DIPLOMARBEIT

# Growth of the bounds for the monodromy matrix of a canonical system

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# Kurzfassung

Diese Arbeit beschäftigt sich mit sogenannten Kanonischen Systemen, einer Klasse von zweidimensionalen Differentialgleichungen, die von einem komplexen Parameter $z$ und einer lokal integrierbaren matrixwertigen Funktion $H$, dem Hamiltonian, abhängig sind.
Konkret wird die Frage gestellt, wie sich die Monodromiematrix, wie wir im Falle eines gar integrierbaren Hamiltonians auf einem kompakten Intervall die Fundamentale Lösungsmatrix am rechten Intervallrand bezeichnen, in Abhängigkeit von $z$ verhält. Unter bestimmten Voraussetzungen liefert die schon lang bekannte Krein-de Branges Formel eine Abschätzung für die Norm der Fundamentallösung, diese Abschätzung liefert jedoch für Hamiltonians mit fast überall verschwindender Determinante keine sehr verwertbare Aussage.
Ganz anders verhält es sich mit einem Satz von Roman Romanov, der sich speziell solchen Hamiltonians widmet und mit dem die Norm der Monodromiematrix nach oben abgeschätzt werden kann. Insbesondere widmen wir uns einer Anwendung dieses Satzes für die Klasse der Hamburger Hamiltonians. Kann das Verhalten solcher Hamiltonians durch sogenannte „regularly varying functions“ beschrieben werden, liefert eine Arbeit von Harald Woracek eine Methode, das asymptotische Wachstum dieser oberen Schranke in Abhängigkeit des komplexen Parameters zu bestimmen.
Da dieser recht allgemeine Ansatz aber nicht immer zu einem genauen Ergebnis, sondern in manchen Randfällen nur eine Eingrenzung dieses asymptotischen Wachstums zulässt, werden wir uns zuletzt Hamburger Hamiltonians widmen, deren Verhalten durch ganz konkrete „regularly varying functions“, genauer durch bestimme „Lindelöf comparison functions“, beschrieben werden kann. Für diese werden wir das asymptotische Wachstum direkt bestimmen und insbesondere die angesprochenen Randfälle ausreizen. So gewinnen wir einen Einblick in die für die Randfälle auftretenden Phänomene und können nicht zuletzt die von Harald Woracek ermittelten Schranken auf ihre Schärfe hin testen.

# Abstract

This paper deals with so-called canonical systems, a class of two-dimensional differential equations which depend on a complex parameter $z$ and a locally integrable matrix-valued function $H$, the Hamiltonian.
Specifically, in case of an integrable Hamiltonian on a compact interval, we ask how the monodromy matrix, as we call the fundamental solution on the right-hand boundary of the interval, behaves as a function of $z$. Under certain conditions, the long known Krein-de Branges formula provides an estimate for the norm of the fundamental solution, but this estimate does not provide very significant information for Hamiltonians with almost everywhere vanishing determinant.
The situation is quite different with a theorem by Roman Romanov, which specifically deals with such Hamiltonians and gives an upper bound for the norm of the monodromy matrix. In particular, we devote ourselves to an application of this theorem for the class of Hamburger Hamiltonians. If the behaviour of such Hamiltonians can be described by so-called regularly varying functions, a paper by Harald Woracek provides a method to determine the asymptotic growth of said upper bound depending on the complex parameter. This rather general approach, however, does not always lead to an exact result, but in some edge cases only yields lower and upper bounds for said asymptotic growth. This is why we finally devote ourselves to Hamburger Hamiltonians whose behaviour can be described by quite concrete regularly varying functions, more precisely by certain Lindelöf comparison functions. For these we will directly determine the asymptotic growth and in particular exhaust the mentioned edge cases. Thereby, we gain an insight into the phenomena possibly occurring for the edge cases, and we are able to test the bounds determined by Harald Woracek with respect to their sharpness.

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# Eidesstattliche Erklärung

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Wien, am 23. Mai 2022

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# Notation

The function $\log^+ \colon \mathbb{R} \to [0,\infty)$ is defined as

$$\log^+(x) \coloneqq \max\{0, \log x\}.$$

Given two functions $f, g\colon [t_0,\infty) \to (0,\infty)$, we will use the following notations.

$$\begin{aligned}
&f \lesssim g :\Longleftrightarrow \exists t_1 \geq t_0, C > 0\colon f(t) \leq Cg(t), \quad t \geq t_1,\\
&f \gtrsim g :\Longleftrightarrow g \lesssim f,\\
&f \asymp g :\Longleftrightarrow f \lesssim g \wedge g \lesssim f.
\end{aligned}$$

Further, we write

$$\begin{aligned}
&f \ll g :\Longleftrightarrow \lim_{t\to\infty} \frac{f(t)}{g(t)} = 0,\\
&f \gg g :\Longleftrightarrow g \ll f,
\end{aligned}$$

and

$$f \sim g :\Longleftrightarrow \lim_{t\to\infty} \frac{f(t)}{g(t)} = 1.$$

Although we will only be dealing with Lebesgue integrals, we will use the convention for Riemann integrals that if $a < b$, we set

$$\int_b^a f(s)\,\mathrm{d}s \coloneqq -\int_a^b f(s)\,\mathrm{d}s.$$

# 1 Introduction

Canonical systems are two-dimensional differential equations of the form

$$y'(t) = zJH(t)y(t),\ t \in (a,b), \tag{1.1}$$

where $z$ denotes a complex number, $J = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$, $-\infty \leq a < b \leq \infty$ and $H$ is a matrix-valued locally integrable function on $(a,b)$ which only vanishes on a null set.
Suppose $-\infty < a$ and that there is $c \in (a,b)$ such that $\int_a^c \|H(t)\| \,\mathrm{d}t < \infty$. Then $H$ can also be viewed as locally integrable on $[a,b)$, and we call $a$ a $L^1$-*boundary point* of $H$. Analogously, if $b < \infty$ and there is $c \in (a,b)$ such that $\int_c^b \|H(t)\| \,\mathrm{d}t < \infty$, $b$ is also called a $L^1$-boundary point of $H$. By $\int a,b \wr$, we denote the interval $(a,b)$ complemented by $H$'s $L^1$-boundary points.
Given a Hamiltonian $H \in L^1_{\text{loc}}\left((a,b), \mathbb{C}^{2\times 2}\right)$ and any $c \in \int a,b \wr$, one can construct what we will denote as the *fundamental solution* of $H$, a function $W_H(t,z)\colon \int a,b \wr \times \mathbb{C} \to \mathbb{C}^{2\times 2}$ such that for all $z \in \mathbb{C}$, $W_H(\text{\textvisiblespace}, z)$ is a fundamental system for (1.1), see Theorem 2.13. We call $c$ the *centre* of $W_H$. As can be seen in the sections 2.2 as well as 2.3, $W_H$ has many nice and useful properties, such as $z \mapsto W_H(t,z)$ being an analytic function for any $t \in \int a,b \wr$, and the rest of this paper will revolve around studying $W_H(t,z)$ as a function of $z$. In particular, if $H \in L^1([a,b], \mathbb{C}^{2\times 2})$ for some compact interval $[a,b]$, we will consider the fundamental solution with centre a and denote $W_H(b,z)$ the *monodromy matrix* of $H$. Our main goal is to estimate the asymptotic growth of $\|W_H(b,z)\|$ for $|z| \to \infty$. The first in this direction happens in section 2.4, where the *exponential type*, which is a way to compare the growth of a function to exponential growth, of $\|W_H(t,z)\|$ will be estimated. For certain Hamiltonians, the Krein-de Branges formula even lets us precisely determine the exponential type, see Theorem 2.38. It shows, however, that if $\det H = 0$ a.e. (almost everywhere), the Krein-de Branges Formula always yields exponential type zero, making it impossible to use this growth scale to compare the growth of monodromy matrices corresponding to Hamiltonians with a.e. vanishing determinant.
As a consequence, we will turn to different, and much younger, theory, starting with a Theorem by Roman Romanov at the beginning of Chapter 3 which is solely focused on Hamiltonians with $\det H = 0$ a.e., see Theorem 3.3. The original version of this Theorem can be found via [Rom17, Theorem 1], but the theorem we will use is a refined and, in fact, stronger version which was developed by Raphael Pruckner and Harald Woracek and can be accessed, in slightly different versions, via [Pru17, Theorem 3.3] and [PW22, Theorem 4.1]. The theorem gives an asymptotic upper bound for $\|W_H(b,z)\|$ for growing $|z|$.
The theorem finds its use in the subsequent Section 3.2. From this point onward, we focus on a certain class of Hamiltonians called *Hamburger Hamiltonians*, which are trace-normed and piece-wise constant in addition to satisfying $\det H = 0$ a.e and are defined in Definition 3.7. Through a clever application of the refined version of Romanov's theorem,

once more developed by Harald Woracek and to be published in [Wor], it is possible to find a computable asymptotic upper bound $B(R)$ for $\max_{|z|=R} \|W_H(b,z)\|$ as long as the Hamburger Hamiltonian satisfies a few regularity conditions, i.e. the Hamiltonian's lengths and angles need to be approximated by certain continuous and nonincreasing functions, see Definition 3.8 and Theorem 3.12.
Within [Wor, Chapter 3], the author applies this method to Hamburger Hamiltonians whose lengths and angles can be approximated by so-called *regularly varying functions*, which are functions with an asymptotic behaviour similar to a power function. In some edge cases, however, this general approach does not yield exact results but only lower and upper bounds for $B(R)$. In order to gain insight into the behaviour of $B(R)$ in these edge cases, Chapter 5 is dedicated to an exact calculation of $B(R)$ when considering more concrete regularly varying functions approximating the decay of the lengths and angles.
For these calculations, however, some theory on regularly varying functions is necessary. Therefore, the first two sections of the previous Chapter 4 are dedicated to some basic theory on regularly varying functions and Karamata's Theorem, respectively. This theorem, which will be essential for our work in Chapter 5, shows how regularly varying functions can be asymptotically integrated.
In Section 4.3, we develop a method to asymptotically invert certain regularly varying functions, the likes of which we are going to work with in the following chapter, and which is therefore, besides Karamata's Theorem, the second key result of Chapter 4.
The regularly varying functions we consider in Chapter 5 are functions of the form $f(t) = t^{\Delta} \log^{\alpha}(t)$. They form a subset of the *Lindelöf comparison functions*, which are introduced in Example 4.6 and are chosen to match what Woracek denotes a *smooth majorisation* in [Wor, Definition 3.6]. As these functions will be given using six parameters $\Delta_l$, $\Delta_\phi$, $\alpha_l$, $\alpha_\phi$, $\mu$ and $\nu$ whereby different restrictions are placed on all parameters, a multitude of cases must be distinguished and very different bounds $B(R)$ occur for different configurations of the parameters, with the most interesting cases being those in which [Wor] does not give an asymptotic bound $B(R)$ or only estimates it from above and below, as he does in [Wor, Theorem 3.12]. When comparing our calculations to those bounds in Section 5.5, it shows that the majority, though not all, are actually being attained within the example we study in the chapter, implying that they are in fact sharp.

# 2 Canonical Systems

This chapter covers some basic theory on (two-dimensional) canonical systems and in particular its fundamental solutions. It is mostly based on lecture notes for a lecture held by Harald Woracek in the Fall of 2019.[1]

## 2.1 The equation

**Definition 2.1.** A canonical system is an equation of the form

$$y'(t) = zJH(t)y(t),\ t \in (a,b), \tag{2.1}$$

where

- $-\infty \le a < b \le \infty$,
- $z \in \mathbb{C}$,
- $H \in L^1_{\text{loc}}\big((a,b), \mathbb{C}^{2\times 2}\big)$,
- $\{t \in (a,b) : H(t) = 0\}$ is a null set,
- $J := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$,
- $y\colon (a,b) \to \mathbb{C}^2$.

We will denote the function $H$ the *Hamiltonian* of the canonical system (2.1).

**Definition 2.2.** For an interval $I \subseteq \mathbb{R}$ and a normed space $X$ we denote by $\mathrm{AC}_{\text{loc}}(I, X)$ the set of all locally absolutely continuous functions of $I$ into $X$. Furthermore, we denote by $\mathrm{BM}_{\text{loc}}(I, X)$ the set of all locally bounded measurable functions of $I$ into $X$.

**Definition 2.3.** Let $H \in L^1_{\text{loc}}\big((a,b), \mathbb{C}^2\big)$. A function $y\colon (a,b) \to \mathbb{C}^2$ is called a *solution of* (2.1) if $y \in \mathrm{AC}_{\text{loc}}$ and $y'(t) = zJH(t)y(t)$ a.e.

**Definition 2.4.** The *integral form* of a canonical system is an integral equation of the form

$$y(t) = y(c) + \int_c^t zJH(s)y(s)\,\mathrm{d}s,\ t \in (a,b), \tag{2.2}$$

for some $c \in (a,b)$ and $a, b, H, J$ as well as $y$ of the same form as in Definition 2.1.

[1] The notes can be found here: https://www.asc.tuwien.ac.at/~woracek/homepage/main.php.

**Definition 2.5.** Let $H \in L^1_{\text{loc}}\left((a,b),\mathbb{C}^2\right)$. A function $y\colon (a,b) \to \mathbb{C}^2$ is called a *solution of* (2.2) if $y \in \text{BM}_{\text{loc}}$ and $y(t) = y(c) + \int_c^t zJH(s)y(s)\,\text{d}s$ holds for some $c \in (a,b)$ and all $t \in (a,b)$.

**Remark 2.6.** By the fundamental theorem of calculus, the following statements are equivalent for any function $y\colon (a,b) \to \mathbb{C}^2$.

(i) $y$ is a solution of (2.1).

(ii) $y$ is a solution of (2.2) for some $c \in (a,b)$.

(iii) $y$ is a solution of (2.2) for all $c \in (a,b)$.

**Definition 2.7.** Let $H \in L^1_{\text{loc}}\left((a,b),\mathbb{C}^2\right)$. We call $a$ an $L^1$*-boundary point* of $H$ if $-\infty < a$ and there exists $c \in (a,b)$ such that $\int_a^c \|H(t)\|\,\text{d}t < \infty$. Analogously, we call $b$ a $L^1$*-boundary point* of $H$ if $b < \infty$ and there exists $c \in (a,b)$ such that $\int_c^b \|H(t)\|\,\text{d}t < \infty$.

**Remark 2.8.** As $H \in L^1_{\text{loc}}\left((a,b),\mathbb{C}^2\right)$, the boundedness of $\int_a^c \|H(t)\|\,\text{d}t$ for *one* $c \in (a,b)$ is equivalent to the boundedness of $\int_a^c \|H(t)\|\,\text{d}t$ for *all* $c \in (a,b)$. The same holds for $\int_c^b \|H(t)\|\,\text{d}t$.

**Definition 2.9.** Let $H_1 \in L^1_{\text{loc}}\left((a_1,b_1),\mathbb{C}^2\right)$ and $H_2 \in L^1_{\text{loc}}\left((a_2,b_2),\mathbb{C}^2\right)$. We call $H_2$ a *reparameterisation* of $H_1$ and write $H_1 \rightleftharpoons H_2$ if there exists an increasing bijection $\phi\colon (a_2,b_2) \to (a_1,b_1)$ such that $\phi \in \text{AC}_{\text{loc}}\left((a_2,b_2),(a_1,b_1)\right)$, $\phi^{-1} \in \text{AC}_{\text{loc}}\left((a_1,b_1),(a_2,b_2)\right)$ and $H_2 = (H_1 \circ \phi) \cdot \phi'$ a.e.

If we want to emphasize the bijection used, we write $H_1 \overset{\phi}{\leftrightharpoons} H_2$ and say that $H_1$ is *reparameterised to $H_2$ via $\phi$*.

**Lemma 2.10.**

(i) *The relation $\leftrightharpoons$ is an equivalence relation on*

$$\bigcup_{-\infty \le a < b \le \infty} L^1_{\text{loc}}\left((a,b),\mathbb{C}^2\right),$$

*the set of all Hamiltonians.*

(ii) *Let $H_1 \in L^1_{\text{loc}}\left((a_1,b_1),\mathbb{C}^2\right)$ and $H_2 \in L^1_{\text{loc}}\left((a_2,b_2),\mathbb{C}^2\right)$ such that $H_1 \overset{\phi}{\leftrightharpoons} H_2$. Then the map*

$$\_\circ\phi\colon \begin{cases} \left[\mathbb{C}^2\right]^{(a_1,b_1)} & \to & \left[\mathbb{C}^2\right]^{(a_2,b_2)} \\ f & \mapsto & f\circ\phi \end{cases}$$

*induces a bijection of the set of solutions of the canonical system with Hamiltonian $H_1$ onto the set of solutions of the canonical system with Hamiltonian $H_2$.*

(iii) *Let again $H_1 \overset{\phi}{\leftrightharpoons} H_2$. Then $b_1$ is an $L^1$-boundary point of $H_1$ if and only if $b_2$ is a $L^1$-boundary point of $H_2$, and the same holds for $a_1$ and $a_2$.*

*Proof.*

(i) The relation $\leftrightharpoons$ is clearly reflexive, as all Hamiltonians $H$ satisfy $H \overset{id}{\leftrightharpoons} H$. To show the symmetry, let $H_1 \overset{\phi}{\leftrightharpoons} H_2$. This leads to

$$\begin{aligned}(H_2 \circ \phi^{-1}) \cdot (\phi^{-1})' &= ([(H_1 \circ \phi) \cdot \phi'] \circ \phi^{-1}) \cdot (\phi^{-1})' \\ &= (H_1 \circ \phi) \circ \phi^{-1} \cdot (\phi' \circ \phi^{-1}) \cdot (\phi^{-1})' \\ &= H_1 \;\; a.e.,\end{aligned}$$

which means $H_2 \overset{\phi^{-1}}{\leftrightharpoons} H_1$. Now let $H_1 \overset{\phi}{\leftrightharpoons} H_2$ and $H_2 \overset{\psi}{\leftrightharpoons} H_3$. We receive

$$\begin{aligned}H_3 &= (H_2 \circ \psi) \cdot \psi' = ([(H_1 \circ \phi) \cdot \phi'] \circ \psi) \cdot \psi' \\ &= (H_1 \circ (\phi \circ \psi)) \cdot (\phi' \circ \psi) \cdot \psi' = H_1 \circ (\phi \circ \psi) \cdot (\phi \circ \psi)' \;\; a.e.,\end{aligned}$$

implying $H_1 \overset{\phi\circ\psi}{\leftrightharpoons} H_3$.

(ii) Let $y$ be a solution of the canonical system with the Hamiltonian $H_1$ and let $c_2, t_2 \in (a_2, b_2)$. As $y$ is a solution, we get

$$\begin{aligned}(y \circ \phi)(t_2) - (y \circ \phi)(c_2) &= \int_{\phi(c_2)}^{\phi(t_2)} zJH_1(s)y(s)\,\mathrm{d}s \\ &= \int_{c_2}^{t_2} zJ(H_1 \circ \phi)(u) \cdot (y \circ \phi)(u) \cdot \phi'(u)\,\mathrm{d}u = \int_{c_2}^{t_2} zJH_2(u) \cdot (y \circ \phi)(u)\,\mathrm{d}u,\end{aligned}$$

showing us that $y \circ \phi$ is a solution of the canonical system with $H_2$. As was already shown in the proof of (i), we have $H_2 \overset{\phi^{-1}}{\leftrightharpoons} H_1$, and it is easy to see that $(\_\circ\phi^{-1})\circ(\_\circ\phi) = id_{[\mathbb{C}^2]^{(a_1,b_1)}}$ and $(\_ \circ \phi) \circ (\_ \circ \phi^{-1}) = id_{[\mathbb{C}^2]^{(a_2,b_2)}}$. Thus, the assertion holds.

(iii) Assume that $b_1$ is an $L^1$-boundary point and let $c_2 \in (a_2, b_2)$. Defining $c_1 := \phi(c_2)$ and bearing Remark 2.8 in mind yields

$$\begin{aligned}\int_{c_2}^{b_2} \|H_2(t)\|\,\mathrm{d}t &= \int_{c_2}^{b_2} \|(H_1 \circ \phi)(t) \cdot \phi'(t)\|\,\mathrm{d}t = \int_{c_2}^{b_2} \|(H_1 \circ \phi)(t)\| \cdot \phi'(t)\,\mathrm{d}t \\ &= \int_{c_1}^{b_1} \|H_1(u)\|\,\mathrm{d}u < \infty,\end{aligned}$$

hence $b_2$ is an $L^1$-boundary point. Using $H_2 \overset{\phi^{-1}}{\leftrightharpoons} H_1$ yields the opposite implication. The assertion concerning $a_1$ and $a_2$ can be shown in the same way.

□

**Definition 2.11.** From now on, whenever we want to formulate a statement about an interval without wanting or being able to determine whether a boundary point is contained

in the interval, we will use the symbols $\int$ for the left and $\wr$ for the right boundary. For example, $\int a, b]$ stands for either $(a, b]$ or $[a, b]$.
In particular, whenever we are referring to a Hamiltonian $H \in L^1_{\text{loc}}((a, b), \mathbb{C}^{2\times 2})$ we denote by $\int a, b\wr$ the interval containing $L^1$-boundary points and excluding non-$L^1$-boundary points of $H$.

**Example 2.12.** Let $H \in L^1_{\text{loc}}((a, b), \mathbb{C}^{2\times 2})$. Given $c \in \int a, b\wr$ and $\delta \in L^1((a, b), \mathbb{R})$ such that $\delta$ is everywhere positive, we set

$$\nu(t) := \begin{cases} \|H(t)\|, & t \in (a, b),\ H(t) \neq 0, \\ \delta(t), & t \in (a, b),\ H(t) = 0, \end{cases} \qquad \phi(t) := \int_c^t \nu(s)\,\mathrm{d}s.$$

The constructed function $\phi$ is an increasing bijection of $[a, b]$ onto $[\phi(a), \phi(b)]$ and is locally absolutely continuous on $(a, b)$. Besides that, as $\phi'(t) = \nu(t) > 0$ a.e., the inverse function $\phi^{-1}$ is locally absolutely continuous on $(\phi(a), \phi(b))$. Setting $\tilde{H} := (H \circ \phi^{-1}) \cdot (\phi^{-1})'$ gives $\tilde{H} \in L^1_{\text{loc}}((\phi(a), \phi(b)), \mathbb{C}^{2\times 2})$ and furthermore $H \overset{\phi^{-1}}{\Longrightarrow} \tilde{H}$.
By construction, we have $H = (\tilde{H} \circ \phi) \cdot \phi'$, which in turn implies $\|H(t)\| = \|\tilde{H}(\phi(t))\| \cdot \nu(t)$ a.e. on $t \in (a, b)$. Therefore, setting $\Delta = \{t \in (a, b) | H(t) \neq 0\}$, we see that $\|\tilde{H}\| = \mathbb{1}_{\phi(\Delta)}$ a.e.

## 2.2 Existence and uniqueness of solutions

**Theorem 2.13.** *Let $-\infty \le a < b \le \infty$, $H \in L^1_{\text{loc}}\left((a, b), \mathbb{C}^{2\times 2}\right)$ and $c \in \int a, b\wr$. Then there exists a function*

$$W_H \colon \int a, b \wr \times \mathbb{C} \to \mathbb{C}^{2\times 2}$$

*with the following properties.*

(i) *For fixed $z \in \mathbb{C}$, the function $W_H(\_, z) \colon [a, b\wr \to \mathbb{C}^{2\times 2}$ is continuous and satisfies*

$$\forall t \in \int a, b\wr : W_H(t, z)J - J = z \int_c^t W_H(s, z)H(s)^T\,\mathrm{d}s. \tag{2.3}$$

(ii) *For fixed $t \in \int a, b\wr$, the function $W_H(t, \_) \colon \mathbb{C} \to \mathbb{C}^{2\times 2}$ is an entire function. The coefficients $W_{H,l}(t)$, $l \in \mathbb{N}$, in its power series expansion*

$$W_H(t, z) = \sum_{l=0}^{\infty} W_{H,l}(t) z^l \tag{2.4}$$

*satisfy the recurrence*

$$W_{H,0}(t) = I, \ \ \forall l \in \mathbb{N}_0 \colon W_{H,l+1}(t) = -\int_c^t W_{H,l}(s)H(s)^T J\,\mathrm{d}s. \tag{2.5}$$

(iii) *For any* $t \in \int a, b\wr$ *and* $z \in \mathbb{C}$, *we have*

$$\|W_H(t,z) - I\| \leq \exp\left(|z| \cdot \left|\int_c^t \|H(s)\| \, \mathrm{d}s\right|\right) - 1. \tag{2.6}$$

Before we proceed to the proof of this theorem, we give a short lemma. A motivation for the estimate given there can be seen as following. Supposing that the function $W_H$ can be written as a power series as described in (2.4) and (2.5), the estimate (2.6) can be rewritten to

$$\begin{aligned} &\|W_H(t,z) - I\| \leq \exp\left(|z| \cdot \left|\int_c^t \|H(s)\| \, \mathrm{d}s\right|\right) - 1 \\ \Leftrightarrow \ &\| \sum_{l=1}^{\infty} W_{H,l}(t) z^l \| \leq \sum_{l=1}^{\infty} \frac{|z|^l \left|\int_c^t \|H(s)\| \, \mathrm{d}s\right|^l}{l!}. \end{aligned}$$

From this inequality, however, it is only natural to expect that also

$$\|W_{H,l}(t)\| \leq \frac{\left|\int_c^t \|H(s)\| \, \mathrm{d}s\right|^l}{l!},$$

which we will prove in the following lemma.

**Lemma 2.14.** *Let* $-\infty \leq a < b \leq \infty$, $H \in L^1_{\text{loc}}((a,b), \mathbb{C}^{2\times 2})$ *and* $c \in \int a, b\wr$. *Then, the recurrence (2.5) yields a well-defined sequence of continuous functions* $W_{H,l}(t) \colon \int a, b\wr \to \mathbb{C}^{2\times 2}$. *Moreover, it holds that*

$$\forall l \in \mathbb{N} \; \forall t \in \int a, b\wr \colon \|W_{H,l}(t)\| \leq \frac{\left|\int_c^t \|H(s)\| \, \mathrm{d}s\right|^l}{l!}. \tag{2.7}$$

*Proof of the Lemma.* The proof is by induction on l. Clearly, the constant function $W_0(H, \_)$ is $\mathrm{AC}_{\text{loc}}(\int a, b\wr)$ and it holds for all $t \in \int a, b\wr$ that

$$\|W_{H,o}(t)\| = \|I\| = \frac{\left|\int_c^t \|H(s)\| \, \mathrm{d}s\right|^0}{0!}.$$

Now, suppose that $W_{H,l}(t) \in \mathrm{AC}_{\text{loc}}(\int a, b\wr)$. It follows that $W_{H,l}(t) H(\_)^T J \in L^1_{\text{loc}}((a,b), \mathbb{C}^{2\times 2})$ and hence

$$W_{H,l+1}(t) = -\int_c^t W_{H,l}(s) H(s)^T J \, \mathrm{d}s \in \mathrm{AC}_{\text{loc}}(\int a, b\wr).$$

Thus, the sequence is well-defined and consists of continuous functions. In order to further prove the asserted estimate, we define the function

$$\varphi(t) := \int_c^t \|H(t)\| \, \mathrm{d}t,$$

for which it holds that $\varphi'(t) = \|H(t)\|$. Moreover, it holds for all $t \in \int a, b \wr$ that

$$\|W_{H,l}(t)\| \leq \frac{|\varphi(t)|^l}{l!}$$

by our induction hypothesis. An easy computation shows

$$\begin{aligned}
\|W_{H,l+1}(t)\| &= \left\| \int_c^t W_{H,l}(s) H(s)^T J \, \mathrm{d}s \right\| \leq \left| \int_c^t \|W_{H,l}(s)\| \left\|H(s)^T\right\| \, \mathrm{d}s \right| \\
&\leq \left| \int_c^t \frac{|\varphi(s)^l|}{l!} \varphi'(s) \, \mathrm{d}s \right| = \left| \int_c^t \frac{\varphi(s)^l}{l!} \varphi'(s) \, \mathrm{d}s \right| = \frac{1}{l!} \left| \int_0^{\varphi(t)} u^l \, \mathrm{d}u \right| \\
&= \frac{|\varphi(t)^{l+1}|}{l!(l+1)} = \frac{\left| \int_c^t \|H(s)\| \, \mathrm{d}s \right|^{l+1}}{(l+1)!}.
\end{aligned}$$

□

**Remark 2.15.** It is easily seen from the proof of the previous lemma that while the estimate (2.7) is shown for the spectral norm, it actually holds for any matrix norm on $\mathbb{C}^{2\times 2}$.

*Proof of the Theorem.* We start by constructing $(W_{H,l})_{l\in\mathbb{N}}$ by the recursion given in (2.5). As we can see from (2.7), the right-hand side of (2.4) converges uniformly for $(t, z) \in [a', b'] \times K$ for any compact interval $[a', b'] \in \int a, b \wr$ and any compact $K \subseteq \mathbb{C}$. Therefore, we can and will use (2.4) as the definition of $W_H(t, z)$. The resulting function satisfies the following conditions.

- Uniformity in $t$ for any given $z \in \mathbb{C}$ implies that $W_H(\_, z)$ is continuous. Now let $t \in \int a, b \wr$, and without loss of generality, assume that $c \leq t$. When restricted to $[c, t]$, the sequence $\left(\left(\sum_{l=0}^N W_{H,l}(s) z^l\right) H(s)^T\right)_{N\in\mathbb{N}}$ is converging a.e. to $W_H(s, z)H(s)^T$ and is dominated as follows.[2]

$$\begin{aligned}
\left\| \left( \sum_{l=0}^N W_{H,l}(s) z^l \right) H(s)^T \right\| &\leq \left\| \sum_{l=0}^N W_{H,l}(s) z^l \right\| \left\|H(s)^T\right\| \leq \sum_{l=0}^\infty \left\| W_{H,l}(s) z^l \right\| \left\|H(s)^T\right\| \\
&\leq \exp\left( |z| \cdot \left| \int_c^s \|H(u)\| \, \mathrm{d}u \right| \right) \left\|H(s)^T\right\| \\
&\leq \underbrace{\exp\left( |z| \cdot \left| \int_c^t \|H(u)\| \, \mathrm{d}u \right| \right)}_{\text{constant}} \underbrace{\left\|H(s)^T\right\|}_{\text{integrable}}
\end{aligned}$$

In particular, all four entries of $\left(\sum_{l=0}^N W_{H,l}(s) z^l\right) H(s)^T$ are bounded by an integrable function and converging a.e. to the corresponding entries of $W_H(s, z)H(s)^T$.

[2] In this calculation, we drop explicit notation of the restrictions, but only consider $s \in [c, t]$.

Therefore, it holds by the Dominated Convergence Theorem that

$$\begin{aligned} z\int_c^t W_H(s,z)H(s)^T\,\mathrm{d}s &= z\int_c^t \left(\sum_{l=0}^{\infty} W_{H,l}(s)z^l\right) H(s)^T\,\mathrm{d}s \\ &= \sum_{l=0}^{\infty}\left(\int_c^t W_{H,l}(s)H(s)^T\,\mathrm{d}s\right) z^{l+1} = \sum_{l=0}^{\infty} W_{l+1}(H,s)Jz^{l+1} = W_H(t,z)J - J. \end{aligned}$$

- Local uniformity in $z$ for any given $t \in \smallint a, b \wr$ implies that $W_H(t, \_)$ is analytic.
- Taking into account (2.7), it holds for all $t \in \smallint a, b \wr$ that

$$\begin{aligned} \|W_H(t,z) - I\| \leq \sum_{l=1}^{\infty}\left(\|W_{H,l}(t)\| \cdot |z|^l\right) &\leq \sum_{l=1}^{\infty} \frac{\left|\int_c^t \|H(s)\|\,\mathrm{d}s\right|^l |z|^l}{l!} \\ &= \exp\left(|z| \cdot \left|\int_c^t \|H(s)\|\,\mathrm{d}s\right|\right) \end{aligned}$$

□

**Definition 2.16.** Let $-\infty \leq a < b \leq \infty$, $H \in L^1_{\mathrm{loc}}((a,b), \mathbb{C}^{2\times 2})$ and $c \in \smallint a, b \wr$. We call the function $W_H(t,z)$ obtained in Theorem 2.13 the *fundamental solution* of $H$ (with *centre* $c$). If, in addition, $b$ is an $L^1$-boundary point, we call $W_H(b,z)$ the *monodromy matrix* of $H$.

**Remark 2.17.** By Theorem 2.13, a fundamental solution $W_H(t,z)$ can be constructed for any $c \in \smallint a, b \wr$. Subsequently, whenever we are given $-\infty \leq a < b \leq \infty$, $H \in L^1_{\mathrm{loc}}((a,b), \mathbb{C}^{2\times 2})$ and $c \in \smallint a, b \wr$, we denote by $W_H(t,z)$ the fundamental solution of $H$ with *centre* $c$, unless otherwise stated.
In the case of Hamiltonians defined on (half-)bounded intervals for which one of the finite boundary points is an $L^1$-boundary point, it is often practical to choose that very point as the centre of their fundamental solutions. For instance, when studying Hamiltonians $H \in L^1([a,b], \mathbb{C}^{2\times 2})$, which we will do in Chapter 3, we will consider their fundamental solutions $W_H$ with centre $a$.

**Remark 2.18.** As we will see from Proposition 2.23 in combination with the construction of the solution in Proposition 2.22, any solution of (2.1) (for $H \in L^1_{\mathrm{loc}}((a,b], \mathbb{C}^{2\times 2})$) and $c \in \smallint a, b])$ can be written as $y(t) = W_H(t,z)^T y(c)$. Therefore, studying the monodromy matrix is a key element of understanding solutions of the differential equation, as it lets us extrapolate to the behaviour of solutions on the right border of the interval. In the upcoming chapters, we will focus on estimating the norm $\|W_H(b,z)\|$ as a function of $z$.

**Remark 2.19.** Let $-\infty \leq a < b \leq \infty$, $H \in L^1_{\mathrm{loc}}((a,b), \mathbb{C}^{2\times 2})$, $c \in \smallint a, b \wr$ and $z \in \mathbb{C}$. Taking Theorem 2.14(i) into account, it is easy to observe that the function $t \mapsto W_H(t,z)$ is locally absolutely continuous. Moreover, it satisfies

$$W_H(c,z) = I, \quad \frac{\partial}{\partial t} W_H(t,z)J = zW_H(t,z)H(t)^T \text{ for a.a. } t \in \smallint a, b \wr\,. \tag{2.8}$$

If $-\infty < a$ and $a$ is an $L^1$-boundary point, we have $W_H(\_, z) \in \mathrm{AC}_{\mathrm{loc}}([a,b), \mathbb{C}^{2\times 2})$. Analogously, in case of $b$ being an $L^1$-boundary point and $b < \infty$, it follows that $W_H(\_, z) \in \mathrm{AC}_{\mathrm{loc}}((a,b], \mathbb{C}^{2\times 2}$. As both $H$ and $W_H$ are mappings into $\mathbb{C}^{2\times 2}$, we can write them as $H(t) = (h_{ij}(t))_{i,j=1}^2$ and $W_H(t,z) = (w_{ij}(t,z))_{i,j=1}^2$. Using this notation, the differential equation in (2.8) is equivalent to the following system of scalar differential equations.

$$\begin{aligned}
\frac{\partial}{\partial t} w_{11}(t,z) &= -z\left(w_{11}(t,z)h_{12}(t) + w_{12}(t,z)h_{22}\right), \\
\frac{\partial}{\partial t} w_{12}(t,z) &= z\left(w_{11}(t,z)h_{11}(t) + w_{12}(t,z)h_{21}\right), \\
\frac{\partial}{\partial t} w_{21}(t,z) &= -z\left(w_{21}(t,z)h_{12}(t) + w_{22}(t,z)h_{22}\right), \\
\frac{\partial}{\partial t} w_{22}(t,z) &= z\left(w_{21}(t,z)h_{11}(t) + w_{22}(t,z)h_{12}\right).
\end{aligned}$$

Further, note that (2.8) also implies that $t \mapsto W_H(t,z)^T$ solves

$$W_H(c,z)^T = I, \quad \frac{\partial}{\partial t} W_H(t,z)^T = zJHW_H(t,z)^T.$$

**Lemma 2.20** (Grönwall Lemma)**.** *Let $-\infty \leq a < b \leq \infty$, $H \in L^1_{\mathrm{loc}}((a,b), \mathbb{C}^{2\times 2})$, $z \in \mathbb{C}$ and let $y$ be a solution of (2.1). Then it holds for all $c, d \in (a,b)$ that*

$$\|y(d)\| \leq \|y(c)\| \exp\left(|z| \cdot \left|\int_c^d \|H(s)\| \,\mathrm{d}s\right|\right).$$

*Proof.* We will perform a proof by cases. For the first case, let $c \leq d$. Being a solution, $y$ satisfies (2.2) for any interval $(\alpha, \beta) \subseteq (a,b)$, especially for the interval (c,d), leading us to

$$\|y(t)\| \leq \|y(c)\| + \int_c^t |z| \cdot \|H(s)\| \cdot \|y(s)\| \,\mathrm{d}s \text{ for any } t \in [d]. \tag{2.9}$$

Adding $\varepsilon > 0$, the function $f\colon [c,d] \to (0,\infty)$ defined by

$$t \mapsto \varepsilon + \|y(c)\| + \int_c^t |z| \cdot \|H(s)\| \cdot \|y(s)\| \,\mathrm{d}s$$

is absolutely continuous and bounded below by $\varepsilon + \|y(c)\| > 0$. Therefore, its logarithm $\log f\colon [c,d] \to (-\infty, \infty)$ is well-defined and also absolutely continuous. The a.a. existing derivative is

$$\frac{\partial}{\partial t} \log f(t) = \frac{f'(t)}{f(t)} = \frac{|z| \cdot \|H(t)\| \cdot \|y(t)\|}{\varepsilon + \|y(c)\| + \int_c^t |z| \cdot \|H(s)\| \cdot \|y(s)\| \,\mathrm{d}s}.$$

Due to (2.9), this means that

$$\frac{\partial}{\partial t} \log f(t) \leq |z| \cdot \|H(t)\|.$$

Integrating both sides and applying (2.9) yields

$$\log \|y(d)\| \leq \log f(d) \leq \log f(c) + \int_c^d |z| \cdot \|H(s)\| \, \mathrm{d}s$$
$$= \log(\varepsilon + \|y(c)\|) + \int_c^d |z| \cdot \|H(s)\| \, \mathrm{d}s,$$

which, by exponentiation and letting $\varepsilon \to 0$, yields the desired inequality. For the second case, assume that $c > d$. We will apply the first case to the canonical system with the following data:

$$\tilde{a} := -b, \; \tilde{b} := -a, \; \tilde{H}(t) := H(-t), \; \tilde{z} := -z. \tag{2.10}$$

Further defining $\tilde{y}(t) := y(-t)$, we see that

$$\tilde{y}'(t) = -y'(-t) = -zJH(-t)y(-t) = \tilde{z}J\tilde{H}(t)\tilde{y(t)},$$

meaning that $\tilde{y}$ is a solution for the canonical system with data 2.9. Therefore, setting $\tilde{c} = -c$ and $\tilde{d} = -d$, resulting in $\tilde{c}, \tilde{d} \in (\tilde{a}, \tilde{b})$ and $\tilde{c} < \tilde{d}$, allows us to apply what we showed in the first case, yielding

$$\|y(d)\| = \|\tilde{y}(\tilde{d})\| \leq \|\tilde{y}(\tilde{c})\| \exp\left(|\tilde{z}| \cdot \left| \int_{\tilde{c}}^{\tilde{d}} \|\tilde{H}(s)\| \, \mathrm{d}s \right| \right)$$
$$= \|y(c)\| \exp\left(|z| \cdot \left| \int_c^d \|H(s)\| \, \mathrm{d}s \right| \right).$$

□

**Proposition 2.21.** *Let* $-\infty \leq a < b \leq \infty$, $H \in L^1_{\mathrm{loc}}((a,b), \mathbb{C}^{2\times 2})$ *and* $c \in \int a, b \wr$.

(i) *For all* $t \in \int a, b \wr$ *and for all* $z \in \mathbb{C}$ *it holds that*

$$\det W_H(t,z) = \exp\left(-z \int_c^t \mathrm{tr}(JH(s)) \, \mathrm{d}s\right).$$

(ii) *Suppose that* $H(t) = H(t)^*$ *a.e. Then for every interval* $[d, d'] \subseteq \int a, b \wr$ *and all* $z, w \in \mathbb{C}$, *it holds that*

$$W_H(d', z)JW_H(d', w)^* - W_H(d, z)JW_H(d, w)^*$$
$$= (z - \overline{w}) \int_d^{d'} W_H(s, z)H(s)^T W_H(s, w)^* \, \mathrm{d}s. \tag{2.11}$$

(iii) *Assume that* $H(t) \in \mathbb{R}^{2\times 2}$ *a.e. Then for all* $t \in \int a, b \wr$ *and for all* $z \in \mathbb{C}$ *it holds that*

$$W_H(t, \overline{z}) = \overline{W_H(t, z)}.$$

*Proof.*

(i) By Theorem 2.13(iii) and Remark 2.19, the mapping $t \mapsto W_H(t,z)$, and therefore also $t \mapsto \det W_H(t,z)$, is locally absolutely continuous on $\rfloor a,b\lfloor$. Furthermore, using the very same remark, we can calculate the function's derivative. In order not to unnecessarily lengthen this calculation, we drop explicit notation of the arguments $t$ and $z$.

$$\begin{aligned}
\frac{\partial}{\partial t}[\det W_H] &= \frac{\partial}{\partial t}[w_{11}w_{22} - w_{12}w_{21}] \\
&= \left(\left[\frac{\partial}{\partial t}w_{11}\right] w_{22} + w_{11}\left[\frac{\partial}{\partial t}w_{22}\right]\right) - \left(\left[\frac{\partial}{\partial t}w_{12}\right] w_{21} + w_{12}\left[\frac{\partial}{\partial t}w_{21}\right]\right) \\
&= [-z\,(w_{11}h_{12}(t) + w_{12}h_{22})]w_{22} + w_{11}[z\,(w_{21}h_{11}(t) + w_{22}h_{21})] \\
&\quad - [z\,(w_{11}h_{11}(t) + w_{12}h_{21})]w_{21} - w_{12}[-z\,(w_{21}h_{12}(t) + w_{22}h_{22})] \\
&= -z(h_{12} - h_{21}(w_{11}w_{22} - w_{12}w_{21}) = -z\mathrm{tr}(JH)\det W_H.
\end{aligned}$$

Now set $\phi(t) := z\int_c^t \mathrm{tr}(JH(s))\,\mathrm{d}s$. As the function $t \mapsto e^{\phi(t)}\det W_H(t,z)$ is absolutely continuous on $[a,c]$, we can compute its a.e. existing derivative, and taking into consideration our previous calculation of $\frac{\partial}{\partial t}[\det W_H]$, we see that $\frac{\partial}{\partial t}\left[e^{\phi(t)}\det W_H(t,z)\right] = 0$ a.e. This circumstance and the fact that $e^{\phi(c)}\det W_H(c,z) = 1$ then lets us deduce that the function is identical to 1, which shows the assertion.

(ii) As $t \mapsto W_H(t,z)JW_H(t,w)^*$ is locally absolutely continuous on $(a,b)$, it holds a.e. that

$$\begin{aligned}
\frac{\partial}{\partial t}&[W_H(t,z)J\; W_H(t,w)^*] \\
&= \left[\frac{\partial}{\partial t}W_H(t,z)J\right]W_H(t,w)^* + W_H(t,z)J\left[\frac{\partial}{\partial t}W_H(t,w)\right]^* \\
&= \left[\frac{\partial}{\partial t}W_H(t,z)J\right]W_H(t,w)^* - W_H(t,z)\left[\frac{\partial}{\partial t}W_H(t,w)J\right]^* \\
&= \left[zW_H(t,z)H(t)^T\right]W_H(t,w)^* - W_H(t,z)\left[wW_H(t,w)H(t)^T\right]^* \\
&= W_H(t,z)\left[zH(t)^T - \overline{w}\overline{H(t)}\right]W_H(t,w)^* \\
&= W_H(t,z)\left[(z-\overline{w})H(t)^T\right]W_H(t,w)^*.
\end{aligned}$$

Now let an interval $[d,d'] \subseteq \rfloor a,b\lfloor$ be given. For any finite interval $[h,h'] \subseteq [d,d']$, the mapping $t \mapsto W_H(t,z)JW_H(t,w)^*$ is absolutely continuous, therefore integrable, and integrating shows our assertion on $[h,h']$. However, as $t \mapsto W_H(t,z)JW_H(t,w)^*$ is also continuous on the entirety of $[d,d']$, we can pass to the limits $h \to d$ and $h' \to d'$, showing us that (2.11) holds for $[d,d']$.

(iii) The last assertion can be shown by looking at the recurrence and power series of $W_H$ given in Theorem 2.13. Inductively, it is easily seen that $H(t)$ mapping exclusively to $\mathbb{R}^{2\times 2}$ means that $W_{H,l}(,,t) \in \mathbb{R}^{2\times 2}$ for all $l \in \mathbb{N}$ and $t \in [a,b\lfloor$, implying that $\sum_{l=0}^N W_{H,l}(t)\overline{z}^l = \overline{\sum_{l=0}^N W_{H,l}(t)z^l}$ for all $N \in \mathbb{N}$. Along with unconditional convergence of both power series, the assertion holds.

□

**Proposition 2.22.** *[Existence Theorem] Let $-\infty \leq a < b \leq \infty$, $H \in L^1_{\mathrm{loc}}((a,b),\mathbb{C}^{2\times 2})$ and $c \in \int a,b\wr$. Let further $z \in \mathbb{C}$ and $\alpha_1,\alpha_2 \in \mathbb{C}$. Then there exists a function $y\colon \int a,b\wr \to \mathbb{C}^2$ with the following properties.*

- *$y$ is continuous and for all $d,\tilde{d} \in \int a,b\wr$ it satisfies*

$$y(\tilde{d}) = y(d) + \int_d^{\tilde{d}} zJH(s)y(s)\,\mathrm{d}s. \tag{2.12}$$

- $y(c) = \begin{pmatrix}\alpha_1\\ \alpha_2\end{pmatrix}$.

- *For all $t \in \int a,b\wr$, it holds that*

$$\|y(t)\| \leq \left\|\begin{pmatrix}\alpha_1\\ \alpha_2\end{pmatrix}\right\| \exp\left(|z|\cdot\left|\int_c^t \|H(s)\|\,\mathrm{d}s\right|\right).$$

*Proof.* Let $W_H(t,z)$ be the fundamental solution described in Theorem 2.13. At first, note that by Proposition 2.21, $W_H(t,z)$ is always a regular matrix. This justifies the following construction. Consider the function

$$y : \begin{cases} \int a,b\wr & \to \quad \mathbb{C}^2 \\ \quad t & \mapsto \quad W_H(t,z)^T\begin{pmatrix}\alpha_1\\ \alpha_2\end{pmatrix} \end{cases}.$$

By Remark 2.19, $y$ is locally absolutely continuous and satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t}y(t) = \left[\frac{\partial}{\partial t}W_H(t,z)^T\right]\begin{pmatrix}\alpha_1\\ \alpha_2\end{pmatrix} = zJH(t)W_H(t,z)^T\begin{pmatrix}\alpha_1\\ \alpha_2\end{pmatrix} = zJH(t)y(t),$$

which is equivalent to (2.12) by Remark 2.6. Further, it clearly solves $y(c) = (\alpha_1,\alpha_2)^T$, and the Grönwall Lemma 2.20 tells us that

$$\|y(t)\| \leq \|y(c)\| \exp\left(|z|\cdot\left|\int_c^t \|H(s)\|\,\mathrm{d}s\right|\right) = \left\|\begin{pmatrix}\alpha_1\\ \alpha_2\end{pmatrix}\right\| \exp\left(|z|\cdot\left|\int_c^t \|H(s)\|\,\mathrm{d}s\right|\right).$$

□

**Proposition 2.23.** *Let $-\infty \leq a < b \leq \infty$ as well as $H \in L^1_{\mathrm{loc}}((a,b),\mathbb{C}^{2\times 2})$ and let $z \in \mathbb{C}$.*

(i) *Let $y_1, y_2$ be solutions of (2.1) and let $c \in \int a,b\wr$. If $y_1(c) = y_2(c)$, then $y_1 = y_2$.*

(ii) *Every solution $y$ of (2.1) has a continuous extension to $\int a,b\wr$, to which we will also refer as $y$.*

*Proof.*

(i) Given two solutions $y_1, y_2$, the function $y := y_1 - y_2$ solves the equation with $y(c) = 0$. By the Grönwall Lemma 2.20, this implies $y = 0$.

(ii) Let $y$ be a solution of (2.1) and let $c \in (a,b)$. Denote by $\tilde{y}$ the solution constructed in the Existence Theorem satisfying $\tilde{y}(c) = y(c)$. By the theorem, $\tilde{y}$ has a continuous extension to $\lbag a,b \rbag$, and by (i), it holds that $y = \tilde{y}$ on $(a,b)$.

□

**Proposition 2.24.** *Let $-\infty \leq a < b \leq \infty$, $H \in L^1_{\mathrm{loc}}((a,b), \mathbb{C}^{2\times 2})$ and $c \in \lbag a,b)$.*

(i) *Let $d \in \lbag a,b \rbag$ and denote by $W_{H|_{[d,b)}}(t,z)$ the fundamental solution for the Hamiltonian $H|_{[d,b)} \in L^1_{\mathrm{loc}}([d,b)$ with centre $d$. Then it holds for all $d' \in [d,b\rbag$ and all $z \in \mathbb{C}$ that*

$$W_H(d',z) = W_H(d,z) W_{H|_{[d,b)}}(d',z).$$

(ii) *Moreover, it holds for all $d \in \lbag a,b \rbag$ and all $z \in \mathbb{C}$ that*

$$\left\| W_H(d,z)^{-1} \right\| \leq \exp\left( |z| \cdot \left| \int_c^d \|H(s)\| \,\mathrm{d}s \right| \right).$$

*Proof.*

(i) Let $\alpha_1, \alpha_2 \in \mathbb{C}$. As the derivative of the function $t \mapsto W_{H|_{[d,b)}}(t,z)^T W_H(d,z)^T$ is $\frac{\partial}{\partial t} W_{H|_{[d,b)}}(t,z)^T W_H(d,z)^T = zJH(t) W_{H|_{[d,b)}}(t,z)^T W_H(d,z)^T$, we see that both

$$t \mapsto W_H(t,z)^T \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} \text{ and } t \mapsto W_{H|_{[d,b)}}(t,z)^T W_H(d,z)^T \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}$$

are solutions of (2.1). Further, we have

$$W_H(d,z)^T \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = W_{H|_{[d,b)}}(d,z)^T W_H(d,z)^T \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix},$$

which, by Proposition 2.23, means that the two functions coincide on $[d,b\rbag$. As $\alpha_1$ and $\alpha_2$ were chosen arbitrarily, we conclude that for all $d' \in [d,b\rbag$ and $z \in \mathbb{C}$, we have

$$W_H(d',z) = W_H(d,z) W_{H|_{[d,b)}}(d',z).$$

(ii) Invertibility of $W_H(d,z)$ is ensured by Proposition 2.21(i). Given $\alpha_1, \alpha_2 \in \mathbb{C}$, the function $y\colon t \mapsto W_H(t,z)^T W_H(d,z)^{-T} \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}$ is a solution of (2.1). Therefore, the Grönwall Lemma 2.20 ensures

$$\begin{aligned} \left\| W_H(d,z)^{-T} \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} \right\| &= \|y(c)\| \leq \|y(d)\| \exp\left( |z| \cdot \left| \int_d^c \|H(s)\| \,\mathrm{d}s \right| \right) \\ &= \left\| \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} \right\| \exp\left( |z| \cdot \left| \int_c^d \|H(s)\| \,\mathrm{d}s \right| \right). \end{aligned}$$

Using the fact that the spectral norm is invariant to transposition yields the asserted estimate.

$\square$

## 2.3 Transformations

**Definition 2.25.** Given a field $\mathbb{K}$, we denote by $\mathrm{GL}(n,\mathbb{K})$ the *general linear group of degree* $n$, meaning the set of all invertible $n \times n$ matrices with entries in $\mathbb{K}$. As the name suggests, $\mathrm{GL}(n,\mathbb{K})$, along with the ordinary matrix multiplication, is a group.

**Definition 2.26.** For $A \in \mathbb{C}^{2\times 2}$ and $Q \in \mathrm{GL}(2,\mathbb{C})$ we denote

$$\circlearrowleft_Q A := Q^{-1}AQ.$$

Further, for mappings $A(\textvisiblespace)$ into $\mathbb{C}^{2\times 2}$, $\circlearrowleft_Q$ is defined point-wise, and given $\alpha \in \mathbb{R}$, we use the abbreviation $\circlearrowleft_\alpha := \circlearrowleft_{\exp(\alpha J)}$.

In the two subsequent lemmas, we discuss the effects of certain transformations applied to a Hamiltonian $H$ on its fundamental solution $W_H$. The claimed equations within those lemmas always refer to two fundamental solutions with the *same* centre $c \in \int a,b\wr$.

**Lemma 2.27.** *Let* $-\infty \le a < b \le \infty$, $H \in L^1_{\mathrm{loc}}((a,b),\mathbb{C}^{2\times 2})$ *and* $c \in \int a,b\wr$.

(i) *For each* $Q \in \mathrm{GL}(2,\mathbb{C})$ *it holds that* $(-JQ^TJ)HQ^{-T} \in L^1_{\mathrm{loc}}(\int a,b\wr,\mathbb{C}^{2\times 2})$ *and that*

$$\forall t \in \int a,b\wr, z \in \mathbb{C}\colon W_{(-JQ^TJ)HQ^{-T}}(t,z) = \circlearrowleft_Q W_H(t,z). \tag{2.13}$$

(ii) *For each* $\alpha \in \mathbb{R}$ *it holds that* $\circlearrowleft_\alpha H \in L^1_{\mathrm{loc}}(\int a,b\wr,\mathbb{C}^{2\times 2})$ *and that*

$$\forall t \in \int a,b\wr, z \in \mathbb{C}\colon W_{\circlearrowleft_\alpha H}(t,z) = \circlearrowleft_\alpha W_H(t,z). \tag{2.14}$$

*Proof.*

(i) Given $z \in \mathbb{C}$, consider the mapping

$$V\colon \begin{cases} \int a,b\wr & \to & \mathbb{C}^{2\times 2} \\ t & \mapsto & \circlearrowleft_Q W_H(t,z) \end{cases}.$$

The function $V$ is continuous on $\int a,b\wr$, locally absolutely continuous on $(a,b)$, satisfies $V(c) = I$ and, utilising the fact that $J(-J) = I$, it holds for a.a. $t \in (a,b)$ that

$$\begin{aligned}\frac{\partial}{\partial t}V(t)J &= Q^{-1}\left[\frac{\partial}{\partial t}W_H(t,z)J\right](-J)QJ = Q^{-1}[zW_H(t,z)H(t)^T](-JQJ) \\ &= z \cdot Q^{-1}W_H(t,z)Q \cdot Q^{-1}H(t)^T(-JQJ) \\ &= z \cdot \circlearrowleft_Q W_H(t,z) \cdot \left[(-JQ^TJH(t)Q^{-T}\right]^T.\end{aligned}$$

By uniqueness of solutions, (2.13) holds.

(ii) At first, we note that $-J(e^{\alpha J})^T J = e^{\alpha J^T} = e^{-\alpha J}$ and $(e^{\alpha J})^{-T} = e^{-\alpha J^T} = e^{\alpha J}$. This leads to

$$(-J(e^{\alpha J})^T J)H(e^{\alpha J})^{-T} = -Je^{-\alpha J}JHe^{\alpha J} = \circlearrowleft_\alpha H.$$

Thus, applying (2.13) with $Q := e^{\alpha J}$ proves the assertion.

□

**Lemma 2.28.** *Let $-\infty \le a < b \le \infty$, $H \in L^1_{\text{loc}}((a,b),\mathbb{C}^{2\times 2})$ and $c \in \int a, b \wr$. For any $\psi \in L^1_{\text{loc}}((a,b),\mathbb{C})$, it holds that also $H + i\psi J \in L^1_{\text{loc}}((a,b),\mathbb{C}^{2\times 2})$. Additionally, when setting $\Psi(t) := \int_c^t \psi(s)\,\mathrm{d}s$, it holds for all $t \in (a,b)$ and all $z \in \mathbb{C}$ that*

$$W_{H+i\psi J}(t,z) = e^{iz\Psi(t)}W_H(t,z).$$

*Proof.* The first assertion, namely that $H + i\psi J \in L^1_{\text{loc}}((a,b),\mathbb{C}^{2\times 2})$, is clear. The second can once more be proven using the uniqueness of solutions. Fix $z \in \mathbb{C}$ and let the function $V$ be given by

$$V\colon \begin{cases} \int a, b \wr & \to \quad \mathbb{C}^{2\times 2} \\ t & \mapsto \quad e^{iz\Psi(t)}W_H(t,z) \end{cases}.$$

Clearly, $V$ is continuous on $\int a, b \wr$ and locally absolutely continuous on $(a,b)$. Furthermore, it clearly holds that $V(c) = I$, and its derivative with respect to $t$ satisfies

$$\begin{aligned} \frac{\partial}{\partial t}V(t)J &= iz\psi(t)e^{iz\Psi(t)}W_H(t,z)J + e^{iz\Psi(t)}\left[\frac{\partial}{\partial t}W_H(t,z)J\right] \\ &= iz\psi(t)e^{iz\Psi(t)}W_H(t,z)J + e^{iz\Psi(t)}zW_H(t,z)H(t)^T \\ &= ze^{iz\Psi(t)}W_H(t,z)\left[H(t) + i\psi(t)J\right]^T \end{aligned}$$

a.e. on $(a,b)$. The uniqueness of solutions from Proposition 2.22 then yields that $V(t) = W_{H+i\psi J}(t,z)$. As $z \in \mathbb{C}$ was chosen arbitrarily, the assertion is shown. □

## 2.4 Estimating the exponential type of fundamental solutions

**Definition 2.29.** Let $f\colon \mathbb{C} \to [0,\infty)$. The *exponential type* of $f$ is defined as

$$\mathfrak{T}_{[\![r]\!]}(f) := \limsup_{|z|\to\infty} \frac{\log^+ f(z)}{|z|},$$

where $\log^+ x := \max\{0, \log x\}$.

**Remark 2.30.** It follows directly from Theorem 2.13(iii) that

$$\mathfrak{T}_{[\![r]\!]}\left(\|W_H(t,\text{␣})\|\right) \le \left|\int_c^t \|H(s)\|\,\mathrm{d}s\right|.$$

**Proposition 2.31.** *Let $-\infty \le a < b \le \infty$, $H \in L^1_{\text{loc}}((a,b),\mathbb{C}^{2\times 2})$ and $c \in \int a, b \wr$. The mapping $t \mapsto \mathfrak{T}_{[\![r]\!]}\left(\|W_H(t,\text{␣})\|\right)$ is continuous on $\int a, b \wr$, locally absolutely continuous on*

*$(a,b)$, and for a.a. $t \in (a,b)$ we have*

$$\frac{\partial}{\partial t}\mathcal{T}_{[\![r]\!]}(\|W_H(t,\llcorner\!\lrcorner)\|) \leq \|H(t)\|.$$

*Furthermore, if $a$ is both an $L^1$-boundary point and $-\infty < a$, the function is $\mathrm{AC}_{\mathrm{loc}}([a,b),\mathbb{R})$. Analogously, if $b$ is both an $L^1$-boundary point and $b < \infty$, it is $\mathrm{AC}_{\mathrm{loc}}((a,b],\mathbb{R})$.*

*Proof.* Let $d, d' \in [a,b\wr, c < d'$. Applying Proposition 2.24, compute

$$\begin{aligned}\log^+\|W_H(d',z)\| &= \log^+\|W_H(d,z)\cdot W_{H|_{[d,b)}}(d',z)\|\\ &\leq \log^+\|W_H(d,z)\| + \log^+\|W_{H|_{[d,b)}}(d',z)\|\\ &\leq \log^+\|W_H(d,z)\| + |z|\cdot\int_d^{d'}\|H(s)\|\,\mathrm{d}s.\end{aligned}$$

Using the second part of the above proposition, we also see that

$$\begin{aligned}\log^+\|W_H(d,z)\| &= \log^+\|W_H(d',z)\cdot W_{H|_{[d,b)}}(d',z)^{-1}\|\\ &\leq \log^+\|W_H(d',z)\| + \log^+\|W_{H|_{[d,b)}}(d',z)^{-1}\|\\ &\leq \log^+\|W_H(d',z)\| + |z|\cdot\int_d^{d'}\|H(s)\|\,\mathrm{d}s.\end{aligned}$$

In combination, these two estimates show that

$$|\mathcal{T}_{[\![r]\!]}(W_H(d,\llcorner\!\lrcorner)) - \mathcal{T}_{[\![r]\!]}(W_H(d',\llcorner\!\lrcorner))| \leq \int_d^{d'}\|H(s)\|\,\mathrm{d}s, \tag{2.15}$$

from which we can directly deduce all stated continuity assertions. Now let $d \in (a,b)$ be a Lebesgue point of $\|H(t)\|$ for which $\frac{\partial}{\partial t}\mathcal{T}_{[\![r]\!]}(\|(H(t,\llcorner\!\lrcorner)\|)$ exists (note that a.a. points in $(a,b)$ satisfy these requirements). Then (2.15), along with an analogous estimate for $d' \leq d$, shows us that the derivative is bounded by $\|H(s)\|$. □

**Lemma 2.32.** *Let $-\infty \leq a < b \leq \infty$, $H \in L^1_{\mathrm{loc}}((a,b),\mathbb{C}^{2\times 2})$ and $c \in \wr a,b\wr$. Then*

$$\forall Q \in \mathrm{GL}(2,\mathbb{C})\colon \mathcal{T}_{[\![r]\!]}(\|W_H(t,\llcorner\!\lrcorner)\|) = \mathcal{T}_{[\![r]\!]}(\|W_{-JQ^{-1}JHQ}(t,\llcorner\!\lrcorner)\|).$$

*Proof.* From the submultiplicativity of the matrix norm as well as Definition 2.26 and Lemma 2.27 we see that

$$\|W_{-JQ^{-1}JHQ}(t,\llcorner\!\lrcorner)\| = \|\circlearrowleft_{Q^{-T}} W_H(t,\llcorner\!\lrcorner)\| \leq \|W_H(t,z)\|\|Q\|\|Q^{-1}\|. \tag{2.16}$$

Further, the fact that $QQ^{-1} = I$ lets us deduce that also

$$\|W_{-JQ^{-1}JHQ}(t,\llcorner\!\lrcorner)\| = \|\circlearrowleft_{Q^{-T}} W_H(t,\llcorner\!\lrcorner)\| \geq \frac{1}{\|Q\|\|Q^{-1}\|}\|W_H(t,z)\|. \tag{2.17}$$

Applying (2.16) and (2.17) pointwise for $z \in \mathbb{C}$ along with taking logarithms and passing to the limit $|z| \to \infty$, we then see that

$$\mathcal{T}_{[\![r]\!]}(\|W_H(t,\lrcorner)\|) \leq \mathcal{T}_{[\![r]\!]}(\|W_{-JQ^{-1}JHQ}(t,\lrcorner)\|) \leq \mathcal{T}_{[\![r]\!]}(\|W_H(t,\lrcorner)\|).$$

□

**Corollary 2.33.** *Let $-\infty \leq a < b \leq \infty$, $H \in L^1_{\text{loc}}((a,b),\mathbb{C}^{2\times 2})$ and $c \in \int a,b \wr$. Then*

$$\forall t \in \int a,b\wr : \mathcal{T}_{[\![r]\!]}(\|W_H(t,\lrcorner)\|) \leq \left| \int_c^t \left[ \inf_{Q\in \mathrm{GL}(2,\mathbb{C})} \|Q^{-1}JH(s)Q\| \right] \mathrm{d}s \right|.$$

*Proof.* For any $Q \in \mathrm{GL}(2,\mathbb{C})$, utilising Proposition 2.31 and Lemma 2.32 yields that

$$\begin{aligned}\frac{\partial}{\partial t}\mathcal{T}_{[\![r]\!]}(\|W_H(t,\lrcorner)\|) &= \frac{\partial}{\partial t}\mathcal{T}_{[\![r]\!]}(\|W_{-JQ^{-1}JHQ}(t,\lrcorner)\|) \\ &\leq \|JQ^{-1}JH(t)Q\| = \|Q^{-1}JH(t)Q\|.\end{aligned}$$

Passing to the infimum over all $Q \in \mathrm{GL}(2,\mathbb{C})$ and integrating proves the assertion. □

The next lemma is based on just linear algebra and does not contain any assertions about Hamiltonians or fundamental solutions, but when combined with Corollary 2.33, it will allow very nice estimates of $\mathcal{T}_{[\![r]\!]}(\|W_H(t,\lrcorner)\|)$.

**Lemma 2.34.** *Let $A \in \mathbb{C}^{2\times 2}$ with* $\mathrm{tr}A = 0$. *Then*

$$\inf_{Q\in \mathrm{GL}(2,\mathbb{C})} \|Q^{-1}AQ\| = \sqrt{|\det A|}.$$

*Proof.* Let $\alpha_1$ and $\alpha_2$ be the eigenvalues of $A$, listed by their algebraic multiplicity. As $\mathrm{tr}A = 0$, there are two cases:

(i) $\alpha_1 = \alpha_2 = 0$ or

(ii) $\alpha_1 = -\alpha_2 \neq 0$.

Consider the first case. As 0 is an eigenvalue of $A$, we have $\det A = 0$. If $A$ is diagonalisable, it follows that $A = 0$ and the assertion holds. Otherwise, let $B \in \mathrm{GL}(2,\mathbb{C})$ such that $B^{-1}AB$ is in Jordan canonical form, meaning that $B^{-1}AB = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Further, for $\delta > 0$ define $Q_\delta := B \cdot \begin{pmatrix} 1 & 0 \\ 0 & \delta \end{pmatrix}$. Clearly, $Q_\delta \in \mathrm{GL}(2,\mathbb{C})$, and it shows that

$$Q_\delta^{-1}AQ_\delta = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{\delta} \end{pmatrix} B^{-1}AB \begin{pmatrix} 1 & 0 \\ 0 & \delta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{\delta} \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & \delta \end{pmatrix} = \begin{pmatrix} 0 & \delta \\ 0 & 0 \end{pmatrix}.$$

This implies that $\inf_{Q\in \mathrm{GL}(2,\mathbb{C})} \|Q^{-1}AQ\| = 0 = \det A$.
Now consider the second case. Here, we have $\det A = -\alpha_1^2$, and since for any $Q \in \mathrm{GL}(2,\mathbb{C})$, the matrix $Q^{-1}AQ$ has the eigenvalues $\alpha_1$ and $-\alpha_1$, it follows from the definition of the spectral norm that

$$\inf_{Q\in \mathrm{GL}(2,\mathbb{C})} \|Q^{-1}AQ\| = |\alpha_1| = \sqrt{|\det A|}.$$

□

**Corollary 2.35.** *Let* $-\infty \leq a < b \leq \infty$*,* $H \in L^1_{\mathrm{loc}}((a,b),\mathbb{C}^{2\times 2})$ *with* $H(t) = H(t)^T$ *a.e. and* $c \in \int a, b \wr$*. Then*

$$\forall t \in \int a, b \wr \colon \mathcal{T}_{[\![r]\!]}(\|W_H(t, \llcorner)\|) \leq \left| \int_c^t \sqrt{|\det H(s)|}\, \mathrm{d}s \right|.$$

*Proof.* $H(t) = H(t)^T$ a.e. implies that $\mathrm{tr} JH(t) = 0$ a.e. Additionally, it holds for all $A \in \mathbb{C}^{2\times 2}$ that $\det JA = \det A$. Applying Corollary 2.33 and Lemma 2.34 for a.a. $s \in (\min\{c,t\}, \max\{c,t\})$ thereafter yields the assertion. □

**Lemma 2.36.** *Let* $-\infty \leq a < b \leq \infty$*,* $H \in L^1_{\mathrm{loc}}((a,b),\mathbb{C}^{2\times 2})$ *with* $H(t) = H(t)^*$ *a.e. as well as* $c \in \int a, b \wr$*, and further assume that either* $H(t) \geq 0$ *or* $H(t) \leq 0$ *a.e. Then it holds for all* $t \in \int a, b \wr$ *and all* $z \in \mathbb{C}$ *that* $\|W_H(t,z)\| \geq 1$*. In particular, if* $b$ *is an* $L^1$*-boundary point, it holds for the monodromy matrix of* $H$ *that* $\|W_H(b,z)\| \geq 1$ *for all* $z \in \mathbb{C}$*.*

*Proof.* At first, we observe that the matrix $-iJ$ is self-adjoint and has the eigenvalues $+1$ and $-1$ with corresponding eigenvectors $(1,-i)^T$ and $(1,i)^T$. Thus, its numerical range is $[-1,1]$. From this, we can deduce that the numerical range of the self-adjoint matrix $W_H(t,z)(-iJ)W_H(t,z)^*$ is contained in $[-\|W_H(t,z)\|^2, \|W_H(t,z)\|^2]$. Applying Proposition 2.21(ii) with $d' = t$, $d = c$ and $w = z$ yields

$$-iJ = W_H(t,z)(-iJ)W_H(t,z)^* - (2\mathrm{Im}\ z) \int_c^t W_H(s,z)H(s)^T W_H(s,z)^*\, \mathrm{d}s.$$

As the integrand is self-adjoint, the same holds for the entire second summand on the right side. Moreover, we have

$$\begin{aligned} &-(2\mathrm{Im}\ z) \int_c^t W_H(s,z)H(s)^T W_H(s,z)^*\, \mathrm{d}s \\ &\qquad \begin{cases} \geq 0 & \textit{if } (\mathrm{Im}\ z \leq 0 \wedge H(t) \geq 0 \text{ a.e.}) \vee (\mathrm{Im}\ z \geq 0 \wedge H(t) \leq 0, \text{ a.e.}), \\ \leq 0 & \textit{if } (\mathrm{Im}\ z \geq 0 \wedge H(t) \geq 0 \text{ a.e.}) \vee (\mathrm{Im}\ z \leq 0 \wedge H(t) \leq 0, \text{ a.e.}). \end{cases} \end{aligned}$$

In the first case, we have $-iJ - W_H(t,z)(-iJ)W_H(t,z)^* \geq 0$ and therefore $-\|W_H(t,z)\|^2 \leq -1$, while the second yields $-iJ - W_H(t,z)(-iJ)W_H(t,z)^* \leq 0$, which in turn implies $\|W_H(t,z)\|^2 \geq 1$. Thus, the assertion holds. □

**Proposition 2.37.** *Let* $-\infty \leq a < b \leq \infty$*,* $H \in L^1_{\mathrm{loc}}((a,b),\mathbb{C}^{2\times 2})$ *with* $H(t) = H(t)^*$ *a.e. as well as* $c \in \int a, b \wr$*, and further assume that either* $H(t) \geq 0$ *or* $H(t) \leq 0$ *a.e. Then*

$$\forall t \in \int a, b \wr, z \in \mathbb{C} \colon \|W_H(t,z)\| \geq \exp\left( |\mathrm{Im}\ z| \cdot \left| \int_c^t \sqrt{|\det H(s)|}\, \mathrm{d}s \right| \right).$$

*Proof.* Define $\psi(t) := \sqrt{|\det H(t)|} \cdot \mathrm{sgn}(t-c)$, $t \in \int a, b \wr$ and set $H_+(t) = H(t) + iJ\psi(t)$ as well as $H_-(t) = H(t) - iJ\psi(t)$. The sesquilinear form induced by $iJ\phi$ is constant equal to 0, resulting in the definiteness of both $H_+$ and $H_-$ being the same as that of $H$.

As $H \in L^1_{\mathrm{loc}}(\rangle a,b\langle, \mathbb{C}^{2\times 2})$, we also have $\psi \in L^1_{\mathrm{loc}}(\rangle a,b\langle, \mathbb{C})$, allowing us to apply Lemma 2.28. Hence, setting $\Psi(t) := \int_c^t \psi(s)\,\mathrm{d}s$, we have

$$W_{H_+}(t,z) = e^{iz\Psi(t)} W_H(t,z), \quad W_{H_-}(t,z) = e^{-iz\Psi(t)} W_H(t,z).$$

Lemma 2.36 tells us that $\|W_{H_+}(t,z)\|, \|W_{H_-}(t,z)\| \geq 1$. Thus,

$$\|W_H(t,z)\| \geq \exp\left(\max\{\mathrm{Re}\,(iz), \mathrm{Re}\,(-iz)\} \cdot \Psi(t)\right),$$

which is just the asserted estimate. □

**Theorem 2.38.** *[Krein-de Branges formula] Let* $-\infty \leq a < b \leq \infty$ *and* $H \in L^1_{\mathrm{loc}}((a,b), \mathbb{R}^{2\times 2})$ *with* $H(t) = H(t)^*$ *a.e., and further assume that either* $H(t) \geq 0$ *or* $H(t) \leq 0$ *a.e. Moreover, let* $c \in \rangle a,b\langle$*. Then*

$$\forall t \in \rangle a,b\langle : \mathcal{T}_{[\![r]\!]}(\|W_H(t,\textvisiblespace)\|) = \left| \int_c^t \sqrt{|\det H(s)|}\,\mathrm{d}s \right|.$$

*Proof.* On the one hand, being a.e. self-adjoint and real-valued, $H$ satisfies the hypothesis of Corollary 2.35, yielding $\mathcal{T}_{[\![r]\!]}(\|W_H(t,\textvisiblespace)\|) \leq \left|\int_c^t \sqrt{|\det H(s)|}\,\mathrm{d}s\right|$. On the other hand, Proposition 2.37 implies that

$$\limsup_{z\in i\mathbb{R}, |z|\to\infty} \frac{\log^+ \|W_H(t,z)\|}{|z|} \geq \left| \int_c^t \sqrt{|\det H(s)|}\,\mathrm{d}s \right|,$$

and therefore $\mathcal{T}_{[\![r]\!]}(\|W_H(t,\textvisiblespace)\|) \geq \left|\int_c^t \sqrt{|\det H(s)|}\,\mathrm{d}s\right|$. □

# 3 Growth theorem

The last chapter ended with Theorem 2.38, the Krein-de Branges formula, which lets us calculate the exponential type $\mathcal{T}_{[\![r]\!]}(\|W_H(t,z)\|)$ for certain Hamiltonians and $t \in \rfloor a, b \lceil$. In particular, when $b$ is a $L^1$-boundary point, we can use the formula to determine the exponential point of the monodromy matrix $W_H(b,z)$. However, the exponential type is a quite rough way to estimate the growth of $\|W_H(b,z)\|$ with respect to $z$, as any growth rate slower than exponential growth, such as growth similar to a polynomial, has exponential type 0.
Moreover, when talking about monodromy matrices of Hamiltonians $H$ with $\det H = 0$ a.e. which satisfy the conditions of Theorem 2.38, a single look to the right side of the Krein-de Branges formula directly tells us that the exponential type of any such monodromy matrix is 0, making it impossible to compare different growth rates using their corresponding exponential types.
In this chapter, we will discuss such Hamiltonians and compute finer upper bounds for the growth of their monodromy matrices $W_H(b,z)$. As we solely focus on Hamiltonians on compact intervals $I = [a,b]$ from now on, we can always construct the fundamental solution using $a$ as its centre. Moreover, to shorten notation, we denote the monodromy matrix to a given Hamiltonian $H$ by $W_H(z) := W_H(b,z)$. Moreover, analogously to

## 3.1 Romanov's Theorem

**Definition 3.1.** Given an interval $I$ and a measurable function $\phi\colon I \to \mathbb{R}$, we denote by $\xi_{\phi(t)}$ the function

$$t \mapsto \begin{pmatrix} \cos(\phi(t)) \\ \sin(\phi(t)) \end{pmatrix}, \quad t \in I.$$

$$t \mapsto \begin{pmatrix} \cos(\phi(t)) \\ \sin(\phi(t)) \end{pmatrix}, \quad t \in I.$$

Correspondingly, when given $\phi \in \mathbb{R}$, we set $\xi_\phi := (\cos(\phi), \sin(\phi))^T$.

**Remark 3.2.** The matrix $\xi_{\phi(t)}\xi_{\phi(t)}^T$ described in Definition 3.1 is of the form

$$\begin{pmatrix} \cos^2(\phi(t)) & \sin(\phi(t))\cos(\phi(t)) \\ \sin(\phi(t))\cos(\phi(t)) & \sin^2(\phi(t)) \end{pmatrix},$$

and it holds that $\det \xi_{\phi(t)}\xi_{\phi(t)}^T = 0$ as well as $\mathrm{tr}\xi_{\phi(t)}\xi_{\phi(t)}^T = 1$ for all $t \in I$.
Now, given any matrix $A \in \mathbb{R}^{2\times 2} \setminus \{0\}$ such that $\det A = 0$ and $A^T = A$, we know that there is $(x,y) \in \mathbb{R}^{2\times 2}$ satisfying $\ker A = \left[(x,y)^T\right]$ and $\left\|(x,y)^T\right\| = 1$. Further, the second

equality implies that there is $\psi \in \mathbb{R}$ such that $(x,y)^T = \xi_\psi$. The other eigenvalue of $A$ is $\mathrm{tr}A \neq 0$, and as eigenvectors to different eigenvalues are orthogonal and the orthogonal complement to $\ker A$ is one-dimensional, we know that setting $\phi := \psi + \pi/2$, $\xi_\phi$ is a normed eigenvector associated with $\mathrm{tr}A$. Now consider the matrix $B := \mathrm{tr}A\xi_\phi\xi_\phi^T$. As $\xi_\phi^T\xi_\psi = 0$ and $\xi_\phi^T\xi_\phi = 1$, $\xi_\phi$ and $\xi_\psi$ are also eigenvectors of $B$ and associated with the same eigenvalues $\mathrm{tr}A$ and 0, respectively. Hence, $A$ and $B$ share the same orthogonal basis of eigenvectors associated to the same eigenvalues and therefore have the same diagonalisation, which gives $A = B$.
This principle can be extended to certain Hamiltonians as follows. Let $I$ be a compact interval and $H \in L^1\left(I, \mathbb{R}^{2\times 2}\right)$ a Hamiltonian on $I$ satisfying $\det H = 0$, $H = H^*$ and $H \geq 0$ a.e. Then there exists a measurable function $\phi\colon I \to \mathbb{R}$ such that $H(t) = \mathrm{tr}H(t) \cdot \xi_{\phi(t)}\xi_{\phi(t)}^T$ a.e. A proof for the two-dimensional case can be found in the lecture notes belonging to a lecture given by Michael Kaltenbäck in 2015,[1] measurable diagonalisations do however also exist for higher dimensions, as is shown within the proof of [SW14, Proposition 3.15].

Measurable diagonalisation of Hamiltonians allows us to formulate the following theorem. It is a refined version of Romanov's Theorem [Rom17, Theorem 1] and was developed by Raphael Pruckner and Harald Woracek. The theorem can be found, in slightly different versions, via [Pru17, Theorem 3.3] as well as [PW22, Theorem 4.1]. We formulate the theorem for a.e. positive semi-definite Hamiltonians, but clearly, the exact same assertion holds for a.e. negative semi-definite Hamiltonians.

**Theorem 3.3.** *Let $I$ be a compact interval and $H \in L^1\left(I, \mathbb{R}^{2\times 2}\right)$ a Hamiltonian on $I$ such that $\det H = 0$, $H = H^*$ and $H \geq 0$ a.e. Write $H$ as $H(t) = \mathrm{tr}H(t) \cdot \xi_{\phi(t)}\xi_{\phi(t)}^T$ with a measurable function $\phi\colon I \to \mathbb{R}$ and denote by $W_H$ the monodromy matrix of $H$. Furthermore, assume that we have*

- *a partition $(y_0, \ldots, y_N)$ of $I$, i.e.*

$$N \in \mathbb{N}, \quad \min I = y_0 < y_1 < \ldots < y_N = \max I,$$

- *rotation parameters $\psi_1, \ldots, \psi_N \in \mathbb{R}$,*
- *distortion parameters $a_1, \ldots, a_N \in (0, 1]$,*

[1]Available here as “Existenz von Lösungen und Operatortheorie im limit circle case (korrigierte Version)”: https://www.asc.tuwien.ac.at/~woracek/2015_WinterSchool-KanSys/WinterSchool2015.html. The existence of measurable diagonalisations is shown in Lemma 5.3.

*and set*

$$A_1 \coloneqq \sum_{j=1}^{N} a_j^2 \int_{y_{j-1}}^{y_j} \cos^2(\phi(t) - \psi_j) \cdot \operatorname{tr} H(t)\, \mathrm{d}t,$$

$$A_2 \coloneqq \sum_{j=1}^{N} \frac{1}{a_j^2} \int_{y_{j-1}}^{y_j} \sin^2(\phi(t = -\psi_j) \cdot \operatorname{tr} H(t)\, \mathrm{d}t,$$

$$A_3 \coloneqq \sum_{j=1}^{N-1} \log\left(\max\left\{\frac{a_j}{a_{j+1}}, \frac{a_{j+1}}{a_j}\right\} \cdot |\cos(\psi_j - \psi_{j+1})| + \frac{|\sin(\psi_j - \psi_{j+1})|}{a_j a_{j+1}}\right),$$

$$A_4 \coloneqq -\log a_1 - \log a_N.$$

*Then*

$$\forall z \in \mathbb{C}\colon \ \log \|W_H(z)\| \leq |z| \cdot (A_1 + A_2) + A_3 + A_4. \tag{3.1}$$

The majority of the proof is split into two lemmas, the first of which mostly utilises the Grönwall Lemma and the multiplicativity of fundamental solutions.

**Lemma 3.4.** *Let $I = [a, b]$ be a compact interval, $H \in L^1(I, \mathbb{C}^{2\times 2})$ and let $W_H(z)$ be its monodromy matrix. Moreover, suppose that we have a partition $(y_0, \dots, y_N)$ of $I$ and matrices $\Omega_1, \dots, \Omega_N \in \mathrm{GL}(2, \mathbb{R})$. Then it holds that*

$$\|W_H(z)\| \leq \exp\left(|z| \sum_{j=1}^{N} \int_{y_{j-1}}^{y_j} \|\Omega_j H(t) J \Omega_j^{-1}\|\, \mathrm{d}t\right) \|\Omega_1^{-1}\| \|\Omega_N\| \prod_{j=1}^{N-1} \|\Omega_j \Omega_{j+1}^{-1}\|. \tag{3.2}$$

*Proof of Lemma 3.4.* For $j \in \{1, N\}$, denote by $W_j(t, z)$ the fundamental solution for the Hamiltonian $H|_{[y_{j-1}, y_j]}$ and by $W_j(z)$ its monodromy matrix $W_j(y_j, z)$. By Proposition 2.24, we then have

$$W_H(z) = W_1(Z) \cdot \ldots \cdot W_N(z),$$

which can be rewritten to

$$W_H(z) = \Omega_1^{-1} \cdot (\Omega_1 W_1(z) \Omega_1^{-1}) \cdot \Omega_1 \Omega_2^{-1} \cdot \ldots \cdot \Omega_{N-1} \Omega_N^{-1} \cdot (\Omega_N W_N(z) \Omega_N^{-1}) \cdot \Omega_N. \tag{3.3}$$

Concurrently, for every fundamental solution $W_j(t, z)$ Lemma 2.27 holds, implying that $\circlearrowleft_{\Omega_j} W_j(t, z) = \circlearrowleft_{\Omega_j} W_{H|_{[y_{j-1}, y_j]}}(t, z) = W_{\circlearrowleft_{\Omega_j} H|_{[y_{j-1}, y_j]}}(t, z)$. Thus, for any $\alpha_1, \alpha_2 \in \mathbb{C}$, the function $t \mapsto \Omega_j^{-T} W_j(t, z)^T \Omega_j^T \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}$ is a solution of the canonical system, and applying the Grönwall Lemma 2.20 yields

$$\begin{aligned} \|\Omega_j^{-T} W_j(z)^T \Omega_j^T \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}\| &= \|\Omega_j^{-T} W_j(y_j, z)^T \Omega_j^T \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}\| \\ &\leq \|\begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}\| \exp\left(|z| \int_{y_{j-1}}^{y_j} \|\Omega_j H(t) \Omega_j^{-1}\|\, \mathrm{d}t\right). \end{aligned}$$

From this, we can conclude that

$$\|\Omega_j W_j(z)\Omega_j^{-1}\| = \|\Omega_j^{-T} W_j(z)\Omega_j^T\| \le \exp\left(|z|\int_{y_{j-1}}^{y_j} \|\Omega_j H(t)\Omega_j^{-1}\|\, \mathrm{d}t\right),$$

which, in combination with (3.3), results in the asserted estimate. □

We will apply this Lemma using the following class of matrices.

**Definition 3.5.** For $a, b > 0$, we denote

$$D(a,b) := \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}.$$

Further, for $a > 0$ and $\psi \in \mathbb{R}$, set

$$\Omega(a,\psi) := D(a,a^{-1})\exp(-\psi J) = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}\begin{pmatrix} \cos\psi & -\sin\psi \\ \sin\psi & \cos\psi \end{pmatrix}$$

For matrices of this particular form, the norms appearing in (3.3) are not hard to calculate, as is shown in the next Lemma.

**Lemma 3.6.** *Let $a, b > 0$ and $\psi, \phi \in \mathbb{R}$.*

(i) $\|\Omega(a,\psi)\| = \|\Omega(a,\psi)^{-1}\| = \max\{a, a^{-1}\}$.

(ii) $\|\Omega(a,\psi)\xi_\phi\xi_\phi^T J\Omega(a,\psi)^{-1}\| = a^2\cos^2(\phi-\psi) + \frac{1}{a^2}\sin^2(\phi-\psi)$.

(iii) *Set*

$$v_+ := \begin{pmatrix} \max\{\frac{a}{b},\frac{b}{a}\}\cdot|\cos(\phi-\psi)| \\ \max\{ab,\frac{1}{ab}\}\cdot|\sin(\phi-\psi)| \end{pmatrix}, \quad v_- := \begin{pmatrix} \min\{\frac{a}{b},\frac{b}{a}\}\cdot|\cos(\phi-\psi)| \\ \min\{ab,\frac{1}{ab}\}\cdot|\sin(\phi-\psi)| \end{pmatrix},$$

*and denote by $\|\_\|_p$. $p \in \{1,2\}$ the $p$-norm on $\mathbb{R}^2$. Then*

$$\begin{aligned} \|v_+\|_2^2 &\le \|\Omega(a,\psi)\Omega(b,\phi)^{-1}\|^2 \\ &= 1 + \|v_+ - v_-\|_2 \cdot \frac{\|v_+ - v_-\|_2 + \|v_+ + v_-\|_2}{2} \le \|v_+\|_1^2. \end{aligned}$$

*Proof of Lemma 3.6.*

(i) As the matrix $\exp(-\psi J)$ is unitary, the spectral norm is invariant to multiplication with it. Therefore, we get

$$\|\Omega(a,\psi)\| = \|D(a,a^{-1})\exp(-\psi J)\| = \|D(a,a^{-1})\| = \max\{a,a^{-1}\},$$

and analogously $\|\Omega(a,\psi)^{-1}\| = \max\{a,a^{-1}\}$.

(ii) At first, we note that

$$J\exp(-\psi J) = \begin{pmatrix} -\sin\psi & -\cos\psi \\ \cos\psi & -\sin\psi \end{pmatrix} = \exp(-\psi J)J$$

and that

$$\xi_\phi \xi_\phi^T = \begin{pmatrix} \cos^2(\phi) & \cos(\phi)\sin(\phi) \\ \sin(\phi)\cos(\phi) & \sin^2(\phi) \end{pmatrix} = \exp(\phi J)\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\exp(-\phi J).$$

Now set $\sigma \coloneqq \phi - \psi$, resulting in

$$\begin{aligned} B &\coloneqq \Omega(a,\psi)\xi_\phi\xi_\phi^T J\Omega(a,\psi)^{-1} = D(a,a^{-1})\exp(-\psi J)\xi_\phi\xi_\phi^T J\exp(\psi J)D(a^{-1},a) \\ &= D(a,a^{-1})\exp(-\psi J)\exp(\phi J)\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\exp(-\phi J)\exp(\psi J)JD(a^{-1},a) \\ &= D(a,a^{-1})\xi_\sigma\xi_\sigma^T JD(a^{-1},a) = \begin{pmatrix} \cos(\sigma)\sin(\sigma) & -a^2\cos^2(\sigma) \\ \frac{1}{a^2}\sin^2(\sigma) & -\cos(\sigma)\sin(\sigma) \end{pmatrix}. \end{aligned}$$

A calculation shows that

$$B^T B = \begin{pmatrix} \frac{1}{a^4}\sin^4(\sigma) + \cos^2(\sigma)\sin^2(\sigma) & * \\ * & a^4\cos^4(\sigma) + \cos^2(\sigma)\sin^2(\sigma) \end{pmatrix},$$

telling us that $\mathrm{tr}(B^T B) = (a^2\cos^2(\sigma) + \frac{1}{a^2}\sin^2(\sigma))^2$. As $\det B = 0$, one of the eigenvalues of $B$ must be 0. Thus, $\|B\| = |\mathrm{tr}B| = \sqrt{\mathrm{tr}B^T B}$.

(iii) We set

$$\begin{aligned} C &\coloneqq \Omega(a,\psi)\Omega(b,\phi)^{-1} = D(a,a^{-1})\exp(\sigma J)D(b^{-1},b) \\ &= \begin{pmatrix} \frac{a}{b}\cos\sigma & -ab\sin\sigma \\ \frac{1}{ab}\sin\sigma & \frac{b}{a}\cos\sigma \end{pmatrix} = \cos\sigma\begin{pmatrix} \frac{a}{b} & 0 \\ 0 & \frac{b}{a} \end{pmatrix} + \sin\sigma\begin{pmatrix} 0 & -ab \\ \frac{1}{ab} & 0 \end{pmatrix}. \end{aligned} \tag{3.4}$$

This yields

$$\begin{aligned} \|C\| &\le |\cos\sigma|\left\|\begin{pmatrix} \frac{a}{b} & 0 \\ 0 & \frac{b}{a} \end{pmatrix}\right\| + |\sin\sigma|\left\|\begin{pmatrix} 0 & -ab \\ \frac{1}{ab} & 0 \end{pmatrix}\right\| \\ &= |\cos\sigma|\max\left\{\frac{a}{b},\frac{b}{a}\right\} + |\sin\sigma|\max\left\{ab,\frac{1}{ab}\right\} = \|v_+\|_1. \end{aligned}$$

Beyond that, the definition of $C$ also shows us that $\det C = 1$ and therefore $\det C^T C = 1$, implying that the eigenvalues of $C^T C$ are the solutions of

$$\lambda + \frac{1}{\lambda} = \mathrm{tr}(C^T C). \tag{3.5}$$

In order to calculate $\mathrm{tr}(C^TC)$, note that

$$C^TC = \begin{pmatrix} \cos^2(\sigma)(\frac{a}{b})^2 + \sin^2(\sigma)(\frac{1}{ab})^2 & * \\ * & \cos^2(\sigma)(\frac{b}{a})^2 + \sin^2(\sigma)(ab)^2 \end{pmatrix},$$

and therefore

$$\mathrm{tr}(C^TC) = \cos^2(\sigma) \cdot \left[ \left(\frac{a}{b}\right)^2 + \left(\frac{b}{a}\right)^2 \right] + \sin^2(\sigma) \cdot \left[ (ab)^2 + \left(\frac{1}{ab}\right)^2 \right].$$

In particular, this calculation shows us that $\mathrm{tr}(C^TC) > 0$, with (3.5) then implying that both eigenvalues are positive. In order to shorten notation, write $\tau = \mathrm{tr}(C^TC)$. The spectral norm of $C^TC$ is its larger eigenvalue. Calculating it yields

$$\begin{aligned} \|C^TC\| &= \frac{\tau}{2} + \sqrt{\frac{\tau^2}{4} - 1} = 1 + \frac{1}{2}\left( (\tau - 2) + \sqrt{(\tau-2)(\tau+2)} \right) \\ &= 1 + (\tau - 2)^{\frac{1}{2}} \cdot \frac{(\tau-2)^{\frac{1}{2}} + (t+2)^{\frac{1}{2}}}{2}. \end{aligned}$$

From our calculation of $\mathrm{tr}(C^TC)$ above, we can now deduce that

$$\begin{aligned} \tau - 2 &= \cos^2(\sigma) \cdot \left(\frac{a}{b} - \frac{b}{a}\right)^2 + \sin^2(\sigma) \cdot \left(ab^2 - \frac{1}{ab}\right)^2 = \|v_+ - v_-\|_2^2, \\ \tau + 2 &= \cos^2(\sigma) \cdot \left(\frac{a}{b} + \frac{b}{a}\right)^2 + \sin^2(\sigma) \cdot \left(ab^2 + \frac{1}{ab}\right)^2 = \|v_+ + v_-\|_2^2. \end{aligned}$$

So far, we have shown every assertion except for the estimate from below. Therefor, observe that

$$f \colon \begin{cases} [1, \infty) & \to & [2, \infty) \\ x & \mapsto & x + \frac{1}{x} \end{cases}$$

is a bijection as well as increasing, continuous and convex. Hence, it's inverse $f^{-1}$ is concave, allowing us to calculate

$$\begin{aligned} \|C\|^2 &= \|C^TC\| = f^{-1}\left(\mathrm{tr}(C^TC)\right) \\ &= f^{-1}\left( \cos^2(\sigma) \cdot f\left( \max\left\{ \left(\frac{a}{b}\right)^2, \left(\frac{b}{a}\right)^2 \right\} \right) \right. \\ &\qquad \left. + \sin^2(\sigma) \cdot f\left( \max\left\{ (ab)^2, \left(\frac{1}{ab}\right)^2 \right\} \right) \right) \\ &\geq \cos^2(\sigma) \cdot \max\left\{ \left(\frac{a}{b}\right)^2, \left(\frac{b}{a}\right)^2 \right\} + \sin^2(\sigma) \cdot \max\left\{ (ab)^2, \left(\frac{1}{ab}\right)^2 \right\} \\ &= \|v_+\|_2^2. \end{aligned}$$

□

Merging what we learned from the past two lemmas now allows us to prove the theorem.

*Proof of Theorem 3.3.* At first, we apply Lemma 3.4 with our given partition and the matrices $\Omega_j := \Omega(a_j, \psi_j)$, $j \in \{1, \dots, N\}$, along with the norm calculations from Lemma 3.6. Taking logarithms yields

$$\begin{aligned}
\log \|W_H(z)\| \leq |z| \sum_{j=1}^{N} \int_{y_{j-1}}^{y_j} &\|\Omega(a_j, \psi_j) H(t) J \Omega(a_j, \psi_j)^{-1}\| \, \mathrm{d}t \\
&+ \sum_{j=1}^{N-1} \log \|\Omega(a_j, \psi_j) \Omega(a_{j+1}, \psi_{j+1})^{-1}\| \\
&+ \log \|\Omega(a_1, \psi_1)^{-1}\| + \log \|\Omega(a_N, \psi_N)\| \\
\leq |z| \sum_{j=1}^{N} &\left( \int_{y_{j-1}}^{y_j} \left( a_j^2 \cos^2(\phi(t) - \psi_j) + \frac{1}{a_j^2} \sin^2(\phi(t) - \psi_j) \right) \cdot \operatorname{tr} H(t) \, \mathrm{d}t \right) \\
&+ \sum_{j=1}^{N-1} \log \left( \max\{ \frac{a_j}{a_{j+1}}, \frac{a_{j+1}}{a_j} \} \cdot |\cos(\psi_j - \psi_{j+1})| + \frac{|\sin(\psi_j - \psi_{j+1})|}{a_j a_{j+1}} \right) \\
&+ \log \frac{1}{a_1} + \log \frac{1}{a_N} \\
= |z|(A_1 + A_2) &+ A_3 + A_4,
\end{aligned}$$

and the proof is complete. □

## 3.2 Growth Theorem for Hamburger Hamiltonians

In this section, we focus on Hamiltonians whose measurable diagonalisations $H(t) = \operatorname{tr} H(t) \cdot \xi_{\phi(t)} \xi_{\phi(t)}^T$ are already given by their construction, allowing us to explicitly calculate the bound given in Theorem 3.3. The Hamiltonians of choice are so-called *Hamburger Hamiltonians*, which are trace-normed and piece-wise constant on the given interval and which are defined in Definition 3.7. Most of the results covered in this section, and in particular its main result, Theorem 3.12, were originally obtained by Harald Woracek and are to be published in [Wor].

**Definition 3.7.** Let $l = (l_j)_{j \in \mathbb{N}}$ be a summable sequence of positive numbers and $\phi = (\phi_j)_{j \in \mathbb{N}}$ a sequence of real numbers. Setting $L := \sum_{j=1}^{\infty}$, we denote by $H_{l,\phi} \in L^1_{\mathrm{loc}}\left([0, L], \mathbb{C}^{2\times 2}\right)$ the Hamiltonian defined as

$$H_{l,\phi}(t) := \xi_{\phi_l} \xi_{\phi_l}^T, \quad \sum_{i=1}^{j-1} l_j \leq t < \sum_{i=1}^{j} l_j, \quad j \in \mathbb{N}.$$

We call Hamiltonians of the form $H_{l,\phi}$ *Hamburger Hamiltonians* and refer to the numbers $l_j$ and $\phi_j$ as their *lengths* and *angles*.

**Definition 3.8.** Let $(l_j)_{j\in\mathbb{N}}$ be a summable sequence of positive numbers and $(\phi_j)_{j\in\mathbb{N}}$ a sequence of real numbers. Then we call a tuple $(d_l, d_\phi, c_l, c_\phi, \psi)$ a *suitable majorisation*, if

(i) $d_l, d_\phi, c_l, c_\phi \colon [1,\infty) \to (0,\infty)$ are continuous and nonincreasing functions, and $\psi \in \mathbb{R}$.

(ii) There is $t_0 \geq 1$ such that $\frac{d_\phi(t)}{d_l(t)}$ is monotone for $t \geq t_0$, and $\int_1^\infty d_l(t)\,\mathrm{d}t < \infty$.

(iii) It holds that

$$\forall j \in \mathbb{N} \colon l_j \leq d_l(j) \wedge |\sin(\phi_{j+1} - \phi_j)| \leq d_\phi(j),$$

as well as

$$\forall N \in \mathbb{N} \colon \sum_{j=N}^{\infty} l_j \leq c_l(N) \wedge \sum_{j=N}^{\infty} l_j \sin^2(\phi_{j+1} - \phi_j) \leq c_\phi(N).$$

(iv) $d_\phi \leq 1$, $\lim_{t\to\infty} c_l(t) = 0$ and $c_\phi \leq c_l$.

**Remark 3.9.** As the functions $d_l, d_\phi, c_l$ and $c_\phi$ are continuous and nonincreasing, there is a unique continuous extension of each of them to $[1,\infty]$ with values in $(0,\infty)$. Similarly, as $\frac{d_\phi}{d_l}$ is eventually monotone, it has a unique continuous extension to $[1,\infty]$ with values in $[0,\infty]$. Hereinafter, we denote by $d_l, d_\phi, c_l, c_\phi$ and $\frac{d_\phi}{d_l}$ these extensions.

Using the described extensions, along with the usual conventions concerning algebra on $[0,\infty]$, allows us to make the following definitions.

**Definition 3.10.** Given a summable sequence $(l_j)_{j\in\mathbb{N}}$ of positive numbers, a sequence $(\phi_j)_{j\in\mathbb{N}}$ of real numbers and a suitable majorisation $(d_l, d_\phi, c_l, c_\phi, \psi)$, we define the functions $e, k, f$ and $\hbar$ as follows.

$$\begin{aligned}
e(t) &:= \frac{2}{(d_l d_\phi)(t)} \colon [1,\infty] \to \left[\frac{2}{(d_l d_\phi)(1)}, \infty\right], \\
k(R) &:= \max\left\{t \in [1,\infty] \colon e(t) \leq R\right\} \colon \left[\frac{2}{(d_l d_\phi)(1)}, \infty\right] \to [1,\infty], \\
f(t) &:= \frac{d_\phi(t)}{d_l(t)} \colon [1,\infty] \to [0,\infty], \\
\hbar(R) &:= \max\left\{t \in [1,\infty] \colon f(t) \leq R\right\} \colon \left[\frac{d_\phi(1)}{d_l(1)}, \infty\right] \to [1,\infty].
\end{aligned}$$

Further, define $g \colon [2,\infty) \times [e(2),\infty) \to (0,\infty)$ as

$$\begin{aligned}
g(t,R) := &\int_1^{\min\{t,k(R)\}} \log\left(R(d_l d_\phi)(s)\right)\,\mathrm{d}s \\
&+ R^{\frac{1}{2}} \int_{\min\{t,k(R)\}}^{\min\{t,\hbar(R)\}} (d_l d_\phi)^{\frac{1}{2}}(s)\,\mathrm{d}s + \int_{\min\{t,\hbar(R)\}}^{t} d_\phi(s)\,\mathrm{d}s,
\end{aligned}$$

and

$$n(t) := (c_l c_\phi)^{\frac{1}{2}}(t) \colon [1,\infty) \to (0,\infty).$$

**Remark 3.11.**

(i) By Definition 3.8 (iii), $d_l d_\phi$ converges to zero for $t \to \infty$. As it is doing so monotonously, $e(t)$ is in fact a bijective function from $[1,\infty)$ to $[e(1),\infty)$ and $k(R)$ is its inverse function.

(ii) If the eventually monotone function $f(t)$ is eventually nonincreasing or just bounded from above, we have $h(R) = \infty$ for sufficiently large $R$.

(iii) $d_\phi$ being bounded from above by 1 implies that $e(t) \geq f(t)$ and consequently $k(R) \leq h(R)$.

(iv) For given $R \geq e(2)$, the mapping $t \to g(t,R)$ is continuous and increasing.

(v) The function $n(t)$ is nonincreasing.

Besides Romanov's Theorem 3.3, the following theorem is the main result of this chapter. Along with the subsequent lemma and corollary, it will form the base of our calculations in Chapter 5.

**Theorem 3.12.** *Let $(l_j)j \in \mathbb{N}$ be a summable sequence of positive numbers and $(\phi_j)_{j\in\mathbb{N}}$ a sequence of real numbers. Further, let $H$ be the Hamburger Hamiltonian with lengths $l_j$ and angles $\phi_j$, and denote by $W_H$ its monodromy matrix. Moreover, assume that $(d_l, d_\phi, c_l, c_\phi, \psi)$ is a suitable majorisation, and let $g(t,R)$ as well as $n(t)$ be defined as in Definition 3.8. Then it holds for $R \geq \frac{2}{(d_l d_\phi)(2)}$ that*

$$\max_{|z|=R} (\log \|W_H(z)\|) \lesssim \min_{t\geq 2} (\max\{g(t,R), R n(t)\} + L(t,R)), \tag{3.6}$$

*where $L(t)$ denotes the remainder term of the form*

$$L(t,R) := 1 + \log^+ R + \log^+ \frac{1}{d_l(t)} + \log^+ \frac{c_l(t)}{c_\phi(t+1)} + \left|\log \frac{d_l(t)}{d_\phi(t)}\right|.$$

*Proof.* The function $g(t,R)$ is positive and nondecreasing with respect to $t$. Consequently, it holds that

$$\min_{t\geq 2} g(t,R) = g(2,R) \geq \log R(d_l d_\phi)(2) > 0.$$

From this, we can conclude that for any fixed $R_0 \geq e(2)$, it holds that

$$\max_{|z|\leq R_0} (\log \|W_H(z)\|) \lesssim \min_{t\geq 2} (\max\{g(t,e(2)), e(2) n(t)\}),$$

which in particular means that (3.6) holds for all $R \leq R_0$. Hence, as $k(R)$ tends to $\infty$ for $R \to \infty$, we can assume $R$ to be sufficiently large such that $f(t) = \frac{d_\phi(t)}{d_l(t)}$ is monotonous

for all $t \geq \bar{k}(R) - 1$ for the remainder of this proof. Moreover, given the case that $\frac{\bar{d}_\phi(t)}{\bar{d}_l(t)}$ is bounded, we assume $R$ to be sufficiently large such that $\frac{1}{R}f(t)$ is bounded by 1.
From this point onward, the proof of the theorem is based on an application of Theorem 3.3, where we will choose the necessary parameters depending on $t$ and $R$.
Due to the design of the Hamburger Hamiltonian, choosing the partition to be a cut-off suggests itself. Setting $N := \lceil t \rceil$, we define the partition $y_j$, $j \in \{0, \ldots, N\}$ by

$$y_j := \begin{cases} \sum_{k=1}^{j} l_k & \textit{if } j \in \{0, \ldots, N-1\}, \\ L & \textit{if } j = N. \end{cases}$$

Similarly, we set the rotation parameters $\psi_j$, $j \in \{1, \ldots, N\}$ to the angles given by $(\phi_j)_{j \in \mathbb{N}}$ and the suitable majorisation, i.e.

$$\psi_j := \begin{cases} \phi_j & \textit{if } j \in \{1, \ldots, N-1\}, \\ \psi & \textit{if } j = N. \end{cases}$$

Further, we set the distortion parameters $a_j$, $j \in \{1, \ldots, N-1\}$ to

$$a_j^2 := \begin{cases} \frac{1}{R\bar{d}_l(j)} & \textit{if } j \leq \bar{k}(R), \\ \left(\frac{\bar{d}_\phi(j)}{R\bar{d}_l(j)}\right)^{\frac{1}{2}} & \textit{if } \bar{k}(R) < j \leq \hslash(R), \\ 1 & \textit{if } \hslash(R) < j, \end{cases}$$

as well as

$$a_N^2 := \left(\frac{\bar{c}_\phi(N)}{\bar{c}_l(N)}\right)^{\frac{1}{2}}.$$

It can be shown that, when choosing the partition and the rotation parameters as we did above, this particular choice of the distortion parameters minimizes the estimates for $A_1, A_2, A_3$ and $A_4$ occurring in $\boxed{2}$.
$\boxed{1}$ In order to apply said Theorem with the parameters given above, the distortion parameters need to satisfy $a_j \leq 1, j \in \{1, \ldots, N\}$. First, the fact that $a_N \leq 1$ follows directly from the fact that $(\bar{d}_l, \bar{d}_\phi, \bar{c}_l, \bar{c}_\phi, \psi)$ is a suitable majorisation and Definition 3.8 (iv). Moreover, it holds for $j > \hslash(R)$ that $a_j = 1$, meaning that the only cases left to consider are $j \in \{1, \ldots, N-1\} \cap [1, \bar{k}(R)]$ and $j \in \{1, \ldots, N-1\} \cap (\bar{k}(R), \hslash(R)]$. In the former instance, we observe that

$$a_j^2 = \frac{1}{R\bar{d}_l(j)} = \frac{1}{R} \cdot \frac{\bar{d}_\phi(j)\bar{e}(j)}{2} \leq \frac{\bar{e}(j)}{2R} \leq \frac{1}{2}$$

since $e(j) \leq e(\bar{k}(R)) \leq R$, while the latter yields

$$a_j^4 = \frac{\bar{d}_\phi(j)}{R\bar{d}_l(j)} = \frac{1}{R}f(j) \leq \frac{1}{R}f(\hslash(R)) \leq 1$$

for unbounded $f(t)$ and

$$a_j^4 = \frac{\bar{d}_\phi(j)}{R\bar{d}_l(j)} = \frac{1}{R}f(j) \leq 1$$

otherwise. In addition to the above, the parameters $a_j$ also have useful monotonicity properties.
We start by noting that the fact that $\text{đ}_l(t)$ is nonincreasing implies

$$a_j \leq a_{j+1},\ j \in \{1, \dots, N-2\} \cap [1, \text{ꝁ}(R) - 1]. \tag{3.7}$$

Further, if $\frac{\text{đ}_\phi(t)}{\text{đ}_l(t)}$ is nonincreasing, we see that

$$a_j \geq a_{j+1},\ j \in \{1, \dots, N-2\} \cap (\text{ꝁ}(R), \hbar(R) - 1], \tag{3.8}$$

while the case of $\frac{\text{đ}_\phi(t)}{\text{đ}_l(t)}$ being nondecreasing results in

$$a_j \leq a_{j+1},\ j \in \{1, \dots, N-2\} \cap (\text{ꝁ}(R), \hbar(R) - 1]. \tag{3.9}$$

In fact, this instance even yields

$$a_j \leq a_{j+1},\ j \in \{1, \dots, N-2\} \cap [1, \hbar(R) - 1]. \tag{3.10}$$

To prove the last assertion, it is sufficient to consider the case when $\lfloor \text{ꝁ}(R) \rfloor + 1 \leq N - 1$, as (3.10) is equivalent to (3.7) otherwise. Under these assumptions, when setting $j := \lfloor \text{ꝁ}(R) \rfloor$, we see that

$$\left(\frac{a_{j+1}}{a_j}\right)^4 = R^2 \text{đ}_l(j)^2 \frac{\text{đ}_\phi(j+1)}{R \text{đ}_l(j+1)} \geq R \text{đ}_l(j)^2 \frac{\text{đ}_\phi(j)}{\text{đ}_l(j)} \geq R \frac{2}{e(j)} \geq 2.$$

Thus, we have $a_{\lfloor \text{ꝁ}(R) \rfloor} \leq a_{\lfloor \text{ꝁ}(R) \rfloor + 1}$, which in combination with (3.7) and (3.9) yields (3.10).
$\boxed{2}$ After showing that the distortion parameters $a_j$ are less or equal to 1, all requirements to the interval, the Hamiltonian, the partition and the parameters are satisfied, and Theorem 3.3 is applicable. Hence, our next step is to calculate $A_i$, $i \in \{1, 2, 3, 4\}$. Using the functions given by our suitable majorisation, it is not hard to find upper bounds of reasonable complexity for $A_1$ as well as $A_2$, and the term $A_4$ is quite simple by design. Only the calculation of $A_3$, or an upper bound thereof, is quite finicky and will require us to distinguish multiple cases.

Due to our special choice of $\psi_j$, the first calculation shows

$$
\begin{aligned}
A_1 &= \sum_{j=1}^{N} a_j^2 \int_{y_{j-1}}^{y_j} \cos^2(\phi(t)-\psi_j)\cdot \operatorname{tr}H(t)\,\mathrm{d}t \\
&= \sum_{j=1}^{N-1} a_j^2 (y_j - y_{j-1}) + a_N^2 \int_{y_{j-1}}^{L} \cos^2(\phi(t)-\psi)\,\mathrm{d}t \\
&= \sum_{j=1}^{N-1} a_j^2 l_j + a_N^2 \sum_{j=N}^{\infty} l_j \cos^2(\phi_j - \psi) \\
&\leq \underbrace{\sum_{j=1}^{N-1} a_j^2 d_l(j)}_{=:\Theta} + a_N^2 c_l(N),
\end{aligned}
\tag{3.11}
$$

while the second one leads to

$$
\begin{aligned}
A_2 &= \sum_{j=1}^{N} \frac{1}{a_j^2} \int_{y_{j-1}}^{y_j} \sin^2(\phi(t)-\psi_j)\cdot \operatorname{tr}H(t)\,\mathrm{d}t = \frac{1}{a_N^2} \sum_{j=N}^{\infty} l_j \sin^2(\phi_j - \psi) \\
&\leq \frac{1}{a_N^2} c_\phi(N),
\end{aligned}
\tag{3.12}
$$

and $A_4$ can be written as

$$
A_4 = -\log a_1 - \log a_N. \tag{3.13}
$$

$A_3$ is of the form

$$
A_3 = \sum_{j=1}^{N-1} \log\left(\max\left\{\frac{a_j}{a_{j+1}}, \frac{a_{j+1}}{a_j}\right\}\cdot |\cos(\psi_j - \psi_{j+1})| + \frac{|\sin(\psi_j - \psi_{j+1})|}{a_j a_{j+1}}\right).
$$

In order to actually compute it, we will use the monotonicity properties of the distortion parameters shown in [1]. Let $m, n \in \mathbb{N}$ such that $1 \leq m < n < N$ and assume that

$a_m \leq a_{m+1} \leq \ldots \leq a_n$. Then, it holds that

$$\begin{aligned}
&\sum_{j=m}^{n-1} \log\left(\max\left\{\frac{a_j}{a_{j+1}}, \frac{a_{j+1}}{a_j}\right\} \cdot |\cos(\psi_j - \psi_{j+1})| + \frac{|\sin(\psi_j - \psi_{j+1})|}{a_j a_{j+1}}\right) \\
&\quad = \sum_{j=m}^{n-1} \log\left(\frac{a_{j+1}}{a_j} \cdot |\cos(\phi_j - \phi_{j+1})| + \frac{|\sin(\phi_j - \phi_{j+1})|}{a_j a_{j+1}}\right) \\
&\quad = \sum_{j=m}^{n-1} \log\frac{a_{j+1}}{a_j} + \log\left(|\cos(\phi_j - \phi_{j+1})| + \frac{|\sin(\phi_j - \phi_{j+1})|}{a_{j+1}^2}\right) \\
&\quad = \log\frac{a_n}{a_m} + \sum_{j=m}^{n-1} \log\left(|\cos(\phi_j - \phi_{j+1})| + \frac{|\sin(\phi_j - \phi_{j+1})|}{a_{j+1}^2}\right) \\
&\quad \leq \log\frac{a_n}{a_m} + \sum_{j=m}^{n-1} \log\left(1 + \frac{d_\phi(j)}{a_j^2}\right).
\end{aligned} \tag{3.14}$$

If we assume that $a_m \geq a_{m+1} \geq \ldots \geq a_n$ instead, a very similar calculation shows

$$\begin{aligned}
&\sum_{j=m}^{n-1} \log\left(\max\left\{\frac{a_j}{a_{j+1}}, \frac{a_{j+1}}{a_j}\right\} \cdot |\cos(\psi_j - \psi_{j+1})| + \frac{|\sin(\psi_j - \psi_{j+1})|}{a_j a_{j+1}}\right) \\
&\quad = \sum_{j=m}^{n-1} \log\left(\frac{a_j}{a_{j+1}} \cdot |\cos(\phi_j - \phi_{j+1})| + \frac{|\sin(\phi_j - \phi_{j+1})|}{a_j a_{j+1}}\right) \\
&\quad = \log\frac{a_m}{a_n} + \sum_{j=m}^{n-1} \log\left(|\cos(\phi_j - \phi_{j+1})| + \frac{|\sin(\phi_j - \phi_{j+1})|}{a_j^2}\right) \\
&\quad \leq \log\frac{a_m}{a_n} + \sum_{j=m}^{n-1} \log\left(1 + \frac{d_\phi(j)}{a_j^2}\right).
\end{aligned} \tag{3.15}$$

Another useful estimate, but independent of monotonic behaviour of the parameters $a_j$, is

$$\begin{aligned}
&\log\left(\max\left\{\frac{a_j}{a_{j+1}}, \frac{a_{j+1}}{a_j}\right\} \cdot |\cos(\psi_j - \psi_{j+1})| + \frac{|\sin(\psi_j - \psi_{j+1})|}{a_j a_{j+1}}\right) \\
&= \log\left(\max\left\{\frac{a_j^2}{a_j a_{j+1}}, \frac{a_{j+1}^2}{a_j a_{j+1}}\right\} \cdot |\cos(\psi_j - \psi_{j+1})| + \frac{|\sin(\psi_j - \psi_{j+1})|}{a_j a_{j+1}}\right) \\
&\leq \log\frac{2}{a_j a_{j+1}},
\end{aligned} \tag{3.16}$$

which holds for all $j \in \{1, \ldots, N-1\}$.

3 $N - 1 \leq \hbar(R)$

In this instance, (3.7) tells us that we can apply (3.14) for $m = 1$ and $n = N - 1$, and combining it with (3.16) along with plugging in the definition of the distortion parameters

$a_j$ gives us

$$
\begin{aligned}
A_3 \leq& \log \frac{a_{N-1}}{a_1} + \sum_{j=1}^{N-2} \log \left(1 + \frac{d_\phi(j)}{a_j^2}\right) \\
&+ \log \left( \max \left\{ \frac{a_{N-1}}{a_N}, \frac{a_N}{a_{N-1} j} \right\} \cdot |\cos(\phi_{N-1} - \psi)| + \frac{|\sin(\phi_{N-1} - \psi)|}{a_{N-1} a_N} \right) \\
\leq& \sum_{j=1}^{N-2} \log \left(1 + \frac{d_\phi(j)}{a_j^2}\right) + \left[ \log \frac{a_{N-1}}{a_1} + \log 2 + \frac{1}{2} \log \frac{R d_l(N-1) c_l(N)^{\frac{1}{2}}}{c_\phi(N)^{\frac{1}{2}}} \right] \\
=& \sum_{j=1}^{N-2} \log \left(1 + R(d_l d_\phi)(j)\right) \\
&+ \left[ \frac{1}{2} \log \frac{d_l(1)}{d_l(N-1)} + \log 2 + \frac{1}{2} \log \frac{R d_l(N-1) c_l(N)^{\frac{1}{2}}}{c_\phi(N)^{\frac{1}{2}}} \right].
\end{aligned}
$$

We now proceed by combining $A_3$ with $\Theta$, the first summand of the estimation of $A_1$ given in (3.11). When plugging in the explicit form of the distortion parameters $a_j$, we see that

$$
\Theta = \sum_{j=1}^{N-1} a_j^2 d_l(j) = \frac{N-1}{R},
$$

and consequently

$$
\begin{aligned}
R\Theta + A_3 \leq N - 1 + &\sum_{j=1}^{N-2} \log \left(1 + R(d_l d_\phi)(j)\right) \\
&+ \left[ \frac{1}{2} \log \frac{d_l(1)}{d_l(N-1)} + \log 2 + \frac{1}{2} \log \frac{R d_l(N-1) c_l(N)^{\frac{1}{2}}}{c_\phi(N)^{\frac{1}{2}}} \right].
\end{aligned}
$$

By assumption, we have $N-1 \leq k(R)$, meaning that $e(j) \leq R$ for all $j \leq N-1$, which is equivalent to $R(d_l d_\phi)(j) \geq 2$ for all $j \leq N-1$. Furthermore, it holds that $N-2 \leq \min\{t, k(R)\}$, which gives

$$
\begin{aligned}
N - 1 + \sum_{j=1}^{N-2} \log \left(1 + R(d_l d_\phi)(j)\right) &\lesssim \sum_{j=1}^{N-2} \log \left(R(d_l d_\phi)(j)\right) \\
&\lesssim \log \left(R(d_l d_\phi)(1)\right) + \int_{j=1}^{N-2} \log \left(R(d_l d_\phi)(t)\right) \, \mathrm{d}t \lesssim g(t, R) + \log R.
\end{aligned}
$$

This estimation ultimately lets us conclude that

$$\begin{aligned}R\Theta + A_3 &\lesssim \not{g}(t,R) + \left[\log R + \frac{1}{2}\log\frac{\not{d}_l(1)}{\not{d}_l(N-1)} + \log 2 + \frac{1}{2}\log\frac{R\not{d}_l(N-1)\not{c}_l(N)^{\frac{1}{2}}}{\not{c}_\phi(N)^{\frac{1}{2}}}\right] \\ &\leq \not{g}(t,R) + \left[\log^+ t + \frac{1}{2}\log^+\frac{1}{\not{d}_l(N-1)} + 1 + \log^+\frac{R\not{d}_l(t-1)^2\not{c}_l(t-1)}{\not{c}_\phi(t)}\right].\end{aligned}$$

$\boxed{4}$ $\frac{\not{d}_\phi(t)}{\not{d}_l(t)}$ nonincreasing and $\hbar(R) < N-1$.

As $\frac{\not{d}_\phi(t)}{\not{d}_l(t)}$ is assumed to be nonincreasing, we have $\hbar(R) = \infty$ for all sufficiently large $R$, and we will assume $R$ to be that large. By (3.7), we can apply (3.14) with $m = 1$ and $n = \lfloor \hbar(R) \rfloor$. Plugging in the definition of the distortion parameters then results in

$$\begin{aligned}&\sum_{j=1}^{\lfloor \hbar(R)\rfloor - 1} \log\left(\max\left\{\frac{a_j}{a_{j+1}}, \frac{a_{j+1}}{a_j}\right\}\cdot|\cos(\psi_j - \psi_{j+1})| + \frac{|\sin(\psi_j-\psi_{j+1})|}{a_j a_{j+1}}\right) \\ &\qquad \leq \log\frac{a_{\lfloor \hbar(R)\rfloor}}{a_1} + \sum_{j=1}^{\lfloor \hbar(R)\rfloor - 1}\log\left(1 + \frac{\not{d}_\phi(j)}{a_j^2}\right) \\ &\qquad = \frac{1}{2}\frac{\not{d}_l(1)}{\not{d}_l(\lfloor \hbar(R)\rfloor)} + \sum_{j=1}^{\lfloor \hbar(R)\rfloor - 1}\log\left(1 + R(\not{d}_l\not{d}_\phi)(j)\right).\end{aligned}$$

Similarly, (3.8) allows us to apply (3.15) with $m = \lfloor \hbar(R)\rfloor + 1$ and $n = N-1$, giving

$$\begin{aligned}&\sum_{j=\lfloor \hbar(R)\rfloor+1}^{N-2} \log\left(\max\left\{\frac{a_j}{a_{j+1}}, \frac{a_{j+1}}{a_j}\right\}\cdot|\cos(\psi_j - \psi_{j+1})| + \frac{|\sin(\psi_j-\psi_{j+1})|}{a_j a_{j+1}}\right) \\ &\qquad \leq \log\frac{a_{\lfloor \hbar(R)\rfloor+1}}{a_{N-2}} + \sum_{j=\lfloor \hbar(R)\rfloor+1}^{N-2}\log\left(1 + \frac{\not{d}_\phi(j)}{a_j^2}\right) \\ &\qquad = \frac{1}{4}\log\frac{\not{d}_\phi(\lfloor \hbar(R)\rfloor+1)\not{d}_l(N-2)}{\not{d}_l(\lfloor \hbar(R)\rfloor+1)\not{d}_\phi(N-2)} + \sum_{j=\lfloor \hbar(R)\rfloor+1}^{N-2}\log\left(1 + \sqrt{R(\not{d}_l\not{d}_\phi)(j)}\right).\end{aligned}$$

The last two calculations cover all but two summands in the definition of $A_3$, namely those with indices $\lfloor \hbar(R)\rfloor$ and $N-1$. For these two, it holds by (3.16) that

$$
\begin{aligned}
&\sum_{j\in\{\lfloor \not{k}(R)\rfloor, N-1\}} \log\left(\max\left\{\frac{a_j}{a_{j+1}}, \frac{a_{j+1}}{a_j}\right\}\cdot |\cos(\psi_j-\psi_{j+1})| + \frac{|\sin(\psi_j-\psi_{j+1})|}{a_j a_{j+1}}\right) \\
&\le \log\frac{2}{a_{\lfloor \not{k}(R)\rfloor} a_{\lfloor \not{k}(R)\rfloor+1}} + \log\frac{2}{a_{N-1}a_N} \\
&= 2\log 2 + \log\left(R^{\frac{1}{2}} \not{d}_l(\lfloor \not{k}(R)\rfloor)^{\frac{1}{2}}\cdot\left(\frac{R\not{d}_l(\lfloor \not{k}(R)\rfloor+1)}{\not{d}_\phi(\lfloor \not{k}(R)\rfloor+1)}\right)^{\frac{1}{4}}\right) \\
&\quad + \log\left(\left(\frac{R\not{d}_l(N-1)}{\not{d}_\phi(N-1)}\right)^{\frac{1}{4}}\left(\frac{\not{c}_l(N)}{\not{c}_\phi(N)}\right)^{\frac{1}{4}}\right) \\
&= 2\log 2 + \frac{1}{4}\log\frac{R^3\not{d}_l(\lfloor \not{k}(R)\rfloor)^2\not{d}_l(\lfloor \not{k}(R)\rfloor+1)}{\not{d}_\phi(\lfloor \not{k}(R)\rfloor+1)} + \frac{1}{4}\log\frac{R\not{d}_l(N-1)\not{c}_l(N)}{\not{d}_\phi(N-1)\not{c}_\phi(N)}.
\end{aligned}
$$

As we plan to combine $A_3$ with $\Theta$ once more, we also need to evaluate under the made assumptions. We have

$$
\Theta = \sum_{j=1}^{N-1} a_j^2 \not{d}_l(j) = \frac{\lfloor \not{k}(R)\rfloor}{R} + \sum_{j=\lfloor \not{k}(R)\rfloor+1}^{N-1}\left(\frac{(\not{d}_l\not{d}_\phi)(j)}{R}\right)^{\frac{1}{2}}, \tag{3.17}
$$

and ultimately get

$$
\begin{aligned}
R\Theta + A_3 &= \lfloor \not{k}(R)\rfloor + \sum_{j=\lfloor \not{k}(R)\rfloor+1}^{N-1}\sqrt{R(\not{d}_l\not{d}_\phi)(j)} + A_3 \\
&\le \lfloor \not{k}(R)\rfloor + \sum_{j=1}^{\lfloor \not{k}(R)\rfloor-1}\log\left(1+R(\not{d}_l\not{d}_\phi)(j)\right) \\
&\quad + \sum_{j=\lfloor \not{k}(R)\rfloor+1}^{N-2}\left(\sqrt{R(\not{d}_l\not{d}_\phi)(j)} + \log\left(1+\sqrt{R(\not{d}_l\not{d}_\phi)(j)}\right)\right) \\
&\quad + \sqrt{R(\not{d}_l\not{d}_\phi)(N-1)} \\
&\quad + \left[\frac{1}{2}\frac{\not{d}_l(1)}{\not{d}_l(\lfloor \not{k}(R)\rfloor)} + \frac{1}{4}\log\frac{\not{d}_\phi(\lfloor \not{k}(R)\rfloor+1)\not{d}_l(N-2)}{\not{d}_l(\lfloor \not{k}(R)\rfloor+1)\not{d}_\phi(N-2)}\right. \\
&\qquad + \frac{1}{4}\log\frac{R^3\not{d}_l(\lfloor \not{k}(R)\rfloor)^2\not{d}_l(\lfloor \not{k}(R)\rfloor+1)}{\not{d}_\phi(\lfloor \not{k}(R)\rfloor+1)} \\
&\qquad \left. + \frac{1}{4}\log\frac{R\not{d}_l(N-1)\not{c}_l(N)}{\not{d}_\phi(N-1)\not{c}_\phi(N)} + 2\log 2\right].
\end{aligned}
$$

In order to estimate everything except for the remainder term with integrals once again, observe that first,

$$\lfloor \Bbbk(R)\rfloor + \sum_{j=1}^{\lfloor \Bbbk(R)\rfloor -1} \log\left(1 + R(d_l d_\phi)(j)\right) \lesssim \sum_{j=1}^{\lfloor \Bbbk(R)\rfloor -1} \log\left(R(d_l d_\phi)(j)\right)$$
$$\leq \log\left(R(d_l d_\phi)(1)\right) + \int_1^{\lfloor \Bbbk(R)\rfloor -1} \log\left(R(d_l d_\phi)(s)\right)\, \mathrm{d}s,$$

and second,

$$\sum_{j=\lfloor \Bbbk(R)\rfloor+1}^{N-2} \left(\sqrt{R(d_l d_\phi)(j)} + \log\left(1+\sqrt{R(d_l d_\phi)(j)}\right)\right) + \sqrt{R(d_l d_\phi)(N-1)}$$
$$\leq \sum_{j=\lfloor \Bbbk(R)\rfloor+1}^{N-1} \left(\sqrt{R(d_l d_\phi)(j)} + \log\left(1+\sqrt{R(d_l d_\phi)(j)}\right)\right)$$
$$\lesssim \sum_{j=\lfloor \Bbbk(R)\rfloor+1}^{N-1} \sqrt{R(d_l d_\phi)(j)} \leq 2 + \int_{\lfloor \Bbbk(R)\rfloor+1}^{N-1} \left(R(d_l d_\phi)(s)\right)^{\frac{1}{2}}\, \mathrm{d}s.$$

Finally, as we have $t \geq N-1 > \Bbbk(R)$, these estimations let us conclude that

$$R\Theta + A_3 \lesssim g(t,R) + \left[\log^+ R + \log^+ \frac{1}{d_l(t)} + \log^+ \frac{d_l(t)}{d_\phi(t)} + \log^+ \frac{d_l(t) c_l(t)}{d_\phi(t) c_\phi(t+1)} + 1\right].$$

[5] $\frac{d_\phi}{d_l}$ nondecreasing and $\Bbbk(R) < N-1 \leq \hbar(R)$
In this instance, (3.10) lets us apply (3.14) with $m=1$ and $n=N-1$. If we further use (3.16) to estimate the last summand of $A_3$, we receive

$$A_3 \leq \log\frac{a_{N-1}}{a_1} + \sum_{j=1}^{N-2} \log\left(1 + \frac{d_\phi(j)}{a_j^2}\right) + \log\frac{2}{a_{N-1}a_N}$$
$$= \frac{1}{4}\log\frac{R d_l(1)^2 d_\phi(N-1)}{d_l(N-1)} + \sum_{j=1}^{\lfloor \Bbbk(R)\rfloor} \log(1 + R(d_l d_\phi)(j))$$
$$+ \sum_{j=\lfloor \Bbbk(R)\rfloor+1}^{N-2} \log\left(1+\sqrt{R(d_l d_\phi)(j)}\right) + \left[\frac{1}{4}\log\frac{R d_l(N-1) c_l(N)}{d_\phi(N-1) c_\phi(N)} + \log 2\right].$$

Beyond that, as (3.17) still holds in this case, we can calculate

$$\begin{aligned}
R\Theta + A_3 &= \lfloor \not{k}(R) \rfloor + \sum_{j=\lfloor \not{k}(R)\rfloor+1}^{N-1} \sqrt{R(\not{d}_l \not{d}_\phi)(j)} + A_3 \\
&\leq \lfloor \not{k}(R) \rfloor + \sum_{j=1}^{\lfloor \not{k}(R) \rfloor} \log(1 + R(\not{d}_l \not{d}_\phi)(j)) \\
&\quad + \sum_{j=\lfloor \not{k}(R)\rfloor+1}^{N-2} \left( \sqrt{R(\not{d}_l \not{d}_\phi)(j)} + \log\left(1 + \sqrt{R(\not{d}_l \not{d}_\phi)(j)}\right)\right) \\
&\quad + \sqrt{R(\not{d}_l \not{d}_\phi)(N-1)} \\
&\quad + \left[ \frac{1}{4} \log \frac{R\not{d}_l(1)^2 \not{d}_\phi(N-1)}{\not{d}_l(N-1)} + \frac{1}{4} \log \frac{R\not{d}_l(N-1)\not{c}_l(N)}{\not{d}_\phi(N-1)\not{c}_\phi(N)} + \log 2 \right],
\end{aligned}$$

and by estimating the occurring sums by integrals as in the previous cases, we see that

$$R\Theta + A_3 \lesssim \not{g}(t, R) + \left[ \log^+ R + \log^+ \frac{\not{d}_\phi(t)}{\not{d}_l(t)} + \log^+ \frac{\not{c}_l(t)}{\not{c}_\phi(t+1)} + 1 \right].$$

6 $\frac{\not{d}_\phi}{\not{d}_l}$ nondecreasing and $\hbar(R) < N - 1$.
Just as in the case before, we can apply (3.14) with $m = 1$ and $n = N - 1$ and estimate the last summand of $A_3$ with (3.16), yielding

$$\begin{aligned}
A_3 &\leq \log \frac{a_{N-1}}{a_1} + \sum_{j=1}^{N-2} \log\left(1 + \frac{\not{d}_\phi(j)}{a_j^2}\right) + \log \frac{2}{a_{N-1} a_N} \\
&= \frac{1}{2}\log(R\not{d}_l(1)) + \sum_{j=1}^{\lfloor \not{k}(R) \rfloor} \log(1 + R(\not{d}_l \not{d}_\phi)(j)) \\
&\quad + \sum_{j=\lfloor \not{k}(R)\rfloor+1}^{\lfloor \hbar(R) \rfloor} \log\left(1 + \sqrt{R(\not{d}_l \not{d}_\phi)(j)}\right) + \sum_{j=\lfloor \hbar(R)\rfloor+1}^{N-1} \log\left(1 + \not{d}_\phi(j)\right) \\
&\quad + \left[ \frac{1}{4} \log \frac{\not{c}_l(N)}{\not{c}_\phi(N)} + \log 2 \right].
\end{aligned}$$

Furthermore, in the current instance it holds that

$$\Theta = \sum_{j=1}^{N-1} a_j^2 \not{d}_l(j) = \frac{\lfloor \not{k}(R) \rfloor}{R} + \sum_{j=\lfloor \not{k}(R)\rfloor+1}^{\lfloor \hbar(R) \rfloor} \left( \frac{(\not{d}_l \not{d}_\phi)(j)}{R} \right)^{\frac{1}{2}} + \sum_{\lfloor \hbar(R)\rfloor+1}^{N-1} \not{d}_l(j),$$

and we have

$$\begin{aligned}
R\Theta + A_3 &= \lfloor \hbar(R)\rfloor + \sum_{j=\lfloor \hbar(R)\rfloor+1}^{\lfloor \hbar(R)\rfloor} \sqrt{R(d_l d_\phi)(j)} + R\cdot \sum_{\lfloor \hbar(R)\rfloor+1}^{N-1} d_l(j) + A_3 \\
&\le \lfloor \hbar(R)\rfloor + \sum_{j=1}^{\lfloor \hbar(R)\rfloor} \log(1+R(d_l d_\phi)(j)) \\
&\quad + \sum_{j=\lfloor \hbar(R)\rfloor+1}^{\lfloor \hbar(R)\rfloor} \left(\sqrt{R(d_l d_\phi)(j)} + \log\left(1+\sqrt{R(d_l d_\phi)(j)}\right)\right) \\
&\quad + R\cdot \sum_{\lfloor \hbar(R)\rfloor+1}^{N-1} d_l(j) + \left[\frac{1}{2}\log(Rd_l(1)) + \frac{1}{4}\log\frac{c_l(N)}{c_\phi(N)} + \log 2\right].
\end{aligned}$$

For $j \ge \hbar(R)$, it holds that $f(j) = \frac{d_\phi(j)}{d_l(j)} \ge R$ and hence $Rd_l(j) \ge d_\phi(j)$. Therefore, $R\cdot\sum_{\lfloor \hbar(R)\rfloor+1}^{N-1} d_l(j) \le \sum_{\lfloor \hbar(R)\rfloor+1}^{N-1} d_\phi(j)$ and by estimating the sums by integrals, we see that

$$R\Theta + A_3 \lesssim g(t,R) + \left[\log^+ R + \log\frac{c_l(t)}{c_\phi(t+1)} + 1\right].$$

□

**Definition 3.13.** From now on, we will denote the bound for $\log\|W_H(z)\|$ given by Theorem 3.12 by $B(R)$, i.e.

$$B(R) := \min_{t\ge 2}\left(\max\{g(t,R), Rn(t)\} + L(t,R)\right).$$

**Lemma 3.14.** *Let $(l_j)_{j\in\mathbb{N}}$ be a summable sequence of positive numbers and $(\phi_j)_{j\in\mathbb{N}}$ a sequence of real numbers. Further, let $(d_l, d_\phi, c_l, c_\phi, \psi)$ be a suitable majorisation and define the functions $g(t,R)$ and $n(t)$ as in Definition 3.10, and let a function $T\colon [e(2),\infty) \to [2,\infty)$ be given. Then, it holds that*

$$\min\{g(T(R),R), Rn(T(R))\} \lesssim B(R).$$

*If we further assume that $(\phi_j)_{j\in\mathbb{N}}$ is not eventually constant and that the functions $\hbar$ and $T$ are bounded by some power, meaning that there is $\alpha > 0$ such that $\hbar(R), T(R) \lesssim R^\alpha$, we have $L(T(R),R) = O(\log R)$ and*

$$\min\{g(T(R),R), Rn(T(R))\} \lesssim B(R) \lesssim \max\{g(T(R),R), Rn(T(R))\}.$$

*Proof.* The function $n(t)$ is nonincreasing and converges to 0, meanwhile, for fixed $R$, $g(t,R)$ is nondecreasing. Further, we have $Rn(2) \asymp R$ and $g(2,R) \lesssim R^{\frac{1}{2}}$ and therefore $g(2,R) \ll Rn(2)$, implying that for all sufficiently large $R$, there must be a unique solution of the equation $g(t,R) = Rn(T)$. Denoting it by $T_0(R)$ and keeping the opposing monotonicity of $\mathfrak{g}$ and $\mathfrak{n}$ in mind, we see that

$$\min\{\mathfrak{g}(t,R), R\mathfrak{n}(t)\} \le \mathfrak{g}(T_0(R) = R\mathfrak{n}(T_0(R)) \le \max\{\mathfrak{g}(t,R), R\mathfrak{n}(t)\} \tag{3.18}$$

for all $t \ge 2$. Now, let $T_1(R) \ge 2$ such that

$$B(R) = \max\{\mathfrak{g}(T_1(R), R), R\mathfrak{n}(T_1(R))\} + L(t_1(R), R).$$

As $L(t,R)$ is always positive (in fact, $L(t,R) \ge 1$), we can deduce from (3.18) that

$$\min\{\mathfrak{g}(T(R), R)R, \mathfrak{n}(T(R))\} \le \mathfrak{g}(T_0(R), R) \le \max\{\mathfrak{g}(T_1(R), R), R\mathfrak{n}(T_1(R))\} < B(R),$$

proving the first assertion. For the second assertion, note that as both $T(R)$ and $\mathfrak{k}(R)$ are bounded from below by 2 and bounded from above by some power, we also know that $e(T(R))$ cannot decrease faster than any power, and hence $\frac{1}{\mathfrak{d}_l(T(R))}$ and $\frac{1}{\mathfrak{d}_\phi(T(R))}$ are also bounded by some power. This shows us that

$$\begin{aligned} L(T(R), R) &= 1 + \log^+ R + \log^+ \frac{1}{\mathfrak{d}_l(T(R))} + \log^+ \frac{\mathfrak{c}_l(T(R))}{\mathfrak{c}_\phi(T(R)+1)} + \left|\log \frac{\mathfrak{d}_l(T(R))}{\mathfrak{d}_\phi(T(R))}\right| \\ &\lesssim \log R. \end{aligned}$$

Further, denoting by $H$ the Hamiltonian with lengths $(l_j)_{j\in\mathbb{N}}$ and angles $(\phi_j)_{j\in\mathbb{N}}$, the latter not being eventually constant means that the entire function $W_H(z)$ is not a polynomial, which in turn implies that $\log R \ll \max_{|z|=R} \log \|W_H(z)\|$ and consequently

$$\log R \ll \max\{\mathfrak{g}(T(R), R), R\mathfrak{n}(T(R))\}$$

by (3.6). Altogether, we can conclude

$$\begin{aligned} B(R) &= \min_{t\ge 2} \left(\max\{\mathfrak{g}(t,R), R\mathfrak{n}(t)\} + L(t,R)\right) \\ &\le \max\{\mathfrak{g}(T(R), R), R\mathfrak{n}(T(R))\} + L(T(R), R) \\ &\sim \max\{\mathfrak{g}(T(R), R), R\mathfrak{n}(T(R))\}. \end{aligned}$$

□

This useful corollary can be deduced directly from the second assertion of Lemma 3.14.

**Corollary 3.15.** *Let $(l_j)_{j\in\mathbb{N}}$ be a summable sequence of positive numbers and $(\phi_j)_{j\in\mathbb{N}}$ a sequence of real numbers which is not eventually constant. Further, let $(\mathfrak{d}_l, \mathfrak{d}_\phi, \mathfrak{c}_l, \mathfrak{c}_\phi, \psi)$ be a suitable majorisation and define the functions $\mathfrak{g}(t,R)$ and $\mathfrak{n}(t)$ as in Definition 3.10. Suppose that there is a function $T\colon [e(2), \infty) \to [2, \infty)$ such that both $\mathfrak{k}$ and $T$ are bounded by some power and that*

$$\mathfrak{g}(T(R), R) \asymp R\mathfrak{n}(T(R)),\ R \ge e(2).$$

*Then,*

$$B(R) \asymp \mathfrak{g}(T(R), R) \asymp R\mathfrak{n}(T(R)).$$

# 4 Regularly Varying Functions

As we are looking to study the application of Theorem 3.3 for certain Hamburger Hamiltonians whose behaviour can be characterised using so-called regularly varying functions in Chapter 5, we first need to look into the definition and some basic properties of these functions. Further, we will prove Karamata's Theorem and show that some regularly varying functions can be inverted asymptotically, both of which are key results concerning our work in said chapter. The first two sections of the current chapter, namely those concerning the basic properties of regularly varying functions and Karamata's Theorem, are based on the lecture notes belonging to a lecture given by Harald Woracek in January 2022.[1] For further theory on regularly varying functions, we refer the interested reader to [BGT87].

## 4.1 Basic Properties

**Definition 4.1.** Let $t_0 > 0$ and $f\colon [t_0, \infty) \to (0, \infty)$ be (Borel-) measurable. If there is $\rho > 0$ such that

$$\lim_{t\to\infty} \frac{f(\lambda t)}{f(t)} = \lambda^\rho \tag{4.1}$$

for all $\lambda > 0$, we call $f$ *regularly varying with index* $\rho$. Further, denote by $RV_\rho$ the set of all regularly varying functions with index $\rho$, and if a function $g$ is regularly varying with index 0, we call it *slowly varying*.

**Remark 4.2.** Looking at (4.1), it is easy to see that for any $\rho, \sigma \in \mathbb{R}$ and $f \in RV_\rho$ it holds that $t^\sigma \cdot f(t) \in RV_{\rho+\sigma}$. In particular, $f$ can be written as $t^\rho \cdot \ell(t)$, where $\ell(t) := \frac{f(t)}{t} \in RV_0$.

**Definition 4.3.** Let $\eta_\infty \in \mathbb{R}, t_0 > 0$ and $\eta, \kappa\colon [t_0, \infty) \to (0, \infty)$ bounded and measurable with $\lim_{t\to\infty} \eta(t) = \lim_{t\to\infty} \kappa(t) = 0$. Then we denote $(t \geq t_0)$

$$s[\eta_\infty, \eta, \kappa](t) := \exp\left(\eta_\infty + \eta(t) + \int_{t_0}^t \kappa(s)\frac{\mathrm{d}s}{s}\right).$$

**Lemma 4.4.** *Any function of the form $s[\eta_\infty, \eta, \kappa]$ as described in Definition 4.3 is slowly varying. Further, the limit (4.1) is even attained locally uniformly in $\lambda$.*

*Proof.* Writing $f(t) := s[\eta_\infty, \eta, \kappa](t)$, we see that $f\colon [t_0, \infty) \to (0, \infty)$ is measurable and positive. Further, when focusing on $\lambda \geq 1$, we see that

$$\frac{f(\lambda t)}{f(t)} = \exp\left(\eta(\lambda t) - \eta(t) + \int_t^{\lambda t} \kappa(t)\frac{\mathrm{d}s}{s}\right)$$

[1] The notes are available here: https://www.asc.tuwien.ac.at/functionalanalysis/?id=seminar.

for $t \geq t_0$, and it holds that

$$\left|\eta(\lambda t) - \eta(t) + \int_t^{\lambda t} \kappa(s) \frac{\mathrm{d}s}{s}\right| \leq \sup_{s \geq t} \eta(s) + \log(\lambda) \sup_{s \geq t} \kappa(s).$$

For $t \to \infty$, the right side is converging to 0 uniformly for $\lambda$ in any interval of the form $[1, \lambda_0]$. Hence, the same holds for the convergence of $\frac{f(\lambda t)}{f(t)}$ to 1. Given an interval of the form $[\frac{1}{\lambda_0}, 1]$ for some $\lambda_0 \geq 1$, applying our observations from above with $\frac{1}{\lambda}$ instead of $\lambda$ once more shows uniform convergence, but this time for $\lambda \in [\frac{1}{\lambda_0}, 1]$. □

**Example 4.5.** Let $t_0 > 0$ and $f\colon [t_0, \infty) \to (0, \infty)$ be continuously differentiable such that

$$\lim_{t \to \infty} \frac{t f'(t)}{f(t)} = 0. \tag{4.2}$$

Then $f$ is slowly varying, which can be seen in the following way. At first, notice that it holds for all positive, continuously differentiable functions that

$$\int \frac{g'(s)}{g(s)} \,\mathrm{d}s = \log(f(s)) + C.$$

This lets us write

$$f(t) = \exp\left(\log(f(t_0)) + \int_{t_0}^{t} \frac{s f'(s)}{f(s)} \frac{\mathrm{d}s}{s}\right),$$

meaning just that $f(t) = s\left[\log(f(t_0)), 0, \frac{s f'(s)}{f(s)}\right](t)$.

**Example 4.6.** For $n \in \mathbb{N}_0$ and sufficiently large $t$, define $\log^{[n]}(t)$ recursively by

$$\log^{[0]}(t) := t, \quad \log^{[n+1]}(t) := \log \circ \log^{[n]}(t).$$

This lets us define the so called *Lindelöf comparison functions*, which are functions of the form

$$f(t) = \prod_{i=0}^{N} \left(\log^{[i]}(t)\right)^{\alpha_i} \tag{4.3}$$

for some $N \in \mathbb{N}_0$ and parameters $a_i \in \mathbb{R}$. Any such function is regularly varying with index $\alpha_0$, which can be shown by utilising Example 4.5 in the following way. At first, note that we can write $f(t) = t^{\alpha_0} \cdot \prod_{i=1}^{N} \left(\log^{[i]}(t)\right)^{\alpha_i}$, with Remark 4.2 making it obvious that $f(t) \in RV_{\alpha_0}$ is equivalent to $\tilde{f}(t) = \prod_{i=1}^{N} \left(\log^{[i]}(t)\right)^{\alpha_i} \in RV_0$. We will now proceed to show the latter.
Clearly, we have $\tilde{f}(t) \in C^{\infty}$, and it further holds for all $n \in \mathbb{N}_0$ that

$$\frac{\mathrm{d}}{\mathrm{d}t} \log^{[n]}(t) = \left(\prod_{j=0}^{n-1} \log^{[j]}(t)\right)^{-1},$$

which can be proven by induction.

Base case: $n = 0$

$$\frac{\mathrm{d}}{\mathrm{d}t}\log^{[0]}(t) = 1 = \left(\prod_{j=0}^{-1}\log^{[j]}(t)\right)^{-1}.$$

Induction step: $n \mapsto n+1$

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\log^{[n+1]}(t) &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\log^{[n]}\circ\log\right)(t) = \left(\prod_{j=0}^{n-1}\log^{[j]}\right)^{-1}\circ\log(t)\cdot\frac{1}{t}\\ &= \left(\prod_{j=0}^{n}\log^{[j]}(t)\right)^{-1}.\end{aligned}$$

Using this fact, it is quite straightforward to show that $\tilde{f}(t)$ satisfies (4.2). We calculate

$$\begin{aligned}\frac{t\tilde{f}'(t)}{\tilde{f}(t)} &= t\frac{\mathrm{d}}{\mathrm{d}t}\left(\log\tilde{f}(t)\right) = t\sum_{i=1}^{N}\frac{\mathrm{d}}{\mathrm{d}t}\left(\alpha_i\log^{[i+1]}(t)\right)\\ &= t\sum_{i=1}^{N}\alpha_i\left(\prod_{j=0}^{i}\log^{[j]}(t)\right)^{-1} = \sum_{i=1}^{N}\alpha_i\left(\prod_{j=1}^{i}\log^{[j]}(t)\right)^{-1}\xrightarrow[t\to\infty]{} 0.\end{aligned}$$

**Example 4.7.** Let $t_0 > 0, \rho \in \mathbb{R}$ and $f\colon [t_0,\infty) \to (0,\infty)$. Then we call $f$ *smoothly varying* at $\infty$ *with index* $\rho$, if $f \in \mathbb{C}^\infty$ and $h(t) := \log f\left(e^t\right)$ satisfies

$$\lim_{t\to\infty} h'(t) = \rho, \quad \lim_{t\to\infty} h^{(n)}(t) = 0,\ n \geq 2. \tag{4.4}$$

Denote by $SV_\rho$ the set of all smoothly varying functions with index $\rho$. As we have $h'(t) = \frac{1}{f(e^t)}\cdot f'(e^t)\cdot e^t$, we can deduce that also $\lim_{t\to\infty}\frac{tf'(t)}{f(t)} = \rho$. Defining $\ell(t) := t^{-\rho}f(t)$, we see that $\ell(t) \in \mathbb{C}^\infty$, $\ell'(t) = (-\rho)t^{-\rho-1}f(t) + t^{-\rho}f'(t)$ and hence

$$\lim_{t\to\infty}\frac{t\ell'(t)}{\ell(t)} = \lim_{t\to\infty}\frac{(-\rho)t^{-\rho}f(t) + t^{1-\rho}f'(t)}{t^{-\rho}f(t)} = -\rho + \lim_{t\to\infty}\frac{tf'(t)}{f(t)} = 0,$$

making $\ell(t)$ slowly varying by Example 4.5, which in turn means that $f(t)$ is regularly varying with index $\rho$. This shows that in fact $SV_\rho \subseteq RV_\rho$.

**Theorem 4.8** (Uniform convergence theorem)**.** *Let $t_0 > 0$ and $f\colon [t_0,\infty) \to (0,\infty)$ be slowly varying. Then*

$$\lim_{t\to\infty}\frac{f(\lambda t)}{f(t)} = 1$$

*locally uniformly for $\lambda \in (0,\infty)$.*

*Proof.* In this scenario, locally uniform convergence means that for any $\lambda \in (0,\infty)$, there is a neighbourhood $U_\lambda$ on which $\sup_{\xi\in U_\lambda}\left|\frac{f(\xi t)}{f(t)}-1\right|$ converges to 0 for $t\to\infty$. Clearly, the existence of such a neighbourhood $U_\lambda$ is equivalent to the existence of a compact neighbourhood $K_\lambda$ with the same properties, and $K_\lambda$ can always be chosen to be a compact interval. Further, as uniformity on a finite number of sets implies uniformity on their union, it is once more equivalent to require uniformity on all intervals of the form $\left[e^{-T},e^{T}\right]$, $T>0$. Finally, the net $\sup_{\xi\in[e^{-T},e^{T}]}\left|\frac{f(\xi t)}{f(t)}-1\right|$ converging to 0 for $t\to\infty$ is equivalent to the convergence of $\frac{f(\lambda_n t_n)}{f(t_n)}$ to 1 for all sequences $(\lambda_n)_{n\in\mathbb{N}}\in\left[e^{-T},e^{T}\right]^{\mathbb{N}}$ and all sequences $(t_n)_{n\in\mathbb{N}}$ satisfying $\lim_{n\to\infty}t_n=\infty$. Summing up, we have found an equivalent formulation of the locally uniform convergence in question, namely

$$\forall T>0, \forall(\lambda_n)_{n\in\mathbb{N}}\in\left[e^{-T},e^{T}\right]^{\mathbb{N}}, \forall(t_n)_{n\in\mathbb{N}}\in[t_0,\infty)^{\mathbb{N}}, \lim_{n\to\infty}t_n=\infty\colon \lim_{n\to\infty}\frac{f(\lambda_n t_n)}{f(t_n)}=1.$$

To prove this, let $T>0, (\lambda_n)_{n\in\mathbb{N}}\in\left[e^{-T},e^{T}\right]^{\mathbb{N}}$ and $(t_n)_{n\in\mathbb{N}}\in[t_0,\infty)^{\mathbb{N}}, \lim_{n\to\infty}t_n=\infty$. Note that we can also write $\lambda_n=e^{\omega_n}$ for $\omega_n:=\log\lambda_n\in[-T,T]$. Now, define the following functions for all $n\in\mathbb{N}$.

$$h_n\colon\begin{cases}[-T,T]\to(0,\infty)\\ y\mapsto\frac{f(e^y t_n)}{f(t_n)}\end{cases},\quad k_n\colon\begin{cases}[-T,T]\to(0,\infty)\\ y\mapsto\frac{f(e^y\lambda_n t_n)}{f(\lambda_n t_n)}\end{cases}.$$

As $f$ is slowly varying, we know that both $(h_n)_n\in\mathbb{N}$ and $(k_n)_n\in\mathbb{N}$ are converging pointwise to 1. Applying Egorov's Theorem (Theorem 7.82 in [Kus14]) now gives us almost uniform convergence, meaning that for any $\varepsilon>0$ there is $A_{h,\varepsilon}\subseteq[-T,T]$ such that $\lambda\left([-T,T]\setminus A_{h,\varepsilon}\right)\le\varepsilon$ and $h_n|_{A_{h,\varepsilon}}\xrightarrow[n\to\infty]{}1$ uniformly, and the same holds for $k_n$. Define $E:=A_{h,\frac{2T}{8}}$ as well as $F:=A_{k,\frac{2T}{8}}$.

Our next goal is to show that for all $n\in\mathbb{N}$, we have $E\cap(\omega_n+F)\neq\emptyset$. In order to do so, note that

$$[-T,T]\cup(\omega_n+[-T,T])\subseteq\begin{cases}[-T+\omega_n,T] & \textit{if } \omega_n\le 0,\\ [-T,T+\omega_n] & \textit{if } \omega_n\ge 0.\end{cases}$$

Hence, it holds that

$$\begin{aligned}\lambda(E\cup(\omega_n+F))\le&\lambda\left([-T,T]\cup(\omega_n+[-T,T])\right)\\ \le&2T+|\omega_n|\le 3T<2\cdot\frac{7}{8}2T\\ \le&\lambda(E)+\lambda(F)=\lambda(E)+\lambda(\omega_n+F),\end{aligned}$$

implying

$$\lambda(E\cap(\omega_n+F))=\lambda(E)+\lambda(\omega_n+F)-\lambda(E\cup(\omega_n+F))>0.$$

What this means is that for all $n\in\mathbb{N}$ there is $y_n\in E$ and $z_n\in F$ such that $y_n=z_n+\omega_n$, showing us that

$$\frac{f(\lambda_n t_n)}{f(t_n)} = \frac{f(\lambda_n t_n)}{f(e^{z_n}\lambda_n t_n)} \cdot \frac{f(e^{y_n} t_n)}{f(t_n)} = (k_n(z_n))^{-1} h_n(y_n) \xrightarrow[n\to\infty]{} 1.$$

As $(\lambda_n)_{n\in\mathbb{N}}$ and $(t_n)_{n\in\mathbb{N}}$ were chosen arbitrarily, the assertion follows. □

**Corollary 4.9.** *Let $t_0 > 0$ and $f\colon [t_0,\infty) \to (0,\infty)$ be slowly varying. Then there exists $T \geq t_0$ such that for all $\alpha \geq T$ it holds that*

$$\inf_{t\in[T,\alpha]} f(t) > 0, \qquad \sup_{t\in[T,\alpha]} f(t) < \infty. \tag{4.5}$$

*Proof.* By Theorem 4.8, we can choose $T$ such that for all $t \geq T$, $\lambda \in [1,2]$ it holds that

$$\frac{1}{2} \leq \frac{f(\lambda t)}{f(t)} \leq 2.$$

Now let $n \in \mathbb{N}$ as well as $t \in [T, 2^n T]$, and choose $m \in \mathbb{N}$ such that $t \in [2^m T, 2^{m+1} T]$. Writing

$$f(t) = \frac{f(t)}{f\left(\frac{t}{2}\right)} \cdot \frac{f\left(\frac{t}{2}\right)}{f\left(\frac{t}{2^2}\right)} \cdot \frac{f\left(\frac{t}{2^2}\right)}{f\left(\frac{t}{2^3}\right)} \cdot \ldots \cdot \frac{f\left(\frac{t}{2^{m-1}}\right)}{f\left(\frac{t}{2^m}\right)} \frac{f\left(\frac{t}{2^m}\right)}{f(T)} \cdot f(T),$$

we see that

$$\frac{1}{2^{m+1}} f(T) \leq f(t) \leq 2^{m+1} f(T).$$

In particular, it holds for all $s \in [T, 2^n T]$ that

$$\frac{1}{2^n} f(T) \leq f(s) \leq 2^n f(T).$$

As $n \in \mathbb{N}$ was chosen arbitrarily, the assertion is shown. □

**Theorem 4.10** (Representation Theorem)**.** *Let $t_0 > 0$ and $f\colon [t_0,\infty) \to (0,\infty)$ be slowly varying. Then there are $\eta_\infty \in \mathbb{R}$ and functions $\eta, \kappa$ as described in Definition 4.3 and there is $T \geq t_0$ such that $f(t) = s[\eta_\infty, \eta, \kappa](t)$, $t \geq T$.*

*Proof.* Choose $T \geq t_0$ as described in Corollary 4.9. Due to the local boundedness of $f$ and the fact that $f$ stays away from 0 on compact intervals, both described in said corollary, we know that $\log f(t)|_{[T,\infty)}$ is both measurable and locally bounded, and hence locally integrable. For the following calculation, it is important to note that for any $\alpha > 0$ and a function $g(u)$ such that the integral $\int_{\alpha a}^{\alpha b} g(u) \frac{du}{u}$ is well-defined, the substitution $s := \frac{u}{\alpha}$ yields

$$\int_{\alpha a}^{\alpha b} g(u) \frac{\mathrm{d}u}{u} = \int_a^b g(s) \alpha \frac{\mathrm{d}s}{\alpha s} = \int_a^b g(s) \frac{\mathrm{d}s}{s}.$$

This lets us write ($t \geq T$)

$$\begin{aligned}\log f(t) &= \int_t^{et} (\log f(t) - \log f(u)) \frac{\mathrm{d}u}{u} \\ &\quad + \int_T^{eT} \log f(u) \frac{\mathrm{d}u}{u} + \left( \int_{eT}^{et} \log f(u) \frac{\mathrm{d}u}{u} - \int_T^{t} \log f(u) \frac{\mathrm{d}u}{u} \right) \\ &= \underbrace{\int_T^{eT} \log f(u) \frac{\mathrm{d}u}{u}}_{=:\eta_\infty} + \underbrace{\int_t^{et} \log \frac{f(t)}{f(u)} \frac{\mathrm{d}u}{u}}_{=:\eta(t)} + \int_T^t \underbrace{\log \frac{f(eu)}{f(u)}}_{=:\kappa(u)} \frac{\mathrm{d}u}{u}.\end{aligned}$$

To show that $\eta$ and $\kappa$ meet the requirements in Definition 4.3, first note that as $\frac{f(et)}{f(t)}$ is positive and tends to 1, $\kappa(t) = \log \frac{f(et)}{f(t)}$ is measurable, bounded and converges to 0. When looking at $\eta(t)$, writing it as $\eta(t) = \log f(t) - \int_t^{et} \log f(u) \frac{\mathrm{d}u}{u}$ along with Corollary 4.9 shows us that $\eta(t)$ is both measurable and locally bounded on $[T, \infty)$. Beyond that, as $\log\left(\frac{f(t)}{f(u)}\right) \xrightarrow[t\to\infty]{} 0$ uniformly for $u \in [t, et]$ as well as $\int_t^{et} \frac{\mathrm{d}u}{u} = 1$ for all $t > 0$, we know that $\eta(t) \xrightarrow[t\to\infty]{} 0$.
Thus, both $\eta$ and $\kappa$ and the real number $\eta_\infty$ are feasible and we can write $f(t) = s[\eta_\infty, \eta, \kappa](t)$, $t \geq T$. □

**Remark 4.11.** As we can see in the proof of the Representation Theorem, the constructed function $s[\eta_\infty, \eta, \kappa](t)$ is of the form

$$s[\eta_\infty, \eta, \kappa](t) = \exp(\eta(t)) \cdot \exp\left( \eta_\infty + \int_T^t \log \frac{f(eu)}{f(u)} \frac{\mathrm{d}u}{u} \right).$$

As $\log f|_{[T,\infty)}(s)$ is locally integrable, so is $\kappa(s) = \log f(es) - \log f(s)$, making $K(t) := \int_T^t \kappa(u) \frac{\mathrm{d}u}{u}$ and thereby $\exp(\eta_\infty + K(t)) = s[\eta_\infty, 0, \kappa](t)$ continuous functions. Moreover, in case of a continuous initial function $f$, $K(t)$ is even continuously differentiable, and the same holds for $\exp(\eta_\infty + K(t)) = s[\eta_\infty, 0, \kappa](t)$.

**Theorem 4.12** (Potter bounds)**.** *Let $t_0 > 0$ and $f \colon [t_0, \infty) \to (0, \infty)$ be slowly varying. Then,*

(i) $\forall \delta > 0 \, \forall A > 1 \, \exists T \geq t_0 \, \forall s, t \geq T$:

$$\frac{f(s)}{f(t)} \leq A \max\left\{ \left(\frac{s}{t}\right)^\delta, \left(\frac{t}{s}\right)^\delta \right\}. \tag{4.6}$$

(ii) *If we further assume that*

$$\inf_{t\in[t_0,\alpha]} f(t) > 0, \qquad \sup_{t\in[t_0,\alpha]} f(t) < \infty,$$

*for all $\alpha \geq t_0$, it even holds that* $\forall \delta > 0\, \exists A > 1\, \forall s, t \geq t_0$:

$$\frac{f(s)}{f(t)} \leq A \max\left\{\left(\frac{s}{t}\right)^{\delta}, \left(\frac{t}{s}\right)^{\delta}\right\}. \tag{4.7}$$

*Proof.*

(i) Let $\delta > 0$ and $A > 1$ and write $f$ as $f(t) = s[\eta_\infty, \eta, \kappa](t)$. Further, choose $T \geq t_0$ such that it holds for all $t \geq T$ that $|\eta(t)| \leq \frac{1}{2}\log A$ as well as $|\kappa(t)| \leq \delta$ and let $s, t \geq T$. Let us suppose for the moment that $s \geq t$. This gives

$$\frac{f(s)}{f(t)} = \exp\left(\eta(s) - \eta(t) + \int_t^s \kappa(r)\frac{\mathrm{d}r}{r}\right) \leq A\exp\left(\delta(\log s - \log t)\right) = A\left(\frac{s}{t}\right)^{\delta}.$$

Conversely, when assuming that $t > s$, a very similar calculation shows that

$$\frac{f(s)}{f(t)} \leq A\left(\frac{t}{s}\right)^{\delta}.$$

(ii) Let again $\delta > 0$ and set $B$ to any number greater than 1. By (i), there is $T$ such that 4.6 holds with $B$ in place of $A$. Furthermore, we define $C := \max\left\{1, \max_{t_0 \leq s,t \leq T} \frac{f(s)}{f(t)}\right\}$ and set $A = BC$. Now let $s, t \geq t_0$. Once more, we begin under the assumption that $s \geq t$. For the following calculation, we need to distinguish three cases: $t \leq s \leq T$, $t \leq T < s$ and $T < t \leq s$. In the first instance, we obtain

$$\frac{f(s)}{f(t)} \leq C < A \leq A\left(\frac{s}{t}\right)^{\delta},$$

in the second

$$\frac{f(s)}{f(t)} = \frac{f(s)}{f(T)} \cdot \frac{f(T)}{f(t)} \leq B\left(\frac{s}{T}\right)^{\delta} \cdot C \leq A\left(\frac{s}{t}\right)^{\delta},$$

and in the third

$$\frac{f(s)}{f(t)} \leq B\left(\frac{s}{t}\right)^{\delta} \leq A\left(\frac{s}{t}\right)^{\delta}.$$

Just like before, the case that $s < t$ can be handled in a very similar way, always yielding $\frac{f(s)}{f(t)} \leq A\left(\frac{t}{s}\right)^{\delta}$, and ergo, the assertion holds.

□

**Corollary 4.13.** *Let $t_0 > 0, \rho \in \mathbb{R}$ and $f\colon [t_0, \infty) \to (0, \infty)$ be regularly varying with index $\rho$. It holds for all $\varepsilon > 0$ that*

$$t^{\rho-\varepsilon} \ll f(t) \ll t^{\rho+\varepsilon}.$$

*Proof.* The function $\ell(t) = t^{-\rho} f(t)$ is slowly varying, meaning that when choosing $\delta := \frac{\varepsilon}{2}$ and $A > 1$ arbitrary, there is $T \geq t_0$ such that (4.6) holds. Consequently, it holds for all

$t \geq T$ that

$$\frac{f(t)}{f(T)} = \left(\frac{t}{T}\right)^{\rho} \frac{\ell(t)}{\ell(T)} \leq \left(\frac{t}{T}\right)^{\rho} \cdot A \left(\frac{t}{T}\right)^{\frac{\varepsilon}{2}} \ll t^{\rho+\varepsilon}$$

and

$$\frac{f(T)}{f(t)} = \left(\frac{T}{t}\right)^{\rho} \frac{\ell(T)}{\ell(t)} \leq \left(\frac{T}{t}\right)^{\rho} \cdot A \left(\frac{T}{t}\right)^{-\frac{\varepsilon}{2}} \ll t^{-(\rho+\varepsilon)}.$$

□

**Proposition 4.14.** *Let $t_0 > 0, \rho \in \mathbb{R}$ and $f\colon [t_0, \infty) \to (0, \infty)$ be slowly varying. Then there exists $T \geq t_0$ and a slowly varying function $g\colon [T, \infty) \to (0, \infty)$ with the Properties described in Example 4.5, i.e. $g$ is continuously differentiable and $\lim_{t\to\infty} \frac{tg'(t)}{g(t)} = 0$, such that $g \sim f$.*

*Proof.* By the Representation Theorem, there is $T_1 \geq t_0$ such that $f$ can be written as $f(t) = s[\eta_\infty, \eta, \kappa](t)$ for $t \geq T_1$. Considering the function $\tilde{g}(t) = s[\eta_\infty, 0, \kappa](t)$, it holds that $\tilde{g}$ is slowly varying and, as $\frac{f(t)}{\tilde{g}(t)} = \exp(\eta(t)) \xrightarrow[t\to\infty]{} 1$, also $f \sim \tilde{g}$. Beyond that, $\tilde{g}$ is continuous, as was described in Remark 4.11.
With $\tilde{g}$ being a slowly varying function, there is again $T \geq T_1$ such that we can once more write $\tilde{g}$ in the form $\tilde{g}(t) = s[\tilde{\eta}_\infty, \tilde{\eta}, \tilde{\kappa}](t)$ for $t \geq T$, where $\tilde{\eta}_\infty, \tilde{\eta}$ and $\tilde{\kappa}$ are constructed just like in the proof of the Representation Theorem. If we then consider the function $g(t) \coloneqq s[\tilde{\eta}_\infty, 0, \tilde{\kappa}](t)$, we see just as above that $g \sim \tilde{g}$ and therefore $g \sim s$. Moreover, as $\tilde{g}$ is continuous, Remark 4.11 tells us that $g$ is continuously differentiable.
The only thing left to prove is that $\lim_{t\to\infty} \frac{tg'(t)}{g(t)} = 0$. In order to do so, observe that

$$\frac{tg'(t)}{g(t)} = \frac{t}{g(t)} \cdot \frac{\mathrm{d}}{\mathrm{d}t} \exp\left(\eta_\infty + \int_T^t \tilde{\kappa}(u) \frac{\mathrm{d}u}{u}\right) = \frac{t}{g(t)} \cdot \left(g(t) \cdot \frac{\tilde{\kappa}(t)}{t}\right) = \tilde{\kappa}(t) \xrightarrow[t\to\infty]{} 0.$$

□

## 4.2 Karamata's Theorem

**Lemma 4.15.** *Let $t_0 \in \mathbb{R}$ and $f, g\colon [t_0, \infty) \to (0, \infty)$ be locally integrable as well as $f \sim g$. Then the following statements hold:*

(i) *If $\int_{t_0}^\infty f(s)\,\mathrm{d}s < \infty$, then*

$$\int_t^\infty f(s)\,\mathrm{d}s \sim \int_t^\infty g(s)\,\mathrm{d}s.$$

(ii) *If $\int_{t_0}^\infty f(s)\,\mathrm{d}s = \infty$, then*

$$\int_{t_0}^t f(s)\,\mathrm{d}s \sim \int_{t_0}^t g(s)\,\mathrm{d}s.$$

*Proof.*

(i) The fact that $f \sim g$ lets us deduce that there is $T_1 \geq t$ such that $g(t) \leq 2f(t)$ for all $t \geq T_1$. Therefore, by the direct comparison test, we obtain $\int_{T_1}^{\infty} g(s)\,\mathrm{d}s < \infty$, and as $g$ is locally integrable, we also have $\int_{t_0}^{\infty} g(s)\,\mathrm{d}s < \infty$. Going further, let $\varepsilon > 0$, then there is $T \geq t_0$ such that $\left|\frac{f(t)}{g(t)} - 1\right| < \varepsilon$ for all $t \geq T$. From this, we can conclude that

$$(1-\varepsilon)\int_t^r g(s)\,\mathrm{d}s \leq \int_t^r f(s)\,\mathrm{d}s \leq (1+\varepsilon)\int_t^r g(s)\,\mathrm{d}s$$

for all $t, r \geq T$. A passage to the limit for $r \to \infty$ then also shows that

$$(1-\varepsilon)\int_t^\infty g(s)\,\mathrm{d}s \leq \int_t^\infty f(s)\,\mathrm{d}s \leq (1+\varepsilon)\int_t^\infty g(s)\,\mathrm{d}s$$

for all $t \geq T$ and hence, since $\varepsilon$ was chosen arbitrarily, $\int_t^\infty f(s)\,\mathrm{d}s \sim \int_t^\infty g(s)\,\mathrm{d}s$, $t \in [t_0, \infty)$.

(ii) Similarly to before, a direct comparison test shows that also $\int_T^\infty g(s)\,\mathrm{d}s = \infty$. Fix $\varepsilon > 0$ and this time $T_1 \geq t_0$ such that $\left|\frac{f(t)}{g(t)} - 1\right| < \frac{\varepsilon}{2}$ for all $t \geq T_1$. Further, let $T \geq T_1$ such that

$$\frac{\int_{t_0}^{T_1} f(s)\,\mathrm{d}s}{\int_{t_0}^{T} g(s)\,\mathrm{d}s} < \frac{\varepsilon}{2} \text{ and } \frac{\int_{t_0}^{T_1} g(s)\,\mathrm{d}s}{\int_{t_0}^{T} f(s)\,\mathrm{d}s} < \frac{\varepsilon}{2}.$$

This leads to

$$\frac{\int_{t_0}^{t} f(s)\,\mathrm{d}s}{\int_{t_0}^{t} g(s)\,\mathrm{d}s} = \frac{\int_{t_0}^{T_1} f(s)\,\mathrm{d}s + \int_{T_1}^{t} f(s)\,\mathrm{d}s}{\int_{t_0}^{t} g(s)\,\mathrm{d}s} < \frac{\varepsilon}{2} + 1 + \frac{\varepsilon}{2} = 1 + \varepsilon$$

as well as

$$\frac{\int_{t_0}^{t} g(s)\,\mathrm{d}s}{\int_{t_0}^{t} f(s)\,\mathrm{d}s} < \frac{\varepsilon}{2} + 1 + \frac{\varepsilon}{2} = 1 + \varepsilon$$

for all $t \geq T$. As $\varepsilon$ was chosen arbitrarily, the proof is complete.

□

Using this Lemma, we are now able to prove one of the key results of this chapter.

**Theorem 4.16** (Karamata's Theorem)**.** *Let $t_0 > 0, f\colon [t_0, \infty) \to (0, \infty)$ be slowly varying and $\rho \in \mathbb{R} \setminus \{0\}$.*

(i) *If $\rho < -1$, it holds that $\int_{t_0}^{\infty} s^\rho f(s)\,\mathrm{d}s < \infty$ and*

$$\int_t^\infty s^\rho f(s)\,\mathrm{d}s \sim -\frac{t^{\rho+1}}{\rho+1} f(t).$$

(ii) *If $\rho > -1$, it holds that $\int_{t_0}^{\infty} s^\rho f(s)\,\mathrm{d}s = \infty$ and*

$$\int_{t_0}^{t} s^\rho f(s)\,\mathrm{d}s \sim \frac{t^{\rho+1}}{\rho+1} f(t).$$

*Proof.* Proposition 4.14 in combination with Lemma 4.15 tells us that without loss of generality, $f$ can be assumed to be continuously differentiable and satisfying $\lim_{t\to\infty} \frac{tf'(t)}{f(t)} = 0$.

(i) In this occurrence, the application of Corollary 4.13 tells us that $t^{\rho+1} f(t) \ll t^0$ and therefore $\lim_{t\to\infty} t^{\rho+1} f(t) = 0$. Moreover, as $\rho < -1$, we can also deduce from the corollary that $\int_{t_0}^{\infty} s^\rho f(s)\,\mathrm{d}s \le \int_{t_0}^{\infty} s^{\rho+\frac{1+\rho}{2}}\,\mathrm{d}s < \infty$. Partial integration then yields

$$\begin{aligned}\int_t^\infty s^\rho f(s)\,\mathrm{d}s &= \frac{s^{\rho+1}}{\rho+1} f(s)\Big|_t^\infty - \int_t^\infty \frac{s^{\rho+1}}{\rho+1} f'(s)\,\mathrm{d}s \\ &= -\frac{t^{\rho+1}}{\rho+1} f(t) - \int_t^\infty \frac{s^\rho}{\rho+1} f(s)\cdot \frac{sf'(s)}{f(s)}\,\mathrm{d}s.\end{aligned}$$

As $\frac{sf'(s)}{f(s)}$ converges to 0, this shows that

$$\int_t^\infty s^\rho f(s)\,\mathrm{d}s \sim \int_t^\infty s^\rho f(s)\left(1 + \frac{sf'(s)}{(\rho+1)f(s)}\right)\mathrm{d}s = -\frac{t^{\rho+1}}{\rho+1} f(t).$$

(ii) We apply Corollary 4.13 once again, this time with the result that $t^\rho f(t) \gg t^{-1}$ and consequently $\int_{t_0}^{\infty} s^\rho f(s) = \infty$. Just like in the first instance, we use partial integration, resulting in

$$\begin{aligned}\int_{t_0}^t s^\rho f(s)\,\mathrm{d}s &= \frac{s^{\rho+1}}{\rho+1} f(s)\Big|_{t_0}^t - \int_{t_0}^t \frac{s^{\rho+1}}{\rho+1} f'(s)\,\mathrm{d}s \\ &= \frac{t^{\rho+1}}{\rho+1} f(t) - \frac{t_0^{\rho+1}}{\rho+1} f(t_0) - \int_{t_0}^t \frac{s^\rho}{\rho+1} f(s)\cdot\frac{sf'(s)}{f(s)}\,\mathrm{d}s.\end{aligned}$$

The fact that $\frac{sf'(s)}{f(s)}$ tends to 0 can be utilised once again and we see that

$$\int_{t_0}^t s^\rho f(s)\,\mathrm{d}s \sim \int_{t_0}^t s^\rho f(s)\left(1 + \frac{sf'(s)}{(\rho+1)f(s)}\right)\mathrm{d}s = \frac{t^{\rho+1}}{\rho+1} f(t) - \frac{t_0^{\rho+1}}{\rho+1} f(t_0) \sim \frac{t^{\rho+1}}{\rho+1} f(t).$$

□

## 4.3 Inverting certain regularly varying functions

**Definition 4.17.** Let $t_0 > 0$ and define the relation $\Xi \subseteq (0,\infty)^{[t_0,\infty)} \times (0,\infty)^{[t_0,\infty)}$ by

$$(f,g) \in \Xi :\Leftrightarrow f(tg(t)) \sim f(t).$$

**Proposition 4.18.** *The relation $\Xi$ satisfies*

(i) $(f_1, g), (f_2, g) \in \Xi \Rightarrow (f_1 f_2, g) \in \Xi$,

(ii) $(f, g) \in \Xi, r \in \mathbb{R} \Rightarrow (f^r, g) \in \Xi$,

(iii) $(f, g) \in \Xi,\ \lim_{t\to\infty} f(t) = \infty, \rho \in \mathbb{R}, \tau \in RV_\rho \Rightarrow (\tau \circ f, g) \in \Xi$,

(iv) $f \in RV_0 \Rightarrow (\log t, f) \in \Xi$,

(v) $(f, g) \in \Xi, r \in \mathbb{R}^+ \Rightarrow (f \circ t^r, t^{r-1}(g \circ t^r)) \in \Xi$.

*Proof.*

(i) As $f_1(t) = f_1(tg(t)) \cdot (1 + o(1))$ and $f_2(t) = f_2(tg(t)) \cdot (1 + o(1))$, we have

$$(f_1 f_2)(t) = f_1(tg(t))(1 + o(1)) \cdot f_2(tg(t))(1 + o(1)) = (f_1 f_2)(tg(t)) \cdot (1 + o(1)).$$

(ii) Again, writing $f(t) = f(tg(t))\cdot(1+o(1))$ shows us that $f^r(t) = f^r(tg(t))\cdot(1+o(1))^r = f^r(tg(t)) \cdot (1 + o(1))$.

(iii) By Remark 4.2, $\tau$ can be written as $\tau(t) = t^\rho \cdot \ell(t)$, where $\ell \in RV_0$. Thus, we get

$$\frac{\tau(f(tg(t)))}{\tau(f(t))} = \frac{(f(tg(t)))^\rho}{(f(t))^\rho} \cdot \frac{\ell(f(tg(t)))}{\ell(f(t))}.$$

The first factor of the right side converges to 1 as $t \to \infty$. Focusing on the second factor, it holds that as $\lim_{t\to\infty} f(t) = \infty$ and $(f, g) \in \Xi$, also $\lim_{t\to\infty} f(tg(t)) = \infty$. Therefore, utilising Potter's Theorem tells us that for any $A > 1, \delta > 0$ there is $r_\delta$ such that

$$\frac{\ell(f(tg(t)))}{\ell(f(t))}, \frac{\ell(f(t))}{\ell(f(tg(t)))} \leq A \max\left\{\frac{f(tg(t))}{f(t)}, \frac{f(t)}{f(tg(t))}\right\}^\delta$$

for all $t > r_\delta$. As $f(tg(t)) \sim f(t)$, choosing $\delta = 1$ shows us that $\lim_{t\to\infty} \frac{\ell(f(tg(t)))}{\ell(f(t))} = 1$.

(iv) We start by noting the fact that

$$\frac{\log(tf(t))}{\log t} = 1 + \frac{\log(f(t))}{\log t}.$$

Additionally, the Representation Theorem tells us that $a$ can be written in the form

$$f(t) = \exp\left(\eta_\infty + \eta(t) + \int_T^t \kappa(s)\frac{\mathrm{d}s}{s}\right),\ t \geq T,$$

for some $T \geq t_0$, where $\eta_\infty \in \mathbb{R}$ and $\eta$ as well as $\kappa$ are bounded measurable functions converging to zero. Thereby,

$$\frac{\log(f(t))}{\log t} = \lim \frac{\eta_\infty + \eta(t) + \int_T^t \kappa(s)\frac{\mathrm{d}s}{s}}{\log t} = 0.$$

(v) As $r \in \mathbb{R}^+$, $t \mapsto t^r$ is an increasing bijection of $\mathbb{R}^+$ onto $\mathbb{R}^+$. Thus, $(f, g) \in \Xi$ implies

$$\lim_{t\to\infty} \frac{f(t \cdot t^{r-1} g(t^r))}{f(t^r)} = \lim_{t\to\infty} \frac{f(t^r g(t^r))}{f(t^r)} = \lim_{t\to\infty} \frac{f(tg(t))}{f(t)} = 1.$$

□

**Proposition 4.19.** *Let $t_0 > 0$ and $f \colon [t_0, \infty) \to (0, \infty)$ be regularly varying with index $\rho \in \mathbb{R}$. Then $f \circ \log \in RV_0$.*

*Proof.* $\log t$ is an increasing bijection of $[1, \infty)$ onto $[0, \infty)$, and as $t \to \infty$, $\frac{\log\lambda + \log t}{\log t}$ converges to 1. By Remark 4.2, $f$ can be written in the form $f(t) = t^\rho \ell(t)$, where $\ell(t) \in RV_0$. This leads to

$$\lim_{t\to\infty} \frac{f(\log(\lambda t))}{f(\log(t))} = \lim_{t\to\infty} \frac{f(\log\lambda + \log t)}{f(\log t)} = \lim_{t\to\infty} \frac{(\log\lambda + \log t)^\rho}{(\log t)^\rho} \cdot \frac{\ell(\log\lambda + \log t)}{\ell(\log t)}.$$

The first factor on the right-hand side clearly converges to 1 as $t \to \infty$, and the limit of the second factor can be calculated using the Representation Theorem (Theorem 4.10), according to which $\ell(t)$ may be written as

$$\ell(t) = c(t) \exp\left( \int_T^t \kappa(s) \frac{\mathrm{d}s}{s} \right), \; t \geq T,$$

for some $T \geq t_0$, where $c(t)$ is a measurable function converging to some $c > 0$ and $\kappa(t)$ converges to 0 as $t \to \infty$. Therefore,

$$\lim_{t\to\infty} \frac{\ell(\log\lambda + \log t)}{\ell(\log t)} = \lim_{t\to\infty} \frac{c(\log\lambda + \log t)}{c(\log t)} \cdot \exp\left( \int_{\log t}^{\log\lambda + \log t} \kappa(s) \frac{\mathrm{d}s}{s} \right) = 1.$$

□

**Remark 4.20.** Let $t_0 > 0$, $\rho \in \mathbb{R}$ and $f \colon [t_0, \infty) \to (0, \infty)$ be regularly varying with index $\rho$. By taking a look at Definition 4.1, it is obvious that $\tilde{f} \colon t \mapsto \frac{1}{f(t)}$ is regularly varying with index $-\rho$.

The following lemma will prove to be essential in Chapter 5.

**Lemma 4.21.** *Let $a$ be regularly varying, $r > 0$ and set*

$$f(t) := t^r \cdot (a \circ \log)(t) \qquad g(t) := \frac{t^{\frac{1}{r}}}{(a \circ \log)^{\frac{1}{r}}(t^{\frac{1}{r}})}.$$

*Then it holds that $(f \circ g)(t) \sim (g \circ f)(t) \sim t$.*

*Proof.* As $a$ is regularly varying, so is $\tilde{a} := a^{\frac{1}{r}}$, and consequently, by Proposition 4.19, $\tilde{a} \circ \log \in RV_0$. Repeated application of Proposition 4.18 tells us that $(\log, \tilde{a} \circ \log) \in \Xi$

therefore $(\tilde{a} \circ \log, \tilde{a} \circ \log) \in \Xi$. Utilising this fact gives

$$(g \circ f)(t) = \frac{t \cdot (a \circ \log)^{\frac{1}{r}}(t)}{(a \circ \log))^{\frac{1}{r}}(t \cdot (a \circ \log)^{\frac{1}{r}}(t))} = t \cdot \frac{(\tilde{a} \circ \log)(t)}{(\tilde{a} \circ \log)(t \cdot (\tilde{a} \circ \log)(t))} \sim t.$$

On the other hand, Remark 4.20 implies that $\frac{1}{\tilde{a} \circ \log} \in RV_0$. By Proposition 4.18, this implies $(a \circ \log, \frac{1}{\tilde{a} \circ \log}) \in \Xi$ and even $((a \circ \log) \circ t^{\frac{1}{r}}, t^{\frac{1}{r}-1}(\frac{1}{\tilde{a} \circ \log} \circ t^{\frac{1}{r}})) \in \Xi$. Hence, we obtain

$$\begin{aligned}(f \circ g)(t) &= \frac{t}{(a \circ \log)(t^{\frac{1}{r}})} \cdot (a \circ \log)(\frac{t^{\frac{1}{r}}}{(a \circ \log)^{\frac{1}{r}}(t^{\frac{1}{r}})}) \\ &= t \cdot \frac{1}{(a \circ \log)(t^{\frac{1}{r}})} \cdot (a \circ \log)(\frac{t \cdot t^{\frac{1}{r}-1}}{(\tilde{a} \circ \log)(t^{\frac{1}{r}})}) \sim t.\end{aligned}$$

□

**Example 4.22.** Let $c > 0$, $\Delta > 0$ and $\alpha \in \mathbb{R}$ and define $f(t) := c \cdot t^{\Delta} \log^{\alpha}(t)$. Then

$$g(t) := \frac{t^{\frac{1}{\Delta}}}{c^{\frac{1}{\Delta}} \cdot \log^{\frac{\alpha}{\Delta}}(t^{\frac{1}{\Delta}})} \asymp \left(\frac{t}{\log^{\alpha} t}\right)^{\frac{1}{\Delta}}$$

satisfies $(f \circ g)(t) \sim (g \circ f)(t) \sim t$.

**Definition 4.23.** Let $f(t)$ be a function of the form described in Example 4.22. We say that the function $g(t)$ defined within the example *asymptotically inverts* $f(t)$ and also write $f^{-1}(t)$ instead of $g(t)$.

# 5 An application of the Growth Theorem for certain Hamburger Hamiltonians

In this chapter, we will focus on Hamburger Hamiltonians for which the functions quantifying the decay, as described in Section 3.2, are of the form

$$\begin{aligned} d_l(t) &:= t^{-\Delta_l} \log^{-\alpha_l}(t), \\ d_\phi(t) &:= t^{-\Delta_\phi} \log^{-\alpha_\phi}(t), \\ c_l(t) &:= t^{1-\Delta_l-\nu} \log^{-\alpha_l}(t), \\ c_\phi(t) &:= t^{1-\Delta_l-\mu} \log^{-\alpha_l}(t), \end{aligned} \tag{5.1}$$

for $t \geq e$, where $\Delta_l \geq 1$, $\Delta_\phi \geq 0$, $\mu \geq \nu \geq 0$ and $\alpha_l, \alpha_\phi \in \mathbb{R}$. Further, if $\Delta_l = 1$, we demand that $\alpha_l > 1$ in order to ensure integrability of $d_l(t)$, and if $\Delta_\phi = 0$, we only consider the case that $\alpha_\phi \geq 0$, as $d_\phi(t)$ must be nonincreasing.
As all of these functions are clearly regularly varying, the results obtained by Harald Woracek in [Wor, Chapter 3] are applicable. In said chapter, the author is determines $B(R)$ for general regularly varying functions with certain indices. In some cases however, for example if $\Delta_\phi = \frac{\mu+\nu}{2}$, the general case does not allow the calculation of precise bounds $B(R)$ (up to $\asymp$). Therefore, it is our goal to explicitly calculate $B(R)$ for all feasible configurations of $\Delta_l, \Delta_\phi, \alpha_l, \alpha_\phi, \mu$ and $\nu$ with respect to the functions given in (5.1) in order to get an insight into the cases where [Wor] only gives upper and lower bounds and to test whether those given bounds are actually attained or not, which would hint that narrower bounds could be possible.
Depending on the interaction of the parameters $\Delta_l, \Delta_\phi, \alpha_l, \alpha_\phi, \mu$ and $\nu$, a multitude of different bounds occurs. In the tables below, an overview of the occurring bounds is given. As we already mentioned, the instance that $\Delta_\phi = \frac{\mu+\nu}{2}$ is one of the key edge cases, with very different bounds occurring for $\Delta_\phi < \frac{\mu+\nu}{2}$ and $\Delta_\phi > \frac{\mu+\nu}{2}$. Therefore, the overview is split corresponding to these three cases.

First case: $\Delta_\phi < \frac{\mu+\nu}{2}$. By Lemma 5.16, it holds that

$$B(R) \asymp R^{\frac{1}{\Delta_l + \frac{\mu+\nu}{2}}} (\log R)^{1-\frac{\alpha_l+2}{2\Delta_l+\mu+\nu}}.$$

Second case: $\Delta_\phi > \frac{\mu+\nu}{2}$. In the subsequent tables, whenever the bound does not depend on the value of a certain parameter (within its feasible range), we write "–" instead. Moreover, as the explicit calculations of the asymptotic bounds $B(R)$ are distributed over several lemmas and theorems, the corresponding lemma/theorem to any case can be found in the rightmost column.

| $\Delta_l + \Delta_\phi$ | $\Delta_l$ | $\mu+\nu$ | $\alpha_l + \alpha_\phi$ | $B(R)$ | Proof |
|---|---|---|---|---|---|
| $< 2$ | 1 | 0 | – | $\asymp R \log^{-\alpha_l}(R)$ | Lemma 5.17 |
| $< 2$ | $> 1$ | – | – | $\asymp R^{\frac{1-\Delta_\phi+\frac{\mu+\nu}{2}}{\Delta_l-\Delta_\phi+\mu+\nu}} \cdot (\log R)^{\frac{\alpha_\phi(1-\Delta_l-\frac{\mu+\nu}{2})-\alpha_l(1-\Delta_\phi+\frac{\mu+\nu}{2})}{\Delta_l-\Delta_\phi+\mu+\nu}}$ | Lemma 5.16 |
| $< 2$ | – | $> 0$ | – | $\asymp R^{\frac{1-\Delta_\phi+\frac{\mu+\nu}{2}}{\Delta_l-\Delta_\phi+\mu+\nu}} \cdot (\log R)^{\frac{\alpha_\phi(1-\Delta_l-\frac{\mu+\nu}{2})-\alpha_l(1-\Delta_\phi+\frac{\mu+\nu}{2})}{\Delta_l-\Delta_\phi+\mu+\nu}}$ | Lemma 5.16 |
| $> 2$ | – | – | – | $\asymp R^{\frac{1}{\Delta_l+\Delta_\phi}} \log^{-\frac{\alpha_l+\alpha_\phi}{\Delta_l+\Delta_\phi}}(R)$ | Lemma 5.16 |
| 2 | 1 | 0 | $< 2$ | $\asymp R^{\frac{2-\alpha_\phi}{2+\alpha_l-\alpha_\phi}}$ | Theorem 5.18 |
| 2 | 1 | 0 | 2 | $\asymp R^{\frac{1}{2}} \log R$ | |
| 2 | $> 1$ | – | – | $\asymp R^{\frac{1}{2}} (\log R)^{1-\frac{\alpha_l+\alpha_\phi}{2}}$ | |
| 2 | – | $> 0$ | – | $\asymp R^{\frac{1}{2}} (\log R)^{1-\frac{\alpha_l+\alpha_\phi}{2}}$ | |
| 2 | – | – | $> 2$ | $\asymp R^{\frac{1}{2}} (\log R)^{1-\frac{\alpha_l+\alpha_\phi}{2}}$ | |

Third case: $\Delta_\phi = \frac{\mu+\nu}{2}$. By Theorem 5.19, the following cases need to be distinguished.

| $\alpha_\phi$ | $\Delta_l + \Delta_\phi$ | $\alpha_l + \alpha_\phi$ | $B(R)$ |
|---|---|---|---|
| $< 0$ | – | – | $\max\left\{ R^{\frac{1}{\Delta_l+\Delta_\phi}} (\log R)^{-\frac{\alpha_l+\alpha_\phi}{\Delta_l+\Delta_\phi}+\alpha_\phi}, R^{\frac{1}{\Delta_l+\Delta_\phi}} (\log R)^{\frac{\alpha_l+\alpha_\phi}{(\Delta_l+\Delta_\phi)(\Delta_l+\Delta_\phi-1)}-\frac{\alpha_l}{\Delta_l+\Delta_\phi-1}} \log\log R \right\} \lesssim B(R) \lesssim R^{\frac{1}{\Delta_l+\Delta_\phi}} (\log R)^{-\frac{\alpha_l+\alpha_\phi}{\Delta_l+\Delta_\phi}}$ |
| $= 0$ | – | – | $\asymp R^{\frac{1}{\Delta_l+\Delta_\phi}} \log^{-\frac{\alpha_l}{\Delta_l+\Delta_\phi}}(R)$ |
| $> 0$ | $< 2$ | – | $\asymp R^{\frac{1}{\Delta_l+\Delta_\phi}} (\log R)^{-\frac{\alpha_l-\alpha_\phi}{\Delta_l+\Delta_\phi}-\alpha_\phi}$ |
| $> 0$ | $> 2$ | – | $\asymp R^{\frac{1}{\Delta_l+\Delta_\phi}} \log^{-\frac{\alpha_l+\alpha_\phi}{\Delta_l+\Delta_\phi}}(R)$ |
| $> 0$ | $= 2$ | $\neq 2$ | $\asymp R^{\frac{1}{2}} \log^{-\frac{\alpha_l+\alpha_\phi}{2}}(R)$ |
| $> 0$ | $= 2$ | $= 2$ | $\asymp R^{\frac{1}{2}} \log^{-1}(R) \log\log R$ |

As we can see from these tables, all but one of the lower and upper bounds for $B(R)$ given in [Wor, Theorem 3.12] are actually attained within the example we study in the current chapter. A detailed look into this is given in Section 5.5.

**Remark 5.1.** The calculation of bounds in this chapter heavily depends on Theorem 3.12. In order to apply it, however, the functions $d_l, d_\phi, c_l$ and $c_\phi$ need to satisfy the criteria given in Definition 3.8. In (5.1), the functions are only defined on $[e, \infty)$. They can however be extended to $(0, \infty)$ in a suitable way, as is shown below.

**Lemma 5.2.** *The functions described in (5.1) can be extended to $(0, \infty)$ in a way such that the resulting functions are $C^\infty((0, \infty))$ as well as bounded by 1 and nonincreasing on $[1, \infty)$.*

*Proof.* Using so-called *mollifiers*, the assertion is not very hard to prove. Doing so is quite technical, however, which is why we will only give a proof for $d_l(t)$.
First of all, we need to differentiate three cases: $\alpha_l < 0$, $\alpha_l > 0$ and $\alpha_l = 0$. In the first case, the function $t^{-\Delta_l} \log^{-\alpha_l}(t)$ is well-defined on the entirety of $(0, \infty)$ and bounded on $[0, \infty)$, but it does not satisfy the monotonicity conditions made. In the second case, the function is neither defined for $t = 1$ nor bounded on $(1, \infty)$, and in the third, $d_l(t) = t^{-\Delta_l}$ is already smooth, nonincreasing and bounded by 1 on $[1, \infty)$. Hence, we only need to construct extensions for the first two cases.

$\alpha_l > 0$

Define the function $d(t) := t^{-\Delta_l} \log^{-\alpha_l}(t), t \in (1, \infty)$. Clearly, this function is smooth and nonincreasing, but it is not bounded by 1 and not defined on $(0, 1]$. Here, the mollifiers come into play, namely those defined in Definition 15.8.5 in [Kal21]. There, the author describes the $C^\infty$ function $k_1 \colon \mathbb{R} \to [0, 1]$, which satisfies $k_1(t) = 1, |t| \leq 1$ and $k_1(t) = 0, |t| \geq 2$. Furthermore, it also holds that $k_1(t)$ is nondecreasing for $t \leq 1$ and nonincreasing for $t \geq 1$. As $d(e) = e^{-\Delta_l} < \frac{1}{2}$, there is $t_0 \in (1, e)$ such that $d(t_0) = \frac{1}{2}$. Now, by shifting and scaling $t$, $k_1$ lets us construct a smooth function $k$ satisfying $k(t_0) = 0$ and $k\left(\frac{t_0+e}{2}\right) = k(e) = 1$. Define $\eta \colon R \to R$ as

$$t \mapsto \begin{cases} k(t) & t < e, \\ 1 & t \geq e. \end{cases}$$

By the nature of $k$, we clearly have $\eta \in \mathbb{C}^\infty$ and therefore $\tilde{d}(t) := (\eta \cdot d)(t) \in C^\infty(1, \infty)$. However, as $\eta(t) = 0$ for all $t \leq t_0$, we can also smoothly extend $\tilde{d}$ by 0 for all $t \leq 1$ and receive $\tilde{d}(t) \in C^\infty(\mathbb{R})$. This function is bounded by $\frac{1}{2}$, but not yet nonincreasing on $[1, \infty)$. To achieve this, note that $\tilde{d}(t)$ attains a maximum on the compact set $[t_0, e]$, and as the function is nonincreasing for $t \geq e$, this maximum is global (and by the nature of $k_1$ the function $\tilde{d}$ is nonincreasing on the right side of it). Thus, set $t_1 \in [t_0.e]$ such that $\tilde{d}(t_1) = \max_{t \in \mathbb{R}}\{\tilde{d}(t)\}$. Again using the function $k_1$, shifting and scaling $t$ and further multiplying the resulting function by $1 - \tilde{d}(t)$, we get the smooth function $\kappa \colon \mathbb{R} \to \mathbb{R}$ satisfying $\kappa(t) = 1 - \tilde{d}(t)$ for $0 \leq t \leq t_1$ as well as $\kappa(t) = 0$ for $t \geq e$. Finally, by defining $d_l \colon (0, \infty) \to (0, \infty), d_l(t) = \tilde{d}(t) + \kappa(t)$, we have a smooth function which is of the desired form for $t \geq e$, is equal to 1 for $1 \leq t \leq t_1$ and nonincreasing as well as bounded by 1 for $t \geq t_1$.

$\alpha_l < 0$

In this instance, properly shaping the function $d_l$ is a lot easier compared to the case above. Setting $d(t) := t^{-\Delta_l} \log^{-\alpha_l}(t)$ and fixing $t_1$ such that $d(t_1) = \max_{t \in \mathbb{R}}\{d(t)\}$, it holds that $d(t)$ is increasing for all $0 < t < t_1$ and decreasing for all $t > t_1$. By once more scaling and shifting the mollifier $k_1$ such that the resulting function $k$ satisfies $k(0) = k(t_1) = 1$ as well as $k(e) = 0$, and setting $\kappa(t) = (1 - d(t)) \cdot k(t)$, we can define the smooth function $d_l(t) = d(t) + \kappa(t)$, which again has the required properties.

□

**Remark 5.3.** As all calculations in this chapter are of asymptotic nature, the explicit form of $\not{d}_l(t), \not{d}_\phi(t), \not{c}_l(t)$ and $\not{c}_\phi(t)$ for $1 \leq t < e$ does not matter very much, as long as the functions are of the form necessary for a suitable majorisation. Therefore, throughout this chapter, $\not{d}_l, \not{d}_\phi, \not{c}_l$ and $\not{c}_\phi$ denote extensions of the functions described in (5.1) in the sense of Lemma 5.2.

**Lemma 5.4.** *Let $(l_j)_{j\in\mathbb{N}}$ be a summable sequence of positive numbers, $(\phi_j)_{j\in\mathbb{N}}$ a sequence of real numbers and $\psi \in \mathbb{R}$ such that Definition 3.8 (iii) is satisfied. Then, $(\not{d}_l, \not{d}_\phi, \not{c}_l, \not{c}_\phi, \psi)$ is a smooth majorisation in the sense of [Wor, Definition 3.6].*

*Proof.* When taking a look at Definition 4.6, we see that $\not{d}_l, \not{d}_\phi, \not{c}_l, \not{c}_\phi$ and $\psi$ are in fact Lindelöf comparison functions when restricted to $[e,\infty)$, and as they are positive and smooth on the entirety of their domain, they are in fact regularly varying. Further, when taking a look at (5.1) and Lemma 5.2, we see that the functions already satisfy all additional conditions given in [Wor, Definition 3.6] in comparison to Definition 3.8. Hence, it only remains to check whether $(\not{d}_l, \not{d}_\phi, \not{c}_l, \not{c}_\phi, \psi)$ is a suitable majorisation, i.e. we have to show (i), (ii) and (iv) in Definition 3.8.
As the functions are smooth and nonincreasing by Lemma 5.2, (i) is clearly satisfied. To check (ii), note that $\frac{\not{d}_\phi(t)}{\not{d}_l(t)}$ is monotone for $t \geq e$. Further, $\not{d}_l(t)$ is always integrable for $\Delta_l > 1$, and in the case of $\Delta_l = 1$, the substitution $u := \log t$ shows us that

$$\int_1^\infty \not{d}_l(t)\,\mathrm{d}t \leq (e-1) + \int_e^\infty t^{-1}\log^{-\alpha_l}(t)\,\mathrm{d}t = (e-1) + \int_1^\infty u^{-\alpha_l}\,\mathrm{d}u < \infty \Leftrightarrow \alpha_l > 1.$$

As we required $\alpha_l > 1$ in case of $\Delta_l = 1$, we have in fact $\int_1^\infty \not{d}_l(t)\,\mathrm{d}t < \infty$. Lastly, we observe that by the explicit form of $\not{d}_\phi, \not{c}_l$ and $\not{c}_\phi$ along with Lemma 5.2 to ensure the boundedness of $\not{d}_\phi|_{[1,e]}$, (iv) also holds. □

**Remark 5.5.** For the remainder of this chapter, we make the following assumptions. Let $(l_j)_{j\in\mathbb{N}}$ be a summable sequence of positive numbers, $(\phi_j)_{j\in\mathbb{N}}$ a sequence of real numbers and $\psi \in \mathbb{R}$ such that Definition 3.8 (iii) is satisfied, making $(\not{d}_l, \not{d}_\phi, \not{c}_l, \not{c}_\phi, \psi)$ a smooth majorisation. Let further $H$ be the Hamburger Hamiltonian with lengths $l_j$ and angles $\phi_j$ as described in Definition 3.7, and just like in Definition 3.13, set $B(R)$ to the upper bound given by Theorem 3.12.

**Remark 5.6.** Using the abbreviations $\Delta := \Delta_l + \Delta_\phi$ and $\tilde{\Delta} := \Delta_l - \Delta_\phi$ as well as $\alpha := \alpha_l + \alpha_\phi$ and $\tilde{\alpha} := \alpha_l - \alpha_\phi$, the other functions described in Definition 3.10 are of the form

$$
\begin{aligned}
\mathfrak{e}(t) &:= \frac{2}{(\mathfrak{d}_l\mathfrak{d}_\phi)(t)} = 2t^{\Delta}\log^{\alpha}(t), \quad e \le t,\\
\mathfrak{k}(R) &\sim \left(\frac{R}{2\log^{\alpha}(R^{\frac{1}{\Delta}})}\right)^{\frac{1}{\Delta}} \asymp R^{\frac{1}{\Delta}}\log^{-\frac{\alpha}{\Delta}}(R),\\
\mathfrak{f}(t) &:= \frac{\mathfrak{d}_\phi(t)}{\mathfrak{d}_l(t)} = t^{\tilde{\Delta}}\log^{\tilde{\alpha}}(t), \quad e \le t,\\
\hbar(R) &\sim \begin{cases} \left(\frac{R}{\log^{\tilde{\alpha}}(R^{\frac{1}{\tilde{\Delta}}})}\right)^{\frac{1}{\tilde{\Delta}}} \asymp R^{\frac{1}{\tilde{\Delta}}}\log^{-\frac{\tilde{\alpha}}{\tilde{\Delta}}}(R) & \text{if} \quad \tilde{\Delta} > 0,\\ \exp(R^{\frac{1}{\tilde{\alpha}}}) & \text{if} \quad \tilde{\Delta} = 0,\ \tilde{\alpha} > 0, \end{cases}\\
\hbar(R) &= \infty \quad \text{if } \tilde{\Delta} < 0 \vee (\tilde{\Delta} = 0,\ \tilde{\alpha} > 0), \text{ for sufficiently large } R,\\
\mathfrak{n}(t) &:= (\mathfrak{c}_l\mathfrak{c}_\phi)^{\frac{1}{2}}(t) = t^{1-\Delta_l-\frac{\mu+\nu}{2}}\log^{-\alpha_l}(t), \quad e \le t,
\end{aligned}
$$

where we used Proposition 4.18 to asymptotically invert $\mathfrak{e}(t)$ and $\mathfrak{f}(t)$.

## 5.1 Calculation of thresholds

As $\mathfrak{k}(R)$ is bounded by some power, Corollary 3.15 motivates looking for approximate solutions $T(R)$ of $\mathfrak{g}(t,R) = R\mathfrak{n}(t)$ in order to calculate $B(R)$. Within the proof of Lemma 3.14, we saw that $\mathfrak{g}(2,R) \ll R\mathfrak{n}(2)$ combined with $\mathfrak{g}(t,R)$ being nondecreasing and $R\mathfrak{n}(t)$ being nonincreasing and even tending to 0 with respect to $t$. This leads to the following strategy to calculate $T(R)$. First, we calculate $\mathfrak{g}(\mathfrak{k}(R),R)$ and $R\mathfrak{n}(\mathfrak{k}(R))$ as well as, if $\frac{\mathfrak{d}_\phi(t)}{\mathfrak{d}_l(t)}$ is unbounded, $\mathfrak{g}(\hbar(R),R)$ and $R\mathfrak{n}(\hbar(R))$. To simplify notation, we denote

$$
\begin{aligned}
\sigma_1 &:= \int_1^{\mathfrak{k}(R)} \log(R(\mathfrak{d}_l\mathfrak{d}_\phi)(s))\,\mathrm{d}t = \mathfrak{g}(\mathfrak{k}(R),R), \quad \tau_1 := R\mathfrak{n}(\mathfrak{k}(R)),\\
\sigma_2 &:= R^{\frac{1}{2}}\int_{\mathfrak{k}(R)}^{\hbar(R)} (\mathfrak{d}_l\mathfrak{d}_\phi)^{\frac{1}{2}}(s)\,\mathrm{d}s, \quad \tau_2 := R\mathfrak{n}(\hbar(R)).
\end{aligned}
$$

Note that $\sigma_1 + \sigma_2 = \mathfrak{g}(\hbar(R),R)$. Now if $\sigma_1 \ll \tau_1$, we know that we need to look for $T(R)$ on the right side of $\mathfrak{k}(R)$, while $\tau_1 \ll \sigma_1$ implies that the approximate solution lies on its left. Analogously, $\sigma_1 + \sigma_2 \ll \tau_2$ and $\tau_2 \ll \sigma_1 + \sigma_2$ mean that $T(R)$ lies on the right or the left side of $\hbar(R)$, respectively.
As a first step within this strategy, we dedicate this section to the computation of these four thresholds for all feasible configurations of $\Delta_l$, $\Delta_\phi$, $\alpha_l$, $\alpha_\phi$, $\mu$ and $\nu$.

**Remark 5.7.** In the situation above, coinciding thresholds let us directly calculate $B(R)$. In case of $\sigma_1 \asymp \tau_1$, setting $T(R) := \mathfrak{k}(R)$ yields $B(R) \asymp \sigma_1 \asymp \tau_1$. Analogously, if $\sigma_1 + \sigma_2 \asymp \tau_2$, choosing $T(R) := \mathfrak{k}(R)$ leads to $B(R) \asymp \sigma_1 + \sigma_2 \asymp \tau_2$.

We start by calculating the thresholds. As calculating $\sigma_2$ for all feasible parameter configurations is by far the most work, we deal with this in a separate Proposition.

**Proposition 5.8.** *Under the assumptions made in Remark 5.5, it holds for all $t \geq e$ that*

$$\int_1^t \log(R(d_l \not{d}_\phi)(s))\, \mathrm{d}t \sim t\left(\log\left(\frac{2R}{\not{e}(t)}\right) + \Delta\right) - \log R.$$

*In particular, it holds that*

$$\sigma_1 \asymp R^{\frac{1}{\Delta}} \log^{-\frac{\alpha}{\Delta}}(R) \asymp \not{k}(R). \tag{5.2}$$

*Further, $\tau_1$ is of the form*

$$\tau_1 \asymp R^{\frac{1+\Delta_\phi - \frac{\mu+\nu}{2}}{\Delta}} (\log R)^{\alpha\frac{\Delta_l + \frac{\mu+\nu}{2} - 1}{\Delta} - \alpha_l}, \tag{5.3}$$

*and for $\tau_2$ it holds that*

$$\tau_2 \asymp R^{\frac{1-\Delta_\phi - \frac{\mu+\nu}{2}}{\tilde{\Delta}}} (\log R)^{\tilde{\alpha}\frac{\Delta_l + \frac{\mu+\nu}{2} - 1}{\tilde{\Delta}} - \alpha_l} \tag{5.4}$$

*if $\Delta_l - \Delta_\phi > 0$ and*

$$\tau_2 \asymp \exp\left(\left(1 - \Delta_l - \frac{\mu+\nu}{2}\right) R^{\frac{1}{\tilde{\alpha}}}\right) R^{-\frac{\alpha_\phi}{\tilde{\alpha}}} \tag{5.5}$$

*if $\Delta_l - \Delta_\phi = 0$ and $\alpha_l - \alpha_\phi > 0$.*

*Proof.* To prove the first assertion, we calculate

$$\begin{aligned}
\int_1^t \log(R(d_l \not{d}_\phi)(s))\, \mathrm{d}t &= \int_1^t \log R\, \mathrm{d}t + \int_1^e \log((\not{d}_l \not{d}_\phi)(t))\, \mathrm{d}t - \int_e^t \Delta \log s + \alpha \log\log s\, \mathrm{d}s \\
&\sim (t-1)\log R - \Delta\left[(s\log s - s)\Big|_e^t\right] - \alpha \int_e^t \log\log s\, \mathrm{d}s \\
&\sim (t-1)\log R + \Delta(t - t\log t) - \alpha t \log\log t \\
&= t\left(\log\left(Rt^{-\Delta}\log^{-\alpha}(t)\right) + \Delta\right) - \log R \\
&= t\left(\log\left(\frac{2R}{\not{e}(t)}\right) + \Delta\right) - \log R,
\end{aligned} \tag{5.6}$$

where we applied the second case of Karamata's Theorem to estimate the integral in the second line. Inserting $t = \not{k}(R)$, this calculation yields

$$\begin{aligned}
\mathscr{g}(\mathscr{k}(R), R) &= \int_1^{\mathscr{k}(R)} \log(R(d_l \mathscr{d}_\phi)(s))\,\mathrm{d}t \\
&= \mathscr{k}(R)\left(\log\left(\frac{2R}{\mathscr{e}(\mathscr{k}(R))}\right) + \Delta\right) - \log R \\
&\asymp \mathscr{k}(R)\,(\log 2 + \Delta) - \log R \\
&\asymp \left(\frac{R}{log^{\alpha}(R)}\right)^{\frac{1}{\Delta}} = R^{\frac{1}{\Delta}} \log^{-\frac{\alpha}{\Delta}}(R) \asymp \mathscr{k}(R).
\end{aligned}$$

For $\tau_1$, we calculate

$$\begin{aligned}
R\mathscr{n}(\mathscr{k}(R)) &= R\mathscr{k}(R)^{1-\Delta_l-\frac{\mu+\nu}{2}} \log^{-\alpha_l}(\mathscr{k}(R)) \\
&\asymp R^{1+\frac{1-\Delta_l-\frac{\mu+\nu}{2}}{\Delta}} (\log R)^{-\alpha\frac{1-\Delta_l-\frac{\mu+\nu}{2}}{\Delta}} \left(\frac{1}{\Delta}\log R - \frac{\alpha}{\Delta}\log\log R\right)^{-\alpha_l} \\
&\asymp R^{\frac{1+\Delta_\phi-\frac{\mu+\nu}{2}}{\Delta}} (\log R)^{\alpha\frac{\Delta_l+\frac{\mu+\nu}{2}-1}{\Delta}-\alpha_l}
\end{aligned}$$

To determine $\tau_2$, we start with the computation

$$\tau_2 = R(\mathscr{n}(\hbar(R))^{\frac{1}{2}} = R(\hbar(R))^{1-\Delta_l-\frac{\mu+\nu}{2}} \log^{-\alpha_l}(\hbar(R)).$$

In the event that $\Delta_l - \Delta_\phi > 0$, the explicit form of $\hbar(R)$ is substantially different to it's form if $\Delta_l - \Delta_\phi = 0$, as it is either growing similar to a power function or exponentially with respect to $R$. The former case results in

$$\begin{aligned}
\tau_2 &= R(\hbar(R))^{1-\Delta_l-\frac{\mu+\nu}{2}} \log^{-\alpha_l}(\hbar(R)) \\
&\asymp R\left(\frac{R}{\log^{\tilde{\alpha}}(R)}\right)^{\frac{1-\Delta_l-\frac{\mu+\nu}{2}}{\tilde{\Delta}}} (\log R - \tilde{\alpha}\log\log R)^{-\alpha_l} \\
&\sim R^{\frac{1-\Delta_\phi-\frac{\mu+\nu}{2}}{\tilde{\Delta}}} (\log R)^{\tilde{\alpha}\frac{\Delta_l+\frac{\mu+\nu}{2}-1}{\tilde{\Delta}}-\alpha_l},
\end{aligned}$$

while the latter yields

$$\begin{aligned}
\tau_2 &= R(\hbar(R))^{1-\Delta_l-\frac{\mu+\nu}{2}} \log^{-\alpha_l}(\hbar(R)) \sim R\exp(R^{\frac{1}{\tilde{\alpha}}})^{1-\Delta_l-\frac{\mu+\nu}{2}} R^{-\frac{\alpha_l}{\tilde{\alpha}}} \\
&= \exp\left(\left(1-\Delta_l-\frac{\mu+\nu}{2}\right) R^{\frac{1}{\tilde{\alpha}}}\right) R^{-\frac{\alpha_\phi}{\tilde{\alpha}}}.
\end{aligned}$$

□

**Remark 5.9.** As we can see from (5.2) and (5.3), $\Delta_\phi < \frac{\mu+\nu}{2}$ implies that $\tau_1 \ll \sigma_1$, while $\Delta_\phi > \frac{\mu+\nu}{2}$ results in $\sigma_1 \ll \tau_1$.

We will now calculate the possible values of the last remaining threshold, namely $\sigma_2$. Not all parameter configurations allow for an exact (up to $\asymp$) computation, but since we only need to know which threshold is bigger asymptotically in order to apply the strategy outlined at the beginning of this section, estimates from below or above are often sufficient.

**Proposition 5.10.** *In the situation of the previous proposition, $\sigma_2$ can be estimated in the following ways.*
*If we have $\Delta_l - \Delta_\phi > 0$ and $\neg(\Delta_\phi = 0 \wedge \alpha_\phi = 0)$, the following cases occur.*

- $\Delta_l + \Delta_\phi < 2$
$$\sigma_2 \asymp R^{\frac{1-\Delta_\phi}{\Delta}} (\log R)^{\tilde{\alpha}\frac{\Delta-2}{2\Delta} - \frac{\alpha}{2}} \tag{5.7}$$
- $\Delta_l + \Delta_\phi > 2$
$$\sigma_2 \lesssim \sigma_1 \asymp R^{\frac{1}{\Delta}} \log^{-\frac{\alpha}{\Delta}}(R) \tag{5.8}$$
- $\Delta_l + \Delta_\phi = 2$
  - $\alpha_l + \alpha_\phi < 2,\ \Delta_\phi > 0$
$$\sigma_2 \asymp R^{\frac{1}{2}} (\log R)^{1-\frac{\alpha}{2}} \tag{5.9}$$
  - $\alpha_l + \alpha_\phi < 2,\ \Delta_\phi = 0, \alpha_\phi > 0$
$$R^{\frac{1}{2}} \ll \sigma_2 \ll R^{\frac{1}{2}} (\log R)^{1-\frac{\alpha}{2}} \tag{5.10}$$
  - $\alpha_l + \alpha_\phi > 2,\ \Delta_\phi > 0$
$$\sigma_2 \asymp R^{\frac{1}{2}} (\log R)^{1-\frac{\alpha}{2}} \tag{5.11}$$
  - $\alpha_l + \alpha_\phi > 2,\ \Delta_\phi = 0, \alpha_\phi > 0$
$$\sigma_2 \ll R^{\frac{1}{2}} (\log R)^{1-\frac{\alpha}{2}} \tag{5.12}$$
  - $\alpha_l + \alpha_\phi = 2, \Delta_\phi > 0$
$$\sigma_2 \asymp R^{\frac{1}{2}} \tag{5.13}$$
  - $\alpha_l + \alpha_\phi = 2, \Delta_\phi = 0, \alpha_\phi > 0$
$$\sigma_2 \ll R^{\frac{1}{2}} \tag{5.14}$$

*Assuming that $\Delta_\phi = 0 \wedge \alpha_\phi = 0$, we can conclude that*
$$\sigma_2 \lesssim \sigma_1 \asymp R^{\frac{1}{\Delta_l}} \log^{-\frac{\alpha_l}{\Delta_l}}(R). \tag{5.15}$$
*In the case of $\Delta_l - \Delta_\phi = 0$ and $\alpha_l - \alpha_\phi > 0$, the following estimates hold.*

- $\Delta_l + \Delta_\phi > 2$
$$\sigma_2 \lesssim R^{\frac{1}{\Delta}} \log^{-\frac{\alpha}{\Delta}}(R) \tag{5.16}$$
- $\Delta_l + \Delta_\phi = 2$

- $\alpha_l + \alpha_\phi < 2$
$$\sigma_2 \asymp R^{\frac{1-\alpha_\phi}{\tilde{\alpha}}} \tag{5.17}$$
- $\alpha_l + \alpha_\phi > 2$
$$\sigma_2 \asymp R^{\frac{1}{2}} (\log R)^{1-\frac{\alpha}{2}} \tag{5.18}$$
- $\alpha_l + \alpha_\phi = 2$
$$\sigma_2 \asymp R^{\frac{1}{2}} \log R \tag{5.19}$$

*Proof.* In all cases, we need to calculate

$$\sigma_2 = R^{\frac{1}{2}} \int_{\Bbbk(R)}^{\hbar(R)} (d_l d_\phi)^{\frac{1}{2}}(s)\, \mathrm{d}s = R^{\frac{1}{2}} \int_{\Bbbk(R)}^{\hbar(R)} s^{-\frac{\Delta}{2}} \log^{-\frac{\alpha}{2}}(s)\, \mathrm{d}s.$$

At first, we focus on the case that $\tilde{\Delta} > 0$.

- $\Delta_l + \Delta_\phi < 2$

  Applying the second case of Karamata's Theorem, see that

$$\int_1^t s^{-\frac{\Delta}{2}} \log^{-\frac{\alpha}{2}}(s)\, \mathrm{d}s \sim (1 - \frac{\Delta}{2}) t^{1-\frac{\Delta}{2}}.$$

  As both $\Bbbk(R) = t^{\frac{1}{\Delta}} \log^{-\frac{\alpha}{\Delta}}(t)$ and $\hbar(R) = t^{\frac{1}{\tilde{\Delta}}} \log^{-\frac{\tilde{\alpha}}{\tilde{\Delta}}}(t)$ are eventually strictly increasing and tend to infinity,[1] this means that

$$\begin{aligned}
&R^{\frac{1}{2}} \int_{\Bbbk(R)}^{\hbar(R)} s^{-\frac{\Delta}{2}} \log^{-\frac{\alpha}{2}}(s)\, \mathrm{d}s \\
&\sim R^{\frac{1}{2}} \left( (\hbar(R))^{1-\frac{\Delta}{2}} \log^{-\frac{\alpha}{2}}(\hbar(R)) - (\Bbbk(R))^{1-\frac{\Delta}{2}} \log^{-\frac{\alpha}{2}}(\Bbbk(R)) \right) \\
&\asymp R^{\frac{1}{2}} \left( R^{\frac{1}{\tilde{\Delta}}(1-\frac{\Delta}{2})} \log(R)^{-\frac{\tilde{\alpha}}{\tilde{\Delta}}(1-\frac{\Delta}{2})} \left( \frac{1}{\tilde{\Delta}} \log R - \frac{\tilde{\alpha}}{\tilde{\Delta}} \log\log R \right)^{-\frac{\alpha}{2}} \right. \\
&\qquad \left. - R^{\frac{1}{\Delta}(1-\frac{\Delta}{2})} \log(R)^{-\frac{\alpha}{\Delta}(1-\frac{\Delta}{2})} \left( \frac{1}{\Delta} \log R - \frac{\alpha}{\Delta} \log\log R \right)^{-\frac{\alpha}{2}} \right) \\
&\asymp R^{\frac{1}{2}} \left( R^{\frac{2-\Delta}{2\tilde{\Delta}}} (\log R)^{-\tilde{\alpha}\frac{2-\Delta}{2\tilde{\Delta}} - \frac{\alpha}{2}} - R^{\frac{2-\Delta}{2\Delta}} (\log R)^{-\alpha\frac{2-\Delta}{2\Delta} - \frac{\alpha}{2}} \right) = (*).
\end{aligned}$$

  Here, if either $\Delta_\phi > 0$ or $\Delta_\phi = 0 \wedge \alpha_\phi > 0$, it holds that either $\tilde{\Delta} < \Delta$ (in the former case) or $\tilde{\Delta} = \Delta \wedge \tilde{\alpha} < \alpha$ (in the latter case) and therefore $R^{\frac{2-\Delta}{2\Delta}} (\log R)^{-\alpha\frac{2-\Delta}{2\Delta} - \frac{\alpha}{2}} \ll R^{\frac{2-\Delta}{2\tilde{\Delta}}} (\log R)^{-\tilde{\alpha}\frac{2-\Delta}{2\tilde{\Delta}} - \frac{\alpha}{2}}$. Consequently, we can further calculate

$$\begin{aligned}
(*) &\sim R^{\frac{1}{2}} \left( R^{\frac{2-\Delta}{2\tilde{\Delta}}} (\log R)^{-\tilde{\alpha}\frac{2-\Delta}{2\tilde{\Delta}} - \frac{\alpha}{2}} \right) \\
&= R^{\frac{1-\Delta_\phi}{\tilde{\Delta}}} (\log R)^{\tilde{\alpha}\frac{\Delta-2}{2\tilde{\Delta}} - \frac{\alpha}{2}}.
\end{aligned}$$

[1] As $\Delta_l \geq 1$ and $\Delta_l + \Delta_\phi < 2$ we certainly have that $\Delta_l - \Delta_\phi > 0$.

- $\Delta_l + \Delta_\phi > 2$

  This time, we apply the first case of Karamata's Theorem, yielding

$$\begin{aligned}\sigma_2 &= R^{\frac{1}{2}} \int_{\text{\textit{k}}(R)}^{\hbar(R)} s^{-\frac{\Delta}{2}} \log^{-\frac{\alpha}{2}}(s)\, \mathrm{d}s \\ &\lesssim R^{\frac{1}{2}} \frac{1}{\frac{\Delta}{2}-1} \text{\textit{k}}(R)^{1-\frac{\Delta}{2}} \log^{-\frac{\alpha}{2}}(\text{\textit{k}}(R)) \\ &\asymp R^{\frac{1}{2}} \left(\frac{R}{\log^{\alpha}(R)}\right)^{\frac{1}{\Delta}(1-\frac{\Delta}{2})} (\log R - \alpha \log\log R)^{-\frac{\alpha}{2}} \\ &\sim R^{\frac{1}{\Delta}} \log^{-\frac{\alpha}{\Delta}}(R).\end{aligned} \tag{5.20}$$

  Moreover, note that as $\sigma_1 \asymp R^{\frac{1}{\Delta}} \log^{-\frac{\alpha}{\Delta}}(R)$ by Proposition 5.8, the above calculation implies $\sigma_2 \lesssim \sigma_1$.

- $\Delta_l + \Delta_\phi = 2$

  We use the substitution $u := \log s$ to obtain

$$\sigma_2 = R^{\frac{1}{2}} \int_{\text{\textit{k}}(R)}^{\hbar(R)} s^{-1} \log^{-\frac{\alpha}{2}}(s)\, \mathrm{d}s = R^{\frac{1}{2}} \int_{\log \text{\textit{k}}(R)}^{\log \hbar(R)} u^{-\frac{\alpha}{2}}\, \mathrm{d}s = (*)$$

  and calculate:

  - $\alpha < 2$, $\Delta_\phi > 0$

$$\begin{aligned}(*) &= R^{\frac{1}{2}} \frac{u^{1-\frac{\alpha}{2}}}{1-\frac{\alpha}{2}}\Big|_{\log \text{\textit{k}}(R)}^{\log \hbar(R)} \asymp R^{\frac{1}{2}} \left((\log \hbar(R))^{1-\frac{\alpha}{2}} - (\log \text{\textit{k}}(R))^{1-\frac{\alpha}{2}}\right) \\ &= R^{\frac{1}{2}} \left(\tilde{\Delta}^{\frac{\alpha}{2}-1} (\log R - \tilde{\alpha} \log\log R)^{1-\frac{\alpha}{2}} - 2^{\frac{\alpha}{2}-1} (\log R - \alpha \log\log R)^{1-\frac{\alpha}{2}}\right) \\ &= R^{\frac{1}{2}} \left(\tilde{\Delta}^{\frac{\alpha}{2}-1} (\log R(1+o(1))^{1-\frac{\alpha}{2}} - 2^{\frac{\alpha}{2}-1} (\log R(1+o(1))^{1-\frac{\alpha}{2}}\right) \\ &\asymp R^{\frac{1}{2}} (\log R)^{1-\frac{\alpha}{2}}.\end{aligned}$$

  - $\alpha < 2$, $\Delta_\phi = 0$, $\alpha_\phi > 0$

$$\begin{aligned}(*) &= R^{\frac{1}{2}} \frac{u^{1-\frac{\alpha}{2}}}{1-\frac{\alpha}{2}}\Big|_{\log \text{\textit{k}}(R)}^{\log \hbar(R)} \asymp R^{\frac{1}{2}} \left((\log \hbar(R))^{1-\frac{\alpha}{2}} - (\log \text{\textit{k}}(R))^{1-\frac{\alpha}{2}}\right) \\ &= R^{\frac{1}{2}} \left(2^{\frac{\alpha}{2}-1} (\log R - \tilde{\alpha} \log\log R)^{1-\frac{\alpha}{2}} - 2^{\frac{\alpha}{2}-1} (\log R - \alpha \log\log R)^{1-\frac{\alpha}{2}}\right) \\ &\asymp R^{\frac{1}{2}} \underbrace{\left((\log R - \tilde{\alpha} \log\log R)^{1-\frac{\alpha}{2}} - (\log R - \alpha \log\log R)^{1-\frac{\alpha}{2}}\right)}_{=:\Lambda}.\end{aligned}$$

    As $\tilde{\alpha} < \alpha$ holds by our assumptions, $\Lambda$ is positive and tends to infinity for growing $R$. However, it also clearly holds that

$$\lim_{R\to\infty} \frac{(\log R - \tilde{\alpha}\log\log R)^{1-\frac{\alpha}{2}}}{(\log R)^{1-\frac{\alpha}{2}}} - \frac{(\log R - \alpha\log\log R)^{1-\frac{\alpha}{2}}}{(\log R)^{1-\frac{\alpha}{2}}} = 0,$$

and we can conclude that

$$R^{\frac{1}{2}} \ll \sigma_2 \ll R^{\frac{1}{2}}(\log R)^{1-\frac{\alpha}{2}}.$$

- $\alpha > 2$, $\Delta_\phi > 0$

$$\begin{aligned}(*) &= R^{\frac{1}{2}} \frac{u^{1-\frac{\alpha}{2}}}{1-\frac{\alpha}{2}}\Big|_{\log \mathchar'26\mkern-9mu k(R)}^{\log \hbar(R)} \asymp R^{\frac{1}{2}}\left((\log \mathchar'26\mkern-9mu k(R))^{1-\frac{\alpha}{2}} - (\log \hbar(R))^{1-\frac{\alpha}{2}}\right) \\ &\asymp R^{\frac{1}{2}}(\log R)^{1-\frac{\alpha}{2}}.\end{aligned}$$

- $\alpha > 2$, $\Delta_\phi = 0$, $\alpha_\phi > 0$

$$\begin{aligned}(*) &= R^{\frac{1}{2}} \frac{u^{1-\frac{\alpha}{2}}}{1-\frac{\alpha}{2}}\Big|_{\log \mathchar'26\mkern-9mu k(R)}^{\log \hbar(R)} \asymp R^{\frac{1}{2}}\left((\log \mathchar'26\mkern-9mu k(R))^{1-\frac{\alpha}{2}} - (\log \hbar(R))^{1-\frac{\alpha}{2}}\right) \\ &\asymp R^{\frac{1}{2}}\underbrace{\left((\log R - \alpha\log\log R)^{1-\frac{\alpha}{2}} - (\log R - \tilde{\alpha}\log\log R)^{1-\frac{\alpha}{2}}\right)}_{=:\Gamma}.\end{aligned}$$

$\Gamma$ is positive and tends to zero for growing $R$, and as

$$\lim_{R\to\infty} \frac{(\log R - \alpha\log\log R)^{1-\frac{\alpha}{2}}}{(\log R)^{1-\frac{\alpha}{2}}} - \frac{(\log R - \tilde{\alpha}\log\log R)^{1-\frac{\alpha}{2}}}{(\log R)^{1-\frac{\alpha}{2}}} = 0,$$

we know that

$$\sigma_2 \ll R^{\frac{1}{2}}(\log R)^{1-\frac{\alpha}{2}}.$$

- $\alpha = 2$, $\Delta_\phi > 0$

$$\begin{aligned}(*) &= R^{\frac{1}{2}}\int_{\log \mathchar'26\mkern-9mu k(R)}^{\log \hbar(R)} u^{-1}\,\mathrm{d}s = R^{\frac{1}{2}}\left(\log\log \hbar(R) - \log\log \mathchar'26\mkern-9mu k(R)\right) \\ &= R^{\frac{1}{2}}\log\left(\frac{\log \hbar(R)}{\log \mathchar'26\mkern-9mu k(R)}\right) \asymp R^{\frac{1}{2}}\log\left(\frac{\frac{1}{\tilde{\Delta}}(\log R - \tilde{\alpha}\log\log R)}{\frac{1}{2}(\log R - \alpha\log\log R)}\right) \\ &= R^{\frac{1}{2}}\log\left(\frac{2}{\tilde{\Delta}}\right) + R^{\frac{1}{2}}\log\left(\frac{\log R - \tilde{\alpha}\log\log R}{\log R - \alpha\log\log R}\right) \\ &\asymp R^{\frac{1}{2}}.\end{aligned}$$

– $\alpha = 2$, $\Delta_\phi = 0$, $\alpha_\phi > 0$

$$\begin{aligned}(*) &= R^{\frac{1}{2}} \int_{\log \bar{k}(R)}^{\log \hbar(R)} u^{-1} \,\mathrm{d}s = R^{\frac{1}{2}} \left(\log\log \hbar(R) - \log\log \bar{k}(R)\right) \\ &= R^{\frac{1}{2}} \log\left(\frac{\log \hbar(R)}{\log \bar{k}(R)}\right) \asymp R^{\frac{1}{2}} \log\left(\frac{\frac{1}{2}(\log R - \tilde{\alpha}\log\log R)}{\frac{1}{2}(\log R - \alpha \log\log R)}\right) \\ &= R^{\frac{1}{2}} \log\left(\frac{\log R - \tilde{\alpha}\log\log R}{\log R - \alpha\log\log R}\right) \ll R^{\frac{1}{2}}.\end{aligned}$$

Next, consider case that $\Delta_\phi = 0$ and $\alpha_\phi = 0$. The peculiarity of this instance is that we have $\bar{d}_\phi(t) \equiv 1$, meaning that $\bar{e}(t) = \frac{2}{\bar{d}_l(t)} = 2\bar{f}(t)$ and thereby $\bar{k}(R) \asymp \hbar(R)$. We calculate

$$\begin{aligned}R^{\frac{1}{2}} \int_{\bar{k}(R)}^{\hbar(R)} s^{-\frac{\Delta}{2}} \log^{-\frac{\alpha}{2}}(s) \,\mathrm{d}s &= R^{\frac{1}{2}} \int_{\bar{k}(R)}^{\hbar(R)} s^{-\frac{\Delta_l}{2}} \log^{-\frac{\alpha_l}{2}}(s) \,\mathrm{d}s \\ &\leq R^{\frac{1}{2}} (\hbar(R) - \bar{k}(R)) \cdot (\bar{k}(R))^{-\frac{\Delta_l}{2}} \log^{-\frac{\alpha_l}{2}}(\bar{k}(R)) \\ &\lesssim R^{\frac{1}{2}} (\bar{k}(R))^{1-\frac{\Delta_l}{2}} \left(\frac{1}{\Delta_l}\log R - \frac{\alpha_l}{\Delta_l}\log\log R\right)^{-\frac{\alpha_l}{2}} \\ &\asymp R^{\frac{1}{2}} R^{\frac{1}{\Delta_l} - \frac{1}{2}} \log^{\frac{\alpha_l}{2} - \frac{\alpha_l}{\Delta_l}}(R) \,(\log R)^{-\frac{\alpha_l}{2}} \\ &\asymp R^{\frac{1}{\Delta_l}} \log^{-\frac{\alpha_l}{\Delta_l}}(R) \asymp \sigma_1.\end{aligned}$$

Let us continue by looking into the case that $\tilde{\Delta} = 0$ as well as $\tilde{\alpha} > 0$. Here, $f(t)$ is of the form $f(t) = \log^{\tilde{\alpha}}(t)$, meaning that it's inverse is $h(R) = \exp(R^{\frac{1}{\tilde{\alpha}}})$. Also note that if $\tilde{\Delta} = 0$, the case that $\Delta_l + \Delta_\phi < 2$ does not occur.

- $\Delta_l + \Delta_\phi > 2$

  As the upper integration bound in (5.20) did not matter for the estimation, we can repeat the same calculations and receive

$$\sigma_2 \lesssim R^{\frac{1}{\Delta}} \log^{-\frac{\alpha}{\Delta}}(R).$$

- $\Delta_l + \Delta_\phi = 2$

  Just as for $\Delta_l - \Delta_\phi > 0$, we use the substitution $u := \log s$,

$$\sigma_2 = R^{\frac{1}{2}} \int_{\bar{k}(R)}^{\hbar(R)} s^{-1} \log^{-\frac{\alpha}{2}}(s) \,\mathrm{d}s = R^{\frac{1}{2}} \int_{\log \bar{k}(R)}^{\log \hbar(R)} u^{-\frac{\alpha}{2}} \,\mathrm{d}s = (*) \tag{5.21}$$

  leading to:

- $\alpha < 2$ [2]

$$(*) = R^{\frac{1}{2}} \frac{u^{1-\frac{\alpha}{2}}}{1-\frac{\alpha}{2}}\Big|_{\log \underline{h}(R)}^{\log \hbar(R)} \asymp R^{\frac{1}{2}} \left( (\log \hbar(R))^{1-\frac{\alpha}{2}} - (\log \underline{h}(R))^{1-\frac{\alpha}{2}} \right)$$
$$\asymp R^{\frac{1}{2}} \left( R^{\frac{1}{\tilde{\alpha}}\cdot(1-\frac{\alpha}{2})} - 2^{\frac{\alpha}{2}-1}(\log R - \alpha \log\log R)^{1-\frac{\alpha}{2}} \right)$$
$$\sim R^{\frac{1}{2}+\frac{1-\frac{\alpha}{2}}{\tilde{\alpha}}} = R^{\frac{1-\alpha_\phi}{\tilde{\alpha}}},$$

- $\alpha > 2$ [2]

$$(*) = R^{\frac{1}{2}} \frac{u^{1-\frac{\alpha}{2}}}{1-\frac{\alpha}{2}}\Big|_{\log \underline{h}(R)}^{\log \hbar(R)} \asymp R^{\frac{1}{2}} \left( (\log \underline{h}(R))^{1-\frac{\alpha}{2}} - (\log \hbar(R))^{1-\frac{\alpha}{2}} \right)$$
$$\asymp R^{\frac{1}{2}} \left( 2^{\frac{\alpha}{2}-1}(\log R - \alpha \log\log R)^{1-\frac{\alpha}{2}} - R^{\frac{1}{\tilde{\alpha}}\cdot(1-\frac{\alpha}{2})} \right)$$
$$\asymp R^{\frac{1}{2}}(\log R)^{1-\frac{\alpha}{2}} - R^{\frac{1-\alpha_\phi}{\tilde{\alpha}}} \sim R^{\frac{1}{2}}(\log R)^{1-\frac{\alpha}{2}},$$

- $\alpha = 2$

$$(*) = R^{\frac{1}{2}} \int_{\log \underline{h}(R)}^{\log \hbar(R)} u^{-1}\,\mathrm{d}s = R^{\frac{1}{2}} \left(\log\log \hbar(R) - \log\log \underline{h}(R)\right)$$
$$\sim R^{\frac{1}{2}} \left( \log(R^{\frac{1}{\tilde{\alpha}}}) - \log\left(\frac{1}{2}\log\left(\frac{R}{\log^\alpha(R)}\right)\right)\right) \asymp R^{\frac{1}{2}} \log R.$$

□

## 5.2 Bounds

In this section, we calculate $B(R)$ for different configurations of the thresholds $\sigma_1$ and $\tau_1$ as well as $\sigma_2$ and $\tau_2$. However, the question of how the relationship between these thresholds depends on the interaction between $\Delta_l$, $\Delta_\phi$, $\alpha_l$, $\alpha_\phi$, $\mu$ and $\nu$, which formed the foundation of the last chapter, will not be treated here, but will be taken up again in Section 5.4 and, to a lesser degree, Section 5.3, where we will merge the results of the Sections 5.1 and 5.2 to directly compute $B(R)$ from the given parameters.

**Lemma 5.11.** *Under the assumptions made in the preliminary section of this chapter, let $\Delta_\phi = \frac{\mu+\nu}{2}$ as well as $\tau_1 \ll \sigma_1$. Then*

$$\max\left\{ R^{\frac{1}{\Delta}}(\log R)^{-\frac{\alpha}{\Delta}+\alpha_\phi}, R^{\frac{1}{\Delta}}(\log R)^{\frac{\alpha}{\Delta(\Delta_l+\Delta_\phi-1)}-\frac{\alpha_l}{\Delta-1}} \log\log R \right\} \lesssim B(R) \lesssim R^{\frac{1}{\Delta}}(\log R)^{-\frac{\alpha}{\Delta}}.$$

*Proof.* First, note that as $\sigma_1 \asymp R^{\frac{1}{\Delta}} \log^{-\frac{\alpha_l+\alpha_\phi}{\Delta}}(R)$ and

$$\tau_1 \asymp R^{\frac{1+\Delta_\phi-\frac{\mu+\nu}{2}}{\Delta}}(\log R)^{\alpha\frac{\Delta_l+\frac{\mu+\nu}{2}-1}{\Delta}-\alpha_l} = R^{\frac{1}{\Delta}}(\log R)^{\alpha\frac{\Delta-1}{\Delta}-\alpha_l} = R^{\frac{1}{\Delta}}(\log R)^{-\frac{\alpha}{\Delta}+\alpha_\phi}$$

[2] Note that for $\tilde{\alpha} > 0$ we have $\frac{1}{2} \gtrless \frac{1-\alpha_\phi}{\tilde{\alpha}} \Leftrightarrow \alpha \gtrless 2$.

by Proposition 5.8, we see that $\tau_1 \ll \sigma_1$ implies $\alpha_\phi < 0$. In particular, this means that $\Delta_\phi > 0$ by our assumptions in (5.1) and hence $\Delta > 1$. Further, by Lemma 3.14, we have

$$R^{\frac{1}{\Delta}}(\log R)^{-\frac{\alpha}{\Delta}+\alpha_\phi} \asymp \tau_1 \lesssim B(R) \lesssim \sigma_1 \asymp R^{\frac{1}{\Delta}} \log^{-\frac{\alpha_l+\alpha_\phi}{\Delta}}(R).$$

In order to approximate $B(R)$, we need to look for approximate solutions of $\mathfrak{g}(t,R) = R\mathfrak{n}(t)$ on the left side of $\mathfrak{k}(R)$. Consider $T := R^{\frac{1}{\Delta}}(\log R)^{\frac{\alpha}{\Delta(\Delta-1)}-\frac{\alpha_l}{\Delta-1}}$, for which it holds that

$$R\mathfrak{n}(T) \asymp \mathfrak{k}(R).$$

Here, note that $\frac{\alpha}{\Delta(\Delta-1)} - \frac{\alpha_l}{\Delta-1} < -\frac{\alpha}{\Delta}$ is equivalent to $\alpha_l < \alpha$, which in turn holds true by our assumption that $\alpha_\phi < 0$. Hence, $T$ indeed lies on the left side of $\mathfrak{k}(R)$. Further, a calculation shows that

$$\begin{aligned} \log T &= \frac{1}{\Delta}\log R + \left(\frac{\alpha}{\Delta(\Delta-1)} - \frac{\alpha_l}{\Delta-1}\right)\log\log R \\ &= \frac{1}{\Delta}\log R + \frac{\alpha - \Delta\alpha_l}{\Delta(\Delta-1)}\log\log R, \end{aligned}$$

as well as

$$\log\log T \sim \log\log R - \log\Delta.$$

With these approximations, we can calculate

$$\begin{aligned} \mathfrak{g}(T,R) &\sim T\log(RT^{-\Delta}\log^{-\alpha}T) + \Delta) - \log R \\ &= T\left(\log R - \Delta\log T - \alpha\log\log T + \Delta\right) - \log R \\ &= T\left(\log R - \log R - \frac{\alpha - \Delta\alpha_l}{\Delta-1}\log\log R - \alpha\log\log R + \alpha\log\Delta + \Delta\right) - \log R \\ &= T\left(-\frac{-\Delta\alpha_l + \Delta\alpha}{\Delta-1}\log\log R + \alpha\log\Delta + \Delta\right) - \log R \\ &= T\left(-\frac{\Delta\alpha_\phi}{\Delta-1}\log\log R + \alpha\log\Delta + \Delta\right) - \log R \\ &= R^{\frac{1}{\Delta}}(\log R)^{\frac{\alpha}{\Delta(\Delta-1)}-\frac{\alpha_l}{\Delta-1}}\left(-\frac{\Delta\alpha_\phi}{\Delta-1}\log\log R + \alpha\log\Delta + \Delta\right) - \log R \\ &\asymp R^{\frac{1}{\Delta}}(\log R)^{\frac{\alpha}{\Delta(\Delta-1)}-\frac{\alpha_l}{\Delta-1}}\log\log R. \end{aligned}$$

As we have already seen at be beginning of the proof, it holds that $\frac{\alpha}{\Delta(\Delta-1)} - \frac{\alpha_l}{\Delta-1} < -\frac{\alpha}{\Delta}$ and therefore $\mathfrak{g}(T,R) \ll R\mathfrak{n}(T)$. Thus, Lemma 3.14 tells us that

$$R^{\frac{1}{\Delta}}(\log R)^{\frac{\alpha}{\Delta(\Delta-1)}-\frac{\alpha_l}{\Delta-1}}\log\log R \asymp \mathfrak{g}(T,R) \lesssim B(R) \lesssim R\mathfrak{n}(T) \asymp R^{\frac{1}{\Delta}}\log^{-\frac{\alpha_l+\alpha_\phi}{\Delta}}(R),$$

and the assertion is shown. □

**Lemma 5.12.** *Let* $\Delta_l + \Delta_\phi < 2$ *as well as* $\sigma_1 \ll \tau_1$. *Then,*

$$B(R) \asymp R^{\frac{1-\Delta_\phi+\frac{\mu+\nu}{2}}{\Delta_l-\Delta_\phi+\mu+\nu}} (\log R)^{\frac{\alpha_\phi(1-\Delta_l-\frac{\mu+\nu}{2})-\alpha_l(1-\Delta_\phi+\frac{\mu+\nu}{2})}{\Delta_l-\Delta_\phi+\mu+\nu}}. \tag{5.22}$$

*Proof.* Note that as $\Delta_l + \Delta_\phi < 2$, it follows that $\Delta_l > \Delta_\phi$, making $\frac{d_\phi(t)}{d_l(t)}$ unbounded. By [Wor, Lemma 3.11], it holds for $t \in (\hbar(R), \hbar(R)]$ that

$$\begin{aligned} g(t,R) &\asymp R^{\frac{1}{\Delta}} \log^{-\frac{\alpha}{\Delta}}(R) + R^{\frac{1}{2}} \int_{\hbar(R)}^{t} s^{-\frac{\Delta}{2}} \log^{-\frac{\alpha}{2}}(s)\, \mathrm{d}s \\ &\asymp R^{\frac{1}{2}} t^{1-\frac{\Delta}{2}} \log^{-\frac{\alpha}{2}}(t). \end{aligned}$$

Define $p(t) := t^{\tilde{\Delta}+\mu+\nu} \log^{\tilde{\alpha}} t$ and set

$$T(R) := p^{-1}(R) \asymp \left(\frac{R}{\log^{\tilde{\alpha}}(R)}\right)^{\frac{1}{\tilde{\Delta}+\mu+\nu}}.$$

Note here that $\sigma_1 \ll \tau_1$ implies that $\Delta_l - \Delta_\phi + \mu + \nu \leq \Delta_l + \Delta_\phi$ and thereby $\hbar(R) \lesssim T(R)$. Further, it also holds that $T(R) \lesssim \hbar(R)$.[3] Writing $T := T(R)$, it holds that

$$\begin{aligned} g(T(R),R) &= R^{\frac{1}{2}} T^{1-\frac{\Delta}{2}} \log^{-\frac{\alpha}{2}}(T) \\ &= R^{\frac{1}{2}} R^{\frac{1-\frac{\Delta}{2}}{\tilde{\Delta}+\mu+\nu}} (\log R)^{-\frac{\tilde{\alpha}}{\tilde{\Delta}+\mu+\nu}(1-\frac{\Delta}{2})} \left(\frac{1}{\tilde{\Delta}+\mu+\nu}(\log R - \tilde{\alpha}\log\log R)\right)^{-\frac{\alpha}{2}} \\ &\asymp R^{\frac{\tilde{\Delta}+\mu+\nu+2-\Delta}{2(\tilde{\Delta}+\mu+\nu)}} (\log R)^{-\frac{\tilde{\alpha}}{\tilde{\Delta}+\mu+\nu}(1-\frac{\Delta}{2})-\frac{\alpha}{2}} \\ &= R^{\frac{1-\Delta_\phi+\frac{\mu+\nu}{2}}{\Delta_l-\Delta_\phi+\mu+\nu}} (\log R)^{\frac{\alpha_\phi(1-\Delta_l-\frac{\mu+\nu}{2})-\alpha_l(1-\Delta_\phi+\frac{\mu+\nu}{2})}{\Delta_l-\Delta_\phi+\mu+\nu}}, \end{aligned}$$

and

$$\begin{aligned} Rn(T(R)) &= RT^{1-\Delta_l-\frac{\mu+\nu}{2}} \log^{-\alpha_l}(T) \\ &= R^{1+\frac{1-\Delta_l-\frac{\mu+\nu}{2}}{\tilde{\Delta}+\mu+\nu}} (\log R)^{\frac{-\tilde{\alpha}}{\tilde{\Delta}+\mu+\nu}(1-\Delta_l-\frac{\mu+\nu}{2})} \left(\frac{1}{\tilde{\Delta}+\mu+\nu})(\log R - \tilde{\alpha}\log\log R)\right)^{-\alpha_l} \\ &\asymp R^{\frac{1-\Delta_\phi+\frac{\mu+\nu}{2}}{\tilde{\Delta}+\mu+\nu}} (\log R)^{\frac{-\tilde{\alpha}}{\tilde{\Delta}+\mu+\nu}(1-\Delta_l-\frac{\mu+\nu}{2})-\alpha_l} \\ &= R^{\frac{1-\Delta_\phi+\frac{\mu+\nu}{2}}{\Delta_l-\Delta_\phi+\mu+\nu}} (\log R)^{\frac{\alpha_\phi(1-\Delta_l-\frac{\mu+\nu}{2})-\alpha_l(1-\Delta_\phi+\frac{\mu+\nu}{2})}{\Delta_l-\Delta_\phi+\mu+\nu}}, \end{aligned}$$

which completes the proof. □

**Lemma 5.13.** *Let* $\sigma_1 \ll \tau_1$ *as well as* $\Delta_l + \Delta_\phi > 2$. *Then,*

$$B(R) \asymp R^{\frac{1}{\Delta}} \log^{-\frac{\alpha}{\Delta}}(R). \tag{5.23}$$

[3] In case of $\mu = \nu = 0$, we have $T(R) \asymp \hbar(R)$ and otherwise $T(R) \ll \hbar(R)$.

*Proof.* We start by providing a proof for the case that $\frac{d_\phi(t)}{d_l(t)}$ is unbounded. At first, focus on the case that $\Delta_l - \Delta_\phi > 0$, where we have

$$\sigma_1 + \sigma_2 \gtrsim \sigma_1 \asymp R^{\frac{1}{\Delta_l+\Delta_\phi}} \log^{-\frac{\alpha_l+\alpha_\phi}{\Delta_l+\Delta_\phi}} \tag{5.24}$$

as well as

$$\tau_2 \asymp R^{\frac{1-\Delta_\phi-\frac{\mu+\nu}{2}}{\Delta_l-\Delta_\phi}} (\log R)^{\tilde{\alpha}\frac{\Delta_l+\frac{\mu+\nu}{2}-1}{\tilde{\Delta}}-\alpha_l}.$$

From

$$\begin{aligned} &\frac{1}{\Delta_l+\Delta_\phi} \gtrless \frac{1-\Delta_\phi}{\Delta_l-\Delta_\phi} \\ \Leftrightarrow\ &\Delta_l - \Delta_\phi \gtrless \Delta_l + \Delta_\phi - \Delta_\phi(\Delta_l+\Delta_\phi) \\ \Leftrightarrow\ &-2\Delta_\phi \gtrless -\Delta_\phi(\Delta_l+\Delta_\phi), \end{aligned}$$

we see that $\tau_2 \ll \sigma_1 + \sigma_2$ as long as $\Delta_\phi > 0$, and it is easy to see that the same holds if $\Delta_\phi = 0$ and $\mu + \nu > 0$. Looking at the case that $\Delta_\phi = \mu + \nu = 0$, we need to compare the exponents of the logarithmic terms. Note that as $d_\phi(t) = t^{-\Delta_\phi} \log^{-\alpha_\phi}(t)$ must be nonincreasing, $\Delta_\phi = 0$ means that $\alpha_\phi \geq 0$. As

$$\begin{aligned} &-\frac{\alpha_l+\alpha_\phi}{\Delta_l} \gtrless \tilde{\alpha}\frac{\Delta_l-1}{\Delta_l} - \alpha_l \\ \Leftrightarrow\ &-\frac{\alpha_l}{\Delta_l} - \frac{\alpha_\phi}{\Delta_l} \gtrless \alpha_l\left(1 - \frac{1}{\Delta_l} - 1\right) + \alpha_\phi\left(\frac{1}{\Delta_l} - 1\right) \\ \Leftrightarrow\ &\alpha_\phi \gtrless \alpha_\phi\frac{2}{\Delta_l}, \end{aligned}$$

the case that $\alpha_\phi > 0$ again implies $\tau_2 \ll \sigma_1 + \sigma_2$. If however $\alpha_\phi = 0$, we obtain $\sigma_1 + \sigma_2 \asymp \tau_2$, and choosing $T(R) := \hbar(R)$ yields $B(R) \asymp \sigma_1 + \sigma_2 \asymp \tau_2 \asymp R^{\frac{1}{\Delta}} \log^{-\frac{\alpha}{\Delta}}(R)$.

Now we look into the case that $\Delta_l - \Delta_\phi = 0 \wedge \alpha_l - \alpha_\phi > 0$. In this instance, (5.24) still holds, while $\tau_2$ is of the form

$$\tau_2 \asymp \exp\left(\left(1 - \Delta_l - \frac{\mu+\nu}{2}\right) R^{\frac{1}{\tilde{\alpha}}}\right) R^{-\frac{\alpha_\phi}{\tilde{\alpha}}}.$$

As $\Delta_l - \Delta_\phi = 0$ and $\Delta_l + \Delta_\phi > 2$, we have $\Delta_l + \frac{\mu+\nu}{2} \geq \Delta_l > 1$ and therefore $\tau_2 \ll \sigma_1 + \sigma_2$.

Summing up, when assuming $\frac{d_\phi(t)}{d_l(t)}$ to be unbounded, we had one sole occurrence of $\sigma_1 + \sigma_2 \asymp \tau_2$, namely the case that $\Delta_\phi = \mu = \alpha_\phi = 0$, for which we already calculated $T(R)$ and $B(R)$. In all other cases, we obtained $\tau_2 \ll \sigma_1 + \sigma_2$, which implies that we have to look for $T(R)$ in between $\hbar\!\!\!/(R)$ and $\hbar(R)$.

By the first case of Karamata's Theorem, it holds for $t \in (\mathcal{k}(R), \hbar(R)]$ that

$$\begin{aligned}
\mathcal{g}(t,R) &= \sigma_1 + R^{\frac{1}{2}} \int_{\mathcal{k}(R)}^{t} s^{-\frac{\Delta}{2}} \log^{-\frac{\alpha}{2}}(s)\,\mathrm{d}s \\
&\lesssim R^{\frac{1}{\Delta}} \log^{-\frac{\alpha}{\Delta}}(R) + R^{\frac{1}{2}} (\mathcal{k}(R))^{1-\frac{\Delta}{2}} \log^{-\frac{\alpha}{2}}(\mathcal{k}(R)) \\
&\asymp R^{\frac{1}{\Delta}} \log^{-\frac{\alpha}{\Delta}}(R) + R^{\frac{1}{2}+\frac{1}{\Delta}(1-\frac{\Delta}{2})} \log^{-\frac{\alpha}{\Delta}(1-\frac{\Delta}{2})-\frac{\alpha}{2}}(R) \\
&\asymp R^{\frac{1}{\Delta}} \log^{-\frac{\alpha}{\Delta}}(R) \asymp \sigma_1,
\end{aligned} \tag{5.25}$$

and therefore $\mathcal{g}(t,R) \asymp R^{\frac{1}{\Delta}} \log^{-\frac{\alpha}{\Delta}}(R)$. Define $\mathcal{q}(t) := t^{\Delta_l + \frac{\mu+\nu}{2} - 1} \log^{\alpha_l}(t)$ and set

$$\begin{aligned}
T(R) &:= \mathcal{q}^{-1}\left(R^{1-\frac{1}{\Delta}} \log^{\frac{\alpha}{\Delta}}(R)\right) \asymp \left(\frac{R^{1-\frac{1}{\Delta}} \log^{\frac{\alpha}{\Delta}}(R)}{\left((1-\frac{1}{\Delta})\log R + \frac{\alpha}{\Delta}\log\log R\right)^{\alpha_l}}\right)^{\frac{1}{\Delta_l + \frac{\mu+\nu}{2} - 1}} \\
&\asymp \left(R^{1-\frac{1}{\Delta}} (\log R)^{\frac{\alpha}{\Delta}-\alpha_l}\right)^{\frac{1}{\Delta_l + \frac{\mu+\nu}{2} - 1}}.
\end{aligned}$$

We further calculate

$$\begin{aligned}
R\mathcal{n}(T) &= RT^{1-\Delta_l-\frac{\mu+\nu}{2}} \log^{-\alpha_l}(T) \\
&\asymp R\left(R^{1-\frac{1}{\Delta}} (\log R)^{\frac{\alpha}{\Delta}-\alpha_l}\right)^{\frac{1-\Delta_l-\frac{\mu+\nu}{2}}{\Delta_l+\frac{\mu+\nu}{2}-1}} (\Delta_l - 1)^{\alpha_l} \\
&\quad \cdot \left((1-\frac{1}{\Delta})\log R + (\frac{\alpha}{\Delta} - \alpha_l)\log\log R\right)^{-\alpha_l} \\
&\asymp R^{\frac{1}{\Delta}} (\log R)^{-\frac{\alpha}{\Delta}+\alpha_l-\alpha_l} \\
&= R^{\frac{1}{\Delta}} \log^{-\frac{\alpha}{\Delta}}(R).
\end{aligned}$$

At last, we also need to look at the case that $\frac{d_\phi(t)}{d_l(t)}$ is bounded. Here, (5.25) holds for all $t \gtrsim \mathcal{k}(R)$, and choosing $\mathcal{q}(t)$ as well as $T(R) := \mathcal{q}^{-1}\left(R^{1-\frac{1}{\Delta}} \log^{\frac{\alpha}{\Delta}}(R)\right)$ just as before yields

$$\mathcal{g}(T,R) \asymp R\mathcal{n}(T,R) \asymp R^{\frac{1}{\Delta}} \log^{-\frac{\alpha}{\Delta}}(R)$$

once more, and the proof is complete. □

In the ensuing Lemma, the following estimation of $\log(t+1)$ for small $t$ will be of great use.

**Remark 5.14.** By the definition of the natural logarithm, it holds for all $1 > t > 0$ that

$$\log(1+t) = \int_1^{1+t} \frac{\mathrm{d}s}{s}, \quad \text{and} \quad \log(1-t) = \int_1^{1-t} \frac{\mathrm{d}s}{s}.$$

From these equations, we can deduce that

$$\frac{t}{1+1} \leq \log(1+t) \leq t, \quad \text{and} \quad t \leq \log(t-1) \leq \frac{t}{t-1},$$

and, more particularly, $\log(t+1) \sim t$ for $t \to 0$.

**Lemma 5.15.** *Let $\Delta_l + \Delta_\phi = 2$ as well as $\sigma_1 \ll \tau_1$ and let either $\frac{d_\phi(t)}{d_l(t)}$ be bounded or $\tau_2 \ll \sigma_1 + \sigma_2$.*

(i) *If $\alpha_l + \alpha_\phi < 2$, then*

$$B(R) \asymp \begin{cases} R^{\frac{2-\alpha_\phi}{2+\alpha_l-\alpha_\phi}} & \text{if} \quad \Delta_l = 1, \mu = \nu = 0, \\ R^{\frac{1}{2}}(\log R)^{1-\frac{\alpha}{2}} & \text{if} \quad 1 < \Delta_l + \frac{\mu+\nu}{2} < 2, \\ R^{\frac{1}{2}}(\log R)^{-\frac{\alpha}{2}} & \text{if} \quad \Delta_l + \frac{\mu+\nu}{2} = 2. \end{cases} \tag{5.26}$$

(ii) *If $\alpha_l + \alpha_\phi > 2$, then*

$$B(R) \asymp \begin{cases} R^{\frac{1}{2}}(\log R)^{1-\frac{\alpha}{2}} & \text{if} \quad \Delta_l = 1, \mu = \nu = 0, \\ R^{\frac{1}{2}}(\log R)^{1-\frac{\alpha}{2}} & \text{if} \quad 1 < \Delta_l + \frac{\mu+\nu}{2} < 2, \\ R^{\frac{1}{2}}(\log R)^{-\frac{\alpha}{2}} & \text{if} \quad \Delta_l + \frac{\mu+\nu}{2} = 2. \end{cases} \tag{5.27}$$

(iii) *If $\alpha_l + \alpha_\phi = 2$, then*

$$B(R) \asymp \begin{cases} R^{\frac{1}{2}} \log R & \text{if} \quad \Delta_l = 1, \mu = \nu = 0, \\ R^{\frac{1}{2}} & \text{if} \quad 1 < \Delta_l + \frac{\mu+\nu}{2} < 2, \\ R^{\frac{1}{2}} \frac{\log\log R}{\log R} & \text{if} \quad \Delta_l + \frac{\mu+\nu}{2} = 2. \end{cases} \tag{5.28}$$

*Proof.* We provide a proof for the case that $\frac{d_\phi(t)}{d_l(t)}$ is unbounded. Just like in the last paragraph of the proof of the previous lemma, choosing the same $T(R)$ for the bounded case (individually for each configuration of the parameters) leads the same estimate of $B(R)$. In order to shorten and simplify our notation, denote $\gamma := \frac{\mu+\nu}{2} + \Delta_l - 1$ whenever $1 < \Delta_l + \frac{\mu+\nu}{2}$. As $\sigma_1 \ll \tau_1$ implies $\Delta_\phi \geq \frac{\mu+\nu}{2}$, it holds for $\gamma$ that

$$0 < \gamma = \Delta_l + \frac{\mu+\nu}{2} - 1 \leq \Delta_l + \Delta_\phi - 1 = 1.$$

Further, it is important to note that as Definition 3.8 requires $c_l(t) = t^{1-\Delta_l-\nu} \log^{-\alpha_l}(t) \to 0$, the case that $\Delta_l + \nu = 1$ means that we must have $\alpha_l > 0$. Let us now start to distinguish the different cases given in the lemma.

It holds for $t \in (\hbar(R), \hbar(R)]$ that

$$g(t,R) \asymp R^{\frac{1}{2}} \log^{-\frac{\alpha}{2}}(R) + R^{\frac{1}{2}} \int_{\hbar(R)}^{t} s^{-1} \log^{-\frac{\alpha}{2}}(s) \, \mathrm{d}s = (*)$$

(i) Substituting $u \coloneqq \log s$ yields

$$\begin{aligned}
(*) &= R^{\frac{1}{2}} \log^{-\frac{\alpha}{2}}(R) + R^{\frac{1}{2}} \int_{\log \mathscr{k}(R)}^{\log t} u^{-\frac{\alpha}{2}} \, \mathrm{d}u \\
&\asymp R^{\frac{1}{2}} \log^{-\frac{\alpha}{2}}(R) + R^{\frac{1}{2}} \frac{u^{1-\frac{\alpha}{2}}}{1-\frac{\alpha}{2}} \Big|_{\log \mathscr{k}(R)}^{\log t} \\
&\asymp R^{\frac{1}{2}} \log^{-\frac{\alpha}{2}}(R) + R^{\frac{1}{2}} \left( (\log t)^{1-\frac{\alpha}{2}} - (\log \mathscr{k}(R))^{1-\frac{\alpha}{2}} \right) \\
&\asymp R^{\frac{1}{2}} \log^{-\frac{\alpha}{2}}(R) + R^{\frac{1}{2}} \left( (\log t)^{1-\frac{\alpha}{2}} - \left( \frac{1}{2} \log R - \frac{\alpha}{2} \log\log R \right)^{1-\frac{\alpha}{2}} \right).
\end{aligned} \tag{5.29}$$

$\underline{\Delta_l = 1, \mu = \nu = 0}$ We start by setting

$$T = T(R) \coloneqq \exp\left( R^{\frac{1}{2+\tilde{\alpha}}} \right).$$

Note that as $\alpha_l > 0$ and $\alpha < 2$, we have $\tilde{\alpha} = 2\alpha_l - \alpha > -2$. A calculation shows

$$\begin{aligned}
\mathscr{g}(T,R) &\asymp R^{\frac{1}{2}} \log^{-\frac{\alpha}{2}}(R) + R^{\frac{1}{2}} \left( (\log T)^{1-\frac{\alpha}{2}} - \left( \frac{1}{2} \log R - \frac{\alpha}{2} \log\log R \right)^{1-\frac{\alpha}{2}} \right) \\
&= R^{\frac{1}{2}} \log^{-\frac{\alpha}{2}}(R) + R^{\frac{1}{2}} \left( R^{\frac{1-\frac{\alpha}{2}}{2+\tilde{\alpha}}} - \left( \frac{1}{2} \log R - \frac{\alpha}{2} \log\log R \right)^{1-\frac{\alpha}{2}} \right) \\
&= R^{\frac{1}{2}} \log^{-\frac{\alpha}{2}}(R) + R^{\frac{2+\tilde{\alpha}+2-\alpha}{4+2\tilde{\alpha}}} - R^{\frac{1}{2}} \left( \frac{1}{2} \log R - \frac{\alpha}{2} \log\log R \right)^{1-\frac{\alpha}{2}} \\
&\sim R^{\frac{2-\alpha_\phi}{2+\tilde{\alpha}}} = R^{\frac{2-\alpha_\phi}{2+\alpha_l-\alpha_\phi}},
\end{aligned}$$

where $\frac{1}{2} < \frac{2-\alpha_\phi}{2+\tilde{\alpha}} < 1$. Additionally, we have

$$\begin{aligned}
R\mathscr{n}(T) &= RT^{1-\Delta_l-\frac{\mu+\nu}{2}} \log^{-\alpha_l}(T) = R \log^{-\alpha_l}(T) \\
&= R \cdot R^{-\frac{\alpha_l}{2+\tilde{\alpha}}} = R^{2+\tilde{\alpha}-\alpha_l} 2 + \tilde{\alpha} = R^{\frac{2-\alpha_\phi}{2+\tilde{\alpha}}}.
\end{aligned}$$

$\underline{1 < \Delta_l + \frac{\mu+\nu}{2} \leq 2}$ Let the function $\mathscr{p}$ be defined as

$$\mathscr{p}(t) \coloneqq t^{2\gamma} \log^{2\alpha_l}(t) \left( (\log t)^{1-\frac{\alpha}{2}} + \log^{-\frac{\alpha}{2}}(R) - \left( \frac{1}{2} \log R - \frac{\alpha}{2} \log\log R \right)^{1-\frac{\alpha}{2}} \right)^2.$$

By Lemma 4.21, we have

$$\begin{aligned}
p^{-1}(t) &\asymp \frac{t^{\frac{1}{2\gamma}}}{\log^{\frac{\alpha_l}{\gamma}}(t^{\frac{1}{2\gamma}})\Big((\log(t^{\frac{1}{2\gamma}}))^{1-\frac{\alpha}{2}} + \log^{-\frac{\alpha}{2}}(R) - \big(\frac{1}{2}\log R - \frac{\alpha}{2}\log\log R\big)^{1-\frac{\alpha}{2}}\Big)^{\frac{1}{\gamma}}} \\
&\asymp \frac{t^{\frac{1}{2\gamma}}}{\left(\frac{1}{2\gamma}\log t\right)^{\frac{\alpha_l}{\gamma}}\left(\left(\frac{1}{2\gamma}\log t\right)^{1-\frac{\alpha}{2}} + \log^{-\frac{\alpha}{2}}(R) - \big(\frac{1}{2}\log R - \frac{\alpha}{2}\log\log R\big)^{1-\frac{\alpha}{2}}\right)^{\frac{1}{\gamma}}} \\
&\asymp \frac{t^{\frac{1}{2\gamma}}}{\log^{\frac{\alpha_l}{\gamma}}(t)\left(\left(\frac{1}{2\gamma}\log t\right)^{1-\frac{\alpha}{2}} + \log^{-\frac{\alpha}{2}}(R) - \big(\frac{1}{2}\log R - \frac{\alpha}{2}\log\log R\big)^{1-\frac{\alpha}{2}}\right)^{\frac{1}{\gamma}}}.
\end{aligned}$$

Choosing

$$\begin{aligned}
&T = T(R) := p^{-1}(R) \\
&\asymp \frac{R^{\frac{1}{2\gamma}}}{\log^{\frac{\alpha_l}{\gamma}}(R)\left(\left(\frac{1}{2\gamma}\log R\right)^{1-\frac{\alpha}{2}} + \log^{-\frac{\alpha}{2}}(R) - \big(\frac{1}{2}\log R - \frac{\alpha}{2}\log\log R\big)^{1-\frac{\alpha}{2}}\right)^{\frac{1}{\gamma}}}
\end{aligned}$$

lets us calculate

$$\begin{aligned}
\log T &\sim \frac{1}{2\gamma}\log R - \frac{\alpha_l}{\gamma}\log\log R - \frac{1}{\gamma}\log\left(\left(\frac{1}{2\gamma}\log R\right)^{1-\frac{\alpha}{2}} + \log^{-\frac{\alpha}{2}}(R)\right. \\
&\qquad\qquad \left. - \left(\frac{1}{2}\log R - \frac{\alpha}{2}\log\log R\right)^{1-\frac{\alpha}{2}}\right) \\
&\sim \frac{1}{2\gamma}\log R.
\end{aligned}$$

This leads to

$$\begin{aligned}
g(T,R) &\asymp R^{\frac{1}{2}}\log^{-\frac{\alpha}{2}}(R) + R^{\frac{1}{2}}\left((\log T)^{1-\frac{\alpha}{2}} - \left(\frac{1}{2}\log R - \frac{\alpha}{2}\log\log R\right)^{1-\frac{\alpha}{2}}\right) \\
&\sim R^{\frac{1}{2}}\log^{-\frac{\alpha}{2}}(R) + R^{\frac{1}{2}}\left(\left(\frac{1}{2\gamma}\log R\right)^{1-\frac{\alpha}{2}} - \left(\frac{1}{2}\log R\right)^{1-\frac{\alpha}{2}}\right) \\
&\asymp \begin{cases} R^{\frac{1}{2}}(\log R)^{1-\frac{\alpha}{2}} & \text{if} \quad 0<\gamma<1, \\ R^{\frac{1}{2}}(\log R)^{-\frac{\alpha}{2}} & \text{if} \quad \gamma = 1. \end{cases}
\end{aligned}$$

Additionally, we also have

$$\begin{aligned}
Rn(T) &= RT^{-\gamma}\log^{-\alpha_l}(T) \\
&\asymp R^{\frac{1}{2}}\log^{\alpha_l}(R)\left(\left(\frac{1}{2\gamma}\log R\right)^{1-\frac{\alpha}{2}}+\log^{-\frac{\alpha}{2}}(R)-\left(\frac{1}{2}\log R-\frac{\alpha}{2}\log\log R\right)^{1-\frac{\alpha}{2}}\right) \\
&\quad\cdot\log^{-\alpha_l}(R) \\
&\sim R^{\frac{1}{2}}\left(\left(\frac{1}{2\gamma}\log R\right)^{1-\frac{\alpha}{2}}+\log^{-\frac{\alpha}{2}}(R)-\left(\frac{1}{2}\log R\right)^{1-\frac{\alpha}{2}}\right) \\
&\asymp \begin{cases} R^{\frac{1}{2}}(\log R)^{1-\frac{\alpha}{2}} & \text{if} \quad 0<\gamma<1, \\ R^{\frac{1}{2}}(\log R)^{-\frac{\alpha}{2}} & \text{if} \quad \gamma=1. \end{cases}
\end{aligned}$$

As $\gamma = \Delta_l + \frac{\mu+\nu}{2} - 1$, this proves the asserted estimate.

(ii) Using the same substitution as above, we receive

$$\begin{aligned}
(*) &= R^{\frac{1}{2}}\log^{-\frac{\alpha}{2}}(R)+R^{\frac{1}{2}}\int_{\log \hbar(R)}^{\log t} u^{-\frac{\alpha}{2}}\,\mathrm{d}u \\
&\asymp R^{\frac{1}{2}}\log^{-\frac{\alpha}{2}}(R)+R^{\frac{1}{2}}\frac{u^{1-\frac{\alpha}{2}}}{1-\frac{\alpha}{2}}\Big|_{\log \hbar(R)}^{\log t} \\
&\asymp R^{\frac{1}{2}}\log^{-\frac{\alpha}{2}}(R)+R^{\frac{1}{2}}\left(\log \hbar(R))^{1-\frac{\alpha}{2}}-((\log t)^{1-\frac{\alpha}{2}}\right) \\
&\asymp R^{\frac{1}{2}}\log^{-\frac{\alpha}{2}}(R)+R^{\frac{1}{2}}\left(\left(\frac{1}{2}\log R-\frac{\alpha}{2}\log\log R\right)^{1-\frac{\alpha}{2}}-(\log t)^{1-\frac{\alpha}{2}}\right).
\end{aligned} \tag{5.30}$$

<u>$\Delta_l = 1, \mu = \nu = 0$</u> In this instance, define

$$T = T(R) := \exp\left(\left(\frac{R}{(\log R)^{2-\alpha}}\right)^{\frac{1}{2\alpha_l}}\right).$$

This leads to

$$\begin{aligned}
\mathfrak{g}(T,R) &\asymp R^{\frac{1}{2}}\log^{-\frac{\alpha}{2}}(R)+R^{\frac{1}{2}}\left(\left(\frac{1}{2}\log R-\frac{\alpha}{2}\log\log R\right)^{1-\frac{\alpha}{2}}-(\log t)^{1-\frac{\alpha}{2}}\right) \\
&\asymp R^{\frac{1}{2}}\log^{-\frac{\alpha}{2}}(R)+R^{\frac{1}{2}}\left(\left(\frac{1}{2}\log R-\frac{\alpha}{2}\log\log R\right)^{1-\frac{\alpha}{2}}-\left(\frac{R}{(\log R)^{2-\alpha}}\right)^{\frac{1-\frac{\alpha}{2}}{2\alpha_l}}\right) \\
&\asymp R^{\frac{1}{2}}(\log R)^{1-\frac{\alpha}{2}},
\end{aligned}$$

as $\frac{1-\frac{\alpha}{2}}{2\alpha_l} < 0$. Further, it holds that

$$R\mathfrak{n}(T) = R\log^{-\alpha_l}(T) = R\left(\frac{R}{(\log R)^{2-\alpha}}\right)^{-\frac{\alpha_l}{2\alpha_l}} = R^{\frac{1}{2}}(\log R)^{1-\frac{\alpha}{2}}.$$

$\underline{1 < \Delta_l + \frac{\mu+\nu}{2} \leq 2}$ Similarly to before, we set

$$\mathfrak{q}(t) := t^{2\gamma}\log^{2\alpha_l}(t)\left(-(\log t)^{1-\frac{\alpha}{2}} + \log^{-\frac{\alpha}{2}}(R) + \left(\frac{1}{2}\log R - \frac{\alpha}{2}\log\log R\right)^{1-\frac{\alpha}{2}}\right)^2,$$

which yields

$$\mathfrak{q}^{-1}(t) \asymp \frac{t^{\frac{1}{2\gamma}}}{\log^{\frac{\alpha_l}{\gamma}}(t)\left(-\left(\frac{1}{2\gamma}\log t\right)^{1-\frac{\alpha}{2}} + \log^{-\frac{\alpha}{2}}(R) + \left(\frac{1}{2}\log R - \frac{\alpha}{2}\log\log R\right)^{1-\frac{\alpha}{2}}\right)^{\frac{1}{\gamma}}},$$

and just as before, we set

$$\begin{aligned} T = T(R) &:= \mathfrak{q}^{-1}(R) \\ &\asymp \frac{R^{\frac{1}{2\gamma}}}{\log^{\frac{\alpha_l}{\gamma}}(R)\left(-\left(\frac{1}{2\gamma}\log R\right)^{1-\frac{\alpha}{2}} + \log^{-\frac{\alpha}{2}}(R) + \left(\frac{1}{2}\log R - \frac{\alpha}{2}\log\log R\right)^{1-\frac{\alpha}{2}}\right)^{\frac{1}{\gamma}}}, \end{aligned}$$

from which we can conclude that

$$\log T \sim \frac{1}{2\gamma}\log R$$

in the same way as above. Now, we calculate

$$\begin{aligned} \mathfrak{g}(T,R) &\asymp R^{\frac{1}{2}}\log^{-\frac{\alpha}{2}}(R) + R^{\frac{1}{2}}\left(\left(\frac{1}{2}\log R - \frac{\alpha}{2}\log\log R\right)^{1-\frac{\alpha}{2}} - (\log T)^{1-\frac{\alpha}{2}}\right) \\ &\sim R^{\frac{1}{2}}\log^{-\frac{\alpha}{2}}(R) + R^{\frac{1}{2}}\left(\left(\frac{1}{2}\log R\right)^{1-\frac{\alpha}{2}} - \left(\frac{1}{2\gamma}\log R\right)^{1-\frac{\alpha}{2}}\right) \end{aligned}$$

as well as

$$\begin{aligned} R\mathfrak{n}(T) &= RT^{-\gamma}\log^{-\alpha_l}(T) \\ &\sim R^{\frac{1}{2}}\log^{\alpha_l}(R)\left(-\left(\frac{1}{2\gamma}\log R\right)^{1-\frac{\alpha}{2}}+\log^{-\frac{\alpha}{2}}(R)\right. \\ &\qquad\left.+\left(\frac{1}{2}\log R-\frac{\alpha}{2}\log\log R\right)^{1-\frac{\alpha}{2}}\right)\left(\frac{1}{2\gamma}\log R\right)^{-\alpha_l} \\ &\asymp R^{\frac{1}{2}}\left(-\left(\frac{1}{2\gamma}\log R\right)^{1-\frac{\alpha}{2}}+\log^{-\frac{\alpha}{2}}(R)+\left(\frac{1}{2}\log R\right)^{1-\frac{\alpha}{2}}\right), \end{aligned}$$

which lets us make the same conclusions as in (i).

(iii) The substitution from the calculation of (5.13) shows that

$$(*)\asymp R^{\frac{1}{2}}\log^{-1}(R)+R^{\frac{1}{2}}(\log\log t-\log\log\mathfrak{k}(R)). \tag{5.31}$$

$\underline{\Delta_l=1,\mu+\nu=0}$ In this instance, we use

$$T=T(R):=\exp\left(\left(\frac{R}{\log^2(R)}\right)^{\frac{1}{2\alpha_l}}\right).$$

With this choice of $T$, we see that

$$\begin{aligned} \mathfrak{g}(T,R) &\asymp R^{\frac{1}{2}}\log^{-1}(R)+R^{\frac{1}{2}}(\log\log t-\log\log\mathfrak{k}(R)) \\ &= R^{\frac{1}{2}}\log^{-1}(R)+R^{\frac{1}{2}}\left(\frac{1}{2\alpha_l}(\log R-2\log\log R)\right. \\ &\qquad\left.-\log\left(\frac{1}{2}(\log R-2\log\log R)\right)\right) \\ &\asymp R^{\frac{1}{2}}\log R \end{aligned}$$

and

$$R\mathfrak{n}(T)=R\log^{-\alpha_l}(T)=R\left(\frac{R}{\log^2(R)}\right)^{-\frac{\alpha_l}{2\alpha_l}}=R^{\frac{1}{2}}\log R.$$

$\underline{1<\Delta_l+\frac{\mu+\nu}{2}<2}$ Define

$$\mathfrak{r}(t):=t^{2\gamma}\log^{2\alpha_l}(t)\left(\log\log t+\log^{-1}(R)+\log 2-\log\log R\right)^2.$$

By Lemma 4.21, we have

$$\begin{aligned}
r^{-1}(t) &\asymp \frac{t^{\frac{1}{2\gamma}}}{\log^{\frac{\alpha_l}{\gamma}}(t^{\frac{1}{2\gamma}})\left(\log\log\left(t^{\frac{1}{2\gamma}}\right)+\log^{-1}(R)+\log 2-\log\log R\right)^{\frac{1}{\gamma}}} \\
&\asymp \frac{t^{\frac{1}{2\gamma}}}{\left(\frac{1}{2\gamma}\right)^{\frac{\alpha_l}{\gamma}}\log^{\frac{\alpha_l}{\gamma}}(t)\left(\log\frac{1}{2\gamma}\log\log(t)+\log^{-1}(R)+\log 2-\log\log R\right)^{\frac{1}{\gamma}}} \\
&\asymp \frac{t^{\frac{1}{2\gamma}}}{\log^{\frac{\alpha_l}{\gamma}}(t)\left(\log\log(t)+\log^{-1}(R)-\log\gamma-\log\log R\right)^{\frac{1}{\gamma}}}.
\end{aligned}$$

Setting

$$\begin{aligned}
T = T(R) \coloneqq r^{-1}(R) &\asymp \frac{R^{\frac{1}{2\gamma}}}{\log^{\frac{\alpha_l}{\gamma}}(R)\left(\log\log R+\log^{-1}(R)-\log\gamma-\log\log R\right)^{\frac{1}{\gamma}}} \\
&= \frac{R^{\frac{1}{2\gamma}}}{\log^{\frac{\alpha_l}{\gamma}}(R)\left(\log^{-1}(R)-\log\gamma\right)^{\frac{1}{\gamma}}}
\end{aligned}$$

and calculating $g(T,R)$ as well as $Rn(T)$ shows that

$$\begin{aligned}
g(T,R) &= R^{\frac{1}{2}}\log^{-1}(R)+R^{\frac{1}{2}}\left(\log\log T-\log\log k(R)\right) \\
&\asymp R^{\frac{1}{2}}\log^{-1}(R)+R^{\frac{1}{2}}\log\left(\frac{\log T}{\log k(R)}\right) \\
&\asymp R^{\frac{1}{2}}\log^{-1}(R) \\
&\quad + R^{\frac{1}{2}}\log\left(\frac{\frac{1}{2\gamma}\left(\log R-2\alpha_l\log\log R-2\log\left(\log^{-1}(R)-\log\gamma\right)\right)}{\frac{1}{2}\left(\log R-2\log\log R\right)}\right) \\
&\asymp R^{\frac{1}{2}}\log^{-1}(R)-R^{\frac{1}{2}}\log\gamma+R^{\frac{1}{2}}\log\left(\frac{\log R-2(\alpha_l-1)\log\log R}{\log R-2\log\log R}\right) \\
&\asymp R^{\frac{1}{2}},
\end{aligned}$$

as $\log\gamma < 0$ and

$$\begin{aligned}
Rn(T) &= RT^{-\gamma}\log^{-\alpha_l}(T) \\
&\asymp R\frac{\log^{\alpha_l}(R)\left(\log^{-1}(R)-\log\gamma\right)}{R^{\frac{1}{2}}}\cdot\gamma^{\alpha_l} \\
&\quad\cdot\left(\frac{1}{2}\log R-\alpha_l\log\log R-\log\left(\log^{-1}(R)-\log\gamma\right)\right)^{-\alpha_l} \\
&\asymp R^{\frac{1}{2}}\left(\log^{-1}(R)-\log\gamma\right) \\
&\asymp R^{\frac{1}{2}}.
\end{aligned}$$

$\underline{\Delta_l + \frac{\mu+\nu}{2} = 2}$ Just as above, we have

$$(*) \asymp R^{\frac{1}{2}} \log^{-1}(R) + R^{\frac{1}{2}} (\log\log t - \log\log \hbar(R)),$$

where $\hbar(R) = R^{\frac{1}{2}} \log^{-1}(R)$. Define the function

$$\mathfrak{s}(t) := t \log^{\alpha_l}(t) \left(\log\log t - \log\log(\hbar(R))\right),$$

and note that by Lemma 4.21 we have

$$\mathfrak{s}^{-1}(t) \asymp \frac{t}{\log^{\alpha_l}(t)(\log\log t - \log\log(\hbar(R)))}.$$

Set

$$\begin{aligned} T = T(R) := \mathfrak{s}^{-1}\left(R^{\frac{1}{2}}\right) &\asymp \frac{R^{\frac{1}{2}}}{\log^{\alpha_l}\left(R^{\frac{1}{2}}\right)\left(\log\log\left(R^{\frac{1}{2}}\right) - \log\log\left(\hbar\left(R\right)\right)\right)} \\ &\asymp \frac{R^{\frac{1}{2}}}{\log^{\alpha_l}(R)\left(\log\log\left(R^{\frac{1}{2}}\right) - \log\log\left(\hbar(R)\right)\right)}. \end{aligned}$$

A quick calculation in combination with the utilisation of Remark 5.14 now shows that

$$\begin{aligned} R\mathfrak{n}(T) &= RT^{1-\Delta_l-\frac{\mu+\nu}{2}} \log^{-\alpha_l}(T) = RT^{-1}\log^{-\alpha_l}(T) \\ &= R^{\frac{1}{2}} \log^{\alpha_l}(R)\left(\log\log\left(R^{\frac{1}{2}}\right) - \log\log\left(\hbar(R)\right)\right) \cdot \log^{-\alpha_l}(T) \\ &= R^{\frac{1}{2}} \frac{\log^{\alpha_l}(R)}{\log^{\alpha_l}(T)} \log\left(\frac{\log\left(R^{\frac{1}{2}}\right)}{\log\left(\hbar(R)\right)}\right) \\ &\asymp R^{\frac{1}{2}} \log\left(\frac{\log\left(R^{\frac{1}{2}}\right)}{\log\left(\hbar(R)\right)}\right) = R^{\frac{1}{2}} \log\left(\frac{\frac{1}{2}\log R}{\frac{1}{2}\log R - \log\log R}\right) \\ &= R^{\frac{1}{2}} \log\left(1 + \frac{\log\log R}{\frac{1}{2}\log R - \log\log R}\right) \sim R^{\frac{1}{2}} \frac{\log\log R}{\frac{1}{2}\log R - \log\log R} \\ &\asymp R^{\frac{1}{2}} \frac{\log\log R}{\log R - \log\log R} \sim R^{\frac{1}{2}} \frac{\log\log R}{\log R}. \end{aligned}$$

Calculating $\mathscr{g}(T,R)$ is significantly more work. We start with the computation

$$\begin{aligned}
\mathscr{g}(T,R) &\asymp R^{\frac{1}{2}}\log^{-1}(R) + R^{\frac{1}{2}}(\log\log T - \log\log \mathscr{k}(R)) \\
&= R^{\frac{1}{2}}\log^{-1}(R) + R^{\frac{1}{2}}\log\left(\frac{\log T}{\log \mathscr{k}(R)}\right) \\
&= R^{\frac{1}{2}}\log^{-1}(R) \\
&+ R^{\frac{1}{2}}\log\left(\frac{\frac{1}{2}\log R - \alpha_l \log\log R - \log\left(\log\log\left(R^{\frac{1}{2}}\right) - \log\log\left(R^{\frac{1}{2}}\log^{-1}(R)\right)\right)}{\frac{1}{2}\log R - \log\log R}\right) \\
&= R^{\frac{1}{2}}\log^{-1}(R) \\
&+ R^{\frac{1}{2}}\underbrace{\log\left(1 + \frac{(1-\alpha_l)\log\log R - \log\left(\log\log\left(R^{\frac{1}{2}}\right) - \log\log\left(R^{\frac{1}{2}}\log^{-1}(R)\right)\right)}{\frac{1}{2}\log R - \log\log R}\right)}_{\Omega}.
\end{aligned}$$

The task is now to estimate $\log(\Omega)$. For simplicity of notation, we write $c := 1-\alpha_l$. It is important to note that as we assumed

$$R^{\frac{1}{2}}\log^{-1}(R) \asymp \sigma_1 \ll \tau_1 \asymp R^{\frac{1}{2}}(\log R)^{\frac{\alpha}{\Delta}-\alpha_l} = R^{\frac{1}{2}}(\log R)^{1-\alpha_l},$$

it holds that $c = 1-\alpha_l > -1$. Consider the second summand in the nominator defining $\Omega$, which can be approximated as

$$\begin{aligned}
\log\left(\log\log\left(R^{\frac{1}{2}}\right) - \log\log\left(R^{\frac{1}{2}}\log^{-1}(R)\right)\right) &= \log\log\left(\frac{\frac{1}{2}\log R}{\frac{1}{2}\log R - \log\log R}\right) \\
&= \log\log\left(1 + \frac{\log\log R}{\frac{1}{2}\log R - \log\log R}\right) \sim \log\left(\frac{\log\log R}{\frac{1}{2}\log R - \log\log R}\right)
\end{aligned}$$

by Remark 5.14. Therefore, the entire nominator can be approximated in the following way.

$$\begin{aligned}
&c\log\log R - \log\left(\log\log\left(R^{\frac{1}{2}}\right) - \log\log\left(R^{\frac{1}{2}}\log^{-1}(R)\right)\right) \\
&\sim c\log\log R - \log\left(\frac{\log\log R}{\frac{1}{2}\log R - \log\log R}\right) \\
&= \log\left(\log^c(R)\cdot\frac{\frac{1}{2}\log R - \log\log R}{\log\log R}\right) \\
&\sim \log\left(\frac{\log^{1+c}(R)}{\log\log R}\right).
\end{aligned}$$

As we have

$$\frac{\mathrm{d}}{\mathrm{d}R}\log\left(\frac{\log^{1+c}(R)}{\log\log R}\right)=\frac{(1+c)\log\log R-1}{R\log R\log\log R}$$

as well as

$$\frac{\mathrm{d}}{\mathrm{d}R}(1+c)\log\log R=\frac{1+c}{R\log R},$$

L'Hôpital's rule shows that

$$\log\left(\frac{\log^{1+c}(R)}{\log\log R}\right)\sim(1+c)\log\log R.$$

Consequently, it holds that

$$\log(\Omega)\sim\log\left(1+\frac{\log\left(\frac{\log^{1+c}(R)}{\log\log R}\right)}{\frac{1}{2}\log R-\log\log R}\right)\sim\log\left(1+\frac{(1+c)\log\log R}{\frac{1}{2}\log R-\log\log R}\right)$$
$$\sim\frac{(1+c)\log\log R}{\frac{1}{2}\log R-\log\log R}\asymp\frac{\log\log R}{\frac{1}{2}\log R-\log\log R}\asymp\frac{\log\log R}{\log R}.$$

Finally, as

$$\frac{\log\log R}{\log R}\gg\log^{-1}(R),$$

we can conclude that

$$g(T,R)\asymp R^{\frac{1}{2}}\log^{-1}(R)+R^{\frac{1}{2}}\log(\Omega)\sim R^{\frac{1}{2}}\log(\Omega)\asymp R^{\frac{1}{2}}\frac{\log\log R}{\log R}.$$

□

## 5.3 Generic Cases

We begin the calculation of $B(R)$ in dependence of $\Delta_l$, $\Delta_\phi$, $\alpha_l$, $\alpha_\phi$, $\mu$ and $\nu$ by looking at the cases named A, B and C in the third chapter of [Wor], in which the application of Theorem 3.9 of said paper allows for a direct calculation.

**Lemma 5.16.** *In the situation described in the preliminary section of this chapter, the bound $B(R)$ can be approximated in the following ways.*

A *If $\Delta_\phi<\frac{\mu+\nu}{2}$, it holds that*

$$B(R)\asymp R^{\frac{1}{\Delta_l+\frac{\mu+\nu}{2}}}(\log R)^{1-\frac{\alpha_l+2}{2\Delta_l+\mu+\nu}}.$$

B *If $\Delta_\phi>\frac{\mu+\nu}{2}$ and $\Delta_l+\Delta_\phi>2$, it holds that*

$$B(R)\asymp R^{\frac{1}{\Delta_l+\Delta_\phi}}\log^{-\frac{\alpha}{\Delta_l+\Delta_\phi}}(R).$$

C *If $\Delta_\phi > \frac{\mu+\nu}{2}$, $\Delta_l + \Delta_\phi < 2$ and $\neg(\Delta_l = 1 \wedge \mu = \nu = 0)$, it holds that*

$$B(R) \asymp R^{\frac{1-\Delta_\phi+\frac{\mu+\nu}{2}}{\tilde{\Delta}+\mu+\nu}} (\log R)^{\frac{\alpha_\phi(1-\Delta_l-\frac{\mu+\nu}{2})-\alpha_l(1-\Delta_\phi+\frac{\mu+\nu}{2})}{\tilde{\Delta}+\mu+\nu}}.$$

*Proof.*

A The function $\jmath(t)$ defined in Theorem 3.5 in [Wor] is of the form

$$\jmath(t) = t^{\Delta_l+\frac{\mu+\nu}{2}} \log^{\frac{\alpha_l}{2}}(t), \; e \leq t,$$

and thus by Example 4.22

$$\jmath^{-1}(R) \asymp R^{\frac{1}{\Delta_l+\frac{\mu+\nu}{2}}} \log^{-\frac{\alpha_l}{2\Delta_l+\mu+\nu}}(R).$$

This leads to

$$\begin{aligned}
\jmath^{-1}\left(\frac{R}{\log R}\right) &\asymp \left(\frac{R}{\log R}\right)^{\frac{1}{\Delta_l+\frac{\mu+\nu}{2}}} (\log R - \log\log R)^{-\frac{\alpha_l}{2\Delta_l+\mu+\nu}} \\
&\asymp R^{\frac{1}{\Delta_l+\frac{\mu+\nu}{2}}} (\log R)^{-\frac{1}{\Delta_l+\frac{\mu+\nu}{2}}-\frac{\alpha_l}{2\Delta_l+\mu+\nu}} \\
&= R^{\frac{1}{\Delta_l+\frac{\mu+\nu}{2}}} (\log R)^{-\frac{\alpha_l+2}{2\Delta_l+\mu+\nu}},
\end{aligned}$$

and subsequently

$$B(R) \asymp R^{\frac{1}{\Delta_l+\frac{\mu+\nu}{2}}} (\log R)^{1-\frac{\alpha_l+2}{2\Delta_l+\mu+\nu}}.$$

B By in [Wor, Theorem 3.5], it holds that

$$B(R) \asymp \mathscr{k}(R) \asymp R^{\frac{1}{\Delta}} \log^{-\frac{\alpha}{\Delta}}(R) = R^{\frac{1}{\Delta_l+\Delta_\phi}} \log^{-\frac{\alpha}{\Delta_l+\Delta_\phi}}(R).$$

C To determine $B(R)$ in the third case, we calculate

$$m(t) := t^2 \frac{(d_l d_\phi)(t)}{(c_l c_\phi)(t)} = \frac{t^{2-\Delta} \log^{-\alpha}(t)}{t^{2-2\Delta_l-\mu-\nu} \log^{-2\alpha_l}(t)} = t^{\tilde{\Delta}+\mu+\nu} \log^{\tilde{\alpha}}(t) \tag{5.32}$$

for $e \leq t$, with Example 4.22 implying that

$$m^{-1}(R) \asymp R^{\frac{1}{\tilde{\Delta}+\mu+\nu}} \log^{-\frac{\tilde{\alpha}}{\tilde{\Delta}+\mu+\nu}}(R),$$

and resulting in

$$\begin{aligned} B(R) &\asymp R(n \circ m^{-1})(R) = R \cdot n\left(R^{\frac{1}{\tilde{\Delta}+\mu+\nu}} \log^{-\frac{\tilde{\alpha}}{\tilde{\Delta}+\mu+\nu}}(R)\right) \\ &= R \cdot R^{\frac{1-\Delta_l-\frac{\mu+\nu}{2}}{\tilde{\Delta}+\mu+\nu}} \log^{-\frac{\tilde{\alpha}\cdot(1-\Delta_l-\frac{\mu+\nu}{2})}{\tilde{\Delta}+\mu+\nu}}(R) \\ &\quad \cdot \left(\frac{\log R}{\tilde{\Delta}+\mu+\nu} - \frac{\tilde{\alpha}}{\tilde{\Delta}+\mu+\nu}\log\log(R)\right)^{-\alpha_l} \\ &\asymp R^{\frac{1-\Delta_\phi+\frac{\mu+\nu}{2}}{\tilde{\Delta}+\mu+\nu}} (\log R)^{-\frac{\tilde{\alpha}\cdot(1-\Delta_l-\frac{\mu+\nu}{2})}{\tilde{\Delta}+\mu+\nu}-\alpha_l} \\ &= R^{\frac{1-\Delta_\phi+\frac{\mu+\nu}{2}}{\tilde{\Delta}+\mu+\nu}} (\log R)^{\frac{\alpha_\phi(1-\Delta_l-\frac{\mu+\nu}{2})-\alpha_l(1-\Delta_\phi+\frac{\mu+\nu}{2})}{\tilde{\Delta}+\mu+\nu}}. \end{aligned}$$

□

## 5.4 Edge Cases

Next, we will focus on the cases where [Wor] does not give an approximation of the bound $B(R)$ for general regularly varying decay functions. Hence, we will try to find estimates in the more specific case of the decay functions given in (5.1). In order to do so, we will consider all feasible configurations of $\Delta_l$, $\Delta_\phi$, $\alpha_l$, $\alpha_\phi$, $\mu$ and $\nu$ not yet covered in Lemma 5.16 and try to calculate matching bounds by applying the computations made in the Sections 5.1 as well as 5.2.

**Lemma 5.17.** *Given the case that $\Delta_\phi > \frac{\mu+\nu}{2}$, $\Delta_l + \Delta_\phi < 2$, $\Delta_l = 1$ and $\mu = \nu = 0$, it holds that*

$$B(R) \asymp R\log^{-\alpha_l}(R). \tag{5.33}$$

*Proof.* Note that we certainly have $\tilde{\Delta} > 0$, which implies the unboundedness of $\frac{d_\phi(t)}{d_l(t)}$. By Proposition 5.10, we have

$$\begin{aligned} \sigma_2 &\asymp R^{\frac{1-\Delta_\phi}{\tilde{\Delta}}} (\log R)^{\tilde{\alpha}\frac{\tilde{\Delta}-2}{2\tilde{\Delta}}-\frac{\alpha}{2}} = R(\log R)^{\tilde{\alpha}\frac{\Delta_\phi-1}{2(1-\Delta_\phi)}-\frac{\alpha}{2}} \\ &= R(\log R)^{-\frac{\tilde{\alpha}}{2}-\frac{\alpha}{2}} = R\log^{-\alpha_l}(R) \end{aligned}$$

and therefore

$$\sigma_1 + \sigma_2 \asymp R^{\frac{1}{\Delta}} \log^{-\frac{\alpha}{\Delta}}(R) + R\log^{-\alpha_l}(R) \asymp R\log^{-\alpha_l}(R),$$

as well as

$$\tau_2 \asymp R^{\frac{1-\Delta_\phi-\frac{\mu+\nu}{2}}{\tilde{\Delta}}} (\log R)^{\tilde{\alpha}\frac{\Delta_l+\frac{\mu+\nu}{2}-1}{\tilde{\Delta}}-\alpha_l} = R\log^{-\alpha_l}(R).$$

Hence, by Remark 5.7, we can conclude that setting $T(R) := \hbar(R)$ leads to $B(R) \asymp \sigma_1 + \sigma_2 \asymp \tau_2 \asymp R\log^{-\alpha_l}(R)$. □

**Theorem 5.18.** *In the instance that* $\Delta_l + \Delta_\phi = 2$ *as well as* $\Delta_\phi > \frac{\mu+\nu}{2}$*, the following bounds occur.*
*If* $\Delta_l = \Delta_\phi = 1,\ \mu = \nu = 0$ *and* $\alpha_l + \alpha_\phi < 2$*, it holds that*

$$B(R) \asymp R^{\frac{2-\alpha_\phi}{2+\alpha_l-\alpha_\phi}}.$$

*If* $\Delta_l = \Delta_\phi = 1,\ \mu = \nu = 0$ *and* $\alpha_l + \alpha_\phi = 2$*, it holds that*

$$B(R) \asymp R^{\frac{1}{2}} \log R.$$

*In any other feasible case, it holds that*

$$B(R) \asymp R^{\frac{1}{2}} (\log R)^{1-\frac{\alpha_l+\alpha_\phi}{2}}.$$

*Proof.* At first, consider the case that $\frac{d_\phi(t)}{d_l(t)}$ is bounded, which is equivalent to $\tilde{\Delta} = 0$ and $\tilde{\alpha} \leq 0$, as $\Delta_l + \Delta_\phi = 2$ implies $\Delta_\phi \leq \tilde{\Delta}$. If $\mu + \nu > 0$, Lemma 5.15 tells us that $B(R) \asymp R^{\frac{1}{2}} (\log R)^{1-\frac{\alpha}{2}}$. In case of $\mu = \nu = 0$, we have to differentiate the cases $\alpha < 2, \alpha > 2$ and $\alpha = 2$, with the corresponding bounds $B(R) \asymp R^{\frac{2-\alpha_\phi}{2+\alpha_l-\alpha_\phi}}, B(R) \asymp R^{\frac{1}{2}} (\log R)^{1-\frac{\alpha}{2}}$ and $B(R) \asymp R^{\frac{1}{2}} \log R$ being given by Lemma 5.15.
We now turn to the case that $\tilde{\Delta} > 0$, which implies the unboundedness of $\frac{d_\phi(t)}{d_l(t)}$. By Proposition 5.8, we have

$$\tau_2 \asymp R^{\frac{1-\Delta_\phi-\frac{\mu+\nu}{2}}{\Delta_l-\Delta_\phi}} (\log R)^{\tilde{\alpha}\frac{\Delta_l+\frac{\mu+\nu}{2}-1}{\tilde{\Delta}}-\alpha_l}.$$

Further, as $\Delta_\phi > \frac{\mu+\nu}{2} \geq 0$, Proposition 5.8 in combination with Proposition 5.10 implies that $\sigma_1+\sigma_2 \asymp R^{\frac{1}{2}} (\log R)^{1-\frac{\alpha}{2}}$. In consequence, it clearly holds for $\mu+\nu > 0$ that $\tau_2 \ll \sigma_1+\sigma_2$, and Lemma 5.15 yields the asserted estimation. For $\mu = \nu = 0$, we have to compare the exponents of the logarithmic terms. As

$$\tilde{\alpha}\frac{\Delta_l - 1}{\tilde{\Delta}} - \alpha_l = \tilde{\alpha}\frac{\Delta_l - 1}{2\Delta_l - 2} - \alpha_l = \frac{\tilde{\alpha}}{2} - \alpha_l = -\frac{\alpha}{2},$$

it still holds that $\tau_2 \ll \sigma_1 + \sigma_2$ and Lemma 5.15 is once more applicable.

Lastly, we consider the case that $\tilde{\Delta} = 0$ and $\tilde{\alpha} > 0$, which also results in the unboundedness of $\frac{d_\phi(t)}{d_l(t)}$, but yielding a substantially different function $\hbar(R)$, namely $\hbar(R) = \exp(R^{\frac{1}{\tilde{\alpha}}})$. By Proposition 5.8, we have

$$\tau_2 \asymp \exp\left(\left(1 - \Delta_l - \frac{\mu+\nu}{2}\right) R^{\frac{1}{\tilde{\alpha}}}\right) R^{-\frac{\alpha_\phi}{\tilde{\alpha}}} = \exp\left(\left(-\frac{\mu+\nu}{2}\right) R^{\frac{1}{\tilde{\alpha}}}\right) R^{-\frac{\alpha_\phi}{\tilde{\alpha}}},$$

while by the Propositions 5.8 and 5.10, $\sigma_1 + \sigma_2$ is of the form

$$\begin{aligned} \sigma_1 + \sigma_2 \asymp \sigma_2 \asymp R^{\frac{1-\alpha_\phi}{\alpha_l - \alpha_\phi}} \qquad & \textit{if } \alpha < 2, \\ \sigma_1 + \sigma_2 \asymp \sigma_2 \asymp R^{\frac{1}{2}} (\log R)^{1-\frac{\alpha}{2}} \qquad & \textit{if } \alpha > 2, \\ \sigma_1 + \sigma_2 \asymp \sigma_2 \asymp R^{\frac{1}{2}} \log R \qquad & \textit{if } \alpha = 2. \end{aligned}$$

At first, we look into the case that $\alpha < 2$. If $\mu + \nu > 0$, $\tau_2$ is decreasing at an exponential speed, resulting in $\tau_2 \ll \sigma_1 + \sigma_2$. As $\frac{1-\alpha_\phi}{\alpha_l - \alpha_\phi} > -\frac{\alpha_\phi}{\alpha_l - \alpha_\phi}$ however, $\mu = \nu = 0$ also leads to $\tau_2 \ll \sigma_1 + \sigma_2$. In the first instance, Lemma 5.15 yields $B(R) \asymp R^{\frac{1}{2}} (\log R)^{1-\frac{\alpha}{2}}$, while the second case results in $B(R) \asymp R^{\frac{2-\alpha_\phi}{2+\alpha_l - \alpha_\phi}}$.
With the same arguments used in the former case, $\mu + \nu > 0$ also leads to $\tau_2 \ll \sigma_1 + \sigma_2$ for $\alpha > 2$. In case of $\mu = \nu = 0$, we need a comparison of the exponents $\frac{1}{2}$ and $-\frac{\alpha_\phi}{\tilde{\alpha}}$. As $\frac{1}{2} > -\frac{\alpha_\phi}{\tilde{\alpha}} \Leftrightarrow \tilde{\alpha} > -2\alpha_\phi \Leftrightarrow \alpha > 0$, we have $\tau_2 \ll \sigma_1 + \sigma_2$ once again. In both cases, the application of Lemma 5.15 shows $B(R) \asymp R^{\frac{1}{2}} \log^{1-\frac{\alpha}{2}}(R)$.
To complete the proof, we look into the case that $\alpha = 2$. Just as above, we get $\tau_2 \ll \sigma_1 + \sigma_2$, with Lemma 5.15 yielding $B(R) \asymp R^{\frac{1}{2}}$ for $\mu + \nu > 0$ and $B(R) \asymp R^{\frac{1}{2}} \log R$ otherwise. ☐

**Theorem 5.19.** *Given the situation that $\Delta_\phi = \frac{\mu+\nu}{2}$, $B(R)$ can be approximated as follows. If $\alpha_\phi < 0$, we have*

$$\max \left\{ R^{\frac{1}{\Delta}} (\log R)^{-\frac{\alpha}{\Delta} + \alpha_\phi}, R^{\frac{1}{\Delta}} (\log R)^{\frac{\alpha}{\Delta(\Delta-1)} - \frac{\alpha_l}{\Delta - 1}} \log \log R \right\} \lesssim B(R) \lesssim R^{\frac{1}{\Delta}} (\log R)^{-\frac{\alpha}{\Delta}}. \tag{5.34}$$

*If $\alpha_\phi = 0$, it holds that*

$$B(R) \asymp R^{\frac{1}{\Delta}} \log^{-\frac{\alpha_l}{\Delta}}(R). \tag{5.35}$$

*If $\alpha_\phi > 0$, the following cases occur.*

- $\Delta_l + \Delta_\phi < 2$ *leads to*
$$B(R) \asymp R^{\frac{1}{\Delta}} (\log R)^{-\frac{\tilde{\alpha}}{\Delta} - \alpha_\phi}, \tag{5.36}$$
- $\Delta_l + \Delta_\phi > 2$ *leads to*
$$B(R) \asymp R^{\frac{1}{\Delta}} \log^{-\frac{\alpha}{\Delta}}(R), \tag{5.37}$$
- $\Delta_l + \Delta_\phi = 2$ *and* $\alpha_l + \alpha_\phi \neq 2$ *leads to*
$$B(R) \asymp R^{\frac{1}{2}} \log^{-\frac{\alpha}{2}}(R), \tag{5.38}$$
- $\Delta_l + \Delta_\phi = 2$ *and* $\alpha_l + \alpha_\phi = 2$ *leads to*
$$B(R) \asymp R^{\frac{1}{2}} \log^{-1}(R) \log \log R. \tag{5.39}$$

**Remark 5.20.** The list of bounds stated in the theorem seems quite extensive, however, when leaving out the cases that either $\alpha_\phi < 0$ or $\Delta_l + \Delta_\phi = 2 \wedge \alpha_l + \alpha_\phi = 2$, the remaining bounds are matching perfectly. Setting $\alpha_\phi = 0$ in any of the right-hand sides of (5.36)-(5.38) results in the right-hand side of (5.35), and analogously, setting $\Delta = 2$ in the right-hand side of (5.36) or (5.37) results in (5.38).

*Proof.* By Proposition 5.8, the first two thresholds we need to consider are of the form

$$\sigma_1 \asymp R^{\frac{1}{\Delta}} \log^{-\frac{\alpha}{\Delta}}(R)$$

and

$$\tau_1 \asymp R^{\frac{1+\Delta_\phi-\frac{\mu+\nu}{2}}{\Delta}} (\log R)^{\alpha\frac{\Delta_l+\frac{\mu+\nu}{2}-1}{\Delta}-\alpha_l} = R^{\frac{1}{\Delta}} (\log R)^{-\frac{\alpha}{\Delta}+\alpha_\phi}.$$

Just like in the proof of Theorem 5.18, we need to distinguish a multitude of cases, starting with differentiating the instances $\alpha_\phi = 0$, $\alpha_\phi < 0$ and $\alpha_\phi > 0$. In the first case, we see that $\sigma_1 \asymp \tau_1$ and therefore $T(R) \asymp \hbar(R)$ as well as $B(R) \asymp \sigma_1 \asymp R^{\frac{1}{\Delta}} \log^{-\frac{\alpha_l}{\Delta}}(R)$. In the second case we have $\tau_1 \ll \sigma_1$ and Lemma 5.11 tells us that

$$\max\left\{R^{\frac{1}{\Delta}}(\log R)^{-\frac{\alpha}{\Delta}+\alpha_\phi}, R^{\frac{1}{\Delta}}(\log R)^{\frac{\alpha}{\Delta(\Delta-1)}-\frac{\alpha_l}{\Delta-1}} \log\log R\right\} \lesssim B(R) \lesssim R^{\frac{1}{\Delta}}(\log R)^{-\frac{\alpha}{\Delta}}.$$

Hence, we only need to focus on the case that $\alpha_\phi > 0$ from now on. Here, we have

$$-\frac{\alpha}{\Delta} < -\frac{\alpha}{\Delta} + \alpha_\phi = -\frac{\alpha}{\Delta} + \alpha\frac{\Delta_l + \frac{\mu+\nu}{2}}{\Delta} - \alpha_l = \alpha\frac{\Delta_l + \frac{\mu+\nu}{2} - 1}{\Delta} - \alpha_l,$$

and thus $\sigma_1 \ll \tau_1$.
We start our examination of the instance that $\alpha_\phi > 0$ by taking a closer look into the case that further $\frac{d_\phi(t)}{d_l(t)}$ is bounded, meaning that either $\tilde{\Delta} < 0$ or $\tilde{\Delta} = 0 \wedge \tilde{\alpha} \leq 0$. In particular, this means that $\Delta = \Delta_l + \Delta_\phi \geq 2$.
If $\Delta > 2$, Lemma 5.13 tells us that $B(R) \asymp R^{\frac{1}{\Delta}} \log^{-\frac{\alpha}{\Delta}}(R)$, and in the event that $\Delta = 2$, Lemma 5.15 yields $B(R) \asymp R^{\frac{1}{2}} \log^{-\frac{\alpha}{2}}(R)$ for $\alpha_l + \alpha_\phi \neq 2$ and $B(R) \asymp R^{\frac{1}{2}} \log^{-1}(R) \log\log R$ for $\alpha_l + \alpha_\phi = 2$.

Now consider the instance that $\tilde{\Delta} > 0$, where $\frac{d_\phi(t)}{d_l(t)}$ is unbounded, and we have

$$\tau_2 \asymp R^{\frac{1-\Delta_\phi-\frac{\mu+\nu}{2}}{\tilde{\Delta}}} (\log R)^{\tilde{\alpha}\frac{\Delta_l+\frac{\mu+\nu}{2}-1}{\tilde{\Delta}}-\alpha_l} = R^{\frac{1-2\Delta_\phi}{\tilde{\Delta}}} (\log R)^{\tilde{\alpha}\frac{\Delta_l+\Delta_\phi-1}{\tilde{\Delta}}-\alpha_l} \tag{5.40}$$

by Proposition 5.8. Start by looking into the case that $\Delta < 2$, where we can directly apply Lemma 5.12, which yields

$$B(R) \asymp R^{\frac{1-\Delta_\phi+\frac{\mu+\nu}{2}}{\Delta_l-\Delta_\phi+\mu+\nu}} (\log R)^{\frac{\alpha_\phi(1-\Delta_l-\frac{\mu+\nu}{2})-\alpha_l(1-\Delta_\phi+\frac{\mu+\nu}{2})}{\Delta_l-\Delta_\phi+\mu+\nu}} = R^{\frac{1}{\tilde{\Delta}}}(\log R)^{-\frac{\tilde{\alpha}}{\tilde{\Delta}}-\alpha_\phi}.$$

Next, we look into the case that $\Delta > 2$. From (5.8), we know that $\sigma_2 \lesssim R^{\frac{1}{\Delta}} \log^{-\frac{\alpha}{\Delta}}(R)$, implying that $\sigma_1 + \sigma_2 \asymp \sigma_1 \asymp R^{\frac{1}{\Delta}} \log^{-\frac{\alpha}{\Delta}}(R)$. As $\frac{1-2\Delta_\phi}{\tilde{\Delta}} = \frac{1-2\Delta_\phi}{\Delta_l-\Delta_\phi}$ is strictly decreasing with respect to $\Delta_\phi$, we see that $\tau_2 \ll \sigma_1 + \sigma_2$ for $\Delta_\phi > 0$, with Lemma 5.13 yielding $B(R) \asymp R^{\frac{1}{\Delta}} \log^{-\frac{\alpha}{\Delta}}(R)$. In the case that $\Delta_\phi = \frac{\mu+\nu}{2} = 0$, we once again have to compare the exponents of the logarithmic terms in $\sigma_1 + \sigma_2$ and $\tau_2$, namely $-\frac{\alpha}{\Delta_l}$ and $\tilde{\alpha}\frac{\Delta_l-1}{\Delta_l} - \alpha_l$. It shows that

$$-\frac{\alpha}{\Delta_l} \gtrless \tilde{\alpha}\frac{\Delta_l - 1}{\Delta_l} - \alpha_l \Leftrightarrow 0 \gtrless \alpha_\phi \left(\frac{2}{\Delta_l} - 1\right),$$

and as we assumed that both $\alpha_\phi > 0$ and $\Delta = \Delta_l + 0 > 2$, we can conclude $\tau_2 \ll \sigma_1 + \sigma_2$, with Lemma 5.13 yielding the assertion.
The last remaining instance (under the assumption that $\tilde{\Delta} > 0$) is that $\Delta = 2$. In this case, (5.40) can be simplified to

$$\tau_2 \asymp R^{\frac{1-2\Delta_\phi}{2-2\Delta_\phi}} \log^{\frac{\tilde{\alpha}}{2-2\Delta_\phi} - \alpha_l}(R), \tag{5.41}$$

where it is important to note that $\frac{1-2\Delta_\phi}{2-2\Delta_\phi} \leq \frac{1}{2}$ and $\frac{1-2\Delta_\phi}{2-2\Delta_\phi} = \frac{1}{2} \Leftrightarrow \Delta_\phi = 0$. Meanwhile, the explicit form of $\sigma_2$ depends on whether $\Delta_\phi$ vanishes or not.
Starting with $\Delta_\phi = \frac{\mu+\nu}{2} > 0$, Proposition 5.10 tells us $\sigma_2 \asymp R^{\frac{1}{2}}(\log R)^{1-\frac{\alpha}{2}}$, which results in

$$\sigma_1 + \sigma_2 \asymp R^{\frac{1}{2}}(\log R)^{1-\frac{\alpha}{2}}. \tag{5.42}$$

As $\frac{1-2\Delta_\phi}{2-2\Delta_\phi} < \frac{1}{2}$, this instance results in $\tau_2 \ll \sigma_1 + \sigma_2$, with Lemma 5.15 yielding $B(R) \asymp R^{\frac{1}{2}} \log^{-\frac{\alpha}{2}}(R)$ for $\alpha_l + \alpha_\phi \neq 2$ and $B(R) \asymp R^{\frac{1}{2}} \log^{-1}(R) \log\log R$ for $\alpha_l + \alpha_\phi = 2$. The case that $\Delta_\phi = \mu + \nu = 0$ leads to

$$\tau_2 \asymp R^{\frac{1}{2}}(\log R)^{\tilde{\alpha}\frac{2-1}{\Delta_l} - \alpha_l} = R^{\frac{1}{2}} \log^{-\frac{\alpha}{2}}(R)$$

by (5.4). As $\tau_2 \asymp \sigma_1$, only $\tau_2 \asymp \sigma_1 + \sigma_2$ and $\tau_2 \ll \sigma_1 + \sigma_2$ can occur. By Remark 5.7 and Lemma 5.15, this always yields $B(R) \asymp R^{\frac{1}{2}} \log^{-\frac{\alpha}{2}}(R)$, except for the case that $\alpha_l + \alpha_\phi = 2$, which results in $B(R) \asymp R^{\frac{1}{2}} \log^{-1}(R) \log\log R$.

The only instance remaining for us to complete the proof is that in which $\tilde{\Delta} = 0$ and $\tilde{\alpha} > 0$, once more making $\frac{d_\phi(t)}{d_l(t)}$ unbounded. In contrast to the case before, however, we have

$$\tau_2 \sim \exp\left(\left(1 - \Delta_l - \frac{\mu+\nu}{2}\right) R^{\frac{1}{\tilde{\alpha}}}\right) R^{\frac{-\alpha_\phi}{\tilde{\alpha}}} = \exp\left((1 - 2\Delta_l) R^{\frac{1}{\tilde{\alpha}}}\right) R^{-\frac{\alpha_\phi}{\tilde{\alpha}}},$$

meaning that $\tau_2$ is decreasing at an exponential speed, always resulting in $\tau_2 \ll \sigma_1 + \sigma_2$. Nonetheless, it is still necessary to distinguish a few cases in order to calculate the bounds $B(R)$. If $\Delta < 2$, we can apply Lemma 5.12 and see that

$$\begin{aligned} B(R) &\asymp R^{\frac{1-\Delta_\phi+\frac{\mu+\nu}{2}}{\Delta_l - \Delta_\phi + \mu + \nu}} (\log R)^{\frac{\alpha_\phi(1-\Delta_l - \frac{\mu+\nu}{2}) - \alpha_l(1-\Delta_\phi + \frac{\mu+\nu}{2})}{\Delta_l - \Delta_\phi + \mu + \nu}} = R^{\frac{1}{2\Delta_\phi}} (\log R)^{\frac{\alpha_\phi(1-2\Delta_\phi) - \alpha_l}{2\Delta_\phi}} \\ &= R^{\frac{1}{\Delta}} (\log R)^{-\frac{\tilde{\alpha}}{\Delta} - \alpha_\phi}, \end{aligned}$$

while in case of $\Delta > 2$, Lemma 5.13 yields $B(R) \asymp R^{\frac{1}{2}} \log^{-\frac{\alpha}{\Delta}}(R)$.
Finally, if $\Delta = 2$, Lemma 5.15 once more lets us conclude that $B(R) \asymp R^{\frac{1}{2}} \log^{-\frac{\alpha}{2}}(R)$ if $\alpha_l + \alpha_\phi \neq 2$ and $B(R) \asymp R^{\frac{1}{2}} \log^{-1}(R) \log\log R$ otherwise. □

## 5.5 Comparison to estimates obtained by Harald Woracek

Within [Wor, Theorem 3.12], the author gives lower and upper bounds for $B(R)$ in three scenarios, denoted by $\boxed{\text{D}}$, $\boxed{\text{E}}$ and $\boxed{\text{E}^+}$. To test their accuracy and to gain insight into whether these bounds could be chosen narrower in the situation of the current chapter, we will compare them to the actual values $B(R)$ attains in these three scenarios.

$\boxed{\text{D}}$ In the first of the three cases, the theorem states that $\Delta_\phi = \frac{\mu+\nu}{2}$ leads to

$$\min\{\mathscr{k}(R), R(\mathscr{n} \circ \mathscr{k})(R)\} \lesssim B(R) \lesssim \max\{\mathscr{k}(R), R(\mathscr{n} \circ \mathscr{k})(R)\}.$$

In our terminology, this can be rewritten to

$$\min\{\sigma_1, \tau_1\} \lesssim B(R) \lesssim \max\{\sigma_1, \tau_1\},$$

as $\sigma_1 \asymp \mathscr{k}(R)$. To check whether or not these bounds are actually attained in a nontrivial way in our example, we look into Theorem 5.19. From its proof, we can see that in the given situation of the Theorem, it holds that

$$\mathscr{k}(R) \asymp \sigma_1 \asymp R^{\frac{1}{\Delta}} \log^{-\frac{\alpha}{\Delta}}(R)$$

and

$$R(\mathscr{n} \circ \mathscr{k})(R) = \tau_1 \asymp R^{\frac{1}{\Delta}} (\log R)^{-\frac{\alpha}{\Delta}+\alpha_\phi}.$$

Applying the theorem in the event that $\alpha_\phi > 0, \Delta_l = 1$ and $\Delta_\phi = 0$, we see that

$$B(R) \asymp R(\log R)^{-\tilde{\alpha}-\alpha_\phi} = R(\log R)^{-\alpha_l} = R(\log R)^{-\frac{\alpha}{\Delta}+\alpha_\phi},$$

which means that

$$\mathscr{k}(R) \asymp R \log^{-\alpha}(R) \ll R(\log R)^{-\alpha_l} \asymp B(R) \asymp R(\mathscr{n} \circ \mathscr{k})(R).$$

If we apply the theorem in the case that $\alpha_\phi > 0$, $\Delta_l + \Delta_\phi \geq 2$ and $\neg(\Delta_l + \Delta_\phi = 2 \wedge \alpha_l + \alpha_\phi = 2)$, however, we receive

$$B(R) \asymp R^{\frac{1}{\Delta}} \log^{-\frac{\alpha}{\Delta}}(R),$$

which yields

$$\mathscr{k}(R) \asymp B(R) \ll R(\mathscr{n} \circ \mathscr{k})(R).$$

Hence, both the estimate from below and above in $\boxed{\text{D}}$ are actually attained. Moreover, the instance that $\alpha_\phi > 0$, $\Delta_l + \Delta_\phi = 2$ and $\alpha_l + \alpha_\phi = 2$ shows that the true value of $B(R)$ can also lie truly in between both bounds, as this case leads to

$$B(R) \asymp R^{\frac{1}{2}} \log^{-1}(R) \log\log R$$

and thereby

$$\mathscr{k}(R) \asymp R^{\frac{1}{2}} \log^{-1}(R) \ll R^{\frac{1}{2}} \log^{-1}(R) \log\log R \asymp B(R) \ll R^{\frac{1}{2}} (\log R)^{-1+\alpha_\phi} \asymp R(\mathscr{n} \circ \mathscr{k})(R).$$

E We continue by taking a look at the case that $\Delta_\phi > \frac{\mu+\nu}{2}$, $\Delta_l + \Delta_\phi = 2$ and $\neg(\Delta_l = 1 \wedge \mu = \nu = 0)$. Here, the theorem states that

$$\mathcal{k}(R) \lesssim B(R) \lesssim R^{\frac{1}{2}} \int_1^{m^{-1}(R)} (d_l d_\phi)^{\frac{1}{2}}(s)\, \mathrm{d}s. \tag{5.43}$$

Once again, we ask whether the bounds are being attained non-trivially. Therefor, it is important to note that as $\Delta_l + \Delta_\phi = 2$, the expression $\neg(\Delta_l = 1 \wedge \mu = \nu = 0)$ is equivalent to $\neg(\tilde{\Delta} = 0 \wedge \mu = \nu = 0)$. The function $m(t)$ is the same as in (5.32), meaning that

$$m(t) = t^{\tilde{\Delta}+\mu+\nu} \log^{\tilde{\alpha}}(t),$$

and as $\neg(\tilde{\Delta} = 0 \wedge \mu = \nu = 0)$, Example 4.22 tells us that

$$m^{-1}(R) \asymp R^{\frac{1}{\tilde{\Delta}+\mu+\nu}} \log^{-\frac{\tilde{\alpha}}{\tilde{\Delta}+\mu+\nu}}(R).$$

To actually calculate the integral, we need to distinguish between $\alpha \neq 2$ and $\alpha = 2$ once again. We start with

$$R^{\frac{1}{2}} \int_1^{m^{-1}(R)} (d_l d_\phi)^{\frac{1}{2}}(s)\, \mathrm{d}s = R^{\frac{1}{2}} + R^{\frac{1}{2}} \int_e^{m^{-1}(R)} s^{-1} \log^{-\frac{\alpha}{2}}(s)\, \mathrm{d}s = (*).$$

For $\alpha \neq 2$, this leaves

$$\begin{aligned}(*) &= R^{\frac{1}{2}} + R^{\frac{1}{2}} \int_e^{m^{-1}(R)} s^{-1} \log^{-\frac{\alpha}{2}}(s)\, \mathrm{d}s = R^{\frac{1}{2}} + R^{\frac{1}{2}} \int_1^{\log\left(m^{-1}(R)\right)} u^{-\frac{\alpha}{2}}\, \mathrm{d}u \\ &= R^{\frac{1}{2}} \left(\log(m^{-1}(R))\right)^{1-\frac{\alpha}{2}} \asymp R^{\frac{1}{2}} \left(\frac{1}{\tilde{\Delta}+\mu+\nu} \log R - \frac{\tilde{\alpha}}{\tilde{\Delta}+\mu+\nu} \log\log R\right)^{1-\frac{\alpha}{2}} \\ &\asymp R^{\frac{1}{2}} \left(\log R\right)^{1-\frac{\alpha}{2}},\end{aligned}$$

while $\alpha = 2$ leads to

$$\begin{aligned}(*) &= R^{\frac{1}{2}} + R^{\frac{1}{2}} \int_e^{m^{-1}(R)} s^{-1} \log^{-1}(s)\, \mathrm{d}s = R^{\frac{1}{2}} + R^{\frac{1}{2}} \left(\log\log(m^{-1}(R))\right) \\ &\asymp R^{\frac{1}{2}} \left(\log\left(\frac{1}{\tilde{\Delta}+\mu+\nu} \log R - \frac{\tilde{\alpha}}{\tilde{\Delta}+\mu+\nu} \log\log R\right)\right) \asymp R^{\frac{1}{2}} \log\log R.\end{aligned}$$

As we can see from Theorem 5.18, any case feasible for E results in $B(R) \asymp R^{\frac{1}{2}} (\log R)^{1-\frac{\alpha}{2}}$, meaning that the matching upper bound in (5.43) is indeed attained whenever $\alpha \neq 2$, while the lower bound, $\mathcal{k}(R) \asymp R^{\frac{1}{2}} \log^{-\frac{\alpha}{2}}$, is never reached.

$E^+$ Lastly, let us additionally postulate that

$$\int_1^{\infty} (d_l d_\phi)^{\frac{1}{2}}(s)\, \mathrm{d}s < \infty.$$

Taking the calculations from the last paragraph into consideration, we see that

$$\int_e^t s^{-1} \log^{-\frac{\alpha}{2}}(s)\, \mathrm{d}s = \begin{cases} (\log t)^{1-\frac{\alpha}{2}} - 1 & \text{if } \alpha \neq 2, \\ \log\log t & \text{if } \alpha = 2, \end{cases},$$

where the right side diverges for $\alpha \leq 2$ and converges for $\alpha > 2$. Thus, the necessary assumptions for $\boxed{\mathrm{E}^+}$ are $\Delta_\phi > \frac{\mu+\nu}{2}$, $\Delta_l + \Delta_\phi = 2$, $\neg(\Delta_l = 1 \wedge \mu = \nu = 0)$ and $\alpha > 2$. With these made, [Wor, Theorem 3.12] tells us that

$$B(R) \lesssim \begin{cases} R^{\frac{1}{2}} \int_{\hbar(R)}^{\infty} (d_l d_\phi)^{\frac{1}{2}}(s)\, \mathrm{d}s & \text{if } \mu + \nu > 0, \\ R^{\frac{1}{2}} \int_{\hbar(R)}^{\infty} (d_l d_\phi)^{\frac{1}{2}}(s)\, \mathrm{d}s + R(n \circ \hbar)(R) & \text{if } \mu = \nu = 0. \end{cases}$$

To see whether the bounds are actually attained in our example, we start by calculating

$$\begin{aligned} R^{\frac{1}{2}} \int_{\hbar(R)}^{\infty} (d_l d_\phi)^{\frac{1}{2}}(s)\, \mathrm{d}s &= R^{\frac{1}{2}} \int_{\hbar(R)}^{\infty} s^{-1} \log^{-\frac{\alpha}{2}}(s)\, \mathrm{d}s = \lim_{t\to\infty} R^{\frac{1}{2}} \int_{\log(\hbar(R))}^{\log t} u^{-\frac{\alpha}{2}}\, \mathrm{d}u \\ &= \lim_{t\to\infty} R^{\frac{1}{2}} u^{1-\frac{\alpha}{2}} \Big|_{\log(\hbar(R))}^{\log t} \\ &= \lim_{t\to\infty} \frac{1}{\frac{\alpha}{2} - 1} R^{\frac{1}{2}} \left( (\log(\hbar(R)))^{1-\frac{\alpha}{2}} - (\log t)^{1-\frac{\alpha}{2}} \right) \\ &\asymp R^{\frac{1}{2}} (\log(\hbar(R)))^{1-\frac{\alpha}{2}} \asymp R^{\frac{1}{2}} \left( \frac{1}{2} \log R - \frac{\alpha}{2} \log\log R \right)^{1-\frac{\alpha}{2}} \\ &\asymp R^{\frac{1}{2}} (\log R)^{1-\frac{\alpha}{2}}. \end{aligned}$$

As we already saw previously, this bound is reached in all cases satisfying the assumptions of $\boxed{\mathrm{E}}$ complemented by $\alpha \neq 2$, and therefore especially for $\alpha > 2$. In particular, the bound also holds for $\mu = \nu = 0$, which means that the addition of $R(n \circ \hbar)(R)$ would not be necessary in the particular situation we studied within the current chapter.

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