Diplomarbeit

Finding Hamiltonians Satisfying Initial Values Arising from the Riemann Zeta-Function

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Introduction

The general topic of this thesis is the theory of canonical systems. These are certain two-dimensional linear ordinary differential equations depending on a complex parameter. One of the many questions concerning canonical systems is the so-called inverse problem: Given initial values depending on the complex parameter, does there exist a canonical system with a solution (depending on the complex parameter) that satisfies those initial values?

On the other hand, we consider the Riemann hypothesis claiming that every nontrivial zero of the Riemann zeta-function has real part $1/2$.

As [11] shows, the Riemann hypothesis holds true if the inverse problem can be solved for certain initial values arising from the Riemann zeta-function and if the solution satisfies a certain boundary condition. Omitting this boundary condition and slightly restricting the domain of the complex parameter, [9] proves that the inverse problem arising from the Riemann zeta-function can be solved. Elaborating the proof (including all the necessary preliminary work) of this result is the main purpose of this thesis.

To this end, Chapter 1 gives a short overview of the basic results and concepts concerning canonical systems. Chapter 2 proves the (original) Fredholm alternative and introduces a generalization of the classical Fredholm theory. This generalization is the base of Chapter 3 which culminates in the proof of the central Theorem 3.1.9. Lastly, Chapter 4 sketches out the connection between Theorem 3.1.9 and the Grand Riemann hypothesis.

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Notation

Here, we itemize all possibly unclear symbols and notational conventions used in this thesis.

- $\mathbb{N}$ stands for the positive integers, i.e. $\mathbb{N} := \{1, 2, 3, \ldots\}$.
- In expressions like $\varepsilon > 0$ or $(a, b]$, we omit $\varepsilon, a, b \in \mathbb{R}$ and implicitly define ε, a, b as real numbers.
- "a.e." means almost everywhere with respect to the Lebesgue measure $\lambda_n, n \in \mathbb{N}$.
- By default, "measurable" means measurable with respect to the Borel sets.
- A subset N of $\mathbb{R}^n, n \in \mathbb{N}$ is called a "null set" if $\lambda_n(N) = 0$. Here, λ_n is the outer Lebesgue measure defined on $\mathcal{P}(\mathbb{R}^n)$.
- For $n \in \mathbb{N}$, $\mathbb{C}^n$ and $\mathbb{R}^n$ are endowed with the Euclidean dot product, norm and distance.
- The disjoint union of two sets is denoted by $\uplus$.
- Let $(\Omega_1, \mathcal{A}_1, \mu_1)$ and $(\Omega_2, \mathcal{A}_2, \mu_2)$ be σ-finite measure spaces. We use $\mathcal{A}_1 \otimes \mathcal{A}_2$ as a symbol for the product space, i.e. the smallest σ-algebra containing the semiring
$$\{A_1 \times A_2 : A_1 \in \mathcal{A}_1, A_2 \in \mathcal{A}_2\}.$$
$\mu_1 \otimes \mu_2$ stands for the unique measure on $(\Omega_1 \times \Omega_2, \mathcal{A}_1 \otimes \mathcal{A}_2)$ with $\mu_1 \otimes \mu_2(A_1 \times A_2) = \mu_1(A_1)\mu_2(A_2)$ for arbitrary $A_1 \in \mathcal{A}_1, A_2 \in \mathcal{A}_2$.
- $L^1_{\text{loc}}((t_0, t_1], \mathbb{R}^{2\times 2})$ stands for functions from $(t_0, t_1]$ to $\mathbb{R}^{2\times 2}$ that are L^1_{loc} with respect to the Borel sets and the Lebesgue measure in each entry.
- $AC_{\text{loc}}((t_0, t_1], \mathbb{C}^2)$ stands for functions from $(t_0, t_1]$ to $\mathbb{C}^2$ that are locally absolutely continuous in each entry.
- $C(\mathbb{R})$ stands for $C(\mathbb{R}, \mathbb{C})$, i.e. $\mathbb{C}$ is the default codomain. The same applies to similar terms like $L^1_{\text{loc}}(\mathbb{R})$.
- For a square matrix M, $M \geq 0$ means that M is symmetric and positive definite.
- M^t denotes the transpose of a matrix M.
- With the exception of Section 2.1, integrals are always Lebesgue integrals although they are generally written like Riemann integrals.
- Integrals of vector-valued functions are computed entry by entry.
- $\mathcal{F}$ stands for the complex Fourier transform. That means we define
$$(\mathcal{F}f)(z) := \int_{-\infty}^{\infty} f(t) \exp(-izt) \,\mathrm{d}t$$
for a suitable function $f : \mathbb{R} \to \mathbb{C}$ and suitable $z \in \mathbb{C}$.

Contents

Chapter 1

Canonical Systems

The bedrock of this thesis is the theory of canonical systems. Thus, we open with a general introduction to canonical systems, based on [1, Chapter 1], covering the main definitions and a short discussion on the most important basic results. Everything in this chapter is proved.

1.1 Definition and Solvability

First, we define canonical systems, which are special parameter-dependent ordinary differential equations.

Definition 1.1.1. A matrix-valued function $H \in L^1_{\text{loc}}((t_0, t_1], \mathbb{R}^{2\times 2})$ for $-\infty \le t_0 < t_1 < \infty$ is called a *Hamiltonian* if

$$H(t) \ge 0 \text{ and } H(t) \ne 0 \text{ a.e.}$$

Then, the *canonical system* associated with H is defined as

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} A(t,z) \\ B(t,z) \end{pmatrix} = zJH(t)\begin{pmatrix} A(t,z) \\ B(t,z) \end{pmatrix} \tag{1.1.1}$$

where

$$J := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

and $z \in \mathbb{C}$. $**$

In contrast to the classical theory of ordinary differential equations, we cannot expect existence of continuously differentiable soltutions to (1.1.1) because the Hamiltonian need not be smooth. Hence, we have to resort to a weaker defintion.

Definition 1.1.2. In the setting of Definition 1.1.1, a function

$$\begin{pmatrix} A(.,z) \\ B(.,z) \end{pmatrix} \in AC_{\text{loc}}((t_0, t_1], \mathbb{C}^2) \text{ for each fixed } z \in \mathbb{C}$$

is called a *solution* if (1.1.1) holds a.e. for each fixed $z \in \mathbb{C}$. $**$

The following remark is crucial in the proofs of Lemma 1.1.4 and Theorem 1.1.5.

Remark 1.1.3. Recall that all norms on $\mathbb{C}^d, d \in \mathbb{N}$ are equivalent and that therefore an integrable function $f : [a,b] \to \mathbb{C}^d$ satisfies

$$\Big\| \int_a^b f(s)\,\mathrm{d}s \Big\|_N \le \int_a^b \|f(s)\|_N \,\mathrm{d}s < \infty$$

where $\|.\|_N$ is any norm on $\mathbb{C}^d$. In the setting of Definition 1.1.1, we have

$$\Big\| \int_a^b H(s)\,\mathrm{d}s \Big\| \leq \int_a^b \|H(s)\|\,\mathrm{d}s < \infty$$

and

$$\Big\| \int_a^b H(s)f(s)\,\mathrm{d}s \Big\|_2 \leq \int_a^b \|H(s)\|\,\|f(s)\|_2\,\mathrm{d}s < \infty$$

for $[a,b] \subseteq (t_0,t_1]$ and a measurable and bounded function $f : [a,b] \to \mathbb{C}^2$. //

Under these circumstances, we can solve (1.1.1) for each $z \in \mathbb{C}$ and a constant initial value:

Lemma 1.1.4. *In the setting of Definition 1.1.1, fix $c \in (t_0,t_1]$ and a $v \in \mathbb{C}^2$. Then, the canonical system (1.1.1) has a unique solution*

$$\begin{pmatrix} A(t,z) \\ B(t,z) \end{pmatrix} : (t_0,t_1] \times \mathbb{C} \to \mathbb{C}^2$$

with

$$\begin{pmatrix} A(c,.) \\ B(c,.) \end{pmatrix} \equiv v.$$

Proof. This proof is a slight modification of the common proof of the Picard-Lindelöf theorem using the Banach fixed point theorem, cf. e.g. [2, Chapter 2] or [3, Satz 2.1].

First, fix $z \in \mathbb{C}$.

We only prove the theorem for an interval $[a,t_1]$ with an arbitrary $a \in (t_0,c]$. Then, we can extend the absolutely continuous solutions to a unique locally absolutely continuous solution on

$$\bigcup_{t_0<a\leq c} [a,t_1] = (t_0,t_1]$$

the following way: For a $t \in (t_0,t_1]$ choose an $a \in (t_0,t)$ and take the value at t of the unique solution on $[a,t_1]$. By uniqueness of the solutions on the intervals of the form $[a,t_1]$, this is well-defined and yields the unique solution on $(t_0,t_1]$.[1]

Since uniform convergence preserves measurability, the space $BM([a,t_1],\mathbb{C}^2)$ of measurable and bounded functions from $[a,t_1]$ to $\mathbb{C}^2$ endowed with the uniform norm is a closed subspace of the space of bounded functions from $[a,t_1]$ to $\mathbb{C}^2$ and thus a Banach space.

We easily see that

$$\|f\|_H := \sup_{t\in[a,t_1]} \|f(t)\|_2 \exp\Big(-|z| \overbrace{\Big| \int_c^t \|JH(s)\|\,\mathrm{d}s \Big|}^{L(t):=} \Big)$$

defines an equivalent norm on $BM([a,t_1],\mathbb{C}^2)$. The function

$$K : \begin{cases} BM([a,t_1],\mathbb{C}^2) & \to BM([a,t_1],\mathbb{C}^2) \\ f & \mapsto (t \mapsto v + \int_c^t zJH(s)f(s)\,\mathrm{d}s) \end{cases}$$

[1]Using a standard $1/n, n \in \mathbb{N}$ argument, we only need to choose a countably many times. Since the countable union of null sets is a null set, this shows that the constructed solution indeed meets (1.1.1) a.e.

even maps to $AC([a,t_1],\mathbb{C}^2)$ because, by definition, the integrand is an L^1-function in each entry. For $f,g \in BM([a,t_1],\mathbb{C}^2)$ and a $t \in [a,t_1]$ with $t > c$ we get

$$\begin{aligned}
&\|K(f)(t) - K(g)(t)\|_2 \exp(-|z|L(t)) \leq \\
&\exp(-|z|L(t)) \int_c^t |z|\|JH(s)\|\|f(s)-g(s)\|_2 \overbrace{\exp(-|z|L(s))\exp(|z|L(s))}^{=1} \mathrm{d}s \leq \\
&\exp(-|z|L(t)) \int_c^t \underbrace{|z|\|JH(s)\|\exp(|z|L(s))}_{=\frac{\mathrm{d}}{\mathrm{d}s}\exp(|z|L(s))} \mathrm{d}s \|f-g\|_H \leq \\
&\exp(-|z|L(t))(\exp(|z|L(t))-1)\|f-g\|_H = \\
&\underbrace{(1-\exp(-|z|L(t)))}_{\in[0,1)}\|f-g\|_H.
\end{aligned}$$

The case $c > t$ works similarly. Calculating the supremum over $[a,t_1]$ on both sides yields

$$\|K(f)-K(g)\|_H \leq \underbrace{\Big(1-\exp\Big(-|z|\int_a^{t_1}\|JH(s)\|\,\mathrm{d}s\Big)\Big)}_{\in[0,1)}\|f-g\|_H.$$

Hence, the Banach fixed point theorem says that there exists a unique $f \in BM([a,t_1],\mathbb{C}^2)$ with $K(f) = f$. In particular, we have $f(c) = v$ and $f \in AC([a,t_1],\mathbb{C}^2)$. Therefore, f is a.e. differentiable, and differentiating $K(f) = f$ yields (1.1.1) on $[a,t_1]$. A solution of (1.1.1) meeting the initial condition lies in $BM([a,t_1],\mathbb{C}^2)$ and integrating (1.1.1) shows that it has to be a fixed point of K. This proves uniqueness. ■

We can even show that these unique solutions are entire with respect to z (cf. [1, Theorem 1.1.]).

Theorem 1.1.5. *In the setting of Lemma 1.1.4, the unique solution is continuous, and for arbitrarily fixed $t \in (t_0,t_1]$ the component functions $A(t,.)$ and $B(t,.)$ are entire.*

Proof. This proof is a refined version of the proof of Lemma 1.1.4 (cf. [3, Lemma 2.2, Proposition 2.3]). This means we once again only consider an interval $[a,t_1]$ for an $a \in (t_0,c]$.
We first claim that

$$\|f\|_H := \sup_{z\in\mathbb{C}} \sup_{t\in[a,t_1]} \|f(t,z)\|_2 \exp(-k|z|L(t))$$

for any $k > 1$ defines a complete norm on the space X of all functions $f : [a,t_1]\times\mathbb{C} \to \mathbb{C}^2$ that are continuous, entire in z for every fixed $t \in [a,t_1]$ and meet $\|f\|_H < \infty$. The set of bounded functions from $[a,t_1]\times\mathbb{C}$ to $\mathbb{C}^2$ endowed with the uniform norm is a Banach space. The map

$$f(t,z) \mapsto f(t,z)\exp(-k|z|L(t))$$

is an isometric isomorphism proving that the space of all functions $f : [a,t_1]\times\mathbb{C} \to \mathbb{C}^2$ with $\|f\|_H < \infty$ is a Banach space endowed with the norm $\|.\|_H$. X obviously is a subspace. For a sequence $(f_n)_{n\in\mathbb{N}}$ in X converging to a bounded f, $\|f\|_H < \infty$ with respect to $\|.\|_H$, we consider $[a,t_1]\times K$ for a compact set $K \subseteq \mathbb{C}$. Since convergence with respect to $\|.\|_H$ implies uniform convergence both on $[a,t_1]\times K$ as well as on K for every fixed $t \in [a,t_1]$, we see that f is continuous on $[a,t_1]\times K$ and analytic in z on K for every fixed $t \in [a,t_1]$. The arbitrary choice of $K \subseteq \mathbb{C}$ shows $f \in X$, i.e. $(X,\|.\|_H)$ is a Banach space.

Next, we claim that the function

$$K : \begin{cases} X & \to X \\ f & \mapsto ((t,z) \mapsto v + \int_c^t zJH(s)f(s,z)\,\mathrm{d}s) \end{cases}$$

indeed maps to X.

Fix $f \in X$. For $(t,z_1),(r,z_2) \in [a,t_1] \times \mathbb{C}$, we have

$$\|K(f)(t,z_1) - K(f)(r,z_2)\|_2 \leq \|K(f)(t,z_1) - K(f)(r,z_1)\|_2 + \|K(f)(r,z_1) - K(f)(r,z_2)\|_2 = \Big\| \int_{[r,t]\cup[t,r]} z_1 JH(s)f(s,z_1)\,\mathrm{d}s \Big\|_2 + \Big\| \int_c^r JH(s)(z_1 f(s,z_1) - z_2 f(s,z_2))\,\mathrm{d}s \Big\|_2 \leq \int_a^{t_1} \mathbf{1}_{[r,t]\cup[t,r]}(s)\|JH(s)\|\|z_1 f(s,z_1)\|_2\,\mathrm{d}s + \int_a^{t_1} \|JH(s)\|\|(z_1 f(s,z_1) - z_2 f(s,z_2))\|_2\,\mathrm{d}s. \quad (1.1.2)$$

Fixing (t,z_1) and sending (r,z_2) to (t,z_1), we see that the integrand of the first integral in (1.1.2) tends to 0 pointwise and is dominated by an integrable function because $\|JH(s)\|$ is integrable on $[a,t_1]$ and $f(s,z_1)$ is continuous in s on $[a,t_1]$. Similarly, $z_2 f(s,z_2)$ tends to $z_1 f(s,z_1)$ for every $s \in [a,t_1]$ because $f(s,z)$ is continuous in z on G. The integrand of the first integral in (1.1.2) is dominated by an integrable function because $\|JH(s)\|$ is integrable on $[a,t_1]$ and $\|(z_1 f(s,z_1) - z_2 f(s,z_2))\|_2$ is bounded above by a constant for $s \in [a,t_1]$ and $|z_1 - z_2| < 1$:

$$\|(z_1 f(s,z_1) - z_2 f(s,z_2))\|_2 \leq \|z_1 f(s,z_1)\|_2 + \|(z_1 - z_2)f(s,z_2)\|_2 + \|z_1 f(s,z_2)\|_2 \leq (2|z_1|+1) \sup_{s\in[a,t_1],|z|\leq|z_1|+1} \|f(s,z)\|_2.$$

The dominated convergence theorem yields that (1.1.2) tends to 0 for $(r,z_2) \to (t,z_1)$. This means $K(f)$ is continuous.

For a fixed $s \in [a,t_1]$, the intergrand $zJH(s)f(s,z)$ is analytic and for a fixed $z \in \mathbb{C}$ it is integrable on $[a,t_1]$. Moreover,

$$\|zJH(s)f(s,z)\|_2 \leq \max_{(t,v)\in[a,t_1]\times K} |v|\|A(s)\|\|f(t,v)\|_2$$

is a dominating integrable function for every compact set $K \subseteq \mathbb{C}$. Thus, $T(f)$ is entire in z for every fixed $t \in [a,t_1]$ by Lemma A.1.3.[2]

Assuming $t \in [a,t_1]$ with $t > c$ (the case $c > t$ works similarly) and $z \in \mathbb{C}$, we have

$$\|K(f)(t,z)\|_2 \exp(-k|z|L(t)) \leq \exp(-k|z|L(t))\Big(\|v\|_2 + \int_c^t |z|\|JH(s)\|\|f(s,z)\|_2 \overbrace{\exp(-k|z|L(s))\exp(k|z|L(s))}^{=1}\,\mathrm{d}s\Big) = \exp(-k|z|L(t))\Big(\|v\|_2 + \int_c^t \underbrace{|z|\|JH(s)\|\exp(k|z|L(s))}_{=(1/k)\frac{\mathrm{d}}{\mathrm{d}s}\exp(k|z|L(s))} \underbrace{\|f(s,z)\|_2 \exp(-k|z|L(s))}_{\leq\|f\|_H}\,\mathrm{d}s\Big) \leq \exp(-k|z|L(t))(\|v\|_2 + (1/k)(\exp(k|z|L(t)) - 1)\|f\|_H) = \overbrace{\exp(-k|z|L(t))}^{\in(0,1]}\|v\|_2 + (1/k)\overbrace{(1 - \exp(-k|z|L(t)))}^{\in[0,1)}\|f\|_H. \quad (1.1.3)$$

[2]Recall that a function from an open subset G of $\mathbb{C}$ to $\mathbb{C}^n$ is analytic iff its component functions are analytic.

Applying the supremum over $[a, t_1] \times \mathbb{C}$, we get $\|K(f)\|_H \leq \|v\|_2 + (1/k)\|f\|_H < \infty$ and, all in all, $K(f) \in X$.

For $f, g \in X$, starting (1.1.3) with $\|K(f)(t, z) - K(g)(t, z)\|_2 \exp(-k|z|L(t))$ shows

$$\|T(f) - T(g)\|_H \leq \underbrace{(1/k)}_{<1}\|f - g\|_H.$$

Therefore, the Banach fixed point theorem guarantees the existence of a unique $f \in X$ with $K(f) = f$. In particular, we have $f(c, z) = v$ and $f(., z) \in AC([a, t_1], \mathbb{C}^2)$ for all $z \in \mathbb{C}$. Hence, $f(., z)$ is a.e. differentiable, and differentiating $K(f)(., z) = f(., z)$ yields (1.1.1) on $[a, t_1]$. This means that f is the unique solution from Lemma 1.1.4. ■

1.2 Singular Points and Transformations

In this section, we outline why we prove that the Hamiltonian has determinant 1 and that all points are regular in the main result of this thesis Theorem 3.1.9. This section is based on Sections 2 and 3 of [1, Chapter 1].

Definition 1.2.1. In the setting of Definition 1.1.1, a point $t \in (t_0, t_1]$ is called *singular* if there exists a $v_t \neq 0, v_t \in \mathbb{R}^2$ with $H(s)v_t = 0$ for almost all s locally around t. A point $t \in (t_0, t_1]$ that is not singular is called *regular*. **

Singular points are of particular interest as they are, at least in a certain way, a trivial case, where the canonical system turns into a difference equation. For singular points, the solution not only is analytic in z but also linearly depends on the parameter:

Remark 1.2.2. In the setting of Definition 1.1.1, let $t \in (t_0, t_1]$ be a singular point. Because of $H(s) \geq 0$ on $(t_0, t_1]$, we can diagonalize this matrix, and the symmetry implies that we can always choose two orthogonal eigenvectors. Therefore, we can find an $\alpha \in [0, \pi)$ such that

$$e_\alpha := \begin{pmatrix} \cos(\alpha) \\ \sin(\alpha) \end{pmatrix}$$

is a constant second eigenvector to v_t locally around t. Hence, we find a real-valued function h defined locally around t mapping to the second eigenvalue. This means the Hamiltonian locally has the form $H(s) = h(s)P_\alpha$ where

$$P_\alpha := \begin{pmatrix} \cos^2(\alpha) & \cos(\alpha)\sin(\alpha) \\ \cos(\alpha)\sin(\alpha) & \sin^2(\alpha) \end{pmatrix}$$

is the projection corresponding to the linear space spanned by e_α. Since H is positive semidefinite, nonzero a.e. and L^1_{loc}, the scalar function h is nonnegative, nonzero a.e. and integrable. We now use the following ansatz: $h(s)P_\alpha$ and $h(s')P_\alpha$ commute for any s, s' locally around t. Thus, if the Hamiltonian were smooth, theory of ordinary differential equations would yield the solution

$$\exp\Big(\int_t^s zh(r)JP_\alpha \,\mathrm{d}r\Big)C = \exp\Big(z\int_t^s h(r)\,\mathrm{d}r JP_\alpha\Big)C \tag{1.2.1}$$

where $C \in \mathbb{C}^2$. Looking at the definition of the power series of the matrix exponential, we have to deal with the matrices $(JP_\alpha)^n, n \in \mathbb{N}$. As J rotates by a quarter circle and the orthogonal space to the space spanned by e_α is the space spanned by v, we see that $P_\alpha JP_\alpha = 0$ holds. Hence, the power series terminates after the linear term, i.e. (1.2.1) turns into

$$\Big(I + z\int_t^s h(r)\,\mathrm{d}r JP_\alpha\Big)C. \tag{1.2.2}$$

It can be easily confirmed that (1.2.2) indeed solves the canonical system locally around t. Setting

$$H_s := \int_t^s H(r)\,\mathrm{d}r,$$

(1.2.2) is a step in the difference equation

$$u_s - u_t = zJH_s u_t$$

for $u_t := C$. //

Definition 1.2.3. In the setting of Definition 1.1.1, a function $\varphi : (t_0, t_1] \to (t_2, t_3]$ with $-\infty \leq t_2 < t_3 < \infty$ is called a *change of variable* if

- φ is bijective.
- φ and φ^{-1} are locally absolutely continuous.

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Remark 1.2.4. In the setting of Definition 1.1.1, let $p \in L^1_{\mathrm{loc}}((t_0, t_1], \mathbb{R})$, $p \geq 0, p \neq 0$ a.e. and $c \in (t_0, t_1]$.

$$\vartheta_p : \begin{cases} (t_0, t_1] & \to \mathbb{R} \\ t & \mapsto \int_c^t p(s)\,\mathrm{d}s \end{cases}$$

is then locally absolutely continuous and strictly increasing, i.e. in particular injective. Apparently, the range is an interval $(t_2, t_3]$ with $-\infty \leq t_2 < t_3 < \infty$. //

With some work, we also get local absolute continuity of the inverse of a change of variable by a well-known result. Thus, ϑ_p is a change of variable.

Lemma 1.2.5. *An absolutely continuous function $f : [a, b] \to \mathbb{R}$ with $f' \geq 0, f' \neq 0$ a.e. has an absolutely continuous inverse.*

Proof. An immediate consequence of the assumptions is that f is strictly increasing and bijectively maps onto the interval $[f(a), f(b)]$. Hence, the inverse $f^{-1} : [f(a), f(b)] \to [a, b]$ is strictly increasing and continuous too. In particular, f^{-1} is of bounded variation. Let A be a subset of $[a, b]$ such that f' exists on A and is strictly positive and $B := [a, b] \setminus A$ is a null set. Our assumptions guarantee existence of such an A. By a standard argument, we have differentiability of f^{-1} on $f(A)$ with a strictly positive derivative. Obviously, $[f(a), f(b)] = f(A) \uplus f(B)$ holds and $f(B)$ is a null set by Lemma A.3.2. Therefore, we have found a partition of $[f(a), f(b)]$ into two Lebesgue-measurable sets, of which one is a null set. Any null set $N \subseteq [f(a), f(b)]$ can now be uniquely decomposed into two Lebesgue-measurable sets $N_A \subseteq f(A)$ and $N_B \subseteq f(B)$. Since N_B is mapped to a subset of B, the set $f^{-1}(N_B)$ is a null set. Assuming that $f^{-1}(N_{A,n})$ is a null set for every $n \in \mathbb{N}$ and $N_{A,n} := \{x \in N_A : |(f^{-1})'(x)| \leq n\}$, we get

$$\lambda(f^{-1}(N_A)) = \lambda\Big(\bigcup_{n=1}^{\infty} f^{-1}(N_{A,n})\Big) \leq \sum_{n=1}^{\infty} \lambda(f^{-1}(N_{A,n})) = 0.$$

Thus, f^{-1} has the Luzin N property and Lemma A.3.2 proves that f^{-1} is absolutely continuous. It remains to show that $f^{-1}(N_{A,n})$ is a null set for every $n \in \mathbb{N}$. For this purpose, we adapt the proof of [4, Chapter VII. Lemma 6.3]. Fix any $n \in \mathbb{N}$ and any $\varepsilon > 0$. Let $D_m, m \in \mathbb{N}$ denote the points x of $N_{A,n}$ such that

$$|f^{-1}(x) - f^{-1}(y)| \leq (n + \varepsilon)|x - y|$$

holds for $y \in N_{A,n}$ whenever $|x-y| \le 1/m$. Because of the differentiability of f^{-1} on $N_{A,n}$ with bound n, it is evident that $(D_m)_{m\in\mathbb{N}}$ is an ascending sequence with $N_{A,n} = \bigcup_{m\in\mathbb{N}} D_m$. By definition of the outer Lebesgue measure we find a sequence $(I_k^m)_{k\in\mathbb{N}}$ of intervals covering D_m such that

$$\sum_{k=1}^{\infty} \lambda(I_k^m) \le \lambda(D_m) + \varepsilon \tag{1.2.3}$$

holds and every interval is shorter than $1/m$ for each $m \in \mathbb{N}$. The definition of D_m yields

$$\lambda(f^{-1}(D_m \cap I_k^m)) \le (n+\varepsilon)\lambda(I_k^m)$$

for $m, k \in \mathbb{N}$. In combination with (1.2.3), this shows

$$\lambda(f^{-1}(D_m)) \le \sum_{k=1}^{\infty} \lambda(f^{-1}(D_m \cap I_k^m)) \le (n+\varepsilon) \sum_{k=1}^{\infty} \lambda(I_k^m) \le (n+\varepsilon)(\lambda(D_m)+\varepsilon)$$

for $m \in \mathbb{N}$. As outer measures are continuous from below, first $m \to \infty$ and then $\varepsilon \to 0$ yield

$$\lambda(f^{-1}(N_{A,n})) \le n\lambda(N_{A,n}) \le n\lambda(N) = 0.$$

■

The requirements f has to meet in Lemma 1.2.5 are rather specific. Still, there is no obvious generalization. Recall that every continuous and injective function defined on an interval is either strictly increasing or strictly decreasing. As [5, Chapter IX. Exercise 13] in combination with [6] proves, there exist examples of strictly increasing, absolutely continuous functions with (strictly increasing, continuous) inverse functions that are not absolutely continuous.

Remark 1.2.6. Continuing the considerations of Remark 1.2.4 and using Lemma 1.2.5, we can motivate the name "change of variable". By direct calculation, we can easily show that $u \in AC_{\text{loc}}((t_0,t_1],\mathbb{C}^2)$ solves (1.1.1) iff $u \circ \vartheta_p^{-1}$ solves

$$\frac{\mathrm{d}}{\mathrm{d}t}(u \circ \vartheta_p^{-1})(t) = zJ\frac{1}{p(\vartheta_p^{-1}(t))}H(\vartheta_p^{-1}(t))(u \circ \vartheta_p^{-1})(t)$$

on $(t_2,t_3]$. In particular, ϑ_p identifies the spaces $AC_{\text{loc}}((t_0,t_1],\mathbb{C}^2)$ and $L^1_{\text{loc}}((t_0,t_1],\mathbb{C}^{2\times 2})$ with the spaces $AC_{\text{loc}}((t_2,t_3],\mathbb{C}^2)$ and $L^1_{\text{loc}}((t_2,t_3],\mathbb{C}^{2\times 2})$ respectively.

Since we have $\det H(t) \ge 0$ and $\det H \in L^1_{\text{loc}}((t_0,t_1],\mathbb{R})$, a Hamiltonian with $\det H(t) \ne 0$ a.e. admits the choice $p(t) := \det H(t)$. This transforms the canonical system in such a way that $\det H(t) = 1$ a.e. holds. //

Looking at Hamiltonians with $\det H(t) \ne 0$ a.e., we can thus, without loss of generality, even restrict to Hamiltonians with determinant 1 a.e.

Chapter 2

Fredholm Theory

One of the most important tools used in this thesis is Fredholm theory of integral operators with continuous integral kernels. Hence, we first introduce the classical theory and then adapt it to our purposes. Since this chapter is a fundamental part of this thesis, everything is proved.

2.1 Classical Fredholm Theory

This section is based on [7, Section 24.1] and [8, Chapter V], which follow Fredholm's original approach.

Definition 2.1.1. We are considering the integral equation ($a < b$)

$$x(s) = y(s) + \lambda \int_a^b K(s,t)x(t)\,\mathrm{d}t \tag{2.1.1}$$

for $s \in [a,b]$. $K \not\equiv 0$ and y are given and continuous on $[a,b] \times [a,b]$ and $[a,b]$ respectively, and λ is an element of $\mathbb{C}$. A *solution* of (2.1.1) is a continuous function $x : [a,b] \to \mathbb{C}$ such that (2.1.1) holds for the given K, y and λ. $**$

Now, the main idea is to solve finite systems of linear equations arising from (2.1.1) and to hope that the solutions converge to a solution of (2.1.1).

Remark 2.1.2. For $n \in \mathbb{N}$, define a subdivision of $[a,b]$ into n intervals of length $\delta_n := (b-a)/n$, i.e. $a < s_1 < \cdots < s_n = b$. Write $x_i := x(s_i)$, $y_i := y(s_i)$ and $K_{ij} := K(s_i, s_j)$ for $i, j \in \{1, \ldots, n\}$ (we will not evaluate at a). (2.1.1) turns into

$$x_i = y_i + \lambda \int_a^b K(s_i, t)x(t)\,\mathrm{d}t \text{ for } i = 1, \ldots, n.$$

Approximation of the integral yields

$$x_i = y_i + \lambda\delta_n \sum_{j=1}^{n} K_{ij}x_j \text{ for } i = 1, \ldots, n.$$

Defining $\boldsymbol{x} := (x_1, \ldots, x_n)^t$ and $\boldsymbol{y}, \boldsymbol{K}$ in an analogous manner, we get the equation

$$\begin{aligned} \boldsymbol{x} = \boldsymbol{y} + \lambda\delta_n \boldsymbol{K}\boldsymbol{x} \Leftrightarrow \\ (I - \lambda\delta_n \boldsymbol{K})\boldsymbol{x} = \boldsymbol{y}. \end{aligned}$$

This equation has a unique solution for every $\boldsymbol{y}$ if and only if

$$\begin{aligned} d_n(\lambda) &:= \det(I - \lambda\delta_n \boldsymbol{K}) \\ &= \begin{vmatrix} 1-\lambda\delta_n K_{11} & -\lambda\delta_n K_{12} & \cdots & -\lambda\delta_n K_{1n} \\ -\lambda\delta_n K_{21} & 1-\lambda\delta_n K_{22} & \cdots & -\lambda\delta_n K_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ -\lambda\delta_n K_{n1} & -\lambda\delta_n K_{n2} & \cdots & 1-\lambda\delta_n K_{nn} \end{vmatrix} \\ &\neq 0. \end{aligned}$$

If this is the case, Cramer's rule states

$$\boldsymbol{x} = \frac{1}{d_n(\lambda)} \operatorname{cof}(I - \lambda\delta_n \boldsymbol{K})^t \boldsymbol{y},$$

where cof stands for the cofactor matrix. For $i \in \{1, \ldots, n\}$, this means

$$x_i = \frac{1}{d_n(\lambda)} \Big(C_{ii} y_i + \delta_n \sum_{\substack{1 \le j \le n \\ j \neq i}} \frac{C_{ij}}{\delta_n} y_j \Big) \tag{2.1.2}$$

where C_{ij} is the (i,j)-entry of $\operatorname{cof}(I - \lambda\delta_n \boldsymbol{K})^t$. Rearranging the Leibniz formula for determinants, we get

$$\begin{aligned} d_n(\lambda) = 1 - \lambda \sum_{i=1}^n \delta_n K_{ii} &+ \lambda^2 \sum_{1 \le i < j \le n} \delta_n^2 \begin{vmatrix} K_{ii} & K_{ij} \\ K_{ji} & K_{jj} \end{vmatrix} \\ &- \lambda^3 \sum_{1 \le i < j < k \le n} \delta_n^3 \begin{vmatrix} K_{ii} & K_{ij} & K_{ik} \\ K_{ji} & K_{jj} & K_{jk} \\ K_{ki} & K_{kj} & K_{kk} \end{vmatrix} + \cdots + (-1)^n \lambda^n \delta_n^n \det \boldsymbol{K} \end{aligned}$$

or

$$\begin{aligned} d_n(\lambda) = 1 - \lambda \sum_{i=1}^n \delta_n K_{ii} &+ \frac{\lambda^2}{2!} \sum_{i,j=1}^n \delta_n^2 \begin{vmatrix} K_{ii} & K_{ij} \\ K_{ji} & K_{jj} \end{vmatrix} \\ &- \frac{\lambda^3}{3!} \sum_{i,j,k=1}^n \delta_n^3 \begin{vmatrix} K_{ii} & K_{ij} & K_{ik} \\ K_{ji} & K_{jj} & K_{jk} \\ K_{ki} & K_{kj} & K_{kk} \end{vmatrix} + \cdots + (-1)^n \lambda^n \delta_n^n \det \boldsymbol{K}. \end{aligned} \tag{2.1.3}$$

In a similar manner, we get

$$\frac{C_{ij}}{\delta_n} = \lambda \Big(K_{ij} - \lambda \sum_{k=1}^n \delta_n \begin{vmatrix} K_{ij} & K_{ik} \\ K_{kj} & K_{kk} \end{vmatrix} - \frac{\lambda^2}{2!} \sum_{k,\ell=1}^n \delta_n^2 \begin{vmatrix} K_{ij} & K_{ik} & K_{i\ell} \\ K_{kj} & K_{kk} & K_{k\ell} \\ K_{\ell j} & K_{\ell k} & K_{\ell\ell} \end{vmatrix} - \ldots \Big), \tag{2.1.4}$$

when $i \neq j$. In the case $i = j$, we delete the row and column of the same number in $I - \lambda\delta_n \boldsymbol{K}$ before calculating the determinant. This does not change the structure of $I - \lambda\delta_n \boldsymbol{K}$, so we get an expression structurally similar to (2.1.3). //

Formally sending n to ∞, we can expect that (2.1.3) and (2.1.4) turn into series of integral expressions and that i, j in (2.1.4) become two free variables. Furthermore, $C_{ii}/d_n(\lambda)$ in (2.1.2) converges to 1 and $\delta_n \sum$ turns into an integral over j. This motivates the following

Definition 2.1.3. Considering the integral equation (2.1.1), we define

$$K\begin{pmatrix} u_1, \dots, u_n \\ v_1, \dots, v_n \end{pmatrix} := \begin{vmatrix} K(u_1, v_1) & K(u_1, v_2) & \cdots & K(u_1, v_n) \\ K(u_2, v_1) & K(u_2, v_2) & \cdots & K(u_2, v_n) \\ \vdots & \vdots & \ddots & \vdots \\ K(u_n, v_1) & K(u_n, v_2) & \cdots & K(u_n, v_n) \end{vmatrix} \tag{2.1.5}$$

for $n \in \mathbb{N}$ and $a \le u_i, v_i \le b$ for $1 \le i \le n$. Then, the *Fredholm determinant* is defined (at least formally) as the series

$$d(\lambda) := \sum_{n=0}^{\infty} d_n \lambda^n \text{ for } \lambda \in \mathbb{C}, \tag{2.1.6}$$

where $d_0 := 1$ and

$$d_n := \frac{(-1)^n}{n!} \int_a^b \dots \int_a^b K\begin{pmatrix} u_1, \dots, u_n \\ u_1, \dots, u_n \end{pmatrix} \mathrm{d}u_1 \cdots \mathrm{d}u_n \tag{2.1.7}$$

for $n \in \mathbb{N}$. The *first Fredholm minor* is defined (at least formally) on $[a, b] \times [a, b]$ as the series

$$D_\lambda(s, t) := \sum_{n=0}^{\infty} D_n(s, t) \lambda^n \text{ for } \lambda \in \mathbb{C}, \tag{2.1.8}$$

where $D_0(s, t) := K(s, t)$ and

$$D_n(s, t) := \frac{(-1)^n}{n!} \int_a^b \dots \int_a^b K\begin{pmatrix} s, u_1, \dots, u_n \\ t, u_1, \dots, u_n \end{pmatrix} \mathrm{d}u_1 \cdots \mathrm{d}u_n \tag{2.1.9}$$

for $n \in \mathbb{N}$. **

Remark 2.1.4. Since the determinant is a polynomial of the matrix entries (cf. Leibniz formula), (2.1.5) is continuous on $[a, b]^{2n}$. Therefore, both (2.1.7) and (2.1.9) exist for all $n \in \mathbb{N}$ and $(s, t) \in [a, b] \times [a, b]$. Moreover, merging the integrals in (2.1.9) to one integral over $[a, b]^n$ by the virtue of Fubini's theorem and applying Lemma A.1.1 with respect to the metric space $[a, b] \times [a, b]$ yield the continuity of (2.1.9) on $[a, b] \times [a, b]$ for all $n \in \mathbb{N}$. //

The most important tool that we need to actually prove convergence is:

Hadamard's inequality. *For any matrix $A \in \mathbb{C}^{n \times n}, n \in \mathbb{N}$, we have*

$$|\det A|^2 \le \prod_{i=1}^{n} \sum_{j=1}^{n} |A_{ij}|^2.$$

In particular, $|A_{ij}| \le M \in \mathbb{R}$ for $i, j = 1, \dots, n$ implies

$$|\det A|^2 \le n^n M^{2n}. \tag{2.1.10}$$

Proof. We follow the proof of [8, Theorem 5.2].

Define

$$c_i^2 := \sum_{j=1}^{n} |A_{ij}|^2 \in \mathbb{R} \tag{2.1.11}$$

with $c_i \ge 0$ for $i = 1, \dots, n$. Fix these $c_1, \dots, c_n$ and let A vary such that (2.1.11) holds. Then, A ranges over a compact subset of $\mathbb{C}^{n^2}$. Thus, $|\det A|$ attains its maximum for a V with (2.1.11). By Laplace's expansion, we have

$$\det V = \sum_{j=1}^{n} V_{ij} \tilde{V}_{ij}$$

for $i = 1, \dots, n$, where $\tilde{V}_{ij}$ is the (i,j)-cofactor of V. The Cauchy-Schwarz inequality and (2.1.11) yield

$$|\det V|^2 \le \sum_{j=1}^{n} |V_{ij}|^2 \sum_{j=1}^{n} |\tilde{V}_{ij}|^2 = c_i^2 \sum_{j=1}^{n} |\tilde{V}_{ij}|^2. \tag{2.1.12}$$

for $i = 1, \dots, n$ with equality if and only if $(\overline{V_{i1}}, \dots, \overline{V_{in}})^t$ and $(\tilde{V}_{i1}, \dots, \tilde{V}_{in})^t$ are linearly dependent. Manipulating $(V_{i1}, \dots, V_{in})^t$ does not affect $(\tilde{V}_{i1}, \dots, \tilde{V}_{in})^t$ by the definition of cofactors. If we make $(\overline{V_{i1}}, \dots, \overline{V_{in}})^t$ and $(\tilde{V}_{i1}, \dots, \tilde{V}_{in})^t$ linearly dependent without violating (2.1.11) for V, (2.1.12) becomes an equality without changing the value of the right side. Hence, maximality of $|\det V|$ demands that $(\overline{V_{i1}}, \dots, \overline{V_{in}})^t$ and $(\tilde{V}_{i1}, \dots, \tilde{V}_{in})^t$ are linearly dependent for $i = 1, \dots, n$. This and the fact that the determinant of a matrix with two linearly dependent rows vanishes show by the virtue of Laplace's expansion that

$$\sum_{j=1}^{n} V_{kj} \tilde{V}_{ij} = 0 = \sum_{j=1}^{n} V_{kj} \overline{V_{ij}} \tag{2.1.13}$$

for $k \neq i$. Combining (2.1.12) and (2.1.13) and using the maximality of $|\det V|$, we compute

$$\begin{aligned} |\det A|^2 \le |\det V|^2 &= \det V \overline{\det V} = \det(VV^*) \\ &= \begin{vmatrix} c_1^2 & 0 & \cdots & 0 \\ 0 & c_2^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \cdots & c_n^2 \end{vmatrix} \\ &= c_1^2 c_2^2 \dots c_n^2 \\ &= \prod_{i=1}^{n} \sum_{j=1}^{n} |A_{ij}|^2. \end{aligned}$$

As an obvious consequence, we get (2.1.10) if $|A|$ is uniformly bounded by M. ■

Lemma 2.1.5. *Considering the integral equation (2.1.1), the Fredholm determinant $d(\lambda)$ and the first Fredholm minor $D_\lambda(s,t)$ are entire functions for arbitrary $(s,t) \in [a,b] \times [a,b]$. In addition, the convergence of (2.1.8) is uniformly absolute in (s,t,λ) as an element of $\mathbb{R}^4$ if λ is restricted to a compact subset of $\mathbb{C}$, i.e. (2.1.8) is continuous on $[a,b] \times [a,b] \times \mathbb{C}$. In particular, we have uniform convergence in (s,t) for a fixed $\lambda \in \mathbb{C}$, and $D_\lambda(s,t)$ is continuous on $[a,b] \times [a,b]$ for every $\lambda \in \mathbb{C}$.*

Proof. The proof is based on [8, Theorem 5.3.1].
With $M := \|K\|_\infty > 0$, Hadamard's inequality shows

$$\left| K \begin{pmatrix} u_1, \dots, u_n \\ v_1, \dots, v_n \end{pmatrix} \right| \le n^{\frac{n}{2}} M^n.$$

Applying this inequality to (2.1.7), we get

$$|d_n| \le (b-a)^n \frac{n^{\frac{n}{2}} M^n}{n!} =: c_n \tag{2.1.14}$$

for $n \in \mathbb{N}$. Because of

$$\lim_{n\to\infty} \frac{c_n}{c_{n+1}} = \lim_{n\to\infty} \frac{(n+1) n^{\frac{n}{2}}}{(b-a)M(n+1)^{\frac{n+1}{2}}} = \lim_{n\to\infty} \frac{\sqrt{(n+1)}}{\underbrace{(b-a)M(1+\frac{1}{n})^{\frac{n}{2}}}_{\to (b-a)M\exp(1/2)}} = \infty,$$

the power series with coefficients $(c_n)_{n\in\mathbb{N}}$ has radius of convergence ∞. Hence, (2.1.14) proves that (2.1.6) is entire. Similarly, (2.1.9) turns into

$$|D_{n-1}(s,t)| \leq (b-a)^{n-1}\frac{n^{\frac{n}{2}}M^n}{(n-1)!} \tag{2.1.15}$$

for $(s,t) \in [a,b]\times[a,b]$ and $n \in \mathbb{N}, n \geq 2$ and (2.1.8) is entire. If we confine λ to a compact subset of $\mathbb{C}$, (2.1.15) yields absolute convergence of (2.1.8) with respect to the uniform norm on $\mathbb{R}^4$. Thus, continuity follows since the partial sums are continuous (Remark 2.1.4). ■

Remark 2.1.6. (cf. [8, Theorem 5.5.1]) For $s = t \in [a,b]$, integrating the definition (2.1.9) of $D_{n-1}(s,t)$ for $n \in \mathbb{N}$ with respect to s and applying Fubini's theorem yields

$$\begin{aligned}\int_a^b D_{n-1}(s,s)\,\mathrm{d}s &= \frac{(-1)^{n-1}}{(n-1)!}\int_a^b\int_a^b\dots\int_a^b K\begin{pmatrix}s,u_1,\dots,u_n\\ s,u_1,\dots,u_n\end{pmatrix}\mathrm{d}u_1\cdots\mathrm{d}u_n\,\mathrm{d}s\\ &= \frac{(-1)^{n-1}}{(n-1)!}\int_a^b\dots\int_a^b\int_a^b K\begin{pmatrix}s,u_1,\dots,u_n\\ s,u_1,\dots,u_n\end{pmatrix}\mathrm{d}s\,\mathrm{d}u_1\cdots\mathrm{d}u_n\\ &= -nd_n.\end{aligned}$$

Thus, we get (Lemma 2.1.5 justifies the use of the dominated convergence theorem to interchange integral and series.)

$$d'(\lambda) = \sum_{n=1}^{\infty} nd_n\lambda^{n-1} = -\sum_{n=0}^{\infty}\lambda^n\int_a^b D_n(s,s)\,\mathrm{d}s = -\int_a^b D_\lambda(s,s)\,\mathrm{d}s \tag{2.1.16}$$

for $\lambda \in \mathbb{C}$. //

Returning to Remark 2.1.2 anfd Definition 2.1.3, we are now capable of completely solving (2.1.1) as a result of the following famous theorem (cf. [8, Theorems 5.4.1, 5.4.2, 5.6.1] or [7, pages 264-268]).

Fredholm alternative. *If $d(\lambda) \neq 0$ holds, the integral equation (2.1.1) has the unique continuous solution*

$$x(s) = y(s) + \frac{\lambda}{d(\lambda)}\int_a^b D_\lambda(s,t)y(t)\,\mathrm{d}t \tag{2.1.17}$$

for $s \in [a,b]$. For a zero $\mu \in \mathbb{C}$ of the Fredholm determinant, the equation

$$x(s) = \mu\int_a^b K(s,t)x(t)\,\mathrm{d}t \tag{2.1.18}$$

for $s \in [a,b]$ has a continuous solution $x \not\equiv 0$.

Proof. We begin with the case $d(\lambda) \neq 0$. Continuity of (2.1.17) is an immediate consequence of Lemma A.1.1 as y is continuous by definition and D_λ by Lemma 2.1.5. Next, we prove

$$D_n(s,t) = d_nK(s,t) + \int_a^b D_{n-1}(s,u)K(u,t)\,\mathrm{d}u \tag{2.1.19}$$

for $(s,t) \in [a,b] \times [a,b]$ and $n \in \mathbb{N}$. For this purpose, we apply Laplace's expansion to the first column in (2.1.9):

$$\begin{aligned}
D_n(s,t) &= \frac{(-1)^n}{n!} \int_a^b \cdots \int_a^b \begin{vmatrix} K(s,t) & K(s,u_1) & \cdots & K(s,u_n) \\ K(u_1,t) & K(u_1,u_1) & \cdots & K(u_1,u_n) \\ \vdots & \vdots & \ddots & \vdots \\ K(u_n,t) & K(u_n,u_1) & \cdots & K(u_n,u_n) \end{vmatrix} \mathrm{d}u_1 \cdots \mathrm{d}u_n \\
&= d_n K(s,t) + \frac{(-1)^n}{n!} \int_a^b \cdots \int_a^b \begin{vmatrix} 0 & K(s,u_1) & \cdots & K(s,u_n) \\ K(u_1,t) & K(u_1,u_1) & \cdots & K(u_1,u_n) \\ \vdots & \vdots & \ddots & \vdots \\ K(u_n,t) & K(u_n,u_1) & \cdots & K(u_n,u_n) \end{vmatrix} \mathrm{d}u_1 \cdots \mathrm{d}u_n \\
&= d_n K(s,t) \\
&\quad + \underbrace{\frac{(-1)^n}{n!} \int_a^b \cdots \int_a^b \sum_{j=1}^n (-1)^j K(u_j,t) K\begin{pmatrix} s, u_1, \ldots, u_{j-1}, u_{j+1}, \ldots, u_n \\ u_1, \ldots, u_n \end{pmatrix} \mathrm{d}u_1 \cdots \mathrm{d}u_n}_{Q_n(s,t):=} .
\end{aligned}$$

Renaming $u_j, u_{j+1}, \ldots, u_n$ with $u, u_j, \ldots, u_{n-1}$ in the j-th summand of $Q_n(s,t)$, swapping $j-1$ adjacent columns such that the column with u is the first one and applying Fubini's theorem (cf. Remark 2.1.4), we get

$$\begin{aligned}
Q_n(s,t) &= \frac{(-1)^{n-1}}{n!} \sum_{j=1}^n \int_a^b \int_a^b \cdots \int_a^b K(u,t) K\begin{pmatrix} s, u_1, \ldots, u_{n-1} \\ u, u_1, \ldots, u_{n-1} \end{pmatrix} \mathrm{d}u_1 \cdots \mathrm{d}u_{n-1}\, \mathrm{d}u \\
&= \int_a^b \overbrace{\frac{(-1)^{n-1}}{(n-1)!} \int_a^b \cdots \int_a^b K\begin{pmatrix} s, u_1, \ldots, u_{n-1} \\ u, u_1, \ldots, u_{n-1} \end{pmatrix} \mathrm{d}u_1 \cdots \mathrm{d}u_{n-1}}^{=D_{n-1}(s,u)} K(u,t)\, \mathrm{d}u \\
&= \int_a^b D_{n-1}(s,u) K(u,t)\, \mathrm{d}u.
\end{aligned}$$

Hence, we have (2.1.19). For fixed $(s,t) \in [a,b] \times [a,b]$, we multiply (2.1.19) by λ^n and sum over n. Lemma 2.1.5 guarantees convergence and justifies the use of the dominated convergence theorem to interchange integral and series. All in all, (2.1.19) turns into

$$D_\lambda(s,t) = d(\lambda) K(s,t) + \lambda \int_a^b D_\lambda(s,u) K(u,t)\, \mathrm{d}u. \tag{2.1.20}$$

Repeating the whole proof up to now with application of Laplace's expansion to the first row in (2.1.9), we also have

$$D_\lambda(s,t) = d(\lambda) K(s,t) + \lambda \int_a^b K(s,u) D_\lambda(u,t)\, \mathrm{d}u. \tag{2.1.21}$$

In the case $d(\lambda) \neq 0$, we define the continuous function $H_\lambda(s,t) := D_\lambda(s,t)/d(\lambda)$ on $[a,b] \times [a,b]$, and (2.1.20) and (2.1.21) now read

$$H_\lambda(s,t) - K(s,t) = \lambda \int_a^b H_\lambda(s,u) K(u,t)\, \mathrm{d}u = \lambda \int_a^b K(s,u) H_\lambda(u,t)\, \mathrm{d}u \tag{2.1.22}$$

for $(s,t) \in [a,b] \times [a,b]$. Using (2.1.17) and (2.1.22) and Fubini's theorem (all involved functions are continuous), we get

$$\begin{aligned} y(s) + \lambda \int_a^b K(s,t)x(t)\,\mathrm{d}t &= y(s) + \lambda \int_a^b K(s,t)\Big(y(t) + \frac{\lambda}{d(\lambda)} \int_a^b D_\lambda(t,r)y(r)\,\mathrm{d}r\Big)\,\mathrm{d}t \\ &= y(s) + \lambda \int_a^b K(s,t)y(t)\,\mathrm{d}t + \lambda^2 \int_a^b \int_a^b K(s,t)H_\lambda(t,r)y(r)\,\mathrm{d}r\,\mathrm{d}t \\ &= y(s) + \lambda \int_a^b K(s,t)y(t)\,\mathrm{d}t + \lambda \int_a^b \lambda \int_a^b K(s,t)H_\lambda(t,r)\,\mathrm{d}t\ y(r)\,\mathrm{d}r \\ &= y(s) + \lambda \int_a^b K(s,t)y(t)\,\mathrm{d}t + \lambda \int_a^b (H_\lambda(s,r) - K(s,r))y(r)\,\mathrm{d}r \\ &= y(s) + \lambda \int_a^b H_\lambda(s,r)y(r)\,\mathrm{d}r \\ &= y(s) + \frac{\lambda}{d(\lambda)} \int_a^b D_\lambda(s,r)y(r)\,\mathrm{d}r = x(s) \end{aligned}$$

for $s \in [a,b]$. Hence, (2.1.17) defines a solution of (2.1.1). Conversely, a solution x of (2.1.1) satisfies

$$y(s) = x(s) - \lambda \int_a^b K(s,t)x(t)\,\mathrm{d}t$$

for $s \in [a,b]$. Using this and (2.1.22) and Fubini's theorem (all involved functions are continuous), we get

$$\begin{aligned} y(s) + \frac{\lambda}{d(\lambda)} \int_a^b D_\lambda(s,t)y(t)\,\mathrm{d}t &= x(s) - \lambda \int_a^b K(s,t)x(t)\,\mathrm{d}t + \lambda \int_a^b H_\lambda(s,t)y(t)\,\mathrm{d}t \\ &= x(s) - \lambda \int_a^b K(s,t)x(t)\,\mathrm{d}t + \lambda \int_a^b H_\lambda(s,t)x(t)\,\mathrm{d}t \\ &\quad - \lambda \int_a^b H_\lambda(s,t)\lambda \int_a^b K(t,r)x(r)\,\mathrm{d}r\,\mathrm{d}t \\ &= x(s) - \lambda \int_a^b K(s,t)x(t)\,\mathrm{d}t + \lambda \int_a^b H_\lambda(s,t)x(t)\,\mathrm{d}t \\ &\quad - \lambda \int_a^b \lambda \int_a^b H_\lambda(s,t)K(t,r)\,\mathrm{d}t\ x(r)\,\mathrm{d}r \\ &= x(s) - \lambda \int_a^b K(s,t)x(t)\,\mathrm{d}t + \lambda \int_a^b H_\lambda(s,t)x(t)\,\mathrm{d}t \\ &\quad - \lambda \int_a^b (H_\lambda(s,r) - K(s,r))x(r)\,\mathrm{d}r \\ &= x(s) \end{aligned}$$

for $s \in [a,b]$. Thus, (2.1.17) is the unique solution of (2.1.1).

Now, we assume $d(\mu) = 0$. If there exists a $t \in [a,b]$ with $D_\mu(.,t) \not\equiv 0$, Lemma 2.1.5 shows that $x := D_\mu(.,t)$ is a continuous, nontrivial function on $[a,b]$. Since (2.1.21) then turns into (2.1.18), we have found the desired solution. In the case $D_\mu \equiv 0$, we have to recall that there exists a $(s_0,t_0) \in [a,b] \times [a,b]$ with $K(s_0,t_0) \neq 0$ by definition. $D_\lambda(s_0,t_0)$ is analytic in λ by Lemma 2.1.5. Hence, $D_0(s_0,t_0) = K(s_0,t_0) \neq 0$ implies that μ is a zero of finite multiplicity of $D_\lambda(s_0,t_0)$. Thus, we can find $(s_1,t_1) \in [a,b] \times [a,b]$ such that the multiplicity $m \in \mathbb{N}$ of μ as a zero of $D_\lambda(s_1, t_1)$ is minimal. Expanding $D_\lambda(s,t)$ at μ for every $(s,t) \in [a,b] \times [a,b]$, we get

$$D_\lambda(s,t) = \sum_{n=m}^{\infty} C_n(s,t)(\lambda - \mu)^n, \tag{2.1.23}$$

where we compute the coefficients with Cauchy's integral formula ($R > 0$ arbitrary)

$$C_n(s,t) = \frac{1}{2\pi i} \oint_{\partial U_R(\mu)} \frac{D_\lambda(s,t)}{(\lambda - \mu)^{n+1}} \, \mathrm{d}\lambda \text{ for } n \in \mathbb{N}, n \geq m, \tag{2.1.24}$$

and $C_m \not\equiv 0$ holds. Choosing a parametrization (e.g. $\varphi \mapsto R\exp(i\varphi)$), Lemma 2.1.5 yields continuity of the integrand in (s,t,φ). Therefore, Lemma A.1.1 shows that $C_n(s,t)$ is continuous on $[a,b] \times [a,b]$ for $n \in \mathbb{N}, n \geq m$. Furthermore, the continuity of $D_\lambda(s,t)$ in (s,t,λ) gives rise to a constant M such that (2.1.24) has the upper bound

$$|C_n(s,t)| \leq \frac{M}{R^n} \text{ for } n \in \mathbb{N}, n \geq m.$$

This means that (2.1.23) converges uniformly absolutely for $(s,t,\lambda) \in [a,b] \times [a,b] \times \overline{U_S(\mu)}$, where $0 < S < R$. Because $R > 0$ was chosen arbitrarily, we have uniform convergence of (2.1.23) if λ is confined to a compact subset of $\mathbb{C}$, and the continuity of the partial sums makes (2.1.23) continuous. As a consequence, the dominated convergence theorem justifies interchanging integral and series when we substitute (2.1.23) in (2.1.16) for a fixed $\lambda \in \mathbb{C}$:

$$d'(\lambda) = -\int_a^b \sum_{n=m}^{\infty} C_n(s,s)(\lambda - \mu)^n \, \mathrm{d}s = -\sum_{n=m}^{\infty} (\lambda - \mu)^n \int_a^b C_n(s,s) \, \mathrm{d}s. \tag{2.1.25}$$

Hence, μ is a zero of $d'(\lambda)$ with multiplicity at least m and a zero of $d(\lambda)$ with multiplicity at least $m+1$. Substituting (2.1.23) in (2.1.21), we once again use the dominated convergence theorem to interchange integral and series, and we get an equality of analytic functions expanded at μ:

$$\begin{aligned} \sum_{n=m}^{\infty} C_n(s,t)(\lambda - \mu)^n = d(\lambda)K(s,t) &+ \mu \sum_{n=m}^{\infty} (\lambda - \mu)^n \int_a^b K(s,u)C_n(u,t) \, \mathrm{d}u \\ &+ \sum_{n=m}^{\infty} (\lambda - \mu)^{n+1} \int_a^b K(s,u)C_n(u,t) \, \mathrm{d}u \end{aligned}$$

for $(s,t) \in [a,b] \times [a,b]$. Equating the m-th coefficient and setting $t = t_1$ yield

$$C_m(s,t_1) = \mu \int_a^b K(s,u)C_m(u,t_1) \, \mathrm{d}u$$

for $s \in [a,b]$, and by choice of m, we have $C_m(s,t_1) \not\equiv 0$. Recalling that this function is continuous in s on $[a,b]$ concludes the proof. ■

Remark 2.1.7. In the proof of the Fredholm alternative, we showed (2.1.19) by applying Laplace's expansion to the first column in (2.1.9). Applying it to the first row yields

$$D_n(s,t) = d_n K(s,t) + \int_a^b K(s,u)D_{n-1}(u,t) \, \mathrm{d}u \tag{2.1.26}$$

for $(s,t) \in [a,b] \times [a,b]$ and $n \in \mathbb{N}$. //

2.2 Generalized Fredholm Determinants

This section follows the parts of [9, Section 1] and [10, Section 2] concerning the generalization of the Fredholm determinant very loosely. Unlike the sources, we define and examine the generalized Fredholm determinant and minor assuming as few conditions as possible.

Definition 2.2.1. Let $\mathcal{G}$ stand for the set of all functions K that meet the following conditions:

$(G1)$ $K \in C(\mathbb{R}, \mathbb{R})$

$(G2)$ $K(x) = 0$ for $x \geq 0$.

$**$

We are now able to generalize Section 2.1 in the sense that we use the integral kernels $\mathcal{G}$ and integrate from t to ∞. This gives rise to the new variable t, which turns out to be the main reason why we study Fredholm theory. In contrast to [9] and [10], we do not use the restriction $t \leq 0$, and we do not restrict (x, y) to $(t, \infty) \times (t, \infty)$.

Definition 2.2.2. Let $K \in \mathcal{G}$, $\lambda \in \mathbb{C}$ and $t \in \mathbb{R}$. We define

$$K\begin{pmatrix} x_1, \dots, x_n \\ y_1, \dots, y_n \end{pmatrix} := \begin{vmatrix} K(x_1 + y_1) & K(x_1 + y_2) & \cdots & K(x_1 + y_n) \\ K(x_2 + y_1) & K(x_2 + y_2) & \cdots & K(x_2 + y_n) \\ \vdots & \vdots & \ddots & \vdots \\ K(x_n + y_1) & K(x_n + y_2) & \cdots & K(x_n + y_n) \end{vmatrix} \tag{2.2.1}$$

on $\mathbb{R}^{2n}$ for $n \in \mathbb{N}$. Then, the *generalized Fredholm determinant* is defined (at least formally) as the series

$$d_K(\lambda, t) := \sum_{n=0}^{\infty} d_n(t)\lambda^n \text{ for } \lambda \in \mathbb{C}, \tag{2.2.2}$$

where $d_0 :\equiv 1$ and

$$d_n(t) := \frac{(-1)^n}{n!} \int_t^{\infty} \dots \int_t^{\infty} K\begin{pmatrix} x_1, \dots, x_n \\ x_1, \dots, x_n \end{pmatrix} \mathrm{d}x_1 \cdots \mathrm{d}x_n \tag{2.2.3}$$

for $n \in \mathbb{N}$. The *generalized first Fredholm minor* is defined (at least formally) on $\mathbb{R}^2$ as the series

$$D_K(x, y, \lambda, t) := \sum_{n=0}^{\infty} D_n(x, y, t)\lambda^n \text{ for } \lambda \in \mathbb{C}, \tag{2.2.4}$$

where $D_0(x, y, t) := K(x + y)$ and

$$D_n(x, y, t) := \frac{(-1)^n}{n!} \int_t^{\infty} \dots \int_t^{\infty} K\begin{pmatrix} x, x_1, \dots, x_n \\ y, x_1, \dots, x_n \end{pmatrix} \mathrm{d}x_1 \cdots \mathrm{d}x_n \tag{2.2.5}$$

for $n \in \mathbb{N}$. $**$

Most of the concepts and theorems of Section 2.1 still hold in a generalized manner for a $K \in \mathcal{G}$. We use the rest of this section to prove them.

Lemma 2.2.3. *For a $K \in \mathcal{G}$, (2.2.2) to (2.2.5) exist for $x, y, t \in \mathbb{R}$ and $\lambda \in \mathbb{C}$. For $x, y \geq t \geq 0$, $d_K(\lambda, t) = 1$ and $D_K(x, y, \lambda, t) = K(x+y) = 0$ hold. $D_K(x, y, \lambda, t)$ and (2.2.5) are 0 for $x, y \geq t$ if either $x \geq -t$ or $y \geq -t$ holds.*

Furthermore, d_K and D_K are continuous on $\mathbb{C} \times \mathbb{R}$ and $\mathbb{R}^2 \times \mathbb{C} \times \mathbb{R}$ respectively, and they are both entire in λ. More precisely, (2.2.2) converges uniformly absolutely in (λ, t) as an element of $\mathbb{R}^3$ on every closed cube and (2.2.4) converges uniformly absolutely in (x, y, λ, t) as an element of $\mathbb{R}^5$ on every closed cube.

Proof. Considering (2.2.3) and (2.2.5), we see that $x_1, \dots, x_n$ are bounded below by t. Hence, $x_j \geq -t$ for a $j \in \{1, \dots, n\}$ and condition $(G2)$ imply that (2.2.1) is the determinant of a matrix with one row and one column of zeros and thus vanishes. This means that we can replace the ∞ boundaries in (2.2.3) and (2.2.5) with $-t$ and the other way around if $x, y \geq t$ holds. Since K is a continuous kernel by condition $(G1)$, the whole Section 2.1 holds for a fixed $t \in \mathbb{R}$ if we confine x and y to $[t, -t]$. In particular, we get existence of (2.2.2), (2.2.3) and (2.2.5). For the confined variables, we also have existence of (2.2.4). Moreover, both (2.2.3) and (2.2.5) vanish for $t \geq 0$ and every $n \in \mathbb{N}$. Therefore, we have $d_K(\lambda, t) = 1$ and $D_K(x, y, \lambda, t) = K(x + y) = 0$ by condition $(G2)$ for $x, y \geq t \geq 0$ and $\lambda \in \mathbb{C}$.

By condition $(G2)$, both $K(x + y)$ and (2.2.5) are 0 for $n \in \mathbb{N}$ if $x, y \geq t$ and either $x \geq -t$ or $y \geq -t$ hold. As a consequence, $D_K(x, y, \lambda, t)$ is 0 for $x, y \geq t$ if either $x \geq -t$ or $y \geq -t$ holds. This shows the existence of (2.2.4) for $x, y \geq t$.

Fix any $n \in \mathbb{N}$. Since (2.2.1) is continuous on $\mathbb{R}^{2n}$ and vanishes if any argument is greater than $-t$, Fubini's theorem can be used to merge the integrals of (2.2.3) to one integral over $[t, -t]^n$. Now, continuity on $\mathbb{R}$ follows since the integrand is bounded on every cube and the measure of $[t, -t]^n \setminus [s, -s]^n$ for $s \leq t$ and $[s, -s]^n \setminus [t, -t]^n$ for $s \geq t$ tends to 0 for $s \to t$.

For (2.2.5), we want to show continuity at $(x, y, t) \in \mathbb{R}^3$. Therefore, we consider $(x_1, y_1, s) \in \mathbb{R}^3$ and a $C < 0$ with $C < x, y, t$. Then, we can replace ∞ in (2.2.5) with $-C$. In the case $t \geq s$, we have

$$\begin{aligned}
|D_n(x,y,t) - D_n(x_1,y_1,s)| &\leq \Big| \int_{[t,-C]^n} K\begin{pmatrix} x,\dots \\ y,\dots \end{pmatrix} - K\begin{pmatrix} x_1,\dots \\ y_1,\dots \end{pmatrix} \mathrm{d}x_1 \dots x_n \Big| \\
&\quad + \Big| \int_{[s,-C]^n \setminus [t,-C]^n} K\begin{pmatrix} x_1,\dots \\ y_1,\dots \end{pmatrix} \mathrm{d}x_1 \dots x_n \Big| \\
&\leq \int_{[t,-C]^n} \Big| K\begin{pmatrix} x,\dots \\ y,\dots \end{pmatrix} - K\begin{pmatrix} x_1,\dots \\ y_1,\dots \end{pmatrix} \Big| \mathrm{d}x_1 \dots x_n \\
&\quad + \int_{[s,-C]^n \setminus [t,-C]^n} \Big| K\begin{pmatrix} x_1,\dots \\ y_1,\dots \end{pmatrix} \Big| \mathrm{d}x_1 \dots x_n,
\end{aligned}$$

and in the case $t \leq s$ we have

$$\begin{aligned}
|D_n(x,y,t) - D_n(x_1,y_1,s)| &\leq \Big| \int_{[s,-C]^n} K\begin{pmatrix} x,\dots \\ y,\dots \end{pmatrix} - K\begin{pmatrix} x_1,\dots \\ y_1,\dots \end{pmatrix} \mathrm{d}x_1 \dots x_n \Big| \\
&\quad + \Big| \int_{[t,-C]^n \setminus [s,-C]^n} K\begin{pmatrix} x,\dots \\ y,\dots \end{pmatrix} \mathrm{d}x_1 \dots x_n \Big| \\
&\leq \int_{[t,-C]^n} \Big| K\begin{pmatrix} x,\dots \\ y,\dots \end{pmatrix} - K\begin{pmatrix} x_1,\dots \\ y_1,\dots \end{pmatrix} \Big| \mathrm{d}x_1 \dots x_n \\
&\quad + \int_{[t,-C]^n \setminus [s,-C]^n} \Big| K\begin{pmatrix} x,\dots \\ y,\dots \end{pmatrix} \Big| \mathrm{d}x_1 \dots x_n
\end{aligned}$$

if $C < x_1, y_1, s$ holds. Since $(x_1, y_1, s) \to (x, y, t)$ implies $s \to t$ and since we can assume (x_1, y_1) is in a compact ball around (x, y), the second integrals vanish (both at the same rate) for $(x_1, y_1, s) \to (x, y, t)$ by the argument above that we used for continuity of (2.2.3). As $(x_1, y_1, s) \to (x, y, t)$ implies $(x_1, y_1) \to (x, y)$ and as the integrand is continuous on $\mathbb{R}^{2n+2}$, the first integral tends to 0 by Lemma A.1.1. All in all, $|D_n(x, y, t) - D_n(x_1, y_1, s)| \to 0$ for $(x_1, y_1, s) \to (x, y, t)$.

Let $C < 0$. For $t \in \mathbb{R}$, conditions $(G1)$ and $(G2)$ guarantee that

$$M(t) := 2\min(t, C) \sup\{|K(x + y)| : (x, y) \in [\min(t, C), \infty) \times [\min(t, C), \infty))\}$$

exists in $\mathbb{R}$, and we have

$$M(t) = 2\min(t, C) \sup_{x \in [2\min(t,C),0]} |K(x)|. \tag{2.2.6}$$

It is a well-known fact that this defines a nonnegative, continuous function on $\mathbb{R}$. Hence, Lemma 2.1.5 holds (after replacing ∞ with $\max(-t, -C)$ by condition $(G2)$) with the inequalities

$$|d_n(t)| \leq \frac{n^{\frac{n}{2}} M(t)^n}{n!}$$

and

$$|D_{n-1}(x, y, t)| \leq \frac{n^{\frac{n}{2}} M(t)^{n-1}}{(n-1)!} \sup_{x \in [2\min(t,C),0]} |K(x)|$$

where $t \in \mathbb{R}$, $(x, y) \in [C, -C] \times [C, -C]$ and $n \in \mathbb{N}, n \geq 2$. More precisely, (2.2.2) converges uniformly absolutely in (λ, t) as an element of $\mathbb{R}^3$ on every closed cube in $\mathbb{C} \times \mathbb{R}$ and (2.2.4) converges uniformly absolutely in (x, y, λ, t) as an element of $\mathbb{R}^5$ on every closed cube in $\mathbb{R}^2 \times \mathbb{C} \times \mathbb{R}$. Thus, d_K and D_K are continuous since we have already proved continuity of d_n and D_n for $n \in \mathbb{N}$. In particular, we showed existence of D_K and that d_K and D_K are entire in λ. ■

Remark 2.2.4. If we fix $t \in \mathbb{R}$ and replace $-t$ with ∞ in the integral boundaries by using condition $(G2)$, we get the following versions of (2.1.20) and (2.1.21) for $\lambda \in \mathbb{C}$ and $(x, y) \in [t, -t] \times [t, -t]$:

$$D_K(x, y, \lambda, t) = d_K(\lambda, t)K(x + y) + \lambda \int_t^\infty D_K(x, z, \lambda, t)K(z + y)\,\mathrm{d}z \tag{2.2.7}$$

$$D_K(x, y, \lambda, t) = d_K(\lambda, t)K(x + y) + \lambda \int_t^\infty D_K(z, y, \lambda, t)K(x + z)\,\mathrm{d}z. \tag{2.2.8}$$

We can expand these integral equations to $(x, y) \in [t, \infty) \times [t, \infty)$ because Lemma 2.2.3 and condition $(G2)$ say that both sides are 0 if either x or y is bigger than $-t$. In fact, we can even allow $(x, y) \in \mathbb{R}^2$ because the original proof only uses $(x, y) \in [t, -t] \times [t, -t]$ to justify the use Fubini's theorem with continuity. Therefore, we just have to replace the upper bound of the integral with a $-C > 0$, where $C < x, y, t$ by virtue of condition $(G2)$. Then we can copy the original proof word by word.

Because of (2.1.18) and condition $(G2)$, we get a nontrivial continuous function $f : [t, -t] \to \mathbb{C}$ with

$$f(x) = \mu \int_t^{-t} K(x + y)f(y)\,\mathrm{d}y = \mu \int_t^\infty K(x + y)f(y)\,\mathrm{d}y \tag{2.2.9}$$

for any $\mu \in \mathbb{C}$ with $d_K(\mu, t) = 0$. (We have $t < 0$ in that case by Lemma 2.2.3.) Condition $(G2)$ allows us to extend f with 0 to a $L^2(t, \infty)$ function with (2.2.9).

If we fix $t \in \mathbb{R}$ and replace $-t$ with ∞ in the integral boundaries by using Lemma 2.2.3, we get the following version of (2.1.26) for $n \in \mathbb{N}$ and $(x, y) \in [t, -t] \times [t, -t]$:

$$D_n(x, y, t) = K(x + y)d_n(t) + \int_t^\infty K(x + z)D_{n-1}(z, y, t)\,\mathrm{d}z. \tag{2.2.10}$$

We can expand this integral equation to $(x, y) \in [t, \infty) \times [t, \infty)$ because condition $(G2)$ and Lemma 2.2.3 say that both sides are 0 if either x or y is bigger than $-t$. //

Chapter 3

Finding Hamiltonians Satisfying Certain Nonconstant Initial Values

In this chapter, we prove the main theorem of this thesis [9, Theorem 1.1.] with certain adaptations. This means we mainly follow the path of [9] and [10]. Since this chapter is the core of this thesis, everything is proved.

3.1 Definitions and Main Theorem

Ultimately, we want to give a positive answer to the question wheter we can find a canonical system that has solutions satisfying certain given initial values. The two big differences from Theorem 1.1.5 is that the initial values depend on the parameter z and that the Hamiltonian itself is the unknown we chiefly aim to find. In contrast to [9], we stick to the setting of Chapter 1 for this purpose. That means we do not apply the change of variables $t \mapsto -t$ and do not define H on $[t_0, t_1)$, which would have been deviations from the traditional Definition 1.1.1 of canonical systems. As a consequence, we have to adapt most of the statements beginning with the integral kernels introduced in [9] and [10]. Moreover for the sake of convenience, our condition $(K3)$ is less general.

Definition 3.1.1. Let $\mathcal{K}$ stand for the set of all functions K that meet the following conditions:

$(K1)$ $K \in C(\mathbb{R}, \mathbb{R})$ holds, and there exists a constant $c(K) > 0$ with $|K(x)| \in O(\exp(c(K)|x|))$.

$(K2)$ $K \not\equiv 0$ and $K(x) = 0$ for $x \geq 0$.

$(K3)$ There exists a closed discrete subset Λ_K of $\mathbb{R}$ such that K is continuously differentiable on $\mathbb{R} \setminus \Lambda_K$ with $K' \in L^\infty_{\text{loc}}(\mathbb{R}, \mathbb{R})$. Furthermore, there exists a function $g_{K,x} \in L^\infty(\mathbb{R}, \mathbb{R})$ for every $x \in \mathbb{R}$ such that

$$\Big|\frac{K(x + .) - K(\tilde{x} + .)}{x - \tilde{x}}\Big| \leq g_{K,x} \text{ a.e.}$$

holds locally around x. (The null set can depend on $\tilde{x}$.)

$(K4)$ Consider the integral operator

$$K[t] : \begin{cases} L^2(t, \infty) & \to L^2(t, \infty) \\ f & \mapsto \int_t^\infty K(. + y) f(y)\,\mathrm{d}y \end{cases}.$$

Let $\tau(K) \in [-\infty, 0]$ be the infimum of all $s \in \mathbb{R}$ such that $K[t]$ does not have eigenvalues ± 1

for $s < t \leq 0$.

**

Remark 3.1.2. Obviously, we have $\mathcal{K} \subseteq \mathcal{G}$. Thus, Section 2.2 holds for $K \in \mathcal{K}$. //

Condition $(K4)$ needs some further discussion. Thus, consolidating [9, page 2] and [10, page 4], we formulate the following lemma.

Lemma 3.1.3. *For $K \in \mathcal{K}$, the map $K[t]$ is a compact and self-adjoint operator on $L^2(t,\infty)$ for $t \in \mathbb{R}$. Furthermore, $t \geq 0$ implies $K[t] \equiv 0$ and $\tau(K) < 0$ always holds.*

Proof. As an immediate consequence of condition $(K2)$ the operator vanishes for $t \geq 0$. Hence, we can assume $t < 0$. Since $K(x+y)$ is continuous on $(t,\infty) \times (t,\infty)$, we can apply Fubini's theorem and

$$\int_t^\infty \int_t^\infty |K(x+y)|^2 \,\mathrm{d}x \,\mathrm{d}y = \int_t^{-t} \int_t^{-t} |K(x+y)|^2 \,\mathrm{d}x \,\mathrm{d}y \leq 4t^2 \max_{x \in [2t,0]} |K(x)|^2 \tag{3.1.1}$$

shows that $K(x+y) \in L^2((t,\infty)\times(t,\infty))$. Lemma A.4.2 yields that $K[t]$ is compact on $L^2(t,\infty)$. Because $K(x+y)$ is symmetric and real-valued, and $f(x)\overline{g(y)} \in L^2((t,\infty)\times(t,\infty))$ justifies Fubini's theorem, we have

$$\begin{aligned}
(f, K[t]g)_{L^2} = \int_t^\infty f(x) \overline{\int_t^\infty K(x+y) g(y) \,\mathrm{d}y} \,\mathrm{d}x &= \int_t^\infty \int_t^\infty f(x) K(x+y) \overline{g(y)} \,\mathrm{d}y \,\mathrm{d}x \\
&= \int_t^\infty \int_t^\infty f(x) K(x+y) \,\mathrm{d}x \; \overline{g(y)} \,\mathrm{d}y \\
&= \int_t^\infty \int_t^\infty K(y+x) f(x) \,\mathrm{d}x \; \overline{g(y)} \,\mathrm{d}y \\
&= (K[t]f, g)_{L^2}
\end{aligned}$$

for $f, g \in L^2(t,\infty)$. By applying Hölder's inequality and (3.1.1), we get

$$\begin{aligned}
\|K[t]f\|_{L^2}^2 = \int_t^\infty \Big| \int_t^\infty K(x+y) f(y) \,\mathrm{d}y \Big|^2 \,\mathrm{d}x &\leq \int_t^\infty \Big(\int_t^\infty |K(x+y) f(y)| \,\mathrm{d}y \Big)^2 \,\mathrm{d}x \\
&\leq \int_t^\infty \int_t^\infty |K(x+y)|^2 \,\mathrm{d}y \|f\|_{L^2}^2 \,\mathrm{d}x \\
&\leq 4t^2 \max_{x \in [2t,0]} |K(x)|^2 \|f\|_{L^2}^2
\end{aligned}$$

for $f \in L^2(t,\infty)$. This proves $\lim_{t \to 0} \|K[t]\| = 0$. Together with the fact that the spectral radius is bounded above by the operator norm, this implies $\tau(K) < 0$. ■

We are now capable of defining certain eligible initial values for the searched canonical system. A motivation of this definition is given in Chapter 4.

Definition 3.1.4. Let $\mathcal{E}$ stand for the set of all pairs (E, K) where E is a function from $\mathbb{C}$ to $\mathbb{C} \uplus \{\infty\}$ and K is an element of $\mathcal{E}$ such that the following conditions hold:

$(E1)$ There exists a closed discrete subset Z_E of $\mathbb{C}$ with $0 \notin Z_E$, $\operatorname{Im}(Z_E) \subseteq [-c(K), c(K)]$, $-Z_E = Z_E$ and $\overline{z} \in Z_E$ for all $z \in Z_E$ such that

$$E : \mathbb{C} \setminus Z_E \to \mathbb{C}$$

is analytic, $E \not\equiv 0$ and $E^\#(z) = E(-z)$ for $z \in \mathbb{C} \setminus Z_E$.

$(E2)$ For $z \in \mathbb{C}$ with $\mathrm{Im}(z) > c(K)$ the identity

$$\frac{E^{\#}(z)}{E(z)} = (\mathcal{F}K)(z)$$

holds.

**

For the sake of convenience and readability, we write $E \in \mathcal{E}$ for the rest of this thesis. Although this kernel is generally not unique, we omit the explicit choice of a suitable K and write K_E for an arbitrarily chosen element of $\mathcal{K}$ with $(E, K_E) \in \mathcal{E}$. Before we get to the next definition, condition $(E2)$ needs some further discussion.

Remark 3.1.5. For an $E \in \mathcal{E}$, conditions $(K1)$ and $(K2)$ yield (cf. Definition A.2.1)

$$\begin{aligned}(\mathcal{F}K_E)(z) &= \int_{-\infty}^{\infty} K_E(t)\exp(-izt)\,\mathrm{d}t = \int_{-\infty}^{0} K_E(t)\exp(-izt)\,\mathrm{d}t \\ &= \int_{0}^{\infty} K_E(-t)\exp(izt)\,\mathrm{d}t = (\mathcal{L}K_E(-.))(-iz)\end{aligned}$$

with $\sigma(K_E(-.)) \leq c(K_E)$. By Lemma A.2.2, $\mathcal{F}K_E$ is analytic for $-iz \in \mathbb{C}$ with $\mathrm{Re}(-iz) = \mathrm{Im}(z) > c(K_E)$. //

In the next lemma and throughout the rest of the chapter, we often use the continuous differentiability of integrals with integral kernel K. More precisely, we use:

Remark 3.1.6. As $K \in \mathcal{K}$ is continuous by condition $(K1)$, it is bounded on every closed interval. The same applies to K' because of $K' \in L^{\infty}_{\mathrm{loc}}(\mathbb{R}, \mathbb{R})$ by condition $(K3)$. Both $K(x+y)$ and $K'(x+y)$ vanish for $y > -x$ by condition $(K2)$. We now claim that the parameter integrals of the form

$$\int_{s}^{\infty} K(x+y)f(y)\,\mathrm{d}y = \int_{s}^{-x} K(x+y)f(y)\,\mathrm{d}y \text{ and} \tag{3.1.2}$$

$$\int_{s}^{\infty} K'(x+y)f(y)\,\mathrm{d}y = \int_{s}^{-x} K'(x+y)f(y)\,\mathrm{d}y \tag{3.1.3}$$

are continuous in $x \in \mathbb{R}$ for $f \in L^1_{\mathrm{loc}}(\mathbb{R})$ and $s \in \mathbb{R}$ and that (3.1.3) is the derivative of (3.1.2) with respect to x:

Obviously, both (3.1.2) and (3.1.3) exist for every $x \in \mathbb{R}$, and K, K' in the integrands range over $[x+s, 0]$. Hence, we find a bound $M > 0$ of $|K|, |K'|$ on $[x+s-\varepsilon, 0]$ for any $\varepsilon > 0$. The function $\mathbf{1}_{[s,-x+\varepsilon]}(y)M|f(y)|$ is integrable and an upper bound of the integrands of (3.1.2) and (3.1.3) locally around x. Finally, $K(.+y)$ and $K'(.+y)$ are continuous at x for $x+y \notin \Lambda_K$ by conditions $(K1)$ and $(K3)$. Thus, Lemma A.1.1 yields continuity.

By condition $(K3)$, $K(.+y)$ is differentiable at x for $x+y \notin \Lambda_K$, and, for a suitable $\varepsilon > 0$, the function $\mathbf{1}_{[s,-x+\varepsilon]}(y)g_{K,x}|f(y)|$ is integrable and a.e. an upper bound of

$$\Big|\frac{K(x+.) - K(\tilde{x}+.)}{x-\tilde{x}}\Big||f|$$

on $[s, \infty)$ for $\tilde{x} \in \mathbb{R}$ with $|x-\tilde{x}| < \varepsilon$ Thus, Lemma A.1.2 yields differentiability with the desired derivative. //

Our aim is to construct a Hamiltonian and corresponding solutions in terms of E and K_E. For this purpose, we need the following Lemma (cf. [9, Proposition 2.1.]):

Lemma 3.1.7. *For $K \in \mathcal{K}$ and a fixed $t \in (\tau(K), \infty)$, the integral equations*

$$1 = \Phi_{K,t}(x) + \int_t^\infty K(x+y)\Phi_{K,t}(y)\,\mathrm{d}y \tag{3.1.4}$$

$$1 = \Psi_{K,t}(x) - \int_t^\infty K(x+y)\Psi_{K,t}(y)\,\mathrm{d}y \tag{3.1.5}$$

have unique solutions in $L^1_{\mathrm{loc}}(\mathbb{R})$. Furthermore, these solutions have the following properties:

(*a.*) *Smoothness and real codomain: $\Phi_{K,t}, \Psi_{K,t} \in C^1(\mathbb{R}, \mathbb{R})$.*

(*b.*) *Limit behavior: $\Phi_{K,t}(x) = 1 = \Psi_{K,t}(x)$ for $x > -t$ and $\Phi_{K,t}(x), \Psi_{K,t}(x) \in O(\exp(c'|x|))$ for any $c' > c(K)$.*

(*c.*) *Diagonals do not vanish: $\Phi_{K,t}(t) \neq 0$ and $\Psi_{K,t}(t) \neq 0$.*

In the special case $t \geq 0$, we have $\Phi_{K,t}(t) = 1 = \Psi_{K,t}(t)$ and

$$\Phi_{K,t}(x) = 1 - \int_{x+t}^0 K(y)\,\mathrm{d}y \tag{3.1.6}$$

$$\Psi_{K,t}(x) = 1 + \int_{x+t}^0 K(y)\,\mathrm{d}y. \tag{3.1.7}$$

Proof. Apparently, the claims for $\Phi_{K,t}$ and $\Psi_{K,t}$ have analogous proofs, so we consider $\Phi_{K,t}$ only. We first deal with the case $t < 0$. The equation

$$\mathbf{1}_{[t,-t]}(x) = \tilde{\Phi}_{K,t}(x) + \int_t^\infty K(x+y)\tilde{\Phi}_{K,t}(y)\,\mathrm{d}y \tag{3.1.8}$$

has a unique solution in $L^2(t,\infty)$ because $\mathbf{1}_{[t,-t]} \in L^2(t,\infty)$ and $K[t]+1$ is invertible since -1 is not in the spectrum of $K[t]$ as it is no eigenvalue by condition $(K4)$ and $K[t]$ is compact by Lemma 3.1.3. For K is real, the real part of this solution also is a solution. This shows that $\tilde{\Phi}_{K,t}$ is real. Now, we claim that the continuous function (cf. Remark 3.1.6)

$$\Phi_{K,t}(x) : \begin{cases} \mathbb{R} & \to \mathbb{R} \\ x & \mapsto 1 - \int_{-t}^{-x} K_E(x+y)\,\mathrm{d}y - \int_t^\infty K(x+y)\tilde{\Phi}_{K,t}(y)\,\mathrm{d}y \end{cases} \tag{3.1.9}$$

is a solution of (3.1.4). For $x > -t$ the integrals in (3.1.9) vanish because of condition $(K2)$ and $-x < -t$. Hence, $\Phi_{K,t}(x) = 1$ holds. For $x \in [t, -t]$, the first integral in (3.1.9) vanishes because of $-x \leq -t$ and (3.1.8) yields $\Phi_{K,t}(x) = \tilde{\Phi}_{K,t}(x)$. Finally, for $x > -t$ the integral in (3.1.8) vanishes because of condition $(K2)$ and the left side vanishes, too. Thus, we have $\tilde{\Phi}_{K,t}(x) = 0$ and (3.1.9) turns into

$$\begin{aligned} \Phi_{K,t}(x) &= 1 - \overbrace{\int_{-t}^{-x} K(x+y)\,\mathrm{d}y}^{\in O(|x|\exp(c(K)|x|))} - \overbrace{\int_t^{-t} K(x+y)\tilde{\Phi}_{K,t}(y)\,\mathrm{d}y}^{\in O(\exp(c(K)|x|))} \\ &= 1 - \int_t^{-x} K(x+y)\Phi_{K,t}(y)\,\mathrm{d}y = 1 - \int_t^\infty K(x+y)\Phi_{K,t}(y)\,\mathrm{d}y, \end{aligned}$$

which also shows $\Phi_{K,t}(x) \in O(\exp(c'|x|))$ by the mean value theorem and condition $(K1)$. Differentiability follows from Remark 3.1.6 because $\Phi_{K,t} \in L^1_{\mathrm{loc}}(\mathbb{R}, \mathbb{R})$. So,

$$\frac{\mathrm{d}\Phi_{K,t}(x)}{\mathrm{d}x} = -\int_t^\infty K'(x+y)\Phi_{K,t}(y)\,\mathrm{d}y = -\int_t^{-x} K'(x+y)\Phi_{K,t}(y)\,\mathrm{d}y \tag{3.1.10}$$

holds and this function is also continuous by Remark 3.1.6.
The difference f of two $L^1_{\mathrm{loc}}(\mathbb{R})$ solutions of (3.1.4) solves the equation

$$0 = f(x) + \int_t^\infty K(x+y)f(y)\,\mathrm{d}y. \tag{3.1.11}$$

In particular, Remark 3.1.6 shows continuity of f. Hence, condition $(K2)$ yields $f(x) = 0$ for $x > -t$ and we get $\mathbf{1}_{[t,\infty)}f \in L^2(t,\infty)$. The same argument as used for (3.1.8) shows unique solvability of (3.1.11) on $L^2(t,\infty)$ and therefore $\mathbf{1}_{[t,\infty)}f \equiv 0$. Looking at (3.1.11) once more, we conclude $f \equiv 0$.
Applying integration by parts to (3.1.10), we see

$$\frac{\mathrm{d}\Phi_{K,t}(x)}{\mathrm{d}x} = \int_t^{-x} K(x+y)\frac{\mathrm{d}\Phi_{K,t}(y)}{\mathrm{d}y}\,\mathrm{d}y + K(x+t)\Phi_{K,t}(t) - \overbrace{K(0)}^{=0}\,\Phi_{K,t}(-x). \tag{3.1.12}$$

Assuming $\Phi_{K,t}(t) = 0$ and once again using condition $(K2)$ to replace $-x$ by ∞ in the first integral, $\frac{\mathrm{d}\Phi_{K,t}}{\mathrm{d}x}$ would solve (3.1.11), i.e. $\frac{\mathrm{d}\Phi_{K,t}}{\mathrm{d}x} \equiv 0$ and $\Phi_{K,t} \equiv C \in \mathbb{R}$. We already proved that 1 is attained by $\Phi_{K,t}$. As a consequence, (3.1.4) would simplify to

$$0 = \int_{x+t}^0 K(y)\,\mathrm{d}y$$

for any $x \in \mathbb{R}$, which means $K \equiv 0$. This would obviously contradict condition $(K2)$.↯
Finally, $t \geq 0$ and condition $(K2)$ imply $K(x+y) = 0$ for $x > t$ and $y \geq t$. Hence, a potential solution of (3.1.4) in $L^1_{\mathrm{loc}}(\mathbb{R})$ must meet $\Phi_{K,t}(x) = 1$ for $x > t$, (3.1.4) simplifies to (3.1.6) and uniqueness holds. On the other hand, (3.1.6) defines a solution that clearly satisfies all claimed properties. ■

Remark 3.1.8. Due to Remark 3.1.6, the identity (3.1.10) remains true for $t \geq 0$. Therefore, integration by parts yields (3.1.12) for arbitrary $(t,x) \in (\tau(K),\infty) \times \mathbb{R}$. Similarly, we have

$$\frac{\mathrm{d}\Psi_{K,t}(x)}{\mathrm{d}x} = -\int_t^{-x} K(x+y)\frac{\mathrm{d}\Psi_{K,t}(y)}{\mathrm{d}y}\,\mathrm{d}y - K(x+t)\Psi_{K,t}(t) \tag{3.1.13}$$

for $(t,x) \in (\tau(E),\infty) \times \mathbb{R}$. //
Henceforth, we always refer to the solutions of (3.1.4) and (3.1.5) if we use $\Phi_{K,t}$ and $\Psi_{K,t}$ for an $K \in \mathcal{K}$ and $t \in (\tau(K),\infty)$. If we want to stress the importance of t, we write $\Phi_K(t,x) := \Phi_{K,t}(x)$ and $\Psi_K(t,x) := \Psi_{K,t}(x)$.
With this, we have all the necessary tools to state the main theorem. Unlike [9, Theorem 1.1.] where Lemma 3.2.7 is used, we define the Hamiltonian in terms of Φ_{K_E} and Ψ_{K_E}. The proof of the theorem is the primary subject of the rest of this chapter.

Theorem 3.1.9. *Let E be an element of $\mathcal{E}$. For $t \in (\tau(K_E),\infty)$, define*

$$H_E(t) := \begin{pmatrix} \Phi_{K_E}(t,t)^2 & 0 \\ 0 & \Psi_{K_E}(t,t)^2 \end{pmatrix}. \tag{3.1.14}$$

Then, H_E is a continuous Hamiltonian with determinant 1, and every $t \in (\tau(K_E),\infty)$ is regular. Next, denote

$$A_E(t,z) := A_{E,t}(z) := -\frac{iz}{2}E(z)\int_{-\infty}^t \Psi_{K_E}(t,x)\exp(-izx)\,\mathrm{d}x \tag{3.1.15}$$

$$B_E(t,z) := B_{E,t}(z) := \frac{z}{2}E(z)\int_{-\infty}^t \Phi_{K_E}(t,x)\exp(-izx)\,\mathrm{d}x. \tag{3.1.16}$$

Then, $(A_E,B_E)^t$ is the solution to H with initial values E in the following sense:

(*a.*) *Fixing* $t \in (\tau(K_E), \infty)$*, the terms* $A_{E,t}(z)$ *and* $B_{E,t}(z)$ *are analytic functions for* $\mathrm{Im}(z) > c(K_E)$*, and they can be extended to analytic functions on* $\mathbb{C} \setminus Z_E$ *satisfying:*

$$\begin{aligned} A^{\#}_{E,t}(z) &= A_{E,t}(z) \quad A_{E,t}(-z) = A_{E,t}(z) \\ B^{\#}_{E,t}(z) &= B_{E,t}(z) \quad B_{E,t}(-z) = -B_{E,t}(z) \end{aligned} \tag{3.1.17}$$

(*b.*) *Fixing* $z \in \mathbb{C} \setminus Z_E$*, both* A_E *and* B_E *are continuously differentiable with respect to* t*.*

(*c.*) $(A_E, B_E)^t$ *solves (1.1.1) for* $t \in (\tau(K_E), \infty)$*,* $z \in \mathbb{C} \setminus Z_E$ *and the Hamiltonian from (3.1.14).*

(*d.*) *For* $z \in \mathbb{C} \setminus Z_E$*, we have*

$$\begin{aligned} A_E(0, z) &= 1/2(E(z) + E^{\#}(z)) \\ B_E(0, z) &= i/2(E(z) - E^{\#}(z)) \end{aligned}$$

and $E(z) = A_E(0, z) - iB_E(0, z)$*.*

3.2 Properties of Φ_K and Ψ_K

Before we study the integral equations of Lemma 3.1.7 with respect to the variable t, we focus on the solutions of the following strongly related integral equations (cf. [9, Proposition 2.2]). For this purpose, we only need the kernels $\mathcal{K}$.

The structure of this section is based on [10, Section 2].

Lemma 3.2.1. *For* $K \in \mathcal{K}$ *and a fixed* $t \in (\tau(K), \infty)$ *the integral equations*

$$K(x + t) = \phi_{K,t}(x) + \int_t^\infty K(x + y)\phi_{K,t}(y)\,\mathrm{d}y \tag{3.2.1}$$

$$K(x + t) = \psi_{K,t}(x) - \int_t^\infty K(x + y)\psi_{K,t}(y)\,\mathrm{d}y \tag{3.2.2}$$

have uniqe solutions in $L^1_{\mathrm{loc}}(\mathbb{R})$*. Furthermore, these solutions have the following properties:*

(*a.*) *Continuity and real codomain:* $\phi_{K,t}, \psi_{K,t} \in C(\mathbb{R}, \mathbb{R})$*.*

(*b.*) *Differentiability:* $\phi_{K,t}, \psi_{K,t} \in C^1(\mathbb{R} \setminus (\Lambda_K - t), \mathbb{R})$*.*

(*c.*) *Limit behavior:* $\phi_{K,t}(x) = 0 = \psi_{K,t}(x)$ *for* $x > -t$ *and* $\phi_{K,t}(x), \psi_{K,t}(x) \in O(\exp(c(K)|x|))$*.*

In the special case $t \geq 0$*, we have* $\phi_{K,t}(t) = 0 = \psi_{K,t}(t)$ *and* $\phi_{K,t}(x) = K(x + t) = \psi_{K,t}(x)$*.*

Proof. The proof is very similar to the proof of Lemma 3.1.7. Apparently, the claims for $\phi_{K,t}$ and $\psi_{K,t}$ have analogous proofs, so we consider $\phi_{K,t}$ only. We first deal with the case $t < 0$. The equation

$$K(x + t) = \tilde{\phi}_{K,t}(x) + \int_t^\infty K(x + y)\tilde{\phi}_{K,t}(y)\,\mathrm{d}y \tag{3.2.3}$$

has a unique solution in $L^2(t, \infty)$: As K is continuous by condition $(K1)$ and $K(x+t)$ vanishes for $x > -t$ by condition $(K2)$, $K(. + t)$ lies in $L^2(t, \infty)$. $K[t] + 1$ is invertible since -1 is not in the spectrum of $K[t]$ as it is no eigenvalue by condition $(K4)$ and $K[t]$ is compact by

Lemma 3.1.3. Because K is real, the real part of this solution also is a solution. This shows that $\tilde{\phi}_{K,t}$ is real. Now, we claim that the continuous function (cf. Remark 3.1.6)

$$\phi_{K,t}(x) : \begin{cases} \mathbb{R} & \to \mathbb{R} \\ x & \mapsto K(x+t) - \int_t^\infty K(x+y)\tilde{\phi}_{K,t}(y)\,\mathrm{d}y \end{cases} \tag{3.2.4}$$

is a solution of (3.2.1). For $x > -t$ the integral and K in (3.1.7) vanish because of condition $(K2)$. Hence, $\phi_{K,t}(x) = 0$ and as a consequence $\phi_{K,t} \in L^2(t,\infty)$ holds. Therefore, the unique solvability of (3.2.3) proves $\phi_{K,t} = \tilde{\phi}_{K,t}$ on $[t,\infty)$. Considering this in the integral in (3.2.4), we see that $\phi_{K,t}$ solves (3.2.1) with

$$\phi_{K,t}(x) = \overbrace{K(x+t)}^{\in O(\exp(c(K)|x|))} - \overbrace{\int_t^{-t} K(x+y)\phi_{K,t}(y)\,\mathrm{d}y}^{\in O(\exp(c(K)|x|))},$$

which also shows $\phi_{K,t}(x) \in O(\exp(c(K)|x|))$ by the mean value theorem and condition $(K1)$.
The difference f of two $L^1_{\mathrm{loc}}(\mathbb{R})$ solutions of (3.2.1) solves the equation

$$0 = f(x) + \int_t^\infty K(x+y)f(y)\,\mathrm{d}y. \tag{3.2.5}$$

In particular, Remark 3.1.6 shows continuity of f. Hence, condition $(K2)$ yields $f(x) = 0$ for $x > -t$ and we get $f \in L^2(t,\infty)$. The same argument as used for (3.2.3) shows unique solvability of (3.2.5) on $L^2(t,\infty)$ and therefore $\mathbf{1}_{[t,\infty)}f \equiv 0$. Looking at (3.2.5) once more, we conclude $f \equiv 0$.
Differentiability follows from Remark 3.1.6 because $\phi_{K,t} \in L^1_{\mathrm{loc}}(\mathbb{R},\mathbb{R})$. So,

$$\frac{\mathrm{d}\phi_{K,t}(x)}{\mathrm{d}x} = K'(x+t) - \int_t^\infty K'(x+y)\phi_{K,t}(y)\,\mathrm{d}y$$

holds for $\mathbb{R} \setminus (\Lambda_K - t)$. This function is also continuous on $\mathbb{R} \setminus (\Lambda_K - t)$ by condition $(K3)$ and Remark 3.1.6.
Finally, $t \geq 0$ and condition $(K2)$ imply $K(x+y) = 0$ for $x > t$ and $y \geq t$. Hence, a potential solution of (3.2.1) in $L^1_{\mathrm{loc}}(\mathbb{R})$ must meet $\phi_{K,t}(x) = 0$ for $x > t$, (3.2.1) simplifies to $\phi_{K,t}(x) = K(x+t)$ and uniqueness holds. On the other hand, $\phi_{K,t}(x) = K(x+t)$ defines a solution that clearly satisfies all claimed properties. ■

From now on, we always refer to the solutions of (3.2.1) and (3.2.2) if we use $\phi_{K,t}$ and $\psi_{K,t}$ for an $K \in \mathcal{K}$ and $t \in (\tau(K),\infty)$. If we want to stress the importance of t, we write $\phi_K(t,x) := \phi_{K,t}(x)$ and $\psi_K(t,x) := \psi_{K,t}(x)$.
The most delicate claim of Theorem 3.1.9 is differentiability of A_E and B_E with respect to t. Unfortunately, the integral equations (3.1.4) and (3.1.5) themselves do not just give rise to a "Lemma 3.1.7 for t". Hence, we use Section 2.2 to get a constructive representation of ϕ_K and ψ_K.

Lemma 3.2.2. *For an $K \in \mathcal{K}$ and $x \in \mathbb{R}$, both $\phi_K(.,x)$ and $\psi_K(.,x)$ are elements of the spaces $C((\tau(K),\infty),\mathbb{R})$ and $C^1((\tau(K),\infty) \setminus (\Lambda_K - x),\mathbb{R})$.*

Proof. The proof of the differentiabilty is based on [10, Lemma 2.5].
Apparently, ϕ_K and ψ_K have analogous proofs, so we only consider ϕ_K. Setting and fixing $y = t$ and $\lambda = -1$ for $t \in (\tau(K),\infty)$ in (2.2.7) yields

$$D_K(x,t,-1,t) = d_K(-1,t)K(x+t) - \int_t^\infty D_K(x,z,-1,t)K(z+t)\,\mathrm{d}z \tag{3.2.6}$$

for $x \geq t$. In Remark 2.2.4, we showed that $d_K(-1,t) = 0$ would imply (2.2.9) with $\mu = -1$ for a $f \in L^2(t,\infty)$ with $f \not\equiv 0$, i.e. $K[t]$ would have the eigenvalue -1. This would contradict condition $(K4)$.↯ Thus, we can divide (3.2.6) by $d_K(-1,t)$ and get (3.2.1) for $x \geq t$. As the generalized first Fredholm minor is continuous in x and vanishes for $x \geq -t$ by Lemma 2.2.3, we have

$$\frac{D_K(.,t,-1,t)}{d_K(-1,t)} \in L^2(t,\infty).$$

By Lemma 3.2.1, the same applies to $\phi_{K,t}$. As K is continuous by condition $(K1)$ and $K(x+t)$ vanishes for $x > -t$ by condition $(K2)$, $K(.+t)$ lies in $L^2(t,\infty)$. The operator $K[t]+1$ is invertible since -1 is not in the spectrum of $K[t]$ as it is no eigenvalue by condition $(K4)$ and $K[t]$ is compact by Lemma 3.1.3. Hence, we conclude

$$\frac{D_K(.,t,-1,t)}{d_K(-1,t)} = (K[t]+1)^{-1}K(.+t) = \phi_{K,t}$$

as elements of $L^2(t,\infty)$, and the continuity of both sides shows

$$\frac{D_K(x,t,-1,t)}{d_K(-1,t)} = \phi_{K,t}(x)$$

for $x \geq t$. This and (3.2.1) yield

$$\phi_{K,t}(x) = K(x+t) - \int_t^\infty K(x+y)\frac{D_K(y,t,-1,t)}{d_K(-1,t)}\,\mathrm{d}y \tag{3.2.7}$$

for $x \in \mathbb{R}$. Replacing ∞ with $-t$ and $-s$ for an $s \in (\tau(K),0]$ respectively by Lemma 2.2.3, setting $S := [s,-s] \setminus [t,-t]$, $T := [t,-t] \setminus [s,-s]$ and fixing x, we get

$$\begin{aligned}\Big|\int_t^\infty K(x+y)\frac{D_K(y,t,-1,t)}{d_K(-1,t)}\,\mathrm{d}y - \int_s^\infty K(x+y)\frac{D_K(y,s,-1,s)}{d_K(-1,s)}\,\mathrm{d}y\Big| \leq \\ \int_t^{-t}|K(x+y)|\Big|\frac{D_K(y,t,-1,t)}{d_K(-1,t)} - \frac{D_K(y,s,-1,s)}{d_K(-1,s)}\Big|\,\mathrm{d}y + \int_S|K(x+y)|\Big|\frac{D_K(y,s,-1,s)}{d_K(-1,s)}\Big|\,\mathrm{d}y\end{aligned} \tag{3.2.8}$$

or

$$\begin{aligned}\Big|\int_t^\infty K(x+y)\frac{D_K(y,t,-1,t)}{d_K(-1,t)}\,\mathrm{d}y - \int_s^\infty K(x+y)\frac{D_K(y,s,-1,s)}{d_K(-1,s)}\,\mathrm{d}y\Big| \leq \\ \int_t^{-t}|K(x+y)|\Big|\frac{D_K(y,t,-1,t)}{d_K(-1,t)} - \frac{D_K(y,s,-1,s)}{d_K(-1,s)}\Big|\,\mathrm{d}y + \int_T|K(x+y)|\Big|\frac{D_K(y,t,-1,t)}{d_K(-1,t)}\Big|\,\mathrm{d}y\end{aligned} \tag{3.2.9}$$

depending on whether $s \leq t$ or $s > t$. The second integrals of (3.2.8) and (3.2.9) vanish at the same rate for $s \to t$ because the measures of S and T tend to 0 and the integrand is bounded for s and y locally around t by continuity (condition $(K1)$ and Lemma 2.2.3). As the integrand is continuous, the first integral of (3.2.8) and (3.2.9) tends to 0 for $s \to t$ by Lemma A.1.1. Hence, (3.2.7) proves that $\phi_K(.,x)$ is continuous on $(\tau(K),\infty)$.

Many arguments for continuity in the rest of this proof are tedious and just slight modifications or refinements of arguments that we already discussed in detail. We do not carry those out in full extent and just refer to the respective argument they are based on.

For $n \in \mathbb{N}$, we consider (2.2.3), but instead of only t, we consider integral boundaries $t_1,\dots,t_n \in (\tau(K),\infty)$. Fixing all of these but one t_j for a $j \in \{1,\dots,n\}$, we can interchange the integrals such that the t_j one is the outmost one and merge all the other ones. This is possible by the virtue of Fubini's theorem since the integrand is continuous and vanishes outside a cube. The same argument also allows us to apply Lemma A.1.1 to see that the merged integral is continuous in x_j. Therefore, the fundamental theorem of calculus yields differentiability at t_j with derivative

$$\frac{(-1)^{n-1}}{n!}\int_{t_1}^{\infty}\cdots\int_{t_n}^{\infty} K\begin{pmatrix}x_1,\dots,x_{j-1},t_j,x_{j+1},\dots,x_n\\ x_1,\dots,x_{j-1},t_j,x_{j+1},\dots,x_n\end{pmatrix}\mathrm{d}x_1\cdots\mathrm{d}x_{j-1}\,\mathrm{d}x_{j+1}\cdots\mathrm{d}x_n.$$

An argument similar to the proof of Lemma 2.2.3 shows that this function is continuous on $(\tau(K),\infty)^n$. Hence, the function is continuously partially differentiable, i.e. continuously differentiable, and the chain rule applied together with the function $s\mapsto(s,\dots,s)$ at $t\in(\tau(K),\infty)$ yields the continuous derivative

$$\begin{aligned} d_n'(t) = \frac{(-1)^{n-1}}{n!}\Big(&\int_t^{\infty}\cdots\int_t^{\infty} K\begin{pmatrix}t,x_2,\dots,x_n\\ t,x_2,\dots,x_n\end{pmatrix}\mathrm{d}x_2\cdots\mathrm{d}x_n\\ &+\int_t^{\infty}\cdots\int_t^{\infty} K\begin{pmatrix}x_1,\dots,x_{j-1},t,x_{j+1},\dots,x_n\\ x_1,\dots,x_{j-1},t,x_{j+1},\dots,x_n\end{pmatrix}\mathrm{d}x_1\cdots\mathrm{d}x_{j-1}\,\mathrm{d}x_{j+1}\cdots\mathrm{d}x_n\\ &+\int_t^{\infty}\cdots\int_t^{\infty} K\begin{pmatrix}x_1,\dots,x_{n-1},t\\ x_1,\dots,x_{n-1},t\end{pmatrix}\mathrm{d}x_1\cdots\mathrm{d}x_{n-1}\Big).\end{aligned} \tag{3.2.10}$$

Setting $C=0$ in (2.2.6), this is bounded by

$$|d_n'(t)|\le\frac{n^{\frac{n}{2}}M(t)^{n-1}}{(n-1)!}\sup_{x\in[2t,0]}|K(x)|.$$

Analogously to the proof of Lemma 2.2.3 or Lemma 2.1.5, the series

$$\frac{\mathrm{d}}{\mathrm{d}t}d_K(\lambda,t)\overset{?}{=}\sum_{n=0}^{\infty}d_n'(t)\lambda^n \text{ for } \lambda\in\mathbb{C} \tag{3.2.11}$$

converges uniformly absolutely in (λ,t) on every closed cube in $\mathbb{C}\times(\tau(K),\infty)$, and thus is continuous. It is a well known fact that the uniform convergence of the continuous derivatives of the partial sums implies that the series is continuously differentiable with derivative (3.2.11). Because of the continuity of D_K (Lemma 2.2.3) and condition $(K3)$, we can apply Remark 3.1.6 to (2.2.7) and (2.2.8) and get the partial derivatives

$$\frac{\partial D_K}{\partial y}(x,y,\lambda,t)=d_K(\lambda,t)K'(x+y)+\lambda\int_t^{\infty}D_K(x,z,\lambda,t)K'(z+y)\,\mathrm{d}z \tag{3.2.12}$$

$$\frac{\partial D_K}{\partial x}(x,y,\lambda,t)=d_K(\lambda,t)K'(x+y)+\lambda\int_t^{\infty}D_K(z,y,\lambda,t)K'(x+z)\,\mathrm{d}z$$

for $x+y\notin\Lambda_K$. An argument similar to (3.2.8), (3.2.9) and Remark 3.1.6 proves that these partial derivatives are continuous in all variables as long as $x+y\notin\Lambda_K$ holds. The integrals alone are continuous without this restriction. Fixing $x,y\in\mathbb{R}$ and repeating the argument for (3.2.10) for a $t\in(\tau(K),\infty)$, we get the partial derivative

$$\begin{aligned} \frac{\partial D_n}{\partial t}(x,y,t) = \frac{(-1)^{n-1}}{n!}\Big(&\int_t^{\infty}\cdots\int_t^{\infty} K\begin{pmatrix}x,t,x_2,\dots,x_n\\ y,t,x_2,\dots,x_n\end{pmatrix}\mathrm{d}x_2\cdots\mathrm{d}x_n\\ &+\int_t^{\infty}\cdots\int_t^{\infty} K\begin{pmatrix}x,x_1,\dots,x_{j-1},t,x_{j+1},\dots,x_n\\ y,x_1,\dots,x_{j-1},t,x_{j+1},\dots,x_n\end{pmatrix}\mathrm{d}x_1\cdots\mathrm{d}x_{j-1}\,\mathrm{d}x_{j+1}\cdots\mathrm{d}x_n\\ &+\int_t^{\infty}\cdots\int_t^{\infty} K\begin{pmatrix}x,x_1,\dots,x_{n-1},t\\ y,x_1,\dots,x_{n-1},t\end{pmatrix}\mathrm{d}x_1\cdots\mathrm{d}x_{n-1}\Big)\end{aligned}$$

for $n \in \mathbb{N}$. An argument similar to the proof of Lemma 2.2.3 shows that this function is continuous on $\mathbb{R}^2 \times (\tau(K), \infty)$. Assuming $(x, y) \in [C, -C]$ for a $C < 0$ and considering (2.2.6), this is bounded by

$$|\frac{\partial D_K}{\partial t}(x, y, t)| \leq \frac{(n+1)^{\frac{n+1}{2}} M(t)^{n-1}}{(n-1)!} \Big(\sup_{x \in [2\min(t,C),0]} |K(x)| \Big)^2.$$

Analogously to (3.2.11), the series

$$\frac{\partial D_K}{\partial t}(x, y, \lambda, t) = \sum_{n=0}^{\infty} \frac{\partial D_n}{\partial t}(x, y, t)\lambda^n \text{ for } \lambda \in \mathbb{C}$$

converges uniformly absolutely in (x, y, λ, t) on every closed cube in $\mathbb{R}^2 \times \mathbb{C} \times (\tau(K), \infty)$ and is the continuous partial derivative of D_K with respect to t. Lemma 2.2.3 states that $D_K(x, y, \lambda, t)$ is entire in λ. Together with (2.2.4) and the fact that power series are differentiated term by term, this shows that the partial derivatives of $D_K(x, y, \lambda, t)$ with respect to $\mathrm{Re}(\lambda)$ and $\mathrm{Im}(\lambda)$ are continuous in all variables[1], too. Summing up, $D_K(x, y, \lambda, t)$ is continuously partially differentiable, i.e. continuously differentiable as long as $x + y \notin \Lambda_K$ holds, and the chain rule applied together with the function $s \mapsto (0, s, 0, 0, s)$ at $t \in (\tau(K), \infty)$ yields the continuous derivative

$$\frac{\mathrm{d}}{\mathrm{d}t}(D_K(x, t, \lambda, t)) = \frac{\partial D_K}{\partial y}(x, t, \lambda, t) + \frac{\partial D_K}{\partial t}(x, t, \lambda, t) \tag{3.2.13}$$

for $x + t \notin \Lambda_K$ and $\lambda \in \mathbb{C}$.

Fixing $x \in \mathbb{R}$, we proved that the function

$$\frac{D_K(x, t, -1, t)}{d_K(-1, t)} \tag{3.2.14}$$

is continuously differentiable in t on $(\tau(K), \infty) \setminus (\Lambda_K - x)$. The continuity of the integral in (3.2.12), condition $(K3)$ and the already shown continuity of $d_K(\lambda, t)$ and $\frac{\partial D_K}{\partial t}(x, t, \lambda, t)$ yield that the derivative of (3.2.14) with respect to t is measurable and bounded on every closed cube in $\mathbb{R} \times (\tau(K), \infty)$. Looking at (3.2.6) and using the mean value theorem for definite integrals and condition $(K3)$, we can also show existence of an upper bound of the differential quotient of (3.2.14) similar to the one in condition $(K3)$. Hence, we have all the necessary properties to apply Lemmas A.1.1 and A.1.2. With a differential quotient version of (3.2.8) and (3.2.9) and a version with (3.2.13) in the integrand, we can now differentiate the integral in (3.2.7) and get the continuous derivative

$$\int_t^{\infty} K(x+y) \frac{\mathrm{d}}{\mathrm{d}t} \Big(\frac{D_K(y, t, -1, t)}{d_K(-1, t)} \Big) \, \mathrm{d}y - K(x+t) \frac{D_K(t, t, -1, t)}{d_K(-1, t)}.$$

This and condition $(K3)$ prove $\phi_K(., x) \in C^1((\tau(K), \infty) \setminus (\Lambda_K - x), \mathbb{R})$. ■

Remark 3.2.3. In the proof of Lemma 3.2.2, we showed $d_K(\pm 1, t) \neq 0$ for $t \in (\tau(K), 0]$ for an $K \in \mathcal{K}$. By Lemma 2.2.3, we have $d_K(\pm 1, t) = 1$ for $t \geq 0$ and that $d_K(\pm 1, t)$ is continuous on $(\tau(K), \infty)$. The definition of d_0, (2.2.3) and condition $(K1)$ yield that $d_K(\pm 1, t)$ is a real-valued function. Therefore, the intermediate value theorem proves $d_K(\pm 1, t) > 0$ on $(\tau(K), \infty)$.

We also have seen that $d_K(\lambda, t)$ is continuously differentiable with respect to t for any $\lambda \in \mathbb{C}$ on $(\tau(K), \infty)$ with derivative (3.2.11). //

[1] For a analytic function f, we have $f'(z) = \frac{\partial f}{\partial x}(z) = -i\frac{\partial f}{\partial y}(z)$, where $z = x + iy$.

Remark 3.2.4. For an $K \in \mathcal{K}$, the version of (3.2.7) for $\psi_{K,t}, t \in (\tau(K), 0]$ is

$$\psi_{K,t}(x) = K(x+t) + \int_t^\infty K(x+y) \frac{D_K(y,t,1,t)}{d_K(1,t)} \, \mathrm{d}y \tag{3.2.15}$$

for $x \in \mathbb{R}$. //

Before we return to $\Phi_{K,t}$ and $\Psi_{K,t}$, we need one more property of the diagonals of ϕ_K and ψ_K. They turn out to be the logarithmic derivatives of the generalized Fredholm determinant at ± 1.

Lemma 3.2.5. *For an $K \in \mathcal{K}$ and $t \geq \tau(K)$, the identities*

$$\begin{aligned} -\frac{\mathrm{d}}{\mathrm{d}t} \ln(d_K(-1,t)) &= -\frac{\frac{\mathrm{d}}{\mathrm{d}t} d_K(-1,t)}{d_K(-1,t)} = \phi_K(t,t) \\ \frac{\mathrm{d}}{\mathrm{d}t} \ln(d_K(1,t)) &= \frac{\frac{\mathrm{d}}{\mathrm{d}t} d_K(1,t)}{d_K(1,t)} = \psi_K(t,t) \end{aligned} \tag{3.2.16}$$

and

$$\exp\Big(\int_t^0 \psi_K(s,s) + \phi_K(s,s) \, \mathrm{d}s \Big) = \frac{d_K(-1,t)}{d_K(1,t)} \tag{3.2.17}$$

hold. In particular, the functions $\phi_K(t,t)$ and $\psi_K(t,t)$ are continuous on $(\tau(K), \infty)$.

Proof. For $t \geq 0$, Lemma 3.2.1 yields $\psi_K(t,t) = 0 = \phi_K(t,t)$, and by Lemma 2.2.3, we have $d_K(\pm 1, t) = 1$. Thus, all the claimed properties are obvious.

We now consider the case $t \in (\tau(K), 0)$. Looking at (2.2.1), we can rewrite (3.2.10) for $n \in \mathbb{N}$. As we have to swap the same amount of adjacent rows and adjacent columns if we want to bring the two t entries in K_E to the first column, we have

$$K\begin{pmatrix} x_1, \dots, x_{j-1}, t, x_{j+1}, \dots, x_n \\ x_1, \dots, x_{j-1}, t, x_{j+1}, \dots, x_n \end{pmatrix} = K\begin{pmatrix} t, x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_n \\ t, x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_n \end{pmatrix} \tag{3.2.18}$$

for $j \in \{2, \dots, n-1\}$ and

$$K\begin{pmatrix} x_1, \dots, x_{n-1}, t \\ x_1, \dots, x_{n-1}, t \end{pmatrix} = K\begin{pmatrix} t, x_1, \dots, x_{n-1} \\ t, x_1, \dots, x_{n-1} \end{pmatrix}.$$

Renaming $x_{j-1}, x_{j+1}, \dots, x_n$ with $x_j, \dots, x_{n-1}$ in (3.2.18), we rewrite (3.2.10) to

$$d_n'(t) = \frac{(-1)^{n-1}}{(n-1)!} \int_t^\infty \dots \int_t^\infty K\begin{pmatrix} t, x_1, \dots, x_{n-1} \\ t, x_1, \dots, x_{n-1} \end{pmatrix} \mathrm{d}x_2 \cdots \mathrm{d}x_n = D_{n-1}(t,t,t). \tag{3.2.19}$$

In the case $n = 1$, the definition of d_0 and (3.2.19) yield

$$d_1'(t) = K(t+t) = K(t+t) d_0(t). \tag{3.2.20}$$

For $n \geq 2$, we use Remark 2.2.4. In our case, the recurrence equation (2.2.10) reads

$$d_n'(t) = K(t+t) d_{n-1}(t) + \int_t^\infty K(t+y) D_{n-2}(y,t,t) \, \mathrm{d}y. \tag{3.2.21}$$

We now multiply (3.2.20) by -1 and (3.2.21) by $(-1)^n$ and sum over n. Lemma 2.2.3 and condition $(K1)$ guarantee uniform convergence in y of the continuous partial sums of the integrands since we can replace ∞ with $-t$ in the integral boundary. Hence, the use of the dominated convergence theorem to interchange integral and series is justified. Considering (2.2.2), (2.2.4) and (3.2.11), the recurrence equation (3.2.21) turns into

$$\frac{\mathrm{d}}{\mathrm{d}t}d_K(-1,t) = -K(t+t)d_K(-1,t) + \int_t^\infty K(t+y)D_K(y,t,-1,t)\,\mathrm{d}y.$$

Dividing by $-d_K(-1,t)$ is possible because of Remark 3.2.3, and we get (3.2.7):

$$-\frac{\frac{\mathrm{d}}{\mathrm{d}t}d_K(-1,t)}{d_K(-1,t)} = K(x+t) - \int_t^\infty K(x+y)\frac{D_K(y,t,-1,t)}{d_K(-1,t)}\,\mathrm{d}y = \phi_K(t,t).$$

If we sum (3.2.20) and (3.2.21) over n without multiplying, the recurrence equation (3.2.21) turns into

$$\frac{\mathrm{d}}{\mathrm{d}t}d_K(1,t) = K(t+t)d_K(1,t) + \int_t^\infty K(t+y)D_K(y,t,1,t)\,\mathrm{d}y.$$

Recalling (3.2.15), this yields

$$\frac{\frac{\mathrm{d}}{\mathrm{d}t}d_K(1,t)}{d_K(1,t)} = K(x+t) + \int_t^\infty K(x+y)\frac{D_K(y,t,1,t)}{d_K(1,t)}\,\mathrm{d}y = \psi_K(t,t).$$

As Remark 3.2.3 states that $d_K(\pm 1,t)$ is continuously differentiable on $(\tau(K),\infty)$, we deduce continuity of $\phi_K(t,t)$ and $\psi_K(t,t)$. Furthermore, Remark 3.2.3 shows $d_K(\pm 1,t) > 0$ on $(\tau(K),\infty)$. Hence, the logarithm of the generalized Fredholm determinant is defined. Differentiating it with respect to t, we get (3.2.16). Using these identities, we can calculate

$$\begin{aligned}\exp\Big(\int_t^0 \psi_K(s,s)+\phi_K(s,s)\,\mathrm{d}s\Big) &= \exp\Big(\int_t^0 \frac{\mathrm{d}}{\mathrm{d}s}\ln(d_K(1,s)) - \frac{\mathrm{d}}{\mathrm{d}s}\ln(d_K(-1,s))\,\mathrm{d}s\Big)\\ &= \exp(\ln(d_K(-1,t)) - \ln(d_K(1,t)))\\ &= \frac{d_K(-1,t)}{d_K(1,t)}.\end{aligned}$$

■

We are now ready to return to $\Phi_{K,t}$ and $\Psi_{K,t}$. The next theorem (cf. [9, Proposition 2.3.]) shows a fundamental connection between $\Phi_{K,t}$, $\Psi_{K,t}$ and $\phi_{K,t}$, $\psi_{K,t}$ that allows us to lift properties that we just showed.

Theorem 3.2.6. *Fix any $K \in \mathcal{K}$. Then, the functions $\Phi_K(.,x)$ and $\Psi_K(.,x)$ are continuously differentiable on $(\tau(K),\infty)$ for every $x \in \mathbb{R}$. Moreover, the identities*

$$\phi_K(t,x) = \frac{1}{\Phi_K(t,t)}\frac{\partial}{\partial t}\Phi_K(t,x) = -\frac{1}{\Psi_K(t,t)}\frac{\partial}{\partial x}\Psi_K(t,x) \tag{3.2.22}$$

$$\psi_K(t,x) = \frac{1}{\Phi_K(t,t)}\frac{\partial}{\partial x}\Phi_K(t,x) = -\frac{1}{\Psi_K(t,t)}\frac{\partial}{\partial t}\Psi_K(t,x) \tag{3.2.23}$$

hold for $(t,x) \in (\tau(K),\infty)\times\mathbb{R}$.

Proof. First, we recall (3.1.12) and Remark 3.1.8. For $(t,x) \in (\tau(K),\infty)\times\mathbb{R}$, this means

$$\frac{\mathrm{d}}{\mathrm{d}x}\Phi_K(t,x) = \int_t^{-x} K(x+y)\frac{\mathrm{d}}{\mathrm{d}y}\Phi_K(t,y)\,\mathrm{d}y + K(x+t)\Phi_K(t,t) \tag{3.2.24}$$

Because of Lemma 3.1.7, we have $\Phi_K(t,t) \neq 0$ and continuity of $\frac{\mathrm{d}\Phi_{K,t}}{\mathrm{d}x}(x)$ on $\mathbb{R}$ for a fixed t. In particular, we have $\frac{\mathrm{d}\Phi_{K,t}}{\mathrm{d}x}(x)/\Phi_K(t,t) \in L^1_{\mathrm{loc}}(\mathbb{R})$ for a fixed t. Dividing (3.2.24) by $\Phi_K(t,t)$ and replacing $-x$ with ∞ by condition $(K2)$ yield

$$\frac{1}{\Phi_K(t,t)}\frac{\partial}{\partial x}\Phi_K(t,x) = \int_t^\infty K(x+y)\frac{1}{\Phi_K(t,t)}\frac{\partial}{\partial x}\Phi_K(t,x)\,\mathrm{d}y + K(x+t).$$

Together with (3.2.2) and the uniqeness statement in Lemma 3.2.1, this proves the first equal sign of (3.2.23). Similarly, we get the second one of (3.2.22) via (3.1.13) and (3.2.1) and

$$\frac{1}{\Psi_K(t,t)}\frac{\partial}{\partial x}\Psi_K(t,x) = -\int_t^\infty K(x+y)\frac{1}{\Psi_K(t,t)}\frac{\partial}{\partial x}\Psi_K(t,x)\,\mathrm{d}y - K(x+t).$$

The continuity in t is just a slight modification of the proof of the continuity of ϕ_K and ψ_K in t (Lemma 3.2.2). Hence, we only outline it. As $\Phi_K(t,x) = 1 = \Psi_K(t,x)$ holds for $t \in (\tau(K),\infty)$ and $x > -t$ by Lemma 3.1.7, we can use condition $(K2)$ to rewrite (3.1.4) to

$$\Phi_K(t,x) = 1 - \int_t^{-t} K(x+y)\Phi_K(t,y)\,\mathrm{d}y - \int_{-t}^{-x} K(x+y)\,\mathrm{d}y \tag{3.2.25}$$

and (3.1.5) to

$$\Psi_K(t,x) = 1 + \int_t^{-t} K(x+y)\Psi_K(t,y)\,\mathrm{d}y + \int_{-t}^{-x} K(x+y)\,\mathrm{d}y \tag{3.2.26}$$

for $(t,x) \in (\tau(K),\infty) \times \mathbb{R}$. In the case $x \geq t$, the second integrals vanish because of condition $(K2)$. Thus, (3.2.25) and (3.2.26) are integral equations of the form (2.1.1) if we fix t, and we can represent $\Phi_K(t,x)$ and $\Psi_K(t,x)$ for $x \in [t,-t]$ in terms of the generalized Fredholm determinant and first minor (Fredholm alternative, Lemma 2.2.3 and Remark 3.2.3):

$$\begin{aligned}\Phi_K(t,y) &= 1 - \int_t^{-t} \frac{D_K(y,z,-1,t)}{d_K(-1,t)}\,\mathrm{d}z\\ \Psi_K(t,y) &= 1 + \int_t^{-t} \frac{D_K(y,z,1,t)}{d_K(1,t)}\,\mathrm{d}z.\end{aligned} \tag{3.2.27}$$

We can extend these identities to $x \geq t$ since both sides are 1 for $x \geq -t$ by Lemma 3.1.7 and by Lemma 2.2.3. Using these expressions in the integrands of (3.1.4) and (3.1.5), we can now proof continuity in t with a double integral versions of (3.2.8) and (3.2.9).

The proof of continuous differentiability with respect to t for a fixed $x \in \mathbb{R}$ is just a modified version of the proof of continuous differentiability in Lemma 3.2.2 using (3.2.27) instead of (3.2.14). Thus, we omit it and only give the results:

$$\begin{aligned}\frac{\partial}{\partial t}\Phi_K(t,x) &= -\int_t^\infty K(x+y)\frac{\partial}{\partial t}\Phi_K(t,y)\,\mathrm{d}y + K(x+t)\Phi_K(t,t)\\ \frac{\partial}{\partial t}\Psi_K(t,x) &= \int_t^\infty K(x+y)\frac{\partial}{\partial t}\Psi_K(t,y)\,\mathrm{d}y - K(x+t)\Psi_K(t,t).\end{aligned} \tag{3.2.28}$$

Remark 3.1.6 and condition $(K1)$ even give continuity of (3.2.28) in x on $\mathbb{R}$ for a fixed $t \in (\tau(K),\infty)$. Because of Lemma 3.1.7, we have $\Phi_K(t,t) \neq 0 \neq \Psi_K(t,t)$. The continuity of $\frac{\partial}{\partial t}\Phi_{K,t}(x)$ and $\frac{\partial}{\partial t}\Psi_{K,t}(x)$ on $\mathbb{R}$ for a fixed $t \in (\tau(K),\infty)$ shows

$$\frac{\partial}{\partial t}\frac{\Phi_{K,t}(x)}{\Phi_K(t,t)}, \frac{\partial}{\partial t}\frac{\Psi_{K,t}(x)}{\Psi_K(t,t)} \in L^1_{\mathrm{loc}}(\mathbb{R}).$$

Dividing by $\Phi_K(t,t)$ and $\Psi_K(t,t)$ respectively and comparing (3.2.28) with (3.2.1) and (3.2.2), the uniqeness statement in Lemma 3.2.1 proves the first equal sign of (3.2.22) and the second one of (3.2.23). ■

If we look at (3.1.14) and recall the defintion of a Hamiltonian, the question arises wheter Φ_K and Ψ_K are continuously differentiable on the diagonal. We already proved a continuous variant for ϕ_K and ψ_K in Lemma 3.2.5. Luckily, the answer turns out to be yes:

Lemma 3.2.7. *For an $K \in \mathcal{K}$, the functions $\Phi_K(t,t)$ and $\Psi_K(t,t)$ are continuously differentiable on $(\tau(K), \infty)$. Moreover, the identity*

$$\Psi_K(t,t) = \frac{1}{\Phi_K(t,t)} = \frac{d_K(-1,t)}{d_K(1,t)} \tag{3.2.29}$$

holds on $(\tau(K), \infty)$.

Proof. This proof follows the proof of [9, Theorem 1.4.].

The proof of the continuous differentiability is once again a tedious but not difficult modification of the proof of continuous differentiability in Lemma 3.2.2 using (3.2.27) instead of (3.2.14). Hence, we omit it and only give the results[2]:

$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi_K(t,t) = -\int_t^\infty K(t+y)\frac{\partial}{\partial t}\Phi_K(t,y)\,\mathrm{d}y + \int_t^\infty K(t+y)\frac{\partial}{\partial y}\Phi_K(t,y)\,\mathrm{d}y + 2K(t+t)\Phi_K(t,t)$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\Psi_K(t,t) = \int_t^\infty K(t+y)\frac{\partial}{\partial t}\Psi_K(t,y)\,\mathrm{d}y - \int_t^\infty K(t+y)\frac{\partial}{\partial y}\Psi_K(t,y)\,\mathrm{d}y - 2K(t+t)\Psi_K(t,t)$$

for $t \in (\tau(K), \infty)$. Using (3.2.22) and (3.2.23), we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi_K(t,t) - 2K(2t)\Phi_K(t,t) - \Phi_K(t,t)\int_t^\infty K(t+y)(\psi_K(t,y) - \phi_K(t,y)))\,\mathrm{d}y = 0 \tag{3.2.30}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}\Psi_K(t,t) + 2K(2t)\Psi_K(t,t) + \Psi_K(t,t)\int_t^\infty K(t+y)(\psi_K(t,y) - \phi_K(t,y)))\,\mathrm{d}y = 0. \tag{3.2.31}$$

On the other hand, we can add (3.2.1) and (3.2.2) together and set $x = t$ to get

$$\phi_K(t,t) + \psi_K(t,t) = 2K(2t) + \int_t^\infty K(t+y)(\psi_K(t,y) - \phi_K(t,y)))\,\mathrm{d}y. \tag{3.2.32}$$

Substituting (3.2.32) in (3.2.30) and (3.2.31) yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi_K(t,t) - (\phi_K(t,t) + \psi_K(t,t))\Phi_K(t,t) = 0$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\Psi_K(t,t) + (\phi_K(t,t) + \psi_K(t,t))\Psi_K(t,t) = 0.$$

By Lemma 3.2.5, we have continuity of $\phi_K(t,t)$ and $\psi_K(t,t)$ on $(\tau(K), \infty)$. Therefore using the identity (3.2.17), these ordinary differential equations are solvable with solutions

$$\Phi_K(t,t) = C_1 \exp\Big(-\int_t^0 \psi_K(s,s) + \phi_K(s,s)\,\mathrm{d}s\Big) = C_1\frac{d_K(1,t)}{d_K(-1,t)}$$
$$\Psi_K(t,t) = C_2 \exp\Big(\int_t^0 \psi_K(s,s) + \phi_K(s,s)\,\mathrm{d}s\Big) = C_2\frac{d_K(-1,t)}{d_K(1,t)}$$

[2] We can guess these derivatives with the following ansatz: If Φ_K were continuously differentiable, we could compute the desired directional derivative with respect to $(1,1)^t$ via (3.1.12) and (3.2.28).

for some $C_1, C_2 \in \mathbb{R}$ and $t \in (\tau(K), \infty)$. Setting $t = 0$, Lemma 3.1.7 and Remark 3.2.3 show

$$1 = \Phi_K(0,0) = C_1 \frac{d_K(1,0)}{d_K(-1,0)} = C_1$$
$$1 = \Psi_K(0,0) = C_2 \frac{d_K(-1,0)}{d_K(1,0)} = C_2.$$

■

Remark 3.2.8. Combining Lemma 3.2.1, Lemma 3.2.2, Theorem 3.2.6 and Lemma 3.2.7, we see that $\frac{\partial}{\partial t}\Phi_K(t,x)$, $\frac{\partial}{\partial x}\Phi_K(t,x)$, $\frac{\partial}{\partial t}\Psi_K(t,x)$ and $\frac{\partial}{\partial x}\Psi_K(t,x)$ are continuous in t on $(\tau(K), \infty)$ for a fixed $x \in \mathbb{R}$ and continuous in x on $\mathbb{R}$ for a fixed $t \in (\tau(K), \infty)$. //

We close this section with two useful corollaries of Theorem 3.2.6 and Lemma 3.2.7. They are based on [9, Propositions 2.4. and 2.5.].

Corollary 3.2.9. *For an $K \in \mathcal{K}$, the integral equations*

$$\Psi_K(t,x) = 1 + \frac{1}{\Phi_E(t,t)^2} \int_x^{-t} \frac{\partial}{\partial t}\Phi_K(t,y)\,\mathrm{d}y \tag{3.2.33}$$
$$\Phi_K(t,x) = 1 + \frac{1}{\Psi_E(t,t)^2} \int_x^{-t} \frac{\partial}{\partial t}\Psi_K(t,y)\,\mathrm{d}y \tag{3.2.34}$$

hold for $(t,x) \in (\tau(K), \infty) \times \mathbb{R}$.

Proof. Integrating (3.2.22) and (3.2.23), $\Psi_K(t,-t) = 1 = \Phi_K(t,-t)$ by Lemma 3.1.7 yields

$$\int_x^{-t} \phi_K(t,y)\,\mathrm{d}y = -\frac{1}{\Psi_K(t,t)} \int_x^{-t} \frac{\partial}{\partial y}\Psi_K(t,y)\,\mathrm{d}y = -\frac{1}{\Psi_K(t,t)} + \frac{\Psi_K(t,x)}{\Psi_K(t,t)}$$
$$\int_x^{-t} \psi_K(t,y)\,\mathrm{d}y = \frac{1}{\Phi_K(t,t)} \int_x^{-t} \frac{\partial}{\partial y}\Phi_K(t,y)\,\mathrm{d}y = \frac{1}{\Phi_K(t,t)} - \frac{\Phi_K(t,x)}{\Phi_K(t,t)}.$$

Multiplying by $\Psi_K(t,t)$ and $\Phi_K(t,t)$ respectively and applying (3.2.29), we get

$$\begin{aligned}\Psi_K(t,x) &= 1 + \Psi_K(t,t) \int_x^{-t} \phi_K(t,y)\,\mathrm{d}y = 1 + \frac{1}{\Phi_K(t,t)} \int_x^{-t} \phi_K(t,y)\,\mathrm{d}y \\ \Phi_K(t,x) &= 1 - \Phi_K(t,t) \int_x^{-t} \psi_K(t,y)\,\mathrm{d}y = 1 - \frac{1}{\Psi_K(t,t)} \int_x^{-t} \psi_K(t,y)\,\mathrm{d}y.\end{aligned} \tag{3.2.35}$$

Applying Theorem 3.2.6 shows (3.2.33) and (3.2.34):

$$\Psi_K(t,x) = 1 + \frac{1}{\Phi_K(t,t)} \int_x^{-t} \frac{1}{\Phi_K(t,t)} \frac{\partial}{\partial t}\Phi_K(t,y)\,\mathrm{d}y$$
$$\Phi_K(t,x) = 1 - \frac{1}{\Psi_K(t,t)} \int_x^{-t} -\frac{1}{\Psi_K(t,t)} \frac{\partial}{\partial t}\Psi_K(t,y)\,\mathrm{d}y.$$

■

Corollary 3.2.10. *For an $K \in \mathcal{K}$ and $t \in (\tau(K), \infty)$, we have*

$$\frac{d_K(1,t)}{d_K(-1,t)} = 1 - \int_t^{-t} \phi_K(t,y)\,\mathrm{d}y \tag{3.2.36}$$
$$\frac{d_K(-1,t)}{d_K(1,t)} = 1 + \int_t^{-t} \psi_K(t,y)\,\mathrm{d}y. \tag{3.2.37}$$

Proof. Setting $x = t$ and dividing by $\Psi_K(t,t)$ and $\Phi_K(t,t)$ respectively in (3.2.35), we get

$$1 = \frac{1}{\Psi_K(t,t)} + \int_t^{-t} \phi_K(t,y)\,\mathrm{d}y$$
$$1 = \frac{1}{\Phi_K(t,t)} - \int_t^{-t} \psi_K(t,y)\,\mathrm{d}y.$$

(3.2.36) and (3.2.37) now follow from Lemma 3.2.7. ■

3.3 Proof of Theorem 3.1.9

Before we get to the actual proof, we prove yet another useful corollary of Theorem 3.2.6 and Lemma 3.2.7 (cf. [9, Lemma 2.1.]):

Corollary 3.3.1. *For an $E \in \mathcal{E}$, Fixing $t \in (\tau(K_E), \infty)$, (3.1.15) and (3.1.16) define complex-valued functions for* $\mathrm{Im}(z) > c(K_E)$*. Moreover using the notation of Theorem 3.1.9, we have*

$$A_{E,t}(z) = \frac{d_{K_E}(-1,t)}{d_{K_E}(1,t)} \frac{1}{2} E(z) \Big(\exp(-izt) + \int_{-\infty}^t \phi_{K_E,t}(x) \exp(-izx)\,\mathrm{d}x \Big) \tag{3.3.1}$$
$$B_{E,t}(z) = \frac{d_{K_E}(1,t)}{d_{K_E}(-1,t)} \frac{i}{2} E(z) \Big(\exp(-izt) - \int_{-\infty}^t \psi_{K_E,t}(x) \exp(-izx)\,\mathrm{d}x \Big). \tag{3.3.2}$$

for $(t,z) \in (\tau(K_E), \infty) \times \mathbb{C}$ *with* $\mathrm{Im}(z) > c(K_E)$.

Proof. By condition $(E1)$, $E(z)$ is analytic for $\mathrm{Im}(z) > c(K_E)$ and therefore defines a function. The integrals in (3.1.15), (3.1.16), (3.3.1) and (3.3.2) exist for $\mathrm{Im}(z) > c(K_E)$ and a fixed $t \in (\tau(K_E), \infty)$ because Lemma 3.1.7 and Lemma 3.2.1 yield continuous $\Phi_{K_E,t}(x), \Psi_{K_E,t}(x) \in O(\exp(c'|x|))$ for any $c' > c(K_E)$ and continuous $\phi_{K_E,t}(x), \psi_{K_E,t}(x) \in O(\exp(c(K_E)|x|))$. More precisely, we have, for example,

$$\Big| \int_{-\infty}^t \Psi_{K_E,t}(x) \exp(-izx)\,\mathrm{d}x \Big| \le \int_{-\infty}^t C \exp(c'|x|) |\exp(-izx)|\,\mathrm{d}x \tag{3.3.3}$$
$$\le \int_0^t C \exp(c'x) |\exp(-izx)|\,\mathrm{d}x + \int_0^\infty C \exp(c'x) \exp(\mathrm{Re}(iz)x)\,\mathrm{d}x$$

for a constant $C > 0$. The first integral always exists and the second one for $\mathrm{Im}(-z) = \mathrm{Re}(iz) < -c'$. Ignoring the integral signs, this also proves $\Psi_{K_E,t}(x) \exp(-izx) \to 0$ for $x \to -\infty$ and $\mathrm{Im}(z) > c(K_E)$. Using Theorem 3.2.6 for $(t,z) \in (\tau(K_E), \infty) \times \mathbb{C}$ with $\mathrm{Im}(z) > c(K_E)$, we get

$$\exp(izt) + \int_{-\infty}^t \phi_{K_E}(t,x) \exp(-izx)\,\mathrm{d}x = \exp(-izt) - \int_{-\infty}^t \frac{\frac{\partial}{\partial x} \Psi_{K_E}(t,x)}{\Psi_{K_E}(t,t)} \exp(-izx)\,\mathrm{d}x$$
$$\exp(izt) - \int_{-\infty}^t \psi_{K_E}(t,x) \exp(-izx)\,\mathrm{d}x = \exp(-izt) - \int_{-\infty}^t \frac{\frac{\partial}{\partial x} \Phi_{K_E}(t,x)}{\Phi_{K_E}(t,t)} \exp(-izx)\,\mathrm{d}x.$$

$\Psi_{K_E,t}(x) \exp(-izx) \to 0$ and $\Phi_{K_E,t}(x) \exp(-izx) \to 0$ for $x \to -\infty$ and the existence of all involved integrals allow us to use integration by parts:

$$\exp(izt) + \int_{-\infty}^t \phi_{K_E}(t,x) \exp(-izx)\,\mathrm{d}x = -iz \int_{-\infty}^t \frac{\Psi_{K_E}(t,x)}{\Psi_{K_E}(t,t)} \exp(-izx)\,\mathrm{d}x$$
$$\exp(izt) - \int_{-\infty}^t \psi_{K_E}(t,x) \exp(-izx)\,\mathrm{d}x = -iz \int_{-\infty}^t \frac{\Phi_{K_E}(t,x)}{\Phi_{K_E}(t,t)} \exp(-izx)\,\mathrm{d}x.$$

Substituting (3.2.29) in the right integrals and using the results in (3.3.1) and (3.3.2) yield (3.1.15) and (3.1.16), i.e. A_E and B_E. ■

Now, we get to the core part of this thesis (cf. [9, Section 3]):

Proof. (Theorem 3.1.9)
First, H_E is continuous on $(\tau(K_E), \infty)$ by Lemma 3.2.7 and the same lemma shows

$$\det H_E(t) = \begin{vmatrix} \Phi_{K_E}(t,t)^2 & 0 \\ 0 & \Psi_{K_E}(t,t)^2 \end{vmatrix} = \begin{vmatrix} \Phi_{K_E}(t,t)^2 & 0 \\ 0 & \Phi_{K_E}(t,t)^{-2} \end{vmatrix} = 1$$

for all $t \in (\tau(K_E), \infty)$. A singular point would imply that 0 would locally be an eigenvalue of the Hamiltonian. Hence, the determinant would locally be 0. Therefore, the constant value 1 of the determinant shows that all $t \in (\tau(K_E), \infty)$ are regular points of H_E.
Fix $t \in (\tau(K_E), \infty)$. By Corollary 3.3.1, both (3.1.15) and (3.1.16) define complex-valued functions for $\mathrm{Im}(z) > c(K_E)$. Thus, we want to find two analytic functions on $\mathbb{C} \setminus Z_E$ that coincide with (3.1.14) and (3.1.15) respectively for $\mathrm{Im}(z) > c(K_E)$: We claim that

$$\begin{aligned} \mathbf{1}_{(-\infty,t]}(x)\Psi_{K_E}(t,x) &= \mathbf{1}_{(-t,\infty)}(x) - \mathbf{1}_{(t,\infty)}(x)\Psi_{K_E}(t,x) + \mathbf{1}_{(-\infty,-t]}(x) \\ &\quad + \int_t^\infty K_E(x+y)\Psi_{K_E}(t,y)\,\mathrm{d}y \\ \mathbf{1}_{(-\infty,t]}(x)\Phi_{K_E}(t,x) &= \mathbf{1}_{(-t,\infty)}(x) - \mathbf{1}_{(t,\infty)}(x)\Phi_{K_E}(t,x) + \mathbf{1}_{(-\infty,-t]}(x) \\ &\quad - \int_t^\infty K_E(x+y)\Phi_{K_E}(t,y)\,\mathrm{d}y \end{aligned} \tag{3.3.4}$$

holds for $x \in \mathbb{R}$. In fact, (3.3.4) is just an rearrangement of (3.1.4) and (3.1.5) if we consider these two equations once for $x \leq t$ and once for $x > t$ and then take the sum. By Lemma 3.1.7, we have

$$\Phi_{K_E}(t,x) = 1 = \Psi_{K_E}(t,x)$$

for $x > -t$ which proves

$$\mathbf{1}_{(-t,\infty)}(x) - \mathbf{1}_{(t,\infty)}(x)\Psi_{K_E}(t,x) = -\mathbf{1}_{(t,-t)}(x)\Psi_{K_E}(t,x)$$

and

$$\mathbf{1}_{(-t,\infty)}(x) - \mathbf{1}_{(t,\infty)}(x)\Phi_{K_E}(t,x) = -\mathbf{1}_{(t,-t)}(x)\Phi_{K_E}(t,x).$$

Applying the complex Fourier transform to (3.3.4) for $\mathrm{Im}(z) > c(K_E)$ yields

$$\begin{aligned} \int_{-\infty}^t \Psi_{K_E}(t,x)\exp(-izx)\,\mathrm{d}x &= -\int_t^{-t} \Psi_{K_E}(t,x)\exp(-izx)\,\mathrm{d}x + \int_{-\infty}^{-t}\exp(-izx)\,\mathrm{d}x \\ &\quad + \int_{-\infty}^\infty \exp(-izx)\int_t^\infty K_E(x+y)\Psi_{K_E}(t,y)\,\mathrm{d}y\,\mathrm{d}x \\ \int_{-\infty}^t \Phi_{K_E}(t,x)\exp(-izx)\,\mathrm{d}x &= -\int_t^{-t} \Phi_{K_E}(t,x)\exp(-izx)\,\mathrm{d}x + \int_{-\infty}^{-t}\exp(-izx)\,\mathrm{d}x \\ &\quad - \int_{-\infty}^\infty \exp(-izx)\int_t^\infty K_E(x+y)\Phi_{K_E}(t,y)\,\mathrm{d}y\,\mathrm{d}x \end{aligned} \tag{3.3.5}$$

where application on the left side is justified by Corollary 3.3.1. The first integrals on the right side have integrands continuous in (t, z) on $(t, -t) \times \mathbb{C}$ and analytic in z. Therefore, Lemma A.1.3 applies and they are entire in z. The second integrals can be calculated:

$$\int_{-\infty}^{-t} \exp(-izx)\,\mathrm{d}x = \frac{\exp(izt)}{-iz} - \lim_{s\to-\infty}\frac{\exp(-izs)}{-iz}. \tag{3.3.6}$$

Since $c(K_E) < \mathrm{Im}(z)$ is positive by condition $(K1)$, we have $\mathrm{Re}(-iz) > 0$ and the limit exists and is 0. By Lemma 3.1.7 and condition $(K1)$, the third integrals (considered as double integrals) have integrands that are continuous in (x, y) on $\mathbb{R} \times (t, \infty)$. Looking at (3.1.5), we see that $\Psi_{K_E}(t, x) \in O(\exp(c'|x|))$ for any $c' > c(K)$ from Lemma 3.1.7 implies

$$\int_t^\infty K_E(x+y)\Psi_{K_E}(t,y)\,\mathrm{d}y \in O(\exp(c'|x|))$$

for any $c' > c(K)$. Because $K_E(x+y)$ vanishes for $x > -t$ and $y > t$ by condition $(K2)$, we thus get

$$\begin{aligned}\int_{-\infty}^\infty \int_t^\infty &|\exp(-izx)K_E(x+y)\Psi_{K_E}(t,y)|\,\mathrm{d}y\,\mathrm{d}x \\ &= \int_{-\infty}^{-t} |\exp(-izx)| \int_t^\infty |K_E(x+y)\Psi_{K_E}(t,y)|\,\mathrm{d}y\,\mathrm{d}x \\ &\leq \int_{-\infty}^{-t} C\exp(c'|x|)|\exp(-izx)|\,\mathrm{d}x < \infty\end{aligned}$$

for $\mathrm{Im}(z) > c' > c(K_E)$ and a constant $C > 0$ using an argument similar to the existence proof (3.3.3) in Corollary 3.3.1. Since $c' > c(K_E)$ is arbitrarily chosen, this proves existence of the third integrals and justifies the use of Fubini's theorem to get

$$\begin{aligned}\int_{-\infty}^\infty &\exp(-izx)\int_t^\infty K_E(x+y)\Psi_{K_E}(t,y)\,\mathrm{d}y\,\mathrm{d}x \\ &= \int_t^\infty \int_{-\infty}^\infty \exp(-izx)K_E(x+y)\Psi_{K_E}(t,y)\,\mathrm{d}x\,\mathrm{d}y \\ &= \int_t^\infty \exp(izy)\Psi_{K_E}(t,y)\int_{-\infty}^\infty \exp(-iz(x+y))K_E(x+y)\,\mathrm{d}x\,\mathrm{d}y \\ &= \frac{E^\#(z)}{E(z)}\Big(\int_t^{-t} \exp(izy)\Psi_{K_E}(t,y)\,\mathrm{d}y + \int_{-t}^\infty \overbrace{\exp(izy)\Psi_{K_E}(t,y)}^{=\exp(izy)\text{ by Lemma 3.1.7}}\,\mathrm{d}y\Big) \\ &= \frac{E(-z)}{E(z)}\Big(\int_t^{-t} \exp(izy)\Psi_{K_E}(t,y)\,\mathrm{d}y + \underbrace{\frac{\exp(-izt)}{-iz}\Big)}_{\text{argument similar to (3.3.6)}}\end{aligned}$$

for $\mathrm{Im}(z) > c(K_E)$ by conditions $(E1)$ and $(E2)$. A similar result is obtained for Φ_{K_E}. To sum up, (3.3.5) turns into

$$\begin{aligned}\int_{-\infty}^t \Psi_{K_E}(t,x)\exp(-izx)\,\mathrm{d}x &= -\int_t^{-t} \Psi_{K_E}(t,x)\exp(-izx)\,\mathrm{d}x + \frac{\exp(izt)}{-iz} \\ &\quad + \frac{E(-z)}{E(z)}\Big(\int_t^{-t} \exp(izy)\Psi_{K_E}(t,y)\,\mathrm{d}y + \frac{\exp(-izt)}{-iz}\Big) \\ \int_{-\infty}^t \Phi_{K_E}(t,x)\exp(-izx)\,\mathrm{d}x &= -\int_t^{-t} \Phi_{K_E}(t,x)\exp(-izx)\,\mathrm{d}x + \frac{\exp(izt)}{-iz} \\ &\quad - \frac{E(-z)}{E(z)}\Big(\int_t^{-t} \exp(izy)\Phi_{K_E}(t,y)\,\mathrm{d}y + \frac{\exp(-izt)}{-iz}\Big)\end{aligned} \tag{3.3.7}$$

for $\operatorname{Im}(z) > c(K_E)$. Replacing (3.3.7) in (3.1.15) and (3.1.16) yields

$$
\begin{aligned}
A_E(t,z) &= \frac{E(z)}{2}\Big(iz\int_t^{-t}\Psi_{K_E}(t,x)\exp(-izx)\,\mathrm{d}x + \exp(izt)\Big)\\
&\quad + \frac{E(-z)}{2}\Big(\exp(-izt) - iz\int_t^{-t}\Psi_{K_E}(t,x)\exp(izx)\,\mathrm{d}x\Big)\\
B_E(t,z) &= \frac{i}{2}E(z)\Big(iz\int_t^{-t}\Phi_{K_E}(t,x)\exp(-izx)\,\mathrm{d}x + \exp(izt)\Big)\\
&\quad - \frac{i}{2}E(-z)\Big(\exp(-izt) - iz\int_t^{-t}\Phi_{K_E}(t,x)\exp(izx)\,\mathrm{d}x\Big)
\end{aligned}
\tag{3.3.8}
$$

for $\operatorname{Im}(z) > c(K_E)$. The right-hand sides are analytic on $\mathbb{C}\setminus Z_E$ by condition $(E1)$. Therefore, $A_{E,t}(z)$ and $B_{E,t}(z)$ are analytic functions for $\operatorname{Im}(z) > c(K_E)$ that have analytic continuations with domain $\mathbb{C}\setminus Z_E$. As Ψ_{K_E} and Φ_{K_E} are real-valued by Lemma 3.1.7, condition $(E1)$ and $\exp(\overline{z}) = \overline{\exp(z)}$ for $z\in\mathbb{C}$ imply that (3.1.17) holds.

For a $t\in(\tau(K_E),\infty)$, Corollary 3.3.1, Lemma 3.2.7, Theorem 3.2.6 and the limit bounds in Lemma 3.2.1 justify the application of Lemma A.1.2 in the sense of the proof of Lemma 3.2.2 and integration by parts like in the proof Corollary 3.3.1 for each fixed $z\in\mathbb{C}\setminus Z_E$ to get

$$
\begin{aligned}
\frac{\partial}{\partial t}A_E(t,z) &= -\frac{iz}{2}E(z)\frac{\partial}{\partial t}\int_{-\infty}^{t}\Psi_{K_E}(t,x)\exp(-izx)\,\mathrm{d}x\\
&= -\frac{iz}{2}E(z)\Big(\int_{-\infty}^{t}\frac{\partial}{\partial t}\Psi_{K_E}(t,x)\exp(-izx)\,\mathrm{d}x + \Psi_{K_E}(t,t)\exp(-izt)\Big)\\
&= -\frac{iz}{2}E(z)\Big(-\frac{\Psi_{K_E}(t,t)}{\Phi_{K_E}(t,t)}\int_{-\infty}^{t}\frac{\partial}{\partial x}\Phi_{K_E}(t,x)\exp(-izx)\,\mathrm{d}x + \Psi_{K_E}(t,t)\exp(-izt)\Big)\\
&= -\frac{iz}{2}E(z)\Big(-\frac{\Psi_{K_E}(t,t)}{\Phi_{K_E}(t,t)}iz\int_{-\infty}^{t}\Phi_{K_E}(t,x)\exp(-izx)\,\mathrm{d}x\Big)\\
&= -z\Psi_{K_E}(t,t)^2\Big(\frac{z}{2}E(z)\int_{-\infty}^{t}\Phi_{K_E}(t,x)\exp(-izx)\,\mathrm{d}x\Big) = -z\Psi_{K_E}(t,t)^2 B_E(t,z)
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{\partial}{\partial t}B_E(t,z) &= \frac{z}{2}E(z)\frac{\partial}{\partial t}\int_{-\infty}^{t}\Phi_{K_E}(t,x)\exp(-izx)\,\mathrm{d}x\\
&= \frac{z}{2}E(z)\Big(\int_{-\infty}^{t}\frac{\partial}{\partial t}\Phi_{K_E}(t,x)\exp(-izx)\,\mathrm{d}x + \Phi_{K_E}(t,t)\exp(-izt)\Big)\\
&= \frac{z}{2}E(z)\Big(-\frac{\Phi_{K_E}(t,t)}{\Psi_{K_E}(t,t)}\int_{-\infty}^{t}\frac{\partial}{\partial x}\Psi_{K_E}(t,x)\exp(-izx)\,\mathrm{d}x + \Phi_{K_E}(t,t)\exp(-izt)\Big)\\
&= \frac{z}{2}E(z)\Big(-\frac{\Phi_{K_E}(t,t)}{\Psi_{K_E}(t,t)}iz\int_{-\infty}^{t}\Psi_{K_E}(t,x)\exp(-izx)\,\mathrm{d}x\Big)\\
&= z\Phi_{K_E}(t,t)^2\Big(-\frac{iz}{2}E(z)\int_{-\infty}^{t}\Psi_{K_E}(t,x)\exp(-izx)\,\mathrm{d}x\Big) = z\Phi_{K_E}(t,t)^2 A_E(t,z).
\end{aligned}
$$

In particular, A_E and B_E are continuous in t for each fixed $z\in\mathbb{C}\setminus Z_E$ and hence these identities even show that the derivatives are continuous. Furthermore, we can easily verify that these identities precisely mean that $(A_E,B_E)^t$ solves (1.1.1) for $t\in(\tau(K_E),\infty)$ and $z\in\mathbb{C}\setminus Z_E$ and the Hamiltonian H_E.

Finally setting $t = 0$ in (3.3.8), we have

$$A_E(0, z) = 1/2(E(z) + E^{\#}(z))$$
$$B_E(0, z) = i/2(E(z) - E^{\#}(z))$$

and $E(z) = A_E(0, z) - iB_E(0, z)$ by $E^{\#}(z) = E(-z)$ in condition $(E1)$. ■

Chapter 4

Motivating the Choice of the Initial Values

In this final chapter, we provide a motivation of the definitions of $\mathcal{K}$ and $\mathcal{E}$. Because of the complexity, we only give an overview and cite sources for proofs and collateral reading. The organization of this chapter loosely follows [11].

4.1 The Selberg Class

The definitions of $\mathcal{K}$ and $\mathcal{E}$ are chosen in such a way that they contain kernels arising from special number theoretic functions. Hence, we need the concept of Dirichlet series:

Definition 4.1.1. Let $a : \mathbb{N} \to \mathbb{C}$ be a complex-valued sequence. Then, the (formal) series

$$\sum_{n=1}^{\infty} \frac{a(n)}{n^s} \tag{4.1.1}$$

for $s \in \mathbb{C}$ is called a *Dirichlet series.* $**$

If we use $L(s)$ for a Dirichlet series, we replace $a(n)$ in (4.1.1) with $a_L(n)$.

Definition 4.1.2. The Dirichlet series

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s}$$

is called *Riemann zeta-function.* $**$

The natural question that now arises is wheter Dirichlet series converge and, if they do, wheter they are analytic.

Lemma 4.1.3. *Let $L(\sigma + it)$ be a Dirichlet series for $\sigma, t \in \mathbb{R}$. There exists a $\sigma_0 \in [-\infty, \infty]$ such that $L(\sigma + it)$ converges for $\sigma > \sigma_0$ and diverges for $\sigma < \sigma_0$. On the half plane $\sigma > \sigma_0$, the Dirichlet series even is analytic.*

Proof. e.g. [12, Chapter I] ■

Remark 4.1.4. We can easily verify that $\sigma_0 = 1$ holds true for the Riemann zeta-function. //

Definition 4.1.5. Writing $s = \sigma + it$ with $\sigma, t \in \mathbb{R}$ for $s \in \mathbb{C}$, the *Selberg class* $\mathcal{S}$ is defined as the set of all Dirichlet series

$$L(s) := \sum_{n=1}^{\infty} \frac{a_L(n)}{n^s} \tag{4.1.2}$$

satisfying:

- For $\sigma > 1$, (4.1.2) converges absolutely.
- There exists a minimal $m_L \in \mathbb{N} \uplus \{0\}$ such that $(s-1)^{m_L} L(s)$ can be extended to an entire function of finite order.[1]
- We can find $r_L \in \mathbb{N} \uplus \{0\}, Q_L > 0, \lambda_1^L, \ldots, \lambda_{r_L}^L > 0$ and $\mu_1^L, \ldots, \mu_{r_L}^L \in \mathbb{C}$ with a nonnegative real part and an $\epsilon_L \in \mathbb{C}$ with $|\epsilon_L| = 1$ such that

  $$\Lambda_L(s) = \epsilon_L \Lambda_L^{\#}(1-s)$$

  holds for $s \in \mathbb{C} \setminus \{0, 1\}$ and

  $$\Lambda_L(s) := Q_L^s \prod_{j=1}^{r_L} \Gamma(\lambda_j^L s + \mu_j^L) L(s)$$

  where Γ denotes the gamma function.
- For every $\varepsilon > 0$, we have $a_L(n) \in O(n^{\varepsilon})$. (The constants can depend on ε.)
- For sufficiently large σ, there exists a complex-valued sequence b_L with $b_L(n) = 0$ if n is the power of a prime and $b_L(n) \in O(n^{\theta})$ for a $\theta \in (0, 1/2)$ such that

  $$\sum_{n=1}^{\infty} \frac{b_L(n)}{n^s}$$

  converges and defines a logarithm of L.

**

Again, the most prominent element of $\mathcal{S}$ is the Riemann zeta-function:

Lemma 4.1.6. *The Riemann zeta-function ζ can be extended to an analytic function on $\mathbb{C} \setminus \{1\}$ with a simple pole at 1 with residue 1. Setting*

$$\Lambda_\zeta(s) := \frac{1}{2} \pi^{-s/2} \Gamma\Big(\frac{s}{2}\Big) \zeta(s) \tag{4.1.3}$$

for $s \in \mathbb{C}$ and $\epsilon_\zeta := 1$, $\zeta \in \mathcal{S}$ holds true.

Proof. e.g. [13, Chapters 1 and 2] ■

Definition 4.1.7. For an $L \in \mathcal{S}$, we define

$$\xi_L(s) := s^{m_L} (s-1)^{m_L} \Lambda_L(s)$$

for $s \in \mathbb{C}$. In the case of ζ and Λ_ζ of (4.1.3), we get the *Riemann xi-function*

$$\xi(s) := \frac{1}{2} s(s-1) \pi^{-s/2} \Gamma\Big(\frac{s}{2}\Big) \zeta(s)$$

for $s \in \mathbb{C}$. **

[1] Recall that the order of an entire function f is defined as the infimum of all $r \geq 0$ with $f(z) \in O(\exp(|z|^r))$ as $z \to \infty$.

The “real” elements of the Selberg class now give rise to elements of $\mathcal{K}$ and $\mathcal{E}$.

Theorem 4.1.8. *Let $L \in \mathcal{S}$ be a function with $L \not\equiv 1$, $L(\mathbb{R} \setminus \{1\}) \subseteq \mathbb{R}$ and $\Lambda_L(\mathbb{R} \setminus \{1\}) \subseteq \mathbb{R}$. An example of such a function is ζ. Define*

$$E_L^{\omega,\nu}(z) := \xi_L(\frac{1}{2} + \omega - iz)^\nu$$

for $z \in \mathbb{C}$, $\omega > 0$ and $\nu \in \mathbb{N}$ and define

$$K_L^{\omega,\nu}(x) := \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{E_L^{\omega,\nu}(-u - iv)}{E_L^{\omega,\nu}(u + iv)} \exp(-ix(u + iv))\,\mathrm{d}u$$

for $x, u, v \in \mathbb{R}$, $\omega > 0$ and $\nu \in \mathbb{N}$. Then, $K_L^{\omega,\nu}$ exists and is independent of v for v large enough. If

$$2\omega\nu \sum_{j=1}^{r_L} \lambda_j^L > 1$$

holds true for $\omega > 0$ and $\nu \in \mathbb{N}$, we have $(E_L^{\omega,\nu}, K_L^{\omega,\nu}) \in \mathcal{E}$.

Proof. [11, Propositions 4.1 and 4.2] ■

4.2 The Grand Riemann Hypothesis

Finally, Section 4.1 now shows why finding Hamiltonians for elements of $\mathcal{E}$ in the sense of Theorem 3.1.9 is motivated by the grand Riemann hypothesis.

Grand Riemann hypothesis. *For an $L \in \mathcal{S}$ and $s \in \mathbb{C}$, we have*

$$\xi_L(s) = 0 \implies \mathrm{Re}(s) = \frac{1}{2}.$$

□

Choosing ζ, we get the Riemann hypothesis, i.e. all zeros of the Riemann xi-function have real part 1/2.

Theorem 4.2.1. *Let $L \in \mathcal{S}$ be a function with $L \not\equiv 1$, $L(\mathbb{R} \setminus \{1\}) \subseteq \mathbb{R}$ and $\Lambda_L(\mathbb{R} \setminus \{1\}) \subseteq \mathbb{R}$. If there exists a function $(\nu_\omega)_{\omega>0}$ to $\mathbb{N}$ such that*

$$|(E_L^{\omega,\nu_\omega})^\#(s)| < |(E_L^{\omega,\nu_\omega})(s)| \tag{4.2.1}$$

holds for all $\omega > 0$ and $s \in \mathbb{C}$ with $\mathrm{Im}(s) > 0$, the Grand Riemann hypothesis holds for L.

Proof. [11, Proposition 2.1] ■

Theorem 4.2.2. *In the setting of Theorem 3.1.9, assume that*

- *$(A_E, B_E)^t$ solves (1.1.1) for $t \in (\tau(K_E), 0]$, $z \in \mathbb{C}$ with $\mathrm{Im}(z) > 0$ and the Hamiltonian from (3.1.14).*
- *$(A_E(t,z), B_E(t,z))^t \not\equiv (0,0)^t$ for every $z \in \mathbb{C}$ with $\mathrm{Im}(z) > 0$ as a function of t.*
- *For every $z \in \mathbb{C}$ with $\mathrm{Im}(z) > 0$, we have*

$$\lim_{t \to \tau(K_E)} \frac{\overline{A_E(t,z)} B_E(t,z) - A_E(t,z) \overline{B_E(t,z)}}{(z - \overline{z})\pi} = 0. \tag{4.2.2}$$

Then, $A_{E,t}(z) - iB_{E,t}(z)$ satisfies (4.2.1) for every $t \in (\tau(K_E), 0]$. Setting $t = 0$, we in particular have

$$|E^{\#}(s)| < |E(s)|$$

for all $s \in \mathbb{C}$ with $\operatorname{Im}(s) > 0$.

Proof. [10, Proposition 2.4] ■

Combining Theorem 4.2.2, Theorem 4.2.1, Theorem 4.1.8 and Theorem 3.1.9, we see that the question whether the Grand Riemann hypothesis holds for an $L \in \mathcal{S}$ with $L \not\equiv 1$, $L(\mathbb{R} \setminus \{1\}) \subseteq \mathbb{R}$ and $\Lambda_L(\mathbb{R} \setminus \{1\}) \subseteq \mathbb{R}$ is essentially reduced to the question when (4.2.2) holds for $E_L^{\omega,\nu}$.

Appendix A

Auxiliary Results

In this short appendix, we cite results that are of fundamental importance for this thesis and either are not considered common knowledge for a graduate student or exist in various, slightly different variants, i.e. we quote them to prevent ambiguity. Sources for proofs are cited.

A.1 Integrals Depending on a Parameter

Lemma A.1.1. *Let $(\Omega, \mathcal{A}, \mu)$ be a measure space, (M, d) a metric space, $s \in M$ and $f : M \times \Omega \to \mathbb{C}$ a function with*

- $f(t, .) \in L^1(\Omega, \mathcal{A}, \mu)$ *for all* $t \in M$
- $f(., x)$ *is continuous at* s *for almost every* $x \in \Omega$.
- *There exists a function* $g \in L^1(\Omega, \mathcal{A}, \mu, \mathbb{R})$ *such that*

$$|f(t, .)| \leq g \text{ a.e.}$$

holds locally around s*. (The null set can depend on* t*.)*

Then, the function $F(t) := \int_\Omega f(t, x) \,\mathrm{d}\mu(x)$ *is continuous at* s.

Proof. e.g. [14, Satz 9.34] or [15, Lemma 14.15.3] ■

Lemma A.1.2. *Let $(\Omega, \mathcal{A}, \mu)$ be a measure space, O an open subset of $\mathbb{R}$, $s \in O$ and $f : O \times \Omega \to \mathbb{C}$ a function with*

- $f(t, .) \in L^1(\Omega, \mathcal{A}, \mu)$ *for all* $t \in O$
- $f(., x)$ *is differentiable at* s *for almost every* $x \in \Omega$.
- *There exists a function* $g \in L^1(\Omega, \mathcal{A}, \mu, \mathbb{R})$ *such that*

$$\Big|\frac{f(t, .) - f(s, .)}{t - s}\Big| \leq g \text{ a.e.}$$

holds locally around s*. (The null set can depend on* t*.)*

In particular, the last two items hold if there exist a function $g \in L^1(\Omega, \mathcal{A}, \mu, \mathbb{R})$ and a fixed null set $N \in \mathcal{A}$ such that $\frac{\partial f}{\partial t}(t, x)$ exists locally around s for all $x \in \Omega \setminus N$ and

$$\Big|\frac{\partial f}{\partial t}(t, x)\Big| \leq g(x)$$

holds. Then, the function $F(t) := \int_\Omega f(t,x)\,\mathrm{d}\mu(x)$ *is differentiable at* s *with*

$$F'(s) = \int_\Omega \frac{\partial f}{\partial t}(s,x)\,\mathrm{d}\mu(x).$$

Proof. e.g. [14, Satz 9.36 and Korollar 9.37] or [15, Lemma 14.15.4] ■

Lemma A.1.3. *Let* $(\Omega, \mathcal{A}, \mu)$ *be a measure space,* G *an open subset of* $\mathbb{C}$ *and* $f : G \times \Omega \to \mathbb{C}$ *a function with*

- $f(z, .) \in L^1(\Omega, \mathcal{A}, \mu)$ *for all* $z \in G$
- *There exists a null set* N *such that* $f(., x)$ *is analytic on* G *for every* $x \in \Omega \setminus N$.
- *For every compact set* $K \subseteq G$*, we find a function* $g_K \in L^1(\Omega, \mathcal{A}, \mu, \mathbb{R})$ *such that*

  $$|f(z,x)| \leq g_K(x)$$

  holds for every $z \in K$ *and* $x \in \Omega \setminus N$.

Then, the function $F(z) := \int_\Omega f(z,x)\,\mathrm{d}\mu(x)$ *is analytic on* G *with*

$$F^{(n)}(z) = \int_\Omega \frac{\partial^n f}{\partial z^n}(z,x)\,\mathrm{d}\mu(x).$$

Proof. e.g. [15, Lemma 14.15.5] ■

A.2 Laplace Transform

Definition A.2.1. Let $f : [0,\infty) \to \mathbb{C}$ be measurable. Define $\sigma(f) \in [-\infty, \infty]$ as the infimum of all $s \in \mathbb{R}$ with

$$(t \mapsto \exp(-st)f(t)) \in L^1([0,\infty)).$$

Then, the *Laplace transform* of f is defined as $\mathcal{L}f : \{z \in \mathbb{C} : \mathrm{Re}(z) > \sigma(f)\} \to \mathbb{C}$ with

$$(\mathcal{L}f)z := \int_0^\infty f(t)\exp(-zt)\,\mathrm{d}t.$$

**

Lemma A.2.2. *Let* $f, g : [0,\infty) \to \mathbb{C}$ *be measurable,* $\eta, b \in \mathbb{C}$ *with* $\eta \neq 0$ *and* $a > 0$. *Then, we have:*

- $\mathcal{L}0 = 0$
- $\sigma(\eta f) = \sigma(f)$ *and* $\mathcal{L}(\eta f) = \eta(\mathcal{L}f)$
- $\sigma(f + g) \leq \max(\sigma(f), \sigma(g))$ *and the Laplace transform is linear, i.e.* $(\mathcal{L}f + g)(z) = (\mathcal{L}f)(z) + (\mathcal{L}g)(z)$ *for* $z \in \mathbb{C}$ *with* $\mathrm{Re}(z) > \max(\sigma(f), \sigma(g))$.
- *If* f *is real-valued, the Laplace tranform is real-valued, too. More general,* $(\mathcal{L}f)(\overline{z}) = \overline{(\mathcal{L}f)(z)}$ *holds if* f *is real-valued.*
- $g(t) = f(at)$ *implies* $\sigma(g) = a\sigma(f)$ *and* $(\mathcal{L}g)(z) = \frac{1}{a}(\mathcal{L}f)(\frac{z}{a})$.
- $g(t) = f(t)\exp(tb)$ *implies* $\sigma(g) = \sigma(f) + \mathrm{Re}(b)$ *and* $(\mathcal{L}g)(z) = (\mathcal{L}f)(z - b)$.
- $\mathcal{L}f$ *is analytic.*
- *The Laplace transform is injective on the space of measurable functions from* $[0,\infty)$ *to* $\mathbb{C}$ *with* $\sigma(.) < \infty$.

Proof. e.g. [15, Proposition 17.3.4 and Proposition 17.3.5] ■

A.3 Luzin N Property

Definition A.3.1. A continuous function $f : [a, b] \to \mathbb{R}$ that maps null sets to null sets is said to have the *Luzin N property.* $**$

Lemma A.3.2. *A function $f : [a, b] \to \mathbb{R}$ is absolutely continuous iff it is continuous, of bounded variation and has the Luzin N property.*

Proof. [5, Chapter IX.: 3. Continuous Mappings: Theorems 3 and 4] ■

A.4 Hilbert-Schmidt Integral Operators

Remark A.4.1. Let $(\Omega_1, \mathcal{A}_1, \mu_1)$ and $(\Omega_2, \mathcal{A}_2, \mu_2)$ be σ-finite measure spaces and $f : \Omega_1 \times \Omega_2 \to \mathbb{C}$ a measurable function on the product space $(\Omega_1 \times \Omega_2, \mathcal{A}_1 \otimes \mathcal{A}_2)$. Assuming

$$\int_{\Omega_2} f(x, y) \, \mathrm{d}\mu_2(y) \in \mathbb{C} \tag{A.4.1}$$

for almost every $x \in \Omega_1$ and decomposing $\operatorname{Re} f$ and $\operatorname{Im} f$ into the positive and negative parts, Fubini's theorem for positive measurable functions states that (A.4.1) is measurable in x on $(\Omega_1, \mathcal{A}_1, \mu_1)$ regardless of wheter f is integrable over the product space with $\mu_1 \otimes \mu_2$ or not. //

Lemma A.4.2. *Let $\emptyset \neq D \subseteq \mathbb{R}$ be open and connected and $k \in L^2(D \times D)$. Then,*

$$K : \begin{cases} L^2(D) & \to L^2(D) \\ f & \mapsto (Kf)(x) := \int_D k(x, y) f(y) \, \mathrm{d}y \end{cases}$$

is a compact operator.

Proof. e.g. [16, Theorem 8.83]. ■

Definition A.4.3. The operators discussed in Lemma A.4.2 are called *Hilbert-Schmidt integral operators.* $**$

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