Analysis and numerical simulation of positive and dead core solutions of singular two-point boundary value problems

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Abstract. We investigate the solvability of the Dirichlet boundary value problem

$$u''(t) = \lambda g(u(t)), \quad \lambda \geq 0, \quad u(0) = 1, \quad u(1) = 1,$$

where λ is a nonnegative parameter. We discuss the existence of multiple positive solutions and show that for certain values of λ, there also exist solutions that vanish on a subinterval $[\rho, 1-\rho] \subset (0,1)$, the so-called dead core solutions. In order to illustrate the theoretical findings, we present computational results for $g(u) = 1/\sqrt{u}$, computed using the collocation method implemented in `bvpsuite`, a new version of the standard MATLAB code `sbvp1.0`.

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1 Introduction

Steady-state diffusion and reaction of several chemical species can, under certain conditions, be reduced to the following Dirichlet problem, see [2]:

$$u''(t) = \phi^2 g_\kappa(u(t)), \quad u(-1) = u(1) = 1. \tag{1.1}$$

Here, ϕ is the Thiele modulus, u is the normalized concentration of one of the reactants, and $g_\kappa(u) > 0$ on $(0,1]$ is a function of the form

$$g_\kappa(u) = \frac{u(\delta_0 + \delta_1 u + \cdots + \delta_{p-1}u^{p-1})}{\delta_0 + \delta_1 + \cdots + \delta_{p-1}} \Big(\frac{\kappa+1}{\kappa+u}\Big)^q,$$

where $p \geq 1$ is integer, $q \geq 0$, and δ_i, $0 \leq i \leq p-1$, are real constants. In particular, for Langmuir-Hinshelwood or Michaelis-Menten kinetics, g_κ may have the form

$$g_\kappa(u) = u\Big(\frac{\kappa+1}{\kappa+u}\Big)^q$$

in the case of a spatially homogeneous enzyme-substrate reaction. For small values of κ one might expect a solution of

$$u''(t) = \phi^2 u^{1-q}(t), \quad u(-1) = u(1) = 1,$$

to be a good approximation for the solution of (1.1). This question has been examined in [12] and [13] for $q > 0$.

By substituting $t = \frac{1}{2}(1+s)$ into (1.1), we obtain the boundary value problem

$$u''(t) = 4\phi^2 g_\kappa(u(t)), \quad u(0) = u(1) = 1.$$

The singular[1] problem

$$u''(t) = \frac{\phi^2}{u^\alpha(t)}, \quad u(0) = u(1) = 1,$$

where $\alpha \in (0,1)$, and the previous investigations in [9], [10], and [11], were strongly motivating further development in [1]. Here, the authors discuss singular Dirichlet boundary value problems which can be written as

$$u''(t) = \lambda g(u(t)), \quad \lambda \geq 0, \tag{1.2a}$$

$$u(0) = 1, \quad u(1) = 1, \tag{1.2b}$$

where λ is a nonnegative parameter, function $g \in C(0,1]$ and it becomes unbounded at $u = 0$.

[1]The singularity occurs, when $u(\tau) = 0$ for $\tau \in (0,1)$.

We say that a function $u \in C^2[0,1]$ is *a positive solution of problem* (1.2), if $0 < u \leq 1$ on $[0,1]$, and u satisfies (1.2a) for $t \in [0,1]$ and (1.2b). A function $u \in C^1[0,1]$ is said to be a *dead core solution of problem* (1.2), if there exists a subinterval $[\alpha,\beta] \subset (0,1)$ such that $u(t) = 0$ for $t \in [\alpha,\beta]$, $0 < u \leq 1$ on $[0,1]\setminus[\alpha,\beta]$, $u \in C^1[0,1] \cap C^2([0,1]\setminus\{\alpha,\beta\})$, u satisfies (1.2a) for $t \in [0,1]\setminus[\alpha,\beta]$, and (1.2b) holds. The interval $[\alpha,\beta]$ is called the *dead core* of u. If $\alpha = \beta$, then we say that u is a *pseudo dead core solution of problem* (1.2).
It follows from [1] that if $g \in C(0,1] \cap L_1[0,1]$ is positive and $\lim_{u\to 0^+} g(u) = \infty$ holds, then for any $\lambda \in [0,\infty)$ problem (1.2) has the following solution structure: Problem (1.2) has either positive solutions, pseudo dead core solutions, or dead core solutions. Another possibility is that problem (1.2) has either positive solutions and pseudo dead core solutions or positive solutions and dead core solutions. In addition, for sufficiently small values of λ problem (1.2) has only positive solutions and for sufficiently large values of λ there exist only dead core solutions.

The aim of this paper is twofold.

- First of all, we discuss relations between the values of the parameter λ and the number and types of solutions to problem (1.2), provided that

$$g \in C(0,1] \cap L_1[0,1],\ \ g \text{ is positive, and } \lim_{u\to 0^+} g(u) = \infty, \tag{1.3}$$

  or

$$g \in C^1(0,1] \cap L_1[0,1],\ \ g \text{ is positive and decreasing on } (0,1], \\ \text{and } \lim_{u\to 0^+} g(u) = \infty. \tag{1.4}$$

- Moreover, we compute solutions u to the singular boundary value problem

$$u''(t) = \frac{\lambda}{\sqrt{u(t)}}, \quad \lambda \geq 0, \tag{1.5a}$$

$$u(0) = 1, \quad u(1) = 1. \tag{1.5b}$$

  Note that (1.5a) is a special case of (1.2a).

For further results on existence of positive and dead core solutions to (1.2a), we refer the reader to articles [10] and [11]. Two-point boundary conditions $u(-1) = u(1) = 1$, and two-point boundary conditions involving derivatives $-u'(-1) + \alpha u(-1) = a$, $u'(1) + \alpha u(1) = a$, $\alpha, a > 0$, have been discussed in [10] and [11], respectively. Positive solutions and dead core solutions of the singular problem

$$u''(t) + f(t,u'(t)) = \lambda h(t,u(t)), \quad u'(a) = 0, \ \ \beta u'(b) + \alpha u(b) = A,$$

where $\beta \geq 0$, $\alpha, A > 0$, were studied in [9].

We now recapitulate the main analytical results which are formulated in Theorems 2.10, 2.11, cf. Section 2.3. To this end, we introduce the auxiliary function

$$\varphi(a) := \begin{cases} \displaystyle\int_a^1 \frac{\mathrm{d}t}{\sqrt{\int_a^t g(s)\,\mathrm{d}s}}, & a \in [0,1), \\ 0, & a = 1, \end{cases} \tag{1.6}$$

where the function g satisfies assumption (1.3). If u is a positive solution or a pseudo dead core solution of problem (1.2) and $a := \min\{u(t) : t \in [0,1]\}$, then $\varphi(a) = \sqrt{\frac{\lambda}{2}}$, cf. (2.3) and (2.9). If u is a dead core solution of problem (1.2) and $u(t) = 0$ for $t \in [\rho, 1-\rho] \subset (0,1)$, then $\varphi(0) = \sqrt{2\lambda}\rho$, see (2.12). We now use the function φ to characterize the types of the solutions to problem (1.2) and their multiplicity.

In Theorem 2.10 we deal with the solution structure of problem (1.2). Let (1.3) hold and let $\varphi(a)$, $a \in [0,1]$, be the function defined by (1.6). Then the following statements hold:

- Problem (1.2) has a positive solution if and only if $\lambda = 2\varphi^2(a)$, where $a \in (0,1]$. In addition, for each $a \in (0,1]$ problem (1.2) with $\lambda = 2\varphi^2(a)$ has a unique positive solution u such that

$$\min\{u(t) : 0 \leq t \leq 1\} = a.$$

- For $\lambda = 2\varphi^2(0)$, problem (1.2) has a unique pseudo dead core solution.
- For each $\lambda > 2\varphi^2(0)$, problem (1.2) has a unique dead core solution.

A related result concerning the multiplicity of positive solutions to problem (1.2) is given in Theorem 2.11. Let (1.4) hold and let $\nu := \max\{\varphi(a) : 0 \leq a \leq 1\}$. Then $\nu > \varphi(0)$ and for each $\lambda \in (2\varphi^2(0), 2\nu^2)$ there exist multiple positive solutions of problem (1.2).

The paper consists of two parts. Analytical results are given in Section 2 and the numerical tests are presented in Section 3. In Section 2, we discuss properties of the function φ and an auxiliary function Q_a, cf. (1.6) and (2.2), respectively. We then study the dependence of the positive, pseudo dead core, and dead core solutions of problem (1.2), on the values of λ. Finally, in Example 2.13 we give the complete analysis of the singular problem (1.5). In Section 3, we describe the numerical approach based on the polynomial collocation which we apply to solve the latter problem numerically, and the main features of the Matlab code `bvpsuite` used for the computations. We also discuss the results of the numerical simulation of problem (1.5) which turn out to be in a very good agreement with the theory.

2 Analytical Results

2.1 Auxiliary functions

We first rewrite $\varphi(a)$ given by (1.6) and obtain

$$\varphi(a) = \begin{cases} \displaystyle\int_0^1 \frac{1-a}{\sqrt{\int_a^{(1-a)t+a} g(s)\,\mathrm{d}s}}\,\mathrm{d}t, & a \in [0,1), \\ 0, & a = 1. \end{cases}$$

Properties of φ are now collected in the following two lemmas.

Lemma 2.1. *Let* (1.3) *hold. Then* $\varphi \in C[0,1]$.
Proof. Let $m := \min\{g(u) : 0 < u \le 1\}$. Then, by assumption (1.3), $m > 0$ and consequently,

$$\frac{1}{\sqrt{\int_a^{(1-a)t+a} g(s)\,\mathrm{d}s}} \le \frac{1}{\sqrt{m(1-a)t}},\ t \in (0,1],\ a \in (0,1). \tag{2.1}$$

Hence

$$0 < \int_0^1 \frac{1-a}{\sqrt{\int_a^{(1-a)t+a} g(s)\,\mathrm{d}s}}\,\mathrm{d}t < \sqrt{\frac{1-a}{m}} \int_0^1 \frac{\mathrm{d}t}{\sqrt{t}} = 2\sqrt{\frac{1-a}{m}},$$

which indicates that $\lim_{a\to 1^-} \varphi(a) = 0$ and therefore, φ is continuous at $a = 1$. In order to prove that φ is continuous on $[0,1)$, we define the function p on $(0,1] \times [-1,1)$ by

$$p(t,a) := \begin{cases} \dfrac{1-a}{\sqrt{\int_a^{(1-a)t+a} g(s)\,\mathrm{d}s}}, & (t,a) \in (0,1] \times [0,1), \\ \dfrac{1}{\sqrt{\int_0^t g(s)\,\mathrm{d}s}}, & (t,a) \in (0,1] \times [-1,0). \end{cases}$$

From (2.1) we conclude $p\colon (0,1] \times [-1,1) \to \mathbb{R}$ and $0 < p(t,a) \le \frac{1}{\sqrt{mt}} \in L_1[0,1]$ for all $a \in [-1,1)$. In addition, p is continuous on $(0,1] \times [-1,1)$. Hence, the continuity theorem (see e.g. [8]) guarantees that the function $\psi(a) = \int_0^1 p(t,a)\,\mathrm{d}t$ is continuous on $[-1,1)$. Since $\varphi(a) = \psi(a)$ for $a \in [0,1)$, the function φ is continuous on $[0,1)$ and $\varphi \in C[0,1]$ follows. □

Lemma 2.2. *Let* (1.4) *hold. Then* $\varphi \in C^1(0,1)$ *and* $\lim_{a\to 0^+} \varphi'(a) = \infty$.

Proof. Let

$$q(t,a) := \frac{(1-a)}{\sqrt{\int_a^{(1-a)t+a} g(s)\,\mathrm{d}s}},\ (t,a) \in (0,1] \times [0,1),$$

and $m := \min\{g(u) : 0 < u \le 1\}$. Then, $m > 0$, $q \in C((0,1] \times [0,1))$, and, by (2.1), $q(\cdot, a) \in L_1[0,1]$ for all $a \in [0,1)$. Since

$$\begin{aligned}\frac{\partial q}{\partial a}(t,a) &= -\frac{1}{\sqrt{\int_a^{(1-a)t+a} g(s)\,\mathrm{d}s}}\\ &\quad -\frac{(1-a)[(1-t)g((1-a)t+a) - g(a)]}{2(\int_a^{(1-a)t+a} g(s)\,\mathrm{d}s)^{\frac{3}{2}}},\end{aligned}$$

we see that $\frac{\partial q}{\partial a}$ is continuous on $(0,1] \times (0,1)$.
Let us choose an arbitrary subinterval $[a_1, a_2] \subset (0,1)$. From

$$\begin{aligned}|(1-t)g((1-a)t+a) - g(a)| &= |(1-t)[g(a) + g'(\xi)(1-a)t] - g(a)|\\ &= t|-g(a) + (1-t)(1-a)g'(\xi)|,\end{aligned}$$

where $\xi \in (a, (1-a)t + a)$, it follows

$$|(1-t)g((1-a)t+a) - g(a)| \le Kt,$$

where $t \in [0,1]$, $a \in [a_1, a_2]$, and $K = \max\{g(u) : a_1 \le u \le a_2\} + \max\{|g'(u)| : a_1 \le u \le 1\}$. This, together with estimate (2.1), yields

$$\left|\frac{\partial q}{\partial a}(t,a)\right| \le \frac{1}{\sqrt{(1-a)mt}} + \frac{K}{2m\sqrt{(1-a)mt}} \in L_1[0,1]$$

for $(t,a) \in (0,1] \times [a_1, a_2]$. Since $\varphi(a) = \int_0^1 q(t,a)\,\mathrm{d}t$ for $a \in [0,1)$, we have $\varphi \in C^1(0,1)$ by the continuity theorem and the differentiation theorem (see e.g. [8]). It remains to show that $\lim_{a\to 0^+} \varphi'(a) = \infty$. By condition (1.4) g is decreasing on $(0,1]$ and consequently, $g(a)-(1-t)g((1-a)t+a) \ge tg(a)$. Hence, see (2.1),

$$\begin{aligned}\varphi'(a) &= -\int_0^1 \frac{\mathrm{d}t}{\sqrt{\int_a^{(1-a)t+a} g(s)\,\mathrm{d}s}} + \frac{1-a}{2}\int_0^1 \frac{g(a) - (1-t)g((1-a)t+a)}{(\int_a^{(1-a)t+a} g(s)\,\mathrm{d}s)^{\frac{3}{2}}}\,\mathrm{d}t\\ &> -\frac{1}{\sqrt{(1-a)m}}\int_0^1 \frac{\mathrm{d}t}{\sqrt{t}} + \frac{(1-a)g(a)}{2}\int_0^1 \frac{t}{(\int_a^{(1-a)t+a} g(s)\,\mathrm{d}s)^{\frac{3}{2}}}\,\mathrm{d}t\\ &> -\frac{2}{\sqrt{(1-a)m}} + \frac{(1-a)g(a)}{4(\int_0^1 g(s)\,\mathrm{d}s)^{\frac{3}{2}}},\quad a \in (0,1).\end{aligned}$$

From the above estimate and from the assumptions $\lim_{a\to 0^+} g(a) = \infty$, and $g \in L_1[0,1]$, we finally conclude $\lim_{a\to 0^+} \varphi'(a) = \infty$. □

For each $a \in [0,1)$, let us now define the function Q_a,

$$Q_a(x) := \begin{cases} \displaystyle\int_a^x \frac{\mathrm{d}s}{\sqrt{\int_a^s g(v)\,\mathrm{d}v}}, & x \in (a,1], \\ 0, & x = a. \end{cases} \tag{2.2}$$

Properties of Q_a are stated in the following lemma.

Lemma 2.3. *Let* (1.3) *hold and let* $a \in [0,1)$. *Then* $Q_a \in C[a,1] \cap C^1(a,1]$ *and* Q_a *is increasing on* $[a,1]$.

Proof. Since $\min\{g(u) : 0 \le u \le 1\} = m > 0$, we have $\int_a^s g(v)\,\mathrm{d}v \ge (s-a)m$ for $a \le s \le 1$, and therefore

$$0 < \int_a^x \frac{\mathrm{d}s}{\sqrt{\int_a^s g(v)\,\mathrm{d}v}} \le \frac{1}{\sqrt{m}} \int_a^x \frac{\mathrm{d}s}{\sqrt{s-a}} = 2\sqrt{\frac{x-a}{m}}, \quad x \in (a,1].$$

Hence, $Q_a(x) < \infty$ for $x \in (a,1]$ and $\lim_{x\to a^+} Q_a(x) = 0$. Consequently, Q_a is continuous at $x = a$. Since the function $p(s) = 1/\sqrt{\int_a^s g(v)\,\mathrm{d}v}$ is positive and continuous on $(a,1]$, we can see that $Q_a \in C[a,1] \cap C^1(a,1]$ and Q_a is increasing on $[a,1]$. □

2.2 Dependence of solutions on the parameter λ

The following two lemmas deal with properties of positive and dead core solutions of problem (1.2).

Lemma 2.4. *Let* (1.3) *hold. Let* u *be a positive solution of problem* (1.2) *for some value of* $\lambda > 0$ *and* $a := \min\{u(t) : 0 \le t \le 1\}$. *Then* u *is symmetric with respect to* $t = \frac{1}{2}$,

$$\int_a^1 \frac{\mathrm{d}s}{\sqrt{\int_a^s g(v)\,\mathrm{d}v}} = \sqrt{\frac{\lambda}{2}} \tag{2.3}$$

and

$$\int_a^{u(t)} \frac{\mathrm{d}s}{\sqrt{\int_a^s g(v)\,\mathrm{d}v}} = \sqrt{2\lambda}\left|t - \frac{1}{2}\right| \tag{2.4}$$

for $t \in [0,1]$.

Proof. It follows from the boundary conditions (1.2b) that $u'(\xi) = 0$ for some $\xi \in (0,1)$ and since $u''(t) = \lambda g(u(t)) > 0$ for $t \in [0,1]$, we conclude that $u' < 0$ on $[0,\xi)$, $u' > 0$ on $(\xi, 1]$ and $a = u(\xi)$. By integrating the equality $u''(t)u'(t) = \lambda g(u(t))u'(t)$ over $[t,\xi] \subset [0,\xi]$, we obtain

$$(u'(t))^2 = 2\lambda \int_a^{u(t)} g(v)\,\mathrm{d}v$$

and consequently, since $u' < 0$ on $[0,\xi)$,

$$u'(t) = -\sqrt{2\lambda \int_a^{u(t)} g(s)\,\mathrm{d}s}, \quad t \in [0,\xi].$$

Finally, integrating of

$$\frac{u'(t)}{\sqrt{\int_a^{u(t)} g(s)\,\mathrm{d}s}} = -\sqrt{2\lambda}, \quad t \in [0,\xi),$$

from $t \in [0,\xi)$ to ξ, yields

$$\int_a^{u(t)} \frac{\mathrm{d}s}{\sqrt{\int_a^s g(v)\,\mathrm{d}v}} = \sqrt{2\lambda}(\xi - t), \quad t \in [0,\xi]. \tag{2.5}$$

We now set $t = 0$ in (2.5) and have

$$\int_a^1 \frac{\mathrm{d}s}{\sqrt{\int_a^s g(v)\,\mathrm{d}v}} = \sqrt{2\lambda}\xi. \tag{2.6}$$

Similar reasoning for the interval $[\xi, 1]$ provides

$$\int_a^{u(t)} \frac{\mathrm{d}s}{\sqrt{\int_a^s g(v)\,\mathrm{d}v}} = \sqrt{2\lambda}(t - \xi), \quad t \in [\xi,1]. \tag{2.7}$$

Since $u(1) = 1$, it follows from (2.7) that

$$\int_a^1 \frac{\mathrm{d}s}{\sqrt{\int_a^s g(v)\,\mathrm{d}v}} = \sqrt{2\lambda}(1 - \xi). \tag{2.8}$$

Combining (2.6) and (2.8) yields $\xi = \frac{1}{2}$ and consequently, (2.3) follows. Relation (2.4) holds due to (2.5) and (2.7) with $\xi = \frac{1}{2}$. From (2.4) we can see that u is symmetric with respect to $t = \frac{1}{2}$, $u(t) = u(1-t)$ for $t \in [0,1]$, and this completes the proof. □

Remark 2.5. Let (1.3) hold and let u be a pseudo dead core solution of problem (1.2). Then, by the definition, there exists a unique point $\xi \in (0,1)$ such that $0 = \min\{u(t) : 0 \le t \le 1\} = u(\xi)$. We proceed analogously to the proof of Lemma 2.4 in order to show that $\xi = \frac{1}{2}$,

$$\int_0^1 \frac{\mathrm{d}s}{\sqrt{\int_0^s g(v)\,\mathrm{d}v}} = \sqrt{\frac{\lambda}{2}} \tag{2.9}$$

and

$$\int_0^{u(t)} \frac{\mathrm{d}s}{\sqrt{\int_a^s g(v)\,\mathrm{d}v}} = \sqrt{2\lambda}\left|t - \frac{1}{2}\right|, \quad t \in [0,1]. \tag{2.10}$$

Remark 2.6. If $\lambda = 0$, then $u \equiv 1$ on $[0,1]$ is the unique solution of problem (1.2).

Lemma 2.7. *Let* (1.3) *hold and let u be a dead core solution of problem* (1.2), *for some $\lambda = \lambda_0$. Then there exists a point $\rho \in (0, \frac{1}{2})$ such that $u(t) = 0$ for $t \in [\rho, 1-\rho]$,*

$$\int_0^{u(t)} \frac{\mathrm{d}s}{\sqrt{\int_0^s g(v)\,\mathrm{d}v}} = \begin{cases} \sqrt{2\lambda_0}(\rho - t), & t \in [0,\rho], \\ \sqrt{2\lambda_0}(t - 1 + \rho), & t \in [1-\rho, 1], \end{cases} \tag{2.11}$$

and

$$\int_0^1 \frac{\mathrm{d}s}{\sqrt{\int_0^s g(v)\,\mathrm{d}v}} = \sqrt{2\lambda_0}\rho. \tag{2.12}$$

Moreover, u is the unique dead core solution of problem (1.2) *with $\lambda = \lambda_0$.*

Proof. Since u is a dead core solution of problem (1.2) with $\lambda = \lambda_0$, we know, by the definition that there exists a subinterval $[\rho, \beta] \subset (0,1)$ such that $u \in C^1[0,1] \cap C^2([0,1] \setminus \{\rho, \beta\})$, $u(t) = 0$ for $t \in [\rho, \beta]$ and $0 < u(t) \le 1$ for $t \in [0,1] \setminus [\rho, \beta]$. Hence, $u(\rho) = u'(\rho) = 0$, $u(\beta) = u'(\beta) = 0$ and $u''(t) = \lambda_0 g(u(t)) > 0$ for $t \in [0,1] \setminus [\rho, \beta]$. Consequently, $u' < 0$ on $[0, \rho)$ and $u' > 0$ on $(\beta, 1]$. We can now proceed analogously to the proof of Lemma 2.4 to show

$$\int_0^{u(t)} \frac{\mathrm{d}s}{\sqrt{\int_0^s g(v)\,\mathrm{d}v}} = \sqrt{2\lambda_0}(\rho - t), \ \ t \in [0, \rho], \tag{2.13}$$

and

$$\int_0^{u(t)} \frac{\mathrm{d}s}{\sqrt{\int_0^s g(v)\,\mathrm{d}v}} = \sqrt{2\lambda_0}(t - \beta), \ \ t \in [\beta, 1]. \tag{2.14}$$

Let us set $t = 0$ in (2.13) and $t = 1$ in (2.14). Then the boundary conditions (1.2b) yield

$$\int_0^1 \frac{\mathrm{d}s}{\sqrt{\int_0^s g(v)\,\mathrm{d}v}} = \sqrt{2\lambda_0}\rho, \quad \int_0^1 \frac{\mathrm{d}s}{\sqrt{\int_0^s g(v)\,\mathrm{d}v}} = \sqrt{2\lambda_0}(1-\beta). \tag{2.15}$$

Hence, (2.12) holds and $\beta = 1-\rho$. Finally, (2.11) follows immediately from (2.13) and (2.14).

It remains to verify that u is the unique dead core solution of the boundary value problem (1.2) with $\lambda = \lambda_0$. Let us assume that w is another dead core solution of the above problem. Let $w(t) = 0$ for $t \in [\rho_1, \beta_1]$ and $0 < w(t) \le 1$ for $t \in [0,1] \setminus [\rho_1, \beta_1]$. Then, cf. (2.13) and (2.14),

$$\int_0^{w(t)} \frac{\mathrm{d}s}{\sqrt{\int_0^s g(v)\,\mathrm{d}v}} = \sqrt{2\lambda_0}(\rho_1 - t), \;\; t \in [0, \rho_1], \tag{2.16}$$

$$\int_0^{w(t)} \frac{\mathrm{d}s}{\sqrt{\int_0^s g(v)\,\mathrm{d}v}} = \sqrt{2\lambda_0}(t - \beta_1), \;\; t \in [\beta_1, 1]. \tag{2.17}$$

Setting $t = 0$ in (2.16) and $t = 1$ in (2.17), we obtain

$$\int_0^1 \frac{\mathrm{d}s}{\sqrt{\int_0^s g(v)\,\mathrm{d}v}} = \sqrt{2\lambda_0}\rho_1, \quad \int_0^1 \frac{\mathrm{d}s}{\sqrt{\int_0^s g(v)\,\mathrm{d}v}} = \sqrt{2\lambda_0}(1-\beta_1).$$

Using (2.15) we conclude that $\rho = \rho_1$, $\beta = \beta_1$, and consequently, $u(t) = w(t) = 0$ for $t \in [\rho, 1-\rho]$. Now, $u(t) = w(t)$ for $t \in [0,\rho) \cup (1-\rho,]$ from (2.16), 2.17) and the fact that the function Q_0 is continuous and increasing on $[0,1]$ by Lemma 2.3. Hence, $u \equiv w$ which means that u is the unique dead core solution of problem (1.2) with $\lambda = \lambda_0$. $\square$

The existence of positive, pseudo dead core, and dead core solutions of problem (1.2) is discussed in the following two lemmas.

Lemma 2.8. *Let* (1.3) *hold and let* φ *be given by* (1.6). *Then the following assertions hold:*

(*i*) *For each* $a \in (0,1]$ *problem* (1.2) *with* $\lambda = 2\varphi^2(a)$, *has a unique positive solution* u *such that* $\min\{u(t) : 0 \le t \le 1\} = a$.

(*ii*) *Problem* (1.2) *has a pseudo dead core solution if and only if*

$$\lambda = 2\left(\int_0^1 \frac{\mathrm{d}s}{\sqrt{\int_0^s g(v)\,\mathrm{d}v}}\right)^2. \tag{2.18}$$

This pseudo dead core solution is unique.

Proof. If $a = 1$ then $u \equiv 1$ on $[0,1]$ is the unique positive solution of problem (1.2) with $\lambda = 0 = 2\varphi^2(1)$. This solution satisfies $\min\{u(t) : 0 \le t \le 1\} = 1$. Let us choose an arbitrary $a \in (0,1)$ and set $\lambda_a = 2\varphi^2(a)$. Then

$$\lambda_a = 2\left(\int_a^1 \frac{\mathrm{d}s}{\sqrt{\int_a^s g(v)\,\mathrm{d}v}}\right)^2.$$

We first note that there exists a positive solution of problem (1.2) with $\lambda = \lambda_a$ such that $\min\{u(t) : 0 \le t \le 1\} = a$. Indeed, by Lemma 2.3, Q_a is increasing on $[a,1]$ and $Q_a \in C[a,1] \cap C^1(a,1]$. Since $Q_a(1) = \sqrt{\frac{\lambda_a}{2}}$ and $Q_a(a) = 0$, we see that Q_a maps $[a,1]$ onto $[0, \sqrt{\frac{\lambda_a}{2}}]$. Therefore, the equation

$$Q_a(w(t)) = \sqrt{2\lambda_a}\Big(t - \frac{1}{2}\Big) \tag{2.19}$$

has a unique solution w on the interval $[\frac{1}{2}, 1]$. Furthermore, $w(\frac{1}{2}) = a$, $w(1) = 1$ and w is increasing on $[\frac{1}{2}, 1]$. In addition,

$$w'(t) = \frac{\sqrt{2\lambda_a}}{Q_a'(w(t))} = \sqrt{2\lambda_a \int_a^{w(t)} g(v)\,\mathrm{d}v} \tag{2.20}$$

for $t \in (\frac{1}{2}, 1]$. Hence, $w' \in C(\frac{1}{2}, 1]$ and $\lim_{t \to (\frac{1}{2})^+} w'(t) = 0$. To show that w' is continuous at $t = \frac{1}{2}$ we set $M := \max\{g(s) : a \le s \le 1\} > 0$. Then, cf. (2.19),

$$\sqrt{2\lambda_a}\Big(t - \frac{1}{2}\Big) = \int_a^{w(t)} \frac{\mathrm{d}s}{\sqrt{\int_a^s g(v)\,\mathrm{d}v}} \ge \frac{1}{\sqrt{M}} \int_a^{w(t)} \frac{\mathrm{d}s}{\sqrt{s-a}} = 2\sqrt{\frac{w(t) - a}{M}}$$

and therefore,

$$0 < \frac{w(t) - w(\frac{1}{2})}{t - \frac{1}{2}} = \frac{w(t) - a}{t - \frac{1}{2}} \le \frac{M\lambda_a(2t-1)}{4}, \quad t \in \left(\frac{1}{2}, 1\right].$$

Consequently, $w'(\frac{1}{2}) = \lim_{t \to (\frac{1}{2})^+} (w(t) - w(\frac{1}{2}))/(t - \frac{1}{2}) = 0$, and we have shown that $w \in C^1[\frac{1}{2}, 1]$. Now (2.20) indicates that $w \in C^2(\frac{1}{2}, 1]$ and

$$w''(t) = \sqrt{2\lambda_a} \frac{g(w(t))w'(t)}{2\sqrt{\int_a^{w(t)} g(v)\,\mathrm{d}v}} = \lambda_a g(w(t)), \quad t \in \left(\frac{1}{2}, 1\right].$$

Moreover, by the de l'Hospital rule,

$$\lim_{t \to (\frac{1}{2})^+} \frac{w'(t) - w'(\frac{1}{2})}{t - \frac{1}{2}} = \lim_{t \to (\frac{1}{2})^+} \frac{w'(t)}{t - \frac{1}{2}} = \sqrt{2\lambda_a} \lim_{t \to (\frac{1}{2})^+} \frac{\sqrt{\int_a^{w(t)} g(v)\,\mathrm{d}v}}{t - \frac{1}{2}}$$
$$= \sqrt{2\lambda_a} \lim_{t \to (\frac{1}{2})^+} \frac{g(w(t))w'(t)}{2\sqrt{\int_a^{w(t)} g(v)\,\mathrm{d}v}} = \lambda_a \lim_{t \to (\frac{1}{2})^+} g(w(t)) = \lambda_a g\left(w\left(\frac{1}{2}\right)\right).$$

To summarize, $w''(t) = \lambda_a g(w(t))$ for $t \in [\frac{1}{2}, 1]$.

Let

$$u(t) = \begin{cases} w(\frac{1}{2} - t), & t \in [0, \frac{1}{2}], \\ w(t), & t \in [\frac{1}{2}, 1]. \end{cases} \tag{2.21}$$

It is easy to see that u is a solution of problem (1.2) with $\lambda = \lambda_a$ and $\min\{u(t) : 0 \leq t \leq 1\} = a$. Let z be another solution of problem (1.2) with $\lambda = \lambda_a$ in (1.2a) and let $\min\{z(t) : 0 \leq t \leq 1\} = a$. Then, by Lemma 2.4, z satisfies (2.4) with $\lambda = \lambda_a$ and with u replaced by z. Hence, $Q_a(z(t)) = \sqrt{2\lambda_a}(t - \frac{1}{2})$ for $t \in [\frac{1}{2}, 1]$. Since we know that $w(t)$, $t \in [\frac{1}{2}, 1]$, is the unique solution of equation (2.19) and, by Lemma 2.4, z is symmetric with respect to $t = \frac{1}{2}$, it follows that $z(t) = u(t)$ for $t \in [0, 1]$. This completes the proof of assertion (i).
It remains to show that (ii) holds. Let $\mu > 0$ and consider the equation

$$Q_0(w(t)) = \sqrt{2\mu}\Big(t - \frac{1}{2}\Big), \quad t \in \left[\frac{1}{2}, 1\right]. \tag{2.22}$$

By Lemma 2.3, $Q_0 \in C[0, 1] \cap C^1(0, 1]$ is increasing on $[0, 1]$. Since $Q_0(0) = 0$ and $Q_0(1) = \int_0^1 1/\sqrt{\int_0^s g(v)\mathrm{d}v}\,\mathrm{d}s$, the function Q_0 maps $[0, 1]$ onto the interval $\left[0, \int_0^1 1/\sqrt{\int_0^s g(v)\mathrm{d}v)}\,\mathrm{d}s\right]$. Therefore, equation (2.22) has a unique solution $w \in C[\frac{1}{2}, 1]$ such that $w(1) = 1$, $w(\frac{1}{2}) = 0$ and $w > 0$ on $(\frac{1}{2}, 1]$, if and only if $Q_0(1) = \sqrt{\frac{\mu}{2}}$, or equivalently, $\mu = \lambda$, where λ is given by (2.18). Then

$$w'(t) = \frac{\sqrt{2\lambda}}{Q_0'(w(t))} = \sqrt{2\lambda \int_0^{w(t)} g(s)\,\mathrm{d}s}, \quad t \in \left(\frac{1}{2}, 1\right], \tag{2.23}$$

and consequently, $\lim_{t \to (\frac{1}{2})^+} w'(t) = 0$. Since, see (2.22) with $\mu = \lambda$,

$$\sqrt{2\lambda}(t - \frac{1}{2}) = \int_0^{w(t)} \frac{\mathrm{d}s}{\sqrt{\int_0^s g(v)\,\mathrm{d}v}} = \frac{w(t)}{\sqrt{\int_0^{\xi(t)} g(v)\,\mathrm{d}v}}, \quad t \in \left[\frac{1}{2}, 1\right],$$

by the Mean Value Theorem for integrals, where $0 < \xi(t) < w(t)$, we have

$$\frac{w(t) - w(\frac{1}{2})}{t - \frac{1}{2}} = \frac{w(t)}{t - \frac{1}{2}} = \sqrt{2\lambda \int_0^{\xi(t)} g(s)\,\mathrm{d}s}.$$

Therefore,

$$\lim_{t \to (\frac{1}{2})^+} \frac{w(t) - w(\frac{1}{2})}{t - \frac{1}{2}} = \lim_{t \to (\frac{1}{2})^+} \sqrt{2\lambda \int_0^{\xi(t)} g(s)\,\mathrm{d}s} = 0$$

since $\lim_{t \to (\frac{1}{2})^+} \xi(t) = 0$. Finally, $w \in C^1[\frac{1}{2}, 1]$, and in analogy to the first part of the proof, we can verify that $w''(t) = \lambda g(w(t))$ for $t \in (\frac{1}{2}, 1]$. Consequently, the function u defined on $[0,1]$ in (2.21) is a pseudo dead core solution of the problem (1.2), and due to Remark 2.5, u is unique. □

Lemma 2.9. *Let* (1.3) *hold. Then there exists a unique dead core solution of the problem* (1.2) *for any* λ *in* (1.2a) *satisfying*

$$\lambda > 2\left(\int_0^1 \frac{\mathrm{d}s}{\sqrt{\int_0^s g(v)\,\mathrm{d}v}}\right)^2. \tag{2.24}$$

Proof. We use Lemma 2.7. Let us choose an arbitrary λ satisfying (2.24). Then, there exists a unique point $\rho \in (0, \frac{1}{2})$ such that

$$\sqrt{2\lambda}\rho = \int_0^1 \frac{\mathrm{d}s}{\sqrt{\int_0^s g(v)\,\mathrm{d}v}}.$$

Consider the equation

$$Q_0(w(t)) = \sqrt{2\lambda}(t-1+\rho), \quad t \in [1-\rho, 1], \tag{2.25}$$

where Q_0 is given by (2.2). By Lemma 2.3, $Q_0 \in C[0,1] \cap C^1(0,1]$ is increasing on $[0,1]$. In addition, $Q_0(0) = 0$ and $Q_0(1) = \int_0^1 1/\sqrt{\int_0^s g(v)\,\mathrm{d}v}\,\mathrm{d}s = \sqrt{2\lambda}\rho$. Hence, there exists a unique function $w \in C[1-\rho, 1]$ satisfying (2.25) such that w is increasing on $[1-\rho, 1]$, $w(1-\rho) = 0$, $w(1) = 1$ and

$$w'(t) = \frac{\sqrt{2\lambda}}{Q_0'(w(t))} = \sqrt{2\lambda \int_0^{w(t)} g(v)\,\mathrm{d}v}, \quad t \in (1-\rho, 1].$$

Using this property and the same reasoning as in the proof of Lemma 2.8 (*ii*), we can conclude that $w \in C^1[1-\rho, 1] \cap C^2(1-\rho, 1]$, $w(1-\rho) = w'(1-\rho) = 0$ and $w''(t) = \lambda g(w(t))$ for $t \in (1-\rho, 1]$. For the equation

$$Q_0(w(t)) = \sqrt{2\lambda}(\rho - t), \quad t \in [0, \rho], \tag{2.26}$$

it is easy to show that it has a unique solution w_* on the interval $[0, \rho]$ with $w_*(1-t) = w(t)$ for $t \in [1-\rho, 1]$. Then the function

$$u(t) = \begin{cases} w(1-t), & t \in [0, \rho], \\ 0, & t \in [\rho, 1-\rho], \\ w(t), & t \in [1-\rho, 1], \end{cases}$$

is a dead core solution of problem (1.2). Its uniqueness follows from Lemma 2.7. □

2.3 Main analytical results

We now show the main analytical results formulated in the introduction.

Theorem 2.10. *Let* (1.3) *hold and let* $\varphi(t)$, $t \in [0,1]$, *be the function defined by* (1.6). *Then, the following statements hold:*

(*i*) *Problem* (1.2) *has a positive solution if and only if* $\lambda = 2\varphi^2(a)$, *where* $a \in (0,1]$. *In addition, for each* $a \in (0,1]$ *problem* (1.2) *with* $\lambda = 2\varphi^2(a)$ *has a unique positive solution* u *such that*

$$\min\{u(t) : 0 \le t \le 1\} = a. \tag{2.27}$$

(*ii*) *For* $\lambda = 2\varphi^2(0)$, *problem* (1.2) *has a unique pseudo dead core solution.*

(*iii*) *For each* $\lambda > 2\varphi^2(0)$, *problem* (1.2) *has a unique dead core solution.*

Proof. (*i*) Let problem (1.2) have a positive solution u and define $a := \min\{u(t) : 0 \le t \le 1\}$. Then $\lambda = 2\varphi^2(a)$ by Lemma 2.4, cf. (2.3), and Remark 2.6. Let us now choose an arbitrary $a \in (0,1]$ and set $\lambda = 2\varphi^2(a)$. Then, by Remark 2.6 and Lemma 2.8 (*i*), there exists a unique positive solution u of problem (1.2) satisfying (2.27).

(*ii*) If $\lambda = 2\varphi^2(0)$, then problem (1.2) has a unique pseudo dead core solution by Lemma 2.8 (*ii*).

(*iii*) Lemma 2.9 finally guarantees that for each $\lambda > 2\varphi^2(0)$ problem (1.2) has a unique dead core solution. □

Theorem 2.10 indicates that if for some $\lambda > 0$ problem (1.2) has a pseudo dead core solution or a dead core solution, then these solutions are unique. For positive solutions of problem (1.2) the situation is different. Theorem 2.10, states that for each $\lambda \in \{2\varphi^2(a) : 0 < a \le 1\}$ problem (1.2) may have multiple positive solutions; their number is equal to the number of roots a of the equation $\lambda - 2\varphi^2(a) = 0$. A related result concerning the multiplicity of positive solutions to problem (1.2) is now given in the following theorem.

Theorem 2.11. *Let* (1.4) *hold and let* $\nu := \max\{\varphi(a) : 0 \le a \le 1\}$. *Then* $\nu > \varphi(0)$. *Moreover, for each* $\lambda \in (2\varphi^2(0), 2\nu^2)$, *there exist multiple positive solutions of problem* (1.2).

Proof. By Lemmas 2.1 and 2.2, $\varphi \in C[0,1] \cap C^1(0,1)$ and $\lim_{a\to 0^+} \varphi'(a) = \infty$. Hence, $\nu > \varphi(0)$, and there exists $\xi \in (0,1)$ such that $\varphi(a) > \varphi(0)$ for $a \in (0,\xi)$ and $\varphi(\xi) = \varphi(0)$. Note that $\varphi(1) = 0$. Then, for each $\tau \in (\varphi(0), \nu)$ there exist $0 < a_1 < a_2 < \xi$ such that $\varphi(a_1) = \tau = \varphi(a_2)$. For $\lambda = 2\tau^2$, we have $\lambda = 2\varphi^2(a_1) = 2\varphi^2(a_2)$ and therefore, by Theorem 2.10 (*i*), problem (1.2) has positive solutions u_1, u_2 such that $\min\{u_i(t) : 0 \le t \le 1\} = a_i$ for $i = 1, 2$. □

Corollary 2.12. *Let* (1.4) *hold and let* ν *be defined as in Theorem* 2.11*. Then the following statements hold:*

(*i*) *For* $\lambda = 2\nu^2$ *there exists a unique dead core solution and a positive solution of problem* (1.2)*.*

(*ii*) *For each* $\lambda \in (2\varphi^2(0), 2\nu^2)$ *there exists a unique dead core solution and at least two positive solutions of problem* (1.2)*.*

(*iii*) *For* $\lambda = 2\varphi^2(0)$ *there exists a unique pseudo dead core solution and a positive solution of problem* (1.2)*.*

(*iv*) *For each* $\lambda \in [0, 2\varphi^2(0))$ *problem* (1.2) *has only positive solutions.*

Proof. The assertions follows from Theorems 2.10 and 2.11 and the fact that the equation $\lambda - 2\varphi^2(a) = 0$ has at least one root a in $(0,1)$ if $\lambda = 2\nu^2$ and $\lambda = 2\varphi^2(0)$, and at least two different roots in $(0,1)$ if $\lambda \in (2\varphi^2(0), 2\nu^2)$. □

We conclude the analytical considerations of the paper with an example illustrating the above characterization of the solutions to (1.2).

Example 2.13. We now consider problem (1.5),

$$u''(t) = \frac{\lambda}{\sqrt{u(t)}}, \quad u(0) = 1, \quad u(1) = 1. \tag{2.28}$$

Here, $g(u) = \frac{1}{\sqrt{u}}$ and (1.4) holds. Since

$$\begin{aligned} \varphi(a) &= \int_a^1 \frac{\mathrm{d}s}{\sqrt{\int_a^s g(v)\,\mathrm{d}v}} = \int_a^1 \frac{\mathrm{d}s}{\sqrt{\int_a^s 1/\sqrt{v}\,\mathrm{d}v}} = \frac{1}{\sqrt{2}}\int_a^1 \frac{\mathrm{d}s}{\sqrt{\sqrt{s}-\sqrt{a}}} \\ &= \frac{1}{\sqrt{2}}\Big[-\frac{8}{3}(1-\sqrt{a})^{3/2} + 4\sqrt{1-\sqrt{a}}\Big] \\ &= \frac{4}{3\sqrt{2}}\sqrt{1-\sqrt{a}}(1+2\sqrt{a}), \quad a \in [0,1], \end{aligned}$$

we have $\varphi^2(a) = \frac{8}{9}(1-\sqrt{a})(1+2\sqrt{a})^2$ and $(\varphi^2(a))' = \frac{4}{3\sqrt{a}}(1+2\sqrt{a})(1-2\sqrt{a})$. Hence, $(\varphi^2)'$ vanishes at the unique point $a = \frac{1}{4}$ and therefore, $\max\{\varphi^2(a) : 0 \le a \le 1\} = \varphi^2(\frac{1}{4}) = \frac{16}{9}$, φ^2 is increasing on $[0, \frac{1}{4}]$ and decreasing on $[\frac{1}{4}, 1]$. Since $\varphi^2(0) = \frac{8}{9}$ and $\varphi^2(1) = 0$, the following solution structure follows from Theorems 2.10, 2.11 and Corollary 2.12:

(i) For each $\lambda \in (2\varphi^2(\frac{1}{4}), \infty) = (\frac{32}{9}, \infty)$ there exists only a unique dead core solution of problem (2.28).

(ii) For $\lambda = 2\varphi^2(\frac{1}{4}) = \frac{32}{9}$ there exist a unique dead core solution and a unique positive solution u of problem (2.28) and $\min\{u(t) : 0 \le t \le 1\} = \frac{1}{4}$.

(iii) For each $\lambda \in (2\varphi^2(0), 2\varphi^2(\frac{1}{4})) = (\frac{16}{9}, \frac{32}{9})$ there exist a unique dead core solution and exactly two positive solutions of problem (2.28).

(iv) For $\lambda = 2\varphi^2(0) = \frac{16}{9}$ there exist the unique pseudo dead core solution $u(t) = (1-2t)^{\frac{4}{3}}$ and the unique positive solution u of problem (2.28). Moreover, $\min\{u(t) : 0 \le t \le 1\} = \frac{3}{4}$ (note that $\varphi(0) = \varphi(\frac{3}{4})$).

(v) For each $\lambda \in [0, 2\varphi^2(0)) = [0, \frac{16}{9})$ there exists a unique positive solution of problem (2.28).
Furthermore, for each $a \in [0,1]$ problem (2.28) with $\lambda = \frac{16}{9}(1-\sqrt{a})(1+2\sqrt{a})^2$ has a unique solution u such that $\min\{u(t) : 0 \le t \le 1\} = a$.
The function, cf. Example 3.5 in [1],

$$u(t) = \begin{cases} (1-\frac{3}{2}\sqrt{\lambda}t)^{\frac{4}{3}}, & t \in [0, \frac{2}{3\sqrt{\lambda}}), \\ 0, & t \in [\frac{2}{3\sqrt{\lambda}}, 1-\frac{2}{3\sqrt{\lambda}}], \\ (1-\frac{3}{2}\sqrt{\lambda}(1-t))^{\frac{4}{3}}, & t \in (1-\frac{2}{3\sqrt{\lambda}}, 1], \end{cases} \tag{2.29}$$

is the unique dead core solution of problem (2.28) with $\lambda > \frac{16}{9}$, and the interval $[\frac{2}{3\sqrt{\lambda}}, 1-\frac{2}{3\sqrt{\lambda}}]$ is its dead core.

In order to describe the positive solutions of the problem (2.28), for $\lambda \in (0, \frac{32}{9}] = (0, 2\varphi^2(\frac{1}{4})]$, we define the set $\mathcal{A}_\lambda := \{a \in (0,1) : \frac{16}{9}(1-\sqrt{a})(1+2\sqrt{a})^2 = \lambda\}$. Then $\mathcal{A}_\lambda \neq \emptyset$ for all $\lambda \in (0, \frac{32}{9}]$. From the properties of the function φ^2 it follows that $\mathcal{A}_\lambda$ is one point set for $\lambda \in (0, \frac{16}{9}] \cup \{\frac{32}{9}\}$ and two point set for $\lambda \in (\frac{16}{9}, \frac{32}{9})$. Let $\lambda \in (0, \frac{32}{9}]$. Then, by Lemma 2.4 and Theorems 2.10 and 2.11, positive solutions of problem (2.28) are solutions $u \in C^2[0,1]$ satisfying (2.4) with $a \in \mathcal{A}_\lambda$ and $g(u) = \frac{1}{\sqrt{u}}$. This means that they solve the following equation:

$$\sqrt{\sqrt{u(t)}-\sqrt{a}}(\sqrt{u(t)}+2\sqrt{a}) = \frac{3\sqrt{\lambda}}{2}|t-\frac{1}{2}|, \quad t \in [0,1].$$

If $u \in C^2[0,1]$ is a solution of the above equation with $a \in \mathcal{A}_\lambda$, then u is unique and $\min\{u(t) : 0 \le t \le 1\} = a$.
Moreover, Lemmas 2.4, 2.7 and Remark 2.5 show that all solutions u of problem (2.28) are symmetric with respect to $t = \frac{1}{2}$ and $\min\{u(t) : 0 \le t \le 1\} = u(\frac{1}{2})$. If u is a positive solution or a pseudo dead core solution, then the relation between the value of the parameter λ in equation (1.5a) and the value $a = u(\frac{1}{2})$ is given by the next formula, see (2.3) and (2.9),

$$\lambda = \frac{16}{9}(1-\sqrt{a})(1+2\sqrt{a})^2, \quad a \in [0,1]. \tag{2.30}$$

3 Numerical Approach

Here, we first describe how we approximate solutions of scalar two-point boundary value problems of the form,

$$u''(t) = f(t, u(t)), \quad t \in [0,1],$$
$$u(0) = u_0, \; u(1) = u_1.$$

We assume that the analytical solution u is appropriately smooth and attempt to solve this problem numerically using the collocation method implemented in our Matlab code `bvpsuite`. It is a new version of the general purpose MATLAB code `sbvp`, cf. [4], [5] and [18], which has already been successfully applied to a variety of problems, see for example [14], [15], [16], [19], and [21]. Collocation is a widely used and well-studied standard solution method for two-point boundary value problems, see for example [3] and the references therein. It also proved robust in case of singular boundary value problems.

The code is designed to solve systems of differential equations of arbitrary order. For simplicity of notation we formulate below a problem whose order varies between four and zero, which means that algebraic constraints which do not involve derivatives are also admitted. Moreover, the problem can be given in a fully implicit form,

$$F(t, u^{(4)}(t), u^{(3)}(t), u''(t), u'(t), u(t)) = 0, \quad 0 < t \leq 1, \tag{3.31a}$$
$$b(u^{(3)}(0), u''(0), u'(0), u(0), u^{(3)}(1), u''(1), u'(1), u(1)) = 0. \tag{3.31b}$$

The program can cope with free parameters, $\lambda_1, \lambda_2, \dots, \lambda_k$, which will be computed along with the numerical approximation for u,

$$F(t, u^{(4)}(t), u^{(3)}(t), u''(t), u'(t), u(t), \lambda_1, \lambda_2, \dots, \lambda_k) = 0,\ 0 < t \leq 1, \tag{3.32a}$$
$$b_{aug}(u^{(3)}(0), u''(0), u'(0), u(0), u^{(3)}(1), u''(1), u'(1), u(1)) = 0. \tag{3.32b}$$

provided that the boundary conditions b_{aug} include k additional requirements to be satisfied by u.

The numerical approximation defined by collocation is computed as follows: On a mesh

$$\Delta := \{\tau_i : i = 0, \dots, N\}, \quad 0 = \tau_0 < \tau_1 \dots < \tau_N = 1$$

we approximate the analytical solution by a collocating function,

$$p(t) := p_i(t), \quad t \in [\tau_i, \tau_{i+1}], \quad i = 0, \dots, N-1,$$

where we require $p \in C^{q-1}[0,1]$ in case that the order of the underlying differential equation is q. Here p_i are polynomials of maximal degree $m - 1 + q$ which satisfy the system (3.31a) at the *collocation points*

$$\{t_{i,j} = \tau_i + \rho_j(\tau_{i+1} - \tau_i),\ i = 0, \dots, N-1,\ j = 1, \dots, m\}, \quad 0 < \rho_1 < \dots < \rho_m < 1,$$

and the associated boundary conditions (3.31b). For $y \in \mathbb{R}^n$, $y = (y_1, \dots, y_n)^T$, we have

$$|y| := \max_{1 \leq k \leq n} |y_k|.$$

Let $y \in C[0,1]$, $y : [0,1] \to \mathbb{R}^n$. For $t \in [0,1]$,

$$|y(t)| := \max_{1 \le k \le n} |y_k(t)|$$

and

$$\|y\|_\infty := \max_{0 \le t \le 1} |y(t)|.$$

Classical theory, cf. [3], predicts that the convergence order for the global error of the method is at least $O(h^m)$, where h is the maximal stepsize, $h := \max_i(\tau_{i+1} - \tau_i)$. More precisely, for the global error of p, $\|p - u\|_\infty = O(h^m)$ holds uniformly in t. For certain choices of the collocation points the so-called *superconvergence* order can be observed. In case of the Gaussian points this means that the approximation is exceptionally precise at the meshpoints τ_i, $\max_{\tau_i \in \Delta} |p(\tau_i) - u(\tau_i)|_\infty = O(h^{2m})$.

To make the computations more efficient, an adaptive mesh selection strategy based on an a posteriori estimate for the global error of the collocation solution may be utilized. We use a classical error estimate based on mesh halving. In this approach, we compute the collocation solution $p_\Delta(t)$ on a mesh Δ. Subsequently, we choose a second mesh Δ_2 where in every interval $[\tau_i, \tau_{i+1}]$ of Δ we insert two subintervals of equal length. On this new mesh, we compute the numerical solution based on the same collocation scheme to obtain the collocating function $p_{\Delta_2}(t)$. Using these two quantities, we define

$$\mathcal{E}(t) := \frac{2^m}{1 - 2^m}(p_{\Delta_2}(t) - p_\Delta(t)) \tag{3.33}$$

as an error estimate for the approximation $p_\Delta(t)$. Assume that the global error $\delta(t) := p_\Delta(t) - u(t)$ of the collocation solution can be expressed in terms of the principal error function $e(t)$,

$$\delta(t) = e(t)|\tau_{i+1} - \tau_i|^m + O(|\tau_{i+1} - \tau_i|^{m+1}), \quad t \in [\tau_i, \tau_{i+1}], \tag{3.34}$$

where $e(t)$ is independent of Δ. Then obviously, the quantity $\mathcal{E}(t)$ satisfies $\mathcal{E}(t) - \delta(t) = O(h^{m+1})$ and the error estimate is asymptotically correct. Our mesh adaptation is based on the equidistribution of the global error of the numerical solution. Thus, we define a *monitor function* $\Theta(t) := \sqrt[m]{\mathcal{E}(t)}/h(t)$, where $h(t) := |\tau_{i+1} - \tau_i|$ for $t \in [\tau_i, \tau_{i+1}]$. Now, the mesh selection strategy aims at the equidistribution of

$$\int_{\tilde{\tau}_i}^{\tilde{\tau}_{i+1}} \Theta(s)\, ds$$

on the mesh consisting of the points $\tilde{\tau}_i$ to be determined accordingly, where at the same time measures are taken to ensure that the variation of the stepsizes is restricted and tolerance requirements are satisfied with small computational effort. Details of the mesh selection algorithm and a proof of the fact that our strategy implies that the global error of the numerical solution is *asymptotically equidistributed* are given in [7].

We now discuss numerical results for the problem (2.28). In the following, we will make use of the relation between the parameter λ and the value $u(1/2) =: a$, cf. (2.30),

$$\lambda = \frac{16}{9}(1-\sqrt{a})(1+2\sqrt{a})^2.$$

For the numerical computations we have chosen to use this relation in a rewritten way,

$$\sqrt{\lambda} = \frac{4}{3}(1-\sqrt{a})^{3/2} + 4\sqrt{a}\sqrt{1-\sqrt{a}}, \tag{3.35}$$

see Figure 1 for illustration.

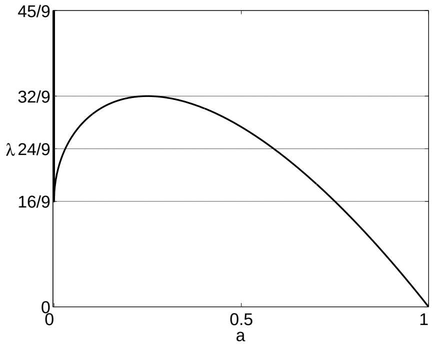

Figure 1: Relation between λ and a.

From the preceding analysis and the form of the graph it is clear that there exists a unique positive solution with $a > 0$, if $\lambda \in [0, \frac{16}{9}) \cup \{\frac{32}{9}\}$, while for $\lambda \in (\frac{16}{9}, \frac{32}{9})$ we should be able to find two positive solutions. For $\lambda = \frac{16}{9}$ there also exists a unique solution with $a = 0$. In addition, Example 2.13 shows that for each $\lambda > \frac{16}{9}$ we have a unique solution with $a = 0$. The solution corresponding to $a = 0$ and $\lambda = \frac{16}{9}$ is the pseudo dead core solution, while the solutions corresponding to $a = 0$ and $\lambda > \frac{16}{9}$ are the dead core solutions.

4 Numerical results

In this section, we show that the collocation code implemented in `bvpsuite` can be used to simulate all solutions to problem (1.5), cf. positive solutions in Section 4.1, pseudo dead core solutions in Section 4.2, and dead core solutions in Section 4.3.

4.1 Positive Solutions

For the numerical treatment we reformulate problem (1.5) as follows,

$$u''(t)\sqrt{u(t)} = \frac{16}{9}(1-\sqrt{a})^3 + 16a(1-\sqrt{a}) + \frac{32}{3}\sqrt{a}(1-\sqrt{a})^2, \quad (4.36a)$$

$$u(0) = 1,\ u(1) = 1,\ a = \zeta, \quad (4.36b)$$

with $\zeta \in (0, 1]$. If not stated otherwise, the initial guess $u(t) \equiv 1$ is used.

We know that unique positive solutions exist for $\lambda \in [0, \frac{16}{9}]$ and for $\lambda = \frac{32}{9}$ and that two positive solutions exist for $\lambda \in (\frac{16}{9}, \frac{32}{9})$. Figure 2 shows the numerical solution, the error estimate and the residual for $\lambda = \frac{32}{9}$. The residual $r(t)$ is calculated by substituting the collocation solution p into the differential equation (4.36a),

$$r(t) := p''(t)\sqrt{p(t)} - \frac{16}{9}(1-\sqrt{a})^3 + 16a(1-\sqrt{a}) + \frac{32}{3}\sqrt{a}(1-\sqrt{a})^2, \quad t \in (0, 1).$$

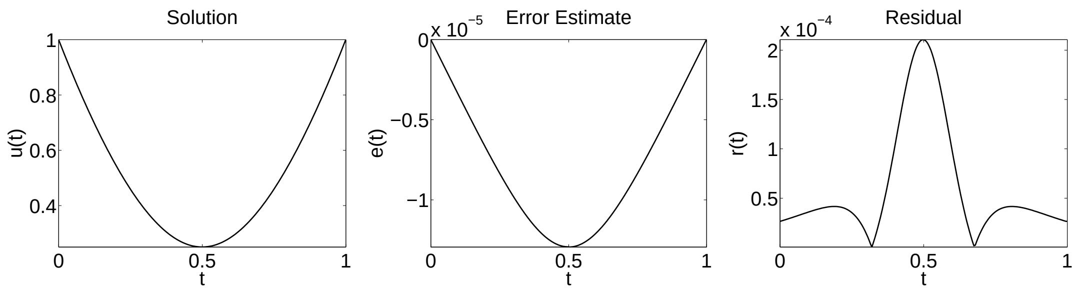

Figure 2: The numerical solution, the error estimate and the residual for $\lambda = \frac{32}{9}$.

Although, the error estimate and the residual indicate that the numerical solution is reasonably accurate, we use another approach to further verify the quality of the calculations. We know that $u(\frac{1}{2}) = a$, where a can be calculated from (3.35). For the numerical solution $\left|p(\frac{1}{2}) - u(\frac{1}{2})\right| = 1.7\ 10^{-7}$ holds. This shows a good accuracy of the approximation.

Moreover, we fix $a = c$, where the constant c can be calculated from the relation between a and λ, $\lambda = \frac{16}{9}(1-\sqrt{a})(1+2\sqrt{a})^2$. Then, we solve the following problem for $u(t)$ and λ, by imposing an additional condition[2] $u\left(\frac{1}{2}\right) = a$:

$$u''(t)\sqrt{u(t)} = \lambda, \quad (4.37a)$$

$$u(0) = 1,\ u(1) = 1,\ u(1/2) = a. \quad (4.37b)$$

[2]Here, λ becomes an additional unknown.

The resulting λ^{num} can now be compared to the exact value $\lambda = \frac{16}{9}(1-\sqrt{a})(1+2\sqrt{a})^2$, and we obtain $|\lambda^{num} - \lambda| \approx 5.8\ 10^{-10}$. Both tests support previous findings about the solution's accuracy.

For $\lambda \in (\frac{16}{9}, \frac{32}{9})$ there exist two positive solutions. Figures 3 and 4 show both solutions for $\lambda = \frac{22}{9}$.

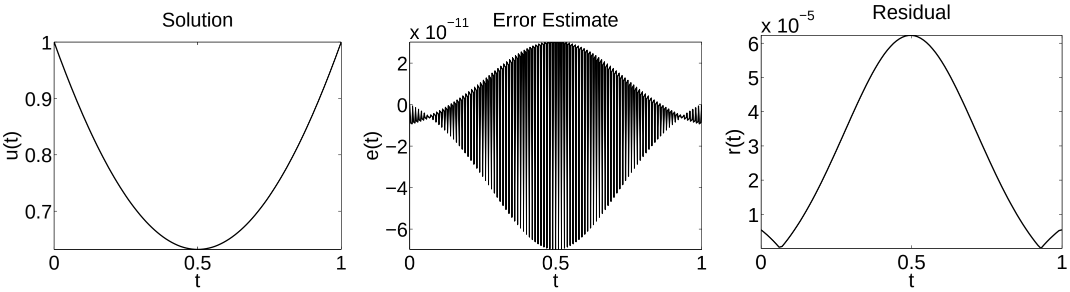

Figure 3: The numerical solution, the error estimate and the residual for $\lambda = \frac{22}{9}$ ($a \approx 0.632$).

Here, $|\lambda^{num} - \lambda| \approx 2.5\ 10^{-10}$.

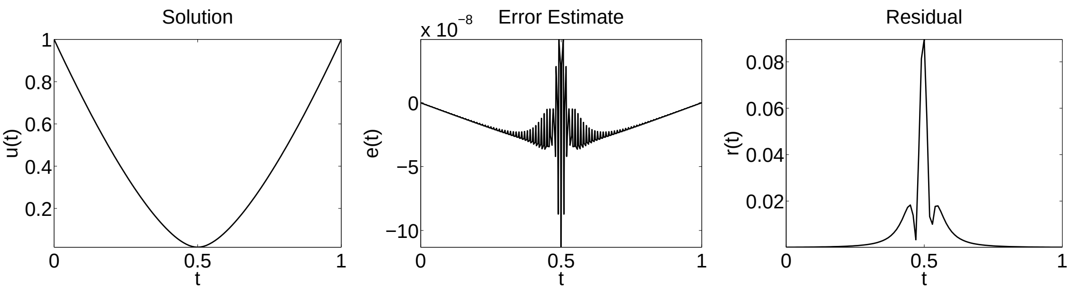

Figure 4: The numerical solution, the error estimate and the residual for $\lambda = \frac{22}{9}$ ($a \approx 0.016$).

For the solution shown in the Figure 3 we obtain $\left|p(\frac{1}{2}) - u(\frac{1}{2})\right| = 5.5\ 10^{-11}$ and for the solution from Figure 4, $\left|p(\frac{1}{2}) - u(\frac{1}{2})\right| = 9.0\ 10^{-8}$ and $|\lambda^{num} - \lambda| \approx 1.7\ 10^{-6}$.

The unique positive solution for $\lambda = \frac{16}{9}$ is shown in Figure (5).

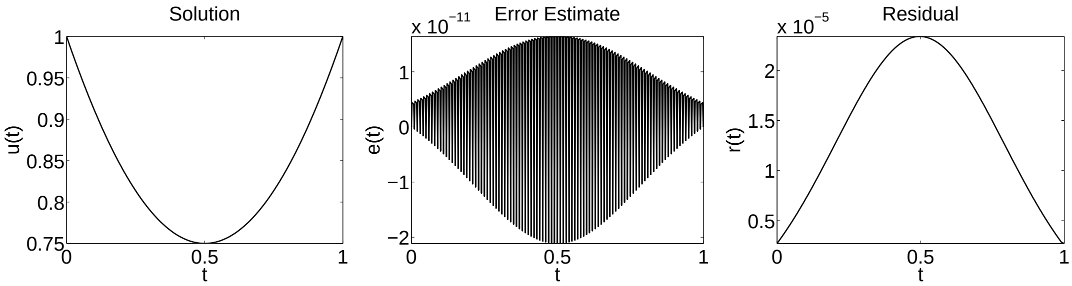

Figure 5: The numerical solution, the error estimate and the residual for $\lambda = \frac{16}{9}$.

Again the positive solution is very accurate, $\left|p(\frac{1}{2}) - u(\frac{1}{2})\right| = 1.7\ 10^{-11}$. Also the error $|\lambda^{num} - \lambda| \approx 9.2\ 10^{-11}$ is very small.

Finally, for $\lambda \in [0, \frac{16}{9})$ there exists a unique positive solution. We have chosen $\lambda = \frac{3}{9}$ and the results are shown in Figure 6.

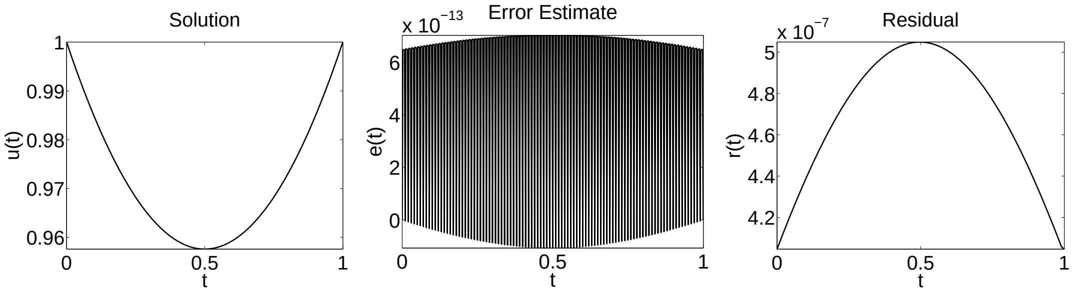

Figure 6: The numerical solution, the error estimate and the residual for $\lambda = \frac{3}{9}$.

Here, $\left|p(\frac{1}{2}) - u(\frac{1}{2})\right| = 8.4\ 10^{-14}$ and $|\lambda^{num} - \lambda| \approx 5.7\ 10^{-13}$.

4.2 Pseudo Dead Core Solution

In order to calculate the pseudo dead core solution for $\lambda = \frac{16}{9}$, we have to rewrite problem (1.5) as follows,

$$u''(t)\sqrt{u(t)}u(t) = \lambda u(t), \quad t \in [0,1], \tag{4.38a}$$

$$u(0) = 1,\ u(1) = 1. \tag{4.38b}$$

Recall, that now $u(1/2) = 0$ and this means that the form of (1.5a) is no more adequate for the numerical treatment. We have run the tests on an equidistant mesh with 500 mesh points and the collocation polynomials of degree 4. The initial approximation is given in Figure 7.

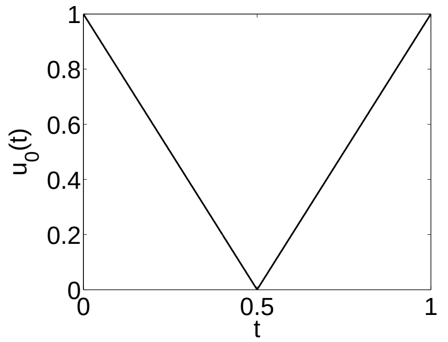

Figure 7: The initial solution approximation for pseudo dead core and dead core solutions of the modified problem (4.38).

The final numerical approximation to the pseudo dead core solution can be found in Figure 8. Due to larger values of higher derivatives of u, especially in the region $t \approx 1/2$, the error estimation procedure becomes unreliable and therefore, we plot the available exact global error $|p(t) - u(t)|$ instead of its estimate. The exact solution $u(t)$ has been calculated using (2.29).

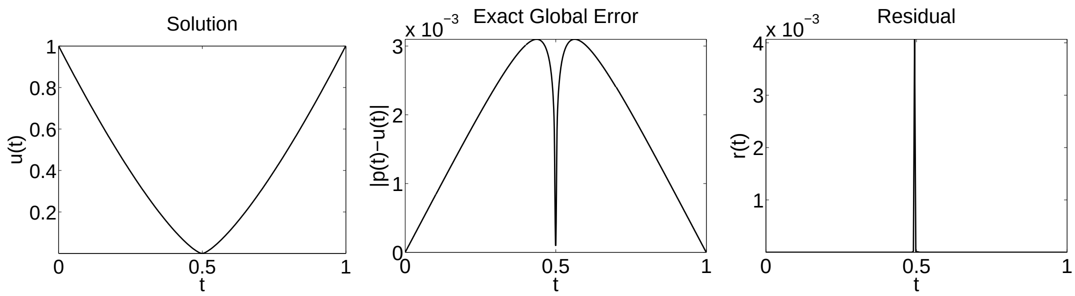

Figure 8: Problem (4.38): The numerical solution, the exact global error $|p(t) - u(t)|$ and the residual for $\lambda = \frac{16}{9}$.

4.3 Dead Core Solutions

In this section, we compute the dead core solutions of problem (4.38) for the values $\lambda = \frac{32}{9}$, $\lambda = \frac{60}{9}$ and $\lambda = \frac{1000}{9}$, cf. Figure 7 for the initial guess. The numerical results are depicted in Figures 9, 10, and 11, respectively. Again we plot the global error $|p(t) - u(t)|$ instead of the error estimate.

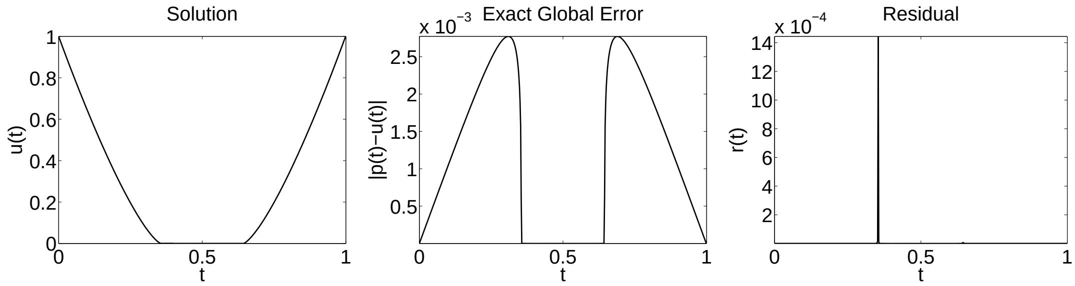

Figure 9: Problem (4.38): The numerical solution, the exact global error $|p(t) - u(t)|$ and the residual for $\lambda = \frac{32}{9}$.

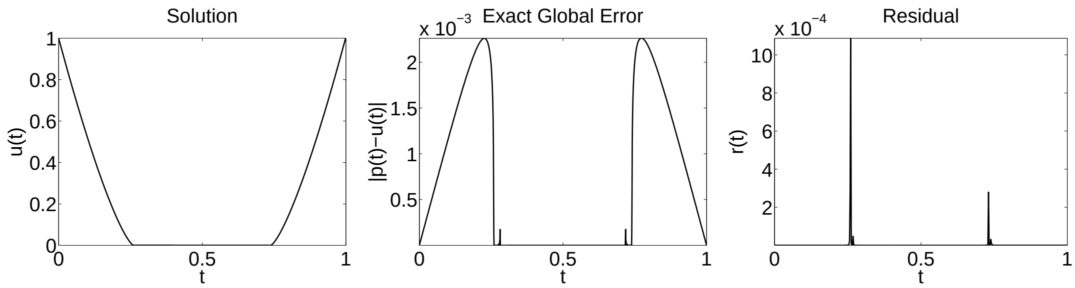

Figure 10: Problem (4.38): The numerical solution, the exact global error $|p(t) - u(t)|$ and the residual for $\lambda = \frac{60}{9}$.

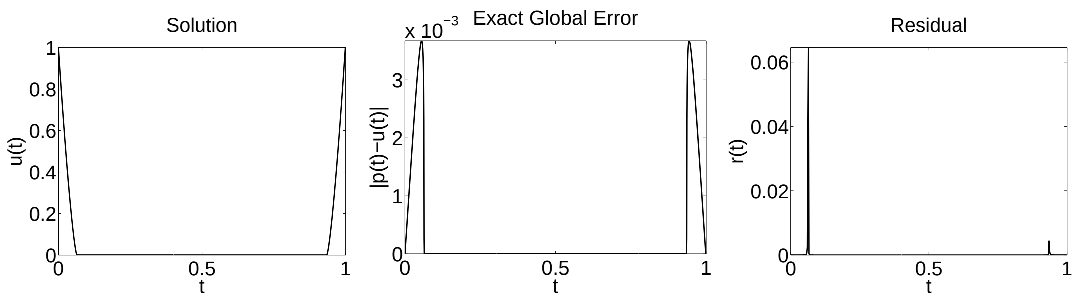

Figure 11: Problem (4.38): The numerical solution, the exact global error $|p(t) - u(t)|$ and the residual for $\lambda = \frac{1000}{9}$.

The global error indicates that the numerical solution is quite accurate. Even for $\lambda = \frac{1000}{9}$ with an extremely difficult solution its quality does not differ much when compared to smaller values of λ.

Since in case of dead core solutions the information on the structure of the exact solution is available, we could in principle use other approaches to solve the problem. However, to carry them out we at least need to know in which region the dead core solution vanishes. Therefore, the following simulations are not possible, in general.

The idea is to split the problem into two subproblems, a left and a right problem, and solve

$$u''(t)\sqrt{u(t)} = \lambda, \quad t \in [0, (2/3\sqrt{\lambda})], \tag{4.39a}$$
$$u(0) = 1, \ u(2/(3\sqrt{\lambda})) = 0, \tag{4.39b}$$

and

$$u''(t)\sqrt{u(t)} = \lambda, \quad t \in [1 - 2/(3\sqrt{\lambda}), 1], \tag{4.40a}$$
$$u(1) = 1, \ u(1 - 2/(3\sqrt{\lambda})) = 0, \tag{4.40b}$$

as a left and right boundary value problem, respectively.

We use collocation at two Gaussian points with both, absolute and relative tolerances set to 10^{-6}. As an initial guess we again choose $u(t) \equiv 1$. Figures 12 and 13 show the solutions, error estimates and residuals for the left and the right problem and $\lambda = \frac{60}{9}$.

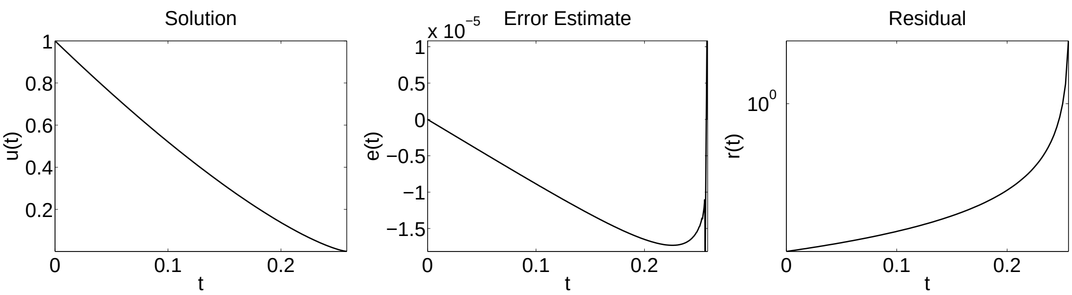

Figure 12: The numerical solution, the error estimate and the residual of the left problem for $\lambda = \frac{60}{9}$.

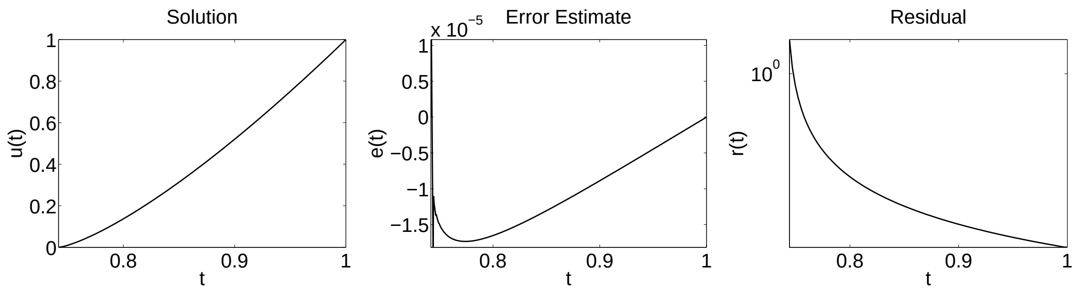

Figure 13: The numerical solution, the error estimate and the residual of the right problem for $\lambda = \frac{60}{9}$.

We can see that in the critical area, where the solution values are small, the values of the residual are large. However, it is well-known that the residual is the so-called *backward error* and its values only indirectly measure the quality of the solution. In general, even when the residual is comparably large, the error does not need to be large, and the solution can be reasonably accurate. To clarify this matter, we solved the problem

$$u''(t)\sqrt{u(t)} = \lambda, \quad t \in [0, t_0],$$
$$u(0) = 1,\ u'(2/(3\sqrt{\lambda})) = 0,\ u(2/(3\sqrt{\lambda})) = 0,$$

for $u(t)$ and for λ. The numerical value λ^{num} has been compared to the exact value $\lambda = \frac{60}{9}$, and we obtained $|\lambda^{num} - \lambda| \approx 2.0\ 10^{-2}$, $|(\lambda^{num} - \lambda)/\lambda| = 3.0\ 10^{-3}$. Fully analogously, we solved the related right problem. The respective values for the errors in λ are comparable. This means that the solution's accuracy can be regarded to be sufficiently good.

As a final approach to calculate the dead core solutions we use an initial value approach. Due to the symmetry of u, it is clear, that $u'(2/(3\sqrt{\lambda})) = 0$ and $u'(1 - 2/(3\sqrt{\lambda})) = 0$ for the left and the right problem, respectively. Therefore, we solve the following left initial value problem

$$u''(t)\sqrt{u(t)} = \lambda, \quad t \in [0, 2/(3\sqrt{\lambda})],$$
$$u'(2/(3\sqrt{\lambda})) = 0,\ u(2/(3\sqrt{\lambda})) = 0,$$

and the right initial value problem

$$u''(t)\sqrt{u(t)} = \lambda, \quad t \in [1 - 2/(3\sqrt{\lambda}), 1],$$
$$u'(1 - 2/(3\sqrt{\lambda})) = 0,\ u(1 - 2/(3\sqrt{\lambda})) = 0.$$

Here, all calculations have been carried out using tolerances set to 10^{-3}, because stricter tolerances would have resulted in much denser grids. Figures 14 and 15 show the results of these computations.

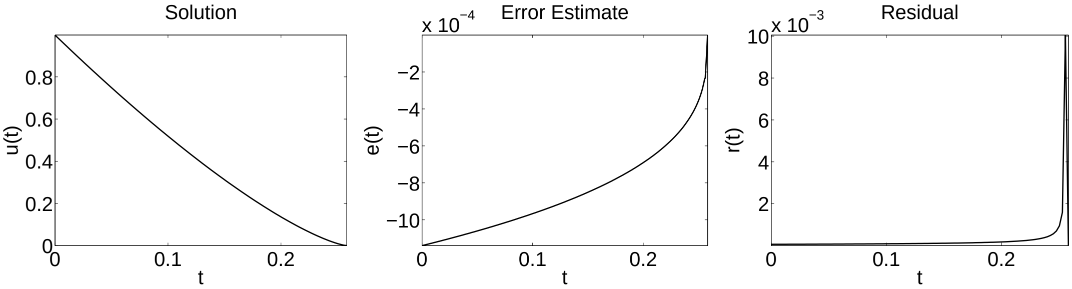

Figure 14: The numerical solution, the error estimate and the residual of the left problem for $\lambda = \frac{60}{9}$.

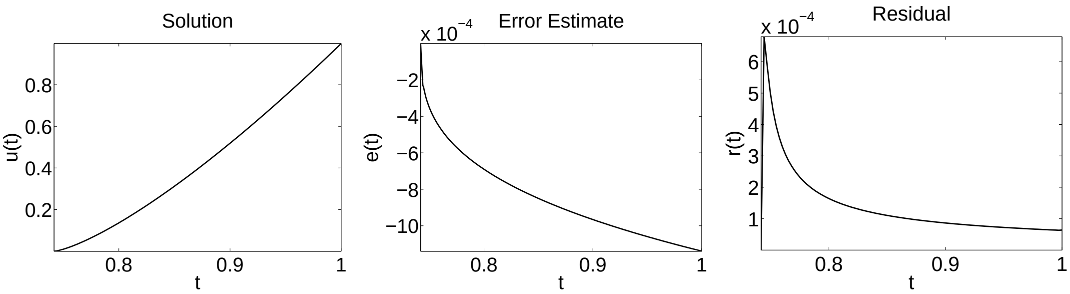

Figure 15: The numerical solution, the error estimate and the residual of the right problem for $\lambda = \frac{60}{9}$.

The solution values $u(0)$ for the left problem and $u(1)$ for the right problem are approximately 0.998 and differ only at the 13th position after decimal point.

Let us denote the numerical solution obtained by solving the right or left boundary value problem by $u_n(t)$, the numerical solution obtained using the initial value approach by $u_i(t)$, and the analytical solution (2.29) by $u_a(t)$. Now, we take a look how these solutions relate to each other. Figures 16 and 17 show $e_1(t) := |u_i(t) - u_a(t)|$, and $e_2(t) := |u_n(t) - u_a(t)|$ for the left and the right problem.

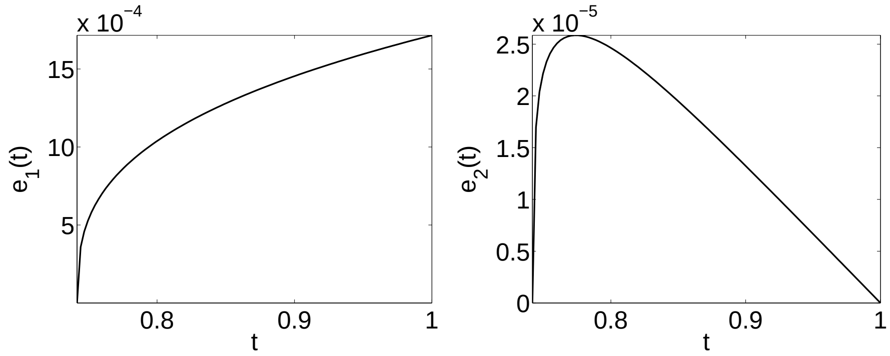

Figure 16: Comparing u_i and u_n originating from different approaches and their global errors: e_1 and e_2 for $\lambda = \frac{60}{9}$ for the left problem.

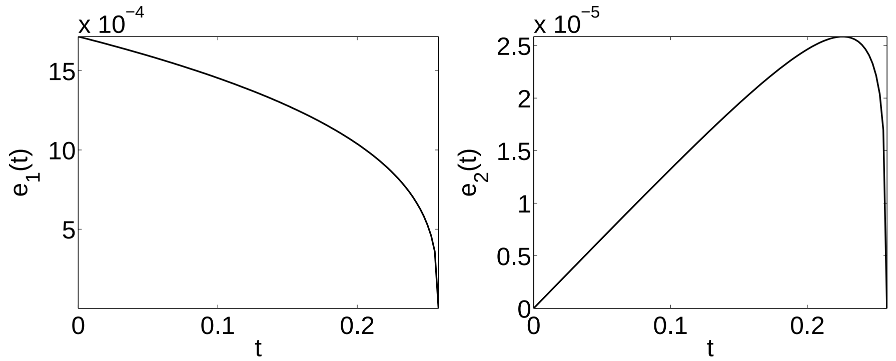

Figure 17: Comparing u_i and u_n originating from different approaches and their global errors: e_1 and e_2 for $\lambda = \frac{60}{9}$ for the right problem.

The above figures suggest that u_i and u_n are both very good approximations for u_a. Moreover, global errors of p, u_i and u_n are comparable.

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