihilator as well, and one of them must be nonzero.

Let σ be an extreme point of Σ, put $\mu = W^{-1}|\sigma|$ and $g = d|\sigma|/d\sigma \in \mathcal{L}^{\perp_1}$. If $\mathcal{L}^{\perp_1}$ is not spanned by g, then there is a nonconstant $f \in g^{-1}\mathcal{L}^{\perp_1}$. Let h be the function $f + 2\|f\|_\infty \geq 0$, normalized to have $\int h \, d|\sigma| = \|\sigma\|$. Then $\|h\|_\infty > 1$ since otherwise

$$\int |1-h| \, d|\sigma| = \int 1-h \, d|\sigma| = \|\sigma\| - \int h \, d|\sigma| = 0, \tag{2.2}$$

meaning that h would have to be a constant σ-a.e. Letting $t = \|h\|_\infty^{-1} < 1$, so that $1 - th \geq 0$, we obtain convex combinations

$$1 = th + (1-t)\frac{1-th}{1-t}, \quad \sigma = t(h\sigma) + (1-t)\left(\frac{1-th}{1-t}\sigma\right). \tag{2.3}$$

As is easily verified from the choice of h and t, this actually exhibits σ as a strict convex combination of elements of Σ, a contradiction. ☐

3 De Branges' Proof

We are now in the position to give de Branges' proof of the necessity of $\mathcal{K}(\mathcal{L}, W) \neq \varnothing$, in the version from [1].

> **Assumptions.** For the rest of this section, fix a lower semicontinuous weight W, an algebraic de Branges space $\mathcal{L} \subset \mathrm{C}_0(W)$ with $\overline{\mathcal{L}} \neq \mathrm{C}_0(W)$, and a measure $\mu \in \mathrm{M}(\mathbb{R})$ with a function $g \in \mathrm{L}^\infty(\mu)$ satisfying the conclusion of de Branges' lemma. Define $\mathfrak{m}(z)$ to be the (everywhere finite and locally bounded) Hall majorant with respect to the norm of $\mathrm{L}^1(\mu)$. Let $\rho_0 : \mathrm{H}(\mathbb{C}) \to \mathrm{C}(\mathbb{R})$ be the restriction operator $\rho_0 F = F|_\mathbb{R}$, and let ρ be the corresponding operator $\mathrm{C}_0(W) \to \mathrm{L}^1(\mu)$.

Note that the results of sections 1.3 and 1.4 apply to $H = \rho\mathcal{L}$ and $B = \mathrm{L}^1(\mu)$. In particular, theorems 1.17 and 1.18 give

3.1 Lemma. *For every $f \in \overline{\rho\mathcal{L}}$, there is $F \in \mathrm{H}(\mathbb{C})$ with $\rho F = f$ and*

$$|F(z)| \leq \|f\|_1 \mathfrak{m}(z), \quad z \in \mathbb{C}. \tag{3.1}$$

The quotient of any two such functions F and G is of bounded type in $\mathbb{C}^\pm$. ☐

3.1 Consequences of de Branges' Lemma

Since $(\rho\mathcal{L})^\perp = \operatorname{span}\{g\}$ implies $\overline{\rho\mathcal{L}} = g^\perp$, we get the following characterization of $\overline{\rho\mathcal{L}}$:

3.2 Lemma. *$f \in \overline{\rho\mathcal{L}}$ iff $\int fg \, d\mu = 0$. In particular, $\overline{\rho\mathcal{L}}$ has codimension 1 in $\mathrm{L}^1(\mu)$.* ☐

We deduce two important properties of the measure μ as proved in [1, p. 891]:

3.3 Lemma. *μ is discrete, and ρ maps $\mathcal{L}$ injectively into $\mathrm{L}^1(\mu)$.*

Proof. Assume that x_0 is an accumulation point of $\operatorname{supp}\mu$, and take an interval $(a,b) \not\ni x_0$ that has at least two points in common with $\operatorname{supp}\mu$.[5] It follows that $\mathrm{L}^1(\mu|_{(a,b)})$ is at least two-dimensional, so that there must be $f \in \overline{\rho\mathcal{L}} \setminus \{0\}$ with $f|_{\operatorname{supp}\mu \setminus (a,b)} = 0$. If F is any entire function with $\rho F = f$, it follows that $\rho_0 F$ must vanish on a set of points accumulating at x_0.[6] The identity theorem gives $F = 0$, and hence $f = 0$, a contradiction.

Now let $F \in \mathcal{L}$ with $\rho F = 0$. By discreteness, this means that $F(x) = 0$ for each $x \in \operatorname{supp}\mu$. If l is the multiplicity of some $x_0 \in \operatorname{supp}\mu$ as zero of F, $G(z) = (z - x_0)^{-l} F(z) \in \mathcal{L}$ is nonzero at exactly one point of $\operatorname{supp}\mu$, which contradicts $\int Gg \, d\mu = 0$. ☐

[5] Recall that the support $\operatorname{supp}\mu$ of μ is the intersection of all closed sets whose complement is a μ-null set.

[6] Argue, for example, as follows: For each k, the set $A_k = \{|\rho_0 F| < 1/k\} \cap \operatorname{supp}\mu$ has complement a μ-null set, so that it must be dense in $\operatorname{supp}\mu$. A_k is also clearly open, so that by the Baire category theorem applied to the complete metric space $\operatorname{supp}\mu$, the intersection $\bigcap A_k = \{\rho_0 F = 0\} \cap \operatorname{supp}\mu$ is dense in $\operatorname{supp}\mu$.

3.2 Analytic Extensions for $\mathrm{L}^1(\mu)$

From the preceding lemma, it follows that ρ has an inverse $\iota : \rho\mathcal{L} \to \mathcal{L}$, which is continuous in the topology of locally uniform convergence by (3.1). Extend ι to a map $\widetilde{\iota} : \overline{\rho\mathcal{L}} \to \mathrm{H}(\mathbb{C})$ by continuity, and let $\widetilde{\mathcal{L}} = \widetilde{\iota}(\overline{\rho\mathcal{L}})$. We show that the properties of ι and $\mathcal{L}$ carry over to $\widetilde{\iota}$ and $\widetilde{\mathcal{L}}$ (cf. [1, pp. 888-890]).

3.4 Proposition. $\rho|_{\widetilde{\mathcal{L}}} = \widetilde{\iota}^{-1}$

This is nontrivial since ρ is in general discontinuous. However, $1_K\rho$ is continuous for any compact K since locally uniform convergence of F_n implies uniform convergence, and hence L^1-convergence of ρF_n on K.

Proof. Since $\widetilde{\mathcal{L}} = \operatorname{ran}\widetilde{\iota}$, it suffices to show $\rho\widetilde{\iota} = \mathrm{Id}_{\overline{\rho\mathcal{L}}}$. Let $\rho F_n \to f \in \overline{\rho\mathcal{L}}$ in $\mathrm{L}^1(\mu)$. Then $\widetilde{\iota}\rho F_n \to \widetilde{\iota} f$ locally uniformly, so that

$$\rho_0\widetilde{\iota} f \leftarrow (\rho_0\widetilde{\iota})\rho F_n = \rho_0(\iota\rho)F_n = \rho_0 F_n \quad \text{pointwise on } \mathbb{R}. \tag{3.2}$$

In particular, $\rho F_n \to \rho\widetilde{\iota} f$ μ-a.e. If we pass to an μ-a.e. convergent sequence $\rho F_n \to f$, it follows that $\rho\widetilde{\iota} f = f$ μ-a.e. ☐

3.5 Proposition. $|\widetilde{\iota} f(z)| \le \|f\|_1 \mathfrak{m}(z), \quad f \in \overline{\rho\mathcal{L}}.$

Proof. We already know that this holds for $f \in \rho\mathcal{L}$, and since the composition $\overline{\rho\mathcal{L}} \xrightarrow{\widetilde{\iota}} \mathrm{H}(\mathbb{C}) \xrightarrow{\chi_z} \mathbb{C}$, where χ_z is the point evaluation at z, is continuous, the statement extends to the closure of $\rho\mathcal{L}$. ☐

Next, we consider division of zeros in $\widetilde{\mathcal{L}}$. To this end, define the difference quotient operator $\mathcal{R}_w[F,G](z) = (z-w)^{-1}(F(z)G(w) - G(z)F(w)), F, G \in \widetilde{\mathcal{L}}$, and the corresponding operator $\mathcal{R}^1_w = \rho\mathcal{R}_w(\widetilde{\iota} \times \widetilde{\iota})$ on $\overline{\rho\mathcal{L}}$. By Proposition 3.4, $\mathcal{R}_w = \widetilde{\iota}\mathcal{R}_w(\rho \times \rho)$ on $\widetilde{\mathcal{L}}$.
It is clear that invariance under $\mathcal{R}_w$ is equivalent to invariance under $R_w H(z) = (z-w)^{-1}H(z)$: $\mathcal{R}_w[F,G] = R_w(G(w)F - F(w)G)$ and conversely, if $F(w) = 0$ and we take G_0 with $G_0(w) = 1$, then $R_w F = \mathcal{R}_w[F, G_0]$.

3.6 Proposition. *$\mathcal{R}_w$ and $\mathcal{R}^1_w$ are continuous.*

Proof. The Schwarz lemma (applied to the function $F(z)G(w) - G(z)F(w)$ on the unit disc $\mathbb{D}(w,1)$ around w) together with the trivial estimate

$$|(z-w)^{-1}F(z)G(w) - G(z)F(w)| \le 2\|F\|_K\|G\|_K, z, w \in K, |z-w| > 1 \tag{3.3}$$

shows that

$$\|\mathcal{R}_w[F,G]\|_K \le \|F(z)G(w) - G(z)F(w)\|_K \le 2\|F\|_K\|G\|_K \tag{3.4}$$

for any compact $K \supset \mathbb{D}(w,1)$. (Here $\|F\|_K := \sup_{z\in K}|F(z)|$.)
The estimate $|(x-w)^{-1}(f(x)\widetilde{\iota} g(w) - g(x)\widetilde{\iota} f(w))| \le \mathfrak{m}(w)(|f(x)| + |g(x)|), |x-w| > 1$ shows that

$$\|1_{\mathbb{R}\setminus\mathbb{D}(w,1)}\mathcal{R}^1_w[f,g]\|_1 \le 2\mathfrak{m}(w)\|f\|_1\|g\|_1, \tag{3.5}$$

which gives the boundedness of $1_{\mathbb{C}\setminus\mathbb{D}(w,1)}\mathcal{R}^1_w$. Since $1_{\mathbb{D}(w,1)}$ is continuous, so is $1_{\mathbb{D}(w,1)}\mathcal{R}^1_w$. ☐

3.7 Lemma. *$\widetilde{\mathcal{L}}$ is an algebraic de Branges space.*

Proof. The proof consists in carrying invariance statements from $\mathrm{H}(\mathbb{C})$ to $\mathrm{L}^1(\mu)$ and back using intertwining relations with ι and ρ.
For invariance under $\sharp$, note that $\rho F^\sharp = \overline{\rho F}, F \in \mathcal{L}$. By definition of ι, it follows that $(\iota f)^\sharp = \iota\overline{f}, f \in \rho\mathcal{L}$. Since $^-$ and $\sharp$ are involutory automorphisms of $\mathrm{L}^1(\mu)$ and $\mathrm{H}(\mathbb{C})$, respectively, $(\widetilde{\iota} f)^\sharp = \widetilde{\iota}\overline{f}, f \in \overline{\rho\mathcal{L}}$ follows by continuity. Since $^-$ leaves $\rho\mathcal{L}$ invariant by continuity, this gives invariance under $\sharp$.

The invariance of $\mathcal{L}$ under $\mathcal{R}_w$ implies that of $\rho\mathcal{L}$ und $\mathcal{R}^1_w$ by $\mathcal{R}^1_w \circ (\rho|_{\mathcal{L}} \times \rho|_{\mathcal{L}}) = \rho|_{\mathcal{L}} \circ \mathcal{R}_w$, and by continuity, we get the invariance of $\overline{\rho\mathcal{L}}$ under $\mathcal{R}^1_w$. The identity

$$\mathcal{R}_w[\widetilde{\mathcal{L}}, \widetilde{\mathcal{L}}] = \tilde{\imath}\mathcal{R}^1_w(\rho \times \rho)[\widetilde{\mathcal{L}}, \widetilde{\mathcal{L}}] = \tilde{\imath}\mathcal{R}^1_w[\overline{\rho\mathcal{L}}, \overline{\rho\mathcal{L}}] \subset \tilde{\imath}\overline{\rho\mathcal{L}} = \widetilde{\mathcal{L}} \tag{3.6}$$

completes the proof. □

3.3 Representation of $\widetilde{\mathcal{L}}$

The construction of a Krein class function will be based on a representation of the elements of $\widetilde{\mathcal{L}}$ in terms of the following functions (cf. [1, pp. 892-893]): For $t_0 \in \operatorname{supp}\mu$ fixed an $t \in \operatorname{supp}\mu \setminus t_0$, $h_t \in \overline{\rho\mathcal{L}}$ and $H_t \in \widetilde{\mathcal{L}}$ are defined by

$$h_t(x) := \begin{cases} -((t-t_0)g(t_0)\mu(t_0))^{-1} & x = t_0, \\ ((t-t_0)g(t)\mu(t))^{-1} & x = t, \\ 0 & x \in \operatorname{supp}\mu \setminus \{t, t_0\}, \end{cases} \qquad H_t = \tilde{\imath}h_t \tag{3.7}$$

3.8 Proposition. *The entire function $(z-t)H_t(z)$ does not depend on $t \in \operatorname{supp}\mu \setminus t_0$.*

Proof. To show $H_t(z) = \frac{z-s}{z-t}H_s(z) = (\operatorname{Id} + (t-s)R_t)H_s(z)$ in $\mathrm{H}(\mathbb{C})$, we prove the analogue in $\mathrm{L}^1(\mu)$:

$$(\operatorname{Id} + (t-s)R^1_t)h_s(x) = h_t(x), \quad x \in \operatorname{supp}\mu. \tag{3.8}$$

This is easily shown for $x \neq s$, and since both functions are in $g^\perp$, they must agree on all of $\operatorname{supp}\mu$. □

3.9 Proposition. $H_t = H_t^\sharp$ *with real simple zeros exactly at* $\operatorname{supp}\mu \setminus \{t, t_0\}$.

Proof. $H_t = H_t^\sharp$ follows from $h_t = \overline{h_t}$. If H_t has a zero $w \in \mathbb{C} \setminus \operatorname{supp}\mu$, we may define

$$G(z) = \frac{z-t}{z-w}H_t(z) = (\operatorname{Id} + (w-t)R_w)H_t(z). \tag{3.9}$$

One easily verifies that G is nonzero on exactly one point of $\operatorname{supp}\mu$, but this contradicts $\int Gg\,d\mu = 0$. Analogously, if H_t has a multiple zero at $w \in \operatorname{supp}\mu$, then we may add a zero at t in exchange for making w a simple zero to get the same contradiction. □

3.10 Lemma. *Each $F \in \widetilde{\mathcal{L}}$ has a representation*

$$F = \sum_{t \neq t_0} F(t)(t-t_0)\mu(t)H_t \tag{3.10}$$

as a locally uniformly convergent series.

Proof. It suffices to prove the analogue in $\mathrm{L}^1(\mu)$, with $f = \rho F$ in place of F. Clearly, $\|h_t\|_1 = 2/(t-t_0)$, so $f\mu \in \ell^1(\operatorname{supp}\mu)$ gives the convergence of

$$\tilde{f} = \sum_{t \neq t_0} f(t)(t-t_0)\mu(t)h_t \tag{3.11}$$

in $\mathrm{L}^1(\mu)$. For $x \in \operatorname{supp}\mu \setminus t_0$, we compute

$$\tilde{f}(x) = f(x)(x-t_0)\mu(x)h_x(x) = f(x), \tag{3.12}$$

so that $f, \tilde{f} \in \widetilde{\mathcal{L}}$ agree on $\operatorname{supp}\mu \setminus t_0$, hence on all of $\operatorname{supp}\mu$. This gives the representation of f. □

3.4 Construction of a Krein Class Function

We can now construct a Krein class function inducing the measure μ (cf. [1, p. 894]), namely $B(z) = (z-t)H_t(z)$. Condition 1) is satisfied by Proposition 3.9. $F/B = (z-t)^{-1}F/H_t$ is in $\mathrm{N}(\mathbb{C}^\pm)$ by Lemma 3.1, and dominated convergence gives

$$F(iy)/B(iy) = \sum_{t\neq t_0} (\mu(t)F(t))\frac{t-t_0}{iy-t} \to 0, \quad y\to\infty. \tag{3.13}$$

Finally,

$$B'(x) = \begin{cases} (x-t_0)H_x(x) & x\in \operatorname{supp}\mu \setminus t_0 \\ (x-t)H_t(x) & x = t_0 \end{cases} = (g(x)\mu(x))^{-1} \tag{3.14}$$

so that

$$\sum_{B(x)=0} \left|\frac{W(x)}{B'(x)}\right| = \int W\, d\mu < \infty. \tag{3.15}$$

4 Proof of Sodin and Yuditskii

Finally, we present a second proof of de Branges' theorem by Sodin and Yuditskii [11, 12] which relies on a normal family argument in $\mathcal{L}$ in place of an extremal construction in $\mathcal{L}^\perp$. Unless we assume W to be finite on a non-discrete set, this approach will only yield a function B satisfying slight weakenings of conditions 2) and 3) in the definition of Krein class functions.

Assumptions. Again, let W be a lower semicontinuous weight, $\mathcal{L}$ an infinite-dimensional non-dense algebraic de Branges subspace of $\mathrm{C}_0(W)$. Denote by $\mathfrak{m}(z)$ the (everywhere finite and continuous) Hall majorant of $\mathcal{L}$ with respect to $\mathrm{C}_0(W)$. For any $F:\mathbb{R}\to\mathbb{C}$, write $\|F\| = \|F/W\|_\infty$. Let $\mathcal{B}$ be the set of all locally uniform limits of $\|\cdot\|$-bounded sequences in $\mathcal{L}$, and $\mathcal{B}_\mathbb{R}$ the real elements of $\mathcal{B}$.

Note that $F\in\mathcal{B}$ does *not* necessarily lie in $\mathrm{C}_0(W)$, but it is clear that $\|F\|<\infty$ and

$$|F(z)| \le \|F\|\mathfrak{m}(z), \quad F\in\mathcal{B}, \tag{4.1}$$

so that any $\|\cdot\|$-bounded subset of $\mathcal{B}$ is a normal family.

We also need an analogue of Theorem 1.18 for $\mathcal{B}$:

4.1 Lemma. *For $F, G\in\mathcal{B}$, the quotient F/G is a meromorphic function of bounded type in $\mathbb{C}^\pm$.*

Proof. Take real $F_n\in\mathcal{L}$ converging locally uniformly to F, and, as in the proof of Theorem 1.18, assume $G\in\mathcal{L}$ with $(R_zG,\phi)\neq 0$ for some $z\in\mathbb{C}\setminus\mathbb{R}$ and $\phi\in\mathcal{L}^\perp$. Again, we may write

$$F_n(z)/G(z) = \widetilde{F}_n(z)/\widetilde{G}(z), \quad \widetilde{F}_n(z) = (R_zF_n,\phi), \widetilde{G}(z) = (R_zG,\phi). \tag{4.2}$$

If we represent ϕ as $W^{-1}\mu$ with $\mu\in\mathrm{M}(\mathbb{R})$ as in Theorem 2.1, we may use dominated convergence to get a corresponding representation of F/G as quotient of $\widetilde{F}(z) = (R_zF,\phi)$ and $\widetilde{G}$. But $\widetilde{F}$ and $\widetilde{G}$ are in $\mathrm{N}(\mathbb{C}^\pm)$ as the Cauchy integrals of $FW^{-1}\,d\mu$ and $GW^{-1}\,d\mu$ by Lemma A.6. ☐

$\mathcal{B}$ is still an algebraic de Branges space:

4.2 Lemma. *If $F\in\mathcal{B}$ and $F(w)=0$, then $G(z) = F(z)/(z-w)\in\mathcal{B}$.*

Proof. Assume that $F\not\equiv 0$, so that for $F_n\in\mathcal{L}$ converging locally uniformly to F, the Hurwitz theorem implies that there is a sequence w_n of zeros of F_n converging to w. The entire functions $G_n(z) = F_n(z)/(z-w_n)$ clearly converge to G locally uniformly on $\mathbb{C}\setminus\mathbb{D}(w,1)$, hence on all of $\mathbb{C}$ by the maximum principle.

Now assume $W \geq c > 0$ in $\mathbb{D}(w, 4)$. Then by the Schwarz lemma,

$$|G_n(x)/W(x)| \leq \|G_n\|_{\partial\mathbb{D}(w_n,3)}/c \leq \|F_n\|_{\partial\mathbb{D}(w_n,3)}/c \leq \|F\|\|\mathfrak{m}\|_{\mathbb{D}(w_n,3)}/c, \quad x \in \mathbb{D}(w_n, 3), \tag{4.3}$$

and clearly $|G_n| \leq |F_n|$ outside $\mathbb{D}(w_n, 1)$. For n so large that $\mathbb{D}(w_n, 1) \subset \mathbb{D}(w, 2) \subset \mathbb{D}(w_n, 3) \subset \mathbb{D}(w, 4)$, it follows that $\|G_n\| \leq c^{-1}\|\mathfrak{m}\|_{\mathbb{D}(w,4)}\|F_n\|$, so that the sequence G_n is also bounded. $\square$

4.3 Remark. While not relevant in the sequel, it is worth noting that any $F \in \mathcal{B} \cap \mathrm{C}_0(W)$ is actually in $\overline{\mathcal{L}}$. To see this take real $F_n \in \mathcal{L}$ with $\|F_n\| \leq 1$ and $F_n \to F$ locally uniformly. From $\|\cdot\|$-boundedness, Theorem 2.1 and $F \in \mathrm{C}_0(W)$, it follows that F_n converge to F weakly in $\mathrm{C}_0(W)$. Komlos' lemma [9, Thm. 3.13] gives us a sequence G_n of convex combinations of the F_n that converges strongly in $\mathrm{C}_0(W)$, so that indeed $F \in \overline{\mathcal{L}}$.

4.1 Chebyshev Sets and ω-Extremality

We shall consider the following notion from [12] of extremality among functions in $\mathcal{B}_\mathbb{R}$:

4.4 Definition. For $\omega \in \mathbb{C}^\pm$, $F \in \mathcal{B}_\mathbb{R}$ is called *ω-extremal* if $F(\omega) \neq 0$ and any function $G \in \mathcal{B}_\mathbb{R}$ with $G(\omega) = F(\omega)$ has $\|G\| \geq \|F\|$.

It is clear that ω-extremal functions exist: It suffices to take $F_0 \in \mathcal{B}_\mathbb{R}$ with $F_0(\omega) \neq 0$, and $F_n \in \mathcal{B}_\mathbb{R}$ with $F_n(\omega) = F_0(\omega)$ and $\|F_n\| \to r = \inf\{\|F\| : F \in \mathcal{B}_\mathbb{R}, F(\omega) = F_0(\omega)\}$, and get a locally uniformly convergent subsequence $F_n \to F \in \mathcal{B}_\mathbb{R}$. Then $F(\omega) = F_0(\omega)$ and

$$|F(x)| \leftarrow |F_n(x)| \leq \|F_n\|W(x) \to rW(x), \quad x \in \mathbb{R}, \tag{4.4}$$

so that $\|F\| = r$ (while we do *not* have $\|F - F_n\| \to 0$).

The following criterion [12, Thm. 1] for ω-extremality in terms of Chebyshev sets will show that ω-extremality does not depend on ω.

4.5 Definition. For $F \in \mathcal{B}_\mathbb{R}$, a discrete set $\Lambda \subset \mathbb{R}$ is called a *Chebyshev set* of F if $F(\lambda) = \pm\|F\|W(\lambda), \lambda \in \Lambda$ with alternating signs. Λ is called *maximal* if it is not contained in any strictly larger Chebyshev set.

We say that two discrete subsets Γ, Λ of $\mathbb{R}$ *interlace* if for $\gamma_1 < \gamma_2 \in \Gamma$, $(\gamma_1, \gamma_2) \cap \Lambda \neq \varnothing$, and for $\lambda_1 < \lambda_2 \in \Lambda$, $(\lambda_1, \lambda_2) \cap \Gamma \neq \varnothing$.

4.6 Definition. For a region $\Omega \subset \mathbb{C}$, $\mathcal{R}(\Omega) = \{F \in \mathrm{H}(\Omega) : \operatorname{Im} F \geq 0\}$.

Note that by the open mapping theorem, $F \in \mathcal{R}(\mathbb{C}^+)$ is either identically zero or $\operatorname{Im} F > 0$ in $\mathbb{C}^+$. For $F \in \mathrm{H}(\mathbb{C})$, let $Z(F)$ denote the set of zeros of F.

4.7 Theorem. *$A \in \mathcal{B}_\mathbb{R}$ is ω-extremal for some $\omega \in \mathbb{C}^+$ iff there is $B \in \mathrm{H}(\mathbb{C})$ real such that*

a) $Z(B)$ is a maximal Chebyshev set of A, and

b) for any $G \in \mathcal{B}_\mathbb{R}$ with $\|G\| \leq \|A\|$, $(A - G)/B \in \mathcal{R}(\mathbb{C}^+)$.

Sufficiency is easy to show: If $G(\omega) = A(\omega)$ and $\|G\| \leq \|A\|$, then $(A - G)/B \in \mathcal{R}(\mathbb{C}^+)$ vanishes at ω, hence in all of $\mathbb{C}$. This argument also establishes uniqueness of ω-extremal functions as soon as the theorem is proved, for which we will need a lemma [12, p. 7]:

4.8 Lemma. *If $A \in \mathcal{B}_\mathbb{R}$ is ω-extremal and $G \in \mathcal{B}_\mathbb{R}$ with $\|G\| < \|A\|$, then $A - G$ has real simple zeros and $Z(A - G)$ interlaces with any maximal Chebyshev set of A.*

Proof. We use the *Markov corrections*

$$A_\delta = \left(1 - \delta\frac{Q}{P}\right) A + \delta\frac{Q}{P}G = A + \delta\frac{Q}{P}(G - A) \in \mathcal{B}_\mathbb{R}, \quad A_\delta(\omega) = A(\omega), \tag{4.5}$$

where $\delta > 0$, $Q(z) = (z-\omega)(z-\overline{\omega})$, and $P(z) = (z-\alpha)(z-\beta)$ with $\alpha, \beta \in Z(A-G)$ either both real or conjugate. From the definition of $\|\cdot\|_W$ and an easy pointwise estimate (taking into account $\|G\| \le \|A\|$!), it follows that for $\Omega \subset \mathbb{R}$,

$$\|1_\Omega A_\delta\| \le (1-t_0)\|1_\Omega A\| + t_0\|1_\Omega G\|, \quad t_0 = \inf\{\delta|Q(x)/P(x)| : x \in \Omega\}, \quad |Q/P| \le \delta^{-1} \text{ on } \Omega. \tag{4.6}$$

1) If α is any nonreal zero of $G-A$ and $\beta = \overline{\alpha}$, $|P/Q|$ is bounded above and below on $\mathbb{R}$, so that for $\delta < \|P/Q\|_\infty^{-1}$, we get $\|A_\delta\| \le (1-t_0)\|A\| + t_0\|G\|$ with $t_0 > 0$, a contradiction.
2) Now assume α is a multiple real zero, and let $\beta = \alpha$. Let $\|G\| < \|A\| - 3\varepsilon$, and take a neigborhood U of α such that $|A-G| < \varepsilon W$ on U. Since Q/P is bounded outside U, (4.6) will clearly hold for $\Omega = \mathbb{R} \setminus U$ and all large δ. While Q/P is not bounded on U, $(G-A)/P$ is, so that for $\delta < \varepsilon\|Q(G-A)/P\|_W^{-1}$,

$$|A_\delta| \le |A| + \delta|Q(G-A)/P| \le |G| + 2\varepsilon W \text{ on } U. \tag{4.7}$$

Together, these estimates give the same contradiction.
3) From $\|G\| < \|A\|$, it is clear that $A-G$ changes sign and hence has a zero between any two consecutive points of a Chebyshev set of A. Conversely, assume $\alpha < \beta$ are consecutive zeros and $\|1_{(\alpha,\beta)}A\| < \|A\| - \varepsilon$. Since $A(\alpha) = G(\alpha), A(\beta) = G(\beta)$ and $\|G\| < \|A\|$, we may even assume $\|1_U A\| < \|A\| - 2\varepsilon$ for a neighborhood U of $[\alpha, \beta]$. Again Q/P is bounded outside U. Inside U, choose $\delta < \varepsilon\|A\|\|(G-A)/P\|_W^{-1}$, so that

$$|A_\delta| \le |A| + \delta|(G-A)/P| < (1-2\varepsilon)\|A\|W + \varepsilon\|A\|W = (1-\varepsilon)\|A\|W, \tag{4.8}$$

again giving a contradiction. ☐

4.9 Lemma. *A has infinitely many zeros.*

Proof. If A has N zeros, then any Chebyshev set Λ of A has at most $N+1$ points, and from the preceding lemma it follows that for $G \in \mathcal{L}$, the $A-G$ has at most $N+2$ zeros. It follows by Lemma A.3 that $(A-G)/A \in \mathrm{N}(\mathbb{C}^\pm)$ is actually a rational function, with the degree of both numerator and denominator bounded independently of G. Consequently, $\mathcal{L}$ must be finite-dimensional, contradicting our assumptions. ☐

4.2 Properties of $\mathcal{R}(\mathbb{C}^+)$

4.10 Lemma. *If $\Gamma, \Lambda \subset \mathbb{R}$ interlace, then there is a function Π meromorphic in $\mathbb{C}$ with simple zeros at Λ and simple poles at Γ such that $\operatorname{Im}\Pi \neq 0$ in $\mathbb{C}^\pm$.*

Proof (cf. [6, p. 308]). If $\Lambda = (\lambda_j), \Gamma = (\gamma_j)$ with $\gamma_j < \lambda_j < \gamma_{j+1}$, we define

$$\Pi(z) = \prod \frac{\gamma_j}{\lambda_j} \cdot \frac{z-\lambda_j}{z-\gamma_j}, \quad z \in \mathbb{C} \setminus \Gamma. \tag{4.9}$$

Since λ_j, γ_j interlace, the alternating series $\sum 1/\lambda_j - 1/\gamma_j$ converges, which gives locally uniform convergence of

$$\sum \left|1 - \frac{\gamma_j}{\lambda_j}\frac{z-\lambda_j}{z-\gamma_j}\right| = \sum \left|\frac{1-z/\gamma_j}{1-z/\gamma_j} - \frac{1-z/\lambda_j}{1-z/\gamma_j}\right| = |z| \sum \frac{1}{|1-z/\gamma_j|}\left(\frac{1}{\lambda_j} - \frac{1}{\gamma_j}\right). \tag{4.10}$$

The locally uniform convergence of Π follows by a standard argument [10, Thm. 15.5].

We compute

$$\arg\frac{1-z/\lambda_j}{1-z/\gamma_j} = \arg(1-z/\lambda_j) - \arg(1-z/\gamma_j) = \arg(z-\lambda_j) - \arg(z-\gamma_j) \ge 0 \tag{4.11}$$

to get

$$0 \le \arg \Pi(z) = \sum \arg(z-\lambda_j) - \arg(z-\gamma_j) \le \sum \arg(z-\lambda_j) - \arg(z-\lambda_{j+1}) = \pi. \tag{4.12}$$

□

Proof (Theorem 4.7). For Λ_1 a maximal Chebyshev set for A, $\Gamma_1 = Z(A)$ and Π_1 as in Lemma 4.10, define $B = A\Pi_1$, so that *a*) is clear. For *b*), let $\|G\| < \|F\|$ and put $\Phi = (A-G)/B$. Since $\Phi = (A/B)(1-G/A)$ with $A/B, G/A \in \mathrm{N}(\mathbb{C}^{\pm})$ (the first function being Π_1^{-1}, the second a quotient of functions in $\mathcal{B}$), we have $\Phi \in \mathrm{N}(\mathbb{C}^{\pm})$. If Π_2 has zeros $\Lambda_2 = Z(B) = \Lambda_1$ and poles $\Gamma_2 = Z(A-G)$, then $\Phi\Pi_2 \in \mathrm{N}(\mathbb{C}^{\pm})$. Since $\Phi\Pi_2$ is zero-free, it follows from Lemma A.8 that $\Phi = \Pi_2^{-1}$, so that $\operatorname{Im}\Phi \ne 0$ in $\mathbb{C}^{\pm}$. □

We cite a similar result on $\mathcal{R}(\mathbb{C}^+)$ without proof:

4.11 Čebotarev Theorem ([6, Thm. VII.2]). *If Φ is real meromorphic in $\mathbb{C}$ with simple poles in $\mathbb{R}$ and $\Phi \in \mathcal{R}(\mathbb{C}^+)$, then Φ has a representation*

$$\Phi(z) = az + b + \sum b_k \left(\frac{1}{z-a_k} - \frac{1}{a_k} \right), \quad \textit{with} \ \sum \frac{b_k}{a_k^2} < \infty, \tag{4.13}$$

where a_k are the poles of Φ, with residues b_k.

4.12 Corollary. $\Phi(iy) = o(y^2)$.

Proof. The above representation gives

$$|\Phi(iy)| \le a|y| + b \sum b_k \frac{|y|}{|iy-a_k||a_k|} \le a|y| + b + |y| \sum \frac{b_k}{a_k^2} = O(y) = o(y^2). \tag{4.14}$$

□

4.13 Lemma. *For $\Omega \subset \mathbb{C}$, $\mathcal{R}(\Omega) \subset \mathrm{N}(\Omega)$.*

Proof. Using the standard isomorphims $\varphi : \mathbb{D} \to \mathbb{C}^+ : z \to i\frac{1+z}{1-z}$ and $\psi = \varphi^{-1}$, we can represent $f \in \mathcal{R}(\Omega)$ as

$$f = \phi \circ \psi \circ f = i \frac{1 + \psi \circ f}{1 - \psi \circ f}, \tag{4.15}$$

which is clearly the quotient of bounded functions. □

4.3 Construction of a Krein Class Function

Let $A \in \mathcal{B}_{\mathbb{R}}$ be ω-extremal for some $\omega \in \mathbb{C}^+$ (hence also for $\overline{\omega}$ by symmetry). As above, let $\Gamma = Z(A)$ and Λ a maximal Chebyshev set for A, take $\Pi \in \mathcal{R}(\mathbb{C}^+)$ with poles Γ and zeros Λ, and let $B = A\Pi$. B is real with real simple zeros by construction, and $A/B = \Pi^{-1} \in \mathcal{R}(\mathbb{C}^{\pm})$, which gives $A/B \in \mathrm{N}(\mathbb{C}^{\pm})$ by the preceding section. For real $F \in \mathcal{L}$ and $G = \|A\|F/\|F\|$, $(A-G)/B \in \mathcal{R}(\mathbb{C}^+)$ by Theorem 4.7 *b*), so that $F/B \in \mathrm{N}(\mathbb{C}^+)$ and $yF(iy)/B(iy) = o(y^3)$ follow from the corresponding properties of A/B and the results of the preceding section.

By Theorem 4.11 (with $\Phi = A/B$, $a_k \in \Lambda$ and $b_k = \operatorname{Res}\Phi(a_k) = A(a_k)/B'(a_k)$),

$$\|A\| \sum_{\lambda\in\Lambda} \frac{W(\lambda)}{(1+\lambda^2)|B'(\lambda)|} = \sum_{\lambda\in\Lambda} \frac{|A(\lambda)|}{(1+\lambda^2)|B'(\lambda)|} < \infty. \tag{4.16}$$

So far, we have obtained a function B satisfying conditions 2) and 3) in the definition of a Krein class function only up to extra factors y^3, λ^2. If we assume $\{W < \infty\}$ to be non-discrete, then there must be infinitely many points $x \in \mathbb{R}$ with $B(x) \ne 0$ and $W(x) < \infty$. For three such points x_1, x_2, x_3, the function $(z-x_1)(z-x_2)(z-x_3)B(z)$ is actually in the Krein class.

A Some Results on Entire Functions

We state a few basic facts about entire functions of exponential and bounded type. All results without a reference are taken from [8, Ch. 6].

A.1 Definition. The *exponential type* of $F \in \mathrm{H}(\mathbb{C})$ is $\tau(F) = \limsup_{z\to\infty} \log|F(z)|/|z|$. F is said to be of exponential type if $\tau(F) < \infty$.

Equivalently, $\tau(F)$ is the infimum of all $t > 0$ such that $F(z)/e^{t|z|}$ is bounded.

A.2 Example. Polynomials have exponential type 0; e^{tz} has exponential type t.

A.3 Lemma. *Every zero-free entire function of exponential type is of the form ce^{tz} for some $t > 0$.*

Proof (cf. [5, pp. 18-19]). If F is zero-free, then $F = \exp\varphi$ with φ entire, and F having exponential type gives an estimate $\operatorname{Re}\varphi(z) \le a|z| + b$. If $\varphi(z) = \sum \gamma_n z^n$, then

$$\frac{1}{2\pi}\int_0^{2\pi} \operatorname{Re}\varphi(Re^{i\theta})e^{-in\theta}\,d\theta = \sum_{k=0}^{\infty} R^k\gamma_k \frac{1}{2\pi}\int_0^{2\pi} e^{ik\theta}e^{-in\theta}\,d\theta - R^k\overline{\gamma}_k\frac{1}{2\pi}\int_0^{2\pi} e^{-ik\theta}e^{-in\theta}\,d\theta \tag{A.1}$$

$$= R^n\gamma_n, \quad n > 0 \tag{A.2}$$

so that $|\gamma_n| \le R^{-n}(aR + b) \to 0$ as $R \to \infty$ for $n > 1$. It follows that φ is linear. ☐

A.4 Definition. $F \in \mathrm{H}(\Omega)$ is of *bounded type* in Ω (write $f \in \mathrm{N}(\Omega)$) if $\log^+|F|$ has a *harmonic majorant* in Ω, i.e. there is a harmonic function h in Ω such that $\log^+|F| \le h$.

A.5 Theorem. $F \in \mathrm{N}(\Omega)$ *iff* $F = f/g$ *with* $f, g \in \mathrm{H}^\infty(\Omega)$ *and* $g(z) \neq 0$ *in* Ω.

A.6 Lemma. *For any complex measure μ with $\int(1+|x|)^{-1}\,d|\mu| < \infty$, the* Cauchy integral

$$F(z) = \int \frac{d\mu(x)}{x - z}, \quad z \in \mathbb{C}\smallsetminus\mathbb{R} \tag{A.3}$$

defines an analytic function $F \in \mathrm{N}(\mathbb{C}^\pm)$.

Proof. By the Jordan decomposition, we may assume μ to be positive. But then $F \in \mathcal{R}(\mathbb{C}^+)$, so that the claim follows from Lemma 4.13. ☐

A.7 Krein's Theorem. $F \in \mathrm{H}(\mathbb{C})$ *is in* $\mathrm{N}(\mathbb{C}^\pm)$ *iff F is of exponential type and has finite logarithmic integral*

$$\int \frac{\log|F(x)|}{1 + x^2}\,dx < \infty. \tag{A.4}$$

In this case,

$$\tau(F) = \max\{\tau_+, \tau_-\}, \quad \tau_\pm = \limsup_{y\to\pm\infty} |y|^{-1}\log|F(iy)|. \tag{A.5}$$

A.8 Lemma. *Every zero-free entire function in* $\mathrm{N}(\mathbb{C}^\pm)$ *is constant.*

Proof. By Lemma A.3 and Krein's theorem, we only have to consider functions of the form $F(z) = e^{tz}$. But $t > 0$ is impossible again by Krein's theorem, since then the logarithmic integral of $F(z)$ is divergent. ☐

A.9 Phragmén-Lindelöf Principle. *Let F be analytic in the angle $A = A_\alpha = \{|\arg z| < \pi/2\alpha\}, \alpha \ge 1$ such that*

1) $\limsup_{z\to a}|F(z)| \le M, a \in \partial A$, *and*
2) $|F(z)| \le C\exp(|z|^\beta), z \in A$ *for some* $\beta < \alpha$.

Then F is bounded by M on A.

Proof. By slightly increasing $\beta < \alpha$, we may assume $|F(z)|/\exp(\eta|z|^\beta) \to 0, |z| \to \infty$ for all $\eta > 0$. On the simply connected region A, we can choose a branch of $G(z) = \exp(\eta z^\beta)$. For $z = x+iy = re^{i\theta} \in A$, we have

$$|G(z)| = \exp(\eta \operatorname{Re} z^\beta) = \exp(\eta r^\beta \cos(\beta\theta)), \quad \beta\theta < \pi/2, \tag{A.6}$$

so that $\exp(\eta r^\beta)/|G(z)|$ is bounded, and $|G(z)| \geq 1$ for $\arg z = \pi/2\alpha$. It follows that $H(z) = F(z)/G(z)$ is bounded by M on ∂A and tends to 0 as $z \to \infty$. Taking R large enough for $|H(Re^{i\theta})| < M, |\theta| < \pi/2\alpha$, the maximum principle gives

$$M \geq |H(z)| = \exp(\eta \operatorname{Re} z^\beta)|F(z)|, |z| < R. \tag{A.7}$$

Since $\exp(\eta \operatorname{Re} z^\beta) \to 1$ as $\eta \to 0$, this proves the claim. □

The following lemma (extracted from [2, Thm. 1.4.3, 6.2.4]) is necessary to complete the interpolation argument in the simple part of the proof of de Branges' theorem.

A.10 Lemma. *If an entire function F is of bounded type in $\mathbb{C}^\pm$ and tends to 0 along the imaginary axis, then F vanishes identically.*

Proof. First note that by (A.5), the assumption implies that F has exponential type 0, so that for $t > 0$ fixed, $G(z) = e^{-tz}F(z)$ is a function of exponential type bounded on each line $\arg z = \theta, |\theta| < \pi/2$, as well as on the imaginary axis by assumption. Therefore, the Phragmén-Lindelöf principle is applicable on the angles $A_\alpha, \alpha < 1$ as well as the first and fourth quadrant, which gives

$$\|G\|_{A_\alpha} \leq \|G\|_{\partial A_\alpha} \leq \max\{\|G\|_{i\mathbb{R}}, \|G\|_{\mathbb{R}+}\}, \quad \alpha < 1. \tag{A.8}$$

Letting $\alpha \searrow 1$, we see that G is actually bounded on the half plane $\{\operatorname{Re} z > 0\}$, so that another application of the Phragmén-Lindelöf principle gives

$$|F(z)| \leq e^{t \operatorname{Re} z}\|G\|_{i\mathbb{R}} = \|F\|_{i\mathbb{R}}, \quad \operatorname{Re} z > 0 \tag{A.9}$$

Letting $t \searrow 0$, it follows that F is bounded by $\|F\|_{i\mathbb{R}}$ on $\operatorname{Re} z > 0$, and the same works for $\operatorname{Re} z < 0$, so that F is actually bounded in $\mathbb{C}$, hence constant. □

References

[1] Baranov, A. and Woracek, H. *De Branges' Theorem on Approximation Problems of Bernstein Type*. In: *J. Inst. Math. Jussieu* 12.4 (2013), pp. 879–899.

[2] Boas, R. P. *Entire Functions*. Academic Press, 1954.

[3] De Branges, L. *The Bernstein Problem*. In: *Proc. Amer. Math. Soc.* 10.5 (1959), pp. 825–832.

[4] De Branges, L. *The Stone-Weierstrass Theorem*. In: *Proc. Amer. Math. Soc.* 10.5 (1959), pp. 822–824.

[5] Koosis, P. *The Logarithmic Integral I*. Cambridge University Press, 1997.

[6] Levin, B. J. *Distribution of Zeros of Entire Functions*. Translations of Mathematical Monographs 5. American Mathematical Society, 1980.

[7] Pitt, L. D. *A General Approach to Approximation Problems of the Bernstein Type*. In: *Adv. Math.* 49 (1983), pp. 264–299.

[8] Rosenblum, M. and Rovnyak, J. *Topics in Hardy Classes and Univalent Functions*. Birkhäuser Advanced Texts. Birkhäuser, 1994.

[9] Rudin, W. *Functional Analysis*. International Series in Pure and Applied Mathematics. McGraw-Hill, 1991.

[10] Rudin, W. *Real and Complex Analysis*. International Series in Pure and Applied Mathematics. McGraw-Hill, 1987.

[11] Sodin, M. and Yuditskii, P. *Another Approach to the de Branges' Theorem on Weighted Polynomial Approximation*. In: *Israel Math. Conf. Proc.* 11 (1997), pp. 221–227.

[12] Sodin, M. and Yuditskii, P. *Chebyshev Alternance and de Branges Theorem 66*. International Symposium on Orthogonal Polynomials and Special Functions – a Complex Analytic Perspective. 2012.

[13] Summers, W. H. *Dual Spaces of Weighted Spaces*. In: *Trans. Amer. Math. Soc.* 151.1 (1970), pp. 323–333.