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Ideals of compact operators

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Introduction

Ideals contained in the algebra of bounded linear operators on a Hilbert space $\mathcal{H}$ is a well discussed functional analytic topic. For $\mathcal{H}$ being separable each proper ideal is contained in the algebra of compact operators on $\mathcal{H}$. Since Calkins's groundbreaking work *Two-sided ideals and congruences in the ring of bounded operators in Hilbert spaces* [Cal41] published in 1941, it is known that operator ideals $\mathscr{S}$ contained in $\mathcal{K}(\mathcal{H})$ correspond bijectively to specific spaces of nonincreasing zero sequences.

Of special interest in the spectral theory are ideals endowed with a symmetric norm, forming Banach spaces. These are called “symmetrically normed ideals” (s.n.-ideals), and examples would be the Schatten–von Neumann classes, which are the operator theoretical counterparts of L^p spaces. It turned out that s.n.-ideals are heavily tied to the s-numbers of their elements. Naturally, the question arises, which spaces of zero sequences correspond bijectively to s.n.-ideals. Up until recently, it was an open question in the field of operator theory. In 2008 N.J. Kalton and F.A. Sukochev answered this question in “Symmetric norms and spaces of operators” [KS08] via the newly introduced uniform Hardy-Littlewood majorization. The main part of this master thesis revises, reshapes N.J. Kalton and F.A. Sukochev results and presents them in a more attractive way. This is achieved by breaking down complex structures to reach a deep level of understanding. Another aim of this work is to be self contained. To this end, the first two chapters introduce the reader to geometry of sequences, the theory of s-numbers and s.n.-ideals. In the first chapter we study some algebraic notions in the space of zero sequences of real numbers, which are basic for all what follows. This includes rearrangement processes, several quasi orders and their interplay, and some operators on sequences. The second chapter gives a brief overview on s-numbers of compact operators, their connection with operator ideals and some theory about symmetrically normed ideals w.r.t. symmetric norming functions. We avoid diving too much into details, for the sake of self-containment. Therefore the presented theory is kept at a basic level and can be found in I.C. Gohberg and M.G. Krein's extensive work *Introduction to the theory of linear nonselfadjoint operators* published in 1969 [GK69]. The subsequent chapter presents, in a more modern style, Calkin's correspondence via replication closed subcones of nonincreasing zero sequences. Chapter 4 consists of the technical core and in chapter 5 Calkin's correspondence between symmetric Banach sequences spaces and s.n.-ideals is established.

Moreover, this master thesis contains an additional chapter, more of preparational nature, that gives a tiny glimpse of a planned survey paper, which aims to present and proof some, almost unknown theorems, discovered by A. A. Mititel and G. I. Russu from the 70's and 80's concerning s.n.-ideals. They give necessary and sufficient conditions when to expect an operator ideal to satisfy the Macaev property (for more details, see the upcoming survey paper).

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"*No one can whistle a symphony. It takes a whole orchestra to play it.*" –Halford E. Luccock

Some basic notation

To start with, let us fix common notation.

- Throughout this work $\mathcal{H}$ denotes an infinite dimensional Hilbert space over $\mathbb{K}$, where $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$.
- The set of all bounded linear operators on $\mathcal{H}$ is denoted by $\mathcal{B}(\mathcal{H})$ and the subset of all compact linear operators is denoted by $\mathcal{K}(\mathcal{H})$.
- $\mathbb{N} = \{1, 2, 3, \ldots\}$ is the set of natural numbers.
- Cardinalities of a set (we will deal only with at most countable cardinalities) are understood as a natural number or ∞. For notational convenience, we will identify a natural number n with the set $\{1, \ldots, n\}$ and the cardinality ∞ with $\mathbb{N}$. This makes expressions like for example "$\forall i \in N$" meaningful.
- Sequences of real numbers are generically denoted as $a = (a_n)_{n\in\mathbb{N}}$.
- c_0 is the Banach space of all sequences a of real numbers with $\lim_{n\to\infty} a_n = 0$ endowed with the supremum norm.
- We denote
$$c_0^+ := \big\{(a_n)_{n\in\mathbb{N}} \in c_0 \;\big|\; \forall n \in \mathbb{N} : a_n \geq 0\big\},$$
$$c_0^\downarrow := \big\{(a_n)_{n\in\mathbb{N}} \in c_0^+ \;\big|\; \forall n \in \mathbb{N} : a_n \geq a_{n+1}\big\}.$$
- The natural pointwise order on sequences of real numbers is
$$a \leq b \quad :\Leftrightarrow \qquad \forall n \in \mathbb{N} : \quad a_n \leq b_n.$$
- We denote
$$|a| := (|a_n|)_{n\in\mathbb{N}}.$$
- $\mathbb{1}_A$ denotes the characteristic sequence of A, i.e.
$$(\mathbb{1}_A)_j := \mathbb{1}_A(j) = \begin{cases} 1 & \text{if } j \in A, \\ 0 & \text{otherwise.} \end{cases}$$
- $\lfloor\,.\,\rfloor$ denote the floor function and $\lceil\,.\,\rceil$ denotes the ceiling function, i.e.
$$\lfloor x \rfloor := \max\big\{m \in \mathbb{Z} \;\big|\; x \geq m\big\}, \quad \lceil x \rceil := \min\big\{m \in \mathbb{Z} \;\big|\; x \leq m\big\}.$$

For completeness we list some properties; none of them requires explicit proof. Recall here that a cone is a subset of a vector space which is closed under linear combinations with nonnegative coefficients.

i) $c_0^\downarrow \subseteq c_0^+$.

ii) c_0^+ and $c_0^\downarrow$ are cones.

iii) $\forall a, b, c \in c_0 : \quad a \leq b \;\Leftrightarrow\; a + c \leq b + c$.

iv) $\forall a, b \in c_0, \beta > 0 : \quad a \leq b \;\Leftrightarrow\; \beta a \leq \beta b$.

v) $\forall a, b \in c_0^+ : \quad a \leq b \;\Rightarrow\; \operatorname{supp} a \subseteq \operatorname{supp} b$.

vi) $a \leq b \Leftrightarrow \exists c \in c_0^+ : \quad b = a + c$

vii) $\forall a \in c_0 : \quad a \leq |a|$.

viii) $\forall a, b \in c_0 : \quad \big||a| - |b|\big| \leq |a + b| \leq |a| + |b|$.

Chapter 1

Geometry of SEQUENCES I

In this chapter we study some algebraic notions in the space of zero sequences of real numbers, which are basic for all what follows. This includes rearrangement processes, several quasi orders and their interplay, and some operators on sequences. Many facts which are stated here are elementary, and we will not elaborate all proof details.

1.1 Rearrangement operators

We saw that there is the obvious map $|\,.\,|$ from c_0 to c_0^+. It is also possible to pass further from c_0^+ to $c_0^\downarrow$. We can "obviously" rearrange the elements $|a_n|$ (and possibly remove some zero terms) to obtain a nonincreasing sequence $(a_n^*)_{n\in\mathbb{N}}$ with the property that every positive number α occurs the same number of times among the values a_n^* as it occurs among the values $|a_n|$. This rearranging process, however, is more involved than is looks on first sight. A sound definition reads as follows.

1.1.1 Definition. Let $a \in c_0$. We define a map $\iota : \mathbb{N} \to \mathbb{N}$ recursively by the following procedure.

- Let $n \in \mathbb{N}$. Then, since a is a zero sequence, the maximum

$$\max\big\{|a_i| \;\big|\; i \in \mathbb{N} \setminus \iota(\{k \in \mathbb{N} \,\big|\, k < n\})\big\}$$

  is attained. Now choose a number $\iota(n) \in \mathbb{N} \setminus \iota(\{k \in \mathbb{N} \,\big|\, k < n\})$ such that $|a_{\iota(n)}|$ equals this maximum.

Having the map ι, we set

$$a_n^* := |a_{\iota(n)}|.$$

The sequence $a^* := (a_n^*)_{n\in\mathbb{N}}$ is called the nonincreasing rearrangement of a.

The nonincreasing rearrangement a^* of a sequence $a \in c_0$ again belongs to c_0, is indeed nonincreasing, and the values $|a_n|$ are listed in a^* according to the number of their occurrances in the sequence $|a|$.

A useful characterisation of the nonincreasing rearrangement is obtained using level sets of a: for each $\delta > 0$ we set

$$L_{>\delta}(a) := \big\{n \in \mathbb{N} \,\big|\, |a_n| > \delta\big\}, \quad L_\delta(a) := \big\{n \in \mathbb{N} \,\big|\, |a_n| = \delta\big\}.$$

Since a is a zero sequence, all these sets are finite.

1.1.2 Lemma. *The following statements hold.*

i) $\forall a \in c_0, \forall n \in \mathbb{N}, \delta > 0 : \quad a_n^* > \delta \;\Leftrightarrow\; \big|L_{>\delta}(a)\big| \geq n.$

ii) $\forall a, b \in c_0 : \quad a^* \leq b^* \;\Leftrightarrow\; \forall \delta > 0 : \big|L_{>\delta}(a)\big| \leq \big|L_{>\delta}(b)\big|.$

iii) $\forall a \in c_0^{\downarrow}, \delta > 0 : \quad L_{>\delta}(a) = \big[1, \big|L_{>\delta}(a)\big|\big] \cap \mathbb{N}.$

iv) $\forall a \in c_0 : \quad |\operatorname{supp} a| = |\operatorname{supp} a^*|.$

Proof.
"i)" : Obviously, holds.
"ii)" : Follows from i)
"iii)" : Is a consequence of $a^* = a$, for all sequences $a \in c_0^{\downarrow}$.
"iv)" : Follows from i). □

For $\zeta \in \ell^\infty$ we denote by M_ζ the multiplication operator with ζ, i.e. $M_\zeta(a) := (\zeta_n a_n)_{n\in\mathbb{N}}$. Some computation rules for the operators $|\,.\,|$, $(\,.\,)^*$, and M_ζ, are:

1.1.3 Lemma. *The following statements hold.*

i) $(c_0)^* = c_0^{\downarrow}$ *and* $.^*\big|_{c_0^{\downarrow}} = \mathrm{id}_{c_0^{\downarrow}}$.

ii) $\forall a \in c_0 : \quad |a|^* = a^*.$

iii) $\forall a \in c_0 : \quad (a^*)^* = a^*.$

iv) $\forall a \in c_0, \beta \in \mathbb{R} : \quad (\beta a)^* = |\beta| a^*.$

v) $\forall a, b \in c_0 : \quad |a| \leq |b| \;\Rightarrow\; a^* \leq b^*.$

vi) $\forall \zeta \in \ell^\infty, a \in c_0 : \quad |M_\zeta a| \leq \|\zeta\|_\infty |a|.$

Let us emphasize that the operator $(\,.\,)^* : c_0 \to c_0$ is not compatible with sums.

Proof. We start with an example to illustrate that $(\,.\,)^*$ is not compatible with sums. Consider the sequences

$$a^1 := (1, 0, 0, 0, \ldots), \quad a^2 := (0, 1, 0, 0, \ldots).$$

Then $(a^1 + a^2)^* = (1, 1, 0, 0, \ldots)$, but $(a^1)^* + (a^2)^* = (2, 0, 0, \ldots)$.
Of the remaining statements we only prove v), all the others are obviously true. Let $a, b \in c_0$ be given with $|a| \leq |b|$. Moreover, let $\delta > 0$ be arbitrary. We have

$$L_{>\delta}(a) = \big\{n \in \mathbb{N} \;\big|\; |a_n| > \delta\big\} \subseteq \big\{n \in \mathbb{N} \;\big|\; |b_n| > \delta\big\} = L_{>\delta}(b),$$

and therefore $|L_{>\delta}(a)| \leq |L_{>\delta}(b)|$. Using Lemma 1.1.2 ii) yields $a^* \leq b^*$. □

Next we show two continuity properties. Here, a family $\mathcal{M} \subseteq c_0$ is called equicontinuous, if (think of a zero sequence as a continuous function on the one-point compactification of $\mathbb{N}$)

$$\forall \varepsilon > 0, \exists n_0 \in \mathbb{N}, \forall n \geq n_0, \forall a \in \mathcal{M} : \quad |a_n| < \varepsilon.$$

1.1.4 Lemma. *The following statements hold.*

i) $(\,.\,)^* : c_0 \to c_0$ *is continuous w.r.t.* $\|\,.\,\|_\infty$.

ii) *Let $(a^i)_{i\in\mathbb{N}}$ be a sequence in c_0 which converges pointwise to some sequence $a \in \mathbb{R}^{\mathbb{N}}$, and assume that the set $\big\{(a^i)^* \;\big|\; i \in I\big\}$ is equicontinuous. Then $a \in c_0$ and*

$$\forall \varepsilon > 0, \exists i_0 \in \mathbb{N}, \forall i \geq i_0 : \quad a^* \leq (a^i)^* + \varepsilon \mathbb{1}_{\mathbb{N}}. \tag{1.1}$$

The condition (1.1) is equivalent to

$$\forall n \in \mathbb{N} : a_n^* \leq \liminf_{i\to\infty} (a^i)_n^*. \tag{1.2}$$

Proof. For the proof of i) assume that $(a^i)_{i\in\mathbb{N}} \to a$ uniformly. Let $\epsilon > 0$, and choose $\delta \in (0, \frac{1}{2}\epsilon)$ such that

$$\{l\delta \mid l \in \mathbb{N}\} \cap \{|a_n| \mid n \in \mathbb{N}\} = \emptyset.$$

Since the first of these sets accumulates only at ∞ and the second only at 0, their distance ϵ' is positive, and we find $i_0 \in \mathbb{N}$ such that $\|a^i - a\|_\infty < \frac{1}{2}\epsilon'$ for all $i \geq i_0$. From this we see that

$$\forall l \in \mathbb{N}, i \geq i_0 : \quad L_{>l\delta}(a) = L_{>l\delta}(a^i).$$

Based on Lemma 1.1.2, it follows that

$$\forall l \in \mathbb{N}, n \in \mathbb{N}, i \geq i_0 : \quad a_n^* \in (l\delta, (l+1)\delta] \Leftrightarrow (a^i)_n^* \in (l\delta, (l+1)\delta],$$

and consequently also

$$\forall n \in \mathbb{N} : \quad a_n^* \leq \delta \;\Leftrightarrow\; (a^i)_n^* \leq \delta.$$

These conditions together cover all values of n, and we conclude that $\|(a^i)^* - a^*\|_\infty < 2\delta$ for all $i \geq i_0$.

We come to the proof of ii). Assume that $(a^i)_{i\in\mathbb{N}}$ is a sequence as in the statement of this item. By pointwise convergence, we have

$$\forall \delta > 0, \forall N \in \mathbb{N}, \exists i_0 \in I, \forall i \geq i_0 : \quad \Big(N \leq \big|L_{>\delta}(a)\big| \Rightarrow N \leq \big|L_{>\delta}(a^i)\big|\Big). \tag{1.3}$$

Let $\varepsilon > 0$ be given. Using again Lemma 1.1.2, our equicontinuity assumption yields $\sup_{i\in I} \big|\{n \in \mathbb{N} \mid |(a^i)_n| > \varepsilon\}\big| < \infty$. Thus there exist only finitely many elements of a whose absolute value exceed ε. We conclude that $a \in c_0$.

Let again $\epsilon > 0$ be given, and let $n \in \mathbb{N}$. Set $\delta := a_n^* - \epsilon$. Then $|L_{>\delta}(a)| \geq n$ by Lemma 1.1.2, and (1.3) provides $i_0 \in \mathbb{N}$ such that for all $i \geq i_0$ also $|L_{>\delta}(a^i)| \geq n$. In turn, $(a^i)_n^* > \delta$, and we conclude that $\liminf_{i\to\infty} (a^i)_n^* \geq a_n^* - \epsilon$. This shows (1.2). To deduce (1.1), choose an index n_0 such that $a_n^* \leq \epsilon$ for all $n \geq n_0$. Then $a_n^* \leq (a^i)_n^* + \varepsilon$ holds for $n \geq n_0$. For the remaining finitely many coordinates, use (1.2) to obtain the required index i_0. The fact that (1.1) implies (1.2) is obvious. □

It turns out practical, and is a significant gain in conceptional clarity, to formalise the concept of rearranging and removing terms of a sequence. We denote

$$\mathrm{Bij}(\mathbb{N}) := \left\{ \iota \subseteq \mathbb{N} \times \mathbb{N} \;\middle|\; \begin{array}{l} \iota \text{ is the graph of a bijective map of some subset} \\ \operatorname{dom}\iota \subseteq \mathbb{N} \text{ onto some subset } \operatorname{ran}\iota \subseteq \mathbb{N} \end{array} \right\}.$$

The set $\mathrm{Bij}(\mathbb{N})$ carries an algebraic structure. Namely, it is a semigroup with the relational composition. This semigroup has a unit element (the function $\mathrm{id}_\mathbb{N}$), contains many idempotents (all functions id_A, where $A \subseteq \mathbb{N}$), and its group of invertible elements is the permutation group $S(\mathbb{N})$ of our base set $\mathbb{N}$. It is invariant under taking relational inverses, and $\iota^{-1} \circ \iota = \mathrm{id}_{\operatorname{dom}\iota}$ and $\iota \circ \iota^{-1} = \mathrm{id}_{\operatorname{ran}\iota}$.

1.1.5 Definition. For each $\iota \in \mathrm{Bij}(\mathbb{N})$ we define an operator $\mathscr{R}_\iota : c_0 \to c_0$ by setting $\mathscr{R}_\iota(a_n)_{n\in\mathbb{N}} := (a'_n)_{n\in\mathbb{N}}$ with

$$a'_n := \begin{cases} a_{\iota(n)} & \text{if } n \in \operatorname{dom}\iota, \\ 0 & \text{otherwise.} \end{cases}$$

Applying $\mathscr{R}_\iota$ means to rearrange the terms a_m with $m \in \operatorname{ran}\iota$, and to remove the terms a_m with $m \in \mathbb{N} \setminus \operatorname{ran}\iota$.

1.1.6 Example.

i) Rearrangement operators can be used to pass to the nonincreasing rearrangement. Revisiting Definition 1.1.1, we observe that the map ι constructed there belongs to $\mathrm{Bij}(\mathbb{N})$, and $a^* = \mathscr{R}_\iota|a|$.

ii) a can be reconstructed from a^* in the sense that

$$a = (M_{\mathbb{1}_A - \mathbb{1}_B} \circ \mathscr{R}_\iota) a^*$$

with suitable $\iota \in \mathrm{Bij}(\mathbb{N})$ and partition $\{A, B\}$ of $\mathbb{N}$.

iii) Rearrangement operators can be used to select elements of a sequence. For a subset $A \subseteq \mathbb{N}$ we have

$$(\mathscr{R}_{\mathrm{id}_A} a)_n = \begin{cases} a_n & \text{if } n \in A, \\ 0 & \text{otherwise.} \end{cases}$$

In other words, $\mathscr{R}_{\mathrm{id}_A} a = M_{\mathbb{1}_A}$.

iv) Rearrangement operators can be used to shift sequences. For a nonnegative integer N, set

$$\iota_N : \begin{cases} \mathbb{N} & \to & \{k \in \mathbb{N} \mid k > N\} \\ n & \mapsto & n + N \end{cases}$$

Then $\iota_N \in \mathrm{Bij}(\mathbb{N})$, and

$$\mathscr{R}_{\iota_N} : (a_1, \ldots, a_N, a_{N+1}, \ldots) \mapsto (a_{N+1}, a_{N+2}, \ldots) \tag{1.4}$$

$$\mathscr{R}_{\iota_N^{-1}} : (a_1, a_2, \ldots) \mapsto (0, \ldots, \underset{\substack{\uparrow \\ N\text{-th place}}}{0}, a_1, a_2, \ldots) \tag{1.5}$$

We denote the left shift (1.4) as $\mathscr{T}^N$, and the right shift (1.5) as $\mathscr{T}^{-N}$.

In the context of realizing the nonincreasing rearrangement, the following observations are sometimes practical.

Some facts and computation rules for the operators $\mathscr{R}_\iota$ are:

1.1.7 Lemma. *The following statements hold.*

i) *Let* $\iota \in \mathrm{Bij}(\mathbb{N})$*. Then* $\mathscr{R}_\iota : c_0 \to c_0$ *is linear and contractive. It is isometric, if and only if* $\mathrm{ran}\,\iota = \mathbb{N}$.

ii) *The map* $\iota \mapsto \mathscr{R}_\iota$ *is compatible with the algebraic structure in the sense that*

$$\forall \iota, \kappa \in \mathrm{Bij}(\mathbb{N}) : \quad \mathscr{R}_\iota \circ \mathscr{R}_\kappa = \mathscr{R}_{\kappa \circ \iota}, \qquad \mathscr{R}_{\mathrm{id}_{\mathbb{N}}} = I. \tag{1.6}$$

In particular, we have $\mathscr{R}_\iota \circ \mathscr{R}_{\iota^{-1}} = M_{\mathbb{1}_{\mathrm{dom}\,\iota}}$ *and* $\mathscr{R}_{\iota^{-1}} \circ \mathscr{R}_\iota = M_{\mathbb{1}_{\mathrm{ran}\,\iota}}$ *for all* $\iota \in \mathrm{Bij}(\mathbb{N})$.

iii) *Let* $\iota \in \mathrm{Bij}(\mathbb{N})$ *with* $\mathrm{ran}\,\iota \cap \mathrm{supp}\,a \neq \emptyset$*, and set* $N := \left|\left\{ j \in \mathbb{N} \setminus \mathrm{ran}\,\iota \;\middle|\; |a_j| \geq \|\mathscr{R}_\iota a\|_\infty \right\}\right|$*. Then*

$$(\mathscr{R}_\iota a)^* \leq \mathscr{T}^N(a^*).$$

In particular, we have $(\mathscr{R}_\iota a)^* \leq a^*$ *for all* $\iota \in \mathrm{Bij}(\mathbb{N})$ *and* $a \in c_0$*, and in this relation equality holds if* $\mathrm{supp}\,a \subseteq \mathrm{ran}\,\iota$.

Further properties are:

iv) $\forall a \in c_0: \quad \operatorname{supp} \mathscr{R}_\iota a = \iota^{-1}(\operatorname{supp} a)$.

v) $\ker \mathscr{R}_\iota = \{a \in c_0 \mid \operatorname{supp} a \cap \operatorname{ran} \iota = \emptyset\}$.

vi) $\operatorname{ran} \mathscr{R}_\iota = \{a \in c_0 \mid \operatorname{supp} a \subseteq \operatorname{dom} \iota\}$.

vii) $\forall a \in c_0, \iota \in \mathrm{Bij}(\mathbb{N}): \quad |\mathscr{R}_\iota a| = \mathscr{R}_\iota |a|$.

viii) $\forall a, b \in c_0, \iota \in \mathrm{Bij}(\mathbb{N})$:

$$a \leq b \ \Rightarrow \ \mathscr{R}_\iota a \leq \mathscr{R}_\iota b.$$

Proof. All statements are obviously true. □

1.1.8 Lemma. *The map constructed in Definition 1.1.1 has the properties*

i) $\mathscr{R}_\iota |a| \in c_0^{\downarrow}$.

ii) $\operatorname{ran} \iota \supseteq \operatorname{supp} a$.

iii) $\operatorname{dom} \iota = \mathbb{N}$.

The first two properties characterize bijective graphs which give the nonincreasing rearrangement, namely: if $\iota \in \mathrm{Bij}(\mathbb{N})$, then $\mathscr{R}_\iota |a| = a^*$ if and only if ι satisfies i) and ii).

Proof. Let a be a zero sequence and ι the map constructed in Definition 1.1.1. The fact that i) holds, follows from

$$|a_{\iota(n+1)}| = \max \{|a_i| \mid i \in \mathbb{N} \backslash \iota(\{k \in \mathbb{N} \mid k < n+1\})\} \leq \max \{|a_i| \mid i \in \mathbb{N} \backslash \iota(\{k \in \mathbb{N} \mid k < n\})\} = |a_{\iota(n)}|.$$

ii) and iii) are also immediately derived from the definition.

Now let $\iota \in \mathrm{Bij}(\mathbb{N})$ be given with $\mathscr{R}_\iota |a| = a^*$. $\mathscr{R}_\iota |a| \in c_0^{\downarrow}$ follows from $a^* \in c_0$. To see that ι satisfies ii), assume the contrary, i.e. there exists a natural number $i \in \operatorname{supp} a$ such that $i \notin \operatorname{ran} \iota$. Let $0 < \varepsilon < |a_i|$ be arbitrary. Then we have the following estimate

$$\begin{aligned} \left|L_{>|a_i|-\varepsilon}(\mathscr{R}_\iota |a|)\right| &= \left|\{j \in \mathbb{N} \mid (\mathscr{R}_\iota |a|)_j > |a_i| - \varepsilon\}\right| = \left|\{j \in \mathbb{N} \mid |a_{\iota(j)}| > |a_i| - \varepsilon\}\right| \\ &= \left|\{j \in \operatorname{ran} \iota \mid |a_j| > |a_i| - \varepsilon\}\right| < \left|\{j \in (\operatorname{ran} \iota \cup \{i\}) \mid |a_j| > |a_i| - \varepsilon\}\right| \\ &\leq \left|\{j \in \mathbb{N} \mid |a_j| > |a_i| - \varepsilon\}\right| = \left|L_{>|a_i|-\varepsilon}(a)\right|. \end{aligned}$$

Since $\left|L_{>|a_i|-\varepsilon}(\mathscr{R}_\iota |a|)\right| < \left|L_{>|a_i|-\varepsilon}(a)\right|$, invoking Lemma 1.1.2 ii) yields the contradiction

$$\mathscr{R}_\iota |a| = a^* = (a^*)^* = (\mathscr{R}_\iota |a|)^* \neq a^*.$$

Now suppose that $\iota \in \mathrm{Bij}(\mathbb{N})$ satisfies i) and ii). Moreover, let $\delta > 0$ be given. Note the obvious fact that $L_{>\delta}(a) \subseteq \operatorname{supp} a$. Then $L_{>\delta}(a) \subseteq \operatorname{ran} \iota$. Consider

$$\begin{aligned} \iota^{-1}\big(L_{>\delta}(a)\big) &= \{\iota^{-1}(i) \mid i \in \operatorname{supp} a, |a_j| > \delta\} = \{j \in \operatorname{dom} \iota \mid \iota(j) \in \operatorname{supp} a, |a_{\iota(j)}| > \delta\} \\ &\subseteq \{j \in \operatorname{dom} \iota \mid |a_{\iota(j)}| > \delta\} = \{j \in \operatorname{dom} \iota \mid (\mathscr{R}_\iota |a|)_j > \delta\} = \{j \in \mathbb{N} \mid (\mathscr{R}_\iota |a|)_j > \delta\} \\ &= L_{>\delta}(\mathscr{R}_\iota |a|) \end{aligned}$$

Since ι is bijective, we obtain $\left|L_{>\delta}(a)\right| \leq \left|L_{>\delta}(\mathscr{R}_{|a|})\right|$. On the other hand, we have

$$\left|L_{>\delta}(\mathscr{R}_{|a|})\right| = \left|\{j \in \mathbb{N} \mid (\mathscr{R}_\iota |a|)_j > \delta\}\right| = \left|\{j \in \operatorname{dom} \iota \mid |a_{\iota(j)}| > \delta\}\right| \leq \left|\{i \in \mathbb{N} \mid |a_i| > \delta\}\right| = \left|L_{>\delta}(a)\right|$$

and we conclude that $\left|L_{>\delta}(a)\right| = \left|L_{>\delta}(\mathscr{R}_{|a|})\right|$. Using Lemma 1.1.2 ii) yields the desired result, $\mathscr{R}_\iota |a| = a^*$. □

1.1.9 Remark. The freedom in the choice of ι left by Lemma 1.1.8 is sometimes of good use. For example, if $|\operatorname{supp} a| < \infty$ and $B \supseteq \operatorname{supp} a$, we can choose $\iota \in \mathrm{Bij}(\mathbb{N})$ satisfying properties i) and ii) from the last lemma, such that in addition $\iota(|B|) = B$.

1.2 Elementary quasi orders on zero sequences

By pushing forward the partial order $\leq$ with the operator $(\,.\,)^*$, we obtain a quasi order.

1.2.1 Definition. For $a, b \in c_0$ we denote

$$a \prec b \quad :\Leftrightarrow \qquad a^* \leq b^*.$$

Moreover, we write

$$a \sim b \quad :\Leftrightarrow \qquad (a \prec b \quad \wedge \quad b \prec a) \quad \Leftrightarrow \quad a^* = b^*.$$

Then indeed $\prec$ is nothing but the inverse image of $\leq$ under the map $(\,.\,)^* \times (\,.\,)^* : c_0 \times c_0 \to c_0^{\downarrow} \times c_0^{\downarrow}$. Some computation rules for $\prec$ are:

1.2.2 Lemma. *The following statements hold.*

i) $\forall a, b \in c_0 : \quad |a| \leq |b| \ \Rightarrow \ a \prec b.$

ii) $\forall a, b \in c_0^{\downarrow} : \quad a \leq b \ \Leftrightarrow \ a \prec b.$

iii) 0 *is the smallest element of* $(c_0, \prec)$.

iv) $\forall a, b \in c_0, \beta > 0 : \quad a \prec b \ \Leftrightarrow \ \beta a \prec \beta b.$

v) $\forall a, b \in c_0 : \quad a \prec b \ \Rightarrow \ |\operatorname{supp} a| \subseteq |\operatorname{supp} b|.$

vi) $\forall a \in c_0 : \quad a \sim |a|.$

vii) $\forall a, b \in c_0^{+}, \iota \in \operatorname{Bij}(\mathbb{N}) :$
$$a \leq b \ \Rightarrow \ \mathscr{R}_\iota a \prec \mathscr{R}_\iota b.$$

viii) $\forall a \in c_0, \iota \in \operatorname{Bij}(\mathbb{N}) : \quad (\mathscr{R}_\iota a) \prec a.$

Proof. Since matters get quickly more and more involved, we give proofs for most of the items in Lemma 1.2.2, even though they seem easy.

"i)" : Let $a, b \in c_0$ with $|a| \leq |b|$. Choose $\iota \in \operatorname{Bij}(\mathbb{N})$ such that $a^* = \mathscr{R}_\iota |a|$. Then we have

$$a^* = \mathscr{R}_\iota |a| \overset{1.1.7\ \text{viii)}}{\leq} \mathscr{R}_\iota |b|.$$

Using Lemma 1.1.7 and Lemma 1.1.3 yields

$$a^* \overset{1.1.3\ \text{iii)}}{=} (a^*)^* \overset{1.1.3\ \text{v)}}{\leq} (\mathscr{R}_\iota |b|)^* \overset{1.1.7\ \text{iii)}}{\leq} b^*$$

"ii)" : Follows from i).
"iii)" : Obviously 0 is the smallest element of $(c_0, \prec)$.
"iv)" : Follows from Lemma 1.1.3 iv).
"v)" : Since for all $a, b \in c_0^{+}$ it holds that $a \leq b$ implies $\operatorname{supp} a \subseteq \operatorname{supp} b$, we obtain $\operatorname{supp} a^* \subseteq \operatorname{supp} b^*$. Keeping in mind that $|\operatorname{supp} a| = |\operatorname{supp} a^*|$, $|\operatorname{supp} b| = |\operatorname{supp} b^*|$ yields v).
"vi)" : Trivial.
"vii)": Let $a, b \in c_0^{+}$ and $\iota \in \operatorname{Bij}(\mathbb{N})$ be given with $a \leq b$. Then invoking Lemma 1.1.7 viii), we have $\mathscr{R}_\iota a \leq \mathscr{R}_\iota b$. Using ii) yields $\mathscr{R}_\iota a \prec \mathscr{R}_\iota b$.
"viii)": Follows from Lemma 1.1.7 iii).

□

Compatibility with sums, as it holds for $\leq$, is lost since $(\,.\,)^*$ is not additive. To see this, consider the example

$$a^1 := (1, 0, 0, 0, \ldots), \quad a^2 := (0, 1, 0, 0, \ldots), \quad b := (2, 0, 0, 0, \ldots).$$

Obviously, $a^1, a^2 \prec b$, but $a^1 + a^2 \not\prec b$.

In the context of compact operators and their s-numbers, another partial order on c_0 occurs.

1.2.3 Definition. For $a, b \in c_0$ we denote

$$a \ll b \quad :\Leftrightarrow \qquad \forall n \in \mathbb{N}: \quad \sum_{j=1}^{n} a_j \leq \sum_{j=1}^{n} b_j.$$

Also this relation can be seen as a pushforward of $\leq$.

1.2.4 Remark. The Cesaro-means operator, which is defined for sequences of real numbers as

$$\mathscr{C}(a_n)_{n\in\mathbb{N}} := \Big(\frac{1}{n}\sum_{j=1}^{n} a_j\Big)_{n\in\mathbb{N}},$$

induces a linear, contractive, order preserving and injective operator on c_0. The relation $\ll$ is the inverse image under $\mathscr{C} \times \mathscr{C} : c_0 \times c_0 \to c_0 \times c_0$ of the pointwise order $\leq$.

Some computation rules for $\ll$ are:

1.2.5 Lemma. *The following statements hold.*

i) $\forall a, b \in c_0: \quad a \leq b \;\Rightarrow\; a \ll b.$

ii) $\forall a, b \in c_0, \beta > 0: \quad a \ll b \;\Leftrightarrow\; \beta a \ll \beta b.$

iii) $\forall a, b, c \in c_0: \quad a \ll b \Leftrightarrow a + c \ll b + c.$

iv) $\forall a \in c_0, \iota \in \mathrm{Bij}(\mathbb{N}): \quad \mathscr{R}_\iota a \ll a^*.$

v) $\forall a, b \in c_0: \quad (a + b)^* \ll a^* + b^*.$

Proof. i) – iii) obviously hold.
"iv)" : Follows from the fact that, for every sequence $a \in c_0$ and finite subset $G \subseteq \mathbb{N}$ it holds that

$$\sum_{j\in G} a_j \leq \sum_{j=1}^{|G|} a_j^*.$$

"v)" : Let $a, b \in c_0$. Choose $\iota \in \mathrm{Bij}(\mathbb{N})$ such that $(a+b)^* = \mathscr{R}_\iota|a+b|$. Since $\mathscr{R}_\iota|a+b| \leq \mathscr{R}_\iota|a| + \mathscr{R}_\iota|b|$ and by iv), $\mathscr{R}_\iota|a| \ll a^*$, $\mathscr{R}_\iota|b| \ll b^*$, it follows that $(a + b)^* \ll a^* + b^*$. □

Compatibility with $(\,.\,)^*$, as it holds for $\leq$ by Lemma 1.1.3 v), is lost. An example is

$$a := (0, 1, 0, \ldots), \quad b := (\frac{1}{2}, \frac{1}{2}, 0, \ldots),$$

where $a \ll b$, but $b^* \ll a^*$ and $a^* \neq b^*$.

Further pushing forward $\ll$ with $(.)^*$, leads to a quasi order known as Hardy-Littlewood majorization.

1.2.6 Definition. For $a, b \in c_0$ we denote

$$a \preccurlyeq b \quad :\Leftrightarrow \qquad a^* \ll b^*.$$

Some computation rules for $\preccurlyeq$ are:

1.2.7 Lemma. *The following statements hold.*

i) $\forall a, b \in c_0: \quad a \prec b \;\Rightarrow\; a \preccurlyeq b.$

ii) $\forall a, b \in c_0^{\downarrow}: \quad a \ll b \;\Leftrightarrow\; a \preccurlyeq b.$

iii) 0 *is the smallest element of* $(c_0, \preccurlyeq)$.

iv) $\forall a, b \in c_0, \beta > 0: \quad a \preccurlyeq b \;\Leftrightarrow\; \beta a \preccurlyeq \beta b.$

v) $\forall a, b \in c_0: \quad a + b \preccurlyeq a^* + b^*.$

vi) $(a \preccurlyeq b \quad \wedge \quad b \preccurlyeq a) \;\Leftrightarrow\; a \sim b.$

vii) $\forall a, b \in c_0:$

$$(b \preccurlyeq a \;\wedge\; b\mathbb{1}_{\operatorname{supp} a} = a) \;\Rightarrow\; b = a.$$

viii) $\forall a \in c_0, \iota \in \operatorname{Bij}(\mathbb{N}): \quad \mathscr{R}_\iota a \preccurlyeq a.$

ix) $\forall a, b, c \in c_0: \quad a \preccurlyeq b \;\Rightarrow\; a + c \preccurlyeq b^* + c^*.$

x) $\forall a, b \in c_0: \quad a \preccurlyeq b \;\Rightarrow\; \|a\|_\infty \leq \|b\|_\infty.$

xi) $\forall a, b \in c_0^{\downarrow}: \quad a \preccurlyeq b \;\Leftrightarrow\; \mathscr{C}a \leq \mathscr{C}b.$

Proof. i) − iv) and ix) − xi) obviously hold.
"v)" : Follows from Lemma 1.2.5 v).
"vi)" : Let $a, b \in c_0$. Obviously, $a \sim b$ implies $a \preccurlyeq b$. Let $a \preccurlyeq b$, $b \preccurlyeq a$ hold. Then we have

$$\sum_{j=1}^{n} a_j^* = \sum_{j=1}^{n} b_j^*,$$

for every natural number n. This implies $a_j^* = b_j^*$ for every $j \in \mathbb{N}$.
"vii)": Let $a, b \in c_0$ with $b \preccurlyeq a$ and $b\mathbb{1}_{\operatorname{supp} a} = a$. Since $b\mathbb{1}_{\operatorname{supp} a} \prec b$, $a = b\mathbb{1}_{\operatorname{supp} a}$ and in particular $a \sim b\mathbb{1}_{\operatorname{supp} a}$, we have $a \preccurlyeq b \preccurlyeq a$ and thus by vi) we obtain $b \sim \mathbb{1}_{\operatorname{supp} a} b$. We conclude that $\operatorname{supp} b = \operatorname{supp} a$.
"viii)": Follows from i) and Lemma 1.2.2 viii).

□

Chapter 2

s-numbers and symmetrically normed ideals

This chapter is all about giving a brief overview on s-numbers of compact operators, their connection with operator ideals and some theory about symmetrically normed ideals w.r.t. symmetric norming functions. We avoid diving too much into details, for the sake of self-containment. Therefore the presented theory is kept at a basic level and can be found in Gohberg-Krein [GK69].

2.1 Compact operators and elementary properties of s-numbers

We recall the definition of compact operators. An operator $T \in \mathcal{B}(\mathcal{H})$ is compact, if the image of the closed unit ball under T is relative compact in $\mathcal{H}$. The set of all compact operators on a Hilbert space $\mathcal{H}$ is denoted by $\mathcal{K}(\mathcal{H})$. It is a well known fact that every compact self-adjoint operator T can be represented via its eigenvalues and eigenvectors, i.e. $T = \sum_{j=1}^{\infty} \lambda_j \langle \cdot, e_j \rangle e_j$. However, there exist compact operators, called Volterra operators, having a spectrum containing only 0. To obtain a similar representation for arbitrary compact operators, one has to study the operator $|T| = (T^*T)^{\frac{1}{2}}$ and its eigenvalues.

2.1.1 Definition. Let $T \in \mathcal{K}(\mathcal{H})$. The Eigenvalues of $|T|$ in decreasing order are called s-numbers and will be denoted by $(s_j(T))_{j=1}^{\infty} = s(T)$.

Via polar decomposition, one can get the Schmidt expansion (or Schmidt series) for compact operators:

2.1.2 Theorem. *(Schmidt expansion) $T \in \mathcal{K}(H)$. Then there exist two orthonormal systems $\{\phi_j \mid j \in \mathbb{N}\}$ and $\{\psi_j \mid j \in \mathbb{N}\}$ such that T admits a representation*

$$T = \sum_{j=1}^{\infty} s_j(T) \langle \cdot, \phi_j \rangle \psi_j.$$

s-numbers satisfy various properties and we state the most basic ones needed in this paper.

2.1.3 Theorem. *Let $T \in \mathcal{K}(\mathcal{H}), A, B \in \mathcal{B}(\mathcal{H})$. Then*

$$\forall j \in \mathbb{N}: \quad s_j(ATB) \leq \|A\| \|B\| s_j(T).$$

2.1.4 Theorem. *s-numbers satisfy the following properties:*

$$s_1(T) = \max_{x\in\mathcal{H}\setminus\{0\}} \frac{\langle |T|x, x\rangle}{\|x\|^2}, \tag{2.1}$$

$$s_{j+1}(T) = \min\Big\{ \max_{x\in\mathcal{L}^\perp\setminus\{0\}} \frac{\langle |T|x, x\rangle}{\|x\|^2} \;\Big|\; \mathcal{L} \text{ is a } j\text{-dimensional linear subspace of } \mathcal{H}\Big\}. \tag{2.2}$$

Considering the Schmidt expansion $T = \sum_{j=1}^{\infty} s_j(T)\langle\cdot,\phi_j\rangle\psi_j$, in fact the maximum in (2.1) is attained for $x = \phi_1$ and the minimum in (2.2) is attained for $\mathcal{L} = \operatorname{span}\{\phi_1,\dots,\phi_j\}$.

2.1.5 Theorem. *Let $S, T \in \mathcal{K}(\mathcal{H})$. Then*

i) *for all $c \in \mathbb{K}$ and $T \in \mathcal{K}(\mathcal{H})$ it holds that*

$$s_j(cT) = |c|s_j(T), \qquad j \in \mathbb{N}.$$

ii) *for all $S, T \in \mathcal{K}(\mathcal{H})$ it holds that*

$$s_{m+n-1}(S+T) \le s_m(S) + s_n(T), \qquad m, n \in \mathbb{N}.$$

holds. In particular

$$s_{2n-1}(S+T) \le s_n(S) + s_n(T), \qquad n \in \mathbb{N}.$$

2.1.6 Theorem. *Let $(T_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{K}(\mathcal{H})$ with $\lim_{n\to\infty} T_n = T$. Then*

$$s(T) = \lim_{n\to\infty} s(T_n),$$

holds w.r.t. $\|\cdot\|_\infty$.

2.1.7 Theorem. *Let $T \in \mathcal{K}(\mathcal{H})$. Denote by $\mathcal{F}_j$ the subspace of $\mathcal{B}(\mathcal{H})$ containing all operators F with $\dim\operatorname{ran} F < j$, for $j \in \mathbb{N}$. Then*

$$s_j(T) = \inf\big\{\|T - F\| \;\big|\; F \in \mathcal{F}_j\big\}, \qquad j \in \mathbb{N}. \tag{2.3}$$

The numbers on the right side in (2.3) are called approximation numbers.

2.1.8 Theorem. *Let $S, T \in \mathcal{K}(H)$. Then $s(S+T) \lll s(S) + s(T)$, i.e.*

$$\sum_{j=1}^{n} s_j(S+T) \le \sum_{j=1}^{n}(s_j(S) + s_j(T)), \qquad n \in \mathbb{N}.$$

2.1.9 Lemma.

i) *For two positive operators $S, T \in \mathcal{B}(\mathcal{H})$ with $S \le T$ the corresponding s-numbers satisfy*

$$s_j(S) \le s_j(T), \quad j \in \mathbb{N}.$$

ii) *For all $T \in \mathcal{K}(\mathcal{H})$ it holds that*

$$s_j(T) = s_j(|T|), \qquad j \in \mathbb{N}.$$

iii) *For all $T \in \mathcal{K}(\mathcal{H})$ and partial isometries $U, V \in \mathcal{B}(\mathcal{H})$ it holds that*

$$s_j(UTV^*) = s_j(T), \qquad j \in \mathbb{N}.$$

2.2 Elementary properties of operator ideals

We recall the notion of operator ideals:

2.2.1 Definition. A subset $\mathscr{S}$ of $\mathcal{B}(\mathcal{H})$ is called an (two-sided) operator ideal if

i) $\mathscr{S}$ is a linear subspace of $\mathcal{B}(\mathcal{H})$.

ii) for all $A, B \in \mathcal{B}(\mathcal{H})$ and $T \in \mathscr{S}$ it holds that $ATB \in \mathscr{S}$.

Additionally, an operator ideal $\mathscr{S}$ is called a proper operator ideal if

iii) $\mathscr{S} \neq \{0\}$ and $\mathscr{S} \neq \mathcal{B}(\mathcal{H})$.

It turns out, that operator ideals, are heavily tied to the s-numbers of their elements.

2.2.2 Proposition. *Let $\mathscr{S}$ be an operator ideal. Furthermore let $S \in \mathcal{B}(\mathcal{H})$ and $T \in \mathscr{S}$. If $s(S) \leq s(T)$, then $S \in \mathscr{S}$.*

One could get an equivalent definition of operator ideals by interchanging item ii) in Definition 2.2.1 with the statement of Proposition 2.2.2.

2.2.3 Theorem. *Let $\mathscr{S}$ be a proper operator ideal. Then*

$$\mathcal{F} \subseteq \mathscr{S} \subseteq \mathcal{K}(\mathcal{H}),$$

holds, where $\mathcal{F}$ denotes the finite rank operators in $\mathcal{B}(\mathcal{H})$.

We recall the notion of diagonal operators. For a sequence $a \in c_0$ and an orthonormal system $\{e_n \mid n \in \mathbb{N}\}$ in $\mathcal{H}$ the operator

$$E_a := \sum_{j=1}^{\infty} a_j \langle \cdot, e_j \rangle e_j,$$

is called the diagonal operator of the sequence a w.r.t. the orthonormal system $(e_n)_{n\in\mathbb{N}}$. We state some obvious properties of these operators without proof.

2.2.4 Lemma. *The following statements hold*

i) $\forall a, b \in c_0 : \quad E_{a+b} = E_a + E_b$.

ii) $\forall \lambda \in \mathbb{R}, a \in c_0 : \quad E_{\lambda a} = \lambda E_a$.

iii) $\forall a \in c_0 : \quad (E_a)^* = E_a$

iv) $\forall a \in c_0 : \quad s(E_a) = a^*$.

To analyse operator ideals, it is convenient to consider subsets, from which the original ideal can be reproduced. We briefly discuss a very useful class of such subsets.

2.2.5 Example. Let $\mathcal{B} := \{e_n \mid n \in \mathbb{N}\}$ be an arbitrary orthonormal basis in $\mathcal{H}$. For an operator ideal $\mathscr{S} \subseteq \mathcal{K}(\mathcal{H})$, let the subset of compact self-adjoint operators consisting out of all diagonal operators w.r.t. the orthonormal basis $\mathcal{B}$ denoted by

$$\mathscr{S}_{\mathcal{B}} := \mathscr{S} \cap \{E_a \mid a \in c_0\},$$

By Proposition 2.2.2 and Lemma 2.1.9 iii), $\mathscr{S}_{\mathcal{B}}$ contains enough information to fully reproduce $\mathscr{S}$, namely by

$$\mathscr{S} = \{T \in \mathcal{K}(\mathcal{H}) \mid E_{s(T)} \in \mathscr{S}_{\mathcal{B}}\}.$$

The set of all diagonal operator w.r.t $\mathcal{B}$ is given by $\mathcal{K}(\mathcal{H})_{\mathcal{B}} = \{E_a \mid a \in c_0\}$ and is in fact closed w.r.t. convergence in the operator norm $\|\cdot\|$. Too see this, let $T \in \mathcal{K}(\mathcal{H})$ and let $(T_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{K}(\mathcal{H})_{\mathcal{B}}$ with $T_n \xrightarrow{\|\cdot\|} T$. Since each T_n is an element of $\mathcal{K}(\mathcal{H})_{\mathcal{B}}$, they admit representations

$$T_n = E_{a^n}, \qquad n \in \mathbb{N},$$

for some sequences $a^n \in c_0$. Since $T_n e_j = a_j^n e_j \xrightarrow{\|\cdot\|} Te_j$, for every $j \in \mathbb{N}$, the limit $\lim_{n\to\infty} a^n =: a$ exists pointwise. Hence, each e_j is an eigenvector of T. The set $\{e_n \mid n \in \mathbb{N}\}$ already contains all eigenvectors of T. Thus, by the spectral theorem for compact self-adjoint operators T admits a representation

$$T := E_a,$$

and therefore $T \in \mathcal{K}(\mathcal{H})_{\mathcal{B}}$.
Now suppose that $\mathscr{S}$ is endowed with a norm $\|\cdot\|_{\mathscr{S}}$ stronger than $\|\cdot\|$, i.e

$$\exists C > 0, \forall T \in \mathscr{S}: \quad \|T\| \leq C\|T\|_{\mathscr{S}},$$

such that $\langle \mathscr{S}, \|\cdot\|_{\mathscr{S}} \rangle$ is a Banach space. Obviously, $\mathscr{S}_{\mathcal{B}}$ is a linear subspace of the operator ideal $\mathscr{S}$ and by the argument above, we immediately obtain that $\langle \mathscr{S}_{\mathcal{B}}, \|\cdot\|_{\mathscr{S}}\big|_{\mathscr{S}_{\mathcal{B}}} \rangle$ is a Banach space.

2.3 Symmetrically normed ideals vs. symmetric norming functions

This section is based on [GK69, Chapter 3] and solely summarises known results about symmetrically normed ideals and symmetric norming functions. We put particular emphasize on the relation between these two notions. The results are presented without proofs, which can be found in [GK69, Chapter 3], unless stated otherwise.

2.3.1 Definition. Let $\mathscr{S}$ be an operator ideal. A norm $\|\cdot\|_{\mathscr{S}}$ on $\mathscr{S}$ is called symmetric if

i) for all $T \in \mathscr{S}$ and $A, B \in \mathcal{B}(\mathcal{H})$ it holds that $\|ATB\|_{\mathscr{S}} \leq \|A\|\|T\|_{\mathscr{S}}\|B\|$.

ii) for all $F \in \mathcal{F}_1$ it holds that $\|F\|_{\mathscr{S}} = \|F\|$.

2.3.2 Definition. Let $\mathscr{S}$ be an operator ideal endowed with a symmetric norm $\|\cdot\|_{\mathscr{S}}$. Then $\langle \mathscr{S}, \|\cdot\|_{\mathscr{S}} \rangle$ is called a symmetrically normed ideal (s.n.-ideal, for short) if it is complete.

2.3.3 Example. Define for each $1 \leq p < \infty$

$$\mathscr{S}_p := \{T \in \mathcal{K}(\mathcal{H}) \mid \sum_{j=1}^{\infty} s_j(T)^p < \infty\} \quad \text{and} \quad \|T\|_p := \big(\sum_{j=1}^{\infty} s_j(T)^p\big)^{\frac{1}{p}}.$$

In fact $\langle \mathscr{S}_p, \|\cdot\|_p \rangle$ are separable s.n.-ideals.

The next proposition shows in particular that symmetric norms depend only on s-numbers.

2.3.4 Proposition. *Let $\mathscr{S}$ be an operator ideal endowed with a symmetric norm $\|\cdot\|_{\mathscr{S}}$. Then for all $S \in \mathcal{B}(\mathcal{H})$ and $T \in \mathscr{S}$ with $s(S) \leq s(T)$ it holds that $S \in \mathscr{S}$ and $\|S\|_{\mathscr{S}} \leq \|T\|_{\mathscr{S}}$.*

2.3.5 Theorem. *Let $\mathscr{S}$ be an operator ideal and let $\|\cdot\|_{\mathscr{S}_1}$ and $\|\cdot\|_{\mathscr{S}_2}$ be two symmetric norms on $\mathscr{S}$ such that $\langle \mathscr{S}, \|\cdot\|_{\mathscr{S}_1} \rangle$ and $\langle \mathscr{S}, \|\cdot\|_{\mathscr{S}_2} \rangle$ are s.n.-ideals. Then $\|\cdot\|_{\mathscr{S}_1}$ and $\|\cdot\|_{\mathscr{S}_2}$ are equivalent norms on $\mathscr{S}$.*

2.3.6 Definition. A function $\Phi : c_{00} \to \mathbb{R}$ is called a norming function if

- $\forall a \in c_{00} \backslash \{0\} : \quad \Phi(a) > 0.$
- $\forall \alpha > 0, a \in c_{00} : \quad \Phi(\alpha a) = \alpha \Phi(a).$
- $\forall a, b \in c_{00} : \quad \Phi(a+b) \leq \Phi(a) + \Phi(b).$
- $\Phi(1, 0, 0, \dots) = 1$

It is called a symmetric norming function (s.n.-function, for short) if it additionally satisfies

- for all $a \in c_{00}$ and permutations π of $\mathbb{N}$ it holds that $\Phi(a) = \Phi(a_\pi)$.

2.3.7 Proposition. *Let Φ be a symmetric norming function. Then Φ has the following properties.*

i) *For all $a, b \in c_{00}$ with $a \lll b$ it follows that $\Phi(a) \leq \Phi(b)$.*

ii) *For all $a \in c_{00}$ it holds that $\Phi(a^*) = \Phi(a)$.*

2.3.8 Example. Define functions

$$\Phi_\infty(a) := a_1^*, \quad \Phi_1(a) := \sum_{j=1}^{\infty} a_j^*, \qquad a \in c_{00}.$$

It is easy to see, that these are indeed s.n.-functions. They are extremal in the sense that for every s.n.-function Φ

$$\Phi_\infty(a) \leq \Phi(a) \leq \Phi_1(a), \qquad a \in c_{00}.$$

2.3.9 Theorem. *There is a one-to-one correspondence between s.n.-functions and symmetric norms on $\mathcal{F}$. It is established by the following assignments:*

i) *For a given symmetric norm $\|\cdot\|_{\mathcal{F}}$ on $\mathcal{F}$ and an arbitrary orthonormal basis $\{e_j \mid j \in \mathbb{N}\}$ one can define a s.n.-function by*

$$\Phi_{\mathcal{F}}(a) := \Big\| \sum_{j=1}^{\infty} a_j^* \langle \cdot, e_j \rangle e_j \Big\|_{\mathcal{F}}, \qquad a \in c_{00}.$$

ii) *For a given s.n.-function Φ one can define a symmetric norm on $\mathcal{F}$ by*

$$\|F\|_\Phi := \Phi(s(F)), \qquad F \in \mathcal{F}.$$

$$\begin{array}{ccc} \|\cdot\|_{\mathcal{F}} & \hookleftarrow & \|\cdot\|_\Phi \\ \downarrow & & \uparrow \\ \Phi_{\mathcal{F}}(\cdot) & \hookrightarrow & \Phi(\cdot) \end{array}$$

Assume we are given a s.n.-function Φ. Since $\Phi(a\mathbb{1}_{[1,n]}) \leq \Phi(a\mathbb{1}_{[1,n+1]})$ holds for $n \in \mathbb{N}$, one can extend the domain of Φ to $\{b \in c_{00} \mid \sup_{n\in\mathbb{N}} \Phi(a\mathbb{1}_{[1,n]}) < \infty\} =: c_\Phi(\mathbb{N})$ in a natural way by

$$\Phi(a) := \sup_{n\in\mathbb{N}} \Phi(a\mathbb{1}_{[1,n]}), \qquad a \in c_\Phi(\mathbb{N}).$$

$c_\Phi(\mathbb{N})$ is called the natural domain of Φ. We recall some properties of $c_\Phi(\mathbb{N})$.

2.3.10 Proposition. *Let Φ be a s.n.-function. Then the natural domain of Φ has the following properties.*

i) $\forall a, b \in c_\Phi(\mathbb{N}): \quad a + b \in c_\Phi(\mathbb{N})$.

ii) $\forall \alpha \in \mathbb{R}, a \in c_\Phi(\mathbb{N}): \quad \alpha a \in c_\Phi(\mathbb{N})$

iii) $\forall a \in c_0, b \in c_\Phi(\mathbb{N}): \; a \lll b \Rightarrow a \in c_\Phi(\mathbb{N})$.

iv) $\forall a \in c_0: \quad a \in c_\Phi(\mathbb{N}) \Leftrightarrow a^* \in c_\Phi(\mathbb{N})$.

2.3.11 Theorem. *Let Φ be a s.n.-function. Then*

$$\mathscr{S}_\Phi := \big\{T \in \mathcal{K}(\mathcal{H}) \;\big|\; s(T) \in c_\Phi(\mathbb{N})\big\}, \quad \|T\|_{\mathscr{S}_\Phi} := \Phi(s(T)), \qquad T \in \mathscr{S}_\Phi,$$

defines a s.n.-ideal.

Denote by $\mathscr{S}_\Phi^o$ the closure of the finite rank operators $\mathcal{F}$ in $\mathscr{S}_\Phi$, i.e. $\mathscr{S}_\Phi^o := \overline{\mathcal{F}}^{\|\cdot\|_{\mathscr{S}_\Phi}}$. Then $\mathscr{S}_\Phi^o$ is a separable s.n.-ideal. It is given as

$$\mathscr{S}_\Phi^o = \big\{T \in \mathscr{S}_\Phi \;\big|\; \lim_{n\to\infty} \Phi(s_{n+1}(T), s_{n+2}(T), \dots) = 0\big\}.$$

We have $\mathscr{S}_\Phi^o \subseteq \mathscr{S}_\Phi$, ad equality holds if and only if $\mathscr{S}_\Phi$ is separable.

2.3.12 Remark.

i) If $\mathscr{S}_\Phi^o \neq \mathscr{S}_\Phi$, there exist infinitely many s.n.-ideals (whose norm is given by $\|\cdot\|_{\mathscr{S}_\Phi}$) between $\mathscr{S}_\Phi^o$ and $\mathscr{S}_\Phi$, see [Rus69b, Theorems 2,3] (proofs are published in [Rus69c]).

ii) Every separable s.n.-ideal is of the form $\mathscr{S}_\Phi^o$ with some s.n.-function Φ. However, there exist s.n.-ideals whose norm is not of the form $\|\cdot\|_{\mathscr{S}_\Phi}$ with a s.n.-function Φ (this is a consequence of [Rus69a, Theorema 2]).

The next theorem addresses the question, when two s.n.-functions $\Phi_1.\Phi_2$ generate the same operator ideal.

2.3.13 Theorem. *We say two s.n.-functions Φ_1, Φ_2 are equivalent if*

$$\big(\sup_{a\in c_{00}} \frac{\Phi_1(a)}{\Phi_2(a)} < \infty\big) \quad \wedge \quad \big(\sup_{a\in c_{00}} \frac{\Phi_2(a)}{\Phi_1(a)} < \infty\big),$$

holds. Let Φ_1, Φ_2 be two s.n.-functions. Then their associated s.n.-ideals $\mathscr{S}_{\Phi_1}, \mathscr{S}_{\Phi_2}$ coincide if and only if their s.n.-functions Φ_1, Φ_2 are equivalent.
Furthermore, let Φ be a s.n.-function. Then $\mathscr{S}_\Phi = \mathcal{K}(\mathcal{H})$ if and only if Φ is equivalent to Φ_∞.

Chapter 3

The Calkin correspondence

This chapter discusses the one-to-one correspondence between solid symmetric subspaces of c_0 and operator ideals in $\mathcal{K}(\mathcal{H})$, called the Calkin correspondence [Cal41]. Beginning with a short section to introduce replication operators and to present a few basic properties of them. The subsequent section establishes a correspondence between solid symmetric subspaces of c_0 and replication closed solid subcones of $c_0^{\downarrow}$ and a correspondence between operator ideals in $\mathcal{K}(\mathcal{H})$ and replication closed solid subcones of $c_0^{\downarrow}$. Proofs are adapted from [Gar67] w.r.t. rearrangement operators and replication closed subcones.

3.1 Replication operator

3.1.1 Definition. Define for every natural number n operators $\mathscr{P}_n : c_0 \to c_0$ by

$$\big((\mathscr{P}_n a)_j\big)_{j\in\mathbb{N}} := (a_{\lceil \frac{j}{n} \rceil})_{j\in\mathbb{N}} = (\underbrace{a_1, \ldots, a_1}_{n\text{-times}}, \underbrace{a_2, \ldots, a_2}_{n\text{-times}}, a_3 \ldots),$$

3.1.2 Lemma. *The following statement holds.*

i) $\forall n \in \mathbb{N}: \quad \mathscr{P}_n c_0^{\downarrow} \subseteq c_0^{\downarrow}$.

ii) $\forall n, k \in \mathbb{N}: \quad \mathscr{P}_{n^k} = \underbrace{\mathscr{P}_n \circ \ldots \circ \mathscr{P}_n}_{k-\text{times}}$.

iii) $\mathscr{P}_1 = \mathrm{id}_{c_0}$.

iv) $\forall n \in \mathbb{N}, a \in c_0^{\downarrow}: \quad \frac{1}{n}\mathscr{P}_n a \lll a$.

v) $\forall n_1, n_2 \in \mathbb{N}, a \in c_0^{\downarrow}: \quad n_1 \le n_2 \;\Rightarrow\; \mathscr{P}_{n_1} a \le \mathscr{P}_{n_2} a$.

vi) $\forall a^1, \ldots, a^k \in c_0: \quad \mathscr{P}_k((a^1)^* + \ldots + (a^k)^*) \ge (a^1 + \ldots + a^k)^*$.

vii) $\forall n_1, \ldots, n_k \in \mathbb{N}, a^1, \ldots, a^k \in c_0^{\downarrow}: \quad \big(\sum_{i=1}^k \mathscr{P}_{n_i} a^i\big)^* = \sum_{i=1}^k \mathscr{P}_{n_i} a^i$.

Proof. Obviously, i), ii), iii), iv), v) and vii) hold.
"vi)" : To see vi) let $a^1, \ldots, a^k \in c_0$ be given. Note that for every natural number $j \in \mathbb{N}$ the set

$$\big\{n \in \mathbb{N} \,\big|\, (a^1)_j^* + \ldots + (a^k)_j^* < |a_n^1 + \ldots + a_n^k|\big\}$$

contains at most $k(j-1)$ elements. We conclude

$$(a^1)_j^* + \ldots + (a^k)_j^* \ge (a^1 + \ldots + a^k)_{k(j-1)+1}^*$$

which leads to

$$\left(\mathscr{P}_k((a^1)^* + \ldots + (a^k)^*)\right)_j = (a^1)^*_{\lceil \frac{j}{k} \rceil} + \ldots + (a^k)^*_{\lceil \frac{j}{k} \rceil} \geq (a^1 + \ldots + a^k)^*_{k(\lceil \frac{j}{k} \rceil - 1)+1} \quad \geq (a^1 + \ldots + a^k)^*_j.$$

□

3.2 The algebraic theory

Using s-numbers one can show that operator ideals correspond bijectively to a certain class of linear subspaces of sequences. This class can be characterised in a neat way, namely by two simple geometric properties.

3.2.1 Definition. Let $\mathscr{E}$ be a linear subspace of c_0. Then $\mathscr{E}$ is called

i) solid, if for all $a \in c_0$ and $b \in \mathscr{E}$ with $|a| \leq |b|$ it holds that $a \in \mathscr{E}$.

ii) symmetric, if for all $a \in \mathscr{E}$ and permutations π of $\mathbb{N}$ it holds that $a_\pi := (a_{\pi(1)}, a_{\pi(2)}, \ldots) \in \mathscr{E}$.

Note the following fact.

3.2.2 Lemma. *A subset $\mathscr{E}$ of c_0 is a solid symmetric subspace of c_0, if and only if*

i) *$\mathscr{E}$ is a linear subspace.*

ii) *$\mathscr{E}$ is solid.*

iii) $\forall a \in \mathscr{E}, \iota \in \mathrm{Bij}(\mathbb{N}) : \quad \mathscr{R}_\iota a \in \mathscr{E}$.

Proof.
"$\Rightarrow$" : Trivially, i) and ii) holds. For iii) let $\iota \in \mathrm{Bij}(\mathbb{N})$ be given. If $|\operatorname{dom} \iota| < \infty$, then there exists a permutation σ of $\mathbb{N}$ such that $\sigma\big|_{\operatorname{dom} \iota} = \iota$. Then we have

$$|\mathscr{R}_\iota a| = \mathscr{R}_\iota |a| \leq \mathscr{R}_\sigma |a|.$$

On the other hand, if $|\operatorname{dom} \iota| = \infty$, we can find two subsets $M_1, M_2 \subseteq \operatorname{dom} \iota$ such that $|M_1| = |M_2| = \infty$ and $\operatorname{dom} \iota = M_1 \cup M_2$ hold. Subsequently, we find two permutations σ_1, σ_2 of $\mathbb{N}$ satisfying

$$\iota(n) = \begin{cases} \sigma_1(n) & n \in M_1, \\ \sigma_2(n) & n \in M_2. \end{cases}$$

Then we have the following pointwise estimate for the rearranged sequence $\mathscr{R}_\iota a$.

$$|\mathscr{R}_\iota a| = \mathscr{R}_\iota |a| = \mathscr{R}_{\sigma_1} |a| \cdot \mathbb{1}_{M_1} + \mathscr{R}_{\sigma_2} |a| \cdot \mathbb{1}_{M_2} \leq \mathscr{R}_{\sigma_1} |a| + \mathscr{R}_{\sigma_2} |a|$$

Note the fact that $a \in \mathscr{E}$ if and only if $|a| \in \mathscr{E}$. Due to $\mathscr{E}$ being solid and symmetric, we have in both cases $|\mathscr{R}_\iota a| \in \mathscr{E}$ and property iii) follows.
"$\Leftarrow$" : This implication is obvious. □

The connection between solid symmetric subspaces of c_0 and ideals of compact operators is established via a third class of objects, namely certain subcones of $c_0^\downarrow$.

3.2.3 Definition. Let $\mathcal{G}$ be a subcone of $c_0^{\downarrow}$. Then $\mathcal{G}$ is called

i) solid, if for all $a \in c_0^{\downarrow}$ and $b \in \mathcal{G}$ with $a \leq b$ it holds that $a \in \mathcal{G}$.

ii) replication closed, if for all $a \in \mathcal{G}$ and $n \in \mathbb{N}$ it holds that $\mathscr{P}_n a \in \mathcal{G}$.

Note the following fact.

3.2.4 Lemma. *A subset $\mathcal{G}$ of $c_0^{\downarrow}$ is a solid replication closed subcone of $c_0^{\downarrow}$, if and only if*

i) $\forall a, b \in \mathcal{G}: \quad a + b \in \mathcal{G}.$

ii) $\forall a \in c_0^{\downarrow}, b \in \mathcal{G}: \quad a \leq b \;\Rightarrow\; a \in \mathcal{G}.$

iii) $\forall a \in \mathcal{G}: \quad \mathscr{P}_2 a \in \mathcal{G}.$

Proof.
"$\Rightarrow$" : Clearly, properties i), ii) and iii) hold.
"$\Leftarrow$" : Let $a \in \mathcal{G}$ and $\lambda > 0$ be given. Choose $n \in \mathbb{N}$ with $n \geq \lambda$, then

$$\lambda a \leq na = \underbrace{a + \ldots + a}_{n\text{-times}} \in \mathcal{G},$$

and hence also $\lambda a \in \mathcal{G}$. Clearly, $\mathcal{G}$ is solid. To see replication closedness of $\mathcal{G}$, let $n \in \mathbb{N}$ be given. Remembering the computation rules for replication operators, cf. Lemma 3.1.2, it follows that

$$\forall a \in c_0^{\downarrow}: \quad \mathscr{P}_n a \leq \mathscr{P}_{2^n} a = (\underbrace{\mathscr{P}_2 \circ \ldots \circ \mathscr{P}_2}_{n\text{-times}}) a \in \mathcal{G},$$

and we obtain $\mathscr{P}_n a \in \mathscr{E}$. □

3.2.5 Theorem. *The assignment*

$$\mathscr{E} \mapsto \mathscr{E} \cap c_0^{\downarrow}$$

establishes a bijection between the set of all solid symmetric subspaces of c_0 and the set of all solid replication closed subcones of $c_0^{\downarrow}$. Its inverse is given by

$$\mathcal{G} \mapsto \{a \in c_0 \mid a^* \in \mathcal{G}\}.$$

Proof.

▷ Let $\mathscr{E}$ be a solid symmetric subspace of c_0. To see that $\mathscr{E} \cap c_0^{\downarrow}$ is indeed a solid replication closed subcone of $c_0^{\downarrow}$ it is sufficient to show that $\mathscr{E} \cap c_0^{\downarrow}$ satisfies i) - iii) from Lemma 3.2.4. Properties i) and ii) clearly hold. To prove iii) let $a \in \mathscr{E} \cap c_0^{\downarrow}$ be given. Define two functions ι_1, ι_2 by

$$\iota_1 : \begin{cases} \{n \in \mathbb{N} \mid n \text{ even}\} \to \mathbb{N} \\ 2n \mapsto n \end{cases} \qquad \iota_2 : \begin{cases} \{n \in \mathbb{N} \mid n \text{ odd}\} \to \mathbb{N} \\ 2n-1 \mapsto n \end{cases}$$

Then, using Lemma 3.2.2, we obtain

$$\mathscr{P}_2 a = (a_1, a_1, a_2, a_2, a_3, \ldots) = (a_1, 0, a_2, 0, \ldots) + (0, a_1, 0, a_2, \ldots) = \underbrace{\mathscr{R}_{\iota_1} a}_{\in \mathscr{E}} + \underbrace{\mathscr{R}_{\iota_2} a}_{\in \mathscr{E}}.$$

Hence, $\mathscr{P}_2 a \in \mathscr{E} \cap c_0^{\downarrow}$.

▷ Let $\mathcal{G}$ be a solid replication closed subcone of $c_0^{\downarrow}$, and define the set

$$\mathscr{E}_{\mathcal{G}} := \{a \in c_0 \mid a^* \in \mathcal{G}\}.$$

To see that $\mathscr{E}_{\mathcal{G}}$ is a linear subspace of c_0, let $a, b \in \mathscr{E}_{\mathcal{G}}$. From $\mathscr{P}_2(a^* + b^*) \in \mathcal{G}$ and Lemma 3.1.2 vi), we obtain $(a+b)^* \in \mathcal{G}$, and thus $a + b \in \mathscr{E}_{\mathcal{G}}$. Clearly, $\mathscr{E}_{\mathcal{G}}$ is closed under scalar multiplication.

Now let $a \in c_0$ and $b \in \mathscr{E}_{\mathcal{G}}$ be arbitrary with $|a| \leq |b|$. Then $a^* \leq b^*$ by Lemma 1.1.3 v), and since $\mathcal{G}$ is solid, we obtain $a^* \in \mathcal{G}$. Thus $a \in \mathscr{E}_{\mathcal{G}}$. For every zero sequence a and permutation π of $\mathbb{N}$ it holds that $(a_\pi)^* = a^*$, and we conclude that $\mathscr{E}_{\mathcal{G}}$ is symmetric.

▷ For some appropriate $\iota \in \mathrm{Bij}(\mathbb{N})$ we have $a^* = \mathscr{R}_\iota |a|$ and $|a| = \mathscr{R}_{\iota^{-1}} a^*$. Invoking Lemma 3.2.2 we know that a is an element of a solid symmetric subspace $\mathscr{E}$ of c_0 if and only if $a^* \in \mathscr{E}$. From this we see that $\mathscr{E} = \mathscr{E}_{\mathscr{E} \cap c_0^{\downarrow}}$. The fact that $\mathcal{G} = \mathscr{E}_{\mathcal{G}} \cap c_0^{\downarrow}$ is clear, and we conclude that

$$\mathscr{E} \mapsto \mathscr{E} \cap c_0^{\downarrow} \quad \text{and} \quad \mathcal{G} \mapsto \mathscr{E}_{\mathcal{G}}$$

are inverse to each other.

Therefore, the correspondence between solid symmetric subspaces of c_0 and solid replication closed subcones of $c_0^{\downarrow}$ is bijective.

□

Coming from the side of operator ideals, we can also connect to solid replication closed cones. This connection proceeds via s-numbers.

3.2.6 Theorem. *The assignment*

$$\mathscr{S} \mapsto \{(s_n(T))_{n \in \mathbb{N}} \mid T \in \mathscr{S}\}$$

establishes a bijection between the set of all operator ideals on $\mathcal{K}(\mathcal{H})$ and the set of all solid replication closed subcones of $c_0^{\downarrow}$. Its inverse is given by

$$\mathcal{G} \mapsto \{T \in \mathcal{K}(\mathcal{H}) \mid s(T) \in \mathcal{G}\}.$$

Proof.

▷ Let $\mathscr{S} \subseteq \mathcal{K}(\mathcal{H})$ be an operator ideal and set

$$\mathcal{G} := \{s(T) \mid T \in \mathscr{S}\}.$$

To see that $\mathcal{G}$ is indeed a solid replication closed subcone of $c_0^{\downarrow}$, it is sufficient to show that $\mathcal{G}$ satisfies i) - iii) from Lemma 3.2.4. To show that property i) holds w.r.t. $\mathcal{G}$, let $a, b \in \mathcal{G}$. The fact that $\mathcal{G} \subseteq c_0^{\downarrow}$, and by Lemma 2.2.4 iv), the diagonal operators E_a, E_b w.r.t. any fixed orthonormal basis in $\mathcal{H}$ satisfy

$$s(E_a) = a, \quad s(E_b) = b.$$

Therefore $E_a, E_b \in \mathscr{S}$. Again, by computation rules of Lemma 2.2.4, we obtain

$$s(E_a + E_b) = (a + b)^* = a + b,$$

leading us to $a + b \in \mathcal{G}$.
To see ii) let $a \in c_0^{\downarrow}$ and $b \in \mathcal{G}$ with $a \leq b$ be given. Again, we consider diagonal operators E_a, E_b w.r.t any fixed orthonormal basis in $\mathcal{H}$. Once more invoking Lemma 2.2.4 iv) gives us

$$s(E_a) = a \leq b = s(E_b).$$

Remembering Proposition 2.2.2, we conclude that $E_a \in \mathscr{S}$, and therefore $a \in \mathcal{G}$.
To prove iii) let $a \in \mathcal{G}$. Define sequences $a^1, a^2 \in c_0$ by

$$a^1 := (a_1, 0, a_2, 0, \ldots), \quad a^2 := (0, a_1, 0, a_2, \ldots).$$

and consider their diagonal operators E_{a^1}, E_{a^2} w.r.t. any fixed orthonormal basis. Keeping in mind that

$$s(E_{a^1}) = (a^1)^* = a, \quad s(E_{a^2}) = (a^2)^* = a,$$

yields $E_{a^1}, E_{a^2} \in \mathscr{S}$ and in particular $E_{a^1} + E_{a^2} \in \mathscr{S}$. Lemma 2.2.4 gives us the final result

$$s(E_{a^1} + E_{a^2}) = s(E_{a^1+a^2}) = (a^1 + a^2)^* = (a_1, a_1, a_2, a_2, \ldots)^* = \mathscr{P}_2 a \in \mathcal{G}.$$

▷ Let $\mathcal{G}$ be a solid replication closed subcone of $c_0^{\downarrow}$, and define

$$\mathscr{S}_{\mathcal{G}} := \{T \in \mathcal{K}(\mathcal{H}) \mid s(T) \in \mathcal{G}\}.$$

First we prove that $\mathcal{G}$ is indeed a linear subspace of $\mathcal{K}(\mathcal{H})$. To this end, let $\lambda \in \mathbb{K}$ and $T \in \mathscr{S}_{\mathcal{G}}$. Since $\mathcal{G}$ is a subcone, it follows that $\lambda T \in \mathscr{S}_{\mathcal{G}}$. Now let $S, T \in \mathscr{S}_{\mathcal{G}}$. We can estimate

$$\begin{aligned}
s(S+T) &= (s_1(S+T), s_2(S+T), \ldots) \\
&= (s_1(S+T), 0, s_3(S+T), 0, \ldots) + (0, s_2(S+T), 0, s_4(S+T), \ldots) \\
&\leq (s_1(S+T), 0, s_3(S+T), 0, \ldots) + (0, s_1(S+T), 0, s_3(S+T), \ldots) \\
&\overset{2.1.5\ \text{ii)}}{\leq} (s_1(S) + s_1(T), s_1(S) + s_1(T), s_2(S) + s_2(T), s_2(S) + s_2(T), \ldots) = \mathscr{P}_2\big(s(S) + s(T)\big),
\end{aligned}$$

which leads to $S + T \in \mathscr{S}_{\mathcal{G}}$.
Now let $T \in \mathscr{S}_{\mathcal{G}}$ and $A, B \in \mathcal{B}(\mathcal{H})$ be arbitrary. By Theorem 2.1.3 we have

$$s(ATB) \leq \|A\|\|B\| s(T).$$

Since $\mathcal{G}$ is a solid subcone of $c_0^{\downarrow}$, we obtain $ATB \in \mathscr{S}_{\mathcal{G}}$.

▷ Invoking Proposition 2.2.2 we know that T is an element of an operator ideal $\mathscr{S}$ if and only if there exists an operator $S \in \mathscr{S}$ such that $s(S) = s(T)$. From this we see that $\mathscr{S}_{\mathcal{G}} = \mathscr{S}$. The fact that $\{T \in \mathcal{K}(\mathcal{H}) \mid s(T) \in \mathscr{S}_{\mathcal{G}}\} = \mathcal{G}$ is clear, and we conclude that

$$\mathscr{S} \mapsto \{(s_n(T))_{n \in \mathbb{N}} \mid T \in \mathscr{S}\} \quad \text{and} \quad \mathcal{G} \mapsto \mathscr{S}_{\mathcal{G}}$$

are inverse to each other.

Therefore, the correspondence between operator ideals and solid replication closed subcones of $c_0^{\downarrow}$ is bijective.

□

3.2.7 Remark. Putting together Theorem 3.2.5 and Theorem 3.2.5, we obtain a bijective correspondence between operator ideals in $\mathcal{K}(\mathcal{H})$ and solid symmetric subspaces of c_0.

This correspondence is given explicitly as follows:

i) $\mathscr{E}_{\mathscr{S}} := \{a \in c_0 \mid E_a \in \mathscr{S}\}$ for $\mathscr{S}$ ideal in $\mathcal{K}(\mathcal{H})$ and diagonal operator E w.r.t. any orthonormal basis.

ii) $\mathscr{S}_{\mathscr{E}} := \{T \in \mathcal{K}(\mathcal{H}) \mid s(T) \in \mathscr{E}\}$ for $\mathscr{E}$ solid symmetric subspace of c_0.

Chapter 4

Geometry of sequences II

This chapter is split into 3 connected parts. The first one, analyzes specific subsets of beginning sections w.r.t. Hardy-Littlewood majorization and characterizes their extreme points. The second section discusses uniform Hardy-Littlewood majorization introduced by J. Kalton und F.A. Sukochev in [KS08] and relates it to the partial orders $\prec, \lll$ from section 1.2. In the third section we consider extended Minkowski functionals associated with beginning sections of each partial order $\prec, \lll, \trianglelefteq$.

4.1 Beginning sections w.r.t. Hardy-Littlewood majorization

We analyze the partial order $\lll$ by clarifying the structure of its beginning sections, in fact, for slightly more general subsets.

4.1.1 Definition. For a subset $B \subseteq \mathbb{N}$ and an element $c \in c_0$ we set

$$\Omega_c^B := \big\{ a \in c_0 \bigm| \operatorname{supp} a \subseteq B, a \lll c \big\}.$$

Then $\Omega_c^{\mathbb{N}}$ is nothing but the beginning section $[0,c]_{\lll}$. To start with we collect some algebraic properties of sets Ω_c^B.

4.1.2 Lemma. *Let $B \subseteq \mathbb{N}$ and $c \in c_0$.*

i) $\Omega_c^B = \Omega_{c^*}^B = \Omega_{c^*\mathbb{1}_{|B|}}^B$.

ii) *For each $\iota \in \operatorname{Bij}(\mathbb{N})$ with $B \subseteq \operatorname{dom}\iota$, the operator $\mathscr{R}_\iota\big|_{\Omega_c^{\iota(B)}}$ is a bijection of $\Omega_c^{\iota(B)}$ onto Ω_c^B.*

iii) *Ω_c^B is absolutely convex.*

Proof.
"i)" : Since $a \lll c$ if and only if $a \lll c^*$, the equality $\Omega_c^B = \Omega_{c^*}^B$ holds. If $|B| = \infty$, the assertion $\Omega_{c^*}^B = \Omega_{c^*\mathbb{1}_{|B|}}^B$ trivially holds. Thus assume $|B| < \infty$. From Lemma 1.1.2 iv) we have for every natural number $N > |\operatorname{supp} a|$ that $a_N^* = 0$. Hence, $a \lll c^*$ if and only if $a \lll c^*\mathbb{1}_{|B|}$.
"ii)" : Let $\iota \in \operatorname{Bij}(\mathbb{N})$ and $a \in \Omega_c^{\iota(B)}$. By Lemma 1.2.7 viii) we have that

$$\mathscr{R}_\iota a \lll a,$$

and in particular $\mathscr{R}_\iota a \lll c$. Since

$$\operatorname{supp} \mathscr{R}_\iota a = \iota^{-1}(\operatorname{supp} a) \subseteq \iota^{-1}(\iota(B)) = B,$$

we obtain that the operator $\mathscr{R}_\iota\big|_{\Omega_c^{\iota(B)}}$ maps $\Omega_c^{\iota(B)}$ onto Ω_c^B. An analogous argumentation yields

$$\mathscr{R}_{\iota^{-1}}\big|_{\Omega_c^B} : \Omega_c^B \to \Omega_c^{\iota(B)}.$$

Due to

$$\mathscr{R}_{\iota^{-1}}\big|_{\Omega_c^B} \circ \mathscr{R}_\iota\big|_{\Omega_c^{\iota(B)}} = (\mathscr{R}_{\iota^{-1}} \circ \mathscr{R}_\iota)\big|_{\Omega_c^{\iota(B)}} = M_{\mathbb{1}_{\operatorname{ran}\iota}}\big|_{\Omega_c^{\iota(B)}} = id_{\Omega_c^{\iota(B)}},$$
$$\mathscr{R}_\iota\big|_{\Omega_c^{\iota(B)}} \circ \mathscr{R}_{\iota^{-1}}\big|_{\Omega_c^B} = (\mathscr{R}_\iota \circ \mathscr{R}_{\iota^{-1}})\big|_{\Omega_c^B} = M_{\mathbb{1}_{\operatorname{dom}\iota}}\big|_{\Omega_c^B} = id_{\Omega_c^B},$$

$\mathscr{R}_\iota\big|_{\Omega_c^{\iota(B)}}$ is a bijection of $\Omega_c^{\iota(B)}$ onto Ω_c^B.

"iii)" : Let $\sum_{i=1}^k \lambda_i c^i$ be an absolute convex combination in Ω_c^B. We estimate

$$\sum_{j=1}^n \big(\sum_{i=1}^k \lambda_i c^i\big)_j^* \overset{1.2.7\ v)}{\leq} \sum_{j=1}^n \sum_{i=1}^k |\lambda_i| (c^i)_j^* = \sum_{i=1}^k |\lambda_i| \sum_{j=1}^n (c^i)_j^* \leq \sum_{i=1}^k |\lambda_i| \sum_{j=1}^n c_j^* \leq \sum_{j=1}^n c_j^*.$$

It follows that $\sum_{i=1}^k \lambda_i c^i \in \Omega_c^B$. □

Our first aim is to show an important topological property of Ω_c^B. To this end we need an elementary fact about pointwise convergent sequences. Thereby, we say that a family of zero sequences $\mathcal{M} \subseteq c_0$ is equicontinuous if it has this property considered as a subset of the space $C(\mathbb{N} \cup \{\infty\})$ of continuous functions on the one-point compactification of $\mathbb{N}$. Explicitly, this means that

$$\forall \varepsilon > 0, \exists n_0 \in \mathbb{N}, \forall n \geq n_0, \forall a \in \mathcal{M} : \quad |a_n| < \varepsilon.$$

4.1.3 Proposition. *Let $B \subseteq \mathbb{N}$ and $c \in c_0$. Then Ω_c^B is a bounded subset of c_0 and is closed in $\mathbb{R}^\mathbb{N}$ w.r.t. pointwise convergence.*

In particular, Ω_c^B is weakly compact in c_0.

Proof. The crucial observation is that $\Omega_c^\mathbb{N} \cap c_0^\downarrow$ is equicontinuous. To see this, let $\varepsilon > 0$ be given. Choose $n_1 \in \mathbb{N}$ such that $c_{n_1}^* \leq \frac{\varepsilon}{2}$. Then, for all $a \in \Omega_c^\mathbb{N} \cap c_0^\downarrow$ and $n \in \mathbb{N}$ with

$$n > \max\{\frac{2}{\varepsilon} \sum_{j=1}^{n_1} c_j^*, n_1\}$$

it holds that

$$n a_n \leq \sum_{j=1}^n a_j \leq \sum_{j=1}^n c_j^* = \sum_{j=1}^{n_1} c_j^* + \sum_{j=n_1+1}^n c_j^* \leq n\frac{\varepsilon}{2} + (n - n_1)\frac{\varepsilon}{2} \leq n\varepsilon.$$

Now let $(a_i)_{i\in I}$ be a net in $\Omega_c^\mathbb{N}$ which converges pointwise to some $a \in \mathbb{R}^\mathbb{N}$. Since $(\Omega_c^B)^* \subseteq \Omega_c^\mathbb{N} \cap c_0^\downarrow$, we may apply lemma 1.1.4 ii). This shows that $a \in c_0$ and that for each $\varepsilon > 0$ we find $i \in I$ for all $N \in \mathbb{N}$ with

$$\sum_{j=1}^N a_j^* \leq \sum_{j=1}^N (a^i)_j^* + N\varepsilon.$$

The sum on the right cannot exceed $\sum_{j=1}^N c_j^* + N\varepsilon$. Since $\varepsilon > 0$ was arbitrary, we conclude that $a \lll c$. The property that $\operatorname{supp} a \subseteq B$ is obviously inherited from the elements a^i. Thus, indeed, Ω_c^B is closed in $\mathbb{R}^\mathbb{N}$ w.r.t. pointwise convergence. Clearly, $\|a\|_\infty \leq \|c\|_\infty$ for all $a \in \Omega_c^\mathbb{N}$.

To show weak compactness in c_0, we pass onto the bidual $c_0'' \cong \ell^\infty$. Since ω^*-convergence in ℓ^∞ implies pointwise convergence, we know that Ω_c^B is ω^*-closed in ℓ^∞. By the Banach-Alaoglu theorem, it is ω^*-compact in the bidual and hence weakly compact in the original space c_0. □

4.1.4 Remark. By the Arzelá-Ascoli theorem the set $c_0^\downarrow \cap \Omega_c^B$ is $\|\cdot\|_\infty$-compact in c_0. The whole set Ω_c^B is certainly $\|\cdot\|_\infty$-closed and $\|\cdot\|_\infty$-bounded. Moreover, if B is infinite and $c \neq 0$, then Ω_c^B is not $\|\cdot\|_\infty$-compact. To see this note that each sequence

$$a^n := (0, \dots, 0, \underset{\substack{\uparrow \\ n\text{-th place}}}{\|c\|_\infty}, 0, \dots, 0), \qquad n \in B,$$

belongs to Ω_c^B. If B is infinite, this family of sequences is not equicontinuous.

For a convex subset C of some linear space, we denote by $\operatorname{Ext} C$ the set of its extreme points. For any subset M of a linear space $\operatorname{conv} M$ denotes the convex hull of M.

4.1.5 Corollary. *Let $B \subseteq \mathbb{N}$ and $c \in c_0$. Then*

$$\Omega_c^B = \overline{\operatorname{conv} \operatorname{Ext}(\Omega_c^B)},$$

where the closure is understood w.r.t. $\|\cdot\|_\infty$.

Proof. The Krein-Milman theorem gives $\Omega_c^B = \overline{\operatorname{conv}(\operatorname{Ext}(\Omega_c^B))}^\omega$. By convexity, the weak closure coincides with the $\|\cdot\|_\infty$-closure. □

To understand the geometry of Ω_c^B, it is left to determine its extreme points.

4.1.6 Theorem. *Let $B \subseteq \mathbb{N}$ and $c \in c_0$. Then*

$$\operatorname{Ext}(\Omega_c^B) = \left\{ a \in c_0 \;\middle|\; \operatorname{supp} a \subseteq B, a \sim c^* \mathbb{1}_{|B|} \right\} \tag{4.1}$$

The essential step towards the proof of this theorem is the following assertion.

4.1.7 Lemma. *Let $a, c \in c_0^\downarrow$ with $a \lll c$ and $a \neq c$, and set*

$$A := \begin{cases} |\operatorname{supp} a| & \text{if } |\operatorname{supp} a| \geq |\operatorname{supp} c|, \\ |\operatorname{supp} a| + 1 & \text{if } |\operatorname{supp} a| < |\operatorname{supp} c|. \end{cases}$$

Then

$$a \notin \operatorname{Ext}(\Omega_c^A).$$

Proof. We start with a preliminary observation. Namely, that for all $n > 1$

$$\sum_{j=1}^{n-1} c_j = \sum_{j=1}^{n-1} a_j \Rightarrow c_n \geq a_n \tag{4.2}$$

$$\Big(\sum_{j=1}^{n-1} c_j > \sum_{j=1}^{n-1} a_j \quad \wedge \quad \sum_{j=1}^{n} c_j = \sum_{j=1}^{n} a_j\Big) \Rightarrow a_{n+1} < a_n \tag{4.3}$$

The first implication follows immediately from $a \lll c$, and the second from $a_{n+1} \leq c_{n+1} \leq c_n < a_n$.

Next we show that

$$\left\{ n \in \mathbb{N} \;\middle|\; a_n \neq c_n \right\} \cap \operatorname{supp} c \neq \emptyset. \tag{4.4}$$

Assume the contrary, i.e. that $a_n = c_n$ holds for all $n \in \operatorname{supp} c$. For each $n \in \mathbb{N}$ with $n > |\operatorname{supp} c|$, it holds that

$$0 \leq \sum_{J=1}^{n} c_j - \sum_{j=1}^{n} a_j = - \sum_{j=|\operatorname{supp} c|+1}^{n} a_j.$$

We conclude that $a_n = 0$ for all $n > |\operatorname{supp} c|$, and in total that $a = c$. This establishes (4.4).

Set

$$k := \min\left\{n \in \mathbb{N} \;\middle|\; a_n \neq c_n\right\}.$$

Then $k \in \operatorname{supp} c$; note here that $\operatorname{supp} c$ (and also $\operatorname{supp} a$) are beginning sections of $\mathbb{N}$. Moreover, if $k > 1$, it also follows that $a_k < c_k \leq c_{k-1} = a_{k-1}$.
We distinguish two cases.

1. Case: $a_{k+1} = 0$
The implication (4.2), and the fact that $c_k \neq a_k$, yields that $a_k < c_k$. Moreover, for all $n \geq k$, we have

$$\sum_{j=1}^{n} c_j - \sum_{j=1}^{n} a_j = \sum_{j=k}^{n} c_j - a_k \geq c_k - a_k > 0.$$

Choose $\varepsilon > 0$ such that

$$0 < \varepsilon < \begin{cases} c_k - a_k & \text{if } k = 1, \\ \min\left\{c_k - a_k, a_{k-1} - a_k\right\} & \text{if } k > 1, \end{cases}$$

and set

$$a^+ := a + \varepsilon \mathbb{1}_{\{k\}}, \quad a^- := a - \varepsilon \mathbb{1}_{\{k\}}.$$

Our choice of ε ensures that

$$\begin{aligned} \text{if } k > 1: &\quad a_k + \varepsilon < a_{k-1}, \quad |a_k - \varepsilon| < a_{k-1}, \\ \forall n \geq k: &\quad \sum_{j=1}^{n} c_j - \sum_{j=1}^{n} a_j^+ \geq c_k - a_k - \varepsilon > 0, \\ \forall n \geq k: &\quad \sum_{j=1}^{n} c_j - \sum_{j=1}^{n} |a_j^-| \geq c_k - a_k - \varepsilon > 0. \end{aligned}$$

The first two relations show that $(a^+)^* = a^+$, $(a^-)^* = |a^-|$, and the second two relations show that $a^\pm \lll c$. Note that $a_n^\pm = a_n$ for all $n < k$, and hence certainly

$$\forall n < k: \quad \sum_{j=1}^{n} c_j \geq \sum_{j=1}^{n} a_j$$

holds. We see that $a^\pm \in \Omega_c^{|\operatorname{supp} a|+1}$, and clearly $a = \frac{1}{2}(a^+ + a^-)$, while $a^+ \neq a^-$.

2. Case: $a_{k+1} > 0$
Set

$$m := \max L_{a_{k+1}}(a).$$

Since $\sum_{j=1}^{k} c_j > \sum_{j=1}^{k} a_j$, we may invoke (4.3) to obtain that

$$\forall n \in L_{a_{k+1}}(a) \backslash \{m\}: \quad \sum_{j=1}^{n} c_j > \sum_{j=1}^{n} a_j.$$

Choose $\varepsilon > 0$ such that

$$\begin{aligned}\forall k \le n < m: \quad 0 <& \varepsilon < \sum_{j=1}^{n} c_j - \sum_{j=1}^{n} a_j,\\ & \varepsilon < a_m - a_{m-1},\\ & \varepsilon < a_{k-1} - a_k, \qquad \text{if } k > 1,\\ & \varepsilon < a_k - a_{k+1}, \qquad \text{if } a_{k+1} < a_k,\end{aligned}$$

and set

$$a^+ := a + \varepsilon \mathbb{1}_{\{k\}} - \varepsilon \mathbb{1}_{\{m\}}, \quad a^- := a - \varepsilon \mathbb{1}_{\{k\}} + \varepsilon \mathbb{1}_{\{m\}}.$$

Our choice of ε ensures that $(a^+)^* = a^+$ and

$$\forall k \le n < m: \quad \sum_{j=1}^{n} c_j \ge \sum_{j=1}^{n} a_j^+, \qquad \forall n \in \mathbb{N}\backslash[k, m): \quad \sum_{j=1}^{n} a_j^+ = \sum_{j=1}^{n} a_j.$$

Furthermore, $(a^-)^* = a^-$ if $a_{k+1} < a_k$, and $(a^-)^* = a^+$ if $a_{k+1} = a_k$. We see that $a^\pm \in \Omega_c^{|\operatorname{supp} a|}$, and clearly $a = \frac{1}{2}(a^+ + a^-)$, while $a^+ \ne a^-$. □

The converse to Lemma 4.1.7 is easy to see.

4.1.8 Lemma. *Let $c \in c_0^\downarrow$. Then*

$$c \in \operatorname{Ext}(\Omega_c^{\mathbb{N}}).$$

In particular $c\mathbb{1}_{|B|} \in \operatorname{Ext}(\Omega_c^B)$ for every subset $B \subseteq \mathbb{N}$.

Proof. Consider a representation of c as a convex combination

$$c\mathbb{1}_{|B|} = \lambda c^1 + (1 - \lambda) c^2$$

with some $\lambda \in (0, 1)$ and $c^1, c^2 \in \Omega_c^B$. Then for each $n \in B$,

$$\sum_{j=1}^{n} c_j = \lambda \sum_{j=1}^{n} c_j^1 + (1 - \lambda) \sum_{j=1}^{n} c_j^2.$$

If $c^1 \lll c$ and $c^2 \lll c$, then both sums on the right cannot exceed the value of the sum on the left. It follows that

$$\forall n \in B: \quad \sum_{j=1}^{n} c_j^1 = \sum_{j=1}^{n} c_j^2 = \sum_{j=1}^{n} c_j,$$

and in total that $c^1 = c^2 = c\mathbb{1}_{|B|}$. □

The proof of Theorem 4.1.6 is now merely a matter of reduction.

Proof (of Theorem 4.1.6). Let $a \in \Omega_c^B$ be given. Since $\Omega_c^B = \Omega_{c^*\mathbb{1}_{|B|}}^B$, we may assume w.l.o.g. that $c = c^*\mathbb{1}_{|B|}$. According to remark 1.1.9, choose $\iota \in \operatorname{Bij}(\mathbb{N})$ such that $\mathscr{R}_\iota |a| = a^*$ holds, and that $\iota(|B|) = B$ if $|\operatorname{supp} a| < \infty$.

To prove the inclusion "⊆" in (4.1), we distinguish two cases.

1. Case: $|\operatorname{supp} a| = |B|$
Since $\iota(|\operatorname{supp} a|) = \operatorname{supp} a$, the linear operator $\mathscr{R}_\iota$ induces a bijection of $\Omega_c^{\operatorname{supp} a}$ onto $\Omega_c^{|\operatorname{supp} a|}$. Thus

$$a \in \operatorname{Ext}(\Omega_c^{\operatorname{supp} a}) \Leftrightarrow a^* \in \operatorname{Ext}(\Omega_c^{|\operatorname{supp} a|}).$$

We have $|\operatorname{supp} a| = |B| \geq |\operatorname{supp} c|$, and Lemma 4.1.7 shows that

$$a^* \neq c \Rightarrow a^* \notin \operatorname{Ext}(\Omega_c^{|\operatorname{supp} a|})$$

Keeping in mind the trivial implication

$$a \in \operatorname{Ext}(\Omega_c^B) \Rightarrow a \in \operatorname{Ext}(\Omega_c^{\operatorname{supp} a}),$$

establishes the inclusion "$\subseteq$" in (4.1).

2. Case: $|\operatorname{supp} a| < |B|$
In this case $\operatorname{supp} a$ is finite, hence $\iota(|B|) = B$, and $|\operatorname{supp} a| + 1 \leq |B|$. $\mathscr{R}_\iota$ induces a bijection of Ω_c^B onto $\Omega_c^{|B|}$, which maps a to a^*. Thus

$$a \in \operatorname{Ext}(\Omega_c^B) \Leftrightarrow a^* \in \operatorname{Ext}(\Omega_c^{|B|}), \tag{4.5}$$

and Lemma 4.1.7 yields

$$a^* \neq c \Rightarrow a^* \notin \operatorname{Ext}(\Omega_c^{|\operatorname{supp} a|+1}) \Rightarrow a^* \notin \operatorname{Ext}(\Omega_c^{|B|}).$$

Again, we have shown the inclusion "$\subseteq$" in (4.1).

For the proof of "$\supseteq$" in (4.1), we assume that $a^* = c$, and we have

$$a = \lambda a^1 + (1-\lambda)a^2$$

with some $\lambda \in (0,1)$ and $a^1, a^2 \in \Omega_c^B$. Let $\zeta \in \{-1,1\}^{\mathbb{N}}$ such that $|a| = M_\zeta a = (\zeta_n a_n)_{n\in\mathbb{N}}$. Then for appropriate $\iota \in \operatorname{Bij}(\mathbb{N})$ we have $a^* = (\mathscr{R}_\iota \circ M_\zeta)a$. Then

$$a^* = \lambda \cdot \mathscr{R}_\iota(M_\zeta a^1) + (1-\lambda) \cdot \mathscr{R}_\iota(M_\zeta a^2),$$

By Lemma 4.1.8 we have a^* is an extreme point of $\Omega_a^{\mathbb{N}}$. Since $\Omega_c^{\mathbb{N}} = \Omega_a^{\mathbb{N}}$ and obviously $\mathscr{R}_\iota(M_\zeta a^1), \mathscr{R}_\iota(M_\zeta a^2) \in \Omega_c^{\mathbb{N}}$, we obtain

$$\mathscr{R}_\iota(M_\zeta a^1) = \mathscr{R}_\iota(M_\zeta a^2) = a^*.$$

Applying $M_\zeta \circ \mathscr{R}_{\iota^{-1}}$ and remembering that $\operatorname{ran} \iota \supseteq \operatorname{supp} a$, yields

$$a^1 \mathbb{1}_{\operatorname{supp} a} = a^2 \mathbb{1}_{\operatorname{supp} a} = a.$$

Moreover, $a^1, a^2 \lll a$. Invoking Lemma 1.2.7 vii) leads to $a^1 = a^2 = a$. We see that also in this case $a \in \operatorname{Ext} \Omega_c^B$. □

An important particular case occurs under a finiteness assumption.

4.1.9 Corollary. *Let $B \subseteq \mathbb{N}$ be finite and let $c \in c_0$. Then each element $a \in \Omega_c^B$ can be written as a convex combination of $|B|+1$ elements $b^1, \ldots, b^{|B|+1}$ with*

$$b^j \sim c^* \mathbb{1}_{|B|}, \qquad j \in \{1, \ldots, |B|+1\}.$$

Proof. The set of all $a \in c_0$ which are supported in B and satisfy $a \sim c^* \mathbb{1}_{|B|}$ is finite. Therefore, its convex hull is compact. Corollary 4.1.5 yields

$$\Omega_c^B = \operatorname{conv}\{a \in c_0 \bigm| \operatorname{supp} a \subseteq B, a \sim c^* \mathbb{1}_{|B|}\}.$$

We may consider Ω_c^B as a subset of the finite dimensional space $\mathbb{R}^{|B|}$. Carathéodory's theorem (see for example [Roc70, Theorem 17.1]) yields the assertion. □

4.1.10 Remark. It is interesting to observe that the extreme points of

$$\Omega_c^{\mathbb{N}} = [0, c]_{\lll}$$

already belong to a much smaller set. Namely, we have $[0, c]_{\prec} \subseteq [0, c]_{\lll}$, and obviously

$$\operatorname{Ext}[0, c]_{\lll} \subseteq [0, c]_{\prec}.$$

4.2 Uniform Hardy-Littlewood majorization

The quasi order $\prec$ is, by its definition as a pushforward with the operator $(\,.\,)^*$, compatible with $(\,.\,)^*$. A big backdraw is that it is not compatible with sums. Hardy-Littlewood majorization $\lll$ is a quasi order containing $\prec$ which is compatible with $(\,.\,)^*$ (just by its definition) and compatible with sums in the sense of Lemma 1.2.7 v). But, for certain purposes, $\lll$ is too large.

The relation of uniform Hardy-Littlewood majorization introduced below is a quasi order which lies in between $\prec$ and $\lll$ and has both compatibilities. This quasi order is of much more complex nature than Hardy-Littlewood majorization.

4.2.1 Definition. Let $\lambda \in \mathbb{N}$. For $a, b \in c_0$ we denote

$$a \trianglelefteq_\lambda b \quad :\Leftrightarrow \qquad \forall n \in \mathbb{N}, r \in \mathbb{N} \cup \{0\} : \quad \sum_{j=\lambda r+1}^{n} a_j^* \leq \sum_{j=r+1}^{n} b_j^*. \tag{4.6}$$

The union over all relations $\trianglelefteq_\lambda$ is called uniform Hardy-Littlewood majorization, and is denoted by $\trianglelefteq$. Explicitly,

$$a \trianglelefteq b \quad :\Leftrightarrow \qquad \exists \lambda \in \mathbb{N} : \quad a \trianglelefteq_\lambda b.$$

Some facts and computation rules for $\trianglelefteq_\lambda$ and $\trianglelefteq$ are:

4.2.2 Lemma. *The previously defined relations are related among each other as*

$$\prec \; = \; \trianglelefteq_1 \subsetneq \trianglelefteq_2 \subsetneq \trianglelefteq_3 \subsetneq \; \ldots \; \subsetneq \trianglelefteq_k \subsetneq \; \ldots \; \subseteq \trianglelefteq \subsetneq \lll . \tag{4.7}$$

The following statements holds.

i) $\forall \lambda \in \mathbb{N}, \forall a, b \in c_0, \beta > 0 : \quad a \trianglelefteq_\lambda b \;\Leftrightarrow\; \beta a \trianglelefteq_\lambda \beta b.$

ii) $\forall \lambda \in \mathbb{N}, \forall a, b \in c_0 : \quad a \trianglelefteq_\lambda b \;\Rightarrow\; |\operatorname{supp} a| \leq \lambda |\operatorname{supp} b|.$

iii) $\forall a, b, c \in c_0 : \quad (a \trianglelefteq_{\lambda_1} b \wedge b \trianglelefteq_{\lambda_2} c) \;\Rightarrow\; a \trianglelefteq_{\lambda_1 \cdot \lambda_2} c.$

iv) $\trianglelefteq$ *is a quasi order.*

Proof. i) – iv) are all trivial.
It is easy to see that all inclusions in (4.7) are strict. For $k \in \mathbb{N}$ with $k \geq 2$ consider sequences

$$a^k := (\underbrace{\frac{1}{k}, \ldots, \frac{1}{k}}_{k\text{-times}}, 0, 0, \ldots), \quad a^\infty := (\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots), \quad b := (1, 0, 0, \ldots).$$

Obviously, $a^k \not\prec b$ for every $k \in \mathbb{N}$. Furthermore, $a^k \trianglelefteq_k b$, $a^\infty \lll b$ but $a^k \ntrianglelefteq_{k-1} b$ and $a^\infty \ntrianglelefteq b$. □

Our goal is to show that $\trianglelefteq$ is indeed compatible with sums. To achieve this, we need a preparatory lemma.

4.2.3 Lemma. *Let $a \in c_0$. Then*

$$\forall n \in \mathbb{N}, r \in \mathbb{N}_0 : \quad \sum_{j=r+1}^{n} a_j^* = \max_{\substack{\iota \in \mathrm{Bij}(\mathbb{N}) \\ |\operatorname{dom}\iota| = n}} \min_{\substack{F \subseteq \operatorname{dom}\iota \\ |F| = n-r}} \sum_{j \in F} \big(\mathscr{R}_\iota |a|\big)_j. \tag{4.8}$$

In particular, the maximum is attained, in fact for $\iota = \kappa\big|_n$, where $\kappa \in \mathrm{Bij}(\mathbb{N})$ gives the nonincreasing rearrangement of a.

Proof.
"$\leq$" : Let $a \in c_0$ and $n \in \mathbb{N}, r \in \mathbb{N}_0$. Furthermore, let $\kappa \in \mathrm{Bij}(\mathbb{N})$ such that $\mathscr{R}_\iota |a| = a^*$. Then

$$\max_{\substack{\iota \in \mathrm{Bij}(\mathbb{N}) \\ |\operatorname{dom}\iota| = n}} \min_{\substack{F \subseteq \operatorname{dom}\iota \\ |F| = n-r}} \sum_{j \in F} \big(\mathscr{R}_\iota |a|\big)_j \geq \min_{\substack{F \subseteq n \\ |F| = n-r}} \sum_{j \in F} \big(\mathscr{R}_\kappa |a|\big)_j = \min_{\substack{F \subseteq n \\ |F| = n-r}} \sum_{j \in F} a_j^* = \sum_{j=r+1}^{n} a_j^*.$$

"$\geq$" : Let $a \in c_0$, $n \in \mathbb{N}, r \in \mathbb{N}_0$ and $\iota \in \mathrm{Bij}(\mathbb{N})$ be arbitrary with $\operatorname{dom}\iota = n$. Let $G_\iota \subseteq n$ with $|G| = n - r$ such that

$$\forall j \in G_\iota, i \in n \backslash G_\iota : \quad \big(\mathscr{R}_\iota |a|\big)_j \leq \big(\mathscr{R}_\iota |a|\big)_i.$$

By Lemma 1.1.7 iii) we obtain

$$\min_{\substack{F \subseteq n \\ |F| = n-r}} \sum_{j \in F} \big(\mathscr{R}_\iota |a|\big)_j = \sum_{j \in G_\iota} \big(\mathscr{R}_\iota |a|\big)_j \leq \sum_{j=1}^{n-r} \big(\mathscr{R}_\iota a\big)_j^* \leq \sum_{j=1}^{n-r} \big(\mathscr{T}^r a^*\big)_j = \sum_{j=r+1}^{n} a_j^*.$$

□

4.2.4 Proposition. *Let $a^1, \ldots, a^k \in c_0$, then*

$$a^1 + \ldots + a^k \trianglelefteq_k (a^1)^* + \ldots + (a^k)^*. \tag{4.9}$$

Proof. Let $n \in \mathbb{N}, r \in \mathbb{N}_0$, $a^1, \ldots, a^k \in c_0$ and $\iota \in \mathrm{Bij}(\mathbb{N})$ with $|\operatorname{dom}\iota| = n$. For each $i \in \{1, \ldots, k\}$ consider subsets $G_\iota^i \subseteq \operatorname{dom}\iota$ with $|G_\iota^i| = r$ such that

$$\sum_{j \in G_\iota^i} \big(\mathscr{R}_\iota |a^i|\big)_j = \max_{\substack{F \subseteq \operatorname{dom}\iota \\ |F| = r}} \sum_{j \in F} \big(\mathscr{R}_\iota |a^i|\big)_j$$

holds. Then choose a set F_ι with $|F_\iota| = n - kr$ satisfying

$$F_\iota \subseteq (\operatorname{dom}\iota) \setminus \big(\bigcup_{i=1}^{k} G_\iota^i\big).$$

Using the identity (4.8) yields the desired result:

$$\begin{aligned}
\sum_{j=kr+1}^{n}(a^1+\ldots+a_k)^*_j &= \max_{\substack{\iota\in\mathrm{Bij}(\mathbb{N})\\ |\operatorname{dom}\iota|=n}} \min_{\substack{F\subseteq\operatorname{dom}\iota\\ |F|=n-kr}} \sum_{j\in F}\big(\mathscr{R}_\iota|a^1+\ldots+a^k|\big)_j \\
&\le \max_{\substack{\iota\in\mathrm{Bij}(\mathbb{N})\\ |\operatorname{dom}\iota|=n}} \sum_{j\in F_\iota}\big(\mathscr{R}_\iota|a^1+\ldots+a^k|\big)_j \le \max_{\substack{\iota\in\mathrm{Bij}(\mathbb{N})\\ |\operatorname{dom}\iota|=n}} \sum_{i=1}^{k}\sum_{j\in F_\iota}\big(\mathscr{R}_\iota|a^i|\big)_j \\
&\le \max_{\substack{\iota\in\mathrm{Bij}(\mathbb{N})\\ |\operatorname{dom}\iota|=n}} \sum_{i=1}^{k}\sum_{j\in\operatorname{dom}\iota\setminus G^i_\iota}\big(\mathscr{R}_\iota|a^i|\big)_j \le \max_{\substack{\iota\in\mathrm{Bij}(\mathbb{N})\\ |\operatorname{dom}\iota|=n}} \sum_{i=1}^{k}\sum_{j=1}^{n-r}\big(\mathscr{R}_\iota a^i\big)^*_j \\
&\overset{\text{1.1.7 iii)}}{\le} \sum_{i=1}^{k}\sum_{j=1}^{n-r}\big(\mathscr{T}^r(a^i)^*\big)_j = \sum_{j=r+1}^{n}\sum_{i=1}^{k}(a^i)^*_j.
\end{aligned}$$

□

To better illustrate the nature of $\trianglelefteq$, we give another – more involved – example that $\trianglelefteq\neq\lll$.

4.2.5 Example. Consider the sequences

$$a := (\frac{1}{2^1},\frac{1}{2^2},\frac{1}{2^3},\ldots),\quad b := (1,\frac{1}{2^1},\frac{1}{4^2},\frac{1}{8^3},\frac{1}{16^4},\ldots) = \big(2^{-(n-1)^2}\big)_{n\in\mathbb{N}}.$$

Obviously, $a\lll b$. To show that $a\ntrianglelefteq b$ holds, we consider $\lambda,r,n\in\mathbb{N}$ with $2\le\lambda<r$ and $0\le\lambda r<n$. We estimate

$$\begin{aligned}
\sum_{j=r+1}^{n} b_j &= \sum_{j=r+1}^{n}\big(\frac{1}{2^{(j-1)}}\big)^{j-1} \le \sum_{j=r+1}^{n}\big(\frac{1}{2^r}\big)^{j-1} = \sum_{j=1}^{n}\big(\frac{1}{2^r}\big)^{j-1} - \sum_{j=1}^{r}\big(\frac{1}{2^r}\big)^{j-1} = \frac{1-(\frac{1}{2^r})^n}{1-\frac{1}{2^r}} - \frac{1-(\frac{1}{2^r})^r}{1-\frac{1}{2^r}} \\
&= \frac{1}{1-\frac{1}{2^r}}\cdot\big(\frac{1}{2^{r^2}}-\frac{1}{2^{nr}}\big) \le \frac{1}{2^{r^2-1}} < \frac{1}{2^{\lambda r+1}}.
\end{aligned}$$

The last inequality holds due to $2\le\lambda<r$. On the other hand, consider the estimates

$$\sum_{j=\lambda r+1}^{n} a_j = \sum_{j=0}^{n}\frac{1}{2^j} - \sum_{j=0}^{\lambda r}\frac{1}{2^j} = 2\big(\frac{1}{2^{\lambda r+1}}-\frac{1}{2^{n+1}}\big) = \frac{1}{2^{\lambda r}}\cdot\big(1-\frac{1}{2^{n-\lambda r}}\big) \ge \frac{1}{2^{\lambda r+1}}$$

Hence, $\sum_{j=\lambda r+1}^{n} a_j > \sum_{j=r+1}^{n} b_j$ and therefore $a\ntrianglelefteq b$.

4.3 Minkowski functionals related to uniform Hardy-Littlewood majorization

After having clarified the structure of beginning sections w.r.t. Hardy-Littlewood majorization, we investigate the beginning sections w.r.t. $\trianglelefteq$ and $\prec$. To start with note the following facts.

4.3.1 Lemma. *Let $c\in c_0$. The sets $[0,c]_\trianglelefteq$ and $\operatorname{conv}[0,c]_\prec$ are absolutely convex. We have*

$$[0,c]_\lll \supsetneq [0,c]_\trianglelefteq \supsetneq \operatorname{conv}[0,c]_\prec \tag{4.10}$$

and

$$[0,c]_\lll = \overline{[0,c]_\trianglelefteq} = \overline{\operatorname{conv}[0,c]_\prec}. \tag{4.11}$$

Proof. The set $\operatorname{conv}[0,c]_{\prec}$ is by definition convex, and by the computation rules Lemma 1.2.2 it is invariant under multiplication with scalars from $[-1,1]$.

To see that $[0,c]_{\trianglelefteq}$ is absolutely convex, consider an element of the form $\sum_{j=1}^{k}\lambda_j c^j$ with some $c^j \trianglelefteq c$ and $\sum_{j=1}^{k}|\lambda_j| \leq 1$. Choose $\lambda \in \mathbb{N}$ such that $c^j \trianglelefteq_\lambda c$ holds for all $j \in \{1,\dots,k\}$. Then

$$\sum_{j=(k\lambda)r+1}^{n}\big(\sum_{i=1}^{k}\lambda_i c^i\big)_j^* \overset{(4.9)}{\leq} \sum_{j=\lambda r+1}^{n}\sum_{i=1}^{k}|\lambda_i|(c^i)_j^* = \sum_{i=1}^{k}|\lambda_i| \sum_{j=\lambda r+1}^{n}(c^i)_j^*$$
$$\overset{(c^i\trianglelefteq_\lambda c)}{\leq} \sum_{i=1}^{k}|\lambda_i| \cdot \sum_{j=r+1}^{n} c_j^* \leq \sum_{j=r+1}^{n} c_j^*.$$

The inclusions "$\supseteq$" in (4.10) certainly hold, and we need to give examples that they are strict. To see $[0,c]_{\lll} \neq [0,c]_{\trianglelefteq}$, consider

$$a := (\frac{1}{2},\frac{1}{4},\frac{1}{8},\dots), \quad c := (1,0,0,\dots). \tag{4.12}$$

Then $a \lll c$ holds, but $a \trianglelefteq c$ is false (if $r \geq 2$, the sum on the right hand side in (4.6) is zero). To see that $[0,c]_{\trianglelefteq} \neq \operatorname{conv}[0,c]_{\prec}$, Let $c \in c_0^{\downarrow}$ be a sequence with $c_j > c_{j+1} \geq 0$ with satisfies the condition

$$\frac{1}{2^{k-1}}\sum_{j=2^{k-1}}^{2^k-1} c_j > c_{2^{k-1}+1},$$

for each $k \in \mathbb{N}$. Now define a sequence $a \in c_0^{\downarrow}$ by

$$a_n := \frac{1}{2^{k-1}} \cdot \sum_{j=2^{k-1}}^{2^k-1} c_j, \quad \text{for } 2^{k-1} \leq n \leq 2^k.$$

Obviously, $a \lll b$. Furthermore, we have for each $k \in \mathbb{N}$ that

$$\sum_{j=1}^{2^k-1} a_j = \sum_{l=1}^{k}\sum_{j=2^{l-1}}^{2^l-1} a_j = \sum_{l=1}^{k}\sum_{j=2^{l-1}}^{2^l-1} c_j = \sum_{j=1}^{2^k-1} c_j \tag{4.13}$$

From the above equality, we can easily deduce $a \trianglelefteq_2 c$. Let $r,n \in \mathbb{N}$. Choose $k \in \mathbb{N}$ with $2^{k-1} \leq 2r < 2^k$. Then, using the fact that $r+1 \leq 2^{k-1}$, we can estimate

$$\sum_{j=2r+1}^{n} a_j = \sum_{j=1}^{n} a_j - \sum_{j=1}^{2r} a_j \leq \sum_{j=1}^{n} c_j - \sum_{j=1}^{2^{k-1}-1} a_j = \sum_{j=1}^{n} c_j - \sum_{j=1}^{2^{k-1}-1} c_j = \sum_{j=2^{k-1}}^{n} c_j \leq \sum_{j=r+1}^{n} c_j.$$

This shows indeed that $a \trianglelefteq_2 c$. To see that $a \notin \operatorname{conv}[0,c]_{\prec}$, assume the contrary, i.e. a can be represented as a convex combination in $[0,c]_{\prec}$, say $a = \sum_{j=1}^{N}\lambda_j c^j$. Due to $c^j \prec c$, we can find $\iota_j \in \operatorname{Bij}(\mathbb{N})$, such that $|c^j| \leq \mathscr{R}_\iota c$. Hence, we have $a \leq \sum_{j=1}^{N}\lambda_j \cdot \mathscr{R}_\iota c$. Since $\mathscr{R}_\iota c \lll c$, and using the equality (4.13) yields

$$\sum_{l=1}^{2^k-1} c_l = \sum_{l=1}^{2^k-1}\sum_{j=1}^{N}\lambda_j \cdot \big(\mathscr{R}_\iota c\big)_l, \qquad k \in \mathbb{N}.$$

Due to c being strictly decreasing, each ι_j leaves $[2^{k-1}, 2^k - 1] \cap \mathbb{N}$ invariant. Choose any $k \in \mathbb{N}$ such that $N < 2^{k-1}$. Then there exists $l_0 \in [2^{k-1}, 2^k - 1] \cap \mathbb{N}$ such that $\iota_j(l_0) \neq 2^{k-1}$, for all $j \in \{1, \dots, N\}$. Hence, we conclude

$$a_{l_0} \leq \sum_{j=1}^{N} \lambda_j \cdot \left(\mathscr{R}_\iota c\right)_{l_0} \leq \sum_{j=1}^{N} \lambda_j c_{2^{k-1}+1} < a_{l_0},$$

and therefore $a \notin \operatorname{conv}[0, c]_{\prec}$.
To see (4.11), it is enough to recall Corollary 4.1.5 and Remark 4.1.10. □

We consider the (extended) Minkowski functionals associated with the various beginning sections, namely

$$|a|_c^{\lll} := \inf\left\{\mu > 0 \,\middle|\, a \in \mu \cdot [0, c]_{\lll}\right\}, \quad |a|_c^{\trianglelefteq} := \inf\left\{\mu > 0 \,\middle|\, a \in \mu \cdot [0, c]_{\trianglelefteq}\right\},$$
$$|a|_c^{\trianglelefteq_\lambda} := \inf\left\{\mu > 0 \,\middle|\, a \in \mu \cdot [0, c]_{\trianglelefteq_\lambda}\right\}, \quad |a|_c^{\prec} := \inf\left\{\mu > 0 \,\middle|\, a \in \mu \cdot \operatorname{conv}[0, c]_{\prec}\right\}.$$

These maps are defined on c_0 and take values in $[0, \infty]$. Due to the above lemma, $|a|_c^{\lll}, |a|_c^{\trianglelefteq}, |a|_c^{\prec}$ are (extended) seminorms.

We use the convex hull of $[0, c]_{\prec}$ in the definition of $|a|_c^{\prec}$ to enforce validity of the triangular inequality. This is not needed for $|a|_c^{\trianglelefteq_\lambda}$, for this Minkowski functional it is enough to know $|\gamma a|_c^{\trianglelefteq_\lambda} = \gamma \cdot |a|_c^{\trianglelefteq_\lambda}$ for all $a \in c_0$ and $\gamma > 0$, which holds just by the definition.

From (4.10) we obtain that

$$\forall c, a \in c_0: \quad |a|_c^{\lll} \leq |a|_c^{\trianglelefteq} \leq |a|_c^{\prec}. \tag{4.14}$$

The first of these inequalities may be strict. For example, the sequences (4.12) satisfy $|a|_c^{\lll} = 1$ and $|a|_c^{\trianglelefteq} = \infty$.

Our aim in this section is to prove the crucial and somewhat surprising fact that in the second inequality in (4.14) always equality holds.

4.3.2 Theorem. *Let $c \in c_0$. Then*

$$\forall a \in c_0: \quad |a|_c^{\trianglelefteq} = |a|_c^{\prec}.$$

The actual proof of this theorem is slightly technical. Before going into the details, we present an outline.

Outline of the argument

We are going to show the following result, from which Theorem 4.3.2 can be deduced easily.

4.3.3 Proposition. *Let $a, c \in c_0^{\downarrow}$ with $a \trianglelefteq c$. Then, for all positive integers $N \geq 2$ there exist $\delta_1, \delta_2 > 0$, and $b^1, \dots, b^N \in c_0$, such that*

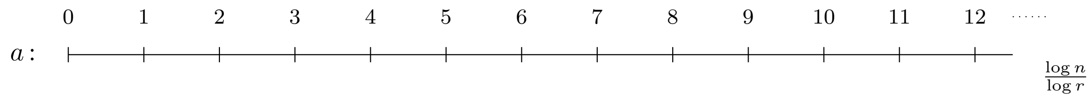
i) $a \in \operatorname{conv}\{b^1, \dots, b^N\}$,

ii) *for all $k \in \{1, \dots, N\}$ and $\gamma > 0$, the element γb^k can be written as a sum of at most $\gamma\delta_1 + \delta_2$ elements of $[0, c]_{\prec}$.*

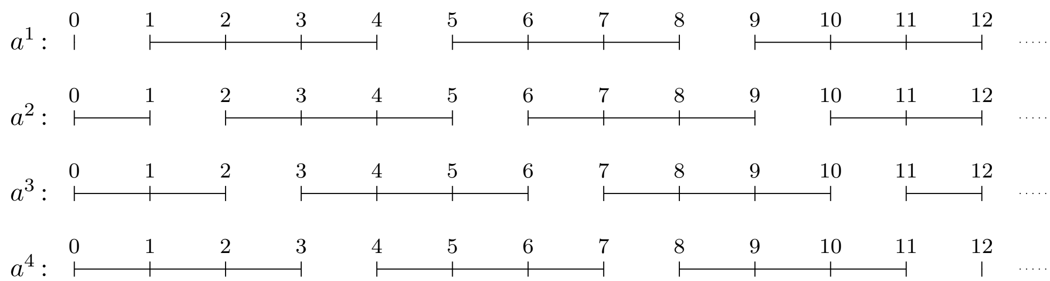
The infimum of all possible choices of δ_1 is equal to 1.

The idea how to prove this proposition is to proceed as follows.

- Split the sequence a into sections with finite length along a geometric progression with some base $r \in \mathbb{N}$, $r \geq 2$ (to illustrate this we draw on a logarithmic scale)



- Truncate a by dropping whole sections along – in the logarithmic scale – arithmetic progressions with some step width $N \in \mathbb{N}$, $N \geq 2$ (in the picture $N = 4$)



Formally, we set

$$I_{k,l} := \mathbb{N} \cap (r^{lN+k}, r^{(l+1)N+k-1}], \quad k \in \{1, \dots, N\}, l \in \{-1, 0, 1, 2, \dots\},$$

and

$$a^k := a \cdot \Big(\sum_{l=-1}^{\infty} \mathbb{1}_{I_{k,l}} \Big), \quad k \in \{1, \dots, N\}.$$

Then a can be recovered from the truncated sequences $a^1, \dots, a^N$ as

$$a = \frac{1}{N-1} \sum_{k=1}^{N} a^k. \tag{4.15}$$

- Prove estimates for truncated sequences a^k, and deduce estimates for a of the form required in Proposition 4.3.3 with some δ_1, δ_2 depending on r and N.

- Let the parameters r and N tend to infinity to push δ_1 towards 1.

The technical core is the third step. In order to work it out efficiently we introduce a notation.

4.3.4 Definition. For $a, c \in c_0$ we denote

$$\big[a\big]_c := \inf \big\{ n \in \mathbb{N} \;\big|\; \exists c^1, \dots, c^n \in [0, c]_{\prec} : \; a = \sum_{j=1}^{n} c^j \big\}$$

The following four lemmata are then the essential ingredients. The first one shows that for finitely supported sequences the number $\big[a\big]_c$ can be controlled in terms of Hardy-Littlewood majorization.

4.3.5 Lemma. *Let $a, c \in c_0^{\downarrow}$. Then*

$$\big[a\big]_c \leq |a|_c^{\lll} + |\operatorname{supp} a| + 1. \tag{4.16}$$

Let in addition $r \in \mathbb{N}$, and recall that $\mathscr{T}^r$ denotes the left shift operator which shifts r times, cf. (1.4). Then

$$\big[\mathscr{T}^r a\big]_c \leq |a|_{\mathscr{T}^r c}^{\lll} + \frac{|\operatorname{supp} a|}{r} + 1. \tag{4.17}$$

The second lemma exhibits the role of uniform Hardy-Littlewood majorization.

4.3.6 Lemma. *Let $r \in \mathbb{N}$, $\lambda \in \mathbb{N}$, and $a, c \in c_0^{\downarrow}$ with $a \trianglelefteq_\lambda c$. Then*

$$\mathscr{T}^{\lambda r} a \lll (1 + (\lambda - 1)r) \cdot \mathscr{T}^r c \qquad \text{and} \qquad \forall m \in \mathbb{N}, m > \lambda : \quad \mathscr{T}^{mr} a \lll \frac{m-1}{m-\lambda} \cdot \mathscr{T}^r c.$$

From this we obtain an estimate for sections of sequences.

4.3.7 Lemma. *Let $a, c \in c_0^{\downarrow}$, let r be a positive odd integer, and let $\lambda \in \mathbb{N}$ with $2\lambda + 1 < r$. Then*

$$\forall k, n \in \mathbb{N}, r^{k+1} < n : \quad \big[a\mathbb{1}_{(r^{k+1}, n]})\big]_{c\mathbb{1}_{(r^k, n]}} \leq \frac{r-3}{r - (2\lambda + 1)} \cdot |a|_c^{\trianglelefteq_\lambda} + \frac{n}{r^k} + 1.$$

Finally, we need a tool to glue together estimates for separate sections.

4.3.8 Lemma. *Let $c \in c_0$ and let $\{B_i \mid i \in I\}$ be a partition of $\mathbb{N}$. Then*

$$\forall a \in c_0 : \quad \big[a\big]_c \leq \sup_{i \in I} \big[a\mathbb{1}_{B_i}\big]_{c\mathbb{1}_{B_i}}$$

Proof details

We start with some computation rules for Minkowski functionals and the function $(a, c) \mapsto \big[a\big]_c$.

4.3.9 Lemma. *The four functions $(a, c) \mapsto |a|_c^{\lll}, |a|_c^{\trianglelefteq}, |a|_c^{\prec}, \big[a\big]_c$ depend only on the nonincreasing rearrangements of a and c, are monotone in a, and anti-monotone in c.*

Written in detail, the following statements hold.

i) $\forall a, c \in c_0 : \quad |a|_c^{\lll} = |a^*|_{c^*}^{\lll},\ |a|_c^{\trianglelefteq} = |a^*|_{c^*}^{\trianglelefteq},\ |a|_c^{\prec} = |a^*|_{c^*}^{\prec},\ \big[a\big]_c = \big[a^*\big]_{c^*}.$

ii) $\forall a^1, a^2, c \in c_0 :$
$$a^1 \lll a^2 \Rightarrow |a^1|_c^{\lll} \leq |a^2|_c^{\lll},$$
$$a^1 \trianglelefteq a^2 \Rightarrow |a^1|_c^{\trianglelefteq} \leq |a^2|_c^{\trianglelefteq},$$
$$a^1 \prec a^2 \Rightarrow \Big(|a^1|_c^{\prec} \leq |a^2|_c^{\prec} \ \wedge\ \big[a^1\big]_c \leq \big[a^2\big]_c\Big).$$

iii) $\forall a, c^1, c^2 \in c_0 :$
$$c^1 \lll c^2 \Rightarrow |a|_{c_1}^{\lll} \geq |a|_{c_2}^{\lll},$$
$$c^1 \trianglelefteq c^2 \Rightarrow |a|_{c_1}^{\trianglelefteq} \geq |a|_{c_2}^{\trianglelefteq},$$
$$c^1 \prec c^2 \Rightarrow \Big(|a|_{c_1}^{\prec} \geq |a|_{c_2}^{\prec} \ \wedge\ \big[a\big]_{c_1} \geq \big[a\big]_{c_2}\Big).$$

Proof. The fact that each of the four functions depends only on c^* and is anti-monotone in the argument c is clear. Moreover, it is clear that the Minkowski functionals related with $\lll$ and $\trianglelefteq$ depend only on a^* and are montone in the argument a.

The crucial observation to prove the remaining assertions, is that (recall that M_ζ denotes the multiplication operator with ζ)

$$\begin{aligned}\forall\iota\in\operatorname{Bij}(\mathbb{N}): &\quad \mathscr{R}_\iota\big(\operatorname{conv}[0,c]_\prec\big)\subseteq\operatorname{conv}[0,c]_\prec,\\ \forall\zeta\in[-1,1]^{\mathbb{N}}: &\quad M_\zeta\big(\operatorname{conv}[0,c]_\prec\big)\subseteq\operatorname{conv}[0,c]_\prec.\end{aligned} \tag{4.18}$$

Note here that the operators $\mathscr{R}_\iota$ and M_ζ are linear.

Each of the elements a and a^* can be written as some appropriate operator $\mathscr{R}_\iota\circ M_\zeta$ applied to the respective other element, and we obtain

$$a\in\operatorname{conv}[0,c]_\prec \;\Leftrightarrow\; a^*\in\operatorname{conv}[0,c]_\prec,$$

from which $|a|_c^\prec=|a^*|_c^\prec$ follows. Again using that the operators $\mathscr{R}_\iota$ and M_ζ are linear, we also see that $\big[a\big]_c=\big[a^*\big]_c$.

Finally, we turn to the remaining monotonicity properties. By what we have shown so far, we may assume w.l.o.g. that $a^1,a^2\in c_0^\downarrow$. Then $a^1\geq 0$ and the premise "$a^1\prec a^2$" means that $a^1\leq a^2$. Set

$$\zeta_j:=\begin{cases}\frac{(a^1)_j}{(a^2)_j} & \text{if } (a^1)_j>0,\\ 0 & \text{otherwise,}\end{cases}$$

then $a^1=M_\zeta a^2$. Yet another time referring to linearity and (4.18), we find that $|a^1|_c^\prec\leq|a^2|_c^\prec$ and $\big[a^1\big]_c\leq\big[a^2\big]_c$. □

Proof (of Lemma 4.3.5). If one of $|a|_c^{\lll}$ and $|\operatorname{supp}a|$ is infinite, there is nothing to prove. Hence, assume that $|a|_c^{\lll}<\infty$ and $|\operatorname{supp}a|<\infty$. Let $\mu>|a|_c^{\lll}$. Then $\frac{1}{\mu}a\in\Omega_c^{\operatorname{supp}a}$, and by Corollary 4.1.9 $\frac{1}{\mu}a$ can be represented as a convex combination with at most $|\operatorname{supp}a|+1$ summands, say

$$\frac{1}{\mu}a=\sum_{j=1}^{|\operatorname{supp}a|+1}\lambda_j a^j$$

with $\lambda_j\in[0,1]$, $\sum_{j=1}^{|\operatorname{supp}a|+1}\lambda_j=1$, and $a^j\sim c^*\mathbb{1}_{|\operatorname{supp}a|}$. Thus we have the representation

$$a=\sum_{j=1}^{|\operatorname{supp}a|+1}\sum_{i=1}^{\lfloor\mu\lambda_j\rfloor}a^j+\sum_{j=1}^{|\operatorname{supp}a|+1}(\mu\lambda_j-\lfloor\mu\lambda_j\rfloor)a^j.$$

Clearly, each of the summands in this representation belongs to $[0,c]_\prec$. The number of summands is

$$\sum_{j=1}^{|\operatorname{supp}a|+1}\lfloor\mu\lambda_j\rfloor+|\operatorname{supp}a|+1\leq\sum_{j=1}^{|\operatorname{supp}a|+1}\mu\lambda_j+|\operatorname{supp}a|+1=\mu+|\operatorname{supp}a|+1.$$

Since $\mu>|a|_c^{\lll}$ was arbitrary, the estimate (4.16) follows.

The second assertion (4.17) is reduced to the readily shown estimate (4.16). This is done by using smashed and replicated sequences instead of shifted ones. In this place recall the replication operator $\mathscr{P}_r$ from Definition 3.1.1.

We start with a general observation. Let $r\in\mathbb{N}$, $b\in c_0^\downarrow$, and define $\mu_j:=b_{jr+1}$, $j\in\mathbb{N}$. For every $j\in\mathbb{N}$ we have

$$\frac{j-1}{r}\geq\Big\lfloor\frac{j-1}{r}\Big\rfloor>\frac{j-1}{r}-1,\qquad\Big\lfloor\frac{j-1}{r}\Big\rfloor+1=\Big\lceil\frac{j}{r}\Big\rceil.$$

Since b is nonincreasing, it follows that

$$b_{j+r} = b_{(\frac{j-1}{r}+1)r+1} \le \underbrace{b_{(\lfloor\frac{j-1}{r}\rfloor+1)r+1}}_{=\mu_{\lceil\frac{j}{r}\rceil}} \le b_{\frac{j-1}{r}r+1} = b_j,$$

and this says that

$$\mathscr{T}^r b \le \mathscr{P}_r \mu \le b.$$

Let us illustrate what happens here (in the picture for $r = 3$):

				μ_1			μ_2			μ_3			
$b:$	b_1	b_2	b_3	$\textcircled{b_4}$	b_5	b_6	$\textcircled{b_7}$	b_8	b_9	$\textcircled{b_{10}}$	b_{11}	b_{12}	$\cdots\cdots$
$\mathscr{P}_r\mu:$	b_4	b_4	b_4	b_7	b_7	b_7	b_{10}	b_{10}	b_{10}	b_{13}	b_{13}	b_{13}	$\cdots\cdots$
$\mathscr{T}^r a:$	b_4	b_5	b_6	b_7	b_8	b_9	b_{10}	b_{11}	b_{12}	b_{13}	b_{14}	b_{15}	$\cdots\cdots$

Now we set $\xi_j := a_{jr+1}$ and $\eta_j := c_{jr+1}$, and use our computation rules and (4.16) to estimate

$$\begin{aligned}\big[\mathscr{T}^r a\big]_c \le \big[\mathscr{P}_r\xi\big]_{\mathscr{P}_r\eta} \le \big[\xi\big]_\eta &\overset{(4.16)}{\le} |\xi|_\eta^{\lll} + |\operatorname{supp}\xi| + 1 \\ &\le |\mathscr{P}_r\xi|_{\mathscr{P}_r\eta}^{\lll} + |\operatorname{supp}\xi| + 1 \le |a|_{\mathscr{T}^r c}^{\lll} + |\operatorname{supp}\xi| + 1.\end{aligned}$$

Finally, observe that $|\operatorname{supp}\xi| \le \frac{|\operatorname{supp} a|}{r}$ (again, since a is nonincreasing). $\square$

Proof (of Lemma 4.3.6). We start with estimating $\mathscr{T}^{\lambda r}a$ by $\mathscr{T}^r c$.

$$\begin{aligned}\sum_{j=1}^{n}(\mathscr{T}^{\lambda r}a)_j = \sum_{j=1+\lambda r}^{n+\lambda r} a_j &\overset{(a\trianglelefteq_\lambda c)}{\le} \sum_{j=1+r}^{n+\lambda r} c_j = \sum_{j=1+r}^{r+n} c_j + \sum_{j=r+n+1}^{\lambda r+n} c_j \\ &\le \sum_{j=1+r}^{r+n} c_j + \frac{(\lambda-1)r}{n}\cdot\underbrace{nc_{r+n+1}}_{\le\sum_{j=1+r}^{r+n}c_j} \le \Big(1+\frac{(\lambda-1)r}{n}\Big)\cdot\sum_{j=1}^{n}(\mathscr{T}^r c)_j\end{aligned}$$

The factor in front of the sum is nonincreasing in n, and its maximal value is $1+(\lambda-1)r$. This shows that $\mathscr{T}^{\lambda r}a \lll (1+(\lambda-1)r)\cdot\mathscr{T}^r c$.

Now let $m > \lambda$ be given. Consider the set

$$M := \Big\{n\in\mathbb{N} \;\Big|\; \sum_{j=1}^{n}(\mathscr{T}^{\lambda r}a)_j > \frac{m-1}{m-\lambda}\cdot\sum_{j=1}^{n}(\mathscr{T}^r c)_j\Big\}.$$

For $n \ge (m-\lambda)r$, we have $\frac{m-1}{m-\lambda} \ge 1+\frac{(\lambda-1)r}{n}$, and hence $M \subseteq [1,(m-\lambda)r)$. Clearly,

$$\forall n\in\mathbb{N}\setminus M: \quad \sum_{j=1}^{n}(\mathscr{T}^{mr}a)_j \le \sum_{j=1}^{n}(\mathscr{T}^{\lambda r}a)_j \le \frac{m-1}{m-\lambda}\cdot\sum_{j=1}^{n}(\mathscr{T}^r c)_j.$$

If $M=\emptyset$, we already obtain the asserted relation $\mathscr{T}^{mr}a \lll \frac{m-1}{m-\lambda}\cdot\mathscr{T}^r c$. Assume that $M\neq\emptyset$, and set $k := \max M$. Then $k < (m-\lambda)r$, and we can estimate for all $n\in\mathbb{N}$ (we need the estimate actually only for $n \in M$)

$$\begin{aligned}\sum_{j=1}^{n} (\mathscr{T}^{mr} a)_j &\le \sum_{j=1}^{n} (\mathscr{T}^{\lambda r + k} a)_j = \sum_{j=1+k}^{n+k} (\mathscr{T}^{\lambda r} a)_j \\ &= \sum_{j=1}^{n+k} (\mathscr{T}^{\lambda r} a)_j - \sum_{j=1}^{k} (\mathscr{T}^{\lambda r} a)_j \le \frac{m-1}{m-\lambda} \cdot \sum_{j=1}^{n+k} (\mathscr{T}^{r} c)_j - \frac{m-1}{m-\lambda} \cdot \sum_{j=1}^{k} (\mathscr{T}^{r} c)_j \\ &= \frac{m-1}{m-\lambda} \cdot \sum_{j=1+k}^{n+k} (\mathscr{T}^{r} c)_j \le \frac{m-1}{m-\lambda} \cdot \sum_{j=1}^{n} (\mathscr{T}^{r} c)_j .\end{aligned}$$

□

Proof (of Lemma 4.3.7). If $|a|_c^{\trianglelefteq_\lambda} = \infty$, there is nothing to prove. Hence, assume that $|a|_c^{\trianglelefteq_\lambda} < \infty$. Let $\mu > 0$ be such that $a \trianglelefteq_\lambda \mu c$. Then also $a\mathbb{1}_{[1,n]} \trianglelefteq_\lambda \mu c \mathbb{1}_{[1,n]}$. Next observe that

$$\begin{aligned}\big(a\mathbb{1}_{(r^{k+1},n]}\big)^* &= \mathscr{T}^{r^{k+1}}\big(a\mathbb{1}_{(r^{k+1},n]}\big) = \mathscr{T}^{r^{k+1}}\big(a\mathbb{1}_{[1,n]}\big), \\ \big(c\mathbb{1}_{(r^{k},n]}\big)^* &= \mathscr{T}^{r^{k}}\big(c\mathbb{1}_{(r^{k},n]}\big) = \mathscr{T}^{r^{k}}\big(c\mathbb{1}_{[1,n]}\big).\end{aligned}$$

Now we can estimate

$$\begin{aligned}\big[a\mathbb{1}_{(r^{k+1},n]}\big]_{c\mathbb{1}_{(r^k,n]}} &= \big[\mathscr{T}^{r\cdot r^k}\big(a\mathbb{1}_{[1,n]}\big)\big]_{\mathscr{T}^{r^k}\left(c\mathbb{1}_{[1,n]}\right)} \overset{(4.17)}{\le} |\mathscr{T}^{\frac{(r-1)}{2}\cdot 2r^k}(a\mathbb{1}_{[1,n]})|^{\lll}_{\mathscr{T}^{2r^k}(c\mathbb{1}_{[1,n]})} + \frac{n}{r^k} + 1 \\ &\overset{4.3.6}{\le} \frac{\frac{r-1}{2} - 1}{\frac{r-1}{2} - \lambda}\mu + \frac{n}{r^k} + 1 \le \frac{r-3}{r-(1+2\lambda)}\mu + \frac{n}{r^k} + 1.\end{aligned}$$

□

The proof of the glueing lemma 4.3.8, is based on the following algebraic fact.

4.3.10 Lemma. *Let* $\{A_i \mid i \in I\}, \{B_i \mid i \in I\} \subseteq \mathcal{P}(\mathbb{N})$ *be two partitions of* $\mathbb{N}$*. Then, for all* $a \in \mathbb{R}^{\mathbb{N}}$ *and* $b \in c_0$*, we have*

$$\big(\forall i \in I: \quad a\mathbb{1}_{A_i} \prec b\mathbb{1}_{B_i}\big) \;\Rightarrow\; \big(a \in c_0 \quad \wedge \quad a \prec b\big). \tag{4.19}$$

Proof. By Lemma 1.1.2 the premise means that

$$\forall \delta > 0, \forall i \in I: \quad \big|L_{>\delta}(a\mathbb{1}_{A_i})\big| \le \big|L_{>\delta}(b\mathbb{1}_{B_i})\big|.$$

Now note that, for each $\alpha > 0$,

$$L_\alpha(a) = \dot{\bigcup_{i\in I}} L_\alpha(a\mathbb{1}_{A_i}),$$

where $\dot{\bigcup}$ denotes a disjoint union. Therefore we have

$$L_{>\delta}(a) = \dot{\bigcup_{i\in I}} \big[L_{>\delta}(a\mathbb{1}_{A_i})\big].$$

From this, and analogous equality for b, we find

$$\forall \delta > 0: \quad \big|L_{>\delta}(a)\big| \le \big|L_{>\delta}(b)\big|,$$

and this means that $a^* \le b^*$. □

Proof (of Lemma 4.3.8). If the right side is infinite, there is nothing to prove. Assume that

$$N := \sup_{i\in I} \big[a\mathbb{1}_{B_i}\big]_{c\mathbb{1}_{B_i}} < \infty,$$

and choose $c^{i,j} \in [0, c\mathbb{1}_{B_i}]_{\prec}$, $i \in I, j \in \{1,\dots,N\}$, such that $a\mathbb{1}_{B_i} = \sum_{j=1}^N c^{i,j}$. Define

$$c^j := \sum_{i\in I} c^{i,j}\mathbb{1}_{B_i} \in \mathbb{R}^{\mathbb{N}}, \qquad j \in \{1,\dots,N\}.$$

Note here that the sets B_i are pairwise disjoint, and hence at each coordinate there is at most one nonzero summand in this sum.

We have

$$\forall i \in I: \quad c^j\mathbb{1}_{B_i} = c^{i,j}\mathbb{1}_{B_i} \prec c^{i,j} \prec c\mathbb{1}_{B_i},$$

and Lemma 4.3.10 implies that $c^j \prec c$. Obviously,

$$a = \sum_{i\in I} a\mathbb{1}_{B_i} = \sum_{i\in I}\sum_{j=1}^N c^{i,j}\mathbb{1}_{B_i} = \sum_{j=1}^N\sum_{i\in I} c^{i,j}\mathbb{1}_{B_i} = \sum_{j=1}^N c^j,$$

and we see that $\big[a\big]_c \le N$. □

Proof (of Proposition 4.3.3). Let $a, c \in c_0^{\downarrow}$ with $a \trianglelefteq c$, and choose $\lambda \in \mathbb{N}$ such that $a \trianglelefteq_\lambda c$.

For each $N \ge 2$, $r > 2\lambda+1$, and $k \in \{1,\dots,N\}$, we consider the partition of $\mathbb{N}$ given by the intervals

$$B_{k,l} := \mathbb{N} \cap (r^{lN+k-1}, r^{(l+1)N+k-1}], \quad l \in \{-1,0,1,2,\dots\}.$$

Then $a^k\mathbb{1}_{B_{k,l}} = a\mathbb{1}_{I_{k,l}}$. Let $\gamma > 0$. For $l \ge 0$ we can estimate

$$\begin{aligned}\big[\gamma a^k\mathbb{1}_{B_{k,l}}\big]_{c\mathbb{1}_{B_{k,l}}} = \big[\gamma a\mathbb{1}_{I_{k,l}}\big]_{c\mathbb{1}_{B_{k,l}}} &\overset{4.3.7}{\le} \frac{r-3}{r-(2\lambda+1)} \cdot |\gamma a|_c^{\trianglelefteq_\lambda} + \frac{r^{(l+1)N+k-1}}{r^{lN+k-1}} + 1 \\ &\le \frac{r-3}{r-(2\lambda+1)} \cdot \gamma + r^N + 1.\end{aligned}$$

For $l = -1$ we proceed as follows. We have $B_{k,-1} = [1, r^{k-1}]$, and $a^k\mathbb{1}_{[1,n]} \le a\mathbb{1}_{[1,n]}$ for all n, and $a \lll c$ which yields $a\mathbb{1}_{[1,n]} \lll c\mathbb{1}_{[1,n]}$ for all n. Thus,

$$\big[\gamma a^k\mathbb{1}_{[1,r^{k-1}]}\big]_{c\mathbb{1}_{[1,r^{k-1}]}} \le \big[\gamma a\mathbb{1}_{[1,r^{k-1}]}\big]_{c\mathbb{1}_{[1,r^{k-1}]}} \overset{(4.16)}{\le} |\gamma a\mathbb{1}_{[1,r^{k-1}]}|^{\lll}_{c\mathbb{1}_{[1,r^{k-1}]}} + r^{k-1} + 1 \le \gamma + r^{N-1} + 1.$$

Lemma 4.3.8 implies that

$$\big[\gamma a^k\big]_c \le \frac{r-3}{r-(2\lambda+1)} \cdot \gamma + r^N + 1.$$

Now remember the representation (4.15), which can be written in the form of a convex combination

$$a = \sum_{k=1}^N \frac{1}{N} \cdot \frac{N}{N-1} a^k.$$

Set $b^k := \frac{N}{N-1}a^k$. Then $a \in \operatorname{conv}\{b^1,\dots,b^N\}$, and for all $\gamma > 0$

$$\big[\gamma b^k\big]_c = \big[\gamma \frac{N}{N-1} \cdot a^k\big]_c \le \gamma \cdot \underbrace{\frac{r-3}{r-(2\lambda+1)}\frac{N}{N-1}}_{=:\delta_1(r,N)} + \underbrace{(r^N+1)}_{=:\delta_2(r,N)}.$$

Obviously, $\lim_{r,N\to\infty} \delta_1(r,N) = 1$. □

It remains to deduce Theorem 4.3.2 from Proposition 4.3.3.

Proof (of Theorem 4.3.2). Let $a, c \in c_0$. Without loss of generality we may assume that $a, c \in c_0^{\downarrow}$.

We know that $|a|_c^{\trianglelefteq} \leq |a|_c^{\prec}$. To prove the reverse inequality, let $a \in c_0$ with $|a|_c^{\trianglelefteq} < \infty$ be given. If $|a|_c^{\trianglelefteq} = 0$, then $a = 0$ and hence also $|a|_c^{\prec} = 0$. We may therefore assume that $|a|_c^{\trianglelefteq} \neq 0$.

Let $\epsilon > 0$. We have

$$\tilde{a} := \frac{a}{(1+\epsilon)|a|_c^{\trianglelefteq}} \trianglelefteq c.$$

Choose, according to Proposition 4.3.3 applied with $\tilde{a}$, data $\delta_1 \leq 1+\epsilon$, $\delta_2 > 0$, $N \geq 2$, and $b^1, \ldots, b^N$, with the properties stated in the proposition.

Let $\gamma > 0$ and $k \in \{1, \ldots, N\}$. Then we can write

$$b^k = \frac{1}{\gamma} \sum_{l=1}^{m} c^l$$

with some $c^l \in [0, c]_{\prec}$ and $m \leq \gamma\delta_1 + \delta_2$. This shows that

$$|b^k|_c^{\prec} \leq \delta_1 + \frac{\delta_2}{\gamma},$$

and therefore also $|\tilde{a}|_c^{\prec} \leq \delta_1 + \frac{\delta_2}{\gamma}$. Since γ was arbitrary, it follows that $|\tilde{a}|_c^{\prec} \leq \delta_1 \leq 1+\epsilon$. From this, we find

$$|a|_c^{\prec} \leq (1+\epsilon)^2 |a|_c^{\trianglelefteq},$$

and since ϵ was arbitrary, it follows that $|a|_c^{\prec} \leq |a|_c^{\trianglelefteq}$. □

Chapter 5

Operator ideals vs. sequence spaces

In the first section of this chapter, we proof specific lemmata and theorems, from [KS08, section 8] for compact operators, which are needed to proof that every symmetric Banach sequence space gives rise to a s.n.-ideal. The second section establishes the Calkin correspondence between symmetric Banach sequence spaces and s.n.-ideals.

5.1 Uniform Hardy-Littlewood majorization of s-numbers of a sum

In this section we prove the following theorem, which gives a triangular inequality for s-numbers.

5.1.1 Theorem. *Let $k \in \mathbb{N}$ and $T_1, \dots, T_k \in \mathcal{K}(\mathcal{H})$. Then*

$$s(T_1 + \ldots + T_k) \trianglelefteq_k s(T_1) + \ldots + s(T_k). \tag{5.1}$$

In fact, we have more generally

$$\forall \alpha_1, \dots, \alpha_k \geq 0, \sum_{i=1}^{k} \alpha_i \leq 1 \; \forall n \in \mathbb{N}, r \in \mathbb{N} \cup \{0\} : \quad \sum_{j=r+1}^{n} s_j(T_1 + \ldots + T_k) \leq \sum_{i=1}^{k} \sum_{j=\lfloor \alpha_i r \rfloor + 1}^{n} s_j(T_i). \tag{5.2}$$

Note that (5.2) implies (5.1) by using $\alpha_i := \frac{1}{k}$.

We prove the theorem first for positive operators (this is a discrete version of [KS08, Lemma 8.5]). The proof is based on a lemma similar to Wielandt [Wie55, Theorem 1] which helps us to understand the sums on the right-hand side of (5.2).

Recall the notion of the trace of an operator: For a compact operator $T \in \mathcal{K}(\mathcal{H})$ with $\sum_{j=1}^{\infty} s_j(T) < \infty$, and an arbitrary orthonormal basis $\{e_i \mid i \in \mathbb{N}\}$ of $\mathcal{H}$, the sum $\sum_{j=1}^{\infty} \langle T e_j, e_j \rangle$ is finite and does not depend on the choice of the orthonormal basis. We denote

$$\operatorname{tr} T := \sum_{j=1}^{\infty} \langle T e_j, e_j \rangle$$

and this number is the trace of T. Note that, for an orthogonal projection $Q \in \mathcal{B}(\mathcal{H})$, the trace of Q equals the dimension of $\operatorname{ran} Q$.

5.1.2 Lemma. *Let $T \in \mathcal{K}(\mathcal{H})$ be positive with Schmidt representation $T = \sum_{j=1}^{\infty} s_j(T)\langle ., \phi_j\rangle\phi_j$ and denote by P_n the orthogonal projection with range* $\operatorname{span}\{\phi_1, \ldots, \phi_n\}$. *Then*

$$\sum_{j=r+1}^{n} s_j(T) = \min\Big\{ \operatorname{tr} QTQ \;\Big|\; Q \text{ orthogonal projection}, Q \le P_n, \operatorname{tr} Q = n-r \Big\}. \tag{5.3}$$

The minimum is attained for Q being the orthogonal projection onto $\operatorname{span}\{\phi_{r+1}, \ldots, \phi_n\}$.

Proof.
"$\geq$" : Let Q be the orthogonal projection onto $\operatorname{span}\{\phi_{r+1}, \ldots, \phi_n\}$. Then $\operatorname{tr} Q = n - r$, and since the trace is independent of the choice of and orthonormal basis, we may evaluate

$$\operatorname{tr} QTQ = \sum_{j=1}^{\infty}\langle QTQ\phi_j, \phi_j\rangle = \sum_{j=r+1}^{n} \langle T\phi_j, \phi_j\rangle = \sum_{j=r+1}^{n} s_j(T).$$

"$\leq$" : Let $r \in \mathbb{N}_0$ be arbitrary but fixed. We proceed by induction on n.

Let $n = r + 1$, and let Q be an orthogonal projection with $Q \le P_n$ and $\operatorname{tr} Q = 1$. Then $\operatorname{ran} Q$ is a one-dimensional subspace of $\operatorname{ran} P_n$, and thus Q admits the representation

$$Q = \langle \cdot, \xi\rangle\xi,$$

with some $\xi \in \operatorname{ran} P_n$, $\|\xi\| = 1$. Therefore we get

$$\operatorname{tr} QTQ = \langle T\xi, \xi\rangle = \sum_{i=1}^{n}\langle T\xi, \langle \xi, \phi_i\rangle\phi_i\rangle = \sum_{i=1}^{n} s_i(T)|\langle \xi, \phi_i\rangle|^2 \ge s_n(T)\underbrace{\sum_{i=1}^{n} |\langle \xi, \phi_i\rangle|^2}_{=1} = s_n(T). \tag{5.4}$$

Hence "$\leq$" in (5.3) holds for $n = r + 1$.

Now let $n \ge r+1$ and assume "$\leq$" in (5.3) holds for n. Let $Q \le P_{n+1}$ be an orthogonal projection with $\operatorname{tr} Q = n + 1 - r$.

1. Case: $Q \le P_n$

Let $E \le Q \le P_n$ be an arbitrary orthogonal projection with $\operatorname{tr} E = n-r$. By the induction hypothesis we estimate

$$\operatorname{tr} QTQ = \operatorname{tr} ETE + \operatorname{tr}(Q - E)T(Q - E) \overset{I.H.}{\ge} \sum_{j=r+1}^{n} s_j(T) + \operatorname{tr}(Q - E)T(Q - E).$$

Note that $(Q - E) \le P_{n+1}$ is an orthogonal projection with a one-dimensional range. The same argument as in (5.4) yields

$$\operatorname{tr}(Q - E)T(Q - E) \ge s_{n+1}(T).$$

2. Case: $Q \not\le P_n$

Let E be the orthogonal projection onto the subspace $\operatorname{ran} Q \cap \operatorname{ran} P_n$. Due to $Q \not\le P_n$, the dimension of the subspace $\operatorname{ran} Q + \operatorname{ran} P_n$ equals $n + 1$. Using the dimension formula we obtain

$$\dim(\operatorname{ran} Q \cap \operatorname{ran} P_n) = \underbrace{\dim \operatorname{ran} Q}_{n+1-r} + \underbrace{\dim \operatorname{ran} P_n}_{n} - \underbrace{\dim(\operatorname{ran} Q + \operatorname{ran} P_n)}_{n+1} = n - r.$$

The same argument as in Case 1 yields

$$\operatorname{tr} QTQ \geq \sum_{j=r+1}^{n+1} s_j(T).$$

□

Proof (of Theorem 5.1.1 for T_i positive). Assume that $T_1, \dots, T_k$ are positive compact operators, and that $\alpha_1, \dots, \alpha_k \geq 0$ satisfy $\sum_{i=1}^{k} \alpha_i \leq 1$. Moreover let $n \in \mathbb{N}, r \in \mathbb{N} \cup \{0\}$ and assume w.l.o.g. that $0 \leq r < n$. The operator $T_1 + \dots + T_k$ is again positive, and its Schmidt representation thus reads as

$$T_1 + \dots + T_k = \sum_{j=1}^{\infty} s_j(T_1 + \dots + T_k)\langle ., \phi_j\rangle \phi_j,$$

with some orthonormal system $\{\phi_j \mid j \in \mathbb{N}\}$. Let E be the orthogonal projection onto span $\{\phi_1, \dots, \phi_n\}$, i.e.

$$E(T_1 + \dots + T_k)E = \sum_{j=1}^{n} s_j(T_1 + \dots + T_k)\langle \cdot, \phi_j\rangle \phi_j.$$

Since $\langle \operatorname{ran} E, \|\cdot\||_{\operatorname{ran} E}\rangle$ is itself a Hilbert space and $ET_iE|_{\operatorname{ran} E} \in \mathcal{K}(\operatorname{ran} E)$, for every $i \in \{1, \dots, k\}$, we obtain the representations

$$ET_iE = \sum_{j=1}^{n} s_j(ET_iE)\langle ., \phi_j^{T_i}\rangle \phi_j^{T_i}, \qquad i \in \{1, \dots, k\},$$

for some orthonormal bases $\{\phi_j^{T_i} \mid j \in \{1, \dots, n\}\}$ of the subspace ran E. Let E_{T_i} denote the orthogonal projection onto the subspace span $\{\phi_1^{T_i}, \dots, \phi_{\lfloor \alpha_i r\rfloor}^{T_i}\}$, for $i \in \{1, \dots, k\}$. Note that these subspaces equal the zero space if $\lfloor \alpha_i r \rfloor = 0$. Then the equations

$$\begin{aligned} \forall j \in \{1, \dots, \lfloor \alpha_i r\rfloor\} : \quad & s_j(ET_iE) = s_j(E_{T_i} T_i E_{T_i}), \\ \forall j \in \{1, \dots, n - \lfloor \alpha_i r\rfloor\} : \quad & s_{j+r}(ET_iE) = s_j\big((E - E_{T_i})T(E - E_{T_i})\big), \end{aligned} \tag{5.5}$$

hold for $i \in \{1, \dots, k\}$. Since

$$\sum_{i=1}^{k} \lfloor \alpha_i r\rfloor \leq \sum_{i=1}^{k} \alpha_i r \leq r,$$

we can find an orthogonal projection F with $\operatorname{tr} F = r$ satisfying

$$\{\phi_j^{T_i} \mid i \in \{1, \dots, k\}, j \in \{1, \dots, \lfloor \alpha_i r\rfloor\}\} \subseteq \operatorname{ran} F \subseteq \operatorname{ran} E.$$

Using Lemma 5.1.2 we get

$$\begin{aligned}
\sum_{j=r+1}^{n} s_j(T_1+\ldots+T_k) &\overset{5.1.2}{\leq} \operatorname{tr}(E-F)(T_1+\ldots+T_k)(E-F) = \sum_{i=1}^{k} \operatorname{tr}(E-F)T_i(E-F) \\
&\leq \sum_{i=1}^{k} \operatorname{tr}(E-E_{T_i})T_i(E-E_{T_i}) = \sum_{i=1}^{k} \sum_{j=1}^{n-\lfloor \alpha_i r\rfloor} s_j((E-E_{T_i})T_i(E-E_{T_i})) \\
&\overset{(5.5)}{=} \sum_{i=1}^{k} \sum_{j=1}^{n-\lfloor \alpha_i r\rfloor} s_{j+\lfloor \alpha_i r\rfloor}(ET_iE) = \sum_{i=1}^{k} \sum_{j=\lfloor \alpha_i r\rfloor+1}^{n} s_j(ET_iE) \\
&\overset{2.1.3}{\leq} \sum_{i=1}^{k} \sum_{j=\lfloor \alpha_i r\rfloor+1}^{n} s_j(T_i).
\end{aligned}$$

□

We give a few auxiliary lemmata which allow to reduce the general case to the readily settled one. Before we present these statements, let us clarify notation.

For $T \in \mathcal{B}(\mathcal{H})$ we denote $\operatorname{Re}(T) := \frac{T+T^*}{2}$ and $|T| = (T^*T)^{\frac{1}{2}}$. Since $\operatorname{Re}(T)$ self-adjoint, there exists a spectral measure E with

$$\operatorname{Re}(T) = \int_{\sigma(\operatorname{Re}(T))} t dE(t). \tag{5.6}$$

We define

$$\operatorname{Re}(T)_+ := \operatorname{Re}(T)E([0,\infty)) = \int_{\sigma(\operatorname{Re}(T))} t\mathbb{1}_{[0,\infty)}(t) dE(t)$$

and call it the positive part of $\operatorname{Re}(T)$. Note that

$$\operatorname{Re}(T)_+ = E([0,\infty)) \operatorname{Re}(T) E([0,\infty)).$$

The polar decomposition an operator $T \in \mathcal{B}(\mathcal{H})$ is the representation

$$T = U|T|,$$

where $|T| = (T^*T)^{\frac{1}{2}}$ and U is a partial isometry (see for example [GGK90, Theorem 6.3]). Recall the property

$$U^*U|T| = U^*T = |T|.$$

The following two lemmata are particular cases of Kosaki [Kos84, Lemma 4, Corollary 5].

5.1.3 Lemma. *Let $T \in \mathcal{B}(\mathcal{H})$. Then there exists a partial isometry $U \in \mathcal{B}(\mathcal{H})$ with*

$$\operatorname{Re}(T)_+ \leq U|T|U^*.$$

Proof. Let $T \in \mathcal{B}(\mathcal{H})$, and write E for the orthogonal projection $E([0,\infty))$ from (5.6). The polar decomposition provides a partial isometry V such that $T = V|T|$. Now define

$$A := \frac{1}{2}E(I+V),$$

and consider the polar decomposition $A|T|^{\frac{1}{2}} = U\big|A|T|^{\frac{1}{2}}\big|$. Since

$$\langle A^*Ax, x\rangle \leq \|x\|^2 = \langle Ix, x\rangle,$$

the inequality $A^*A \leq I$ holds. Using

$$A|T|A^* = U\big|A|T|^{\frac{1}{2}}\big|^2U^* = U(|T|^{\frac{1}{2}}A^*A|T|^{\frac{1}{2}})U^* \leq U|T|U^*,$$

leads to the desired result:

$$\begin{aligned}U|T|U^* - \operatorname{Re}(T)_+ &= U|T|U^* - \frac{1}{2}E(V|T| + |T|V^*)E \geq A|T|A^* - \frac{1}{2}E(V|T| + |T|V^*)E \\ &= \frac{1}{4}E(I+V)|T|(I+V^*)E - \frac{1}{2}E(V|T| + |T|V^*)E \\ &= \frac{1}{4}E\big[(I+V)|T|(I+V^*) - 2V|T| - 2|T|V^*\big]E = \frac{1}{4}E\big[(I-V)|T|(I-V^*)\big]E \\ &\geq 0.\end{aligned}$$

□

5.1.4 Lemma. *Let $k \in \mathbb{N}$ and $T_1, \ldots, T_k \in \mathcal{B}(\mathcal{H})$. Then there exist partial isometries $U_1, \ldots, U_k, W \in \mathcal{B}(\mathcal{H})$ such that*

$$|T_1 + \ldots + T_k| \leq \sum_{i=1}^{k} U_i|W^*T_i|{U_i}^*.$$

Proof. Let $T_1, \ldots, T_k \in \mathcal{B}(\mathcal{H})$, and choose a partial isometry $W \in \mathcal{B}(\mathcal{H})$ with

$$T_1 + \ldots + T_k = W|T_1 + \ldots + T_k|.$$

Then we obtain

$$\begin{aligned}|T_1 + \ldots + T_k| &= \operatorname{Re}|T_1 + \ldots + T_k| = \operatorname{Re}(W^*T_1 + \ldots + W^*T_k) = \operatorname{Re}(W^*T_1) + \ldots + \operatorname{Re}(W^*T_k) \\ &\leq \operatorname{Re}(W^*T_1)_+ + \ldots + \operatorname{Re}(W^*T_k)_+.\end{aligned}$$

From Lemma 5.1.3 we know that there exist partial isometries $U_1, \ldots, U_k \in \mathcal{B}(\mathcal{H})$ such that

$$\operatorname{Re}(W^*T_i)_+ \leq U_i|W^*T_i|{U_i}^*, \qquad i \in \{1, \ldots, k\},$$

and this completes the proof. □

Now we gathered the necessary tools to prove Theorem 5.1.1.

Proof (of Theorem 5.1.1). Let $T_1, \ldots, T_k \in \mathcal{K}(\mathcal{H})$ and $n \in \mathbb{N}, r \in \mathbb{N} \cup \{0\}$ such that $0 \leq r < n$. Moreover, let $U_1, \ldots, U_k$ and W be partial isometries as in Lemma 5.1.4. Using the readily proven fact that (5.2) holds for positive operators, we can estimate

$$\begin{aligned}\sum_{j=r+1}^{n} s_j(T_1 + \ldots + T_k) &\overset{\text{2.1.9 ii)}}{=} \sum_{j=r+1}^{n} s_j(|T_1 + \ldots + T_k|) \overset{\substack{\text{5.1.4 \&} \\ \text{2.1.9 i)}}}{\leq} \sum_{j=r+1}^{n} s_j\Big(\sum_{i=1}^{k} U_i|W^*T_i|{U_i}^*\Big) \\ &\overset{(5.2)}{\leq} \sum_{i=1}^{k} \sum_{j=\lfloor \alpha_i r \rfloor + 1}^{n} s_j(U_i|W^*T_i|{U_i}^*) \overset{\text{2.1.3}}{\leq} \sum_{i=1}^{k} \sum_{j=\lfloor \alpha_i r \rfloor + 1}^{n} s_j(W^*T_i) \\ &\leq \sum_{i=1}^{k} \sum_{j=\lfloor \alpha_i r \rfloor + 1}^{n} s_j(T_i).\end{aligned}$$

□

5.2 Symmetric Banach sequence spaces

We saw in Section 3.2 that operator ideals in $\mathcal{K}(\mathcal{H})$ correspond bijectively to solid symmetric subspaces of c_0, namely via the Calkin correspondence

$$\mathscr{E} \mapsto \mathscr{S}_{\mathscr{E}} = \left\{ T \in \mathcal{K}(\mathcal{H}) \;\middle|\; s(T) \in \mathscr{E} \right\} \quad \text{for} \quad \mathscr{E} \text{ solid symmetric subspace.} \tag{5.7}$$

The question arises which solid symmetric subspaces correspond to symmetrically normed ideals. This question is much more involved than the algebraic theory, and was answered only recently in [KS08].

5.2.1 Definition. Let $\mathscr{E}$ be a linear subspace of c_0 endowed with a norm $\|\cdot\|_{\mathscr{E}}$. Then $\mathscr{E}$ is called a symmetric Banach sequence space if

i) $\mathscr{E}$ is a solid symmetric subspace of c_0.

ii) $\langle \mathscr{E}, \|\cdot\|_{\mathscr{E}} \rangle$ is a Banach space.

iii) for all $a \in c_0$ and $b \in \mathscr{E}$ with $|a| \leq |b|$ it holds that $\|a\|_{\mathscr{E}} \leq \|b\|_{\mathscr{E}}$.

iv) for all $a \in \mathscr{E}$ it holds that $\|a^*\|_{\mathscr{E}} = \|a\|_{\mathscr{E}}$.

v) $\|(1, 0, 0, \ldots)\|_{\mathscr{E}} = 1$.

Our aim in this section is to prove that in the Calkin correspondence symmetric Banach sequence spaces correspond to symmetrically normed ideals. Recall here that E_a denotes the diagonal operator with eigenvalues a w.r.t. some orthonormal basis.

5.2.2 Theorem. *The following two statements hold.*

i) *Given a symmetrically normed ideal $\langle \mathscr{S}, \|\cdot\|_{\mathscr{S}} \rangle$, define*

$$\|a\|_{\mathscr{E}_{\mathscr{S}}} := \|E_a\|_{\mathscr{S}} \quad \textit{for} \quad a \in \mathscr{E}_{\mathscr{S}}.$$

Then $\langle \mathscr{E}_{\mathscr{S}}, \|\cdot\|_{\mathscr{E}_{\mathscr{S}}} \rangle$ is a symmetric Banach sequence space.

ii) *Given a symmetric Banach sequence space $\langle \mathscr{E}, \|\cdot\|_{\mathscr{E}} \rangle$, define*

$$\|T\|_{\mathscr{S}_{\mathscr{E}}} := \|s(T)\|_{\mathscr{E}} \quad \textit{for} \quad T \in \mathscr{S}_{\mathscr{E}}. \tag{5.8}$$

Then $\langle \mathscr{S}_{\mathscr{E}}, \|\cdot\|_{\mathscr{S}_{\mathscr{E}}} \rangle$ is a symmetrically normed ideal.

The fact that every symmetrically normed ideal gives rise to a symmetric Banach sequence space is seen in a straightforward way.

Proof (of Theorem 5.2.2, first item). Assume $\langle \mathscr{S}, \|\cdot\|_{\mathscr{S}} \rangle$ is a symmetrically normed ideal, and consider the associated sequence space $\mathscr{E}_{\mathscr{S}} = \left\{ a \in c_0 \;\middle|\; E_a \in \mathscr{S} \right\}$, where E_a is the diagonal operator of a sequence a w.r.t. to an orthonormal basis $\mathcal{B}$ of $\mathcal{H}$. In Section 3.2 we elaborated the fact that $\mathscr{E}_{\mathscr{S}}$ is a solid symmetric subspace of c_0. Since norms on a symmetric Banach sequence spaces solely depend on the nonincreasing rearrangement of a sequence, it is somewhat natural to define a functional on $\mathscr{E}_{\mathscr{S}}$ by $\|a\|_{\mathscr{E}_{\mathscr{S}}} := \|E_a\|_{\mathscr{S}}$, as the right side only depends on $s(E_a) = a^*$. That $\|\cdot\|_{\mathscr{E}_{\mathscr{S}}}$ is a norm is easy to see and follows from the fact that $\|\cdot\|_{\mathscr{S}}$ is a norm and using computation rules of Lemma 2.2.4. In fact $\mathscr{E}_{\mathscr{S}}$ endowed with $\|\cdot\|_{\mathscr{E}_{\mathscr{S}}}$ is a symmetric Banach sequence space. That $\mathscr{E}_{\mathscr{S}}$ satisfies iv) and v) from Definition 5.2.1 is obvious. To see that iii) holds for $\mathscr{E}_{\mathscr{S}}$, let $a \in c_0$, $b \in \mathscr{E}_{\mathscr{S}}$ with $|a| \leq |b|$. This implies that $a^* \leq b^*$ holds. By Proposition 2.2.2, we have

$$\|a\|_{\mathscr{E}_{\mathscr{S}}} = \|E_a\|_{\mathscr{S}} \leq \|E_b\|_{\mathscr{S}} = \|b\|_{\mathscr{E}_{\mathscr{S}}}.$$

To see completeness of $\mathscr{E}_{\mathscr{S}}$, let $(a^n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $\mathscr{E}_{\mathscr{S}}$. Then $(E_{a^n})_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathscr{S}$. It is here to be mentioned that every symmetric norm on an operator ideal is stronger than the operator norm. This follows from

$$\|T\| = s_1(T) = s_1(T)\cdot\|E_{(1,0,0,\dots)}\|_{\mathscr{S}} = \|E_{(s_1(T),0,0,\dots)}\|_{\mathscr{S}} \le \|E_{s(T)}\|_{\mathscr{S}} = \|T\|_{\mathscr{S}}.$$

Hence, the sequence $(E_{a^n})_{n\in\mathbb{N}}$ converges in the operator norm to some operator $T \in \mathcal{K}(\mathcal{H})$. Now invoking Example 2.2.5 ensures that T is indeed a diagonal operator itself, i.e. there exists a zero sequence a such that $T = E_a$ holds. We conclude that $(a^n)_{n\in\mathbb{N}}$ converges to a in $\mathscr{E}_{\mathscr{S}}$. Hence, $\mathscr{E}_{\mathscr{S}}$ is complete, and in particular $\langle \mathscr{E}_{\mathscr{S}}, \|\cdot\|_{\mathscr{E}_{\mathscr{S}}}\rangle$ is a symmetric Banach sequence space. □

The second item in Theorem 5.2.2 is a deep result. Given a symmetric Banach sequence space $\langle \mathscr{E}, \|\cdot\|_{\mathscr{E}}\rangle$, it is natural to use the stated definition of $\|\cdot\|_{\mathscr{S}_{\mathscr{E}}}$ since norms on symmetrically normed ideals solely depend on s-numbers. Then it has to be proven that indeed a symmetrically normed ideal is obtained in this way. The key steps are to show the triangular inequality for $\|\cdot\|_{\mathscr{S}_{\mathscr{E}}}$ and completeness of the space.

We start with a couple of lemmata. The first two exploit a somewhat surprising viewpoint on solid symmetric subspaces of c_0 and lead to the triangular inequality for $\|\cdot\|_{\mathscr{S}_{\mathscr{E}}}$. The second two contain the technical core for the proof of completeness.

5.2.3 Lemma. *Let $\mathscr{E}$ be a linear subspace of c_0. Then $\mathscr{E}$ is a solid symmetric subspace of c_0 if and only if*

$$\forall a \in c_0, c \in \mathscr{E}: \quad a \trianglelefteq c \;\Rightarrow\; a \in \mathscr{E}.$$

Proof.
"$\Rightarrow$" : Let $a \in c_0$ and $c \in \mathscr{E}$ be given with $a \trianglelefteq c$. Using Theorem 4.3.2 we obtain $|a|_c^{\prec} = |a|_c^{\trianglelefteq} \le 1$. Let $\mu > 1$, then we can write $\frac{1}{\mu}a$ as a convex combination $\frac{1}{\mu}a = \sum_{j=1}^n \lambda_j c^j$ with $c^j \prec c$. Since $\mathscr{E}$ is solid, we obtain $c^j \in \mathscr{E}$, and therefore $a \in \mathscr{E}$.
"$\Leftarrow$" : Let $a \in c_0$ and assume that there exists $b \in \mathscr{E}$ such that either $|a| \le |b|$ or $a = \mathscr{R}_\pi b$ for some permutation π. In both cases it follows that $a \prec b$, and hence that $a \trianglelefteq b$. From this we conclude $a \in \mathscr{E}$. □

5.2.4 Lemma. *Let $\mathscr{E}$ be a symmetric Banach sequence space. Then*

$$\forall a, c \in \mathscr{E}\backslash\{0\}: \quad \|a\|_{\mathscr{E}} \le |a|_c^{\trianglelefteq}\cdot\|c\|_{\mathscr{E}}.$$

In particular, $\|a\|_{\mathscr{E}} \le \|c\|_{\mathscr{E}}$ whenever $a \trianglelefteq c$.

Proof. Let $a, c \in \mathscr{E}\backslash\{0\}$. If $|a|_c^{\trianglelefteq} = \infty$, there is nothing to prove. Hence, assume that $|a|_c^{\trianglelefteq} < \infty$. Remember Theorem 4.3.2, and again choose $\mu > |a|_c^{\trianglelefteq} = |a|_c^{\prec}$ and a convex combination $\frac{1}{\mu}a = \sum_{j=1}^n \lambda_j c^j$ with $c^j \prec c$. Then $\|c^j\|_{\mathscr{E}} \le \|c\|_{\mathscr{E}}$ by the properties of $\|\cdot\|_{\mathscr{E}}$, and we can estimate

$$\|a\|_{\mathscr{E}} = \mu\cdot\|\frac{1}{\mu}a\|_{\mathscr{E}} = \mu\cdot\|\sum_{j=1}^n \lambda_j c^j\|_{\mathscr{E}} \le \mu\cdot\sum_{j=1}^n \lambda_j\|c^j\|_{\mathscr{E}} \le \mu\cdot\sum_{j=1}^n \lambda_j\|c\|_{\mathscr{E}} = \mu\cdot\|c\|_{\mathscr{E}}.$$

Since μ was arbitrary, we see that the statement of the lemma holds. □

5.2.5 Lemma. *Let $a \in c_0^{\downarrow}$, let $n \in N, r \in \mathbb{N}\cup\{0\}$ with $2r \le n$, and let $q \in \mathbb{N}$. Then*

$$\sum_{j=\lfloor\frac{2r}{q}\rfloor+1}^{n} a_j \le 2\sum_{j=r+1}^{n} q a_{\lceil\frac{j}{q}\rceil}.$$

Proof. First note that

$$q \sum_{j=\lfloor \frac{2r}{q} \rfloor+1}^{n} a_j = \sum_{j=q\lfloor \frac{2r}{q} \rfloor+1}^{qn} a_{\lceil \frac{j}{q} \rceil}.$$

We split the sum on the right in two parts, cutting at r:

$$\sum_{j=q\lfloor \frac{2r}{q} \rfloor+1}^{qn} a_{\lceil \frac{j}{q} \rceil} = \sum_{j=q\lfloor \frac{2r}{q} \rfloor+1}^{r} a_{\lceil \frac{j}{q} \rceil} + \sum_{j=r+1}^{qn} a_{\lceil \frac{j}{q} \rceil}.$$

The first sum has at most $q-1$ summands. If it is at all present, i.e., if $q\lfloor \frac{2r}{q} \rfloor + 1 \leq r$, then

$$\left\lceil \frac{r+1}{q} \right\rceil \geq \left\lceil \frac{q\lfloor \frac{2r}{q} \rfloor + 1}{q} \right\rceil = \left\lceil \left\lfloor \frac{2r}{q} \right\rfloor + \frac{1}{q} \right\rceil = \left\lfloor \frac{2r}{q} \right\rfloor + 1 \geq \left\lceil \frac{r+1}{q} \right\rceil,$$

and hence necessarily $\left\lceil \frac{r+1}{q} \right\rceil = \left\lceil \frac{q\lfloor \frac{2r}{q} \rfloor+1}{q} \right\rceil$. Therefore we obtain

$$\sum_{j=q\lfloor \frac{2r}{q} \rfloor+1}^{r} a_{\lceil \frac{j}{q} \rceil} \leq (q-1)a_{\lceil \frac{r+1}{q} \rceil} \leq (q-1) \sum_{j=r+1}^{qn} a_{\lceil \frac{j}{q} \rceil}. \tag{5.9}$$

Putting together, it follows that

$$\sum_{j=\lfloor \frac{2r}{q} \rfloor+1}^{n} a_j \leq q \sum_{j=r+1}^{qn} a_{\lceil \frac{j}{q} \rceil}.$$

It holds that

$$(n-r) \sum_{j=r+1}^{qn} a_{\lceil \frac{j}{q} \rceil} \leq (qn-r) \sum_{j=r+1}^{n} a_{\lceil \frac{j}{q} \rceil},$$

and from this we see that

$$\sum_{j=r+1}^{qn} a_{\lceil \frac{j}{q} \rceil} \leq \frac{n}{n-r} \sum_{j=r+1}^{n} q a_{\lceil \frac{j}{q} \rceil}.$$

Finally, observe that $\frac{n}{n-r} \leq 2$. □

5.2.6 Lemma. *Let $\mathscr{E}$ be a symmetric Banach sequence space. Then, for each $q \in \mathbb{N}$, the replication operator $\mathscr{P}_q$ is bounded with norm at most q.*

In particular, if we have $q_i > 0$ and $a_i \in \mathscr{E}$ with $\sum_{i=1}^{\infty} q_i^2 \|a_i\|_{\mathscr{E}} < \infty$, then the series $\sum_{i=1}^{\infty} q_i \mathscr{P}_{q_i} a_i$ is absolutely convergent w.r.t. $\|\cdot\|_{\mathscr{E}}$.

Proof. We have

$$\|\mathscr{P}_q a\|_{\mathscr{E}} = \|(\underbrace{a_1, \ldots, a_1}_{q \text{ times}}, \underbrace{a_2, \ldots a_2}_{q \text{ times}}, \ldots)\|_{\mathscr{E}} \leq q\|a\|_{\mathscr{E}}.$$

The additional statement is now clear. □

Proof (of Theorem 5.2.2, second item). Assume we are given a symmetric Banach sequence space $\langle \mathscr{E}, \|\cdot\|_{\mathscr{E}} \rangle$, and consider the solid symmetric subspace $\mathscr{S}_{\mathscr{E}}$ endowed with $\|\cdot\|_{\mathscr{S}_{\mathscr{E}}}$ as in (5.8).

It is obvious that $\|T\|_{\mathscr{S}_{\mathscr{E}}} \geq 0$ with equality if and only if $T = 0$, and that $\|\alpha T\|_{\mathscr{S}_{\mathscr{E}}} = |\alpha| \|T\|_{\mathscr{S}_{\mathscr{E}}}$ for all $\alpha \in \mathbb{K}$. For the proof of the triangle inequality, let $S, T \in \mathscr{S}_{\mathscr{E}}$. Since $s(S+T) \trianglelefteq s(S) + s(T)$ by Theorem 5.1.1, the above Lemma 5.2.4 yields

$$\begin{aligned}\|S+T\|_{\mathscr{S}_{\mathscr{E}}} &= \|s(S+T)\|_{\mathscr{E}} \leq |s(S+T)|^{\trianglelefteq}_{s(S)+s(T)} \cdot \|s(S)+s(T)\|_{\mathscr{E}} \leq \|s(S)+s(T)\|_{\mathscr{E}} \\ &\leq \|s(S)\|_{\mathscr{E}} + \|s(T)\|_{\mathscr{E}} = \|S\|_{\mathscr{S}_{\mathscr{E}}} + \|T\|_{\mathscr{S}_{\mathscr{E}}}.\end{aligned}$$

We see that $\|\cdot\|_{\mathscr{S}_{\mathscr{E}}}$ is a norm. The fact that it is a symmetric norm is again clear. First, for every one-dimensional operator it holds that $\|T\| = s_1(T) = \|s(T)\|_{\mathscr{E}}$. Second, for bounded operators A, B we have $s(ATB) \leq \|A\|\|B\|s(T)$, and hence $\|s(ATB)\|_{\mathscr{E}} \leq \|A\|\|B\|\|s(T)\|_{\mathscr{E}}$.

It remains to prove completeness. Assume that $(T_m)_{m\in\mathbb{N}}$ is a Cauchy sequence in $\langle \mathscr{S}_{\mathscr{E}}, \|\cdot\|_{\mathscr{S}_{\mathscr{E}}}\rangle$. Note that $\|\cdot\| \leq \|\cdot\|_{\mathscr{S}_{\mathscr{E}}}$, since

$$\|T\| = s_1(T) = \|(s_1(T), 0, 0, \ldots)\|_{\mathscr{E}} \leq \|s(T)\|_{\mathscr{E}} = \|T\|_{\mathscr{S}_{\mathscr{E}}}.$$

Hence, $(T_m)_{m\in\mathbb{N}}$ is also a Cauchy sequence w.r.t. the operator norm, and therefore converges in the operator norm to some compact operator T. We are going to show that $T \in \mathscr{S}_{\mathscr{E}}$ and is the limit w.r.t. $\|\cdot\|_{\mathscr{S}_{\mathscr{E}}}$ of some subsequence of $(T_m)_{m\in\mathbb{N}}$. From this it is then clear that $T_m \to T$ w.r.t. $\|\cdot\|_{\mathscr{S}_{\mathscr{E}}}$.

Choose a sequence of natural numbers q_i, such that $\sum_{i=1}^{\infty} \frac{1}{q_i} \leq 1$. Since $(T_m)_{m\in\mathbb{N}}$ is a Cauchy-sequence, we can extract a subsequence $(T_{m_j})_{j\in\mathbb{N}}$ with

$$\forall j \geq 2: \quad \|T_{m_j} - T_{m_{j-1}}\|_{\mathscr{S}_{\mathscr{E}}} \leq \frac{1}{q_j^2} \cdot \frac{1}{j^2}.$$

For notational convenience set $T_{m_0} := 0$. Then we can write T as the telescoping series

$$T = \sum_{j=1}^{\infty} \big(T_{m_j} - T_{m_{j-1}}\big), \tag{5.10}$$

which converges w.r.t. the operator norm.

Set $\tilde{T}_j := T_{m_j} - T_{m_{j-1}}$. Let $k \in \mathbb{N}$, then Lemma 5.2.6 yields that the series

$$c^k := \sum_{i=k}^{\infty} q_i \mathscr{P}_{q_i}\big(s(\tilde{T}_i)\big)$$

converges w.r.t. $\|\cdot\|_{\mathscr{E}}$ and, by the completeness of $\mathscr{E}$, thus $c^k \in \mathscr{E}$. Clearly, $\|c^k\|_{\mathscr{E}} \leq \sum_{j=k}^{\infty} \frac{1}{j^2}$ for $k \geq 2$.

We use c^k to estimate s-numbers. Let $n \in \mathbb{N}, r \in \mathbb{N} \cup \{0\}$, and $k, N \in \mathbb{N}$. Then

$$\begin{aligned}\sum_{j=2r+1}^{n} s_j(T_{m_{k+N}} - T_{m_{k-1}}) &= \sum_{j=2r+1}^{n} s_j(\tilde{T}_k + \ldots + \tilde{T}_{k+N}) \overset{5.1.1}{\leq} \sum_{i=k}^{k+N} \sum_{j=\lfloor \frac{r}{q_i} \rfloor +1}^{n} s_j(\tilde{T}_i) \\ &\overset{5.2.5}{\leq} \sum_{i=k}^{k+N} 2 \sum_{j=r+1}^{n} q_i s_{\lceil \frac{j}{q_i} \rceil}(\tilde{T}_i) \leq 2 \sum_{j=r+1}^{n} \sum_{i=k}^{\infty} q_i s_{\lceil \frac{j}{q_i} \rceil}(\tilde{T}_i) = \sum_{j=r+1}^{n} (2c^k)_j.\end{aligned}$$

Letting $N \to \infty$, we conclude that $s(T - T_{m_{k-1}}) \trianglelefteq 2c^k$. By means of Lemma 5.2.3 this implies that $T \in \mathscr{S}_{\mathscr{E}}$. Now Lemma 5.2.4 applies, and yields

$$\|T - T_{m_{k-1}}\|_{\mathscr{S}_{\mathscr{E}}} \leq \|c^k\|_{\mathscr{E}} \leq \sum_{j=k}^{\infty} \frac{1}{j^2} \qquad k \geq 2.$$

We see that $T_{m_k} \to T$ w.r.t. $\|\cdot\|_{\mathscr{S}_{\mathscr{E}}}$. □

Chapter 6

Normed cones and cone maps

This chapter is more or less preparation for a survey paper, which aims to present and proof some, almost unknown theorems, discovered by A. A. Mititel and G. I. Russu from the 70's and 80's concerning s.n.-ideals. Section 6.1 introduces notions like normed cones, cone maps and semigroups of cone maps and establishes the powerful Lemma 6.1.5. The second section discusses some necessary and sufficient condition, when to expect the Cesaro-means and its weighted dual version to be invariant and bounded on specific normed cones. In the final section, an interpolation theorem is established.

6.1 Normed cones and semigroups

Recall that a cone is a subset of a vector space which is closed under linear combinations with non-negative coefficients. Moreover, a map f between two cones $\mathcal{C}_1$ and $\mathcal{C}_2$ is called a cone map, if it is positively homogeneous and additive, i.e.:

- $\forall \lambda > 0, a \in \mathcal{C}: \quad f(\lambda a) = \lambda \cdot f(a).$
- $\forall a, b \in \mathcal{C}: \quad f(a+b) = f(a) + f(b).$

The set of all cone maps mapping from $\mathcal{C}_1$ to $\mathcal{C}_2$ is denoted by $\mathrm{Hom}(\mathcal{C}_1, \mathcal{C}_2)$.

In this chapter we deal with cones which additionally carry a norm, and with bounded maps between such cones.

6.1.1 Definition. A pair $\langle \mathcal{C}, \|\cdot\|_{\mathcal{C}} \rangle$ is called a normed cone, if $\mathcal{C}$ is a cone and the map $\|\cdot\|_{\mathcal{C}} : \mathcal{C} \to [0, \infty)$ satisfies

i) $\forall a \in \mathcal{C}: \quad a = 0 \;\Leftrightarrow\; \|a\|_{\mathcal{C}} = 0.$

ii) $\forall a, b \in \mathcal{C}: \quad \|a+b\|_{\mathcal{C}} \leq \|a\|_{\mathcal{C}} + \|b\|_{\mathcal{C}}.$

iii) $\forall \lambda > 0, a \in \mathcal{C}: \quad \|\lambda a\|_{\mathcal{C}} = \lambda \cdot \|a\|_{\mathcal{C}}.$

If $\mathcal{C}_1$ and $\mathcal{C}_2$ are normed cones and $f \in \mathrm{Hom}(\mathcal{C}_1, \mathcal{C}_2)$, we set

$$\|f\| := \sup \left\{ \|fa\|_{\mathcal{C}_2} \;\middle|\; a \in \mathcal{C}_1, \|a\|_{\mathcal{C}_1} \leq 1 \right\} \in [0, \infty],$$

and say that f is bounded, if $\|f\| < \infty$.

6.1.2 Example. For $1 \leq p < \infty$ let $\mathbb{L}^p$ be the cone

$$\mathbb{L}^p := \left\{ a \in c_0^{\downarrow} \mid \sum_{j=1}^{\infty} a_j^p < \infty \right\}.$$

Naturally, it becomes a normed cone with

$$\|a\|_p := \sum_{j=1}^{\infty} a_j^p.$$

In a similar way, we have the normed cones

$$\mathbb{L}_+^p := \{a \in c_0^{\downarrow} \mid \|a\|_p^+ < \infty\} \quad \text{with} \quad \|a\|_p^+ := \sum_{j=1}^{\infty} j^{\frac{1}{p}-1} a_j,$$
$$\mathbb{L}_w^p := \{a \in c_0^{\downarrow} \mid \|a\|_p^w < \infty\} \quad \text{with} \quad \|a\|_p^w := \inf\{C \geq 0 \mid \forall n \in \mathbb{N} : a_n \leq C n^{-\frac{1}{p}}\}.$$

Note that $\mathbb{L}_+^p \subseteq \mathbb{L}_w^p$, and that $\mathbb{L}_w^p \subseteq \mathbb{L}^{p+\varepsilon}$ for all $\varepsilon > 0$.

Given a cone map $f \in \mathrm{Hom}(c_{00}^{\downarrow}, c_0^{\downarrow})$, we can naturally extend it to a generically much larger cone.

6.1.3 Definition. For $f \in \mathrm{Hom}(c_{00}^{\downarrow}, c_0^{\downarrow})$ and $a \in c_0^{\downarrow}$ we set

$$\bar{f}(a) := \lim_{N\to\infty} \sum_{j=1}^{N} (a_j - a_{j+1}) \cdot f(\mathbb{1}_{[1,j]}) \in [0,\infty]^{\mathbb{N}},$$

and

$$\mathrm{dom}\,\bar{f} := \{a \in c_0^{\downarrow} \mid \bar{f}(a)_1 < \infty\}.$$

Note that for each $a \in c_0^{\downarrow}$ we have $a = \sum_{j=1}^{\infty}(a_j - a_{j+1}) \cdot \mathbb{1}_{[1,j]}$ uniformly, that $\mathrm{dom}\,\bar{f}$ is a cone and $\bar{f} : \mathrm{dom}\,\bar{f} \to c_0^{\downarrow}$ is a cone map.

6.1.4 Definition. Let $\mathcal{C}$ be a cone. A map $\mathcal{F} : \mathbb{N} \to \mathrm{Hom}(\mathcal{C},\mathcal{C})$ is called a semigroup of cone maps, if it is a homomorphism of the monoids $\langle \mathbb{N}, \cdot, 1\rangle$ and $\langle \mathrm{Hom}(\mathcal{C},\mathcal{C}), \circ, \mathrm{id}_{\mathcal{C}}\rangle$. Explicitly, this means that

$$\mathcal{F}(1) = \mathrm{id}_{\mathcal{C}} \quad \text{and} \quad \forall N_1, N_2 \in \mathbb{N} : \quad \mathcal{F}(N_1 N_2) = \mathcal{F}(N_1) \circ \mathcal{F}(N_2).$$

Observe the following, simple but powerful, fact.

6.1.5 Lemma. *Let $\langle \mathcal{C}, \|\cdot\|_{\mathcal{C}}\rangle$ be a normed cone, and let $\mathcal{F} : \mathbb{N} \to \mathrm{Hom}(\mathcal{C},\mathcal{C})$ be a semigroup of cone maps. Assume that*

$$\forall N \in \mathbb{N} : \quad \|\mathcal{F}(N)\| < \infty \quad \textit{and} \quad \|\mathcal{F}(N)\| \leq \|\mathcal{F}(N+1)\|. \tag{6.1}$$

Then, for any fixed $N_0 \in \mathbb{N}\backslash\{1\}$

$$\forall N \in \mathbb{N} : \quad \|\mathcal{F}(N)\| \leq \|\mathcal{F}(N_0)\| \cdot N^{\beta},$$

where $\beta := \frac{\log \|\mathcal{F}(N_0)\|}{\log N_0}$.

Proof. Let $N \in \mathbb{N}$ and choose $k \in \mathbb{N}$ with $N_0^{k-1} \leq N \leq N_0^k$. Then

$$\|\mathcal{F}(N)\| \leq \|\mathcal{F}(N_0^k)\| \leq \|\mathcal{F}(N_0)\|^k = N_0^{\beta k} = N_0^{\beta} \cdot N_0^{\beta(k-1)} \leq N_0^{\beta} \cdot N_0^{\beta N} = \|\mathcal{F}(N_0)\| \cdot N^{\beta}.$$

□

6.1.6 Corollary. *Let $\langle \mathcal{C}, \|\cdot\|_{\mathcal{C}} \rangle$ be a normed cone, and let $\mathcal{F} : \mathbb{N} \to \mathrm{Hom}(\mathcal{C}, \mathcal{C})$ be a semigroup of cone maps which satisfies (6.1). Then the following statements are equivalent.*

i) $\exists \beta < 1 : \quad \|\mathcal{F}(N)\| = O(N^{\beta})$.

ii) $\|\mathcal{F}(N)\| = o(N)$.

iii) $\exists N_0 \in \mathbb{N} : \quad \|\mathcal{F}(N_0)\| < N_0$.

Proof. The implications i) $\Rightarrow$ ii) $\Rightarrow$ iii) are trivial.
"iii) $\Rightarrow$ i)" : Let $\mathbb{N}_0 \in \mathbb{N}$ be given such that $\|\mathcal{F}(N_0)\| < N_0$. Then $\beta := \frac{\log \|\mathcal{F}(N_0)\|}{\log N_0} < 1$, and Lemma 6.1.5 gives us the estimate

$$\|\mathcal{F}(N)\| \leq \|\mathcal{F}(N_0)\| \cdot N^{\beta}.$$

□

6.2 A boundedness criterion

In this section we study some concrete cone maps on particular classes of normed cones.

The cone maps we are interested in are the replication operator $\mathscr{P}_N$, the Cesaro means operator $\mathscr{C}$, and dual versions of those.

6.2.1 Definition.

- The Hardy-operator $\mathscr{H}_n : c_0^{\downarrow} \to c_0^{\downarrow}$ is defined as

$$\big((\mathscr{H}_n a)_j\big)_{j\in\mathbb{N}} = \Big(\frac{1}{n} \sum_{k=(j-1)n+1}^{jn} a_k\Big)_{j\in\mathbb{N}} = \big(\frac{a_1 + \ldots + a_n}{n}, \frac{a_{n+1} + \ldots + a_{2n}}{n}, \ldots\big).$$

- We define a family of summation operators $\mathscr{D}_w$ where $w \in [0,1]$ as

$$\big(\mathscr{D}_w a\big)_n := \sum_{j=n}^{\infty} \Big(\frac{j}{n}\Big)^{w} \frac{a_j}{j},$$

  for $a \in c_0^{\downarrow}$ such that the series converge.

The relevant properties of normed cones are the following.

6.2.2 Definition. We call a normed cone $\langle \mathcal{C}, \|\cdot\|_{\mathcal{C}} \rangle$ which is contained in $c_0^{\downarrow}$

- complete, if for every Cauchy sequence $(a^n)_{n\in\mathbb{N}}$ in $\mathcal{C}$, there exists $a \in \mathcal{C}$ such that $\lim_{n\to\infty} a^n = a$ holds w.r.t. to $\|\cdot\|_{\mathcal{C}}$ and pointwise.
- solidly normed, if $\mathcal{C}$ is solid and

$$\forall a, b \in \mathcal{C} : \quad |a| \leq |b| \Rightarrow \|a\|_{\mathcal{C}} \leq \|b\|_{\mathcal{C}}.$$

- Hardy-monotone, if $\mathcal{C}$ is solid and

$$\forall N \in \mathbb{N} : \quad \|N \cdot \mathscr{H}_N\| \leq \|(N+1) \cdot \mathscr{H}_{N+1}\|.$$

- $\lll$-monotone normed, if

$$\forall a, b \in \mathcal{C} : \quad a \lll b \Rightarrow \|a\|_{\mathcal{C}} \leq \|b\|_{\mathcal{C}}.$$

To start with, we observe some simple facts about these notions.

6.2.3 Lemma. *Let $\langle \mathcal{C}, \|\cdot\|_{\mathcal{C}}\rangle$ be a normed cone which is contained in $c_0^{\downarrow}$ and is solid.*

i) *For all $N \in \mathbb{N}$ and $a \in c_0^{\downarrow}$ we have $\mathscr{H}_N a \leq a$.*

ii) *For all $N \in \mathbb{N}$ and $a \in c_0^{\downarrow}$ we have $(\mathscr{H}_N \circ \mathscr{P}_N)(a) = a$.*

iii) *If $\langle \mathcal{C}, \|\cdot\|_{\mathcal{C}}\rangle$ is $\lll$-monotone normed, then it is Hardy-monotone.*

Proof. The first two assertions are obvious, the third follows since $n\mathscr{H}_n a \lll (n+1)\mathscr{H}_{n+1} a$. □

Our aim in this section is to prove the following boundedness criterion.

6.2.4 Theorem. *Let $\langle \mathcal{C}, \|\cdot\|_{\mathcal{C}}\rangle$ be a replication closed and solidly normed cone in $c_0^{\downarrow}$.*

i) *If $\langle C, \|\cdot\|_{\mathcal{C}}\rangle$ is complete, then*

$$\|\mathscr{P}_N|_{\mathcal{C}}\| = o(N) \quad \Longrightarrow \quad \mathscr{C}(\mathcal{C}) \subseteq \mathcal{C} \ \wedge \ \|\mathscr{C}|_{\mathcal{C}}\| < \infty.$$

ii) *If $\langle C, \|\cdot\|_{\mathcal{C}}\rangle$ is $\lll$-monotone, then*

$$\|\mathscr{P}_N|_{\mathcal{C}}\| = o(N) \quad \Longleftarrow \quad \mathscr{C}(\mathcal{C}) \subseteq \mathcal{C} \ \wedge \ \|\mathscr{C}|_{\mathcal{C}}\| < \infty.$$

iii) *If $\langle C, \|\cdot\|_{\mathcal{C}}\rangle$ is complete and Hardy-monotone, then*

$$\|\mathscr{H}_N|_{\mathcal{C}}\| = o(1) \quad \Longrightarrow \quad \exists w \in (0,1]\ \forall w' \in [0,w] : \ \mathscr{D}_{w'}(\mathcal{C}) \subseteq \mathcal{C} \ \wedge \ \|\mathscr{D}_{w'}|_{\mathcal{C}}\| < \infty.$$

iv) *If $\langle C, \|\cdot\|_{\mathcal{C}}\rangle$ is $\lll$-monotone*

$$\|\mathscr{H}_N|_{\mathcal{C}}\| = o(1) \quad \Longleftarrow \quad \exists w \in (0,1]\ \forall w' \in [0,w] : \ \mathscr{D}_{w'}(\mathcal{C}) \subseteq \mathcal{C} \ \wedge \ \|\mathscr{D}_{w'}|_{\mathcal{C}}\| < \infty.$$

For a cone enjoying all the properties occurring in the theorem, we have a slightly stronger assertion.

6.2.5 Proposition. *Assume that $\langle \mathcal{C}, \|\cdot\|_{\mathcal{C}}\rangle$ is a replication closed complete $\lll$-monotone and solidly normed cone in $c_0^{\downarrow}$. Then the following equivalences hold.*

i) $\|\mathscr{P}_N\| = o(N) \quad \Longleftrightarrow \quad \mathscr{C}(\mathcal{C}) \subseteq \mathcal{C} \ \wedge \ \|\mathscr{C}\| < \infty.$

ii) $\|\mathscr{H}_N\| = o(1) \quad \Longleftrightarrow \quad \mathscr{D}_0(\mathcal{C}) \subseteq \mathcal{C} \ \wedge \ \|\mathscr{D}_0\| < \infty.$

The core of the proof of Theorem 6.2.4 is to show mutual domination relations between the replication operator $\mathscr{P}$ and the Cesaro-means operator $\mathscr{C}$, and between the Hardy-operator $\mathscr{H}$ and the family $\mathscr{D}_w$. We present these relations in the form of two lemmata.

6.2.6 Lemma. *Let $a \in c_0^{\downarrow}$. Then we have*

$$\mathscr{C}a \leq \sum_{j=0}^{\infty} 2^{1-j} \cdot \mathscr{P}_{2^j} a \quad \textit{and} \quad \mathscr{P}_N a \lll \frac{N-1}{\log N} \cdot \mathscr{C}a, \quad N \geq 4,$$

where the limit in the left inequality is pointwise in $[0,\infty]^{\mathbb{N}}$.

Proof.

▷ At first we show that the left statement holds. To this end, let $n \in \mathbb{N}$ be arbitrary. Furthermore, let k be a natural number such that $2^{k-1} \le n \le 2^k$ is satisfied. Then an elementary estimation proves the asserted property:

$$\begin{aligned}\Big(\sum_{j=0}^{\infty} 2^{1-j} \cdot \mathscr{P}_{2^j} a\Big)_n &= \sum_{j=0}^{\infty} 2^{1-j} \big(\mathscr{P}_{2^j} a\big)_n \ge \sum_{j=0}^{\infty} 2^{1-j} \big(\mathscr{P}_{2^j} a\big)_{2^k} = \sum_{j=0}^{k-1} 2^{1-j} a_{2^{(k-j)}} + \sum_{j=k}^{\infty} 2^{1-j} a_1 \\ &= 2^{1-k}\Big(\sum_{j=0}^{k-1} 2^{k-j} a_{2^{(k-j)}} + 2a_1\Big) = 2^{1-k}\Big(\sum_{i=1}^{k} 2^i a_{2^i} + 2a_1\Big) \\ &\ge 2^{1-k}\Big(\sum_{i=1}^{k}\Big(\sum_{j=2^i}^{2^{i+1}-1} a_j\Big) + 2a_1\Big) = 2^{1-k}\Big(\sum_{j=2}^{2^{k+1}-1} a_j + 2a_1\Big) \\ &\ge 2^{1-k} \sum_{j=1}^{2^{k+1}-1} a_j \ge \frac{1}{2^{k-1}} \sum_{j=1}^{2^{k-1}} a_j = \big(\mathscr{C} a\big)_{2^{k-1}} \ge \big(\mathscr{C} a\big)_n.\end{aligned}$$

▷ We continue with an auxiliary notice. For each $N > 1$ and $x \in [1, N]$ we have

$$x = (1-\lambda) \cdot 1 + \lambda \cdot N \quad \text{with} \quad \lambda := \frac{x-1}{N-1} \in [0,1].$$

The function $\log x$ is concave, and it follows that

$$\log x \ge (1-\lambda) \cdot \log 1 + \lambda \cdot \log N = \frac{x-1}{N-1} \cdot \log N$$

Equivalently, we may say that

$$\forall x \in [1, N]: \quad \frac{\log N}{N-1} \cdot x \le \frac{\log N}{N-1} + \log x. \tag{6.2}$$

▷ We give another auxiliary notice. Let $(a^n)_{n\in\mathbb{N}}, (b^n)_{n\in\mathbb{N}}$ be two sequences in $c_0^{\downarrow}$ with $a^n \lll b^n$, for every $n \in \mathbb{N}$. Furthermore, assume that the limits $\lim\limits_{n\to\infty} a^n =: a$, $\lim\limits_{n\to\infty} b^n =: b$ exists w.r.t. convergence in $\|\cdot\|_\infty$. Consider

$$\sum_{j=1}^{N} a_j = \lim_{n\to\infty} \sum_{j=1}^{N} a_j^n \le \lim_{n\to\infty} \sum_{j=1}^{N} b_j^n = \sum_{j=1}^{N} b_j$$

Hence, we conclude that $a \lll b$.

▷ Note that each sequence $a \in c_0^{\downarrow}$ can be uniformly approximated by linear combinations of characteristic sequences in $c_0^{\downarrow}$, namely by

$$a = \lim_{N\to\infty} \sum_{j=1}^{N} (a_j - a_{j+1}) \cdot \mathbb{1}_{[1,j]} = \lim_{N\to\infty} (a_1 - a_{N+1}, a_2 - a_{N+1}, ..., a_N - a_{N+1}, 0, 0, \ldots).$$

By the last step, and since $\mathscr{P}, \mathscr{C}$ are continuous operators on c_0, $\lll$ is compatible with the algebraic operations, it is sufficient to show the assertion for characteristic sequences $\mathbb{1}_{[1,m]}$. To this end, we start with observing that for each $m \in \mathbb{N}$

$$\big(\mathscr{C}\mathbb{1}_{[1,m]}\big)_n = \begin{cases} 1 & \text{if } n \le m, \\ \frac{m}{n} & \text{if } m < n. \end{cases} \qquad \big(\mathscr{C}^2\mathbb{1}_{[1,m]}\big)_n = \begin{cases} 1 & \text{if } n \le m, \\ \frac{m}{n}\big(1 + \sum_{j=m+1}^{n} \frac{1}{j}\big) & \text{if } m < n. \end{cases}$$

From this it follows that

$$\frac{\log N}{N-1}\cdot\mathscr{P}_N\mathbb{1}_{[1,m]}\lll\mathscr{C}\mathbb{1}_{[1,m]}\Leftrightarrow\frac{\log N}{N-1}\cdot\mathbb{1}_{[1,mN]}\lll\mathscr{C}\mathbb{1}_{[1,m]}\overset{1.2.7\ \mathrm{xi)}}{\Leftrightarrow}\frac{\log N}{N-1}\cdot\mathscr{C}\mathbb{1}_{[1,mN]}\le\mathscr{C}^2\mathbb{1}_{[1,m]}$$

$$\Leftrightarrow\begin{cases}\frac{\log N}{N-1}\le 1 & \text{if } n\le m,\\ \frac{\log N}{N-1}\le\frac{m}{n}\big(1+\sum_{j=m+1}^{n}\frac{1}{j}\big) & \text{if } m<n\le mN,\\ \frac{\log N}{N-1}\cdot\frac{mN}{n}\le\frac{m}{n}\big(1+\sum_{j=m+1}^{n}\frac{1}{j}\big) & \text{if } mN<n.\end{cases}$$

The first line holds for all $N\ge 2$. To show the second line, use (6.2) with $x:=\frac{n}{m}$ to compute

$$\frac{\log N}{N-1}\cdot\frac{n}{m}\le\frac{\log N}{N-1}+\log\frac{m}{n}=\frac{\log N}{N-1}-\sum_{j=m+1}^{n}\frac{1}{j}+\rho_{n,m},$$

where the remainder term $\rho_{n,m}$ is subject to $-\frac{1}{2m}<\rho_{n,m}<\frac{1}{2n}$. When $N\ge 4$, we have $\frac{\log N}{N-1}<\frac{1}{2}$, and it follows that indeed

$$\frac{\log N}{N-1}\cdot\frac{n}{m}\le 1+\sum_{j=m+1}^{n}\frac{1}{j}.$$

The third line follows at once from the already established case "$n=mN$".

□

6.2.7 Lemma. *Let $a\in c_0^{\downarrow}$ and $w\in[0,1]$. Then we have*

$$\mathscr{D}_w a\le\sum_{j=0}^{\infty}2^{1+jw}\cdot\mathscr{H}_{2^j}a\quad\textit{and}\quad\mathscr{H}_N a\lll\frac{1}{\log N}\cdot\mathscr{D}_0 a,\qquad N\ge 4,$$

where the limit in the left inequality is pointwise in $[0,\infty]^{\mathbb{N}}$.

Proof.

▷ At first we show that the left statement holds. To this end, let $n\in\mathbb{N}$ and $w\in[0,1]$ be arbitrary. Then an elementary estimation proves the asserted property:

$$\begin{aligned}\big(\mathscr{D}_w a\big)_n&=\frac{a_n}{n}+\sum_{k=n+1}^{\infty}\Big(\frac{k}{n}\Big)^w\frac{a_k}{k}=\frac{a_n}{n}+\sum_{j=0}^{\infty}\frac{1}{n^w}\sum_{k=n2^j+1}^{n2^{j+1}}\frac{a_k}{k^{1-w}}\le a_n+\sum_{j=0}^{\infty}\frac{1}{n^w}\cdot\frac{1}{(n2^j)^{1-w}}\sum_{k=n2^j+1}^{n2^{j+1}}a_k\\&=a_n+\sum_{j=0}^{\infty}2^{jw}\cdot\frac{1}{n}\sum_{i=0}^{n-1}\frac{1}{2^j}\sum_{k=(n+i)2^j}^{(n+i+1)2^j-1}a_k=a_n+\sum_{j=0}^{\infty}2^{jw}\cdot\frac{1}{n}\sum_{i=1}^{n}\frac{1}{2^j}\sum_{k=(n+i-1)2^j+1}^{(n+i)2^j}a_k\\&=a_n+\sum_{j=0}^{\infty}2^{jw}\cdot\frac{1}{n}\sum_{i=1}^{n}\big(\mathscr{H}_{2^j}a\big)_{n+i}\le a_n+\sum_{j=0}^{\infty}2^{jw}\big(\mathscr{H}_{2^j}a\big)_n\le\sum_{j=0}^{\infty}2^{1+jw}\big(\mathscr{H}_{2^j}a\big)_n\\&=\Big(\sum_{j=0}^{\infty}2^{1+jw}\cdot\mathscr{H}_{2^j}a\Big)_n\end{aligned}$$

▷ Again it is enough to show that the asserted inequality holds for each sequence $\mathbb{1}_{[1,m]}$, $m\in\mathbb{N}$, equivalently, that

$$\log N\cdot\big(\mathscr{C}\circ\mathscr{H}_N\big)\le\big(\mathscr{C}\circ\mathscr{D}_0\big)\mathbb{1}_{[1,m]}.\tag{6.3}$$

We compute the involved expressions:

$$\Big(\big(\mathscr{C}\circ\mathscr{H}_N\big)\mathbb{1}_{[1,m]}\Big)_n = \big(\mathscr{C}\mathbb{1}_{[1,m]}\big)_{nN} = \begin{cases} 1 & \text{if } nN\le m,\\ \frac{m}{nN} & \text{if } m<nN.\end{cases}$$

$$\big(\mathscr{D}_0\mathbb{1}_{[1,m]}\big)_n = \sum_{j=n}^{\infty}\frac{1}{j}\cdot\big(\mathbb{1}_{[1,m]}\big)_j = \begin{cases}\sum_{j=n}^{m}\frac{1}{j} & \text{if } n\le m,\\ 0 & \text{if } m<n.\end{cases}$$

$$\begin{aligned}\Big(\big(\mathscr{C}\circ\mathscr{D}_0\big)\mathbb{1}_{[1,m]}\Big)_n &= \frac{1}{n}\sum_{j=1}^{n}\big(\mathscr{D}_0\mathbb{1}_{[1,m]}\big)_j = \frac{1}{n}\sum_{j=1}^{n\min\{m,n\}}\sum_{k=j}^{m}\frac{1}{k}\\ &= \frac{1}{n}\Big(\sum_{j=1}^{\min\{m,n\}}\frac{1}{j}\cdot j + \sum_{j=\min\{m,n\}}^{m}\frac{1}{j}\cdot\min\{m,n\}\Big) = \frac{\min\{m,n\}}{n}\big(1+\sum_{j=\min\{m,n\}}^{m}\frac{1}{j}\big)\\ &= \begin{cases}1+\sum_{j=n+1}^{m}\frac{1}{j} & \text{if } n\le m,\\ \frac{m}{n} & \text{if } m<n.\end{cases}\end{aligned}$$

It follows that

$$(6.3)\text{ holds}\Leftrightarrow\begin{cases}\log N\le 1+\sum_{j=n+1}^{m}\frac{1}{j} & \text{if } n\le\frac{m}{N},\\ \log N\cdot\frac{m}{nN}\le 1+\sum_{j=n+1}^{m}\frac{1}{j} & \text{if } \frac{m}{N}<n\le m,\\ \log N\cdot\frac{m}{nN}\le\frac{m}{n} & \text{if } m<n.\end{cases}$$

The first line holds since

$$1+\sum_{j=n+1}^{m}\frac{1}{j} = 1+\ln\Big(\frac{m}{n}\Big)+\rho_{m,n}\ge 1+\log\Big(\frac{m}{n}\Big)+\rho_{m,n}\ge\log N,$$

and $-\frac{1}{2}<\rho_{m,n}<\frac{1}{2}$. The third line is obviously true. To see the second line, we apply (6.2) with $x:=\frac{m}{n}$. This yields

$$\frac{\log N}{N}\cdot\frac{m}{n}\le\frac{\log N}{N}+\frac{N-1}{N}\log\frac{m}{n}\le\frac{1}{2}+\log\frac{m}{n}\le 1+\rho_{m,n}+\log\frac{m}{n} = 1+\sum_{j=n+1}^{m}\frac{1}{j}.$$

□

Proof (of Theorem 6.2.4).
"i)" : Assume that $\langle C,\|\cdot\|_{\mathcal{C}}\rangle$ is complete and $\|\mathscr{P}_N\| = o(N)$. Obviously the map $N\mapsto\mathscr{P}_N|_{\mathcal{C}}$ is a semigroup of cone maps. Remembering the computation rule v) of Lemma 3.1.2 we see that Corollary 6.1.6 is applicable. This gives us scalars $\beta<1$, $C>0$ such that $\|\mathscr{P}_N|_{\mathcal{C}}\|\le CN^\beta$. Now let $a\in\mathcal{C}$. Then, for each $N\in\mathbb{N}$,

$$\Big\|\sum_{j=0}^{N}2^{1-j}\mathscr{P}_{2^j}a\Big\|_{\mathcal{C}}\le\sum_{j=0}^{N}2^{1-j}\|\mathscr{P}_{2^j}a\|_{\mathcal{C}}\le\sum_{j=0}^{N}2^{1-j}C(2^j)^\beta\le\frac{2C}{1-2^{\beta-1}}\cdot\|a\|_{\mathcal{C}}.$$

Since $\mathcal{C}$ is complete, the limit $\sum_{j=0}^{\infty}2^{1-j}\mathscr{P}_{2^j}a$ exists in $\mathcal{C}$, in particular this limit exists w.r.t. pointwise convergence. Invoking Lemma 6.2.6 yields $\mathscr{C}a\le\sum_{j=0}^{\infty}2^{1-j}\mathscr{P}_{2^j}a$. Since $\mathcal{C}$ is solidly normed, we have

$$\mathscr{C}a\in\mathcal{C}\quad\text{and}\quad\|\mathscr{C}a\|_{\mathcal{C}}\le\frac{2C}{1-2^{\beta-1}}\cdot\|a\|_{\mathcal{C}}.$$

"ii)" : Assume that the Cesaro-means operator leaves $\mathcal{C}$ invariant and is bounded. Invoking Lemma 6.2.6 yields $\mathscr{P}_N a \lll \frac{N-1}{\log N} \cdot \mathscr{C} a$, for $N \geq 4$. Since $\mathcal{C}$ is replication closed and $\lll$-monotone, we obtain $\|\mathscr{P}_N\| = o(N)$.
"iii)" : Assume that $\langle C, \|\cdot\|_{\mathcal{C}}\rangle$ is complete, Hardy-monotone and $\|\mathscr{H}_N\| = o(1)$. Obviously the map $N \mapsto N \cdot \mathscr{H}_N|_{\mathcal{C}}$ is a semigroup of cone maps. Due to $\|\mathscr{H}_N\| = o(1)$ and $\mathcal{C}$ being Hardy-monotone, it follows that the semigroup of cone maps $(N \cdot \mathscr{H}_N)_{N\in\mathbb{N}}$ satisfies (6.1). Thus Corollary 6.1.6 gives us scalars $\beta < 1$, $C > 0$ such that $\|N \cdot \mathscr{H}_N|_{\mathcal{C}}\| \leq CN^{\beta}$. Now let $a \in \mathcal{C}$. Then, for each $N \in \mathbb{N}$, $w \in [0, \beta - 1)$,

$$\Big\|\sum_{j=0}^{N} 2^{1+jw} \cdot \mathscr{H}_{2^j} a\Big\|_{\mathcal{C}} \leq \sum_{j=0}^{\infty} 2^{1+jw} \cdot \|\mathscr{H}_{2^j} a\|_{\mathcal{C}} \leq \sum_{j=0}^{N} 2C(2^j)^{w+\beta-1} \leq \frac{2C}{1-2^{w+\beta-1}} \cdot \|a\|_{\mathcal{C}}.$$

Since $\mathcal{C}$ is complete, the limit $\sum_{j=0}^{\infty} 2^{1+jw} \cdot \mathscr{H}_{2^j} a$ exists in $\mathcal{C}$, in particular this limit exists w.r.t. pointwise convergence. Invoking Lemma 6.2.7 yields $\mathscr{D}_w a \leq \sum_{j=0}^{\infty} 2^{1+jw} \mathscr{H}_{2^j} a$, and since $\mathcal{C}$ is solidly normed, we find

$$\mathscr{D}_w a \in \mathcal{C} \quad \text{and} \quad \|\mathscr{D}_w a\|_{\mathcal{C}} \leq \frac{2C}{1-2^{w+\beta-1}} \cdot \|a\|_{\mathcal{C}}.$$

"iv)" : Assume that there exists $w \in (0, 1]$ such that for each $w' \in [0, w]$ the operator $\mathscr{D}_{w'}$ leaves $\mathcal{C}$ invariant and is bounded. Invoking Lemma 6.2.7 yields $\mathscr{H}_N a \lll \frac{1}{\log N} \cdot \mathscr{D}_0 a$, for $N \geq 4$. Note that $\mathscr{D}_0 a \leq \mathscr{D}_{w'} a$, in particular $\mathscr{D}_0 a \lll \mathscr{D}_{w'} a$. Since $\mathcal{C}$ is $\lll$-monotone, we obtain $\|\mathscr{H}_N\| = o(1)$. □

Proof (of Proposition 6.2.5).
"i)" : The assertion follows from Theorem 6.2.4, i) and ii).
"ii)" : The assumptions in Theorem 6.2.4, iii) and iv) are satisfied and we obtain

$$\|\mathscr{H}_{\mathscr{P}_N}\| = o(1) \ \Leftrightarrow \ (\ \exists w \in (0,1], \forall w' \in [0,w] : \quad \mathscr{D}_{w'}(\mathcal{C}) \subseteq \mathcal{C} \quad \wedge \quad \|\mathscr{D}_{w'}\| < \infty\ ). \tag{6.4}$$

As mentioned above, $\mathscr{D}_0 a \lll \mathscr{D}_{w'} a$ for all $a \in c_0^{\downarrow}$. Thus (6.4) implies the right side in ii). The other implication is obtained via Lemma 6.2.7. □

6.3 An interpolation theorem

Our aim in this section is to prove the following theorem.

6.3.1 Theorem. *Let $\langle \mathcal{C}, \|\cdot\|_{\mathcal{C}}\rangle$ be a replication closed normed cone in $c_0^{\downarrow}$. Consider the following statements.*

i) $\|\mathscr{P}_N|_{\mathcal{C}}\| = o(N)$ *and* $\|\mathscr{H}_N|_{\mathcal{C}}\| = o(1)$.

ii) *There exists $p > 1$ such that the following statements hold*

- *$\mathcal{C}$ is a subcone of $\mathbb{L}_+^p$.*
- *For every cone map $f : c_{00}^{\downarrow} \to c_0^{+}$ satisfying*

$$\exists c_1, c_2 > 0, \forall a \in \{\mathbb{1}_{[1,m]} \mid m \in \mathbb{N}\}, \forall n \in \mathbb{N} : \quad (f(a))_n \leq \min\Big\{\frac{c_1}{n} \cdot \|a\|_1, \frac{c_2}{n^{\frac{1}{p}}} \cdot \|a\|_p^{+}\Big\}, \tag{6.5}$$

it holds that the domain of the pointwise extension $\bar{f}$ contains $\mathcal{C}$ and $\bar{f}$ maps $\mathcal{C}$ boundedly into itself.

iii) *There exists $p > 1$ such that the following statements hold.*

- $\mathcal{C}$ *is a subcone of* $\mathbb{L}^p_+$.
- *Every cone map* $f : \mathcal{C} \to c_0^+$ *satisfying* (6.5)*, maps* $\mathcal{C}$ *boundedly into itself.*

The following statements hold:

▷ *If* $\mathcal{C}$ *is complete, solidly normed and Hardy-monotone, then* i) *implies* ii).

▷ ii) *always implies* iii).

▷ *If* $\mathcal{C}$ *is* $\lll$*-monotone, then* iii) *implies* i).

Observe that the statement in item iii) is a weak-type interpolation property. In fact, (6.5) means that f is $\|\cdot\|_1$-$\|\cdot\|_1^w$–bounded and $\|\cdot\|_p^+$-$\|\cdot\|_p^w$–bounded.

Having available the machinery developed so far, the proof of Theorem 6.3.1 is not anymore difficult. We use Theorem 6.2.4 and, for the implication "i) ⇒ ii)", the following elementary fact.

6.3.2 Lemma. *Let* $c_1, c_2 > 0$*,* $w \in (0,1]$*, and set* $\tilde{c}_1 := \max\{c_1, \frac{c_2}{w}\}$*. Then*

$$\forall m, n \in \mathbb{N}: \quad \min\Big\{c_1 \frac{m}{n}, c_2 \frac{1}{n^w} \sum_{j=1}^{m} j^{w-1}\Big\} \leq \big(\tilde{c}_1 \cdot \mathscr{C} \mathbb{1}_{[1,m]} + c_2 \cdot \mathscr{D}_w \mathbb{1}_{[1,m]}\big)_n.$$

Proof.

▷ The case that $m < n$ is immediate from:

$$c_1 \frac{m}{n} = c_1 \big(\mathscr{C} \mathbb{1}_{[1,m]}\big)_n \leq \tilde{c}_1 \big(\mathscr{C} \mathbb{1}_{[1,m]}\big)_n + c_2 \big(\mathscr{D}_w \mathbb{1}_{[1,m]}\big).$$

▷ Assume that $n \leq m$. Then we estimate the sum by an integral to obtain

$$\begin{aligned} c_2 \frac{1}{n^w} \sum_{j=1}^{m} j^{w-1} &= c_2 \frac{1}{n^w} \sum_{j=1}^{n-1} j^{w-1} + c_2 \big(\mathscr{D}_w \mathbb{1}_{[1,m]}\big)_n \leq c_2 \frac{1}{n^w} \int_0^{n-1} x^{w-1} dx + c_2 \big(\mathscr{D}_w \mathbb{1}_{[1,m]}\big)_n \\ &= c_2 \frac{1}{w n^w} (n-1)^w + c_2 \big(\mathscr{D}_w \mathbb{1}_{[1,m]}\big)_n \leq \tilde{c}_1 + c_2 \big(\mathscr{D}_w \mathbb{1}_{[1,m]}\big)_n \\ &= \tilde{c}_1 \big(\mathscr{C} \mathbb{1}_{[1,m]}\big)_n + c_2 \big(\mathscr{D}_w \mathbb{1}_{[1,m]}\big)_n. \end{aligned}$$

□

Proof (of Theorem 6.3.1).
"i) ⇒ ii)" : Assume that i) holds under the assumption that $\mathcal{C}$ is complete, solid and Hardy-monotone. Then Theorem 6.2.4 tells us that $\mathscr{C}$ maps $\mathcal{C}$ boundedly into itself, and that $\mathscr{D}_{\frac{1}{p}}$ maps $\mathcal{C}$ boundedly into itself for all sufficiently large p. For such p, in particular $\mathcal{C} \subseteq \mathbb{L}^p_+$. Let f be a cone map as in ii). By Lemma 6.3.2 the assumption (6.5) implies that

$$\forall a \in \big\{\mathbb{1}_{[1,m]} \,\big|\, m \in \mathbb{N}\big\}: \quad f(a) \leq \tilde{c}_1 \cdot \mathscr{C} a + c_2 \cdot \mathscr{D}_{\frac{1}{p}} a$$

Both sides of the inequality are cone maps, and hence it holds for all $a \in c_{00}^{\downarrow}$. Passing to the pointwise extension yields

$$\forall a \in c_0^{\downarrow}: \quad \bar{f}(a) \leq \tilde{c}_1 \cdot \mathscr{C} a + c_2 \cdot \mathscr{D}_{\frac{1}{p}} a \tag{6.6}$$

From (6.6) it is obvious that $\bar{f}$ maps $\mathcal{C}$ boundedly into itself.
"ii) $\Rightarrow$ iii)": Assume that ii) holds, and let f be a cone map as in iii). Set $g := f\big|_{c_{00}^{\downarrow}}$. Applying ii) with g yields that $\bar{g}$ maps $\mathcal{C}$ boundedly into itself. We show that $f = \bar{g}\big|_{\mathcal{C}}$. Let $a \in \mathcal{C}$, and decompose

$$a = \underbrace{\sum_{j=1}^{N}(a_j - a_{j+1}) \cdot \mathbb{1}_{[1,j]}}_{=:a^N} + \underbrace{\sum_{j=N+1}^{\infty}(a_j - a_{j+1}) \cdot \mathbb{1}_{[1,j]}}_{=:b^N}$$

Since $b^N \leq a$, and $a^N \in c_{00}^{\downarrow}$, we have $a^N, b^N \in \mathcal{C} \subseteq \mathbb{L}_+^p$ and

$$f(a) = g(a^N) + f(b^N). \tag{6.7}$$

Also we have $b^N \overset{N\to\infty}{\longrightarrow} 0$ uniformly, and $b^N \geq b^{N+1} \geq 0$, for each N. By the monotone convergence theorem, it follows that $\|b^N\|_p^+ \overset{N\to\infty}{\longrightarrow} 0$, and now (6.5) implies

$$\forall n \in \mathbb{N} : \lim_{N\to\infty} (f(b^N))_n = 0.$$

"iii) $\Rightarrow$ i)" : Finally we assume that iii) holds under the assumption that $\mathcal{C}$ is $\lll$-monotone. We check that $\mathscr{C}$ and $\mathscr{D}_{\frac{1}{p}}$ (for every $p > 1$) satisfy (6.5) for every sequence $a \in c_0^{\downarrow}$.

$$\begin{aligned}
\big(\mathscr{C}a\big)_n &= \frac{1}{n}\sum_{j=1}^{n} a_j \leq \frac{1}{n} \cdot \|a\|_1, \\
&= \frac{1}{n}\sum_{j=1}^{n} a_j \leq \frac{1}{n}\sum_{j=1}^{n}\Big(\frac{j}{n}\Big)^{\frac{1}{p}-1} a_j = \frac{1}{n^{\frac{1}{p}}}\sum_{j=1}^{n} j^{\frac{1}{p}-1}a_j \leq \frac{1}{n^{\frac{1}{p}}} \cdot \|a\|_p^+, \\
\big(\mathscr{D}_{\frac{1}{p}}a\big)_n &= \sum_{j=n}^{\infty}\Big(\frac{j}{n}\Big)^{\frac{1}{p}}\frac{a_j}{j} = \frac{1}{n}\sum_{j=n}^{\infty}\Big(\frac{j}{n}\Big)^{\frac{1}{p}-1} a_j \leq \frac{1}{n} \cdot \|a\|_1, \\
&= \sum_{j=n}^{\infty}\Big(\frac{j}{n}\Big)^{\frac{1}{p}}\frac{a_j}{j} = \frac{1}{n^{\frac{1}{p}}}\sum_{j=n}^{\infty} j^{\frac{1}{p}-1}a_j \leq \frac{1}{n^{\frac{1}{p}}} \cdot \|a\|_p^+
\end{aligned}$$

Hence, we may apply iii) with $\mathscr{C}\big|_{\mathcal{C}}$ and, since $\mathcal{C} \subseteq \mathbb{L}_+^p$, also with $\mathscr{D}_{\frac{1}{p}}\big|_{\mathcal{C}}$. By means of Theorem 6.2.4 the statement i) follows. □

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