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Quantum kinetic Fokker-Planck equ.: global-in-time solutions & dispersive effects

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model:

- quantum Fokker-Planck equ. (augmented Caldeira-Leggett)
- t-evolution of open quantum system in one-particle approximation (e.g. quantum diffusion)
- evolution for Wigner function $w(x, v, t)$:
  $$w_t + v \cdot \nabla_x w + \Theta[V]w = Qw; \quad x, v \in \mathbb{R}^d,\ t > 0$$
  $$\Theta[V]w(x, v)$$
  $$= i(2\pi)^{-\frac{d}{2}} \int [V(x + \tfrac{\eta}{2}) - V(x - \tfrac{\eta}{2})]\hat{w}(x, \eta)e^{i\eta v} d\eta$$
- Qw models diffusive effects: quantum FP-/relaxation term/ "Q-Boltzmann"
- evolution for density matrix $\hat{\rho}(t)$:

$$\hat{\rho}_t = \underbrace{-i[H, \hat{\rho}]}_{\text{Hamiltonian}} + \sum_j \underbrace{L_j \hat{\rho} L_j^* - \frac{1}{2}(L_j^* L_j \hat{\rho} + \hat{\rho} L_j^* L_j)}_{\text{dissipative}}$$

applications:

- electron transport in quantum semiconductor; electron-phonon-interaction
  $\longrightarrow$ numerical simulation of resonant tunneling diode
- mean-field model: selfconsistent potential $V(x,t)$:

$$-\Delta V = n = \int w(x,v,t)dv$$

  $\longrightarrow$ open quantum system with mean-field potential

questions:

- well-posedness (existence, uniqueness, regularization)
- steady states (in confinement potential)
- $t \longrightarrow \infty$ behavior (convergence rate to steady state)
- macroscopic scaling limits of WPFP
- numerical methods (deterministic spectral methods; Monte Carlo methods)

Outline:

I) analytical difficulties

II) idea of Wigner-Poisson analysis

III) density matrix analysis

IV) kinetic Wigner-Fokker-Planck equation

- dispersive effects $\Rightarrow$ a-priori estimates
- (iterative) construction of the solution

V) perspectives / open problems

I) Wigner-Poisson-Fokker-Planck

$$\begin{cases} w_t + v \cdot \nabla_x w + \Theta[V]w = Qw \\ w(x,v,t=0) = w_0(x,v) \\ -\Delta V(x,t) = n; \quad n(x,t) = \int w(x,v,t)dv \\ Qw = \underbrace{\sigma \Delta_v w}_{\text{class. diffusion}} + \underbrace{\beta \, \text{div}_v(vw)}_{\text{friction}} + \underbrace{\alpha \Delta_x w + 2\gamma \, \text{div}_x(\nabla_v w)}_{\text{quantum diffusion}} \end{cases}$$

- quantum mechanically "correct" ($\Rightarrow n(x) \geq 0$) if:
  $\alpha\,\sigma - \gamma^2 \geq \frac{\beta^2}{16}$ $\longrightarrow$ classical FP-term not OK

analytical problems:

(1) non-linear potential – definition of $n(x,t)$:
natural setting: $w(.,.,t) \in L^2(\mathbb{R}^{2d})$, usually $w \notin L^1$
$\rightarrow n \stackrel{?}{=} \int w dv$
one possibility [AA-Dhamo-Manzini, Ann.IHP-C '07]:
w in v-weighted space: $w \in L^2((1+|v|^4)dxdv)$ for $d=3$

(2) a-priori estimates for global-in-t solution (beyond $||w(t)||_{L^2}$)

II) Wigner-Poisson analysis:

$$\begin{cases} w_t + v \cdot \nabla_x w + \Theta[V]w = 0 \\ -\Delta V = n = \int w dv \end{cases}$$

(a) reformulation as Schrödinger-Poisson system:

$$w(x,v,t) = \mathcal{F}_{\eta \to v} \sum_{j \in \mathbb{N}} \lambda_j \underbrace{\psi_j(x + \frac{\eta}{2}, t)\overline{\psi_j(x - \frac{\eta}{2}, t)}}_{\text{pure quantum state}}$$

- $\lambda_j \geq 0$ ... occupation probability (const. in t !)

$$\Rightarrow \begin{cases} i\partial_t \psi_j = -\frac{1}{2}\Delta_x \psi_j + V(x,t)\psi_j; \quad j \in \mathbb{N};\ x \in \mathbb{R}^d \\ -\Delta V(x,t) = n(x,t) = \sum_j \lambda_j |\psi_j|^2 \end{cases}$$

- assumption on w_0 : $\lambda_j \geq 0,\ \{\lambda_j\} \in l^1$

  trick:

$$\psi_j \in L^2(\mathbb{R}^d) \Rightarrow n(.,t) \in L^1(\mathbb{R}^d)$$

  [Brezzi-Markowich '91] for 3D

- reformulation impossible for open quantum systems — λ_j not const!

(b) $\frac{i}{2}\Delta_x$ generates a linear evolution group on $H^1(\mathbb{R}^d)$

(c) 3D: $V\psi = \left(\frac{1}{4\pi|x|} * |\psi|^2\right)\psi$
... is local Lipschitz map on $H^1(\mathbb{R}^3)$
$\Rightarrow$ $\exists!$ local-in-t solution of non-linear problem

(d) <u>a-priori-estimates:</u>

$$\|\psi_j(t)\|^2_{L^2} = const$$

... mass conservation

$$\underbrace{\frac{1}{2}\sum_j \lambda_j \|\nabla\psi_j\|^2_{L^2}}_{E_{kin}} + \underbrace{\frac{1}{2}\|\nabla V\|^2_{L^2}}_{E_{pot}} = const$$

... energy conservation

$\Rightarrow$ global-in-t solution: $\psi_j \in C(0,\infty; H^1(\mathbb{R}^3))$

III) density matrix analysis for WPFP

$$w(x,v,t) = \mathcal{F}_{\eta\to v}\rho\left(x+\tfrac{\eta}{2}, x-\tfrac{\eta}{2}, t\right)$$
$$w \in \mathbb{R} \quad \leftrightarrow \quad \rho(x,y) = \overline{\rho(y,x)}$$

- density matrix operator on $L^2(\mathbb{R}^d)$:

$$(\hat\rho f)(x) = \int \rho(x,y) f(y) dy \qquad \text{... self-adjoint}$$

- physical quantum state:

$$\hat\rho \geq 0,\ \hat\rho \in \mathcal{J}_1(L^2(\mathbb{R}^d)) \qquad \text{... trace class operator}$$

eigenvalues: $\{\lambda_j\} \in l^1$
eigenfunctions: $\{\psi_j\}$

(a) definition of particle density:

$$n(x,t) = \text{“}\rho(x,x,t)\text{“} \dots \quad \begin{matrix}\text{formal for } \hat\rho \in \mathcal{J}_2 \\ \text{rigorous for } \hat\rho \in \mathcal{J}_1\end{matrix}$$

$$\|n\|_{L^1(\mathbb{R}^d)} \leq \|\hat\rho\|_{\mathcal{J}_1} = \mathrm{Tr}\hat\rho = const \text{ in } t$$

kinetic energy:
$E_{\mathrm{kin}}(\hat\rho) = -\frac{1}{2}\mathrm{Tr}\,(\Delta\hat\rho) \geq 0$ for $\hat\rho \geq 0$

time evolution:

- if $\rho(x,y) \in L^2(\mathbb{R}^{2d})$ $\Leftrightarrow$ $\hat\rho \in \mathcal{J}_2 = HS$ $\Rightarrow$ PDE:

$$\rho_t = -i(H_x - H_y)\rho - \frac{\beta}{2}(x-y)\cdot(\nabla_x - \nabla_y)\rho +$$
$$+ \left[\alpha|\nabla_x + \nabla_y|^2 - \sigma|x-y|^2 + 2i\gamma(x-y)\cdot(\nabla_x + \nabla_y)\right]\rho$$

$$H_x = -\frac{1}{2}\Delta_x + V(x,t)$$

- if $\hat\rho \in \mathcal{J}_1$ $\Rightarrow$ no "nice" space for kernel $\rho(x,y)$ $\Rightarrow$ no PDE !

$$\frac{\mathrm{d}}{\mathrm{dt}}\hat\rho = -i[\tilde H, \hat\rho] + \underbrace{\sum_k L_k\hat\rho L_k^* - \frac{1}{2}\left(L_k^*L_k\hat\rho + \hat\rho L_k^* L_k\right)}_{=:A(\hat\rho)} \qquad (1)$$

$$\tilde H = -\frac{1}{2}\Delta_x + V(x,t) - i\frac{\beta}{4}\{x, \nabla\}$$

- (1) is in *Lindblad form* [Lindblad '76]
  $\Leftrightarrow$ positivity is preserved: $\hat\rho(t) \geq 0$;
  ("$\Leftarrow$" for complete positivity, only for bounded L_k)

  $\Rightarrow$ formally dissipative in $\mathcal{J}_1$

Examples of Lindblad operators:

1) $L_1 = x \quad \Rightarrow Qw = \frac{1}{2}\Delta_v w$

2) $L_1 = \nabla_x \quad \Rightarrow Qw = \frac{1}{2}\Delta_x w$

3) $Q = QFP \Rightarrow L_k = \alpha_k \cdot x + \beta_k \cdot \nabla_x\,; \quad k = 1, .., 2d$

4)

$$Qw = \frac{w_0 - w}{\tau}, \quad w_0 \leftrightarrow \hat{\rho}_0 = \sum_{j \in \mathbb{N}} \mu_j |\phi_j >< \phi_j|, \ \mathrm{Tr}\,\hat{\rho}_0 = 1$$

$$\Rightarrow L_{jk} = \sqrt{\frac{\mu_k}{\tau}} |\phi_k >< \phi_j|, \quad j, k \in \mathbb{N}$$

L_k is NOT uniquely defined by Q

(b) <u>construction of linear evolution semigroups:</u>

- dissipative open quantum system (V given):

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{dt}}\hat{\rho} &= \mathcal{L}(\hat{\rho}) := -i[\tilde{H},\hat{\rho}] + A(\hat{\rho}), \quad t \geq 0 \\ \hat{\rho}(t=0) &= \hat{\rho}_0 \end{cases}$$

$A(\hat{\rho})$ ... dissipative / Lindblad terms

[E. Davies '76]: $\exists$ a linear C_0-semigroup on $\mathcal{J}_1$ ("minimal solution")

<u>possible problems:</u>

- semigroup <u>not</u> unique
- $\mathcal{D}(\mathcal{L})$ "too small"
- <u>not</u> conservative: Tr $(\hat{\rho}(t)) \leq$ Tr $\hat{\rho}_0$

$\Rightarrow$ need to prove: $\mathcal{D}(\overline{\mathcal{L}})$ is "big enough"

• need to prove: $\mathcal{D}(\overline{\mathcal{L}})$ is “big enough”:

Lemma [AA-Sparber, CMP '04]:
since $\mathcal{L}$ is “quadratic” in x and ∇_x
$\Rightarrow \overline{\mathcal{L}|_{D_\infty}}$ is the “maximum extension” in $\mathcal{J}_1$
$D_\infty \subseteq \mathcal{J}_1$ ... dense subset with C_0^∞-kernels

Proof: for $\hat{\rho} \in \mathcal{D}(\mathcal{L}_{\text{max}}) = \{\hat{\rho} \in \mathcal{J}_1 | \mathcal{L}(\hat{\rho}) \in \mathcal{J}_1\}$:

$$D_\infty \ni \hat{\sigma}_n \overset{n\to\infty}{\longrightarrow} \hat{\rho} \quad \text{in graph norm } \|.\|_{\mathcal{L}}$$

$$\sigma_n(x,y) := \underbrace{\chi_n(x)}_{C_0^\infty\text{-cutoff}} \left[\varphi_n(x) *_x \rho(x,y) *_y \underbrace{\varphi_n(y)}_{C_0^\infty\text{-mollifier}} \right] \chi_n(y)$$

$\Rightarrow$ Theorem: C_0-semigroup $e^{\mathcal{L}t}$ of Davies is unique & trace preserving ($\leftarrow$ first for classical solutions + density argument)

(c) kinetic energy space:

$$
\begin{aligned}
E_{\text{kin}}(\hat\rho) &= -\tfrac{1}{2}\mathrm{Tr}\,(\Delta\hat\rho) \geq 0 \qquad \text{for } \hat\rho \geq 0 \\
\mathcal{E} &:= \{\hat\rho \in \mathcal{J}_1 | E_{\text{kin}}(\hat\rho) < \infty\} \\
\|\hat\rho\|_{\mathcal{E}} &:= \||\, \sqrt{1-\tfrac{1}{2}\Delta}\hat\rho\sqrt{1-\tfrac{1}{2}\Delta}\, \||_{\mathcal{J}_1} \\
&\overset{\hat\rho\geq 0}{=} \ \mathrm{Tr}\,\hat\rho - \tfrac{1}{2}\mathrm{Tr}\,(\Delta\hat\rho)
\end{aligned}
$$

- $e^{\mathcal{L}t}$ is C_0-semigroup in $\mathcal{E}$
  (Davies' semigroup-construction only in $\mathcal{J}_1$
  $\Rightarrow$ prove t-continuity of $\|e^{\mathcal{L}t}\hat\rho\|_{\mathcal{E}}$ explicitly)

- Hartree-term $[V[\hat\rho], \hat\rho]$ ... local Lipschitz map in $\mathcal{E}$ (but not in $\mathcal{J}_1$ $\longrightarrow$ reason for using $\mathcal{E}$)

  Proof:

$$
\begin{aligned}
\|V[\hat\rho]\|_{L^p} &\leq C_q\|n[\hat\rho]\|_{L^q} \\
&\leq C_q\||\hat\rho\||^{\theta}_{\mathcal{J}_1} \cdot E_{\text{kin}}(\hat\rho)^{1-\theta} \\
& \quad 3 < p \leq \infty,\ 1 \leq q \leq 3
\end{aligned}
$$

  generalized Lieb-Thirring inequality [Lions-Paul '93], [A.A., CPDE '96]

  $\Rightarrow$ local-in-t solution

(d) a-priori estimate for total energy:

$$E_{\text{tot}}(\hat{\rho}) := E_{\text{kin}}(\hat{\rho}) + \frac{1}{2}\|\nabla_x V[\hat{\rho}]\|_{L^2}^2$$
$$\frac{\text{d}}{\text{dt}} E_{\text{tot}} = d\,\sigma\,\text{Tr}\,\hat{\rho}_0 - \beta\, E_{\text{kin}}(t) - \alpha\,\|n(t)\|_{L^2}^2$$

Theorem [AA-Sparber, CMP '04]:
∃! global-in-t, trace preserving, finite energy solution of WPFP:

$$\hat{\rho} \in C(0,\infty;\mathcal{E})$$

advantage:

- physical "energy space" $\mathcal{E} := \{\hat{\rho} \in \mathcal{J}_1 | E_{\text{kin}}(\hat{\rho}) < \infty\}$
- physical a-priori estimates
- extension to Hartree-Fock straightforward

draw-back:

- set-up won't generalize to bounded domain problems

IV) kinetic Wigner-Poisson-FP analysis

goals of a-priori estimates for WPFP:

- global-in-time solution
- construction of the solution via fixed point map

difficulties with a-priori estimates:

- the only "trivial" estimate: $\|w(t)\|_2 \le e^{\frac{3}{2}\beta t}\|w_0\|_2,\ t \ge 0$, since $\Theta[V]$ is skew-adjoint
  (but *no* other L^p–estimates)
- mass conservation ($\int\int w(x,v,t)\,dxdv = const$)
  and energy balance ($E_{\text{kin}}(t) = \frac{1}{2}\int\int |v|^2 w(x,v,t)\,dxdv$)
  not usable, since $w \in \mathbb{R}$ (unless $\hat{\rho}(t) \ge 0$ is used)

idea:

- $n[w] = \int w\,dv$ *not* naturally defined, but
- $E = -\nabla V = C\frac{x}{|x|^3} * \int w\,dv$
  can be defined with dispersive regularization of free transport

a-priori estimate for electric field on any $(0, T]$

idea: use dispersive effects of free transport
([Perthame '96] for VP, [Castella '98] for VPFP)

- strategy for Wigner-Poisson:

$$w(x,v,t) = w_0(x-vt,v) - \int_0^t (\Theta[V]w)(x-vs,v,t-s)\,ds$$

split field $E(x,t) = -\nabla V(x,t) = -\frac{1}{4\pi}\frac{x}{|x|^3} * n(x,t)$:

$$\begin{aligned} E_0(x,t) &= -\frac{1}{4\pi}\frac{x}{|x|^3} *_x \int w_0(x-vt,v)\,dv \\ E_1(x,t) &= \frac{1}{4\pi}\frac{x}{|x|^3} *_x \underbrace{\int_0^t \int_{\mathbb{R}^3_v} (\Theta[V]w)(x-vs,v,t-s)\,dv ds}_{=-n_1(x,t)} \end{aligned}$$

- cp. Vlasov-Poisson [Perthame '96]:

$$\begin{aligned} n_1(x,t) &= \int_0^t \int (\nabla_x V \cdot \nabla_v w)(x-vs,v,t-s)\,dv ds \\ &= \mathrm{div}_x \int_0^t s \int (\underbrace{\nabla_x V}_{=-E}\, w)(x-vs,v,t-s)\,dv ds \end{aligned}$$

a-priori estimate for electric field (WP - cont.)

- reformulation of $\Theta[V]$:

$$\Theta[V]w = \mathcal{F}^{-1}_{\eta\to v}\left(\delta V(x,\eta)\hat{w}(x,\eta)\right) = \nabla_x V *_x \Phi(x,v) *_v \nabla_v w,$$

$$\delta V(x,\eta) = V(x+\frac{\eta}{2}) - V(x-\frac{\eta}{2}) = \int_{-\frac{1}{2}}^{\frac{1}{2}} \eta\cdot\nabla_x V(x-r\eta)\,dr$$

1D:

$$\delta V(x,\eta)\hat{w}(x,\eta) = \eta\nabla_x V *_x \underbrace{\left(\frac{1}{|\eta|}\chi_{[-|\eta|/2,|\eta|/2]}\right)(x)}_{=:\hat{\Phi}(x,\eta)}\ \hat{w}(x,\eta)$$

- field in WP: $(E_1)_j(x,t) =$

$$= C\sum_{k=1}^{3}\frac{x_j x_k}{|x|^5} *_x \int_0^t s\int \left(\partial_{x_k} V *_x \Phi *_v w\right)(x-vs,v,t-s)\,dv\,ds$$

Lemma Let $w_0 \in L^2$, $\|\int w_0(x-vt,v)dv\|_{L^q_x} \le t^{-\omega_q}$, $t \le T$
Then, for $0<t\le T$:

$$\|E_1(t)\|_2 \le C\int_0^t s\,ds\left\|\int (\nabla V *_x \Phi *_v w)(x-vs,v,t-s)dv\right\|_{L^2_x}$$

$$\le C\int_0^t \frac{s\,ds}{s^{3/2}}\|(E_0+E_1)(t-s)\|_2\,\|w_0\|_2$$

$$\Rightarrow \|E_0(t)\|_2,\ \|E(t)\|_2 \le Ct^{-\frac{3}{2q}+\frac{5}{4}-\omega_q}, \quad \text{(VP: } E(t)\in L^p_x\text{)}$$

a-priori estimate for electric field (WPFP)

$$w_t = Aw - \Theta[V]w$$
$$-\Delta V(x,t) = n(x,t) = \int w(x,v,t)dv$$

with

$$Aw := -v\cdot\nabla_x w + \sigma\Delta_v w + \beta\ \mathrm{div}_v(vw) + \alpha\Delta_x w + 2\gamma\ \mathrm{div}_x(\nabla_v w)$$

- Wigner-Poisson-Fokker-Planck solution:

$$\begin{aligned} w(x,v,t) &= \iint G(t) w_0(x_0,v_0)dx_0dv_0 \\ &- \int_0^t \iint G(s)(\Theta[V]w)(x_0,v_0,t-s)\,dx_0dv_0ds \end{aligned}$$

with Green's function $G(x,v,x_0,v_0,t)$ of WFP [Sparber-Carrillo-Dolbeault-Markowich '04]

Lemma Let $w_0 \in L^2$, $\|\int w_0(x-vt,v)dv\|_{L^q_x} \le t^{-\omega_q}$, $t \le T$
Then, for $0 < t \le T$:

- $\|E_0(t)\|_p, \|E(t)\|_p \le Ct^{\frac{3}{2p}-\frac{3}{2q}+\frac{1}{2}-\omega_q}$, $2 \le p < 6$
  $p > 2$ by parabolic regularization of $G(t)$
- need for construction of solution: $E(t) \in L^3_x$, integrable in t

Strichartz estimate for free transport equation

[Castella-Perthame '96]:

$$\| \int w_0(x - vt, v)dv \|_{L^q_x} \leq C|t|^{-3(1-\frac{1}{q})} \|w_0\|_{L^1_x(L^q_v)}, \quad t \in \mathbb{R}$$

as typical assumption on initial condition w_0.

• needed for construction of solution:
$1 \leq q < \frac{6}{5} \Rightarrow \|E(t)\|_3 \in L^1(0,T)$

a-priori estimate for potential V on any $(0,T]$

definition of the potential from Poisson equation div$E = n$:

$$V = \frac{C}{|x|} * n = \frac{C}{|x|} * \mathrm{div} E = C \sum_j \frac{x_j}{|x|^3} * E_j$$

Lemma Let $w_0 \in L^2$, $\|\int w_0(x - vt, v)dv\|_{L^q_x} \le t^{-\omega_q}$, $t \le T$
Then, for $0 < t \le T$:

$$\|V(t)\|_p \le Ct^{\frac{3}{2p} - \frac{3}{2q} + 1 - \omega_q}, \; 6 \le p \le \infty$$

$$\Rightarrow \Theta[V(t)] \in \mathcal{B}(L^2(\mathbb{R}^6)), \quad t > 0$$

definition of self-consistent field, potential, density

goal: define nonlinear term $\Theta[V[w]]w$
$\longrightarrow$ construct solution in iterative map

standard definition – pointwise in t, unfeasible for $w \in L^2$:

$$n(x,t) = \int w(x,v,t)dv, \quad V = \frac{C}{|x|} * n, \quad E = -\nabla V$$

alternative definition (for WP) – nonlocal in t, via integral equation/a-priori estimate for E_1 (map $w \mapsto E_1[w]$ nonlinear):

$$E_1[w]_j(x,t) =$$

$$= C \sum_{k=1}^{3} \frac{x_j x_k}{|x|^5} *_x \int_0^t s \int (E[w]_k *_x \Phi *_v w)(x-vs, v, t-s)\, dvds$$

$$V_1[w] = C \sum_j \frac{x_j}{|x|^3} * E_1[w]_j, \qquad (\ n_1[w] = -\Delta V_1[w]\ )$$

- (a-priori) estimate on $E_1[w]$ only depends on $\|w(t)\|_2$
- $E_1[w] \in L^1((0,T); L^2(\mathbb{R}^3))$ well-defined $\forall w \in C([0,T]; L^2(\mathbb{R}^6))$;
  $w(t)$ need *not* be the self-consistent Wigner function !
- 2 definitions coincide for self-consistent solution/fixed point

construction of the WPFP solution on any $[0, T]$

definition of iterative map M
in $B_R = \{w \in C([0,T]; L^2(\mathbb{R}^6)) \mid \|w\| \le R\}$,
$R := e^{3/2\,\beta T}\|w_0\|_2$ (NB: $\|w(t)\|_2 \le e^{3/2\,\beta t}\|w_0\|_2$)

$$w \longmapsto \underbrace{V[w] = V_0 + V_1[w]}_{\in L^1((0,T);L^\infty(\mathbb{R}^3))} \longmapsto Mw := \tilde{w},$$

with

$$\begin{aligned} \tilde{w}_t &= A\tilde{w} + \Theta[V[w]]\tilde{w}, \quad t \in (0,T] \\ \tilde{w}(t=0) &= w_0 \end{aligned}$$

Lemma Let $w_0 \in L^2$, $\|\int w_0(x{-}vt, v)dv\|_{L^q_x} \le t^{-\omega q}$, $t \le T$
Then, M^n is a contraction on B_R for $n = n(T)$ large enough.

$\Rightarrow$ unique WPFP solution

global-in-time solution for WPFP

Theorem [AA-Dhamo-Manzini, Ind.Univ.MJ '07]
Let $w_0 \in L^2_{x,v} \cap L^1_x(L^q_v)$ for some $1 \leq q < \frac{6}{5}$.

Then, WPFP has a unique *mild* solution:

$$w \in C([0,\infty); L^2(\mathbb{R}^6)) \cap C((0,\infty); C_B^\infty(\mathbb{R}^6))$$

$$n, V, E \in C((0,\infty); C_B^\infty(\mathbb{R}^3))$$

• use a-posteriori: regularization of Green's function

V) perspectives / open problems:

(1) steady state of Wigner-Fokker-Planck in (linear) confinement potential
[Sparber-Carrillo-Dolbeault-Markowich '04]:
for $|x|^2$-potential $\longrightarrow$ exponential decay towards $w_\infty(x, v)$ via entropy method (for non-symmetric Fokker-Planck equ.)

- perturbed potential $|x|^2 + \tilde{V}(x)$: work in progress
- general (linear) confinement potential: open

(2) large-time behavior of QFP $\hat{\rho}(t) \longrightarrow \hat{\rho}_\infty$ (e.g. with relative quantum entropy): open

(3) Wigner-Poisson: (purely kinetic) well-posedness open

(4) Wigner-Poisson: well-posedness (for density matrix $\hat{\rho}_0 \in \mathcal{J}_1$) open (e.g. with $\hat{\rho}$–Strichartz inequalities)