ASC Report No. 47/2020

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E. Davoli and M. Friedrich

Institute for Analysis and Scientific Computing
Vienna University of Technology — TU Wien
www.asc.tuwien.ac.at ISBN 978-3-902627-00-1

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Institute for Analysis and Scientific Computing
Vienna University of Technology
Wiedner Hauptstraße 8–10
1040 Wien, Austria

E-Mail: admin@asc.tuwien.ac.at
WWW: http://www.asc.tuwien.ac.at
FAX: +43-1-58801-10196

ISBN 978-3-902627-00-1

TWO-WELL LINEARIZATION FOR SOLID-SOLID PHASE TRANSITIONS

ELISA DAVOLI AND MANUEL FRIEDRICH

Abstract. In this paper we consider nonlinearly elastic, frame-indifferent, and singularly perturbed two-well models for materials undergoing solid-solid phase transitions in any space dimensions, and we perform a simultaneous passage to sharp-interface and small-strain limits. Sequences of deformations with equibounded energies are decomposed via suitable Caccioppoli partitions into the sum of piecewise constant rigid movements and suitably rescaled displacements. These converge to limiting partitions, deformations, and displacements, respectively. Whereas limiting deformations are simple laminates whose gradients only attain two values, the limiting displacements belong to the class of special functions with bounded variation (SBV). The latter feature elastic contributions measuring the distance to simple laminates, as well as jumps associated to two consecutive phase transitions having vanishing distance, and thus not being detected by the limiting deformations. By Γ-convergence we identify an effective limiting model given by the sum of a quadratic linearized elastic energy in terms of displacements along with two surface terms. The first one is proportional to the total length of interfaces created by jumps in the gradient of the limiting deformation. The second one is proportional to twice the total length of interfaces created by jumps in the limiting displacement, as well as by the boundaries of limiting partitions. A main tool of our analysis is a novel two-well rigidity estimate which has been derived in [32] for a model with anisotropic second-order perturbation.

1. Introduction

Solid-solid phase changes are the physical phenomena for which, by strong temperature or pressure variations, a solid can modify its crystalline structure without undergoing any intermediate liquid phase. Well-known examples are temperature-induced phase transitions between martensite and austenite in shape-memory alloys (see, e.g., [14, 19]), the nucleation of different ice forms at elevated pressure, or the mechanisms behind the evolution of graphite into diamond in carbon composites.

In this paper we focus on materials exhibiting exactly two different phases by considering nonlinearly elastic, frame-indifferent, and singularly perturbed two-well models in any space dimensions. Our goal is to perform a simultaneous passage from nonlinear-to-linearized elastic energies and from diffuse-to-sharp interface descriptions of solid-solid phase transitions. We start by introducing the modeling assumptions and discussing the background. Afterwards, we describe our main results.

In the setting of nonlinear elasticity, the coexistence of two phases can be mathematically described by variational two-well problems, based on the study of energy functionals of the form

$$y \in H^1(\Omega;\mathbb{R}^d) \to \int_\Omega W(\nabla y)\,\mathrm{d}x. \tag{1.1}$$

In the expression above, $\Omega \subset \mathbb{R}^d$, $d \in \mathbb{N}$, is a bounded Lipschitz domain, representing the reference configuration of a material undergoing a solid-solid phase transition between phases $A, B \in \mathbb{M}^{d\times d}$. (Here, $\mathbb{M}^{d\times d}$ is the set of real $d \times d$ matrices.) The stored energy density $W\colon \mathbb{M}^{d\times d} \to [0,+\infty)$ in (1.1) is a nonlinear, frame-indifferent function whose zero set has the two-well structure

$$\{F \in \mathbb{M}^{d\times d}\colon\ W(F) = 0\} = SO(d)A \cup SO(d)B,$$

with $SO(d)$ denoting the set of proper rotations in $\mathbb{M}^{d\times d}$. The model in (1.1) is disadvantaged by a quite unphysical drawback. In fact, whenever A and B are rank-one connected, low energy sequences for generic boundary value problems are known to possibly exhibit highly oscillatory behaviors. In order to prevent this effect, 'phenomenological' higher order regularizations are often incorporated in the energy functional. These may be interpreted as surface energies penalizing the transition between different energy wells. A concrete example is provided by the following *diffuse-interface model*, where transitions between the two wells $SO(d)A$ and $SO(d)B$ are controlled by augmenting (1.1) via a second-order singular perturbation:

$$y \in H^2(\Omega;\mathbb{R}^d) \to I_\varepsilon(y) := \frac{1}{\varepsilon^2}\int_\Omega W(\nabla y)\,\mathrm{d}x + \varepsilon^2 \int_\Omega |\nabla^2 y|^2\,\mathrm{d}x. \tag{1.2}$$

The competition between the two energy contributions in (1.2) is tailored by the smallness parameter $\varepsilon > 0$, which introduces a length scale into the problem. (We adopt it with exponent 2 since this will have notational advantages in the following.) As ε tends to zero, the higher-order perturbation becomes more singular, and I_ε behaves more similarly to a *sharp-interface model*. Roughly speaking, in fact, low-energy sequences for I_ε exhibit transition layers between different phases of width ε^2 (see, e.g., [11, 13, 18, 50, 58]).

Energy functionals as (1.2) are naturally linked to the study of classical Cahn-Hilliard-Modica-Mortola energies, cf. [46, 56, 57], which in turn are strongly connected to the theory of minimal surfaces and to the modeling of liquid-liquid phase transitions. As the width ε of transition layers tends to zero, the behavior of Modica-Mortola energies has been shown to approach, in the sense of Γ-convergence (see [16, 29] for an overview), that of a surface energy being proportional to the length of the interfaces between the different phases. Amidst the extensive literature, we single out the seminal contributions [12, 15, 36, 61, 64, 65] for a characterization of both scalar and vectorial Modica-Mortola energies, the results [51] for an analysis of local minimizers, [5, 10] for extensions to the multiwell scenario, and the recent contribution [28] for the case of spatially dependent wells. We finally mention [66] for related models for Lithium-Ion batteries.

The study of analogous sharp-interface limits in the solid-solid setting has been initiated by S. Conti, I. Fonseca, and G. Leoni in [23], neglecting the effects of frame indifference. In dimension two, the frame-indifferent purview has been characterized by S. Conti and B. Schweizer for two rank-one connected wells A and B, first in a linearized setting in [26], and then in the the fully nonlinear framework of (1.2) in [25, 27]. We also mention the contributions [52, 53] for related microscopic models for two-dimensional martensitic transformations.

The first analysis of sharp-interface limits for singularly perturbed frame-indifferent energies in higher dimensions $d > 2$ has been obtained in our previous work [32], for a slightly modified version of the model (1.2) where the energy contains a further anisotropic perturbation. More specifically, when the two wells have exactly one rank-one connection, after rotation, we can assume without loss of generality that $B - A = \kappa \mathrm{e}_d \otimes \mathrm{e}_d$ for $\kappa > 0$. Then, our model reads as follows:

$$y \in H^2(\Omega;\mathbb{R}^d) \to E_{\varepsilon,\eta}(y) := I_\varepsilon(y) + \eta^2 \int_\Omega \Big(|\nabla^2 y|^2 - |\partial^2_{dd} y|^2\Big)\,\mathrm{d}x \tag{1.3}$$

for $\eta > 0$. Owing to the additional anisotropic perturbation, our analysis is restricted to the case of exactly one rank-one connection. We stress that this additional energy term does not affect frame indifference, and penalizes only transitions in the direction orthogonal to the rank-one connection $\mathrm{e}_d \otimes \mathrm{e}_d$, while still allowing for phase transitions between the two different energy wells. We refer to [44, 45, 49, 67] for studies of related models involving anisotropic perturbations.

In [32] we have shown that, for a suitable choice of η (dependent on ε), the functionals in (1.3) Γ-converge as $\varepsilon \to 0$ (in the L^1-topology) to the sharp-interface limit

$$\mathcal{E}_0(y) := \begin{cases} K\,\mathcal{H}^{d-1}(J_{\nabla y}) & \text{if } \nabla y \in BV(\Omega; R\{A,B\}) \text{ for some } R \in SO(d),\\ +\infty & \text{otherwise in } L^1(\Omega;\mathbb{R}^d), \end{cases} \tag{1.4}$$

where K corresponds to the energy of optimal transitions between the two phases (see (3.4) for the exact expression). Roughly speaking, limiting deformations are necessarily piecewise affine with $J_{\nabla y}$ consisting of hyperplanes orthogonal to e_d intersected with Ω (see [34] and Figure 1). We point out that, up to a possibly different constant K, the model in (1.4) is the same as the one identified in [25]. An essential ingredient in [32] is a novel *two-well rigidity estimate* (see Theorem 3.2 below). It provides stronger estimates with respect to previous results in the literature (see e.g. [22, 25, 47, 54]) by introducing a *phase indicator*, which allows to identify the predominant phase in each point of Ω.

In this paper we further build upon this new rigidity estimate to combine the perspective of deriving sharp-interface limits for phase transitions with the passage from nonlinear-to-linearized elastic energies.

In fact, triggered by the availability of rigidity estimates (mainly [42]) the derivation of effective linearized models has attained a great deal of attention over the last years. Their interest originates from the observation that they generally provide good approximations of the behavior of nonlinear models for deformations that are 'close' to rigid movements in a suitable sense. In fact, under the assumption that A is the identity matrix Id, a formal asymptotic expansion shows that, by considering deformations y of the form $y = \mathrm{id} + \varepsilon u$ for a smooth displacement u, there holds

$$\int_\Omega W(\nabla y)\,\mathrm{d}x = \int_\Omega W(\mathrm{Id} + \varepsilon \nabla u)\,\mathrm{d}x \sim \frac{\varepsilon^2}{2}\int_\Omega D^2W(\mathrm{Id})\,\nabla u : \nabla u\,\mathrm{d}x + o(\varepsilon^2),$$

where D^2W denotes the second-order differential of W and $o(\varepsilon^2)/\varepsilon^2 \to 0$ as ε tends to zero. In other words, the leading order behavior of the energy W is completely encoded by the quadratic form of linearized elasticity $\frac{1}{2}\int_\Omega D^2W(\mathrm{Id})\nabla u : \nabla u\,\mathrm{d}x$. Whereas ε^2 is related to the width of transition layers, as explained above, the parameter ε represents the typical order of elastic strains. This heuristic argument has been made rigorous by G. Dal Maso, M. Negri, and D. Percivale in the seminal paper [31] for single-well energies under standard growth conditions. An extension to the case of weaker growth conditions has been the subject of [2]. We further refer to related studies on atomistic systems [17, 63], homogenization [43, 59], viscoelasticity [39], plasticity [55], or fracture [37, 38, 60].

Some of the aforementioned linearization results have been generalized to the multiwell setting for wells approaching the identity as $\varepsilon \to 0$, see e.g. [1, 48, 62]. For fixed wells (independent of ε), results are limited to [3] (see [4] for an atomistic counterpart). There, the authors consider a stronger higher-order perturbation compared to the ones in (1.2) and (1.3). In particular, they characterize, under appropriate boundary conditions, linearization around one of the two wells, i.e., a crucial feature is that *only one phase* (say, the identity) is present in the limiting model. This is an effect of the stronger higher-order perturbation that, roughly speaking, *prevents the occurrence of macroscopic phase transitions* in the effective functional. In mathematical terms, their penalization is chosen in a specific way to ensure compactness and convergence of rescaled displacements $u = (y - \mathrm{id})/\varepsilon$ in suitable Sobolev norms.

The main novelty of this work consists in providing a new perspective on solid-solid phase transitions, allowing simultaneously to have phase changes present in the limit, as well as to perform a 'pointwise dependent' linearization that keeps track of the different 'predominant phases' in each region of the body. We consider here sequences of energies of the form (1.3) for suitable ε-dependent η (see Remark 3.1 below for details), denoted by $\mathcal{E}_\varepsilon$ in the following. We point out that η is chosen to be 'big enough' to guarantee that our quantitative rigidity estimate in Theorem 3.2 provides enough compactness properties, but also 'small enough' so that the limiting behavior of the energies is not affected by the anisotropic perturbation and no second-order derivatives of the deformations are involved in the limiting description. We refer to [32, Remark 4.5 and paragraph before Theorem 1.1] for a discussion of this point.

Our first result consists in showing that to every sequence of deformations $\{y^\varepsilon\}_\varepsilon \subset H^2(\Omega;\mathbb{R}^d)$ with equibounded $\mathcal{E}_\varepsilon$-energies we can associate a limiting deformation $y \in H^1(\Omega;\mathbb{R}^d)$, with $\nabla y \in BV(\Omega; R\{A,B\})$ for some $R \in SO(d)$, a limiting displacement $u \in SBV^2_{\mathrm{loc}}(\Omega;\mathbb{R}^d)$ (see Appendix A), and a limiting Caccioppoli partition $\mathcal{P} = \{P_j\}_j$. The jump set of u is the (at most) countable union of hyperplanes orthogonal to e_d and intersected with Ω, and the components of $\mathcal{P}$ are given by the intersection of Ω with d-dimensional stripes having sides orthogonal to e_d.

The full statement of our result is quite technical: for this reason we present here a simplified version and refer to Theorem 3.3 for the precise formulation.

Theorem 1.1 (Simplified compactness result). *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^d$, $d \geq 2$, such that all its slices orthogonal to the e_d-direction are connected (see* H8.*). Let W satisfy* H1.-H4. *Let $\{y^\varepsilon\}_\varepsilon \subset H^2(\Omega;\mathbb{R}^d)$ be such that $\sup_{\varepsilon>0}\mathcal{E}_\varepsilon(y^\varepsilon) < +\infty$. Then, to every deformation y^ε we can associate a rotation $R^\varepsilon \in SO(d)$, a Caccioppoli partition $\mathcal{P}^\varepsilon = \{P_j^\varepsilon\}_j$, phase indicators $\mathcal{M}^\varepsilon = \{M_j^\varepsilon\}_j \subset \{A,B\}$, and translations $\mathcal{T}^\varepsilon = \{t_j^\varepsilon\}_j \subset \mathbb{R}^d$, as well as a limiting triple $(y,u,\mathcal{P})$ with $\nabla y \in BV(\Omega; R\{A,B\})$ such that*

$$\begin{aligned} &R^\varepsilon \to R, \\ &P_j^\varepsilon \to P_j \quad \textit{in measure for all } j, \end{aligned}$$

$$y^\varepsilon - \frac{1}{\mathcal{L}^d(\Omega)}\int_\Omega y^\varepsilon(x)\,\mathrm{d}x \to y \quad \textit{strongly in } H^1(\Omega;\mathbb{R}^d),$$
$$u^\varepsilon \to u \quad \textit{in measure in } \Omega, \textit{ and } \nabla u^\varepsilon \rightharpoonup \nabla u \textit{ weakly in } L^2_{\mathrm{loc}}(\Omega;\mathbb{M}^{d\times d}),$$

where u^ε *denote rescaled displacement fields associated to* $\mathcal{P}^\varepsilon, \mathcal{M}^\varepsilon$, $\mathcal{T}^\varepsilon$, *and* R^ε, *defined by*

$$u^\varepsilon := \frac{y^\varepsilon - \sum_j (R^\varepsilon M_j^\varepsilon\, x + t_j^\varepsilon)\chi_{P_j^\varepsilon}}{\varepsilon}. \tag{1.5}$$

The assumptions on W are classical regularity and coercivity conditions for two-well nonlinear elastic energies, cf. Subsection 2.1. In particular, the statement shows that sequences of deformations with equibounded energies can be decomposed into the sum of piecewise constant rigid movements $\sum_j (R^\varepsilon M_j^\varepsilon\, x + t_j^\varepsilon)\chi_{P_j^\varepsilon}$ and scaled displacements u^ε. The limiting quantities $(y, u, \mathcal{P})$ play different roles in the description of the effective model: roughly speaking, the limiting deformation y encodes the two different phases, which are in general still present in the limit, and correspondingly indicates the surfaces where phase transitions occur. The limiting displacement u and the partition $\mathcal{P}$, instead, keep track of the situation in which in the limiting model two neighboring areas are in the same phase but at level ε they were separated by small intermediate regions in the opposite phase having asymptotically vanishing width as $\varepsilon \to 0$, see Figure 3 below for an illustration. More specifically, intermediate layers of width comparable to ε (i.e., the order of elastic strains) are encoded by the jump set of u and widths asymptotically larger than ε are associated to the boundary of the partition $\partial P_j \cap \Omega$, $P_j \in \mathcal{P}$. Finally, u features also elastic displacements.

In particular, Theorem 1.1 motivates the notion of *admissible triples* as the collection of triples $(y, u, \mathcal{P})$ that are attained in the sense of the convergences in Theorem 1.1, starting from a sequence of deformations $\{y^\varepsilon\}_\varepsilon$. In what follows, we will refer to the convergence properties in Theorem 1.1 as *tripling of the variables.* See also [37] for a related notion of convergence.

The second step of our analysis consists in providing a characterization of admissible limiting triples $(y, u, \mathcal{P})$. For ease of presentation, we collect our findings in a simplified statement and refer to Subsection 3.3 for the precise formulation of the results.

Theorem 1.2 (Simplified characterization of limiting triples)**.** *Let* $(y, u, \mathcal{P})$ *be an admissible triple for the sequence* $\{y^\varepsilon\}_\varepsilon$. *Then,*

- y *and* $\mathcal{P}$ *are uniquely defined;*
- u *is uniquely defined up to piecewise translations of the form* $\sum_j t_j\chi_{P_j}$, $\{t_j\}_j \subset \mathbb{R}^d$, *and global (infinitesimal) rotations;*
- $J_{\nabla y} \subset \bigcup_{j=1}^\infty \partial P_j \cap \Omega$*;*
- *the jump of* u *is constant on every connected component of its jump set.*

The non-uniqueness of the displacement field is simply a consequence of the possible non-uniqueness in the definition of u^ε, see (1.5). The last point of the statement represents a 'laminate structure' of limiting displacement fields. This regularity of u is achieved thanks to the anisotropic penalization in (1.3) and neglects branching phenomena, see also Remark 3.10 for more details.

Denoting by $\mathcal{A}$ the class of all admissible limiting triples $(y, u, \mathcal{P})$, our main contribution consists in showing that the asymptotic behavior of the energies $\mathcal{E}_\varepsilon$ is described by the functional

$$\mathcal{E}_0^{\mathcal{A}}(y,u,\mathcal{P}) := \frac{1}{2}\int_\Omega D^2W(\nabla y(x))\nabla u(x){:}\nabla u(x)\,\mathrm{d}x + K\mathcal{H}^{d-1}(J_{\nabla y}) + 2K\mathcal{H}^{d-1}\Big(\big(J_u \cup \big(\bigcup\nolimits_j \partial P_j \cap \Omega\big)\big) \setminus J_{\nabla y}\Big) \tag{1.6}$$

for every $(y, u, \mathcal{P}) \in \mathcal{A}$. We point out that the constant K in (1.6) is the same one as in (1.4). We observe that $\mathcal{E}_0^{\mathcal{A}}$ reduces to (1.4) for $u = 0$ and $\mathcal{P}$ coinciding with the collection of connected components of the two sets $\{x \in \Omega\colon\ \nabla y(x) \in SO(d)A\}$ and $\{x \in \Omega\colon\ \nabla y(x) \in SO(d)B\}$. Analogously, $\mathcal{E}_0^{\mathcal{A}}$ coincides with the quadratic form of linearized elasticity, and hence with the limiting model in [3] for $u \in H^1(\Omega;\mathbb{R}^d)$, for the trivial partition $\mathcal{P}$ consisting only of Ω, and for a deformation y with $\nabla y = \mathrm{Id}$ in Ω. In this sense, our limiting description combines both the effects of the sharp-interface characterizations [25, 32] and those of the multiwell linearization [3]. In contrast to these results, it features an additional surface term: as described above, the jump of u and the boundary of the partition encode small intermediate layers in the opposite phase at level ε with width bigger than or comparable to ε which induce two 'consecutive phase transitions', see Figure 3. Our Γ-convergence result is proven under the compatibility condition that this additional term enters the energy with double cost with respect to single phase transitions, i.e., we suppose that

$$K^A_{\mathrm{dp}} = K^B_{\mathrm{dp}} = 2K, \tag{1.7}$$

where K^A_{dp} and K^B_{dp} represent, roughly speaking, the energy necessary for performing these double-phase transitions at level ε. (The subscript 'dp' stands for 'double profile'. We refer to (3.26) for their precise expression.) Our main result reads as follows:

Theorem 1.3. *Let Ω be a bounded strictly star-shaped domain (see (2.7)) satisfying the further connectedness assumption in* H8. *Let W satisfy* H1.-H7. *and assume that the compatibility condition in (1.7) holds true. Then, $\mathcal{E}_\varepsilon$ Γ-converges to $\mathcal{E}_0^{\mathcal{A}}$ in the topology provided by the tripling of the variables in Theorem 1.1.*

We refer to Subsection 2.1 and Subsection 3.1 for the formulation of H1.-H7. The difference between our result and the Γ-convergence analyses in [25, 32] and [3] is mostly in the adopted topology. In [25, 32] an effective energy is identified in the strong L^1-topology for deformations y. The result in [3], instead, is derived in the weak H^1-topology for rescaled displacements $(y - \mathrm{id})/\varepsilon$. Our model combines this 'global' point of view with a 'local' one: the limiting Caccioppoli partition plays the role of identifying subdomains where the small-strains approximation of linearized elasticity, encoded by the limiting displacement u, is well posed. Finally, the surface-energy term associated to the jump set of u and to $\mathcal{P}$ keeps track of the multiple phase changes that the material had to undergo at level ε on regions with vanishing widths.

We stress here that the focus of our study is not on minimization problems and their convergence but rather on the identification of the limiting energy functional. For completeness, we also mention that the case of incompatible wells, i.e., the setting where A and B have no rank-one connections, is not included in our analysis but would be much simpler to handle. Indeed, the limiting model would linearize around just one of the two phases, leading to a limiting description analogous to [3].

We point out that the lower bound in Theorem 1.3 holds under no further assumptions on the two profile energies, i.e., the compatibility condition (1.7) is only needed for the construction of recovery sequences. In Subsection 6.5 we present a self-contained discussion showing that, under an additional assumption on the energy density (see (3.27) below) optimal profiles are one-dimensional and the compatibility condition in (1.7) is indeed satisfied. This assumption is fulfilled, e.g., when the energy only depends on the distance of the deformation gradient from the two wells, see (3.28).

We close the introduction with some comments on the proof structure. The proof of Theorem 1.1 relies on a series of intermediate results: all statements involving limiting rotations, partitions, and deformations are essentially proven in Proposition 4.2. The sequence of translations and the limiting displacements are first exhibited on subsets of Ω and eventually on Ω itself in Propositions 4.5 and 4.6, respectively. Finally, a further delicate construction is needed to ensure uniqueness of the limiting Caccioppoli partition. This is based on a certain *selection principle*, see (3.17) below. Indeed, without such a requirement, there might be different possible choices for the limiting partition, see the discussion in Example 3.4 for an in-depth analysis of this point. Key ingredients for the compactness analysis are the two-well rigidity estimate recalled in Theorem 3.2 and a characterization of the two phase regions established in [32, Proposition 3.7], see also Proposition 4.1.

The statements collected in Theorem 1.2 are the subject of three different propositions. In particular, the uniqueness properties of limiting deformation, displacement, and partition are proven in Proposition 3.6. This latter one is shown to be a consequence of the selection principle described above. The characterization of the jump set of ∇y is contained in Proposition 3.7, whereas that of the jump set of u is the subject of Proposition 3.8.

As customary in Γ-convergence analysis, the proof of Theorem 1.3 consists in first showing that $\mathcal{E}_0^{\mathcal{A}}$ provides a lower bound for the limiting behavior of the energies $\mathcal{E}_\varepsilon$ (see Theorem 3.13), and then in showing that this lower bound is indeed optimal (see Theorem 3.14). The proof of the liminf inequality essentially relies on providing a characterization of the double-profile energies $K^M_{\rm dp}$, $M \in \{A, B\}$. An important point is to show that optimal double phase transitions are, a priori, energetically more expensive than gluing together two optimal profiles performing each a single phase transition in an energetically optimal way (in other words, $K^M_{\rm dp} \geq 2K$), see Proposition 6.2. The key ingredients for proving the upper bound are explicit constructions of local recovery sequences performing energetically optimal single and double phase transitions, see Propositions 6.4 and 6.5. Both sequences are constructed starting from a delicate slicing argument introduced in [32] and recalled in Proposition 6.13 below. In addition, they are chosen so that they coincide with isometries far from the interfaces, and they can then be 'glued together' in the proof of Theorem 3.14.

The paper is organized as follows: in Section 2 we review the state-of-the-art and perform an overview of the main mathematical difficulties. In Section 3 we describe our model and state the main results. Sections 4 and 5 are devoted to the proofs of the compactness theorem and to the characterization of limiting triples, respectively. The proof of Theorem 1.3 is the subject of Section 6.

1.1. **Notation.** In what follows, we fix $d \in \mathbb{N}$, $d \geq 2$, and we consider a bounded Lipschitz domain $\Omega \subset \mathbb{R}^d$. We will denote points $x \in \mathbb{R}^d$ as $x = (x', x_d)$, with $x' \in \mathbb{R}^{d-1}$ and $x_d \in \mathbb{R}$. In the whole paper we use standard notations for Sobolev spaces, as well as for $BV(\Omega)$ and $SBV(\Omega)$. We refer the reader to [7] for definitions and main results. Some basic properties of special functions of bounded variation and Caccioppoli partitions are recalled in Appendix A. We will omit the target space of our functions whenever this is clear from the context. The identity map on $\mathbb{R}^d$ will be denoted by id or, with a slight abuse of notation, simply by x. For $m \in \mathbb{N}$, the m-dimensional Lebesgue and Hausdorff measures of a set will be indicated by $\mathcal{L}^m$ and $\mathcal{H}^m$, respectively. By $\fint_\Omega$ we denote the average integral $\frac{1}{\mathcal{L}^d(\Omega)}\int_\Omega$.

We denote by $e_1, \ldots, e_d$ and e_{ij}, $i, j = 1, \ldots, d$, the standard basis in $\mathbb{R}^d$ and $\mathbb{M}^{d\times d}$, respectively. We will use the notation Id for the identity matrix in $\mathbb{M}^{d\times d}$ and denote by $SO(d) \subset \mathbb{M}^{d\times d}$ the set of proper rotations. The sets of symmetric and skew-symmetric matrices are indicated by $\mathbb{M}^{d\times d}_{\rm sym}$ and $\mathbb{M}^{d\times d}_{\rm skew}$, respectively. In what follows, we will adopt the Frobenius scalar product between matrices $F : G := \mathrm{Tr}(F^T G)$ for every $F, G \in \mathbb{M}^{d\times d}$, and we will use the symbol $|\cdot|$ for the associated Frobenius norm. For every set $S \subset \mathbb{R}^d$, we indicate by χ_S its characteristic function, defined as $\chi_S(x) = 1$ if $x \in S$ and $\chi_S(x) = 0$ otherwise. Given two sets $S_1, S_2 \in \mathbb{R}^d$, we denote by $S_1 \triangle S_2$ their symmetric difference. Inclusions of sets $S_1 \subset S_2$ are always intended up to sets of negligible measure, i.e., $\mathcal{L}^d(S_1 \setminus S_2) = 0$. By $B_\rho(x)$ we denote the d-dimensional ball of radius $\rho > 0$ and center $x \in \mathbb{R}^d$.

2. State-of-the-art, heuristics, and challenges

In this section we recall the state-of-the-art for sharp-interface limits in the theory of solid-solid phase transitions, and for derivations of linearized models from nonlinear elastic energies. We additionally highlight the main open questions and difficulties.

2.1. **Models in nonlinear elasticity for two-well energies.** To every *deformation* $y \in H^1(\Omega; \mathbb{R}^d)$ we associate the elastic energy

$$\int_\Omega W(\nabla y)\,\mathrm{d}x,$$

where $W\colon \mathbb{M}^{d\times d} \to [0, +\infty)$ is a map representing the *stored-energy density*, and satisfying the following properties:

H1. (Regularity) W is continuous;
H2. (Frame indifference) $W(RF) = W(F)$ for every $R \in SO(d)$ and $F \in \mathbb{M}^{d\times d}$;
H3. (Two-well structure) $W(A) = W(B) = 0$, where $A = \mathrm{Id}$, and $B = \mathrm{diag}\,(1, 1, \ldots, 1, 1 + \kappa)$, for $\kappa > 0$;
H4. (Coercivity) there exists a constant $c_1 > 0$ such that

$$W(F) \geq c_1 \mathrm{dist}^2(F, SO(d)\{A, B\}) \quad \text{for every } F \in \mathbb{M}^{d\times d};$$

H5. (Quadratic behavior around the two wells) there exists $\delta_W > 0$ such that W is of class C^2 in
$$\{F \in \mathbb{M}^{d\times d} : \operatorname{dist}(F, SO(d)\{A, B\}) < \delta_W\}.$$

H6. (Growth condition from above) there exists a constant $c_2 > 0$ such that
$$W(F) \le c_2 \operatorname{dist}^2(F, SO(d)\{A, B\}) \quad \text{for every } F \in \mathbb{M}^{d\times d}.$$

Assumptions H1.-H5. are standard requirements on stored-energy densities in nonlinear elasticity. We note that after an affine change of variables one can always assume that the two wells have the form given in H3., see [34, Discussion before Proposition 5.1 and Proposition 5.2]. Specifically, the choice $\kappa > 0$ amounts to the case of exactly one rank-one connection between A and B, namely to the setting in which the only solution of $B - RA = a \otimes \nu$ with $R \in SO(d)$, $a, \nu \in \mathbb{R}^d$, and $|\nu| = 1$ is given by $R = \mathrm{Id}$, $\nu = e_d$, and $a = \kappa e_d$.

We point out that assumption H6. is not compatible with the impenetrability condition
$$W(F) \to +\infty \quad \text{as} \quad \det F \to +0, \tag{2.1}$$
which is usually enforced to model a blow-up of the elastic energy under strong compressions. In the derivation of sharp-interface limits for solid-solid phase transitions [25, 26, 32], however, condition H6. is instrumental for the construction of recovery sequences. (Note that, in dimension two, by means of a more elaborated construction performed in [27], assumption H6. may be dropped.)

In order to model solid-solid phase transitions, we analyze a nonlinear energy given by the sum of a suitable rescaling of the elastic energy and a singular perturbation. For every $\varepsilon > 0$, we consider the functional $E^P_\varepsilon \colon H^2(\Omega;\mathbb{R}^d) \to [0,+\infty)$ defined by
$$E^P_\varepsilon(y) := \frac{1}{\varepsilon^2}\int_\Omega W(\nabla y)\,\mathrm{d}x + \int_\Omega P_\varepsilon(\nabla^2 y)\,\mathrm{d}x, \tag{2.2}$$
where $P_\varepsilon : \mathbb{R}^{d\times d\times d} \to [0,+\infty)$ is a function which depends on the small parameter ε. In the following subsections, we will specify the choice of P_ε according to different modeling assumptions.

The parameter ε in the definition above represents the typical order of the strain, whereas ε^2 is related to the size of transition layers [11, 13, 18, 50, 58]. The first term in the right-hand side of (2.2) favors deformations y whose gradient is close to the two wells of W, whereas the second term penalizes transitions between two different values of the gradient.

In the following, we will call A and B the *phases*. Regions of the domain where ∇y is in a neighborhood of $SO(d)A$ will be called *A-phase regions* of y and accordingly we will speak of *B-phase regions*.

2.2. **Review of existing results.** We now continue by recalling some results about sharp-interface limits and derivation of linearized models. The exact setting of the paper and our main results can be found in Section 3. There, we will also recall a more recent result on sharp-interface limits which we proved in [32], and which represents the departure point of our analysis.

A sharp-interface limit for a model of solid-solid phase transitions. Classical singularly perturbed two-well problems are described by energies of the form
$$I_\varepsilon(y) := \frac{1}{\varepsilon^2}\int_\Omega W(\nabla y)\,\mathrm{d}x + \varepsilon^2\int_\Omega |\nabla^2 y|^2\,\mathrm{d}x \tag{2.3}$$
for every $y \in H^2(\Omega;\mathbb{R}^d)$, corresponding to the choice $P_\varepsilon(G) = \varepsilon^2|G|^2$, $G \in \mathbb{R}^{d\times d\times d}$, in (2.2). This subsection is devoted to a presentation of the analysis performed by S. Conti and B. Schweizer [25] which addresses the sharp-interface limit of this model in dimension two, as ε tends to zero. Although in [25] also the case of two rank-one connections is considered, we focus here on compatible wells having exactly one rank-one connection (see assumption H3.).

Denote by $\mathscr{Y}(\Omega)$ the class of admissible limiting deformations, defined as
$$\mathscr{Y}(\Omega) := \bigcup_{R\in SO(d)} \mathcal{Y}_R(\Omega), \quad \text{where } \mathcal{Y}_R(\Omega) := \big\{y \in H^1_\#(\Omega;\mathbb{R}^d) \colon \nabla y \in BV(\Omega; R\{A,B\})\big\} \text{ for } R \in SO(d), \tag{2.4}$$

where $H^1_{\#}(\Omega;\mathbb{R}^d) := \{y \in H^1(\Omega;\mathbb{R}^d)\colon\ \fint_\Omega y\,\mathrm{d}x = 0\}$. For every open subset $\Omega' \subset \Omega$, we will adopt the notation $\mathscr{Y}(\Omega')$ to indicate the corresponding admissible deformations. In [25, Proposition 3.2] the authors established the following compactness result.

Lemma 2.1 (Compactness)**.** *Let $d \in \mathbb{N}$, $d \geq 2$, and let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain. Let W satisfy assumptions* H1.–H4. *Then, for all sequences $\{y^\varepsilon\}_\varepsilon \subset H^2(\Omega;\mathbb{R}^d)$ for which*

$$\sup_{\varepsilon>0} I_\varepsilon(y^\varepsilon) < +\infty,$$

there exists a map $y \in \mathscr{Y}(\Omega)$ such that, up to the extraction of a subsequence (not relabeled), there holds

$$y^\varepsilon - \fint_\Omega y^\varepsilon(x)\,\mathrm{d}x \to y \quad \textit{strongly in } H^1(\Omega;\mathbb{R}^d).$$

The limiting deformations y have the structure of a simple laminate. Indeed, G. Dolzmann and S. Müller [34] have shown that for $y \in \mathcal{Y}_R(\Omega)$ the essential boundary of the set $T := \{x \in \Omega\colon\ \nabla y(x) \in RA\}$ consists of subsets of hyperplanes that intersect $\partial\Omega$ and are orthogonal to e_d, and that y is affine on balls whose intersection with ∂T has zero $\mathcal{H}^{d-1}$-measure, cf. Figure 1 (see also Appendix A for the definition of essential boundary for a set of finite perimeter).

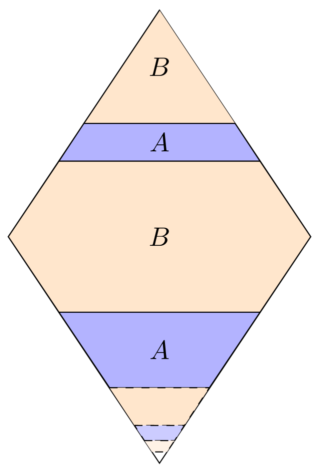

Figure 1. The gradient of a limiting deformation $y \in \mathcal{Y}_{\mathrm{Id}}(\Omega)$, in the case in which $B - A = \kappa e_d \otimes e_d$.

We now introduce the limiting sharp-interface energy. Denoting by $Q := (-\frac{1}{2}, \frac{1}{2})^d$ the d-dimensional unit cube centered in the origin and with sides parallel to the coordinate axes, we consider the *optimal-profile energy*

$$K_0 := \inf\Big\{\liminf_{\varepsilon\to 0} I_\varepsilon(y^\varepsilon, Q)\colon\ \lim_{\varepsilon\to 0}\|y^\varepsilon - y_0^+\|_{L^1(Q)} = 0\Big\}, \tag{2.5}$$

where $y_0^+ \in H^1_{\mathrm{loc}}(\mathbb{R}^d;\mathbb{R}^d)$ is the continuous function with $\nabla y_0^+ = A\chi_{\{x_d>0\}} + B\chi_{\{x_d<0\}}$ and $y_0^+(0) = 0$. (Here, $\chi_{\{x_d>0\}}$ and $\chi_{\{x_d<0\}}$ denote the characteristic functions of the two halfplanes $\{x_d > 0\}$ and $\{x_d < 0\}$, respectively.) Note that K_0 corresponds to the energy of an optimal phase transition from A to B, and that it is invariant under changing the roles of the two phases, i.e., invariant by replacing y_0^+ with the function $y_0^- \in H^1_{\mathrm{loc}}(\mathbb{R}^d;\mathbb{R}^d)$ satisfying $y_0^-(0) = 0$ and $\nabla y_0^- = B\chi_{\{x_d>0\}} + A\chi_{\{x_d<0\}}$.

The sharp-interface limiting functional $I_0\colon L^1(\Omega;\mathbb{R}^d) \to [0,+\infty]$ is defined as

$$I_0(y) := \begin{cases} K_0\mathcal{H}^{d-1}(J_{\nabla y}) & \text{if } y \in \mathscr{Y}(\Omega) \\ +\infty & \text{otherwise.}\end{cases} \tag{2.6}$$

In [25, Theorem 3.1] it was proved that, in the two-dimensional setting, I_0 is the variational limit of the sequence $\{I_\varepsilon\}_\varepsilon$ in the sense of Γ-convergence. (For an exhaustive treatment of Γ-convergence we refer the reader to [16, 29].)

Theorem 2.2 (Γ-convergence in dimension $d = 2$)**.** *Let $d = 2$, let $\Omega \subset \mathbb{R}^2$ be a bounded, strictly star-shaped Lipschitz domain, and let W satisfy* H1.–H4. *and* H6. *Then*

$$\Gamma - \lim_{\varepsilon \to 0} I_\varepsilon = I_0$$

with respect to the strong L^1-topology.

We recall that an open set Ω is strictly star-shaped if there exists a point $x_0 \in \Omega$ such that

$$\{tx + (1-t)x_0 : t \in (0,1)\} \subset \Omega \quad \text{for every } x \in \partial\Omega. \tag{2.7}$$

Here and in the sequel, we follow the usual convention that convergence of the continuous parameter $\varepsilon \to 0$ stands for convergence of arbitrary sequences $\{\varepsilon_i\}_i$ with $\varepsilon_i \to 0$ as $i \to \infty$, see [16, Definition 1.45]. In [27], the same Γ-convergence result as in Theorem 2.2 has been obtained by dropping H6. via a more elaborate construction allowing to incorporate an impenetrability condition of the form (2.1).

The result in Theorem 2.2 is limited to the two-dimensional setting due to the limsup inequality: the definition of sequences with optimal energy approximating a limit that has multiple flat interfaces relies on a deep technical construction. This so-called $H^{1/2}$*-rigidity on lines* (see [25, Section 3.3]) is only available in dimension $d = 2$. We also refer to a recent related study for microscopic models of two-dimensional martensitic transformations [53]. The issue of dimensionality has been overcome in [32] by considering a slightly modified model, see Subsection 3.1 for details.

Linearization around the identity for multiwell energies. In the context of multiwell linearization, R. Alicandro, G. Dal Maso, G. Lazzaroni, and M. Palombaro [3] investigated a multiwell energy $F_\varepsilon \colon H^2(\Omega;\mathbb{R}^d) \to [0,+\infty)$ of the form

$$F_\varepsilon(y) := \frac{1}{\varepsilon^2}\int_\Omega W(\nabla y)\,\mathrm{d}x + \varepsilon^{2-\gamma_d(r)}\int_\Omega |\nabla^2 y|^2\,\mathrm{d}x \tag{2.8}$$

for $r \in [1,2]$ and a suitable function $\gamma_d \colon [1,2] \to (0,+\infty)$, where for $d = 2$ it holds $\gamma_2(r) = r$, cf. [3, Equation (1.9)]. Here, the singular higher order term penalizes transitions between different wells in a stronger way with respect to (2.3). This corresponds to the choice $P_\varepsilon(G) = \varepsilon^{2-\gamma_d(r)}|G|^2$, $G \in \mathbb{R}^{d\times d\times d}$, in (2.2). In [3], the problem is studied in arbitrary dimension for a finite number of different wells and under very general growth conditions for the elastic energy and the second-order penalization. There, also the influence of external forces, under different scalings of the singular perturbation, is thoroughly discussed. For a simple exposition, however, we present only the basic case here and we specify the result to our two phases A and B.

First, [3, Theorem 2.3] along with the well-known rigidity estimate in [42] yields the following compactness result.

Lemma 2.3 (Compactness)**.** *Let $d \in \mathbb{N}$, $d \geq 2$, and $r \in (1,2]$. Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain. Let W satisfy assumptions* H1.–H4. *Then, for all sequences $\{y^\varepsilon\}_\varepsilon \subset H^2(\Omega;\mathbb{R}^d)$ satisfying $\sup_{\varepsilon>0} F_\varepsilon(y^\varepsilon) < +\infty$ we find rotations $R^\varepsilon \in SO(d)$, translations $t^\varepsilon \in \mathbb{R}^d$, and phases $M^\varepsilon \in \{A,B\}$ such that*

$$\sup_{\varepsilon>0} \left\| \frac{y^\varepsilon - (R^\varepsilon M^\varepsilon x + t^\varepsilon)}{\varepsilon} \right\|_{W^{1,r}(\Omega)} < +\infty.$$

Additionally imposing Dirichlet boundary conditions of the form $y^\varepsilon = \mathrm{id} + \varepsilon g$ on a part of the boundary with $g \in W^{1,\infty}(\Omega;\mathbb{R}^d) \cap H^2(\Omega;\mathbb{R}^d)$, one can choose $R^\varepsilon = \mathrm{Id}$, $t^\varepsilon = 0$, and $M^\varepsilon = A = \mathrm{Id}$ in the above result, see [3, Theorem 1.8]. This implies the uniform bound $\sup_{\varepsilon>0} \|u^\varepsilon\|_{W^{1,r}(\Omega)} < +\infty$ for the *rescaled displacement fields*

$$u^\varepsilon := \frac{y^\varepsilon - \mathrm{id}}{\varepsilon}. \tag{2.9}$$

In other words, for sequences with bounded F_ε-energy, Lemma 2.3 together with prescribed boundary conditions ensures compactness in $W^{1,r}$ for rescaled displacement fields. We write the nonlinear energy in terms of the displacement fields by setting $\hat{F}_\varepsilon(u) = F_\varepsilon(\mathrm{id} + \varepsilon u)$ for $u \in H^2(\Omega;\mathbb{R}^d)$.

Formally, the effective linearized energy $F_0\colon W^{1,r}(\Omega;\mathbb{R}^d)\to[0,+\infty]$ can be calculated by a Taylor expansion, and has the structure

$$F_0(u):=\begin{cases}\int_\Omega \mathcal{Q}_{\rm lin}\big(\mathrm{Id},e(u)\big)\,\mathrm{d}x & \text{if } u\in H^1(\Omega;\mathbb{R}^d),\\ +\infty & \text{otherwise.}\end{cases}\tag{2.10}$$

where $\mathcal{Q}_{\rm lin}\colon SO(d)\{A,B\}\times\mathbb{M}^{d\times d}\in[0,+\infty)$ is the quadratic form

$$\mathcal{Q}_{\rm lin}(RM,F):=\frac{1}{2}D^2W(RM)F:F\tag{2.11}$$

for every $R\in SO(d)$, $M\in\{A,B\}$, and $F\in\mathbb{M}^{d\times d}$. Note that frame indifference (see H2.) implies that the energy only depends on the symmetric part $e(u):=\frac{1}{2}((\nabla u)^T+\nabla u)$ of the strain, see (2.10). More generally, in view of H4., one can check that (cf. (5.3) below)

$$\mathcal{Q}_{\rm lin}(RM,SRM)=0\quad\text{if and only if}\quad R\in SO(d),\ M\in\{A,B\},\text{ and } S\in\mathbb{M}^{d\times d}_{\rm skew}.\tag{2.12}$$

The relation of $\hat{F}_\varepsilon$ and F_0 has been made rigorous by Γ-convergence (see [3, Theorem 1.9]).

Theorem 2.4 (Passage from nonlinear to linearized energies by Γ-convergence)**.** *Let $d\in\mathbb{N}$, $d\geq 2$, and $r\in(1,2]$. Let $\Omega\subset\mathbb{R}^d$ be a bounded Lipschitz domain. Let W satisfy assumptions* H1.–H5. *Then*

$$\Gamma-\lim_{\varepsilon\to 0}\hat{F}_\varepsilon=F_0$$

with respect to the weak $W^{1,r}$-topology.

2.3. **Phase transitions and linearization: Heuristics and challenges.** Our goal is to combine the above two approaches and to identify a model which allows both for phase transitions and for the passage to linearized energies in terms of rescaled displacement fields. As a first observation, we note that the setting in (2.8) is more specific than the one considered in (2.3) in the sense that deformations with finite energy are essentially in *one* phase, A or B, see Lemma 2.3. Imposing certain boundary conditions, one can always infer that the same phase, e.g. $A=\mathrm{Id}$, is predominant. Then it is indeed meaningful to perform a linearization around the identity. This differs significantly from the laminate structure of the limiting configurations obtained in Lemma 2.1, where different phases may be active and phase transitions between the different phase regions occur, see Figure 1. In (2.8), the second-order penalization is so strong that basically phase transitions in the limit $\varepsilon\to 0$ are forbidden. In the following, we discuss some of the challenges in more detail (we concentrate on the planar case $d=2$ for simplicity), and then describe the approach adopted in this work.

(a) Volume of the minority phase. In the model (2.8), the B-phase region, i.e., the set where the deformation gradient ∇y^ε takes values in a neighborhood of $SO(d)B$, denoted by T^ε_B in the following, has small $\mathcal{L}^2$-measure. Heuristically, this property can be seen as follows. From the boundedness of the energy and H4. one can deduce, for a suitable definition of T^ε_B, that

$$\mathcal{H}^1(\partial T^\varepsilon_B\cap\Omega)\leq C\|\mathrm{dist}(\nabla y^\varepsilon,SO(2))\|_{L^2(\Omega)}\|\nabla^2 y^\varepsilon\|_{L^2(\Omega)}\leq C\varepsilon\varepsilon^{\frac{\gamma_2(r)}{2}-1}=\varepsilon^{\frac{r}{2}},\tag{2.13}$$

where in the last step we used $\gamma_2(r)=r$, see below (2.8). (We refer to [32, Proof of Proposition 3.7, Step 1] for details on the first inequality.) By the (relative) isoperimetric inequality we obtain

$$\min\{\mathcal{L}^2(T^\varepsilon_B),\mathcal{L}^2(\Omega\setminus T^\varepsilon_B)\}\leq C\varepsilon^r.$$

Assuming that T^ε_B is the *minority phase*, i.e., the minimum is attained for T^ε_B, we get

$$\mathcal{L}^2(T^\varepsilon_B)\leq C\varepsilon^r.\tag{2.14}$$

This scaling of the area of the minority phase excludes phase transitions of the form given in Figure 2(a) where both $\mathcal{L}^2(T^\varepsilon_B)$ and $\mathcal{L}^2(\Omega\setminus T^\varepsilon_B)$ are bounded uniformly from below. It is worth mentioning that the calculation (2.13) for the model (2.3) (corresponding to $r=0$) would give $\mathcal{H}^1(\partial T^\varepsilon_B)\leq C$. This reflects the fact that phase transitions in the limit $\varepsilon\to 0$ are possible in that framework, see Lemma 2.1, Figure 1, and Figure 2(a).

(b) Criticality of the scaling. For compactness of rescaled displacement fields $u^\varepsilon=(y^\varepsilon-\mathrm{id})/\varepsilon$, see (2.9), we necessarily need $\mathcal{L}^2(T^\varepsilon_B)\to 0$ as otherwise $|\nabla u^\varepsilon|\to+\infty$ on a set of positive measure. More

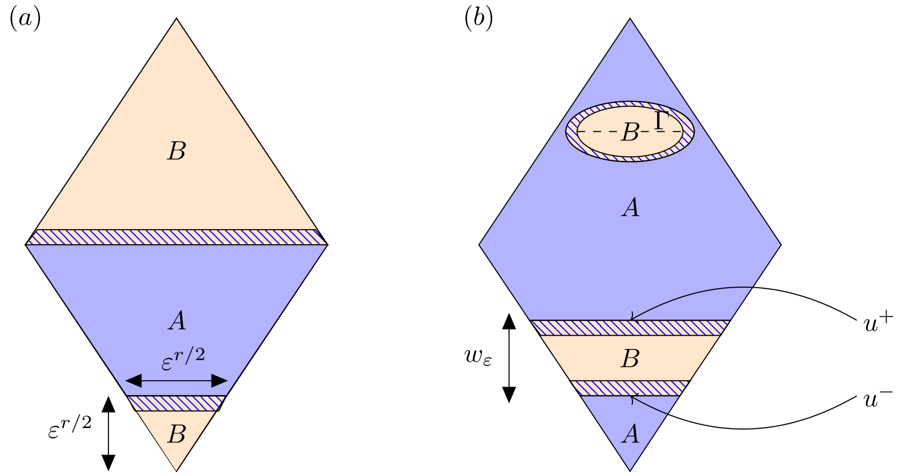

FIGURE 2. (a) Illustration of the A and B phase regions of a deformation y^ε with finite energy (2.3) in dimension $d = 2$. The shadowed regions, where a transition of the gradient between $SO(2)A$ and $SO(2)B$ occurs, are horizontal reflecting the laminate structure of configurations with bounded energy. For the energy (2.8), the phase transition at the lower boundary is possible, whereas the transition in the upper part would lead to unbounded energies as $\varepsilon \to 0$, cf. (2.13). (b) In the upper part of the figure, we depict a minority island centered around a segment Γ, which may have length ~ 1 in the e_1-direction, but width at most $\sim \varepsilon$, cf. (2.15). Such a set necessarily has curved boundaries and is also penalized by the elastic energy in a neighborhood of the island. In the lower part, the phenomenon described in (2.16) is illustrated.

precisely, since $|\nabla u^\varepsilon| \sim 1/\varepsilon$ on T_B^ε, it turns out that the bound in (2.14) is sharp in order to derive the uniform estimate $\|\nabla u^\varepsilon\|_{L^r(\Omega)} \le C$ of Lemma 2.3.

Recall that (2.14) was derived from (2.13) via the isoperimetric inequality. One may ask if this estimate is sharp, i.e., if the scaling $\varepsilon^{2-\gamma_2(r)} = \varepsilon^{2-r}$ of the penalization in (2.8) is really necessary to obtain (2.14). For a small region near the boundary of Ω whose boundary in Ω is a short straight line of length $\sim \varepsilon^{\frac{r}{2}}$ (see Figure 2(a)) the scaling is indeed critical. (We also refer to Example 3.2 in [3].) As the interface between the two phases is horizontal, such a transition is only realizable close to the boundary. For small inclusions of the B-phase in the interior, so-called *minority islands*, this is impossible, see Figure 2(b).

(c) Minority islands. The situation for such minority islands is indeed quite different. In dimension two and without a strong second-order penalization, merely under the assumption that in a neighborhood N of the island the quantity $\int_N |\nabla^2 y^\varepsilon| \, \mathrm{d}x$ is smaller than a universal constant independent of ε, S. CONTI and B. SCHWEIZER [25, Proposition 2.1] derived the remarkable bound

$$\mathcal{L}^2(T_B^\varepsilon) \le C \int_\Omega \mathrm{dist}(\nabla y^\varepsilon, SO(2)\{A, B\}) \, \mathrm{d}x \le C\varepsilon, \tag{2.15}$$

where the last step follows from the boundedness of the elastic energy. Roughly speaking, they showed that minority islands, although possibly being long in the e_1-direction (the direction orthogonal to the rank-one connection), have width at most $\sim \varepsilon$ in the e_2-direction, cf. Figure 2(b). Their result is indeed sharp in the sense that they provide a configuration with a minority island of length ~ 1 and width $\sim \varepsilon$ such that the energy (2.3) is bounded uniformly in ε, see [26, Remark 6.1]. A d-dimensional analogue has been provided in [32, Remark 3.9].

(d) Internal jumps. This phenomenon excludes compactness in $W^{1,r}$ for every $r > 1$, even if for a sequence $\{y^\varepsilon\}_\varepsilon$ there is only a single minority island of width ε in the e_2-direction around a 1-dimensional horizontal set Γ. Indeed, in that scenario the strain $|\nabla u^\varepsilon|$ of the rescaled displacement fields u^ε (see (2.9)) would scale like $1/\varepsilon$ on a set of $\mathcal{L}^2$-measure $\sim \varepsilon$, and one could expect no Sobolev compactness. On the contrary, it would be natural for u^ε to converge to an SBV function which jumps on Γ. In the following, we will refer to the setting described above as that of *internal jumps*. We again recall that this issue is excluded in the model (2.8) by the bound (2.14).

(e) Double phase transitions. A similar phenomenon may occur in the presence of a B-phase layer with width $w_\varepsilon \sim \varepsilon$ as indicated in the lower part of Figure 2(b) which corresponds to two 'consecutive phase transitions'. Heuristically, denoting by $y^\varepsilon_+(x')$, $y^\varepsilon_-(x')$, $u^\varepsilon_+(x')$, and $u^\varepsilon_-(x')$ the traces of y^ε and u^ε on the upper and lower boundary (with respect to the e_2-direction) of such a layer, one expects that $y^\varepsilon_+(x') \approx y^\varepsilon_-(x') + w_\varepsilon B e_2$, and thus

$$\lim_{\varepsilon\to 0} \big(u^\varepsilon_+(x') - u^\varepsilon_-(x')\big) = \lim_{\varepsilon\to 0} \frac{y^\varepsilon_+(x') - y^\varepsilon_-(x') - w_\varepsilon e_2}{\varepsilon} = const. = \kappa \lim_{\varepsilon\to 0} \frac{w_\varepsilon}{\varepsilon} e_2, \tag{2.16}$$

where we recall (2.9) and the fact that $(B-A)e_2 = \kappa e_2$, see H3. Consequently, the limiting function would jump with constant jump height $\kappa \lim_{\varepsilon\to 0} \frac{w_\varepsilon}{\varepsilon} e_2$. Interestingly, the jump height is essentially determined by w_ε, i.e., by the width of the B-phase layer. Let us also mention an additional problem occurring if $w_\varepsilon \gg \varepsilon$: in this latter setting the sequence of rescaled displacement fields would not even converge to an SBV function, cf. (2.16).

The perspective of the present work. The goal of the present contribution is to overcome the above mentioned issues. In particular, building upon a novel two-well rigidity estimate proved in [32] for a model augmented by a suitable anisotropic second-order penalization (see Subsection 3.1), we will introduce a *generalized definition of the rescaled displacement fields* which takes the presence of the two phases A and B in different parts of the domain into account. Roughly speaking, these displacement fields will measure the distance of the deformations y^ε from suitable rigid movements which may be different on the components of a partition of Ω induced by the A and B phase regions. This more flexible definition will allow us to carry out the following tasks in any dimension $d \geq 2$:

- derive a linearization result for configurations where both phases are present, in particular where phase transitions occur;
- obtain compactness results in a piecewise Sobolev setting for generalized rescaled displacements, despite the presence of minority islands with macroscopic length;
- identify an effective limiting model comprising linearized elastic energies and contributions for single and double phase transitions.

In our investigation, however, we do not take the presence of internal jumps into account for this would lead to a considerably more involved limiting energy, see Remark 3.10 for a discussion in that direction. From a modeling point of view, this amounts to excluding the presence of minority islands of width $\sim \varepsilon$ (see Figure 2(b)), whereas minority islands of width $\ll \varepsilon$ are allowed. In our model, this will be achieved by considering a suitable anisotropic second-order penalization.

3. The model and main results

In this section we introduce our model with a refined singular perturbation, state the rigidity estimate proved in [32], and present our main results.

3.1. A model with a refined singular perturbation and its sharp-interface limit.

In this subsection we present the exact mathematical setting of this paper and recall our previous work [32]. We analyze a nonlinear energy given by the sum of the non-convex elastic energy, a singular perturbation, and a higher-order penalization in the direction orthogonal to the rank-one connection. To be precise, for each $\varepsilon, \eta > 0$, we consider the functional

$$E_{\varepsilon,\eta}(y) := \frac{1}{\varepsilon^2}\int_\Omega W(\nabla y)\,\mathrm{d}x + \varepsilon^2 \int_\Omega |\nabla^2 y|^2\,\mathrm{d}x + \eta^2 \int_\Omega (|\nabla^2 y|^2 - |\partial^2_{dd} y|^2)\,\mathrm{d}x \tag{3.1}$$

for every $y \in H^2(\Omega;\mathbb{R}^d)$. This corresponds to the choice

$$P_\varepsilon(G) = \varepsilon^2|G|^2 + \eta^2 \sum_{i=1}^d \sum_{\substack{(j,k)\in\{1,\dots,d\}^2,\\ (j,k)\neq(d,d)}} |G_{ijk}|^2, \quad G \in \mathbb{R}^{d\times d\times d},$$

in (2.2). Note that (3.1) coincides with the energy functional in (2.3) when $\eta = 0$. In what follows, we will study the asymptotic behavior of the energies

$$\mathcal{E}_\varepsilon := E_{\varepsilon,\bar\eta_{\varepsilon,d}}, \tag{3.2}$$

where $\{\bar\eta_{\varepsilon,d}\}_\varepsilon$ is defined by

$$\bar\eta_{\varepsilon,d} := \varepsilon^{-1+\alpha(d)}, \qquad \text{with} \qquad \alpha(d) := 1/(2d). \tag{3.3}$$

We refer to Remark 3.1 below for details on the choice of the parameter. We denote the restriction of $\mathcal{E}_\varepsilon$ to a subset $\Omega' \subset \Omega$ by $\mathcal{E}_\varepsilon(y,\Omega')$. In [32, Proposition 4.3, Theorem 4.4, and Remark 4.5] we have shown that the asymptotic behavior of the energies $\mathcal{E}_\varepsilon$ is described (via Γ-convergence in the strong L^1-topology) by the sharp-interface model $\mathcal{E}_0\colon L^1(\Omega;\mathbb{R}^d) \to [0,+\infty]$, given by

$$\mathcal{E}_0(y) := \begin{cases} K\,\mathcal{H}^{d-1}(J_{\nabla y}) & \text{if } y \in \mathscr{Y}(\Omega), \\ +\infty & \text{otherwise,} \end{cases}$$

where the *optimal-profile energy* is defined by

$$K := \inf\Big\{ \liminf_{\varepsilon\to 0} \mathcal{E}_\varepsilon(y^\varepsilon, Q) : \lim_{\varepsilon\to 0} \|y^\varepsilon - y_0^+\|_{L^1(Q)} = 0 \Big\}. \tag{3.4}$$

Here, $Q = (-\frac{1}{2},\frac{1}{2})^d$ again denotes the d-dimensional unit cube centered in the origin, y_0^+ was defined below (2.5), and for the definition of $\mathscr{Y}(\Omega)$ we refer to (2.4). Note that (3.4) is the counterpart to (2.5) for the model in (3.1). From the definition of the optimal-profile energy and the fact that the penalization in (3.1) (with $\eta = \bar\eta_{\varepsilon,d}$) is stronger than the one in (2.3), we deduce the inequality $K \geq K_0$. As pointed out in [32, Remark 4.5], the additional penalization term in (3.1) with respect to (2.3) does not affect the qualitative behavior of the sharp-interface limit, only the constant in (3.4) may change. Moreover, the fact that $\bar\eta_{\varepsilon,d} \ll \varepsilon^{-1}$ guarantees that, asymptotically when passing to a linearized strain regime, the resulting model does not feature second-order derivatives, see [32, Introduction] and Remark 3.11 below.

We mention that anisotropic singular perturbations have already been used in related problems, see e.g. [49, 67]. In the present context, the role of the perturbation is twofold: (1) It allows us to use the two-well rigidity estimate proved in [32], see Theorem 3.2 below. (2) As discussed at the end of Subsection 2.3, the penalization simplifies the analysis by excluding the formation of *internal jumps* for limiting displacement fields, see Remark 3.10 below for more details. We remark that this anisotropy is the reason why we study the case of exactly one rank-one connection.

Remark 3.1 (Choice of the penalization constant)**.** We briefly mention that the result in [32] is slightly more general in the sense that it holds also for penalization constants $\{\eta_{\varepsilon,d}\}_\varepsilon$ with $\eta_{\varepsilon,d} \ll \bar\eta_{\varepsilon,d}$, see [32, (4.5)], i.e., our choice of the penalization constant here is 'less sharp'. For the sake of simplicity rather than generality, we prefer to work with (3.3) since it simplifies many estimates in the following. (In particular, the statement of the rigidity estimate in Theorem 3.2 below becomes simpler.)

Let us now recall the two-well rigidity result which is the fundamental ingredient for the proof of the aforementioned Γ-convergence result and, at the same time, is instrumental for our work. More precisely, in the present paper, besides yielding properties on optimal sequences in (3.4) necessary for deriving the sharp-interface limit, this estimate plays additionally a pivotal role for showing compactness of sequences with equibounded energies and for providing an optimal lower bound for the asymptotic behavior of the sequence $\{\mathcal{E}_\varepsilon\}_\varepsilon$. We present here a slightly simplified version of [32, Theorem 3.1] with $p = 2$ and $\bar\eta_{\varepsilon,d}$ in place of η.

Theorem 3.2 (Two-well rigidity estimate)**.** *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^d$ with $d \geq 2$, and let $\{\bar\eta_{\varepsilon,d}\}_\varepsilon$ be as in (3.3). Suppose that W satisfies* H1.–H4. *Let $E > 0$. Then for each $\Omega' \subset\subset \Omega$ there* *exists a constant $C = C(\Omega, \Omega', \kappa, c_1, E) > 0$ such that for every $y \in H^2(\Omega;\mathbb{R}^d)$ with $\mathcal{E}_\varepsilon(y) \le E$ there exist a rotation $R \in SO(d)$ and a phase indicator $\Phi \in BV(\Omega; \{A, B\})$ satisfying*

$$\|\nabla y - R\Phi\|_{L^2(\Omega')} \le C\varepsilon \quad \text{and} \quad |D\Phi|(\Omega) \le C. \tag{3.5}$$

Additionally, the choice of the rotation R and the phase indicator Φ is independent of the set $\Omega' \subset\subset \Omega$. If Ω is a paraxial cuboid, (3.5) holds on the entire domain Ω for a constant $C = C(\Omega, \kappa, c_1, E) > 0$.

We point out that the result in [32, Theorem 3.1] is more general. Indeed, it is stated for any $\eta \ge \varepsilon$ and for a range of integrability exponents. The present version for the choice $\eta = \bar{\eta}_{\varepsilon,d}$ is the counterpart to the simplified version [32, Theorem 1.1] on general bounded Lipschitz domains, and for a non-sharp choice of $\alpha(d)$. We refer to [32, Section 3] for additional motivation on this estimate, in particular for a comparison with other quantitative rigidity estimates for multiwell energies.

The focus of this contribution is on a Γ-convergence analysis of the energies $\mathcal{E}_\varepsilon$ in a topology different from the one specified above. It will lead to a limiting model simultaneously keeping track both of sharp interfaces between the two phases and of linearization effects. The precise topology for our Γ-convergence result is detailed in Subsection 3.2 below, and the Γ-limit is presented in Subsection 3.4. Due to the necessity of linearizing nonlinear elastic energies, we additionally need a local Lipschitz condition for the construction of recovery sequences: besides the assumptions H1.-H6. stated in Subsection 2.1, we also require

H7. (Local Lipschitz condition) there exists a constant $c_3 > 0$ such that

$$|W(F_1) - W(F_2)| \le c_3(1 + |F_1| + |F_2|)|F_1 - F_2| \quad \text{for all} \quad F_1, F_2 \in \mathbb{M}^{d\times d}.$$

Moreover, for simplicity we will assume that

H8. (Geometric condition) for all $t \in \mathbb{R}$ the set $\Omega \cap \{x_d = t\}$ is connected (whenever nonempty).

The latter condition is only needed for the compactness result in Theorem 3.3 and could be dropped at the expense of more elaborated arguments, see Remark 4.3 for details.

3.2. **Compactness.** This subsection is devoted to our main compactness result. Our approach consists in decomposing sequences of deformations $\{y^\varepsilon\}_\varepsilon$ with equibounded $\mathcal{E}_\varepsilon$-energies into the sum of two parts:

(a) Piecewise rigid movements, where 'piecewise' refers to associated Caccioppoli partitions induced by the A and B phase regions. These converge to the limit y of the deformations $\{y^\varepsilon\}_\varepsilon$.
(b) Displacements, rescaled by ε, whose strain is equibounded in L^2. These converge to a limiting displacement field, which is piecewise Sobolev, with possible jumps with normal in e_d-direction.

In order to formulate the main result of this subsection, we need to introduce some notation. Denote by $\mathscr{P}(\Omega)$ the following collection of *Caccioppoli partitions* of Ω:

$$\mathscr{P}(\Omega) := \Big\{\mathcal{P} = \{P_j\}_j \text{ partition of } \Omega\colon \ \bigcup\nolimits_j \partial P_j \cap \Omega \subset \bigcup\nolimits_{i\in\mathbb{N}} (\mathbb{R}^{d-1} \times \{s_i\}) \cap \Omega \text{ for } \{s_i\}_i \subset \mathbb{R}\Big\}. \tag{3.6}$$

We point out that the partitions can be finite or may consist of countably many sets. (For simplicity, we do not specify the index set corresponding to the indices j.) The definition above implies that $\bigcup_j \partial P_j \cap \Omega$ consists of subsets of hyperplanes orthogonal to e_d, which extend up to the boundary of Ω. Note that every Caccioppoli partition on the bounded domain Ω induces an ordered one just by a permutation of the indices. For this reason, throughout the paper we always tacitly assume that partitions are ordered. We will say that $P^\varepsilon \to P$ in measure as $\varepsilon \to 0$ if $\chi_{P^\varepsilon} \to \chi_P$ in L^1. Let $\mathscr{U}(\Omega)$ be the set of *displacements* whose jump sets are the union of countably many subsets of hyperplanes orthogonal to e_d, i.e.,

$$\mathscr{U}(\Omega) := \Big\{u \in SBV^2_{\rm loc}(\Omega;\mathbb{R}^d)\colon \ J_u \subset \bigcup\nolimits_{i\in\mathbb{N}} (\mathbb{R}^{d-1} \times \{s_i\}) \cap \Omega \text{ for } \{s_i\}_i \subset \mathbb{R}\Big\}. \tag{3.7}$$

For basic properties of Caccioppoli partitions and SBV functions we refer to Appendix A. In particular, the essential boundary of a set is indicated by ∂^*. For sets $\Omega' \subset \Omega$ and $S \subset \Omega$, we denote by $\pi_d(S)$ the orthogonal projection of S onto the e_d-axis, and define the *layer set*

$$L_{\Omega'}(S) = \Omega' \cap \big(\mathbb{R}^{d-1} \times \pi_d(S)\big). \tag{3.8}$$

We now state our main compactness result. Recall the definition of $\mathcal{Y}_R(\Omega)$ in (2.4).

Theorem 3.3 (Compactness). *Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain satisfying* H8. *Assume that W satisfies assumptions* H1.–H4.*, and let $\{y^\varepsilon\}_\varepsilon \subset H^2(\Omega;\mathbb{R}^d)$ be a sequence of deformations satisfying the uniform energy estimate*

$$\sup_{\varepsilon>0} \mathcal{E}_\varepsilon(y^\varepsilon) \le C_0 < +\infty. \tag{3.9}$$

Then, up to the extraction of a subsequence (not relabeled), the following holds:

(a) (Piecewise rigidity) There exist Caccioppoli partitions $\mathcal{P}^\varepsilon := \{P_j^\varepsilon\}_j$ of Ω such that

$$\mathcal{H}^{d-1}\big(\bigcup\nolimits_j \partial^* P_j^\varepsilon\big) \le C, \tag{3.10}$$

$$\sum\nolimits_j \min\big\{\mathcal{L}^d(\Omega' \cap P_j^\varepsilon), \mathcal{L}^d(L_{\Omega'}(P_j^\varepsilon) \setminus P_j^\varepsilon)\big\} \le C_{\Omega'}\,\varepsilon^p \quad \text{for every } \Omega' \subset\subset \Omega, \tag{3.11}$$

for some $p = p(d) \in (1,2)$, where C depends only on C_0 and Ω, and $C_{\Omega'}$ additionally on Ω'. There exist associated rotations $R^\varepsilon \in SO(d)$, as well as collections of phase indicators $\mathcal{M}^\varepsilon := \{M_j^\varepsilon\}_j$, with $M_j^\varepsilon \in \{A, B\}$ for every j and ε, such that

$$\sup_{\varepsilon>0} \|\nabla y^\varepsilon - \sum\nolimits_j R^\varepsilon M_j^\varepsilon \chi_{P_j^\varepsilon}\|_{L^2(\Omega')} \le C_{\Omega'}\,\varepsilon \quad \text{for every } \Omega' \subset\subset \Omega. \tag{3.12}$$

(b) (Limiting deformation and partition) There exist a limiting rotation $R \in SO(d)$, a limiting deformation $y \in \mathcal{Y}_R(\Omega)$, and a limiting partition $\mathcal{P} = \{P_j\}_j \in \mathscr{P}(\Omega)$ such that

$$R^\varepsilon \to R, \tag{3.13}$$

$$P_j^\varepsilon \to P_j \quad \text{in measure for all } j, \tag{3.14}$$

$$y^\varepsilon - \fint_\Omega y^\varepsilon(x)\,\mathrm{d}x \to y \quad \text{strongly in } H^1(\Omega;\mathbb{R}^d), \tag{3.15}$$

$$\sum\nolimits_j R^\varepsilon M_j^\varepsilon \chi_{P_j^\varepsilon} \overset{*}{\rightharpoonup} \nabla y \quad \text{weakly* in } BV(\Omega;\mathbb{M}^{d\times d}). \tag{3.16}$$

(c) (Displacements) We find collections of constants $\mathcal{T}^\varepsilon := \{t_j^\varepsilon\}_j \subset \mathbb{R}^d$, associated to $\mathcal{P}^\varepsilon$, satisfying

$$\frac{|t_i^\varepsilon - t_j^\varepsilon|}{\varepsilon} \to +\infty \quad \text{for all } i \ne j \text{ with } \mathcal{L}^d(P_i), \mathcal{L}^d(P_j) > 0, \text{ and } \lim_{\varepsilon\to 0} M_i^\varepsilon = \lim_{\varepsilon\to 0} M_j^\varepsilon, \tag{3.17}$$

and defining the rescaled displacement fields associated to $\mathcal{P}^\varepsilon, \mathcal{M}^\varepsilon$, $\mathcal{T}^\varepsilon$, and R^ε by

$$u^\varepsilon := \frac{y^\varepsilon - \sum_j (R^\varepsilon M_j^\varepsilon\, x + t_j^\varepsilon)\chi_{P_j^\varepsilon}}{\varepsilon}, \tag{3.18}$$

there exists $u \in \mathscr{U}(\Omega)$ such that

$$u^\varepsilon \to u \quad \text{in measure in } \Omega, \tag{3.19}$$

$$\nabla u^\varepsilon \rightharpoonup \nabla u \quad \text{weakly in } L^2_{\mathrm{loc}}(\Omega;\mathbb{M}^{d\times d}). \tag{3.20}$$

In view of our compactness result, sequences of deformations having equibounded energies decompose into the sum of piecewise rigid movements with gradients $\sum_j R^\varepsilon M_j^\varepsilon \chi_{P_j^\varepsilon}$, reflecting also the different phases A and B, and scaled SBV-displacements u^ε whose gradients are uniformly bounded in $L^2_{\mathrm{loc}}(\Omega;\mathbb{M}^{d\times d})$. Let us comment on the compactness result and on some of the proof ideas.

The definition of the piecewise rigid movements, as well as (3.10)-(3.12), follow from the geometric two-well rigidity result recalled in Theorem 3.2. In particular, (3.11) shows that each component has either small volume or coincides (up to a small set) with a 'layer' of Ω'. (We also refer to Figure 4 below for a 2d illustration.) At this point, the passage to subdomains is necessary and in (3.12) we control the quantities only in L^2_{loc}, cf. (3.5). If Ω is a paraxial cuboid, this passage can be avoided, see Remark 4.3 for details in that direction. Let us also emphasize that the rotation R^ε is defined *globally*, i.e., it is independent of the components of the partition $\mathcal{P}^\varepsilon$.

Standard compactness results (see Theorem A.1) imply (3.13)-(3.14), whereas (3.15) follows from Lemma 2.1, and for (3.16) we also take (3.12) into account. The global point of view for phase transitions given in Lemma 2.1 is combined with a local one in (3.17)-(3.20): the Caccioppoli partitions play the role of identifying subdomains where the small-strain displacement fields defined in (3.18) satisfy good compactness properties (3.19)-(3.20).

In this context, condition (3.17) represents a selection principle for the Caccioppoli partitions. (Note that $\lim_{\varepsilon\to 0} M^\varepsilon_k$ for $k = i, j$ is well defined by (3.13), (3.14), and (3.16).) Loosely speaking, it implies that two regions of the domain in the same phase, say phase A, are represented in the limit by two different sets P_i and P_j if and only if along the sequence $\{\mathcal{P}^\varepsilon\}_\varepsilon$ there is a layer contained in the B-phase region lying between P^ε_i and P^ε_j whose width is asymptotically (as $\varepsilon \to 0$) much larger than ε, cf. the discussion below (2.16). We emphasize that, without the selection principle (3.17), there might be different possible choices for the limiting partition, as the following example shows.

Example 3.4 (Non-uniqueness of limiting partition)**.** The choice of different partitions at level ε is not equivalent. In particular, different ε-decompositions determine different limiting displacements and Caccioppoli partitions, which may contain a different 'amount of information'. To clarify this, consider the following two-dimensional example. (For related examples, we refer to [38, Example 2.5] or [37, Example 2.4]). Let

$$\Omega = (0,1)\times(0,2), \quad \Omega_1 = (0,1)\times(0,1), \quad \Omega_2 = (0,1)\times(1,2)$$

and for $\varepsilon > 0$ and $l \in \{1/2, 1, 2\}$ consider the sets

$$\Omega^{\varepsilon,l}_3 = (0,1)\times(1-\varepsilon^l, 1+\varepsilon^l), \quad \Omega^{\varepsilon,l}_1 = \Omega_1 \setminus \Omega^{\varepsilon,l}_3, \quad \Omega^{\varepsilon,l}_2 = \Omega_2 \setminus \Omega^{\varepsilon,l}_3.$$

We define three different example sequences according to the value of l: first, define $\tilde{y}^{\varepsilon,l} \in H^1(\Omega;\mathbb{R}^2)$ by

$$\tilde{y}^{\varepsilon,l}(x) := \begin{cases} x & x \in \Omega^{\varepsilon,l}_1 \\ Bx - \kappa(1-\varepsilon^l)e_2 & x \in \Omega^{\varepsilon,l}_3 \\ x + 2\kappa\varepsilon^l e_2 & x \in \Omega^{\varepsilon,l}_2 \end{cases}$$

for every $x \in \Omega$, where κ is given in H3., and then

$$y^{\varepsilon,l} := \tilde{y}^{\varepsilon,l} * \frac{1}{\varepsilon^4}\varphi(\cdot/\varepsilon^2),$$

where $\varphi\colon \mathbb{R}^2 \to \mathbb{R}^2$ is a standard mollifier with $\mathrm{supp}(\varphi) \subset B_1(0)$. One can check that $\sup_{\varepsilon>0} \mathcal{E}_\varepsilon(y^{\varepsilon,l}) < \infty$. There are two natural alternative decompositions of the maps $y^{\varepsilon,l}$, namely

$$(1) \;\; y^{\varepsilon,l} = (R^{\varepsilon,l} M^{\varepsilon,l}_1 x + t^{\varepsilon,l}_1)\chi_{P^{\varepsilon,l}_1} + \varepsilon u^{\varepsilon,l}, \qquad \text{and} \qquad (2) \;\; y^{\varepsilon,l} = \sum\nolimits_{j=1}^3 (\hat{R}^{\varepsilon,l}\hat{M}^{\varepsilon,l}_j x + \hat{t}^{\varepsilon,l}_j)\chi_{\hat{P}^{\varepsilon,l}_j} + \varepsilon \hat{u}^{\varepsilon,l},$$

where $R^{\varepsilon,l} = \hat{R}^{\varepsilon,l} = \mathrm{Id}$ and the Caccioppoli partitions, phases, and constant translations are defined as

(1) $P^{\varepsilon,l}_1 = \Omega, \quad M^{\varepsilon,l}_1 = A, \quad t^{\varepsilon,l}_1 = 0,$

(2) $\hat{P}^{\varepsilon,l}_j = \Omega^{\varepsilon,l}_j, \quad \hat{M}^{\varepsilon,l}_1 = \hat{M}^{\varepsilon,l}_2 = A, \ \hat{M}^{\varepsilon,l}_3 = B, \qquad \hat{t}^{\varepsilon,l}_1 = 0, \ \hat{t}^{\varepsilon,l}_2 = 2\kappa\varepsilon^l e_2 - b\varepsilon, \ \hat{t}^{\varepsilon,l}_3 = -\kappa(1-\varepsilon^l)e_2,$

respectively, where $b \in \mathbb{R}^2$ is some arbitrary translation. This leads to the different limiting displacement fields and Caccioppoli partitions

$$\begin{aligned} &(1) \;\; u^l = 0\cdot\chi_{\Omega_1} + s^l e_2 \chi_{\Omega_2}, && P^l_1 = \Omega, \\ &(2) \;\; \hat{u}^l = 0\cdot\chi_{\Omega_1} + b\chi_{\Omega_2}, && \hat{P}^l_1 = \Omega_1, \ \hat{P}^l_2 = \Omega_2, \ \hat{P}^l_3 = \emptyset, \end{aligned}$$

where $s^l := 2\kappa \lim_{\varepsilon\to 0}\varepsilon^{l-1}$ for $l \in \{1/2, 1, 2\}$.

In case (2), where the sets Ω_1 and Ω_2 are split in the limiting partition, the limiting displacement does not provide any information on the behavior of the deformations at the ε-level. Note that the translation $b \in \mathbb{R}^2$ just expresses the non-uniqueness of the limiting configuration and does not have any physically reasonable interpretation, see Proposition 3.6 below. On the contrary, in case (1) the jump height of the limiting displacement on $\partial\Omega_1 \cap \partial\Omega_2$ provides information on the width of the intermediate layer $\Omega^{\varepsilon,l}_3$ where the deformation is in phase B: The jump heights $s^2 = 0$ and $s^1 = 2\kappa$ express that the width is of order $\ll \varepsilon$ and $\sim \varepsilon$, respectively. As $s^{1/2} = \infty$, we observe that $u^{1/2} \notin \mathscr{U}(\Omega)$. Thus, alternative (1) is not allowed in the case $l = 1/2$ and the sets Ω_1 and Ω_2 have to be split in the limiting partition. The observation that coarser partitions provide more information suggests to define the partition 'as coarse as possible'. This intuition is exactly reflected in the selection principle (3.17): for $l = 1, 2$ we apply Case (1) and only for $l = 1/2$ we apply Case (2). □

As a consequence of Theorem 3.3, we introduce the following notion of convergence.

Definition 3.5. (i) We say that a sequence of deformations $\{y^\varepsilon\}_\varepsilon$ is *asymptotically represented* by a limiting triple $(y, u, \mathcal{P}) \in \mathscr{Y}(\Omega) \times \mathscr{U}(\Omega) \times \mathscr{P}(\Omega)$, and write

$$y^\varepsilon \to (y, u, \mathcal{P}),$$

if there are sequences $\{R^\varepsilon\}_\varepsilon$, $\{\mathcal{P}^\varepsilon\}_\varepsilon$, $\{\mathcal{M}^\varepsilon\}_\varepsilon$, and $\{\mathcal{T}^\varepsilon\}_\varepsilon$ such that (3.10)–(3.20) hold.
(ii) We call a sequence of quadruples $(R^\varepsilon, \mathcal{P}^\varepsilon, \mathcal{M}^\varepsilon, \mathcal{T}^\varepsilon)$ *admissible for* $\{y^\varepsilon\}_\varepsilon$ if (3.10)–(3.20) are satisfied.
(iii) We call a triple $(y, u, \mathcal{P}) \in \mathscr{Y}(\Omega) \times \mathscr{U}(\Omega) \times \mathscr{P}(\Omega)$ *admissible for* $\{y^\varepsilon\}_\varepsilon$ if $\{y^\varepsilon\}_\varepsilon$ is asymptotically represented by $(y, u, \mathcal{P})$.

Although we use the notation $\to$ and call $(y, u, \mathcal{P})$ a limiting triple, it is clear that Definition 3.5 cannot be understood as a convergence in the usual sense. In particular, a specific feature of our limiting model is that in the limit $\varepsilon \to 0$ a *tripling of the variables* occurs. Another crucial aspect is given by the fact that along the sequence a characterization in terms of quadruples is needed. Let us highlight the relation between the quadruples and the limiting triples: the deformation $y \in \mathscr{Y}(\Omega)$ is determined by the rotation R^ε, the partitions $\mathcal{P}^\varepsilon$, and the phases $\mathcal{M}^\varepsilon$, see (3.16). For the displacement field u we additionally need the translations $\mathcal{T}^\varepsilon$, see (3.18)-(3.19). Finally, the limiting partition $\mathcal{P}$ is directly related to $\mathcal{P}^\varepsilon$ by (3.14).

We will now proceed with a more specific characterization of the admissible limiting triples for a sequence $\{y^\varepsilon\}_\varepsilon$.

3.3. **Characterization of admissible limiting triples.** In this subsection, our aim is to give a complete characterization of all limiting triples $(y, u, \mathcal{P})$ which are admissible for a sequence $\{y^\varepsilon\}_\varepsilon$ considered in Theorem 3.3. This, in turn, specifies the domain of our effective energy discussed in the next subsection. Below we will see that the choice of the deformation y and the partition $\mathcal{P}$ is unique. On the other hand, however, we see that u is not determined uniquely:

Consider admissible quadruples $\{(R^\varepsilon, \mathcal{P}^\varepsilon, \mathcal{M}^\varepsilon, \mathcal{T}^\varepsilon)\}_\varepsilon$ for a sequence $\{y^\varepsilon\}_\varepsilon$ which is asymptotically represented by a triple $(y, u, \mathcal{P})$, where $\mathcal{T}^\varepsilon = \{t^\varepsilon_j\}_j$. Then, we find another sequence of admissible quadruples $\{(\hat{R}^\varepsilon, \hat{\mathcal{P}}^\varepsilon, \hat{\mathcal{M}}^\varepsilon, \hat{\mathcal{T}}^\varepsilon)\}_\varepsilon$ by setting $\hat{R}^\varepsilon = \exp(-\varepsilon S) R^\varepsilon$ for $S \in \mathbb{M}^{d\times d}_{\rm skew}$, $\hat{\mathcal{P}}^\varepsilon = \mathcal{P}^\varepsilon$, $\hat{\mathcal{M}}^\varepsilon = \mathcal{M}^\varepsilon$, and $\hat{\mathcal{T}}^\varepsilon = \{\hat{t}^\varepsilon_j\}_j$ with $\hat{t}^\varepsilon_j = t^\varepsilon_j - \varepsilon t_j$ for some $t_j \in \mathbb{R}^d$ for all j. (Here, exp denotes the matrix exponential.) In view of (3.16) and (3.18)–(3.19), a short computation yields that this sequence of quadruples will give the limiting triple $(y, \hat{u}, \mathcal{P})$ with

$$\hat{u}(x) = u(x) + \sum\nolimits_j t_j \chi_{P_j}(x) + S\,\nabla y(x)\,x \qquad \text{for all } x \in \Omega. \tag{3.21}$$

To take this ambiguity of the limiting description into account, for a given deformation $y \in \mathscr{Y}(\Omega)$ and a given Caccioppoli partition $\mathcal{P} = \{P_j\}_j \in \mathscr{P}(\Omega)$, we introduce the set

$$\mathscr{T}(y, \mathcal{P}) = \Big\{ T\colon \Omega \to \mathbb{R}^d \text{ such that } T(x) = \sum\nolimits_j t_j \chi_{P_j}(x) + S\,\nabla y(x)\,x, \quad t_j \in \mathbb{R}^d,\ S \in \mathbb{M}^{d\times d}_{\rm skew} \Big\} \tag{3.22}$$

of corresponding piecewise translations combined with global infinitesimal rotations. We obtain the following characterization.

Proposition 3.6 (Characterization of admissible limiting triples)**.** *Let $\{y^\varepsilon\}_\varepsilon$ be a sequence as in Theorem 3.3. Let $(y^1, u^1, \mathcal{P}^1)$ and $(y^2, u^2, \mathcal{P}^2)$ be two admissible triples. Then the following assertions hold:*

(i) *$y^1 = y^2$ and $\mathcal{P}^1 = \mathcal{P}^2$ (up to possible reorderings of the sets).*
(ii) *There exists $T \in \mathscr{T}(y^1, \mathcal{P}^1) = \mathscr{T}(y^2, \mathcal{P}^2)$ such that $u^1 - u^2 = T$.*
(iii) *For each $\tilde{T} \in \mathscr{T}(y^1, \mathcal{P}^1)$, the triple $(y^1, u^1 + \tilde{T}, \mathcal{P}^1)$ is admissible.*

Property (i) states that the limiting deformation is uniquely determined. It follows from (3.15). The corresponding property for the partition is a consequence of the selection principle in (3.17). Without such a condition other choices are possible, see Example 3.4 for more details. Property (ii) states that the admissible displacement fields for a sequence $\{y^\varepsilon\}_\varepsilon$ are determined uniquely up to piecewise translations and a global (infinitesimal) rotation. This non-uniqueness has been illustrated in (3.21).

The next result characterizes the jump sets involved in admissible limiting triples.

Proposition 3.7 (Admissible limiting triples; jump set and partition)**.** *Let $\{y^\varepsilon\}_\varepsilon$ be a sequence as in Theorem 3.3. Then for each admissible triple $(y,u,\mathcal{P})$ in the sense of Definition 3.5 there holds*

$$J_{\nabla y} \subset \bigcup\nolimits_j \partial P_j \cap \Omega.$$

There are examples of sequences $\{y^\varepsilon\}_\varepsilon$ such that the inclusion is strict.

The fact that the inclusion may be strict can be seen in Case (2) of Example 3.4 (corresponding to $l = 1/2$). We also note by Proposition 3.6(iii) that there is always an admissible displacement field u with $\bigcup_j \partial P_j \cap \Omega \subset J_u$. This inclusion might be strict, see Case (1) in Example 3.4 with $l = 1$. We proceed with a result which specifies the jump heights of admissible limiting displacement fields. For $u \in \mathscr{U}(\Omega)$, the normal on J_u is given by $\nu_u = e_d$. We denote by u^+ and u^- the corresponding one-sided limits of u and we let $[u] := u^+ - u^-$.

Proposition 3.8 (Admissible limiting displacement fields; jump heights)**.** *Let $(y,u,\mathcal{P})$ be an admissible triple in the sense of Definition 3.5 and let $R \in SO(d)$ be such that $y \in \mathcal{Y}_R(\Omega)$. We have*

(i) $[u](x)$ *constant for* $\mathcal{H}^{d-1}$*-a.e.* $x \in (\mathbb{R}^{d-1} \times \{t\}) \cap \Omega$ *for all* $t \in \mathbb{R}$ *with* $J_u \cap (\mathbb{R}^{d-1} \times \{t\}) \neq \emptyset$,

(ii) $[u](x) \in [0,+\infty)Re_d$ *for* $\mathcal{H}^{d-1}$*-a.e.* $x \in \big(J_u \setminus \bigcup_j \partial P_j\big) \cap \{\nabla y = RA\}$,

(iii) $-[u](x) \in [0,+\infty)Re_d$ *for* $\mathcal{H}^{d-1}$*-a.e.* $x \in \big(J_u \setminus \bigcup_j \partial P_j\big) \cap \{\nabla y = RB\}$.

Roughly speaking, property (i) is a consequence of the geometry of the A and B phase regions induced by the rigidity estimate. We refer to (3.11) and to Figure 4 for an illustration. We also refer to the discussion on the jump height in (2.16). In particular, (i) implies that the jump set consists of subsets of hyperplanes orthogonal to e_d, which extend up to the boundary of Ω. Some intuition for point (ii) has been provided in (2.16), see also Case (1) in Example 3.4 with $l = 1$. Point (iii) is similar by changing the roles of the phases A and B. Note that (ii) and (iii) are well defined by Proposition 3.7.

Definition 3.9. In view of Theorem 3.3, Proposition 3.7, and Proposition 3.8, we introduce the set of admissible limiting triples

$$\mathcal{A} := \Big\{(y,u,\mathcal{P}) \in \mathscr{Y}(\Omega)\times\mathscr{U}(\Omega)\times\mathscr{P}(\Omega):\ J_{\nabla y} \subset \bigcup\nolimits_{j=1}^\infty \partial P_j \cap \Omega,\ u \text{ satisfies (i)–(iii) in Proposition 3.8}\Big\}.$$

Remark 3.10 (Internal jumps)**.** As discussed already heuristically in Subsection 2.3, the choice of the penalization factor (3.3) simplifies the analysis by excluding the formation of *internal jumps* for limiting displacement fields, see Proposition 3.8(i). This allows us to formulate our limiting model for displacements in a piecewise Sobolev setting. Let us mention that without such a requirement the domain of the limiting model is expected to be the space of generalized functions of bounded variation $GSBD^2(\Omega)$ introduced in [30], with an additional constraint on the jump sets of admissible functions. Note that this phenomenon is not just a technical mathematical issue, but is related to *branching*, i.e., to the presence of microstructures near interfaces, see e.g. [20, 21, 33, 49, 67]. Particularly, see [21] for a simplified scalar model in SBV addressing the low volume-fraction of one phase, and dealing with the problem of internal jumps. (We also refer to [33] for some extensions to a vectorial model in the geometrically linear setting, and to [24] for a corresponding scaling law in the case of a martensitic nucleus embedded in an austenitic matrix.)

3.4. **The effective limiting model and Γ-convergence.** This subsection is devoted to the identification of the effective limiting model. We start by introducing the limiting energy functional. We preliminarily recall that, in view of assumption H5., the stored energy density W is C^2 in a neighborhood of the set $SO(d)\{A,B\}$. We also recall the quadratic form $\mathcal{Q}_{\rm lin}$ defined in (2.11), Definition 3.9, and the asymptotic optimal-profile energy in (3.4). We define the functional

$$\mathcal{E}_0^{\mathcal{A}}(y,u,\mathcal{P}) := \int_\Omega \mathcal{Q}_{\rm lin}(\nabla y, \nabla u)\,\mathrm{d}x + K\mathcal{H}^{d-1}(J_{\nabla y}) + 2K\mathcal{H}^{d-1}\Big(\big(J_u \cup \big(\bigcup\nolimits_j \partial P_j \cap \Omega\big)\big) \setminus J_{\nabla y}\Big) \tag{3.23}$$

for every $(y,u,\mathcal{P}) \in \mathcal{A}$. Note that the elastic term is well defined as $\nabla y(x) \in SO(d)\{A,B\}$ for a.e. $x \in \Omega$.

We briefly compare this energy to the limiting models in Subsection 2.2 and explain the relation to $\mathcal{E}_\varepsilon$ introduced in (3.2). First, the elastic energy is more general than the one in (2.10) as it accounts for the two different phases indicated by ∇y. Moreover, in contrast to (2.6), the functional contains two surface terms: the jumps of ∇y represent the energy associated to single phase transitions between A- and B-phases, already appearing in (2.6). The second surface term corresponds to two 'consecutive phase transitions', i.e., two transitions with a small intermediate layer whose width vanishes as $\varepsilon \to 0$, which remain undetected by y. More generally speaking, by relaxation in the limit $\varepsilon \to 0$, the first term (single transition) and the second term (double transition) effectively correspond to an odd and an even number of consecutive phase transitions, respectively, cf. Figure 3. Note that the second surface term enters the energy with double cost with respect to single phase transitions. This term itself has two contributions: recalling the selection principle for the partition in (3.17), small intermediate layers of width $\sim \varepsilon$ are associated to J_u in the limit $\varepsilon \to 0$ and layers with asymptotically much larger width are encoded by the partition $\mathcal{P}$. Layers of width $\ll \varepsilon$ do not affect the limiting energy. This is illustrated in Example 3.4.

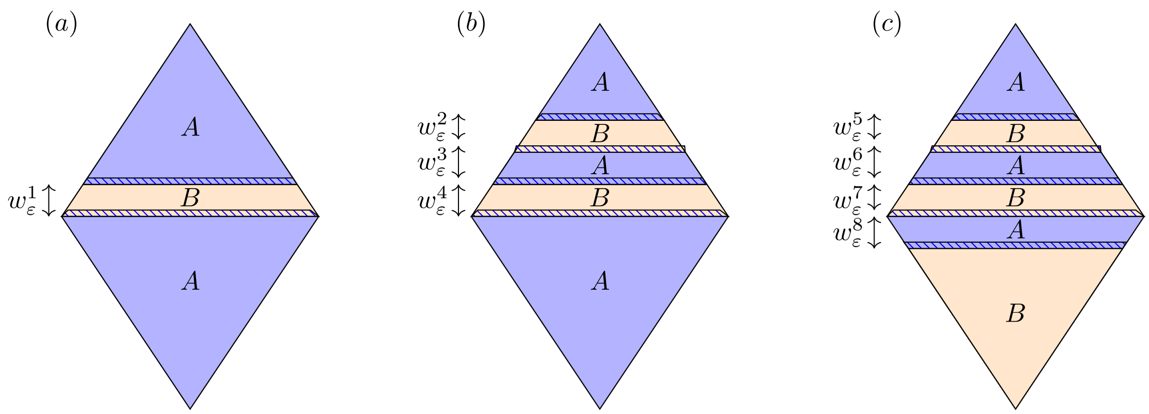

FIGURE 3. Illustration of situations corresponding to even and odd numbers of consecutive phase transitions. We assume that $w_\varepsilon^i \to 0$ as $\varepsilon \to 0$ and that $\liminf_{\varepsilon\to 0} w_\varepsilon^i/\varepsilon > 0$ for $i = 1,\dots,8$. The shaded regions describe the areas in which the phase transitions occur. (a) We depict here the case of two phase transitions: the intermediate phase has infinitesimal width w_ε^1 and thus disappears in the limit. Its presence at level ε, though, still affects $\mathcal{E}_0^{\mathcal{A}}$. Indeed, in the second surface term, the length of the interface between the two limiting A-regions will enter the energy with density $2K$. (b) The case of three intermediate phases is depicted. Although being different from (a) on the level ε, this situation leads to the same effective energy. In this sense, two intermediate phases 'compensate each other' in the limit. Note that the jump height of the limiting function is determined by w_ε^2 and w_ε^4 only. (c) We illustrate here the situation of five phase transitions: the energy contribution is accounted for in $\mathcal{E}_0^{\mathcal{A}}$ by the first surface term, i.e., the length of the interface between the limiting A- and B-regions, reflected by $J_{\nabla y}$, will enter the energy only with density K.

Remark 3.11 (Second-gradient terms)**.** The effective model described in (3.23) does not contain second-gradient terms neither in y nor in u. Indeed, the choice $\bar{\eta}_{\varepsilon,d} \ll \varepsilon^{-1}$ guarantees that the effects of higher-order contributions, in particular of their anisotropic part, enter the limiting energy only in terms of the value of the constant K, but no dependence on second-order derivatives persists in the model after the limiting passage.

The main contribution of this paper consists in showing that the sequence $\{\mathcal{E}_\varepsilon\}_\varepsilon$ is asymptotically described by $\mathcal{E}_0^{\mathcal{A}}$, in the sense of Γ-convergence in the topology introduced in Definition 3.5. As a preliminary observation, we note that the limiting energy is invariant under changes of the asymptotic representative.

Remark 3.12 (Energy invariance for different asymptotic representatives)**.** Suppose that a sequence $\{y^\varepsilon\}_\varepsilon$ is asymptotically represented by two triples $(y^1, u^1, \mathcal{P}^1), (y^2, u^2, \mathcal{P}^2) \in \mathcal{A}$. Then, $\mathcal{E}_0^{\mathcal{A}}(y^1, u^1, \mathcal{P}^1) = \mathcal{E}_0^{\mathcal{A}}(y^2, u^2, \mathcal{P}^2)$. This follows from (2.12), (3.22), Proposition 3.6, and (3.23).

Our first result shows that $\mathcal{E}_0^{\mathcal{A}}$ provides a lower bound for the asymptotic behavior of the energy functionals $\{\mathcal{E}_\varepsilon\}_\varepsilon$.

Theorem 3.13 (Γ-liminf inequality)**.** *Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain satisfying* H8. *Let W satisfy assumptions* H1.–H5.*, let $(y, u, \mathcal{P}) \in \mathcal{A}$, and let $\{y^\varepsilon\}_\varepsilon \subset H^2(\Omega;\mathbb{R}^d)$ be such that $y^\varepsilon \to (y, u, \mathcal{P})$. Then*
$$\liminf_{\varepsilon\to 0} \mathcal{E}_\varepsilon(y^\varepsilon) \geq \mathcal{E}_0^{\mathcal{A}}(y, u, \mathcal{P}).$$

Our second result is the proof that the lower bound identified in Theorem 3.13 is optimal. For the construction of recovery sequences we need slightly stronger assumptions: we require that the set is strictly star-shaped (see (2.7)), we assume H6. and H7., and we need a specific condition for the asymptotic optimal-profile energy. In order to state our result, we need some additional notation. Define the set of sequences
$$\mathcal{W}_d := \big\{ \{w_\varepsilon\}_\varepsilon : \ w_\varepsilon \in (0,\infty), \ w_\varepsilon \to 0, \ \liminf_{\varepsilon\to 0}\,(w_\varepsilon/\varepsilon) > 0 \big\}, \tag{3.24}$$
and define the functions
$$y^A_{\rm dp} := e_d \chi_{\{x_d>0\}}, \qquad y^B_{\rm dp} := -e_d \chi_{\{x_d>0\}}. \tag{3.25}$$
For $M \in \{A, B\}$, we introduce the *double-profile energy*
$$K^M_{\rm dp} := \sup_{h>0} \sup_{\{w_\varepsilon\}_\varepsilon \in \mathcal{W}_d} \inf \Big\{ \limsup_{\varepsilon\to 0} \mathcal{E}_\varepsilon\big(y^\varepsilon, Q' \times (-h,h)\big) : \ \frac{y^\varepsilon - Mx}{w_\varepsilon} \to y^M_{\rm dp} \text{ in measure in } Q' \times (-h,h) \Big\}, \tag{3.26}$$
where here and in the following $Q' := (-\frac{1}{2}, \frac{1}{2})^{d-1} \subset \mathbb{R}^{d-1}$. We defer a discussion about the definition of $K^M_{\rm dp}$, and proceed with the Γ-limsup inequality.

Theorem 3.14 (Γ-limsup inequality)**.** *Let $\Omega \subset \mathbb{R}^d$ be a bounded, strictly star-shaped, Lipschitz domain in $\mathbb{R}^d$ satisfying* H8. *Let W satisfy assumptions* H1.–H7.*, and suppose that $K^A_{\rm dp} = K^B_{\rm dp} = 2K$. Let $(y, u, \mathcal{P}) \in \mathcal{A}$. Then there exists $\{y^\varepsilon\}_\varepsilon \subset H^2(\Omega;\mathbb{R}^d)$ such that $y^\varepsilon \to (y, u, \mathcal{P})$ in $\mathcal{A}$, and*
$$\limsup_{\varepsilon\to 0} \mathcal{E}_\varepsilon(y^\varepsilon) \leq \mathcal{E}_0^{\mathcal{A}}(y, u, \mathcal{P}).$$

The notion of strictly star-shaped sets will allow us to reduce the constructions to the case of finitely many phase transitions, similarly to the investigation in [25]. The additional assumptions H6. and H7. are instrumental to control the nonlinear elastic energies of the recovery sequence, whenever the gradient is away from the two wells. We now address definition (3.26) and explain the condition $K^A_{\rm dp} = K^B_{\rm dp} = 2K$.

First, in order to understand the role of the sequences $\mathcal{W}_d$ defined in (3.24), recall the setting in Figure 3(a). The case in which, locally at level ε, two portions of the domain in the same phase are separated by an intermediate region in the opposite phase, is reflected by an energy contribution in the limiting functional $\mathcal{E}_0^{\mathcal{A}}$ whenever the width of the 'intermediate layer' behaves asymptotically as one of the sequences in $\mathcal{W}_d$. We recall that, if $\liminf_{\varepsilon\to 0} (w_\varepsilon/\varepsilon) \in (0,+\infty)$, this is encompassed by the jump set of the limiting displacement u, whereas the opposite scenario is captured by the limiting partition $\mathcal{P}$.

Intuitively, the value $K^A_{\rm dp}$ in (3.26) provides an upper bound for the energy of an optimal profile which contains two phase transitions, first from A to B and then from B to A, with an intermediate layer in the B-phase of width $\{w_\varepsilon\}_\varepsilon$, see Figure 3(a). The interpretation of $K^B_{\rm dp}$ is the same after interchanging the roles of the phases. The *compatibility condition* $K^A_{\rm dp} = K^B_{\rm dp} = 2K$ is needed in the construction of recovery sequences. On the one hand, it seems a natural condition as K and $K^A_{\rm dp}, K^B_{\rm dp}$ correspond to the case of one and two phase transitions, respectively. On the other hand, for general densities W we are able to prove only one inequality and the other inequality only under extra assumptions on W. More precisely, we have the following.

Proposition 3.15 (Relation of K, $K^A_{\rm dp}$, and $K^B_{\rm dp}$: inequality)**.** *The values K, $K^A_{\rm dp}$, and $K^B_{\rm dp}$ introduced in (3.4) and (3.26) satisfy $\min\{K^A_{\rm dp}, K^B_{\rm dp}\} \geq 2K$.*

We now discuss an additional assumption on W which implies equality. Assume that the energy density additionally satisfies

$$W(F) \geq W\big(\mathrm{Id} + (|Fe_d| - 1)e_{dd}\big) \qquad \text{for all } F \in \mathbb{M}^{d\times d}. \tag{3.27}$$

As we will show in Lemma 6.16, this condition ensures that optimal profiles are one-dimensional. It can be understood as a generalization of condition (H_3) in [23] where one-dimensionality of profiles has been discussed for a two-well problem without frame indifference. Note that this condition is compatible with frame indifference. A model case is a situation where the energy only depends on the distance of the two wells, i.e.,

$$W(F) = \phi\big(\mathrm{dist}(F, SO(d)A), \mathrm{dist}(F, SO(d)B)\big) \qquad \text{for all } F \in \mathbb{M}^{d\times d}, \tag{3.28}$$

where $\phi\colon ([0,\infty))^2 \to [0,\infty)$ is a smooth function with $c_1(\min\{t_1,t_2\})^2 \leq \phi(t_1,t_2) \leq c_2(\min\{t_1,t_2\})^2$ for all $t_1, t_2 \in [0,\infty)$ which is increasing in both entries. We refer to (6.107) below for details.

Given condition (3.27), we are able to show the following.

Proposition 3.16 (Relation of K, $K^A_{\rm dp}$, and $K^B_{\rm dp}$: equality)**.** *Suppose that (3.27) holds. The values K, $K^A_{\rm dp}$, and $K^B_{\rm dp}$ introduced in (3.4) and (3.26) satisfy $K^A_{\rm dp} = K^B_{\rm dp} = 2K$.*

We do not have an explicit example, but we conjecture that for certain energy densities one might indeed have $\min\{K^A_{\rm dp}, K^B_{\rm dp}\} > 2K$. Moreover, in contrast to (2.6) and (3.4), we cannot apply a symmetry argument to show that $K^B_{\rm dp}$ equals $K^A_{\rm dp}$. In general, $K^A_{\rm dp}$ and $K^B_{\rm dp}$ might be different.

Intuitively, $\min\{K^A_{\rm dp}, K^B_{\rm dp}\} > 2K$ means that two optimal profiles in (3.4) cannot be combined to a competitor in (3.26) without essentially increasing the energy. In any case, if e.g. $K^A_{\rm dp} > 2K$, the energy would probably depend on the width of the intermediate B-layer and the limiting energy (3.23) would necessarily also depend on the jump height of u. We do not pursue this more complicated case here, but only provide a result under the aforementioned compatibility condition. In this case, the cost of a double phase transition always equals $2K$, independently of the width of the intermediate layer.

This concludes the presentation of our results. The remainder of the paper is devoted to the proofs. The proof of Theorem 3.3 is the subject of Section 4. In particular, the limiting deformations, rotations, and partitions are identified in Proposition 4.2, whereas the limiting displacement fields are exhibited in Proposition 4.5 and Proposition 4.6. The remaining part of the proof of Theorem 3.3 consists in showing that partitions and translations at the ε-level can be chosen so that the selection principle in (3.17) holds true. The characterization of limiting triples described in Subsection 3.3 is provided in Section 5. Theorems 3.13 and 3.14 are proven in Subsections 6.1 and 6.2.

The main step of the proof of the lower bound in Theorem 3.13 consists in showing that in the 'bulk part' of the domain and around the different limiting interfaces the asymptotic behavior of the energies can be bounded from below by the elastic energy and by the two surface terms, respectively. Key ingredients are the notions of optimal-profile and double-profile energy functions (see (6.3) and (6.5)), as well as Propositions 6.1–6.2, providing a characterization of the local behavior of the energy around the different limiting interfaces. The former was proven in [32, Propostion 4.6]. The proof of the latter is carried out in Subsection 6.3.

The proof of Theorem 3.14 relies on two main intermediate results, which are proven in Subsection 6.4: (1) in Proposition 6.4 we generalize [32, Proposition 4.7] to construct local recovery sequences around single phase transitions; (2) in Proposition 6.5 we prove the corresponding result for double phase transitions. Eventually, in Subsection 6.5 we show that under (3.27) optimal profiles for single phase transitions are one-dimensional (see Lemma 6.16), and that $K^A_{\rm dp} = K^B_{\rm dp} = 2K$ (see Proposition 3.16).

4. Compactness analysis

This section is devoted to the proof of our compactness result in Theorem 3.3. We proceed in several steps: we first identify sequences of rotations, phase indicators, and partitions, as well as a limiting deformation and partition such that (3.10)-(3.16) hold, see Proposition 4.2. Then, Proposition 4.5 and Proposition 4.6 are devoted to the construction of (sequences of) translations and the definition of displacement fields, see (3.18)–(3.20), first on subsets of Ω and eventually on Ω itself. Finally, a further delicate construction is needed to show that by a suitable choice of the partitions and the translations also the selection principle (3.17) can be guaranteed.

In what follows, we will use the notion of sets of finite perimeter and Caccioppoli partitions. We refer to Appendix A for basic properties. Before we start, we recall the two-well rigidity estimate in Theorem 3.2 and point out that the result hinges on the following characterization of the two phase regions (see [32, Proposition 3.7 and Remark 3.8]). We refer to Figure 4 for a two-dimensional visualization.

Proposition 4.1 (Decomposition into phases)**.** *Let Φ be the phase indicator identified in Theorem 3.2, and define $T := \{\Phi = A\}$. Then*

$$\begin{aligned}
&\text{(i)} \quad \mathcal{H}^{d-1}(\partial^* T \cap \Omega) \le c\mathcal{E}_\varepsilon(y),\\
&\text{(ii)} \quad \int_{\partial^* T\cap\Omega} |\langle \nu_T, e_i\rangle| \,\mathrm{d}\mathcal{H}^{d-1} \le c\varepsilon^{2-\alpha(d)}\,\mathcal{E}_\varepsilon(y) \quad \textit{for} \quad i = 1,\dots,d-1,\\
&\text{(iii)} \quad \int_{-\infty}^{+\infty} \mathcal{H}^{d-2}\Big(\big(\mathbb{R}^{d-1}\times\{t\}\big)\cap\partial^*T\cap\Omega\Big)\,\mathrm{d}t \le c\varepsilon^{2-\alpha(d)}\,\mathcal{E}_\varepsilon(y),
\end{aligned} \tag{4.1}$$

where ν_T denotes the outer normal to T, $\partial^* T$ its essential boundary, $\alpha(d)$ is the quantity introduced in (3.3), and $\mathcal{E}_\varepsilon$ is the energy functional defined in (3.1)–(3.2).

We point out that the statement in [32, Proposition 3.7] is more general but reduces to the proposition above for the choice $\eta = \bar\eta_{\varepsilon,d}$ (see (3.3)).

In the proof of the compactness result, the set T will be the starting point for constructing the partitions. Properties (4.1)(i),(ii) are crucial to show (3.10) and to pass to a limiting partition in $\mathscr{P}(\Omega)$ by compactness. Item (4.1)(iii) is instrumental to prove (3.11).

We now start by identifying the limiting deformation and limiting partition. Recall the definition of $\mathcal{Y}_R(\Omega)$ and $\mathscr{P}(\Omega)$ in (2.4) and (3.6), respectively.

Proposition 4.2 (Deformations and partitions)**.** *Let Ω be a bounded Lipschitz domain satisfying H8. Suppose that W fulfills H1.–H4. Let $\{y^\varepsilon\}_\varepsilon \subset H^2(\Omega;\mathbb{R}^d)$ be a sequence of deformations satisfying (3.9). Then, we find a sequence of triples $(R^\varepsilon, \mathcal{P}^\varepsilon, \mathcal{M}^\varepsilon)$, a limiting rotation $R \in SO(d)$, a limiting deformation $y \in \mathcal{Y}_R(\Omega)$, and a limiting partition $\mathcal{P} = \{P_j\}_j \in \mathscr{P}(\Omega)$ such that (3.10)-(3.16) hold after extracting a subsequence. The components of $\mathcal{P}$ are connected.*

We point out that in Theorem 3.3 the components are not connected in general. At this intermediate stage, however, constructing the partition with this additional property is instrumental for the definition of displacement fields in Propositions 4.5 and 4.6 below as it allows to apply Poincaré inequalities on each component.

Proof of Proposition 4.2. Let $\{y^\varepsilon\}_\varepsilon \subset H^2(\Omega;\mathbb{R}^d)$ be a sequence of deformations satisfying (3.9). We denote the orthogonal projection of Ω onto the e_d-axis by the interval (a, b).

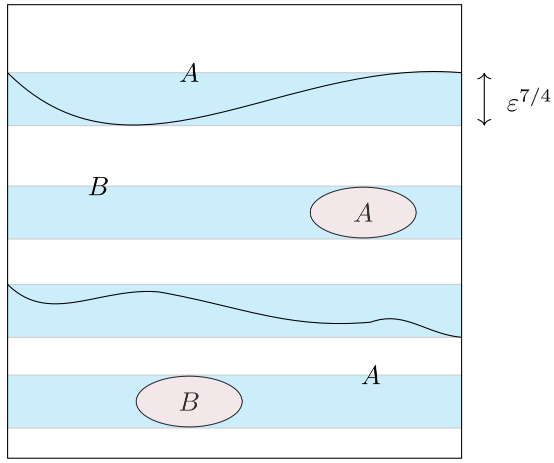

FIGURE 4. A visualization of phase regions in dimension $d = 2$. The (anisotropic) second-order penalization guarantees that phase transitions occur inside cylindrical layers of height $\varepsilon^{7/4}$. (Note that $\alpha(d) = 1/4$ for $d = 2$.) Additionally, $\varepsilon^{7/4}$ is an upper bound on the height of minority islands in the e_2-direction. In other words, connected components of the phase regions have either small volume or coincide (up to a small set) with a layer of Ω. In higher dimensions, a similar interpretation is possible, up to higher order terms.

Step 1: Preliminary estimates. First, we apply Theorem 3.2 to obtain sequences of rotations $\{R^\varepsilon\}_\varepsilon \subset SO(d)$ and phase indicators $\{\Phi^\varepsilon\}_\varepsilon \subset BV(\Omega; \{A, B\})$ such that for all $\Omega' \subset\subset \Omega$

$$\|\nabla y^\varepsilon - R^\varepsilon \Phi^\varepsilon\|_{L^2(\Omega')} = \|\nabla y^\varepsilon - (R^\varepsilon A \chi_{T^\varepsilon} + R^\varepsilon B \chi_{\Omega \setminus T^\varepsilon})\|_{L^2(\Omega')} \leq C_{\Omega'} \varepsilon \quad \text{and} \quad |D\Phi^\varepsilon|(\Omega) \leq C, \tag{4.2}$$

where $T^\varepsilon = \{\Phi^\varepsilon = A\}$ denotes the *A-phase regions*, see Proposition 4.1, $C_{\Omega'}$ depends on Ω', and C is related to C_0 in (3.9).

In the following, we will need to apply the relative isoperimetric inequality on sections of the form $\Omega \cap \{x_d = t\}$, $t \in (a, b)$. In general, the involved constant may depend on t. As a remedy, we pass to suitable subsets of Ω with properties independent of t: for $\varepsilon > 0$, we can choose $\Omega^\varepsilon \subset\subset \Omega$ with Lipschitz boundary, satisfying H8., and

$$\sup\nolimits_{\varepsilon>0} \mathcal{H}^{d-1}(\partial\Omega^\varepsilon) < +\infty, \qquad \lim_{\varepsilon \to 0} d_H(\Omega, \Omega^\varepsilon) = 0 \tag{4.3}$$

(d_H denotes the Hausdorff distance) such that for each $t \in (a, b)$ and each set of finite perimeter $E \subset \Omega^\varepsilon \cap \{x_d = t\}$ there holds

$$\min\Big\{\mathcal{H}^{d-1}(E),\ \mathcal{H}^{d-1}\Big((\Omega^\varepsilon \cap \{x_d = t\}) \setminus E\Big)\Big\} \leq \varepsilon^{-\alpha(d)} \big(\mathcal{H}^{d-2}(\partial^* E \cap \Omega^\varepsilon)\big)^{\frac{d-1}{d-2}}, \tag{4.4}$$

where $\alpha(d)$ is defined in (3.3). (For $d = 2$, the left hand side has to be interpreted as zero if $\mathcal{H}^0(\partial^* E \cap \Omega^\varepsilon) = 0$.) Indeed, these sets can be constructed as follows.

For fixed $\rho > 0$, let $\Omega^\rho \subset\subset \Omega$ be a Lipschitz domain satisfying H8. which is a finite union of cylindrical sets of the form $\omega \times (h^-, h^+)$ for $\omega \subset \mathbb{R}^{d-1}$ Lipschitz, i.e., there are only a finite number of different shapes for $\Omega^\rho \cap \{x_d = t\}$, denoted by $\omega_i \times \{t\}$ for Lipschitz domains ω_i, $i = 1, \ldots, N^\rho$. (We do not include ρ in the notation for simplicity.) Given $t \in (a, b)$, choose ω_i such that $\omega_i \times \{t\} = \Omega^\rho \cap \{x_d = t\}$ and consider $E \subset \Omega^\rho \cap \{x_d = t\}$. Then we can apply the relative isoperimetric inequality on ω_i to obtain (4.4) for a constant C_i^ρ depending on ω_i in place of $\varepsilon^{-\alpha(d)}$ and Ω^ρ in place of Ω^ε. (See [35, Theorem 2, Section 5.6.2]; note that the theorem in the reference above is stated and proved in a ball, but that the argument only relies on Poincaré inequalities, and thus easily extends to bounded Lipschitz domains.) Choose an infinitesimal sequence $\{\rho_k\}_k \subset (0, +\infty)$ and a corresponding strictly decreasing infinitesimal sequence $\{\varepsilon_k\}_k \subset (0, +\infty)$ such that the sequence $\{\Omega^{\rho_k}\}_k$ satisfies (4.3) (with Ω^{ρ_k} in place of Ω^ε) and

$$\max_{i=1,\ldots,N^{\rho_k}} C_i^{\rho_k} \leq \varepsilon_k^{-\alpha(d)}.$$

To conclude, we apply the following diagonalization argument: for $\varepsilon \in [\varepsilon_k, \varepsilon_{k-1}]$ we set $\rho^\varepsilon := \rho_{k-1}$. The claim follows by considering the sets $\Omega^\varepsilon := \Omega^{\rho^\varepsilon}$.

Step 2: Construction of auxiliary partitions. We start the actual proof by constructing a finite partition of $T^\varepsilon \cap \Omega^\varepsilon$ as follows: we define $f^\varepsilon \colon (a,b) \to (0,+\infty)$ by

$$f^\varepsilon(t) = \mathcal{H}^{d-1}(\{x_d = t\} \cap T^\varepsilon \cap \Omega^\varepsilon) \quad \text{ for } t \in (a,b). \tag{4.5}$$

We observe that $f^\varepsilon \in BV((a,b))$, and that its total variation can be estimated by

$$|Df^\varepsilon|(a,b) \le \mathcal{H}^{d-1}(\partial^* T^\varepsilon \cup \partial\Omega^\varepsilon). \tag{4.6}$$

In fact, for any $\psi \in C_c^\infty(\Omega)$ with $\psi = 1$ on Ω^ε, we get by Fubini's theorem that

$$\begin{aligned}|Df^\varepsilon|(a,b) &= \sup_{\substack{\varphi \in C_c^1(a,b)\\ \|\varphi\|_{L^\infty(a,b)} \le 1}} \int_{(a,b)} f^\varepsilon \varphi' \,\mathrm{d}t = \sup_{\substack{\varphi \in C_c^1(a,b)\\ \|\varphi\|_{L^\infty(a,b)} \le 1}} \int_\Omega \chi_{T^\varepsilon \cap \Omega^\varepsilon}(x', x_d)\ \varphi'(x_d) \,\mathrm{d}(x', x_d),\\ &= \sup_{\varphi \in C_c^1(a,b), \|\varphi\|_{L^\infty(a,b)} \le 1} \int_\Omega \chi_{T^\varepsilon \cap \Omega^\varepsilon}(x', x_d)\ \mathrm{div}\big(\psi(x)\varphi(x_d)e_d\big) \,\mathrm{d}(x', x_d),\end{aligned}$$

where we write $x = (x', x_d)$ with $x' \in \mathbb{R}^{d-1}$ and $x_d \in \mathbb{R}$. Therefore, we obtain

$$|Df^\varepsilon|(a,b) \le \sup_{\varphi \in C_c^1(\Omega;\mathbb{R}^d),\, \|\varphi\|_{L^\infty(\Omega)} \le 1} \int_\Omega \chi_{T^\varepsilon \cap \Omega^\varepsilon}\ \mathrm{div}(\varphi) \,\mathrm{d}x = |D\chi_{T^\varepsilon \cap \Omega^\varepsilon}|(\Omega).$$

Then, (4.6) follows from [7, (3.29), (3.62)].

We set $p := 1 + \frac{3}{2d(2d-3)} \in (1,2)$. (The choice becomes clear later.) Choose $\sigma_\varepsilon \in (\varepsilon^p/2, \varepsilon^p)$ such that

$$\mathcal{H}^0\big(\partial^*\{f^\varepsilon \le \sigma_\varepsilon\} \cap (a,b)\big) \le 2\varepsilon^{-p} \int_{\varepsilon^p/2}^{\varepsilon^p} \mathcal{H}^0\big(\partial^*\{f^\varepsilon \le s\} \cap (a,b)\big) \,\mathrm{d}s \le 2\varepsilon^{-p} |Df^\varepsilon|(a,b), \tag{4.7}$$

where the last step follows from the coarea formula for BV functions (see [7, Theorem 3.40]). We choose $a < d_1 < d_2 < \ldots < d_{m-1} < b$ such that $\partial^*\{f^\varepsilon \le \sigma_\varepsilon\} \cap (a,b) = \{d_j\}_{j=1}^{m-1}$, where $m - 1 \le 2\varepsilon^{-p}|Df^\varepsilon|(a,b)$ by (4.7). We define a finite partition of $T^\varepsilon \cap \Omega^\varepsilon$ consisting of the sets

$$\tilde{P}_j^\varepsilon = T^\varepsilon \cap \Omega^\varepsilon \cap \{d_{j-1} < x_d < d_j\}, \qquad j = 1, \ldots, m, \tag{4.8}$$

where we let $d_0 = a$ and $d_m = b$. In view of the definition in (4.5), we can estimate the 'upper' and 'lower' boundary of $\tilde{P}_j^\varepsilon$ by $\mathcal{H}^{d-1}((\partial^*\tilde{P}_j^\varepsilon \cap \Omega^\varepsilon) \setminus \partial^* T^\varepsilon) \le 2\sigma_\varepsilon \le 2\varepsilon^p$. Therefore, since $m - 1 \le 2\varepsilon^{-p}|Df^\varepsilon|(a,b)$ by (4.7), (4.6) yields

$$\sum\nolimits_{j=1}^m \mathcal{H}^{d-1}(\partial^*\tilde{P}_j^\varepsilon) \le 2m\varepsilon^p + \mathcal{H}^{d-1}(\partial^* T^\varepsilon \cup \partial\Omega^\varepsilon) \le 5\mathcal{H}^{d-1}(\partial^* T^\varepsilon \cup \partial\Omega^\varepsilon) + 2\varepsilon^p. \tag{4.9}$$

We repeat the above procedure for $\Omega^\varepsilon \setminus T^\varepsilon$ in place of T^ε and obtain a finite partition of $\Omega^\varepsilon \setminus T^\varepsilon$ which we denote by $\{\tilde{P}_j^\varepsilon\}_{j=m+1}^n$. Repeating the argument in (4.9) we get $\sum_{j=m+1}^n \mathcal{H}^{d-1}(\partial^*\tilde{P}_j^\varepsilon) \le 5\mathcal{H}^{d-1}(\partial^* T^\varepsilon \cup \partial\Omega^\varepsilon) + 2\varepsilon^p$. We set $\tilde{P}_{n+1}^\varepsilon = (\Omega \setminus \Omega^\varepsilon) \cap T^\varepsilon$ and $\tilde{P}_{n+2}^\varepsilon = \Omega \setminus (\Omega^\varepsilon \cup T^\varepsilon)$. Since $\mathcal{H}^{d-1}(\partial\Omega) < +\infty$, by (4.3), (3.9), and Proposition 4.1(i) we conclude

$$\sum\nolimits_{j=1}^{n+2} \mathcal{H}^{d-1}(\partial^*\tilde{P}_j^\varepsilon) \le 10\mathcal{H}^{d-1}(\partial^* T^\varepsilon \cup \partial\Omega^\varepsilon) + 4\varepsilon^p + \mathcal{H}^{d-1}(\partial(\Omega \setminus \Omega^\varepsilon)) + 2\mathcal{H}^{d-1}(\partial^* T^\varepsilon \cap (\Omega \setminus \Omega^\varepsilon)) \le C \tag{4.10}$$

for a constant $C > 0$ independent of ε. For later purposes, we note that each set $\tilde{P}_j^\varepsilon$ is either contained in T^ε or in $\Omega \setminus T^\varepsilon$.

Step 3: Limiting rotation, deformation, and partition. Up to the extraction of a subsequence (not relabeled), we may assume that

$$R^\varepsilon \to R \in SO(d),$$

i.e., we directly have (3.13). Applying Lemma 2.1, up to passing to a further subsequence, we find $y \in \mathscr{Y}(\Omega)$, see (2.4), such that (3.15) holds. By (4.2) we get that there exists $\Phi \in BV(\Omega; \{A, B\})$ such that

$$\Phi^\varepsilon \rightharpoonup^* \Phi \quad \text{weakly* in } BV(\Omega; \{A, B\})$$

and hence almost everywhere in Ω. By (4.2), (3.13), and (3.15) we then get $y \in \mathcal{Y}_R(\Omega)$.

By (4.10) and the compactness theorem for Caccioppoli partitions (Theorem A.1) we obtain a limiting partition $\tilde{\mathcal{P}} := \{\tilde{P}_j\}_j$ such that $\tilde{P}_j^\varepsilon \to \tilde{P}_j$ in measure for all indices j (up to a subsequence). Note that the components $\{\tilde{P}_j\}_j$ are possibly not indecomposable. Therefore, we let $\mathcal{P} = \{P_j\}_j$ be the partition consisting of the connected components of $\{\tilde{P}_j\}_j$. (This partition exists due to [8, Theorem 1], see also Appendix A.) By the lower semicontinuity of the Hausdorff measure and (4.10) we also deduce

$$\sum\nolimits_j \mathcal{H}^{d-1}(\partial^* P_j) = \sum\nolimits_j \mathcal{H}^{d-1}(\partial^* \tilde{P}_j) \le C. \tag{4.11}$$

We close this step of the proof by showing that $\mathcal{P} \in \mathscr{P}(\Omega)$. Clearly, by the definition of $\mathcal{P}$, it suffices to prove $\tilde{\mathcal{P}} \in \mathscr{P}(\Omega)$. To this end, it suffices to show that

$$\nu_{\tilde{P}_j}(x) = \pm e_d \quad \text{for} \quad \mathcal{H}^{d-1}\text{-a.e. } x \in \partial^* \tilde{P}_j \cap \Omega, \tag{4.12}$$

where $\nu_{\tilde{P}_j}$ denotes the outer unit normal to $\tilde{P}_j$. Let $\Omega' \subset\subset \Omega$. Fix $i \in \{1, \dots, d-1\}$. Since the function $\varphi(\nu) = |\langle \nu, e_i \rangle|$ is BV-elliptic (see [7, Theorem 5.20, Example 5.23]), lower semicontinuity results for sets of finite perimeter [6, Theorem 2.1] imply

$$\int_{\partial^* \tilde{P}_j \cap \Omega'} |\langle \nu_{\tilde{P}_j}, e_i \rangle| \, \mathrm{d}\mathcal{H}^{d-1} \le \liminf_{\varepsilon \to 0} \int_{\partial^* \tilde{P}_j^\varepsilon \cap \Omega'} |\langle \nu_{\tilde{P}_j^\varepsilon}, e_i \rangle| \, \mathrm{d}\mathcal{H}^{d-1}. \tag{4.13}$$

For ε sufficiently small we have $\Omega' \subset \Omega^\varepsilon$, see (4.3). Then the definition of $\tilde{P}_j^\varepsilon$ (see (4.8)) implies

$$\liminf_{\varepsilon \to 0} \int_{\partial^* \tilde{P}_j^\varepsilon \cap \Omega'} |\langle \nu_{\tilde{P}_j^\varepsilon}, e_i \rangle| \, \mathrm{d}\mathcal{H}^{d-1} \le \liminf_{\varepsilon \to 0} \int_{\partial^* T^\varepsilon \cap \Omega} |\langle \nu_{T^\varepsilon}, e_i \rangle| \, \mathrm{d}\mathcal{H}^{d-1} \tag{4.14}$$

since $\nu_{\tilde{P}_j^\varepsilon}(x) = \pm e_d$ for $\mathcal{H}^{d-1}$-a.e. $x \in \partial^* \tilde{P}_j^\varepsilon \setminus \partial^* T^\varepsilon$. In view of Proposition 4.1(ii) and (3.9), recalling the definition of $\alpha(d) = 1/(2d)$ in (3.3), we obtain by (4.13)–(4.14)

$$\int_{\partial^* \tilde{P}_j \cap \Omega'} |\langle \nu_{\tilde{P}_j}, e_i \rangle| \, \mathrm{d}\mathcal{H}^{d-1} = 0 \quad \text{for every } i = 1, \dots, d-1.$$

Thus, (4.12) holds since $\Omega' \subset\subset \Omega$ was arbitrary. Therefore, $\tilde{\mathcal{P}} \in \mathscr{P}(\Omega)$ and then also $\mathcal{P} \in \mathscr{P}(\Omega)$.

Step 4: Definition of the sequence of partitions and phase indicators. We now define the partitions $\mathcal{P}^\varepsilon$ and the phase indicators $\mathcal{M}^\varepsilon$, and show (3.10), (3.12), (3.14), and (3.16). The proof of (3.11) is deferred to Step 5 below. Let $\mathcal{P}^\varepsilon = \{P_j^\varepsilon\}_j$ be the partition consisting of the nonempty components of

$$\{\tilde{P}_k^\varepsilon \cap P_j\colon\ j, k \in \mathbb{N}\}. \tag{4.15}$$

Since $\tilde{P}_k^\varepsilon \to \tilde{P}_k$ for all indices k and $P_j \subset \tilde{P}_k$ for some k, we clearly get that (3.14) holds. Additionally, property (3.10) follows from (4.10)–(4.11).

Recall that each component of $\mathcal{P}^\varepsilon$ is contained in T^ε or $\Omega \setminus T^\varepsilon$, see the sentence below (4.10). We define the sequence $\mathcal{M}^\varepsilon = \{M_j^\varepsilon\}_{j,\varepsilon}$ by $M_j^\varepsilon = A$ for all j such that $P_j^\varepsilon \subset T^\varepsilon$, and $M_j^\varepsilon = B$ otherwise. Then (3.12) follows from (4.2). This along with (3.15) also implies

$$\sum\nolimits_j R^\varepsilon M_j^\varepsilon \chi_{P_j^\varepsilon} \to \nabla y \quad \text{strongly in } L^2_{\mathrm{loc}}(\Omega; \mathbb{M}^{d \times d}). \tag{4.16}$$

Due to (3.10), we have $\sum_j \mathcal{H}^{d-1}(\partial^* P_j^\varepsilon) \le C$, which yields

$$\big|D\big(\sum\nolimits_j R^\varepsilon M_j^\varepsilon \chi_{P_j^\varepsilon}\big)\big|(\Omega) \le C.$$

This along with (4.16) and a BV compactness argument yields (3.16).

Step 5: Proof of (3.11). It remains to prove (3.11). Choose $\Omega' \subset\subset \Omega$ and let ε be sufficiently small such that $\Omega' \subset \Omega^\varepsilon$, see (4.3). We show (3.11) only for the components of $\mathcal{P}^\varepsilon$ which are contained in $T^\varepsilon \cap \Omega^\varepsilon$ since for components contained in $\Omega^\varepsilon \setminus T^\varepsilon$ the argument is the same. Denote by $\pi_d(P_j^\varepsilon)$ the orthogonal projection of P_j^ε onto the e_d-axis. In view of (4.7)–(4.8), (4.15), and the fact that $\mathcal{P} \in \mathscr{P}(\Omega)$, we can decompose the collection of components into the two sets

$$\mathcal{J}_1^\varepsilon = \big\{P_j^\varepsilon \subset T^\varepsilon \cap \Omega^\varepsilon\colon\ \mathcal{H}^{d-1}(P_j^\varepsilon \cap \{x_d = t\}) \le \sigma_\varepsilon \text{ for a.e. } t \in \pi_d(P_j^\varepsilon)\big\},$$

$$\mathcal{J}_2^\varepsilon = \big\{P_j^\varepsilon \subset T^\varepsilon \cap \Omega^\varepsilon\colon \ \mathcal{H}^{d-1}(P_j^\varepsilon \cap \{x_d = t\}) > \sigma_\varepsilon \text{ for a.e. } t \in \pi_d(P_j^\varepsilon)\big\}. \tag{4.17}$$

First, since $\sigma_\varepsilon \le \varepsilon^p$, we clearly get by Fubini's theorem that

$$\sum\nolimits_{P_j^\varepsilon \in \mathcal{J}_1^\varepsilon} \mathcal{L}^d(\Omega^\varepsilon \cap P_j^\varepsilon) \le (b-a)\sigma_\varepsilon \le C\varepsilon^p, \tag{4.18}$$

where C only depends on Ω. We now consider the components in $\mathcal{J}_2^\varepsilon$. We let

$$I_j^\varepsilon = \big\{t \subset \pi_d(P_j^\varepsilon)\colon \ \mathcal{H}^{d-1}\big((\Omega^\varepsilon \setminus P_j^\varepsilon) \cap \{x_d = t\}\big) > \sigma_\varepsilon\big\} \quad \text{for every } j \in \mathcal{J}_2^\varepsilon. \tag{4.19}$$

Since $\sigma_\varepsilon \le \varepsilon^p$, we get

$$\sum\nolimits_{P_j^\varepsilon \in \mathcal{J}_2^\varepsilon} \int_{\pi_d(P_j^\varepsilon)\setminus I_j^\varepsilon} \mathcal{H}^{d-1}\big((\Omega^\varepsilon \setminus P_j^\varepsilon) \cap \{x_d = t\}\big)\,\mathrm{d}t \le (b-a)\sigma_\varepsilon \le C\varepsilon^p. \tag{4.20}$$

On the other hand, for a.e. $t \in I_j^\varepsilon$ we get by (4.4), applied for $E = P_j^\varepsilon \cap \{x_d = t\}$, and by (4.17), (4.19) that

$$\begin{aligned}\sigma_\varepsilon &\le \min\big\{\mathcal{H}^{d-1}(P_j^\varepsilon \cap \{x_d = t\}), \mathcal{H}^{d-1}\big((\Omega^\varepsilon \setminus P_j^\varepsilon) \cap \{x_d = t\}\big)\big\}\\ &\le \varepsilon^{-\alpha(d)}\big(\mathcal{H}^{d-2}(\partial^*(P_j^\varepsilon \cap \{x_d = t\}) \cap \Omega^\varepsilon)\big)^{\frac{d-1}{d-2}}.\end{aligned}$$

As $\sigma_\varepsilon \ge \varepsilon^p/2$, we find $1/2 \le (1/2)^{(d-2)/(d-1)} \le \varepsilon^{-(\alpha(d)+p)(d-2)/(d-1)}\mathcal{H}^{d-2}(\partial^* P_j^\varepsilon \cap \{x_d = t\} \cap \Omega^\varepsilon)$. Integrating over I_j^ε and summing over the components $\mathcal{J}_2^\varepsilon$, we get

$$\sum\nolimits_{P_j^\varepsilon \in \mathcal{J}_2^\varepsilon} \mathcal{L}^1(I_j^\varepsilon) \le C\varepsilon^{\frac{-(\alpha(d)+p)(d-2)}{d-1}} \sum\nolimits_j \int_a^b \mathcal{H}^{d-2}\big(\partial^* P_j^\varepsilon \cap \{x_d = t\} \cap \Omega^\varepsilon\big)\,\mathrm{d}t.$$

We recall (4.15) and the fact that $\mathcal{P} \in \mathscr{P}(\Omega)$. Moreover, we have $\bigcup_j \partial^* \tilde{P}_j^\varepsilon \cap \Omega^\varepsilon \cap \{x_d = t\} \subset \partial^* T^\varepsilon \cap \Omega \cap \{x_d = t\}$ for a.e. $t \in (a,b)$, where $\mathcal{H}^{d-1}$-a.e. $x \in \partial^* T^\varepsilon$ is contained in the boundary of at most two different components, see (4.8). Then, (3.9) and Proposition 4.1(iii) yield

$$\sum_{P_j^\varepsilon \in \mathcal{J}_2^\varepsilon} \mathcal{L}^1(I_j^\varepsilon) \le C\varepsilon^{\frac{-(\alpha(d)+p)(d-2)}{d-1}} \int_{-\infty}^{\infty} \mathcal{H}^{d-2}(\partial^* T^\varepsilon \cap \{x_d = t\} \cap \Omega)\,\mathrm{d}t \le C\varepsilon^{-(\alpha(d)+p)(d-2)/(d-1)}\varepsilon^{2-\alpha(d)},$$

where $C > 0$ depends on C_0. Recalling $p = 1 + \frac{3}{2d(2d-3)}$ and $\alpha(d) = 1/(2d)$, this yields $\sum_{P_j^\varepsilon \in \mathcal{J}_2^\varepsilon} \mathcal{L}^1(I_j^\varepsilon) \le C\varepsilon^p$ by an elementary computation. This along with (4.20) and the fact that $\mathcal{H}^{d-1}(\Omega^\varepsilon \cap \{x_d = t\}) \le (\mathrm{diam}(\Omega))^{d-1}$ for all $t \in (a,b)$ yields

$$\begin{aligned}\sum\nolimits_{P_j^\varepsilon \in \mathcal{J}_2^\varepsilon} \mathcal{L}^d(L_{\Omega^\varepsilon}(P_j^\varepsilon) \setminus P_j^\varepsilon) &\le (\mathrm{diam}(\Omega))^{d-1} \sum\nolimits_{P_j^\varepsilon \in \mathcal{J}_2^\varepsilon} \mathcal{L}^1(I_j^\varepsilon)\\ &\quad + \sum\nolimits_{P_j^\varepsilon \in \mathcal{J}_2^\varepsilon} \int_{\pi_d(P_j^\varepsilon)\setminus I_j^\varepsilon} \mathcal{H}^{d-1}\big((\Omega^\varepsilon \setminus P_j^\varepsilon) \cap \{x_d = t\}\big)\,\mathrm{d}t \le C\varepsilon^p, \end{aligned} \tag{4.21}$$

where the constant C depends only on Ω and C_0, and $L_{\Omega^\varepsilon}(P_j^\varepsilon)$ is defined in (3.8). By combining (4.18) and (4.21) we get (3.11) since $\Omega^\varepsilon \supset \Omega'$ (for ε small enough). This concludes the proof. □

Remark 4.3 (Geometry of Ω)**.** (i) Condition H8. could be dropped at the expense of more elaborated estimates. First, in (3.11), $L_{\Omega'}(P_j)$ would have to be replaced by the connected components of $L_{\Omega'}(P_j)$ which intersect P_j. Accordingly, the isoperimetric inequality (4.4), applied in Step 5 of the proof, would need to be applied separately in each of the components of $\Omega^\varepsilon \cap \{x_d = t\}$ to get an estimate along the lines of (4.21).

(ii) The passage to a subdomain in (3.11)–(3.12) is not needed if Ω is a paraxial cuboid: in this case, Theorem 3.2 can be replaced by an equivalent statement directly on Ω, see [32, Theorem 3.1 and Remark 3.2]. Moreover, the isoperimetric inequality (4.4) in Step 5 can be performed on the (identical) cuboids $\Omega \cap \{x_d = t\}$ of dimension $d-1$.

Recall the definition of $\mathscr{U}(\Omega)$ in (3.7). The next step will be to identify limiting displacement fields for subsets $\Omega' \subset\subset \Omega$. Before that, we state an elementary local property of partitions that we will use several times.

Lemma 4.4 (Local property of partitions)**.** *Let $K \subset\subset \Omega$. Then, for each $\mathcal{P} \in \mathscr{P}(\Omega)$, the set K only intersects a finite number of sets contained in $\mathcal{P}$.*

Proof. The result is a direct consequence of the compactness of K, and of the definition of $\mathscr{P}(\Omega)$. □

Proposition 4.5 (Rescaled displacement fields on subdomains)**.** *Consider the setting of Proposition 4.2. Let $\Omega' \subset\subset \Omega$, and denote by $\{P_j\}_{j=1}^N$ the components of $\mathcal{P}$ which intersect Ω', see Lemma 4.4. Then there exist $u \in \mathscr{U}(\Omega')$ with $J_u \subset \bigcup_j \partial P_j$ and collections of constants $\{t_j^\varepsilon\}_{j=1}^N$ for $\varepsilon > 0$ such that the rescaled displacements $u^\varepsilon\colon \Omega' \to \mathbb{R}^d$ defined by*

$$u^\varepsilon(x) := \varepsilon^{-1} \sum\nolimits_{j=1}^N \big(y^\varepsilon(x) - (R^\varepsilon M_j^\varepsilon\, x + t_j^\varepsilon)\big)\chi_{P_j^\varepsilon}(x) + \varepsilon^{-1} \sum\nolimits_{j>N} \big(y^\varepsilon(x) - R^\varepsilon M_j^\varepsilon\, x\big)\chi_{P_j^\varepsilon}(x) \tag{4.22}$$

for $x \in \Omega'$ satisfy (up to a subsequence, not relabeled)

$$u^\varepsilon \to u \quad \text{in measure in } \ \Omega', \qquad\qquad \nabla u^\varepsilon \rightharpoonup \nabla u \quad \text{weakly in } L^2(\Omega'; \mathbb{M}^{d\times d}). \tag{4.23}$$

We note that the second addend in (4.22) is intended to be zero if $\{P_j^\varepsilon\}_j$ consists only of N components.

Proof. First, we recall that the components $\{P_j\}_{j=1}^N$ are connected by definition, that $\mathcal{H}^{d-1}(\partial P_j \setminus \partial^* P_j) = 0$, and that $\nu_{P_j} = \pm e_d$ for $\mathcal{H}^{d-1}$-a.e. $x \in \partial P_j \cap \Omega$, where the latter two properties follow from the fact that $\mathcal{P} \in \mathscr{P}(\Omega)$. Possibly choosing another set $\Omega' \subset\subset \Omega'' \subset\subset \Omega$ we can assume that the sets $P_j \cap \Omega''$, $j = 1, \ldots, N$, are connected and have Lipschitz boundary. Clearly, it suffices to show the statement for Ω'' in place of Ω'. For simplicity, we still denote this set by Ω'.

Let $(R^\varepsilon, \mathcal{P}^\varepsilon, \mathcal{M}^\varepsilon)$ be the triples identified in Proposition 4.2. By (3.12) we get

$$\Big\| \sum\nolimits_j \big(\nabla y^\varepsilon - R^\varepsilon M_j^\varepsilon\big)\, \chi_{P_j^\varepsilon} \Big\|_{L^2(\Omega')} \le C\varepsilon \tag{4.24}$$

for a constant $C > 0$ depending on Ω'.

Step 1: Poincaré estimate on each component. Since $P_j \cap \Omega'$ is connected with Lipschitz boundary, we can choose an increasing sequence of smooth connected sets $K_n \subset\subset P_j \cap \Omega'$ such that $\mathcal{L}^d((P_j \cap \Omega') \setminus K_n) \to 0$ as $n \to \infty$. The sets can be chosen such that the functions

$$f_j^{n,\varepsilon}(x) := \varepsilon^{-1}(y^\varepsilon(x) - R^\varepsilon\, M_j^\varepsilon\, x - t_j^{n,\varepsilon}) \quad \text{ for every } x \in K_n, \tag{4.25}$$

for suitable $t_j^{n,\varepsilon} \in \mathbb{R}^d$, satisfy a Poincaré estimate

$$\|f_j^{n,\varepsilon}\|_{L^p(K_n)} \le C \|\nabla f_j^{n,\varepsilon}\|_{L^p(K_n)}, \tag{4.26}$$

where the constant C depends on P_j, but is independent of ε and n. By (3.14) and (3.11) we get $P_j^\varepsilon \cap \Omega' \to P_j \cap \Omega'$ and $L_{\Omega'}(P_j^\varepsilon) \to P_j \cap \Omega'$ in measure as $\varepsilon \to 0$. The latter and the fact that $K_n \subset\subset \Omega' \cap P_j$ show that $K_n \subset L_{\Omega'}(P_j^\varepsilon)$ for ε small enough (depending on n). Thus, by using again (3.11) and $\mathcal{L}^d((P_j^\varepsilon \cap \Omega') \triangle (P_j \cap \Omega')) \to 0$, we get

$$\mathcal{L}^d(K_n \setminus P_j^\varepsilon) \le \mathcal{L}^d\big(K_n \setminus L_{\Omega'}(P_j^\varepsilon)\big) + \mathcal{L}^d\big(L_{\Omega'}(P_j^\varepsilon) \setminus P_j^\varepsilon\big) = \mathcal{L}^d\big(L_{\Omega'}(P_j^\varepsilon) \setminus P_j^\varepsilon\big) \le C\varepsilon^p \tag{4.27}$$

for ε small enough depending on n, where $p = p(d) \in (1, 2)$ is fixed. Let L be a sufficiently large constant (independent of ε, n) such that

$$\mathrm{dist}(F, SO(d)\{A, B\}) \ge |F - R^\varepsilon M_j^\varepsilon|/2 \quad \text{ for all } \ F \in \mathbb{M}^{d\times d} \text{ with } |F - R^\varepsilon M_j^\varepsilon| \ge L.$$

Then, $\|\nabla f_j^{n,\varepsilon}\|_{L^p(K_n)}$ can be controlled by

$$\begin{aligned}
&\|\nabla f_j^{n,\varepsilon}\|_{L^p(P_j^\varepsilon \cap K_n)} + \|\nabla f_j^{n,\varepsilon}\|_{L^p((K_n \setminus P_j^\varepsilon) \cap \{|\nabla y^\varepsilon - R^\varepsilon M_j^\varepsilon| \le L\})} + \|\nabla f_j^{n,\varepsilon}\|_{L^p((K_n \setminus P_j^\varepsilon) \cap \{|\nabla y^\varepsilon - R^\varepsilon M_j^\varepsilon| > L\})} \\
&\le \frac{1}{\varepsilon}\|\nabla y^\varepsilon - R^\varepsilon M_j^\varepsilon\|_{L^p(P_j^\varepsilon \cap \Omega')} + \frac{L}{\varepsilon}\big(\mathcal{L}^d(K_n \setminus P_j^\varepsilon)\big)^{\frac{1}{p}} + \frac{2}{\varepsilon}\|\mathrm{dist}(\nabla y^\varepsilon, SO(d)\{A, B\})\|_{L^p(\{|\nabla y^\varepsilon - R^\varepsilon M_j^\varepsilon| > L\})}.
\end{aligned}$$

Using Hölder's inequality for $p < 2$, (4.24), (4.27), as well as (3.1), (3.9) together with H4. we obtain the uniform estimate $\|\nabla f_j^{n,\varepsilon}\|_{L^p(K_n)} \le C$ for $C > 0$ independent of n and ε. Then (4.26) yields

$$\|f_j^{n,\varepsilon}\|_{W^{1,p}(K_n)} \le C. \tag{4.28}$$

We now show that the translations $\{t_j^{n,\varepsilon}\}_\varepsilon$ and thus the functions $\{f_j^{n,\varepsilon}\}_\varepsilon$ can actually be chosen *independently* of n. Recall that $K_n \supset K_1$ for all $n \in \mathbb{N}$. In view of (4.25) and (4.28), we have

$$\varepsilon^{-1}|t_j^{n,\varepsilon} - t_j^{m,\varepsilon}|\,\mathcal{L}^d(K_1) \le \|f_j^{\varepsilon,n}\|_{L^1(K_n)} + \|f_j^{\varepsilon,m}\|_{L^1(K_m)} \le C \tag{4.29}$$

for every $m, n \in \mathbb{N}$, where the constant C is independent of n, m, and ε. Thus, for every $\varepsilon > 0$ we get that $\{t_j^{n,\varepsilon}\}_n$ is a bounded sequence, and up to the extraction of a subsequence (not relabeled) there exists t_j^ε such that

$$t_j^{n,\varepsilon} \to t_j^\varepsilon \quad \text{as } n \to +\infty. \tag{4.30}$$

The constants t_j^ε are the ones from the statement of the proposition. By (4.29) we get $\varepsilon^{-1}|t_j^{n,\varepsilon} - t_j^\varepsilon| \le C$ for a constant $C > 0$ independent of n and ε. This along with (4.28) yields that the functions

$$f_j^\varepsilon(x) := \varepsilon^{-1}(y^\varepsilon(x) - R^\varepsilon\, M_j^\varepsilon\, x - t_j^\varepsilon) \quad \text{ for every } x \in P_j^\varepsilon, \tag{4.31}$$

satisfy for all $n \in \mathbb{N}$ and all ε small enough (depending on n)

$$\|f_j^\varepsilon\|_{W^{1,p}(K_n)} \le C,$$

where the constant $C > 0$ is independent of ε and n. Thus, by a compactness and a diagonal argument there exists a function $f_j \in W^{1,p}(P_j \cap \Omega'; \mathbb{R}^d)$ such that (up to a subsequence)

$$f_j^\varepsilon \rightharpoonup f_j \quad \text{ weakly in } W^{1,p}(P_j \cap \Omega'; \mathbb{R}^d). \tag{4.32}$$

Step 2: Definition of the limiting displacement field. Recall the functions f_j identified in (4.32) and the constants t_j^ε from (4.30). We set $u := \sum_{j=1}^N f_j \chi_{P_j}$ on Ω' and define u^ε as in (4.22). Below we will show that indeed $u \in \mathscr{U}(\Omega')$, see (3.7), but now we first confirm (4.23). In view of (4.31), we get that $u^\varepsilon = f_j^\varepsilon$ on $P_j^\varepsilon \cap \Omega'$. We claim that, up to a further subsequence, there holds

$$\begin{aligned} &\text{(i)} \;\; u^\varepsilon \to f_j = u \quad \text{ in measure on } P_j \cap \Omega' \quad \text{for all } j = 1, \dots, N, \\ &\text{(ii)} \;\; \nabla u^\varepsilon \rightharpoonup \nabla u \quad \text{weakly in } L^2(\Omega'; \mathbb{M}^{d\times d}). \end{aligned} \tag{4.33}$$

In fact, (4.32) along with (3.14) and $u^\varepsilon = f_j^\varepsilon$ on $P_j^\varepsilon \cap \Omega'$ implies measure convergence on $P_j \cap \Omega'$. This yields (i). To see (ii), we use (4.22) and (4.24) to get

$$\nabla u^\varepsilon = \varepsilon^{-1} \sum\nolimits_j \big(\nabla y^\varepsilon - R^\varepsilon M_j^\varepsilon\big)\, \chi_{P_j^\varepsilon} \rightharpoonup g$$

weakly in $L^2(\Omega'; \mathbb{M}^{d\times d})$ for a suitable function g. Again by (4.32) we get $g = \nabla f_j$ on each $P_j \cap \Omega'$, and therefore $g = \nabla u$ a.e. on Ω'. This yields (ii). Clearly, (4.33) implies (4.23).

It remains to check that $u \in \mathscr{U}(\Omega')$. Recall that only the components P_j, $j = 1, \dots, N$, intersect Ω'. Since $f_j \in W^{1,p}(P_j \cap \Omega'; \mathbb{R}^d)$ for all $j = 1, \dots, N$, we get $J_u \subset \bigcup_{j=1}^N \partial P_j$. Thus, we find $\mathcal{H}^{d-1}(J_u) < +\infty$ since $\mathcal{P}$ is a Caccioppoli partition. More precisely, as $\mathcal{P} \in \mathscr{P}(\Omega)$, the jump set of u is contained in $(d-1)$-dimensional hyperplanes orthogonal to e_d. It thus remains to show that $u \in SBV^2(\Omega'; \mathbb{R}^d)$. First, note that $\nabla u \in L^2(\Omega'; \mathbb{M}^{d\times d})$ by (4.33)(ii). Since each $P_j \cap \Omega'$ has Lipschitz boundary, we get that $u|_{P_j \cap \Omega'} \in H^1(P_j \cap \Omega'; \mathbb{R}^d)$, and the trace of u on $\partial P_j \cap \Omega'$ exists. As the number of sets $\{P_j\}_j$ intersecting Ω' is finite, we obtain $u \in SBV^2(\Omega'; \mathbb{R}^d)$ by applying [7, Theorem 3.84]. ☐

We next show that the translations can be defined so that there exists a limiting rescaled displacement field on the whole domain Ω.

Proposition 4.6 (Rescaled displacement fields)**.** *Consider the setting of Proposition 4.2. Then there exist collections of constants $\mathcal{T}^\varepsilon = \{t_j^\varepsilon\}_j$ for $\varepsilon > 0$ and $u \in \mathscr{U}(\Omega)$ with $J_u \subset \bigcup_j \partial P_j$ such that the rescaled displacements u^ε defined in (3.18) satisfy (3.19)–(3.20).*

Proof. Consider a sequence $\{\Omega_n\}_n$ of open sets, compactly contained in Ω, satisfying $\Omega_n \subset \Omega_{n+1}$ for every $n \in \mathbb{N}$, and such that $\mathcal{L}^d(\Omega \setminus \Omega_n) \to 0$ as $n \to \infty$. We denote by $\{\mathcal{P}^\varepsilon\}_\varepsilon$ and $\mathcal{P}$ the partitions identified in Proposition 4.2. In view of Lemma 4.4, we can reorder the partition $\mathcal{P} = \{P_j\}_j$ in a specific way and can choose integers $N_1 \le N_2 \le \dots$ such that $\{P_j\}_{j=1}^{N_n}$ indicate the components of $\mathcal{P}$ which intersect Ω_n. For each $n \in \mathbb{N}$, the translations given by Proposition 4.5 (with Ω_n in place of Ω') are denoted by

$\{t_j^{\varepsilon,n}\}_{j=1}^{N_n}$. The displacement fields on Ω_n defined in (4.22) are denoted by $u^{\varepsilon,n}$. We denote their limits by $u^n \in \mathscr{U}(\Omega_n)$ and recall that $J_{u^n} \subset \bigcup_j \partial P_j$. By a diagonal argument, we may suppose that there exists a subsequence of ε (not relabeled) such that (4.23) holds for all $n \in \mathbb{N}$, i.e.,

$$u^{\varepsilon,n} \to u^n \quad \text{in measure in } \Omega_n, \qquad \nabla u^{\varepsilon,n} \rightharpoonup \nabla u^n \quad \text{weakly in } L^2(\Omega_n;\mathbb{M}^{d\times d}). \tag{4.34}$$

Now it is elementary to check that for each $n \in \mathbb{N}$ there holds

$$\lim_{\varepsilon\to 0} \varepsilon^{-1}(t_j^{\varepsilon,n} - t_j^{\varepsilon,n+1}) \quad \text{exists and is finite for all } 1 \le j \le N_n. \tag{4.35}$$

Indeed, this follows from $\mathcal{L}^d(P_j \cap \Omega_n) > 0$ for all $1 \le j \le N_n$, and the fact that

$$\varepsilon^{-1}(t_j^n - t_j^{n+1})\chi_{P_j^\varepsilon \cap \Omega_n} = (u^{\varepsilon,n+1} - u^{\varepsilon,n})\chi_{P_j^\varepsilon \cap \Omega_n} \to (u^{n+1} - u^n)\chi_{P_j \cap \Omega_n}$$

in measure, see (3.14) and (4.22)–(4.23), as well as (4.34).

We define the collection of translations $\mathcal{T}^\varepsilon = \{t_j^\varepsilon\}_j$ as follows: for each j, choose $n \in \mathbb{N}$ such that $N_{n-1} < j \le N_n$, and set $t_j^\varepsilon = t_j^{\varepsilon,n}$, where we define $N_0 = 0$ for convenience. We define $u^\varepsilon\colon \Omega \to \mathbb{R}^d$ as in (3.18). By recalling the definition of $u^{\varepsilon,n}$ in (4.22), we get that the restriction of u^ε on Ω_n, for $n \in \mathbb{N}$, satisfies

$$u^\varepsilon = u^{\varepsilon,n} + \sum\nolimits_{j=1}^{N_n} \varepsilon^{-1}(t_j^{\varepsilon,n} - t_j^\varepsilon)\chi_{P_j^\varepsilon \cap \Omega_n} - \sum\nolimits_{j>N_n} \varepsilon^{-1} t_j^\varepsilon \chi_{P_j^\varepsilon \cap \Omega_n} \quad \text{on } \Omega_n.$$

We introduce the function $v^n \in \mathscr{U}(\Omega_n)$ by

$$v^n = u^n + \sum\nolimits_{j=1}^{N_n} \Big(\lim_{\varepsilon\to 0} \varepsilon^{-1}(t_j^{\varepsilon,n} - t_j^\varepsilon)\Big)\chi_{P_j\cap\Omega_n}, \tag{4.36}$$

which is well defined by (4.35) and the fact that $t_j^\varepsilon = t_j^{\varepsilon,m}$ for the index $1 \le m \le n$ such that $N_{m-1} < j \le N_m$. In view of (3.14), (4.34), and the fact that $P_j \cap \Omega_n = \emptyset$ for all $j > N_n$, we then get

$$u^\varepsilon \to v^n \quad \text{in measure on } \Omega_n, \qquad \nabla u^\varepsilon \rightharpoonup \nabla v^n \quad \text{weakly in } L^2(\Omega_n;\mathbb{M}^{d\times d}). \tag{4.37}$$

This also shows that $v^n = v^m$ on Ω_n for all $n \le m$. This observation allows to define the function $u\colon \Omega \to \mathbb{R}^d$ by $u = v^n$ on Ω_n for all $n \in \mathbb{N}$. The fact that $J_{u^n} \subset \bigcup_j \partial P_j$ along with (4.36) also yields $J_u \subset \bigcup_j \partial P_j$. Clearly, we get $u \in \mathscr{U}(\Omega)$ since $v^n \in \mathscr{U}(\Omega_n)$ for all $n \in \mathbb{N}$. Finally, by (4.37) and the fact that $u = v^n$ on Ω_n we get that u^ε satisfies (3.19)–(3.20). This concludes the proof. □

We conclude this section with the proof of Theorem 3.3. Given the above constructions, it remains to show that the partitions and translations can be chosen in a specific way such that also the selection principle (3.17) is satisfied. Although the realization of this is very technical, the main idea is quite simple: whenever two components violate (3.17), they are combined, and they are replaced by a single component in the partition.

Proof of Theorem 3.3. Let $\{y^\varepsilon\}_\varepsilon \subset H^2(\Omega;\mathbb{R}^2)$ be a sequence of deformations satisfying (3.9). Consider a sequence $\{\Omega_n\}_n$ of open sets compactly contained in Ω, satisfying $\Omega_n \subset \Omega_{n+1}$ for every $n \in \mathbb{N}$, and such that $\mathcal{L}^d(\Omega \setminus \Omega_n) \to 0$. We will prove that, after extracting a subsequence in ε (not relabeled), for each $n \in \mathbb{N}$ there exists a sequence of quadruples $(R^\varepsilon, \mathcal{P}^{\varepsilon,n}, \mathcal{M}^{\varepsilon,n}, \mathcal{T}^{\varepsilon,n})$ with $\mathcal{P}^{\varepsilon,n} = \{P_j^{\varepsilon,n}\}_j$, $\mathcal{M}^{\varepsilon,n} = \{M_j^{\varepsilon,n}\}_j$, $\mathcal{T}^{\varepsilon,n} = \{t_j^{\varepsilon,n}\}_j$ and limiting triples $(y, u^n, \mathcal{P}^n) \in \mathscr{Y}(\Omega)\times\mathscr{U}(\Omega)\times\mathscr{P}(\Omega)$ such that (3.10)–(3.16) and (3.18)–(3.20) hold, and additionally we have

$$\frac{|t_i^{\varepsilon,n} - t_j^{\varepsilon,n}|}{\varepsilon} \to +\infty \quad \text{for all } i \neq j \ \text{ with } P_i^n \cap \Omega_n \neq \emptyset,\ P_j^n \cap \Omega_n \neq \emptyset, \text{ and } \lim_{\varepsilon\to0} M_i^{\varepsilon,n} = \lim_{\varepsilon\to0} M_j^{\varepsilon,n}, \tag{4.38}$$

where $\{P_j^n\}_j$ denote the components of the limiting partition $\mathcal{P}^n$. Note that the deformation y and the rotations R^ε can be chosen independently of $n \in \mathbb{N}$. Moreover, we will see that the objects can be constructed such that for each $n \ge m$ and each $\varepsilon > 0$ we have

(i) for all j there exists l_j such that $P_j^{\varepsilon,m} \subset P_{l_j}^{\varepsilon,n}$,

(ii) for all j we have $M_{l_j}^{\varepsilon,n} = M_j^{\varepsilon,m}$ with l_j given in (i),

(iii) if $\mathcal{L}^d(P_j^{\varepsilon,m} \cap \Omega_m) > 0$, then $t_{l_j}^{\varepsilon,n} = t_j^{\varepsilon,m}$ with l_j given in (i),

$$\text{(iv)}\ \ u^{\varepsilon,n} = u^{\varepsilon,m} \text{ on } \Omega_m \quad \text{and} \quad \nabla u^{\varepsilon,n} = \nabla u^{\varepsilon,m} \text{ on } \Omega, \tag{4.39}$$

where $u^{\varepsilon,n}$ denote the rescaled displacement fields given in (3.18) for the quadruples $(R^\varepsilon, \mathcal{P}^{\varepsilon,n}, \mathcal{M}^{\varepsilon,n}, \mathcal{T}^{\varepsilon,n})$. We defer the proof to Step 2 below and first show that this implies Theorem 3.3 for a suitable diagonal sequence (Step 1).

Step 1: Extracting a diagonal sequence. First, we find by (3.19) on Ω_n and Ω_m, and by (4.39)(iv) that for all $n \ge m$ there holds $u^n = u^m$ on Ω_m and $\nabla u^n = \nabla u^m$ on Ω. This observation allows to define the function $u\colon \Omega \to \mathbb{R}^d$ by $u = u^n$ on Ω_n for all $n \in \mathbb{N}$. Clearly, we get $u \in \mathscr{U}(\Omega)$ since $u^n \in \mathscr{U}(\Omega)$ for all $n \in \mathbb{N}$. In particular, there holds for all $n \in \mathbb{N}$

$$u = u^n \quad \text{on } \ \Omega_n, \qquad \nabla u = \nabla u^n \quad \text{on } \Omega. \tag{4.40}$$

As $\mathcal{P}^{\varepsilon,n}$ is a coarsening of $\mathcal{P}^{\varepsilon,m}$ for all $n \ge m$ by (4.39)(i), we get that $\mathcal{P}^n$ is a coarsening of $\mathcal{P}^m$ for all $n \ge m$ by (3.14). This gives $\sum_j \mathcal{H}^{d-1}(\partial P_j^n) \le \sum_j \mathcal{H}^{d-1}(\partial P_j^1) < +\infty$ for all $n \in \mathbb{N}$. By Theorem A.1 there exists a partition $\mathcal{P} = \{P_j\}_j$ such that $P_j^n \to P_j$ in measure for all $j \in \mathbb{N}$. Note that this convergence also implies $\mathcal{P} \in \mathscr{P}(\Omega)$. This and (3.14) for each $m \in \mathbb{N}$ yield

$$\lim_{n\to\infty} \sum\nolimits_j \mathcal{L}^d(P_j^n \triangle P_j) = 0, \qquad \lim_{\varepsilon\to 0} \sum\nolimits_j \mathcal{L}^d(P_j^{\varepsilon,m} \triangle P_j^m) = 0 \quad \text{for all } m \in \mathbb{N}, \tag{4.41}$$

where $\triangle$ denotes the symmetric difference of two sets, see below Theorem A.1. Thus, by Attouch's diagonalization lemma [9, Lemma 1.15 and Corollary 1.16], we can choose a diagonal sequence $\{n(\varepsilon)\}_\varepsilon$ such that

$$P_j^{\varepsilon,n(\varepsilon)} \to P_j \quad \text{ in measure as } \varepsilon \to 0 \text{ for all indices } j. \tag{4.42}$$

We now define the triples $\mathcal{P}^\varepsilon = \mathcal{P}^{\varepsilon,n(\varepsilon)}$, $\mathcal{M}^\varepsilon = \mathcal{M}^{\varepsilon,n(\varepsilon)}$, and $\mathcal{T}^\varepsilon = \mathcal{T}^{\varepsilon,n(\varepsilon)}$, and check that (3.10)–(3.20) hold for the limiting triple $(y, u, \mathcal{P})$.

First, (3.10)–(3.11) follow directly from the corresponding properties of the partitions $\mathcal{P}^{\varepsilon,n}$. We observe that (4.39)(i),(ii) yield

$$\sum\nolimits_j R^\varepsilon M_j^{\varepsilon,n(\varepsilon)} \chi_{P_j^{\varepsilon,n(\varepsilon)}} = \sum\nolimits_j R^\varepsilon M_j^{\varepsilon,1} \chi_{P_j^{\varepsilon,1}}.$$

This implies (3.12), (3.13), (3.15), and (3.16) by using the corresponding properties for the triple $(R^\varepsilon, \mathcal{P}^{\varepsilon,1}, \mathcal{M}^{\varepsilon,1})$. Property (3.14) follows from (4.42).

Consider the rescaled displacement fields $u^{\varepsilon,n(\varepsilon)}$ defined in (3.18). For each $m \in \mathbb{N}$ we have $u^{\varepsilon,n(\varepsilon)} \to u^m = u$ in measure on Ω_m by (4.39)(iv), (4.40), and (3.19) for m. As m was arbitrary, we get (3.19). In a similar fashion, (3.20) follows also by taking into account (4.39)(iv), (4.40), and (3.20) for each m.

It remains to check (3.17). To this end, we fix $i \neq j$ such that $\mathcal{L}^d(P_i), \mathcal{L}^d(P_j) > 0$, and $\lim_{\varepsilon\to 0} M_i^{\varepsilon,n(\varepsilon)} = \lim_{\varepsilon\to 0} M_j^{\varepsilon,n(\varepsilon)}$. In view of (4.41)–(4.42), we can fix $m \in \mathbb{N}$ (independently of ε) and $\varepsilon_0 = \varepsilon_0(m) > 0$ such that for all $0 < \varepsilon \le \varepsilon_0$ we have for $k = i, j$

$$\text{(i)}\ \ \mathcal{L}^d(P_k^m \cap \Omega_m) > 0,\ \mathcal{L}^d(P_k^{\varepsilon,m} \cap \Omega_m) > 0 \qquad \text{and} \qquad \text{(ii)}\ \ \mathcal{L}^d(P_k^{\varepsilon,n(\varepsilon)} \triangle P_k^{\varepsilon,m}) \le \frac{1}{2}\mathcal{L}^d(P_k^{\varepsilon,m}). \tag{4.43}$$

(To see (ii), we use $\mathcal{L}^d(P_k^{\varepsilon,n(\varepsilon)} \triangle P_k^{\varepsilon,m}) \le \mathcal{L}^d(P_k^{\varepsilon,n(\varepsilon)} \triangle P_k) + \mathcal{L}^d(P_k \triangle P_k^m) + \mathcal{L}^d(P_k^m \triangle P_k^{\varepsilon,m}) \to 0$ and $\mathcal{L}^d(P_k^{\varepsilon,m}) \to \mathcal{L}^d(P_k^m)$ as $\varepsilon \to 0$.) Possibly by passing to a smaller ε_0, we can also suppose that $n(\varepsilon) \ge m$ for all $\varepsilon \le \varepsilon_0$. By (4.39)(i) for $n = n(\varepsilon)$ we find a component $P_{l_k}^{\varepsilon,n(\varepsilon)}$ which contains $P_k^{\varepsilon,m}$ up to an $\mathcal{L}^d$-negligible set for $k = i, j$. By (4.43)(ii) we necessarily have that $\mathcal{L}^d(P_k^{\varepsilon,n(\varepsilon)} \cap P_k^{\varepsilon,m}) > 0$. Thus, $k = l_k$. This along with (4.43)(i) and (4.39)(ii),(iii) shows $M_k^{\varepsilon,n(\varepsilon)} = M_k^{\varepsilon,m}$ and $t_k^{\varepsilon,n(\varepsilon)} = t_k^{\varepsilon,m}$ for $k = i, j$. Then, also $\lim_{\varepsilon\to 0} M_i^{\varepsilon,m} = \lim_{\varepsilon\to 0} M_j^{\varepsilon,m}$ and therefore, taking also (4.38), (4.43)(i) into account, we finally get

$$\lim_{\varepsilon\to 0} \frac{|t_i^{\varepsilon,n(\varepsilon)} - t_j^{\varepsilon,n(\varepsilon)}|}{\varepsilon} = \lim_{\varepsilon\to 0} \frac{|t_i^{\varepsilon,m} - t_j^{\varepsilon,m}|}{\varepsilon} = +\infty.$$

Step 2: Coarsening scheme. We inductively construct sequences of quadruples $(R^\varepsilon, \mathcal{P}^{\varepsilon,n}, \mathcal{M}^{\varepsilon,n}, \mathcal{T}^{\varepsilon,n})$ and limiting triples $(y, u^n, \mathcal{P}^n)$ for $n \in \mathbb{N}$ such that (3.10)–(3.16), (3.18)–(3.20), and (4.38)–(4.39) hold.

We start with $n = 1$. We apply Proposition 4.6 to obtain rotations R^ε and triples $(\hat{\mathcal{P}}^\varepsilon, \hat{\mathcal{M}}^\varepsilon, \hat{\mathcal{T}}^\varepsilon)$, as well as a limiting triple $(y, \hat{u}, \hat{\mathcal{P}})$ such that (3.10)–(3.16) and (3.18)–(3.20) hold. We write $\hat{\mathcal{P}}^\varepsilon = \{\hat{P}^\varepsilon_j\}_j$, $\hat{\mathcal{M}}^\varepsilon = \{\hat{M}^\varepsilon_j\}_j$, and $\hat{\mathcal{T}}^\varepsilon = \{\hat{t}^\varepsilon_j\}_j$. We modify the triples to get sequences which also satisfy (4.38).

Coarsening scheme for $n = 1$. We construct $\mathcal{P}^{\varepsilon,1}$, $\mathcal{T}^{\varepsilon,1}$, and $\mathcal{M}^{\varepsilon,1}$, as well as the limiting partition $\mathcal{P}^1$ and the limiting displacement u^1 by the following iterative scheme: suppose that two components $\hat{P}_i$ and $\hat{P}_j$ of $\hat{\mathcal{P}}$ with $i \neq j$ violate (4.38) on Ω_1, i.e.,

$$\liminf_{\varepsilon\to 0} \varepsilon^{-1}|\hat{t}^\varepsilon_i - \hat{t}^\varepsilon_j| < +\infty, \qquad \lim_{\varepsilon\to 0} \hat{M}^\varepsilon_i = \lim_{\varepsilon\to 0} \hat{M}^\varepsilon_j, \qquad \hat{P}_i \cap \Omega_1 \neq \emptyset, \qquad \hat{P}_j \cap \Omega_1 \neq \emptyset. \tag{4.44}$$

First, by passing to a subsequence in ε (not relabeled), we get $\hat{M}^\varepsilon_i = \hat{M}^\varepsilon_j$ for all ε. Now, we replace $\hat{P}_i$ and $\hat{P}_j$ in $\hat{\mathcal{P}}$ by $P^1_* := \hat{P}_i \cup \hat{P}_j$. In a similar fashion, we replace $\hat{P}^\varepsilon_i$ and $\hat{P}^\varepsilon_j$ in $\hat{\mathcal{P}}^\varepsilon$ by $P^{\varepsilon,1}_* := \hat{P}^\varepsilon_i \cup \hat{P}^\varepsilon_j$ for each $\varepsilon > 0$. Accordingly, on the set $P^{\varepsilon,1}_*$ we introduce the translation $t^{\varepsilon,1}_* = \hat{t}^\varepsilon_i$ and the phase $M^{\varepsilon,1}_* := \hat{M}^\varepsilon_i = \hat{M}^\varepsilon_j$ for each $\varepsilon > 0$. In view of Lemma 4.4, only finitely many components of $\hat{\mathcal{P}}$ intersect Ω_1. Thus, we can repeat this construction at most a finite number of times until, for the resulting partition $\mathcal{P}^1$ and the triples $(\mathcal{P}^{\varepsilon,1}, \mathcal{M}^{\varepsilon,1}, \mathcal{T}^{\varepsilon,1})$, each pair of components P^1_i and P^1_j satisfies (4.38). This concludes the construction in the case $n = 1$. (The definition of the resulting displacement field u^1 will be indicated below.)

We check that (3.10)–(3.16), (3.18)–(3.20), and (4.38) are satisfied. First, (4.38) clearly holds true by construction. To confirm the other properties, we assume for simplicity that the above coarsening scheme was applied only once for two sets $\hat{P}_i$ and $\hat{P}_j$ intersecting Ω_1 since the general case follows by induction. First, (3.13) and (3.15) are not affected by the modification, and therefore still hold. Since the function $\sum_j R^\varepsilon \hat{M}^\varepsilon_j \chi_{\hat{P}^\varepsilon_j}$ remains unchanged by construction, also (3.12) and (3.16) are still satisfied. To see (3.10) and (3.14), it suffices to recall that $P^{\varepsilon,1}_* = \hat{P}^\varepsilon_i \cup \hat{P}^\varepsilon_j$ which implies that $P^{\varepsilon,1}_* \to P^1_* = \hat{P}_i \cup \hat{P}_j$ in measure. We now show (3.11) for $\Omega' \subset\subset \Omega$. As $\mathcal{L}^d(\hat{P}_k \cap \Omega_1) > 0$ for $k = i, j$, for ε small enough, (3.11) and (3.14) (for $\hat{\mathcal{P}}^\varepsilon$) imply $\mathcal{L}^d(\Omega' \cap \hat{P}^\varepsilon_k) \geq \mathcal{L}^d(L_{\Omega'}(\hat{P}^\varepsilon_k) \setminus \hat{P}^\varepsilon_k)$ for $k = i, j$. This also yields $\mathcal{L}^d(\Omega' \cap P^{\varepsilon,1}_*) \geq \mathcal{L}^d(L_{\Omega'}(P^{\varepsilon,1}_*) \setminus P^{\varepsilon,1}_*)$ for ε small enough. Therefore, since $\mathcal{L}^d(L_{\Omega'}(P^{\varepsilon,1}_*) \setminus P^{\varepsilon,1}_*) \leq \sum_{k=i,j} \mathcal{L}^d(L_{\Omega'}(\hat{P}^\varepsilon_k) \setminus \hat{P}^\varepsilon_k)$, (3.11) holds, as well. We now finally introduce the limiting displacement field and check (3.19)–(3.20). We observe

$$u^{\varepsilon,1} - \hat{u}^\varepsilon = \varepsilon^{-1}(\hat{t}^\varepsilon_j - \hat{t}^\varepsilon_i)\chi_{\hat{P}^\varepsilon_j}$$

where $u^{\varepsilon,1}$ and $\hat{u}^\varepsilon$ are the corresponding displacement fields defined in (3.18) with respect to the quadruples $(R^\varepsilon, \mathcal{P}^{\varepsilon,1}, \mathcal{M}^{\varepsilon,1}, \mathcal{T}^{\varepsilon,1})$ and $(R^\varepsilon, \hat{\mathcal{P}}^\varepsilon, \hat{\mathcal{M}}^\varepsilon, \hat{\mathcal{T}}^\varepsilon)$, respectively. By (4.44) we obtain $\varepsilon^{-1}(\hat{t}^\varepsilon_j - \hat{t}^\varepsilon_i) \to t_0 \in \mathbb{R}^d$, possibly passing to a subsequence (not relabeled). This implies that $u^{\varepsilon,1}$ converges in measure to

$$u^1 := \hat{u} + t_0\chi_{\hat{P}_j} \in \mathscr{U}(\Omega) \tag{4.45}$$

and gives (3.19). Finally, (3.20) follows from $\nabla u^{\varepsilon,1} = \nabla \hat{u}^\varepsilon$ and $\nabla u^1 = \nabla \hat{u}$.

Now suppose that the quadruples $(R^\varepsilon, \mathcal{P}^{\varepsilon,n-1}, \mathcal{M}^{\varepsilon,n-1}, \mathcal{T}^{\varepsilon,n-1})$ and the limiting triple $(y, u^{n-1}, \mathcal{P}^{n-1})$ in step $n-1$ have been constructed such that (3.10)–(3.16), (3.18)–(3.20), and (4.38) hold, and (4.39) is satisfied up to step $n-1$. We define the objects in step n as follows: if (4.38) holds with respect to the set Ω_n, we simply set $(\mathcal{P}^{\varepsilon,n}, \mathcal{M}^{\varepsilon,n}, \mathcal{T}^{\varepsilon,n}) = (\mathcal{P}^{\varepsilon,n-1}, \mathcal{M}^{\varepsilon,n-1}, \mathcal{T}^{\varepsilon,n-1})$, and observe that all properties are automatically satisfied.

If (4.38) is violated, the strategy is to apply the coarsening scheme described above to modify the partitions and translations such that all properties, in particular (4.38)–(4.39), are fulfilled.

Coarsening scheme for general n. If two components P^{n-1}_i and P^{n-1}_j violate (4.38) (with respect to the set Ω_n), we combine them to one component $P^n_* := P^{n-1}_i \cup P^{n-1}_j$ and similarly we define $P^{\varepsilon,n}_* := P^{\varepsilon,n-1}_i \cup P^{\varepsilon,n-1}_j$ for all $\varepsilon > 0$. Moreover, we define the phase $M^{\varepsilon,n}_* = M^{\varepsilon,n-1}_i = M^{\varepsilon,n-1}_j$ for all $\varepsilon > 0$. Concerning the translation $t^{\varepsilon,n}_*$, we proceed as follows: we observe that at most one of the two sets P^{n-1}_i and P^{n-1}_j intersects Ω_{n-1}. Indeed, it is not possible that both sets intersect Ω_{n-1} as (4.38) holds by construction in step $n-1$, and we assumed that P^{n-1}_i and P^{n-1}_j violate (4.38) with respect to $\Omega_n \supset \Omega_{n-1}$.

Suppose that (at most) P_i^{n-1} intersects Ω_{n-1}. We define $t_*^{\varepsilon,n} := t_i^{\varepsilon,n-1}$. We repeat this procedure (at most a finite number of times, cf. Lemma 4.4) until all pairs of components satisfy (4.38).

Then, for the resulting quadruple, (4.38) is satisfied by construction. Exactly as before in the step $n = 1$, we can check that (3.10)-(3.16) and (3.18)-(3.20) hold. Finally, let us confirm (4.39): (i) follows from the fact that in the procedure we iteratively have combined two components. Similarly, (ii) is a consequence of the fact that only sets with the same phase are combined. Finally, (iii) and (iv) follow from the definition of the translations in the coarsening scheme and the fact that, if two components are combined, at least one did not intersect Ω_{n-1}.

We perform this coarsening scheme for each $n \in \mathbb{N}$. Note that in each step we pass to a further subsequence in ε (not relabeled). Then, (4.38)–(4.39) follow for each $n \in \mathbb{N}$ for a suitable diagonal sequence. □

Remark 4.7 (Local properties of jump sets)**.** For later purposes, we remark that each $K \subset\subset \Omega$ intersects only a finite number of $(d-1)$-dimensional hyperplanes orthogonal to e_d which intersect J_u. This can be seen as follows: the construction of the displacement fields in the previous proof shows that $J_{u^n} \subset \bigcup_j \hat{P}_j$ for all $n \in \mathbb{N}$. This follows from (4.45) and the fact that $J_{\hat{u}} \subset \bigcup_j \hat{P}_j$, see Proposition 4.6 for $\hat{u}$ and $\hat{P}_j$ in place of u and P_j, respectively. Therefore, also $J_u \subset \bigcup_j \hat{P}_j$ by (4.40). The desired property now follows from Lemma 4.4.

We close this section by mentioning that the definition and construction of the partition in the previous proof is inspired by [37, Section 5] where in a different context partitions with a property of type (3.17) are called *coarsest partitions*.

5. Analysis of admissible limiting configurations

This section is devoted to the proofs of Proposition 3.6, Proposition 3.7, and Proposition 3.8. We first show that limiting deformations and partitions are uniquely identified whereas limiting displacements may differ by global infinitesimal rotations and piecewise translations.

Proof of Proposition 3.6. Let $\{y^\varepsilon\}_\varepsilon$ be a sequence as in Theorem 3.3 and let $(y^1, u^1, \mathcal{P}^1)$, $(y^2, u^2, \mathcal{P}^2)$ be two admissible triples. We start with the proof of (i). First, $y^1 = y^2$ follows directly from (3.15). In what follows, we thus simply denote the deformation by y. Suppose by contradiction that the two partitions $\mathcal{P}^1 = \{P_j^1\}_j$ and $\mathcal{P}^2 = \{P_j^2\}_j$ are different. Up to reordering we may assume that $P_1^1 \cap P_1^2$ and $P_2^1 \cap P_1^2$ have positive $\mathcal{L}^d$-measure.

Let $(R^{\varepsilon,1}, \mathcal{P}^{\varepsilon,1}, \mathcal{M}^{\varepsilon,1}, \mathcal{T}^{\varepsilon,1})$ and $(R^{\varepsilon,2}, \mathcal{P}^{\varepsilon,2}, \mathcal{M}^{\varepsilon,2}, \mathcal{T}^{\varepsilon,2})$ be sequences of quadruples converging to the limiting triples $(y, u^1, \mathcal{P}^1)$ and $(y, u^2, \mathcal{P}^2)$, respectively, in the sense of (3.10)–(3.20). By (3.13) we have $\lim_{\varepsilon\to 0} R^{\varepsilon,1} = \lim_{\varepsilon\to 0} R^{\varepsilon,2} = R \in SO(d)$, where R is such that $y \in \mathcal{Y}_R(\Omega)$. By (3.14), (3.16), and the fact that $P_1^1 \cap P_1^2$ and $P_2^1 \cap P_1^2$ have positive $\mathcal{L}^d$-measure, we then obtain for all ε small enough

$$M_1^{\varepsilon,1} = M_2^{\varepsilon,1} = M_1^{\varepsilon,2}. \tag{5.1}$$

Since the rescaled displacement fields $u^{\varepsilon,1}$ and $u^{\varepsilon,2}$, defined in (3.18) with respect to the two different quadruples, converge in measure in Ω by (3.19), we observe that also

$$\frac{1}{\varepsilon}\Big(\sum\nolimits_j (R^{\varepsilon,1}M_j^{\varepsilon,1}\,x + t_j^{\varepsilon,1})\chi_{P_j^{\varepsilon,1}} - \sum\nolimits_j (R^{\varepsilon,2}M_j^{\varepsilon,2}\,x + t_j^{\varepsilon,2})\chi_{P_j^{\varepsilon,2}}\Big)$$

converges in measure in Ω. In view of (3.14), (5.1), and the fact that $P_1^1 \cap P_1^2$ and $P_2^1 \cap P_1^2$ have positive $\mathcal{L}^d$-measure, we obtain

$$|R^{\varepsilon,1} - R^{\varepsilon,2}| + |t_1^{\varepsilon,1} - t_1^{\varepsilon,2}| + |t_2^{\varepsilon,1} - t_1^{\varepsilon,2}| \le C\varepsilon \tag{5.2}$$

uniformly in ε for some $C > 0$. This is an elementary property for affine mappings. (See, e.g., [40, Lemma 3.4]; the function ψ therein can be chosen as in [41, Remark 2.2].) By the triangle inequality this particularly yields $|t_1^{\varepsilon,1} - t_2^{\varepsilon,1}| \le C\varepsilon$. This, however, contradicts (3.17) in view of (5.1). This concludes the proof of (i).

In the following, we denote the unique partition by $\mathcal{P} = \{P_j\}_j$ to simplify notation. We now show (ii). To this end, fix P_j with positive measure. In view of (3.14) and (3.16), we find $M_j^{\varepsilon,1} = M_j^{\varepsilon,2}$ for ε small enough. As $u^{\varepsilon,1} - u^{\varepsilon,2}$ converges in measure in Ω by (3.19), we thus obtain $|R^{\varepsilon,1} - R^{\varepsilon,2}| \le C\varepsilon$ and $|t_j^{\varepsilon,1} - t_j^{\varepsilon,2}| \le C_j\varepsilon$ for a constant $C > 0$ depending only on Ω, and some $C_j > 0$ depending on j but not on ε, see (5.2) for a similar argument. Using the formula (see [42, (3.20)])

$$\Big|\frac{(FR^T)^T + FR^T}{2} - \mathrm{Id}\Big| = \mathrm{dist}(F, SO(d)) + \mathrm{O}(|F-R|^2) \quad \text{for } F \in \mathbb{M}^{d\times d},\ R \in SO(d), \tag{5.3}$$

we obtain $S^\varepsilon \in \mathbb{M}^{d\times d}_{\rm skew}$ with $|S^\varepsilon| \le C$ such that

$$R^{\varepsilon,2} - R^{\varepsilon,1} = (R^{\varepsilon,2}\,(R^{\varepsilon,1})^T - \mathrm{Id})\,R^{\varepsilon,1} = (\varepsilon S^\varepsilon + \mathrm{O}(\varepsilon^2))\,R^{\varepsilon,1}.$$

Thus, possibly passing to a subsequence (not relabeled), we find $S \in \mathbb{M}^{d\times d}_{\rm skew}$ and for each $j \in \mathbb{N}$ with $\mathcal{L}^d(P_j) > 0$ a constant $t_j \in \mathbb{R}^d$ such that $\varepsilon^{-1}(t_j^{\varepsilon,2} - t_j^{\varepsilon,1}) \to t_j$ and $\varepsilon^{-1}(R^{\varepsilon,2} - R^{\varepsilon,1}) \to SR$, where $R \in SO(d)$ is such that $y \in \mathcal{Y}_R(\Omega)$. In particular, note that S is independent of the component P_j. By (3.16), (3.18)–(3.19), and the fact that $M_j^{\varepsilon,1} = M_j^{\varepsilon,2}$ for ε small enough we get for almost every $x \in P_j$

$$u^1(x) - u^2(x) = \lim_{\varepsilon\to 0}\big(u^{\varepsilon,1}(x) - u^{\varepsilon,2}(x)\big) = \lim_{\varepsilon\to0}\frac{1}{\varepsilon}\Big((R^{\varepsilon,2} - R^{\varepsilon,1})M_j^{\varepsilon,1}\,x + t_j^{\varepsilon,2} - t_j^{\varepsilon,1}\Big) = S\,\nabla y(x)\,x + t_j.$$

Recalling the definition in (3.22) we obtain (ii).

We finally show (iii). To this end, fix $\tilde{T} \in \mathscr{T}(y,\mathcal{P})$, say $\tilde{T}(x) = \sum_j \tilde{t}_j\chi_{P_j}(x) + \tilde{S}\,\nabla y(x)\,x$ for $x \in \Omega$. We have to show that $(y, u^1 + \tilde{T}, \mathcal{P})$ is an admissible triple. Recall that the quadruples $(R^{\varepsilon,1}, \mathcal{P}^{\varepsilon,1}, \mathcal{M}^{\varepsilon,1}, \mathcal{T}^{\varepsilon,1})$ converge to $(y, u^1, \mathcal{P})$ in the sense of (3.10)-(3.20).

We let $\bar{\mathcal{P}}^\varepsilon = \mathcal{P}^{\varepsilon,1}$, $\bar{\mathcal{M}}^\varepsilon = \mathcal{M}^{\varepsilon,1}$ and define $\bar{\mathcal{T}}^\varepsilon = \{\bar{t}_j^\varepsilon\}_j$ by $\bar{t}_j^\varepsilon = t_j^{\varepsilon,1} - \varepsilon\tilde{t}_j$ for all indices j. Moreover, we let $\bar{R}^\varepsilon \in SO(d)$ be such that $|\bar{R}^\varepsilon - (\mathrm{Id} - \varepsilon\tilde{S})R^{\varepsilon,1}| = \mathrm{dist}((\mathrm{Id} - \varepsilon\tilde{S})R^{\varepsilon,1}, SO(d))$, which by (5.3) (for $F = (\mathrm{Id} - \varepsilon\tilde{S})R^{\varepsilon,1}$ and $R = R^{\varepsilon,1}$) implies

$$\bar{R}^\varepsilon = (\mathrm{Id} - \varepsilon\tilde{S})\,R^{\varepsilon,1} + \mathrm{O}(\varepsilon^2). \tag{5.4}$$

We now see that $(\bar{R}^\varepsilon, \bar{\mathcal{P}}^\varepsilon, \bar{\mathcal{M}}^\varepsilon, \bar{\mathcal{T}}^\varepsilon)$ converges to $(y, u^1 + \tilde{T}, \mathcal{P})$ in the sense of (3.10)-(3.20). Indeed, as $|\bar{R}^\varepsilon - R^{\varepsilon,1}| \le C\varepsilon$, the properties (3.10)-(3.16) are satisfied. Property (3.17) follows from the corresponding property for $\mathcal{T}^{\varepsilon,1}$ and the definition of $\bar{\mathcal{T}}^\varepsilon$. Define $\bar{u}^\varepsilon$ as in (3.18). To confirm (3.19), we calculate for almost every $x \in P_j$ using (3.16) and (5.4)

$$\begin{aligned}\lim_{\varepsilon\to0}\big(\bar{u}^\varepsilon(x) - u^{\varepsilon,1}(x)\big) &= \lim_{\varepsilon\to0}\frac{1}{\varepsilon}\big((R^{\varepsilon,1} - \bar{R}^\varepsilon)\,M_j^{\varepsilon,1}\,x + t_j^{\varepsilon,1} - \bar{t}_j^\varepsilon\big) = \lim_{\varepsilon\to0}\frac{1}{\varepsilon}(R^{\varepsilon,1} - \bar{R}^\varepsilon)\,M_j^{\varepsilon,1}\,x + \tilde{t}_j \\ &= \tilde{S}\,\nabla y(x)\,x + \tilde{t}_j.\end{aligned}$$

Using (3.19) for u^1, we find $\bar{u}^\varepsilon \to u^1 + \tilde{T}$ in measure on the bounded set Ω. This yields (3.19). Finally, (3.20) follows from a similar computation. ☐

We proceed by characterizing the jump set of the gradients of limiting deformations.

Proof of Proposition 3.7. As $y \in \mathcal{Y}_R(\Omega)$, we recall that $\partial\{x \in \Omega\colon \nabla y(x) \in RA\}$ consists of subsets of hyperplanes orthogonal to e_d, see below Lemma 2.1. Now, assume by contradiction that $J_{\nabla y} \not\subset \bigcup_j \partial P_j \cap \Omega$. Then, by $\mathcal{P} \in \mathscr{P}(\Omega)$ and Lemma 4.4, we find a stripe $D := \{t_0 - \rho < x_d < t_0 + \rho\} \cap \Omega'$, with $\Omega' \subset\subset \Omega$, $t_0 \in \mathbb{R}$, and $\rho > 0$ small, such that $D \subset P_j$ for some $j \in \mathbb{N}$ and (up to reflection) $D \cap \{x_d > t_0\} \subset \{\nabla y = RA\}$, $D \cap \{x_d < t_0\} \subset \{\nabla y = RB\}$. In view of (3.13)–(3.14), however, this contradicts (3.16). To see that the inclusion might be strict, we refer to Case (2) in Example 3.4 with $l = 1/2$. ☐

We conclude this section with a characterization of the jump heights of limiting displacements.

Proof of Proposition 3.8. We first observe that it suffices to show that, if $\Omega' \subset\subset \Omega$, then the result holds for every $x \in \Omega'$. Consider a (subset of a) hyperplane $S := \{x_d = t_0\} \cap \Omega'$ with $\mathcal{H}^{d-1}\big(S \cap J_u\big) > 0$. We distinguish two situations:

$$\text{(a)}\quad \mathcal{H}^{d-1}\Big(S\cap\bigcup\nolimits_j \partial P_j\Big)=0 \quad\text{and}\quad \text{(b)}\quad \mathcal{H}^{d-1}\Big(S\cap\bigcup\nolimits_j \partial P_j\Big)>0.$$

To simplify notation, we set without restriction $t_0=0$. We start with Case (a). Choose another set Ω'' with $\Omega'\subset\subset\Omega''\subset\subset\Omega$. As $\mathcal{P}\in\mathscr{P}(\Omega)$, by Lemma 4.4 and Remark 4.7 we find $\rho>0$ small enough such that the cylindrical set $D:=\omega\times(-\rho,\rho)$, $\omega\subset\mathbb{R}^{d-1}$, satisfies $D\cap\{x_d=0\}=S$, is contained in a single component P_j, is contained in Ω'', and satisfies

$$J_u\cap D\subset S=\{x_d=0\}\cap\Omega'. \tag{5.5}$$

By Proposition 3.7, it is not restrictive to concentrate on the case $\nabla y=RA$ on $D\subset P_j$, which corresponds to proving properties (i) and (ii) of the statement. Analogously, property (iii) may be derived after some modifications in the notation.

Step 1: Case (a), property (ii). Let $(R^\varepsilon,\mathcal{P}^\varepsilon,\mathcal{M}^\varepsilon,\mathcal{T}^\varepsilon)$ be sequences of quadruples converging to $(y,u,\mathcal{P})$ in the sense of (3.10)–(3.20), and define u^ε as in (3.18). Assume also that $\mathcal{J}^\varepsilon$ is the (at most countable) set of indices for the partition $\mathcal{P}^\varepsilon$. We denote by $\mathcal{J}_1^\varepsilon$ the indices with $\mathcal{L}^d(\Omega''\cap P_j^\varepsilon)\le\mathcal{L}^d(L_{\Omega''}(P_j^\varepsilon)\setminus P_j^\varepsilon)$, and we let $\mathcal{J}_2^\varepsilon=\mathcal{J}^\varepsilon\setminus\mathcal{J}_1^\varepsilon$. By (3.11), (3.14), (3.19), (3.20), Fubini's theorem, and Fatou's lemma we get that for $\mathcal{H}^{d-1}$-a.e. $x'\in\omega$ there exists a sequence $\{\varepsilon_k\}_k\subset(0,+\infty)$ with $\varepsilon_k\to0$ such that for a.e. $0<\rho'<\rho$ we have

$$\begin{aligned}
&\text{(i)}\quad (x',-\rho'),(x',\rho')\in P_j^{\varepsilon_k}\ \text{for all } k \text{ large enough},\quad u^{\varepsilon_k}(x',\pm\rho')\to u(x',\pm\rho')\ \text{as } k\to\infty,\\
&\text{(ii)}\quad \sum_{j\in\mathcal{J}_1^\varepsilon}\mathcal{L}^1\big(P_j^{\varepsilon_k}\cap(\{x'\}\times(-\rho',\rho'))\big)+\sum_{j\in\mathcal{J}_2^\varepsilon}\mathcal{L}^1\Big(\big(L_{\Omega''}(P_j^{\varepsilon_k})\setminus P_j^{\varepsilon_k}\big)\cap\big(\{x'\}\times(-\rho',\rho')\big)\Big)\le\bar C(x')\,\varepsilon_k^p,\\
&\text{(iii)}\quad \int_{-\rho'}^{\rho'}|\nabla u^{\varepsilon_k}(x',t)|^2\,\mathrm{d}t\le\bar C(x'),
\end{aligned}\tag{5.6}$$

where $\bar C(x')>0$ depends on Ω'' and x', but is independent of ρ' and $\{\varepsilon_k\}_k$. We point out that in general the sequence $\{\varepsilon_k\}_k$ depends on x'. For later purposes, however, we note that, for a.e. pair of points $x_1',x_2'\in\omega$, we can choose a single sequence $\{\varepsilon_k\}_k$ such that (5.6) holds.

Fix $x'\in\omega$ and $0<\rho'<\rho$ such that (5.6) is satisfied. For notational simplicity, we drop the subscript k of the corresponding sequence $\{\varepsilon_k\}_k$ and we omit the dependence on x'. Define

$$\mathcal{B}^\varepsilon(x';\rho'):=\Big\{t\in(-\rho',\rho'):\ \sum\nolimits_j M_j^\varepsilon\chi_{P_j^\varepsilon}(x',t)=B\Big\}. \tag{5.7}$$

By the fundamental theorem of calculus, in view of the definition of u^ε in (3.18), we get

$$\begin{aligned}
y^\varepsilon(x',\rho')-y^\varepsilon(x',-\rho')&=\int_{-\rho'}^{\rho'}\partial_d y^\varepsilon(x',t)\,\mathrm{d}t\\
&=\varepsilon\int_{-\rho'}^{\rho'}\partial_d u^\varepsilon(x',t)\,\mathrm{d}t+\mathcal{L}^1(\mathcal{B}^\varepsilon(x';\rho'))\,R^\varepsilon B\,e_d+\big(2\rho'-\mathcal{L}^1(\mathcal{B}^\varepsilon(x';\rho'))\big)R^\varepsilon A\,e_d.
\end{aligned}$$

Thus, by (5.6)(iii) and Hölder's inequality we find

$$\varepsilon^{-1}\big|y^\varepsilon(x',\rho')-y^\varepsilon(x',-\rho')-2\rho'R^\varepsilon A\,e_d-\mathcal{L}^1(\mathcal{B}^\varepsilon(x';\rho'))\,R^\varepsilon(B-A)\,e_d\big|\le(2\bar C(x')\rho')^{1/2}. \tag{5.8}$$

Since $\nabla y=RA$ on $D\subset P_j$, we get $M_j^\varepsilon=A$ for ε sufficiently small by (3.16). Thus, by (3.18) and (5.6)(i), we also have

$$\varepsilon^{-1}\big(y^\varepsilon(x',\rho')-y^\varepsilon(x',-\rho')-2\rho'R^\varepsilon A\,e_d\big)=u^\varepsilon(x',\rho')-u^\varepsilon(x',-\rho')$$

for every ε sufficiently small. Recall the definition of κ in H3. By (3.13), (5.6)(i), and (5.8), up to passing to a further subsequence (depending on ρ'), we get that $\ell(x';\rho'):=\lim_{\varepsilon\to0}\varepsilon^{-1}\mathcal{L}^1(\mathcal{B}^\varepsilon(x';\rho'))\ge0$ exists, is finite, and satisfies

$$|u(x',\rho')-u(x',-\rho')-\kappa\,\ell(x';\rho')\,Re_d|\le(2\bar C(x')\rho')^{1/2}. \tag{5.9}$$

Here, we used that $\bar{C}(x')$ is independent of ε. On the other hand, the fundamental theorem of calculus for the limiting displacement together with (5.5) yields

$$|u(x',\rho') - u(x',-\rho') - [u](x',0)| \le \int_{-\rho'}^{\rho'} |\partial_d u(x',t)|\,\mathrm{d}t \le (2\bar{C}(x')\rho')^{1/2}, \tag{5.10}$$

where the last inequality follows by (5.6)(iii), Hölder's inequality, and a lower semicontinuity argument. By combining (5.9) and (5.10) we deduce

$$\big|[u](x',0) - \kappa\,\ell(x';\rho')\,Re_d\big| \le 2(2\bar{C}(x')\rho')^{1/2}. \tag{5.11}$$

Property (ii) in Case (a) now follows by recalling that $\ell(x';\rho') \ge 0$, by the fact that $\bar{C}(x')$ may depend on x' but is independent of ρ', and by considering a sequence $\rho' \to 0$ such that (5.6) holds. (We briefly note that property (iii) corresponds to $\nabla y = RB$ on $D \subset P_j$. This case can be treated along similar lines, by interchanging the roles of A and B.)

Step 2: Case (a), property (i). We now show property (i) by contradiction, where without restriction we treat the case $\nabla y = RA$ on $D \subset P_j$. If the statement were wrong, we would find $x'_1, x'_2 \in \omega$ and $0 < \rho' < \rho$ such that for each x'_i, $i = 1,2$, (5.6) holds (with x'_i in place of x', for a single sequence $\{\varepsilon_k\}_k$) and such that

$$\big|[u](x'_1,0) - [u](x'_2,0)\big| \ge 5(2\bar{C}\rho')^{1/2}, \tag{5.12}$$

where we set $\bar{C} = \max_{i=1,2}\bar{C}(x'_i)$. We again drop the index k of the sequence $\{\varepsilon_k\}_k$. Define $\mathcal{B}^\varepsilon(x'_i;\rho')$ as in (5.7) for $i = 1,2$. Repeating the reasoning in Step 1, see particularly (5.11), we find $|[u](x'_i,0) - \kappa\,\ell(x'_i;\rho')\,Re_d| \le 2(2\bar{C}\rho')^{1/2}$ for $i = 1,2$, where the limits $\ell(x'_i;\rho') := \lim_{\varepsilon\to 0}\varepsilon^{-1}\mathcal{L}^1(\mathcal{B}^\varepsilon(x'_i;\rho'))$ can again be assumed to exist after passage to a subsequence (not relabeled). By the triangle inequality and (5.12), we find $\kappa|\ell(x'_1;\rho') - \ell(x'_2;\rho')| \ge (2\bar{C}\rho')^{1/2}$. This implies

$$\inf_{\varepsilon>0}\varepsilon^{-1}\big|\mathcal{L}^1(\mathcal{B}^\varepsilon(x'_1;\rho')) - \mathcal{L}^1(\mathcal{B}^\varepsilon(x'_2;\rho'))\big| > 0.$$

In view of the definition (5.7), this contradicts (5.6)(ii) since $p > 1$. This concludes the proof of (i) and of Case (a).

Step 3: Case (b), property (i). To complete the proof of the proposition, it remains to show assertion (i) in Case (b). (Note that assertions (ii) and (iii) are trivial in this case.) In this situation, possibly passing to a smaller ρ, by Lemma 4.4 we get that the set $D = \omega \times (-\rho,\rho)$ considered in Case (a), see before (5.5), only intersects two components P_{j_1} and P_{j_2}, with $D \cap P_{j_1} = D \cap \{x_d < 0\}$ and $D \cap P_{j_2} = D \cap \{x_d > 0\}$. In a similar fashion to (5.6), in view of (3.11), (3.14), (3.19), and (3.20), Fatou's lemma yields that for $\mathcal{H}^{d-1}$-a.e. $x' \in \omega$ there exists an infinitesimal sequence $\{\varepsilon_k\}_k$ such that for a.e. $0 < \rho' < \rho$ there holds

$$(x',-\rho') \in P^{\varepsilon_k}_{j_1},\quad (x',\rho') \in P^{\varepsilon_k}_{j_2} \quad \text{for all } k \text{ large enough}, \qquad u^{\varepsilon_k}(x',\pm\rho') \to u(x',\pm\rho') \text{ as } k \to \infty, \tag{5.13}$$

and properties (ii) and (iii) of (5.6) are satisfied. Given $x' \in \omega$ and $0 < \rho' < \rho$, arguing exactly as in the proof of (5.8) in Case (a), we find (we again drop the index k and the dependence on x' in the sequel)

$$\varepsilon^{-1}\big|y^\varepsilon(x',\rho') - y^\varepsilon(x',-\rho') - 2\rho' R^\varepsilon A\,e_d - \mathcal{L}^1(\mathcal{B}^\varepsilon(x';\rho'))\,R^\varepsilon(B-A)\,e_d\big| \le (2\bar{C}(x')\rho')^{1/2},$$

where $\mathcal{B}^\varepsilon(x';\rho')$ is defined in (5.7). By (3.16), for ε sufficiently small, we may assume that $M^\varepsilon_j = M_j$ for $j = j_1, j_2$. Thus, in view of (3.18) and (5.13), we get

$$\varepsilon^{-1}\big(y^\varepsilon(x',\rho') - y^\varepsilon(x',-\rho') - \rho' R^\varepsilon\,(M_{j_1} + M_{j_2})\,e_d\big) - \varepsilon^{-1}(t^\varepsilon_{j_2} - t^\varepsilon_{j_1}) = u^\varepsilon(x',\rho') - u^\varepsilon(x',-\rho').$$

This along with the previous estimate entails

$$\big|u^\varepsilon(x',\rho') - u^\varepsilon(x',-\rho') - v_\varepsilon(x';\rho')\big| \le (2\bar{C}(x')\rho')^{1/2}, \tag{5.14}$$

where for brevity we have set

$$v_\varepsilon(x';\rho') := \varepsilon^{-1}\mathcal{L}^1(\mathcal{B}^\varepsilon(x';\rho'))\,R^\varepsilon(B-A)\,e_d + \varepsilon^{-1}\rho' R^\varepsilon\,(2A - (M_{j_1} + M_{j_2}))\,e_d - \varepsilon^{-1}(t^\varepsilon_{j_2} - t^\varepsilon_{j_1}). \tag{5.15}$$

Then (5.13) and (5.14) show that there exists a constant vector $v(x';\rho') \in \mathbb{R}^d$ depending on ρ' and x' such that, up to the extraction of a subsequence (not relabeled), there holds $v_\varepsilon(x';\rho') \to v(x';\rho')$. By using (5.5) and (5.6)(iii), we get that (5.10) also holds in the present situation. Then, similar to the proof of (5.11) in Case (a), we obtain by (5.13) and (5.14)

$$|[u](x',0) - v(x';\rho')| \leq 2(2\bar{C}(x')\rho')^{1/2}. \tag{5.16}$$

The proof of property (i) is now obtained by contradiction by following the lines of the proof in Case (a): suppose that there were $x_1', x_2' \in \omega$ and $0 < \rho' < \rho$ such that for each x_i', $i = 1, 2$, (5.13) and (5.6) (ii),(iii) hold (with x_i' in place of x'), and the two points are such that $\big|[u](x_1',0) - [u](x_2',0)\big| \geq 5(2\bar{C}\rho')^{1/2}$, where as before $\bar{C} := \max_{i=1,2} \bar{C}(x_i')$. By (5.16) this yields $\big|v(x_1';\rho') - v(x_2';\rho')\big| \geq (2\bar{C}\rho')^{1/2}$. In view of (5.15), this however contradicts (5.6)(ii). This concludes the proof. $\square$

6. Derivation of the effective linearized energy

This section is devoted to the proof of our Γ-convergence result for the sequence of energies $\mathcal{E}_\varepsilon = E_{\varepsilon,\bar{\eta}_{\varepsilon,d}}$ introduced in (3.1) (with $\bar{\eta}_{\varepsilon,d}$ from (3.3)) and the limiting energy $\mathcal{E}_0^{\mathcal{A}}$ defined in (3.23). In Subsections 6.1 and 6.2 we prove Theorems 3.13 and 3.14, respectively. A key ingredient for the liminf inequality is a characterization of the double-profile energy $K^M_{\rm dp}$ (see (3.26)), in particular its connection to the optimal-profile counterpart K (see (3.4)). This result is subject of Proposition 6.2 and is proven in Subsection 6.3. The proof of the limsup inequality is performed under the additional assumption that

$$K^M_{\rm dp} = 2K \quad \text{ for } M \in \{A, B\}, \tag{6.1}$$

and essentially relies on Propositions 6.4 and 6.5. The latter provide constructions of local recovery sequences around interfaces performing a single and a double phase transition, respectively, and coinciding with isometries far from the interfaces. Their proofs are contained in Subsection 6.4. Finally, in Subsection 6.5 we show that, under the additional assumption in (3.27), condition (6.1) can be verified. This hinges on the property that in this case optimal profiles for single phase transitions are one dimensional, see Lemma 6.16.

6.1. **The liminf inequality.** In this subsection we show that the functional $\mathcal{E}_0^{\mathcal{A}}$ is a lower bound for the asymptotic behavior of the energy functionals $\mathcal{E}_\varepsilon$. As a preparation, we introduce the notion of optimal-profile and double-profile energy functions, and we state their main properties.

Consider $\omega \subset \mathbb{R}^{d-1}$ open and bounded, and let $h > 0$. For brevity, we use the following notation for cylindrical sets

$$D_{\omega,h} := \omega \times (-h, h). \tag{6.2}$$

We define the *optimal-profile energy function*

$$\mathcal{F}(\omega; h) = \inf\Big\{ \liminf_{\varepsilon\to 0} \mathcal{E}_\varepsilon(y^\varepsilon, D_{\omega,h}) : \ \lim_{\varepsilon \to 0} \|y^\varepsilon - y_0^+\|_{L^1(D_{\omega,h})} = 0 \Big\} \tag{6.3}$$

for every $\omega \subset \mathbb{R}^{d-1}$ and $h > 0$, where y_0^+ was defined below (2.5). As mentioned there, due to the invariance of the energy functionals $\mathcal{E}_\varepsilon$ under the operation $Ty(x) = -y(-x)$, the optimal-profile energy is independent of the direction in which the transition between the two phases A and B occurs, i.e., in (6.3) we can replace y_0^+ by the continuous function $y_0^- \in H^1_{\rm loc}(\mathbb{R}^d;\mathbb{R}^d)$ with $y_0^-(0) = 0$ and $\nabla y_0^- = B\chi_{\{x_d>0\}} + A\chi_{\{x_d<0\}}$. We refer to [26, Lemma 3.2] for details. We start with the property that the optimal-profile energy is independent of h and depends on ω only in terms of $\mathcal{H}^{d-1}(\omega)$. The following characterization has been proved in [32, Proposition 4.6].

Proposition 6.1 (Optimal-profile energy function)**.** *For all $h > 0$ and all open, bounded sets $\omega \subset \mathbb{R}^{d-1}$ with $\mathcal{H}^{d-1}(\partial\omega) = 0$ there holds $\mathcal{F}(\omega; h) = K\,\mathcal{H}^{d-1}(\omega)$, where K is the constant from (3.4).*

In a similar fashion, we investigate properties of the double-profile energy given in (3.26). Recall $\mathcal{W}_d$ in (3.24). We define the set of functions jumping on the interface by

$$\mathcal{U}_{\rm dp}(D_{\omega,h}) := \big\{u \in SBV^2_{\rm loc}(D_{\omega,h};\mathbb{R}^d) \colon \ \mathcal{H}^{d-1}(J_u) > 0, \ J_u \subset \omega \times \{0\}\big\}. \tag{6.4}$$

Then, for $M \in \{A, B\}$, we define the *double-profile energy function*

$$\mathcal{F}^M_{\rm dp}(\omega; h) = \inf_{u \in \mathcal{U}_{\rm dp}(D_{\omega,h})} \inf_{\{w_\varepsilon\}_\varepsilon \in \mathcal{W}_d} \inf \Big\{ \liminf_{\varepsilon \to 0} \mathcal{E}_\varepsilon(y^\varepsilon, D_{\omega,h}) : \ \frac{y^\varepsilon - Mx}{w_\varepsilon} \to u \text{ in measure in } D_{\omega,h} \text{ as } \varepsilon \to 0 \Big\}, \tag{6.5}$$

for every $\omega \subset \mathbb{R}^{d-1}$ and $h > 0$. The double-profile energy can be characterized as follows.

Proposition 6.2 (Double-profile energy function)**.** *For all $h > 0$, all open, bounded sets $\omega \subset \mathbb{R}^{d-1}$ with $\mathcal{H}^{d-1}(\partial\omega) = 0$, and for $M \in \{A, B\}$ there holds*

$$K^M_{\rm dp}\, \mathcal{H}^{d-1}(\omega) \geq \mathcal{F}^M_{\rm dp}(\omega, h) \geq 2K\, \mathcal{H}^{d-1}(\omega), \tag{6.6}$$

where K and $K^M_{\rm dp}$ are defined in (3.4) and (3.26), respectively.

Note that the result in particular implies Proposition 3.15. Moreover, in the case $2K = K^M_{\rm dp}$, equality holds in (6.6). (We refer to Subsection 6.5 for a setting in which this condition is fulfilled). We defer the proof of Proposition 6.2 to Subsection 6.3 below. At this stage, we only mention that it is achieved in two steps: we first show that $\mathcal{F}^M_{\rm dp}(\omega, h)$ is independent of h and depends on ω only in terms of $\mathcal{H}^{d-1}(\omega)$, see Proposition 6.6 below. Then, in a second step we address the connection between $\mathcal{F}^M_{\rm dp}(Q', 1)$, $K^M_{\rm dp}$, and $2K$, see Proposition 6.7. We now proceed with the proof of the liminf inequality.

Proof of Theorem 3.13. Let $(y, u, \mathcal{P}) \in \mathcal{A}$, see Definition 3.9, and let $y^\varepsilon \to (y, u, \mathcal{P})$ in the sense of Definition 3.5, i.e., there are sequences $\{R^\varepsilon\}_\varepsilon$, $\{\mathcal{P}^\varepsilon\}_\varepsilon$, $\{\mathcal{M}^\varepsilon\}_\varepsilon$, and $\{\mathcal{T}^\varepsilon\}_\varepsilon$ such that (3.10)–(3.20) hold. Suppose that $y \in \mathcal{Y}_R(\Omega)$ for $R \in SO(d)$, see (2.4). To simplify the exposition, we suppose that $⨍_\Omega y^\varepsilon \,{\rm d}x = 0$, i.e., by (3.15) we get

$$y^\varepsilon \to y \quad \text{strongly in } H^1(\Omega; \mathbb{R}^d). \tag{6.7}$$

By Proposition 3.6(iii), Proposition 3.8(i), and Remark 3.12, possibly passing to another displacement field being admissible for the sequence $\{y^\varepsilon\}_\varepsilon$, we may without restriction assume that

$$\bigcup\nolimits_j \partial P_j \cap \Omega \subset J_u. \tag{6.8}$$

As Ω has Lipschitz boundary, by the definition of the set $\mathcal{A}$ in Definition 3.9 and by Proposition 3.8(i) there exist sequences $\{\omega^y_i\}_i$, $\{\omega^u_i\}_i$ of Lipschitz domains in $\mathbb{R}^{d-1}$ and sequences $\{\alpha^y_i\}_i$, $\{\alpha^u_i\}_i$ of real numbers such that

$$J_{\nabla y} = \bigcup\nolimits_{i \in \mathbb{N}} \omega^y_i \times \{\alpha^y_i\} \quad \text{and} \quad J_u \setminus J_{\nabla y} = \bigcup\nolimits_{i \in \mathbb{N}} \omega^u_i \times \{\alpha^u_i\}. \tag{6.9}$$

Let $\delta > 0$. We can find $I_y, I_u \in \mathbb{N}$ such that

$$\mathcal{H}^{d-1}(J_{\nabla y}) - \delta \leq \sum\nolimits_{i=1}^{I_y} \mathcal{H}^{d-1}\big(\omega^y_i \times \{\alpha^y_i\}\big), \qquad \mathcal{H}^{d-1}(J_u \setminus J_{\nabla y}) - \delta \leq \sum\nolimits_{i=1}^{I_u} \mathcal{H}^{d-1}\big(\omega^u_i \times \{\alpha^u_i\}\big). \tag{6.10}$$

Moreover, we choose $h > 0$ such that the cylindrical sets (see (6.2)) $\alpha^y_i e_d + D_{\omega^y_i, h}$, $i = 1, \ldots, I_y$, and $\alpha^u_i e_d + D_{\omega^u_i, h}$, $i = 1, \ldots, I_u$, are pairwise disjoint, and do not intersect the interfaces $\{\omega^y_i \times \{\alpha^y_i\}\}_{i > I_y}$ and $\{\omega^u_i \times \{\alpha^u_i\}\}_{i > I_u}$. The latter is possible due to $J_{\nabla y} \subset \bigcup_j \partial P_j \cap \Omega$ (see definition of $\mathcal{A}$), Lemma 4.4, and Remark 4.7 which imply that the interfaces $\{\omega^y_i \times \{\alpha^y_i\}\}_{i > I_y}$ and $\{\omega^u_i \times \{\alpha^u_i\}\}_{i > I_u}$ can only accumulate at $\partial\Omega$, see [26, Proof of Proposition 3.1] for details, and the lower part of Figure 1 for an illustration.

By possibly passing to a smaller $h > 0$ (not relabeled), we can choose $\tilde{\omega}^y_i \subset\subset \omega^y_i$ and $\tilde{\omega}^u_i \subset\subset \omega^u_i$ with Lipschitz boundary such that

$$\begin{aligned} \mathcal{H}^{d-1}(\omega^y_i) &\leq \mathcal{H}^{d-1}(\tilde{\omega}^y_i) + \delta / I_y \quad \text{for } i = 1, \ldots, I_y, \\ \mathcal{H}^{d-1}(\omega^u_i) &\leq \mathcal{H}^{d-1}(\tilde{\omega}^u_i) + \delta / I_u \quad \text{for } i = 1, \ldots, I_u, \end{aligned} \tag{6.11}$$

and such that

$$D^y_i := \alpha^y_i e_d + D_{\tilde{\omega}^y_i, h} \subset\subset \Omega \text{ for } i = 1, \ldots, I_y, \qquad D^u_i := \alpha^u_i e_d + D_{\tilde{\omega}^u_i, h} \subset\subset \Omega \text{ for } i = 1, \ldots, I_u,$$

see Figure 5 below.

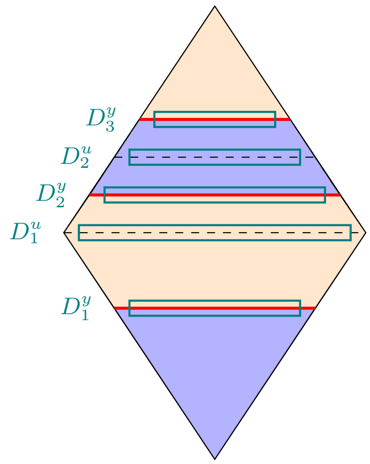

FIGURE 5. A visualization of the different interfaces and sets under (6.8). The phase regions associated to A and B are colored in blue and orange, respectively. The cylindrical sets $\{D_i^y\}_{i=1,\dots,\mathcal{I}_y}$ and $\{D_i^u\}_{i=1,\dots,\mathcal{I}_u}$ are drawn in green. The corresponding interfaces in $J_{\nabla y}$ and J_u are highlighted with thick red and dashed black lines, respectively.

Moreover, it is also not restrictive to assume that

$$\sum\nolimits_{i=1}^{I_y} \mathcal{L}^d(D_i^y) + \sum\nolimits_{i=1}^{I_u} \mathcal{L}^d(D_i^u) \leq \delta. \tag{6.12}$$

We define the set

$$\Omega_\delta := \big\{x \in \Omega\colon\ \mathrm{dist}(x, \partial\Omega) > \delta\big\} \setminus \Big(\bigcup\nolimits_{i=1}^{I_y} D_i^y \cup \bigcup\nolimits_{i=1}^{I_u} D_i^u\Big). \tag{6.13}$$

The main steps of the proof will consist in estimating the surface energies by

$$\begin{aligned} &\text{(i)} \quad \liminf_{\varepsilon\to 0} \mathcal{E}_\varepsilon\Big(y^\varepsilon, \bigcup\nolimits_{i=1}^{I_y} D_i^y\Big) \geq K\big(\mathcal{H}^{d-1}(J_{\nabla y}) - 2\delta\big), \\ &\text{(ii)} \quad \liminf_{\varepsilon\to 0} \mathcal{E}_\varepsilon\Big(y^\varepsilon, \bigcup\nolimits_{i=1}^{I_u} D_i^u\Big) \geq 2K\big(\mathcal{H}^{d-1}(J_u \setminus J_{\nabla y}) - 2\delta\big), \end{aligned} \tag{6.14}$$

and the elastic energy by

$$\liminf_{\varepsilon\to 0} \mathcal{E}_\varepsilon(y^\varepsilon, \Omega_\delta) \geq \int_{\Omega_\delta} \mathcal{Q}_{\mathrm{lin}}(\nabla y, \nabla u)\,\mathrm{d}x, \tag{6.15}$$

where the quadratic form $\mathcal{Q}_{\mathrm{lin}}$ is defined in (2.11). Once these estimates have been settled, in view of (3.23), we then indeed obtain $\liminf_{\varepsilon\to 0} \mathcal{E}_\varepsilon(y^\varepsilon, \Omega) \geq \mathcal{E}_0^{\mathcal{A}}(y, u, \mathcal{P})$ by letting $\delta \to 0$, by taking (6.8) as well as (6.12)–(6.13) into account, and by using monotone convergence. Let us now prove (6.14) and (6.15).

Step 1: Proof of (6.14)(i). By (6.7), $y \in \mathcal{Y}_R(\Omega)$, (6.9), and the fact that the sets $\{D_i^y\}_i$ are pairwise disjoint and contain only one interface, we get for each $i = 1, \dots, I_y$ that

$$R^{-1}y^\varepsilon(\cdot + \alpha_i^y e_d) \to y_0^+ \qquad \text{or} \qquad R^{-1}y^\varepsilon(\cdot + \alpha_i^y e_d) \to y_0^- \qquad \text{in } L^1(D_{\tilde{\omega}_i^y,h}; \mathbb{R}^d).$$

Therefore, by H2., (6.3), and the comment thereafter we obtain

$$\liminf_{\varepsilon\to 0} \mathcal{E}_\varepsilon\Big(y^\varepsilon, \bigcup\nolimits_{i=1}^{I_y} D_i^y\Big) \geq \sum\nolimits_{i=1}^{I_y} \liminf_{\varepsilon\to 0} \mathcal{E}_\varepsilon\Big(R^{-1}y^\varepsilon(\cdot + \alpha_i^y e_d), D_{\tilde{\omega}_i^y,h}\Big) \geq \sum\nolimits_{i=1}^{I_y} \mathcal{F}(\tilde{\omega}_i^y; h).$$

Then, by Proposition 6.1 and (6.10)–(6.11) we get

$$\liminf_{\varepsilon\to 0} \mathcal{E}_\varepsilon\Big(y^\varepsilon, \bigcup\nolimits_{i=1}^{I_y} D_i^y\Big) \geq K \sum\nolimits_{i=1}^{I_y} \mathcal{H}^{d-1}(\tilde{\omega}_i^y) \geq K\big(\mathcal{H}^{d-1}(J_{\nabla y}) - 2\delta\big).$$

This shows (6.14)(i).

Step 2: Proof of (6.14)(ii). By (6.9) and the fact that the cylindrical sets are chosen to be pairwise disjoint and to contain only one interface we know that ∇y is constant on each D_i^u, $i = 1 \dots, I_u$. We choose $M_i \in \{A, B\}$ such that $\nabla y = RM_i$ on D_i^u. We will distinguish two cases, indicated by the index sets

$$\mathcal{I}_1 := \Big\{ i = 1 \dots I_u \colon\, (\omega_i^u \times \{\alpha_i^u\}) \cap \bigcup\nolimits_j \partial P_j \cap \Omega = \emptyset \Big\}, \qquad \mathcal{I}_2 := \{1, \dots, I_u\} \setminus \mathcal{I}_1. \tag{6.16}$$

Step 2(a): $i \in \mathcal{I}_1$. In view of (6.9), (6.16), and the fact that the cylindrical sets are pairwise disjoint and contain only one interface, we get $D_i^u \subset P_k$ for some index k. Then by (3.14), (3.16), (3.18), and (3.19) we get as $\varepsilon \to 0$

$$\varepsilon^{-1}\big(y^\varepsilon - R^\varepsilon M_i\, x - t_k^\varepsilon\big) \to u \qquad \text{in measure in } D_i^u. \tag{6.17}$$

As the cylindrical sets are pairwise disjoint and contain only one interface, we find $u(\cdot + \alpha_i^u e_d) \in \mathcal{U}_{\rm dp}(D_{\tilde\omega_i^u,h})$ (recall (6.4)). We define the function

$$\bar y^\varepsilon(x) := (R^\varepsilon)^T y^\varepsilon(x + \alpha_i^u e_d) - (R^\varepsilon)^T t_k^\varepsilon - M_i \alpha_i^u e_d$$

for $x \in D_{\tilde\omega_i^u,h}$, and we note by (3.13) and (6.17) that $\varepsilon^{-1}(\bar y^\varepsilon - M_i x) \to \bar u$ in measure in $D_{\tilde\omega_i^u,h}$, where $\bar u := R^T u(\cdot + \alpha_i^u e_d) \in \mathcal{U}_{\rm dp}(D_{\tilde\omega_i^u,h})$. Then, the sequences $\{\bar y^\varepsilon\}_\varepsilon$ and $\{w_\varepsilon\}_\varepsilon \in \mathcal{W}_d$ defined by $w_\varepsilon := \varepsilon$ for all ε are admissible in (6.5). Thus, by the translational and rotational invariance of the energy we get

$$\liminf_{\varepsilon \to 0} \mathcal{E}_\varepsilon\big(y^\varepsilon, D_i^u\big) = \liminf_{\varepsilon \to 0} \mathcal{E}_\varepsilon\big(\bar y^\varepsilon, D_{\tilde\omega_i^u,h}\big) \geq \mathcal{F}_{\rm dp}^{M_i}(\tilde\omega_i^u; h). \tag{6.18}$$

Step 2(b): $i \in \mathcal{I}_2$. In this case, by (6.9) and the fact that the cylindrical sets are pairwise disjoint and contain only one interface, D_i^u intersects two components P_k and P_l, namely $\tilde\omega_i^u \times (\alpha_i^u - h, \alpha_i^u) \subset P_k$ and $\tilde\omega_i^u \times (\alpha_i^u, \alpha_i^u + h) \subset P_l$. As before, we have $\nabla y = RM_i$ on D_i^u. Let $w_\varepsilon := |t_k^\varepsilon - t_l^\varepsilon|$, where $t_k^\varepsilon, t_l^\varepsilon$ are the elements from the translations $\mathcal{T}^\varepsilon$ corresponding to the sets P_k^ε and P_l^ε. By (3.14), (3.16), (3.18), and (3.19) we get as $\varepsilon \to 0$

$$\varepsilon^{-1}\big(y^\varepsilon - R^\varepsilon M_i\, x - t_j^\varepsilon\big) \to u \qquad \text{in measure in } D_i^u \cap P_j \text{ for } j \in \{k, l\}. \tag{6.19}$$

By (3.17) we find $w_\varepsilon/\varepsilon \to \infty$. Moreover, for a.e. $x_k \in D_i^u \cap P_k$ and a.e. $x_l \in D_i^u \cap P_l$, by multiplying (6.19) with ε and using (6.7) we get $\limsup_{\varepsilon \to 0} |t_k^\varepsilon - t_l^\varepsilon| \leq |y(x_k) - y(x_l)| + |M_i||x_k - x_l|$. This implies that $\lim_{\varepsilon \to 0} w_\varepsilon = \lim_{\varepsilon \to 0} |t_k^\varepsilon - t_l^\varepsilon| = 0$ as y is continuous. Thus, $\{w_\varepsilon\}_\varepsilon \in \mathcal{W}_d$, see (3.24). By possibly passing to a subsequence (not relabeled), we may suppose that $(t_l^\varepsilon - t_k^\varepsilon)/w_\varepsilon \to t_0 \in \mathbb{R}^d$. We check that

$$\frac{y^\varepsilon - (R^\varepsilon M_i\, x + t_k^\varepsilon)}{w_\varepsilon} \to t_0 \chi_{\{x_d \geq \alpha_i^u\}} \qquad \text{in measure in } D_i^u. \tag{6.20}$$

In fact, by (6.19) and $\varepsilon/w_\varepsilon \to 0$, we first get

$$w_\varepsilon^{-1}\big(y^\varepsilon - R^\varepsilon M_i\, x - t_k^\varepsilon\big) = (\varepsilon/w_\varepsilon)\, \varepsilon^{-1}\big(y^\varepsilon - R^\varepsilon M_i\, x - t_k^\varepsilon\big) \to 0 \qquad \text{in measure in } D_i^u \cap P_k,$$

and by again using (6.19), $\varepsilon/w_\varepsilon \to 0$, as well as $(t_l^\varepsilon - t_k^\varepsilon)/w_\varepsilon \to t_0$ we find

$$w_\varepsilon^{-1}\big(y^\varepsilon - R^\varepsilon M_i\, x - t_k^\varepsilon\big) = (\varepsilon/w_\varepsilon)\, \varepsilon^{-1}\big(y^\varepsilon - R^\varepsilon M_i\, x - t_l^\varepsilon\big) + w_\varepsilon^{-1}\big(t_l^\varepsilon - t_k^\varepsilon\big) \to t_0$$

in measure in $D_i^u \cap P_l$. Now, by (6.20) and by arguing along the lines of (6.17)–(6.18) we can define a sequence $\{\bar y^\varepsilon\}_\varepsilon$ via rotation and shifting such that $\{\bar y^\varepsilon\}_\varepsilon$ and $\{w_\varepsilon\}_\varepsilon \in \mathcal{W}_d$ are admissible in (6.5). Then, we deduce

$$\liminf_{\varepsilon \to 0} \mathcal{E}_\varepsilon\big(y^\varepsilon, D_i^u\big) \geq \mathcal{F}_{\rm dp}^{M_i}(\tilde\omega_i^u; h). \tag{6.21}$$

We now conclude the proof of (6.14)(ii) as follows: combining (6.18), (6.21), and Proposition 6.2 we get

$$\liminf_{\varepsilon \to 0} \mathcal{E}_\varepsilon\Big(y^\varepsilon, \bigcup\nolimits_{i=1}^{I_u} D_i^u\Big) \geq \sum\nolimits_{i=1}^{I_u} \mathcal{F}_{\rm dp}^{M_i}(\tilde\omega_i^u; h) \geq 2K \sum\nolimits_{i=1}^{I_u} \mathcal{H}^{d-1}(\tilde\omega_i^u).$$

Then, (6.14)(ii) follows from (6.10)–(6.11).

Step 3: Proof of (6.15). We start by recalling the definition of u^ε in (3.18) and by noting that (3.20) implies

$$\int_{\Omega_\delta} |\nabla u^\varepsilon|^2 \,\mathrm{d}x \leq C_\delta \qquad \text{for all } \varepsilon > 0, \tag{6.22}$$

where $C_\delta > 0$ depends on the set Ω_δ defined in (6.13), and thus on δ. We now define two small exceptional sets: first, we let $\alpha \in (0,1)$, and we define the set of large linearized strains by

$$\Omega^\varepsilon_{\rm strain} := \{x \in \Omega_\delta \colon |\nabla u^\varepsilon(x)| \geq \varepsilon^{-\alpha}\}. \tag{6.23}$$

By Chebyshev's inequality and (6.22) we estimate

$$\mathcal{L}^d\big(\Omega^\varepsilon_{\rm strain}\big) \leq \varepsilon^{2\alpha}\int_{\Omega_\delta} |\nabla u^\varepsilon|^2\,{\rm d}x \leq C_\delta \varepsilon^{2\alpha}. \tag{6.24}$$

Moreover, by (3.16) and by the continuous embedding of $BV(\Omega;\mathbb{M}^{d\times d})$ into $L^1(\Omega;\mathbb{M}^{d\times d})$ we find a sequence $\{\delta_\varepsilon\}_\varepsilon \subset (0,+\infty)$ such that $\delta_\varepsilon \to 0$ and

$$\lim_{\varepsilon\to 0}\frac{1}{\delta_\varepsilon}\int_\Omega \big|\sum\nolimits_j (R^\varepsilon M^\varepsilon_j)\chi_{P^\varepsilon_j} - \nabla y\big|\,{\rm d}x = 0. \tag{6.25}$$

Then, we define the set

$$\Omega^\varepsilon_{\rm phase} := \Big\{x \in \Omega_\delta \colon \big|\sum\nolimits_j (R^\varepsilon M^\varepsilon_j)\chi_{P^\varepsilon_j}(x) - \nabla y(x)\big| \geq \delta_\varepsilon\Big\} \tag{6.26}$$

of points where the phases along the sequence differ by at least δ_ε from the phases in the limit. Clearly, (6.25) entails

$$\lim_{\varepsilon\to 0}\mathcal{L}^d(\Omega^\varepsilon_{\rm phase}) \leq \lim_{\varepsilon\to 0}\frac{1}{\delta_\varepsilon}\int_\Omega \big|\sum\nolimits_j (R^\varepsilon M^\varepsilon_j)\chi_{P^\varepsilon_j} - \nabla y\big|\,{\rm d}x = 0. \tag{6.27}$$

By combining (6.24) and (6.27) we find

$$\lim_{\varepsilon\to 0}\mathcal{L}^d(\Omega_\delta \setminus \Omega^\varepsilon_{\rm good}) = 0, \qquad \text{where } \ \Omega^\varepsilon_{\rm good} := \Omega_\delta \setminus \big(\Omega^\varepsilon_{\rm strain} \cup \Omega^\varepsilon_{\rm phase}\big). \tag{6.28}$$

By (3.1) and the definition in (3.18) we get

$$\mathcal{E}_\varepsilon(y^\varepsilon,\Omega_\delta) \geq \frac{1}{\varepsilon^2}\int_{\Omega_\delta} W(\nabla y^\varepsilon)\,{\rm d}x \geq \frac{1}{\varepsilon^2}\sum\nolimits_j \int_{\Omega^\varepsilon_{\rm good}\cap P^\varepsilon_j} W\big(R^\varepsilon M^\varepsilon_j + \varepsilon\nabla u^\varepsilon(x)\big)\,{\rm d}x. \tag{6.29}$$

By assumptions H2., H3., and H5. we can perform a Taylor expansion and write

$$W(RM+F) = \frac{1}{2}D^2W(RM)F : F + \omega_W(F)$$

for all $F \in \mathbb{M}^{d\times d}$ with $|F| < \delta_W$, where $\omega_W \colon \mathbb{M}^{d\times d} \to \mathbb{R}$ satisfies

$$\lim_{\rho\to 0^+}\eta_W(\rho) = 0, \qquad \text{where } \ \eta_W(\rho) := \sup\Big\{\frac{\omega_W(F)}{|F|^2} : |F| \leq \rho\Big\}. \tag{6.30}$$

This expansion along with (6.23), (6.29), and the fact that $\Omega^\varepsilon_{\rm good} \cap \Omega^\varepsilon_{\rm strain} = \emptyset$ yields for ε small enough

$$\begin{aligned}
\mathcal{E}_\varepsilon(y^\varepsilon,\Omega_\delta) &\geq \sum\nolimits_j \int_{\Omega^\varepsilon_{\rm good}\cap P^\varepsilon_j}\Big(\frac{1}{2}D^2W(R^\varepsilon M^\varepsilon_j)\nabla u^\varepsilon : \nabla u^\varepsilon + \frac{1}{\varepsilon^2}\omega_W(\varepsilon\nabla u^\varepsilon)\Big)\,{\rm d}x \\
&= \sum\nolimits_j \int_{\Omega^\varepsilon_{\rm good}\cap P^\varepsilon_j}\Big(\frac{1}{2}D^2W(R^\varepsilon M^\varepsilon_j)\nabla u^\varepsilon : \nabla u^\varepsilon + |\nabla u^\varepsilon|^2\frac{\omega_W(\varepsilon\nabla u^\varepsilon)}{|\varepsilon\nabla u^\varepsilon|^2}\Big)\,{\rm d}x \\
&\geq \sum\nolimits_j \frac{1}{2}\int_{\Omega^\varepsilon_{\rm good}\cap P^\varepsilon_j} D^2W(R^\varepsilon M^\varepsilon_j)\nabla u^\varepsilon : \nabla u^\varepsilon\,{\rm d}x - \eta_W\big(\varepsilon^{1-\alpha}\big)\|\nabla u^\varepsilon\|^2_{L^2(\Omega^\varepsilon_{\rm good})}.
\end{aligned}$$

Then, by (6.22) and (6.30) we get

$$\liminf_{\varepsilon\to 0}\mathcal{E}_\varepsilon(y^\varepsilon,\Omega_\delta) \geq \liminf_{\varepsilon\to 0}\sum\nolimits_j \frac{1}{2}\int_{\Omega^\varepsilon_{\rm good}\cap P^\varepsilon_j} D^2W(R^\varepsilon M^\varepsilon_j)\nabla u^\varepsilon : \nabla u^\varepsilon\,{\rm d}x. \tag{6.31}$$

By H5., (6.26), and the fact that $\Omega^\varepsilon_{\rm good} \cap \Omega^\varepsilon_{\rm phase} = \emptyset$ we find

$$\Big|\sum\nolimits_j \int_{\Omega^\varepsilon_{\rm good}\cap P^\varepsilon_j}\big(D^2W(R^\varepsilon M^\varepsilon_j) - D^2W(\nabla y)\big)\nabla u^\varepsilon : \nabla u^\varepsilon\,{\rm d}x\Big| \leq \hat\delta_\varepsilon \int_{\Omega^\varepsilon_{\rm good}} |\nabla u^\varepsilon|^2\,{\rm d}x,$$

where $\{\hat{\delta}_\varepsilon\}_\varepsilon \subset (0,+\infty)$ is a sequence depending on W and $\{\delta_\varepsilon\}_\varepsilon$, which satisfies $\hat{\delta}_\varepsilon \to 0$. This along with (6.22) and (6.31) yields

$$\liminf_{\varepsilon\to 0} \mathcal{E}_\varepsilon(y^\varepsilon, \Omega_\delta) \geq \liminf_{\varepsilon\to 0} \frac{1}{2}\int_{\Omega^\varepsilon_{\rm good}} D^2W(\nabla y)\nabla u^\varepsilon : \nabla u^\varepsilon \,\mathrm{d}x. \tag{6.32}$$

In view of (3.20) and (6.28), there holds $\nabla u^\varepsilon \chi_{\Omega^\varepsilon_{\rm good}} \rightharpoonup \nabla u$ weakly in $L^2(\Omega_\delta;\mathbb{M}^{d\times d})$. Note that $D^2W(RM)$ is positive semidefinite for $M \in \{A,B\}$ by H2. and H3. Thus, by (6.32) and the weak lower semicontinuity of convex integral functionals, we conclude that

$$\liminf_{\varepsilon\to 0} \mathcal{E}_\varepsilon(y^\varepsilon, \Omega_\delta) \geq \frac{1}{2}\int_{\Omega_\delta} D^2W(\nabla y)\nabla u : \nabla u \,\mathrm{d}x.$$

This along with the definition in (2.11) shows (6.15). This concludes the proof. □

6.2. **The limsup inequality.** In this subsection we prove the optimality of the lower bound identified in Theorem 3.13, under the additional condition that $2K = K^M_{\rm dp}$, for $M \in \{A,B\}$, cf. (3.4) and (3.26). We first collect some basic properties of the elastic energy density.

Lemma 6.3 (Elementary properties of the energy density)**.** *Let $W\colon \mathbb{M}^{d\times d} \to [0,+\infty)$ satisfy assumptions H1.–H5. and H7. Let $0 < \delta \leq \delta_W/2$, where δ_W is the constant introduced in* H5. *We define $\mathcal{V}_\delta = \{F \in \mathbb{M}^{d\times d}\colon \operatorname{dist}(F, SO(d)\{A,B\}) < \delta\}$. Then there exists a constant $C > 0$ only depending on W, a constant $C_\delta > 0$ additionally depending on δ, and $\rho_\delta > 0$ with $\rho_\delta \to 0$ as $\delta \to 0$ such that*

(i) $W(F+G) \leq W(F) + C\sqrt{W(F)}|G| + \frac{1}{2}D^2W(F)\,G : G + \rho_\delta|G|^2$ *for all $F \in \mathcal{V}_\delta$, $G \in B_\delta(0)$,*

(ii) $W(F+G) \leq W(F) + C_\delta\sqrt{W(F)}|G|$ *for all $F \in \mathbb{M}^{d\times d}\setminus \mathcal{V}_\delta$, $G \in B_\delta(0)$,*

where $B_\delta(0) \subset \mathbb{M}^{d\times d}$ denotes the open ball centered at 0 with radius δ.

The proof of this lemma is postponed to the end of this subsection.

We proceed with the construction of local recovery sequences around the interfaces. To this end, recall the definition of K in (3.4). Let y_0^+ and y_0^- be the maps defined right after (2.5). We recall the notion of cylindrical sets from (6.2) and the definition of strictly star-shaped domains in (2.7). We start by stating the local construction of recovery sequences for a single phase transition.

Proposition 6.4 (Local recovery sequence for single phase transition)**.** *Let $d \in \mathbb{N}$, $d \geq 2$. Let $\Omega \subset \mathbb{R}^d$ be a bounded, strictly star-shaped Lipschitz domain. Let $\omega' \subset \mathbb{R}^{d-1}$ be a bounded Lipschitz domain and $h > 0$ such that $\partial\omega'\times(-h,h)$ does not intersect Ω. Then, there exist sequences $\{v_\varepsilon^+\}_\varepsilon, \{v_\varepsilon^-\}_\varepsilon \subset H^2(D_{\omega',h}\cap\Omega;\mathbb{R}^d)$ with*

$$v_\varepsilon^\pm \to y_0^\pm \quad \text{in } H^1(D_{\omega',h}\cap\Omega;\mathbb{R}^d), \tag{6.33}$$

such that

$$\lim_{\varepsilon\to 0} \mathcal{E}_\varepsilon(v_\varepsilon^\pm, D_{\omega',h}\cap\Omega) = K\,\mathcal{H}^{d-1}\big((\omega'\times\{0\})\cap\Omega\big), \tag{6.34}$$

and for ε sufficiently small we have

$$v_\varepsilon^\pm = \begin{cases} I^\pm_{1,\varepsilon}\circ y_0^\pm & \text{if } x_d \geq 3h/4,\\ I^\pm_{2,\varepsilon}\circ y_0^\pm & \text{if } x_d \leq -3h/4,\end{cases} \tag{6.35}$$

where $\{I^\pm_{1,\varepsilon}\}_\varepsilon$ and $\{I^\pm_{2,\varepsilon}\}_\varepsilon$ are sequences of isometries which converge to the identity as $\varepsilon \to 0$.

We emphasize that the above statement means that for *any* sequence $\{\varepsilon_i\}_i$ converging to zero a local recovery sequence can be constructed. The crucial point is that the sequence $\{v_\varepsilon^\pm\}_\varepsilon$ is rigid away from the interface. This will allow us to appropriately 'glue together' local recovery sequences around different interfaces.

The next result provides a local construction of recovery sequences for the case in which two consecutive phase transitions create small intermediate layers at level ε between two portions of the material in the same phase, cf. Figure 3. Owing to the compatibility condition that $2K = K^M_{\rm dp}$, for $M \in \{A,B\}$, cf. (3.4) and (3.26), this provides a double energetic contribution. Recall the mappings $y^M_{\rm dp}$ defined in (3.25).

Proposition 6.5 (Local recovery sequence for double phase transitions)**.** *Let $d \in \mathbb{N}$, $d \geq 2$. Let $\Omega \subset \mathbb{R}^d$ be a bounded, strictly star-shaped Lipschitz domain. Let $\omega' \subset \mathbb{R}^{d-1}$ be a bounded Lipschitz domain and $h > 0$ such that $\partial\omega' \times (-h,h)$ does not intersect Ω. Let $M \in \{A, B\}$ and suppose that the constant $K^M_{\rm dp}$ defined in (3.26) satisfies $K^M_{\rm dp} = 2K$. Then, for every $\{w_\varepsilon\}_\varepsilon \subset \mathcal{W}_d$ there exists a sequence $\{v^M_\varepsilon\}_\varepsilon \subset H^2(D_{\omega',h} \cap \Omega; \mathbb{R}^d)$ with*

$$\frac{v^M_\varepsilon - Mx}{w_\varepsilon} \to y^M_{\rm dp} \qquad \text{in measure on } \Omega \cap D_{\omega',h} \tag{6.36}$$

such that

$$\lim_{\varepsilon\to 0} \mathcal{E}_\varepsilon(v^M_\varepsilon, \Omega \cap D_{\omega',h}) = 2K\,\mathcal{H}^{d-1}\big((\omega' \times \{0\}) \cap \Omega\big), \qquad v^M_\varepsilon = \begin{cases} I^M_{1,\varepsilon} \circ Mx & \text{if } x_d \geq 3h/4 \\ I^M_{2,\varepsilon} \circ Mx & \text{if } x_d \leq -3h/4, \end{cases} \tag{6.37}$$

where $\{I^M_{1,\varepsilon}\}_\varepsilon$ and $\{I^M_{2,\varepsilon}\}_\varepsilon$ are sequences of isometries converging to the identity as $\varepsilon \to 0$.

We defer the proofs of Propositions 6.4 and 6.5 to Subsection 6.4. (Let us mention that in the special case $\Omega = D_{\omega',h}$ the statement in Proposition 6.4 has already been proven in [32, Proposition 4.7], and here we address the generalization to strictly star-shaped Lipschitz domains Ω.) We continue with the proof of the limsup inequality. As a final preparation, we introduce the following convention: we say that a sequence of functions $\{v^\varepsilon\}_\varepsilon$ converges to v *up to translation* if there exist $\{\alpha_\varepsilon\}_\varepsilon \subset \mathbb{R}$ and $\{b_\varepsilon\}_\varepsilon \subset \mathbb{R}^d$ such that

$$v^\varepsilon(\cdot - \alpha_\varepsilon e_d) - b_\varepsilon \to v \tag{6.38}$$

with respect to a given topology. In a similar fashion, we say that two functions v_1, v_2 coincide *up to translation* if $v_2 = v_1(\cdot - \alpha e_d) - b$ for $\alpha \in \mathbb{R}$ and $b \in \mathbb{R}^d$.

Proof of Theorem 3.14. Let $(y, u, \mathcal{P}) \in \mathcal{A}$. Without loss of generality, after a rotation, we can assume that $y \in \mathcal{Y}_{\rm Id}(\Omega)$. Moreover, similarly to the proof of Theorem 3.13, it is also not restrictive to assume that

$$J_{\nabla y} \subset \bigcup\nolimits_j \partial P_j \cap \Omega \subset J_u. \tag{6.39}$$

In fact, the first inclusion always holds true by Definition 3.9, and by using Proposition 3.8(i) we may pass to another displacement field of the form $\tilde{u} = u + \mathcal{T}(y, \mathcal{P})$, see (3.22), such that the second inclusion holds for $\tilde{u}$ in place of u. In view of Remark 3.12, this does not affect the energy and we observe that a recovery sequence $\{y^\varepsilon\}_\varepsilon$ for $(y, \tilde{u}, \mathcal{P})$ in the sense of Definition 3.5 is also admissible for the original triple $(y, u, \mathcal{P})$ by Proposition 3.6(iii). As a further preliminary remark, we observe that by a diagonal argument it suffices to find for every $\delta > 0$ a recovery sequence $\{y^\varepsilon\}_\varepsilon$ for $(y, u, \mathcal{P})$ such that

$$\limsup_{\varepsilon\to 0} \mathcal{E}_\varepsilon(y^\varepsilon) \leq \mathcal{E}^{\mathcal{A}}_0(y, u, \mathcal{P}) + \delta. \tag{6.40}$$

In this context, we point out that the asymptotic representation introduced in Definition 3.5 is based on the convergences (3.10)–(3.20) which themselves are metrizable, i.e., diagonal arguments are applicable.

For convenience of the reader, we start with a short outline of the proof: in Steps 1–2 we explain that it is not restrictive to treat only problems with a finite number of interfaces and that one can assume ∇u to be smooth. In Step 3 we construct local approximate sequences around the interfaces. These are then 'glued together' to obtain an auxiliary recovery sequence $\{\tilde{y}^\varepsilon\}_\varepsilon$ converging to y, and capturing correctly the surface energy of the limiting triple $(y, u, \mathcal{P})$, see Step 4. To recover the displacement field u in the limit and to estimate the elastic contributions correctly, we then perturb $\{\tilde{y}^\varepsilon\}_\varepsilon$ by adding a term of order ε. We check that this new sequence $\{y^\varepsilon\}_\varepsilon$ indeed satisfies $y^\varepsilon \to (y, u, \mathcal{P})$ (Step 5) and $\limsup_{\varepsilon\to 0} \mathcal{E}_\varepsilon(y^\varepsilon) \leq \mathcal{E}^{\mathcal{A}}_0(y, u, \mathcal{P})$ (Step 6). Finally, Step 7 is devoted to some technical estimates.

Step 1: Reduction to a finite number of interfaces. Using the star-shapedness of the domain (say, with respect to the origin) along with Remark 4.7, one can apply a scaling argument to reduce the problem to limiting configurations where J_u consists of a finite number of disjoint interfaces orthogonal to e_d. For details on this argument we refer to [26, Proof of Proposition 5.1] and also [32, Proof of Theorem 4.4], Step I). We just mention that, for $\rho > 1$, one considers rescaled triples $(y_\rho, u_\rho, \mathcal{P}_\rho)$ of the form $y_\rho(x) = \rho y(x/\rho)$, $u_\rho(x) = \rho u(x/\rho)$, and $P_j^\rho = \rho P_j \cap \Omega$ for each component $P_j^\rho \in \mathcal{P}_\rho$. This sequence satisfies $\mathcal{E}_0^{\mathcal{A}}(y_\rho, u_\rho, \mathcal{P}_\rho) \to \mathcal{E}_0^{\mathcal{A}}(y, u, \mathcal{P})$ as $\rho \to 1$. The geometrical intuition is that, since infinitely many interfaces can only occur close to the boundary (see also the lower part of Figure 1), a rescaling allows to reduce the study to a finite number of interfaces. It suffices to construct recovery sequences for $(y_\rho, u_\rho, \mathcal{P}_\rho)$ since a recovery sequence for $(y, u, \mathcal{P})$ can then be obtained by a diagonal argument.

Summarizing, by (6.39) we can suppose that there exist finitely many Lipschitz domains $\omega_i \subset \mathbb{R}^{d-1}$ and $\alpha_i \in \mathbb{R}$ for $i = 1, \dots, I$ such that

$$J_{\nabla y} \cup \bigcup\nolimits_j (\partial P_j \cap \Omega) \cup J_u = J_u = \bigcup\nolimits_{i=1}^{I} (\omega_i \times \{\alpha_i\}). \tag{6.41}$$

Since Ω is star-shaped, we have that $\Omega \setminus J_u$ is the union of $I+1$ connected components which we indicate as $\{B_i\}_{i=1}^{I+1}$. The sets are ordered such that $\partial B_i \cap \partial B_{i+1} = \omega_i \times \{\alpha_i\}$ for $i = 1, \dots, I$, and the outer normal to B_i on $\partial B_i \cap \partial B_{i+1}$ is given by e_d (see Figure 6 below).

Step 2: Reduction to displacement fields with smooth gradient. In a similar fashion, we can also suppose that $u \in \mathscr{U}(\Omega)$ has a smooth gradient: by Proposition 3.8 we find $\{b_i\}_{i=1}^{I+1} \subset \mathbb{R}e_d$ such that the mapping

$$u' := u - \sum\nolimits_{i=1}^{I+1} b_i \chi_{B_i} \tag{6.42}$$

satisfies $u' \in H^1(\Omega;\mathbb{R}^d)$. Choose a smooth sequence $\{u'_k\}_k \subset C^\infty(\overline{\Omega};\mathbb{R}^d)$ approximating u' in $H^1(\Omega;\mathbb{R}^d)$ and observe that $u_k := u'_k + \sum_{i=1}^{I+1} b_i \chi_{B_i} \in \mathscr{U}(\Omega)$ satisfies $u_k \to u$ in $L^1(\Omega;\mathbb{R}^d)$ and $\nabla u_k \to \nabla u$ in $L^2(\Omega;\mathbb{M}^{d\times d})$. Again by a diagonal argument and by using that the limiting energy $\mathcal{E}_0^{\mathcal{A}}$ is continuous with respect to the strong L^2-convergence of displacement-gradients (see (3.23)), it suffices to construct recovery sequences for displacement fields $u \in \mathscr{U}(\Omega)$ such that $\nabla u \in C^\infty(\overline{\Omega};\mathbb{M}^{d\times d})$.

Step 3: Local construction of the approximate recovery sequence. We now start with the construction of recovery sequences around the interfaces. For brevity, we set $J_{\mathcal{P}} = \bigcup_j \partial P_j \cap \Omega$. In view of (6.39) and (6.41), we can write

$$J_{\nabla y} = \bigcup\nolimits_{i\in\mathcal{I}_y} (\omega_i \times \{\alpha_i\}), \qquad J_{\mathcal{P}} \setminus J_{\nabla y} = \bigcup\nolimits_{i\in\mathcal{I}_{\mathcal{P}}} (\omega_i \times \{\alpha_i\}), \qquad J_u \setminus (J_{\nabla y} \cup J_{\mathcal{P}}) = \bigcup\nolimits_{i\in\mathcal{I}_u} (\omega_i \times \{\alpha_i\}), \tag{6.43}$$

where $\mathcal{I}_y$, $\mathcal{I}_{\mathcal{P}}$, and $\mathcal{I}_u$ are three pairwise disjoint index sets with $\mathcal{I}_y \cup \mathcal{I}_{\mathcal{P}} \cup \mathcal{I}_u = \{1, \dots, I\}$.

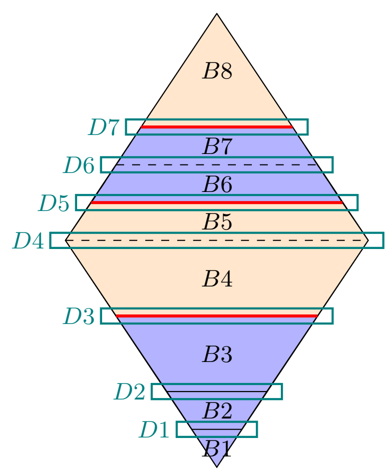

FIGURE 6. A visualization of the different interfaces and sets after the rescaling in Step 1 and under (6.39). The phase regions associated to A and B are colored in blue and orange, respectively. The interfaces associated to the sets $\mathcal{I}_y$ and $\mathcal{I}_u$ are highlighted with thick red and dashed black lines, respectively. The remaining interfaces correspond to the set $\mathcal{I}_{\mathcal{P}}$. The connected components of $\Omega \setminus J_u$ are indicated as $\{B_i\}_{i=1}^{8}$, whereas the cylindrical sets $\{D_i\}_{i=1}^{7}$ around the interfaces (see (6.44)) are drawn in green.

As explained in [26, Proof of Proposition 5.1], we can choose Lipschitz domains $\omega_i' \supset\supset \omega_i$ as well as $h > 0$ such that the sets $\partial\omega_i' \times (\alpha_i - h, \alpha_i + h)$ do not intersect Ω, the different cylindrical sets $D_i := \alpha_i e_d + D_{\omega_i',h}$ are pairwise disjoint, and one has

$$(\omega_i' \times \{\alpha_i\}) \cap \Omega = \omega_i \times \{\alpha_i\}. \tag{6.44}$$

We again refer to Figure 6 for an illustration. We now distinguish the cases of the three index sets $\mathcal{I}_y$, $\mathcal{I}_\mathcal{P}$, and $\mathcal{I}_u$: first, we fix $i \in \mathcal{I}_y$. As the sets D_i are pairwise disjoint, we get that on $D_i \cap \Omega$ the function y coincides with y_0^+ or y_0^- up to translation (recall convention (6.38)). Thus, by Proposition 6.4 we can find a sequence $\{v_\varepsilon^+\}_\varepsilon$ or $\{v_\varepsilon^-\}_\varepsilon$ such that (6.33) holds up to translation, (6.34)–(6.35) are satisfied, and the sequence $\{v_\varepsilon^+\}_\varepsilon$ or $\{v_\varepsilon^-\}_\varepsilon$ converges to y in $L^1(D_i \cap \Omega; \mathbb{R}^d)$.

For $i \in \mathcal{I}_\mathcal{P} \cup \mathcal{I}_u$, we observe that y coincides up to translation with Mx on $D_i \cap \Omega$ for some $M \in \{A, B\}$. If $i \in \mathcal{I}_\mathcal{P}$, we apply Proposition 6.5 for the sequence $w_\varepsilon = \sqrt{\varepsilon}$. If $i \in \mathcal{I}_u$, we apply Proposition 6.5 for $w_\varepsilon = |b_{i+1} - b_i|\varepsilon$, cf. (6.42). In this context, we also note that by Proposition 3.8, the fact that $\nu_u = e_d$ on J_u, and the ordering of the sets $\{B_i\}_{i=1}^{I+1}$ (see Step 1), we have $(b_{i+1} - b_i)\chi_{\{x_d > 0\}} = |b_{i+1} - b_i| y_{\rm dp}^M$ with $y_{\rm dp}^M$ defined in (3.25). In both cases, we obtain a sequence $\{v_\varepsilon\}_\varepsilon \subset H^2(D_i \cap \Omega; \mathbb{R}^d)$ such that (6.36) holds up to translation, (6.37) is fulfilled, and $v_\varepsilon \to y$ in measure on $D_i \cap \Omega$. More precisely, (6.36) and the definition of $\{w_\varepsilon\}_\varepsilon$ in each case imply

$$\begin{aligned} &\text{(i)} \ \ \varepsilon^{-1}\big(v_\varepsilon - y\big) \to (b_{i+1} - b_i)\chi_{\{x_d \geq \alpha_i\}} && \text{on } D_i \cap \Omega \text{ for } i \in \mathcal{I}_u, \\ &\text{(ii)} \ \ \varepsilon^{-1/2}\big(v_\varepsilon - y\big) \to y_{\rm dp}^M(\cdot - \alpha_i e_d), && \text{on } D_i \cap \Omega \text{ for } i \in \mathcal{I}_\mathcal{P}, \end{aligned} \tag{6.45}$$

where both properties hold in the sense of measure convergence.

For convenience, we denote this local sequence by $\{v_\varepsilon^i\}_\varepsilon \subset H^2(D_i \cap \Omega; \mathbb{R}^d)$ for each $i = 1, \ldots, I$. For later purposes, by using Lemma 2.1 we note that

$$v_\varepsilon^i \to y \quad \text{strongly in } H^1(D_i \cap \Omega; \mathbb{R}^d) \text{ for all } i = 1, \ldots, I. \tag{6.46}$$

Step 4: Global construction of the recovery sequence. Recall that $\Omega \setminus J_u = \bigcup_{i=1}^{I+1} B_i$, and let $B_i' := B_i \setminus \bigcup_{j=1}^I D_j$ for all $i = 1, \ldots, I+1$. Owing to Propositions 6.4 and 6.5, using (6.46), and arguing as in [25, Proof of Proposition 3.5], we then choose iteratively isometries $\{I_i^\varepsilon\}_{i=1}^I$ and $\{\hat{I}_i^\varepsilon\}_{i=1}^{I+1}$ such that all isometries converge to the identity as $\varepsilon \to 0$, and setting

$$\tilde{y}^\varepsilon := I_i^\varepsilon \circ v_\varepsilon^i \quad \text{on } D_i \cap \Omega \qquad \text{and} \qquad \tilde{y}^\varepsilon := \hat{I}_i^\varepsilon \circ y \quad \text{on } B_i',$$

the maps $\tilde{y}^\varepsilon \colon \Omega \to \mathbb{R}^d$ satisfy $\{\tilde{y}^\varepsilon\}_\varepsilon \subset H^2(\Omega; \mathbb{R}^d)$ and

$$\tilde{y}^\varepsilon \to y \text{ strongly in } H^1(\Omega; \mathbb{R}^d). \tag{6.47}$$

Moreover, by (6.45) we get that

$$\begin{aligned} &\text{(i)} \ \ \varepsilon^{-1}\big(\tilde{y}^\varepsilon - I_i^\varepsilon \circ y\big) \to (b_{i+1} - b_i)\chi_{\{x_d \geq \alpha_i\}} && \text{if } i \in \mathcal{I}_u, \\ &\text{(ii)} \ \ \varepsilon^{-1/2}\big(\tilde{y}^\varepsilon - I_i^\varepsilon \circ y\big) \to y_{\rm dp}^{M_i}(\cdot - \alpha_i e_d), && \text{if } i \in \mathcal{I}_\mathcal{P}, \end{aligned} \tag{6.48}$$

where both convergences hold in measure in $D_i \cap \Omega$, and $M_i \in \{A, B\}$ is such that $\nabla y = M_i$ on $D_i \cap \Omega$ if $i \in \mathcal{I}_\mathcal{P}$. Note that, up to translations, it is not restrictive to suppose that $⨍_\Omega \tilde{y}^\varepsilon \, \mathrm{d}x = 0$. By construction we have

$$\nabla \tilde{y}^\varepsilon \in SO(d)\{A, B\} \qquad \text{and} \qquad \nabla^2 \tilde{y}^\varepsilon = 0 \qquad \text{on } \bigcup\nolimits_{i=1}^{I+1} B_i' = \Omega \setminus \bigcup\nolimits_{i=1}^{I} D_i \tag{6.49}$$

for every ε. Thus, again by the properties of the sequences $\{v_\varepsilon^i\}_\varepsilon$ obtained from Propositions 6.4 and 6.5, we get by (6.39), (6.43), (6.44), and (6.49) that

$$\begin{aligned} \limsup_{\varepsilon \to 0} \mathcal{E}_\varepsilon(\tilde{y}^\varepsilon) &= \lim_{\varepsilon \to 0} \sum\nolimits_{i=1}^I \mathcal{E}_\varepsilon(v_\varepsilon^i, D_i) \leq K \sum\nolimits_{i \in \mathcal{I}_y} \mathcal{H}^{d-1}(\omega_i \times \{\alpha_i\}) + 2K \sum\nolimits_{i \in \mathcal{I}_\mathcal{P} \cup \mathcal{I}_u} \mathcal{H}^{d-1}(\omega_i \times \{\alpha_i\}) \\ &= K\, \mathcal{H}^{d-1}(J_{\nabla y}) + 2K\, \mathcal{H}^{d-1}\Big(\big(J_u \cup \big(\bigcup\nolimits_j \partial P_j \cap \Omega\big)\big) \setminus J_{\nabla y}\Big). \end{aligned}$$

By (3.23) we then conclude that

$$\limsup_{\varepsilon\to 0} \mathcal{E}_\varepsilon(\tilde{y}^\varepsilon) + \int_\Omega \mathcal{Q}_{\rm lin}(\nabla y, \nabla u)\,\mathrm{d}x \le \mathcal{E}_0^{\mathcal{A}}(y,u,\mathcal{P}). \tag{6.50}$$

So far, we have constructed a sequence $\{\tilde{y}^\varepsilon\}_\varepsilon \subset H^2(\Omega;\mathbb{R}^d)$ satisfying $\tilde{y}^\varepsilon \to y$ strongly in $H^1(\Omega;\mathbb{R}^d)$ and (6.50). In view of (6.50), we can apply Theorem 3.3 to obtain a limiting triple $(\tilde{y},\tilde{u},\tilde{\mathcal{P}})$ such that $\tilde{y}^\varepsilon \to (\tilde{y},\tilde{u},\tilde{\mathcal{P}})$ in the sense of Definition 3.5. We also note by (3.15), (6.47), and $\fint_\Omega \tilde{y}^\varepsilon\,\mathrm{d}x = 0$ that $\tilde{y} = y$. Then, by (6.39), (6.50), and Theorem 3.13 we find

$$\int_\Omega \mathcal{Q}_{\rm lin}(\nabla y, \nabla\tilde{u})\,\mathrm{d}x + 2K\mathcal{H}^{d-1}\Big(\big(J_{\tilde{u}} \cup \big(\bigcup\nolimits_j \partial\tilde{P}_j \cap \Omega\big)\big) \setminus J_{\nabla y}\Big) \le 2K\mathcal{H}^{d-1}(J_u \setminus J_{\nabla y}). \tag{6.51}$$

We write $\tilde{\mathcal{P}} = \{\tilde{P}_j\}_j$. We will prove that

$$\begin{aligned} &\text{(i)} \quad \bigcup\nolimits_j \partial\tilde{P}_j \cap \Omega = \bigcup\nolimits_j \partial P_j \cap \Omega, \\ &\text{(ii)} \quad J_u = J_{\tilde{u}} \cup \big(\bigcup\nolimits_j \partial\tilde{P}_j \cap \Omega\big). \end{aligned} \tag{6.52}$$

In particular, (i) yields $\mathcal{P} = \tilde{\mathcal{P}}$. We defer the proof of (6.52) to Step 7 below and now proceed with the construction of the recovery sequence. Note that in general $\tilde{u} \neq u$, and therefore we need to perturb $\{\tilde{y}^\varepsilon\}_\varepsilon$ to obtain a sequence such that the rescaled displacement fields converge to u. To this end, for each $\varepsilon > 0$ we let

$$y^\varepsilon := \tilde{y}^\varepsilon + \varepsilon u', \tag{6.53}$$

where u' is the (smooth) function corresponding to u defined in (6.42). We now check that $y^\varepsilon \to (y,u,\mathcal{P})$ in the sense of Definition 3.5 (Step 5) and then compute the energy of the sequence (Step 6).

Step 5: Convergence to the limiting triple. The goal of this step is to show that $y^\varepsilon \to (y,u,\mathcal{P})$ in the sense of Definition 3.5. Owing to (6.39) and recalling $y \in \mathcal{Y}_{\rm Id}(\Omega)$, we choose $M_j \in \{A,B\}$ such that $\nabla y = M_j$ on each component P_j. Similarly to (6.42), by the fact that $J_{\tilde{u}} \subset J_u$ (see (6.52)) and Proposition 3.8(i) applied for $\tilde{u}$ we find $\{\tilde{b}_i\}_{i=1}^{I+1} \subset \mathbb{R}e_d$ such that $\tilde{u}' := \tilde{u} - \sum_{i=1}^{I+1}\tilde{b}_i\chi_{B_i} \in H^1(\Omega;\mathbb{R}^d)$. By (6.51) and (6.52)(ii) we get $\int_\Omega \mathcal{Q}_{\rm lin}(\nabla y,\nabla\tilde{u})\,\mathrm{d}x = \int_\Omega \mathcal{Q}_{\rm lin}(\nabla y,\nabla\tilde{u}')\,\mathrm{d}x = 0$. Note that $F \mapsto \mathcal{Q}_{\rm lin}(M, FM)$ is positive definite on $\mathbb{M}^{d\times d}_{\rm sym}$ by (2.12). Therefore, by Korn's and Poincaré's inequalities and the fact that $\tilde{u}' \in H^1(\Omega;\mathbb{R}^d)$, it is elementary to check that $\tilde{u}' = \sum_j (SM_jx + \tilde{s}_j)\chi_{P_j}$ for some $S \in \mathbb{M}^{d\times d}_{\rm skew}$ and suitable $\{\tilde{s}_j\}_j \subset \mathbb{R}^d$. (Note that the skew symmetric matrix S here is necessarily independent of the set P_j as $\tilde{u}' \in H^1(\Omega;\mathbb{R}^d)$.) Consequently, we get

$$\tilde{u} = \sum\nolimits_j (SM_jx + \tilde{s}_j)\chi_{P_j} + \sum\nolimits_{i=1}^{I+1}\tilde{b}_i\chi_{B_i}. \tag{6.54}$$

Since $\{B_i\}_{i=1}^{I+1}$ is a refinement of the partition $\{P_j\}_j$ (see (6.41) and Figure 6), we find for each $i = 1,\ldots,I+1$ a corresponding index j_i such that $B_i \subset P_{j_i}$. For $i \in \mathcal{I}_{\mathcal{P}} \cup \mathcal{I}_u$, this implies

$$[\tilde{u}] = \tilde{b}_{i+1} + \tilde{s}_{j_{i+1}} - (\tilde{b}_i + \tilde{s}_{j_i}) \ \text{ on } \omega_i \times \{\alpha_i\} = \partial B_i \cap \partial B_{i+1}, \tag{6.55}$$

where $j_i = j_{i+1}$ if $i \in \mathcal{I}_u$, cf. (6.43). Let $\{(R^\varepsilon, \mathcal{P}^\varepsilon, \mathcal{M}^\varepsilon, \mathcal{T}^\varepsilon)\}_\varepsilon$ be the quadruples given by Theorem 3.3 for $\{\tilde{y}^\varepsilon\}_\varepsilon$ such that (3.10)–(3.20) hold. In particular, (3.14) and (3.18)–(3.19) yield

$$\varepsilon^{-1}\big(\tilde{y}^\varepsilon - (R^\varepsilon M_j^\varepsilon\, x + t_j^\varepsilon)\big) \to \tilde{u} \qquad \text{in measure on } P_j \text{ for every } j. \tag{6.56}$$

Fix $i \in \mathcal{I}_u$ as defined in (6.43), and recall that $D_i \cap \Omega \subset P_j$ for some index j. By (6.48)(i), the fact that $\nabla y = M_j$ on P_j, and by a compactness argument for affine mappings we find (for a subsequence, not relabeled) $\varepsilon^{-1}(I^\varepsilon \circ y - (R^\varepsilon M_j^\varepsilon\, x + t_j^\varepsilon)) \to S_iM_jx + d_i$ pointwise almost everywhere on $D_i \cap \Omega$ for suitable $S_i \in \mathbb{M}^{d\times d}_{\rm skew}$ and $d_i \in \mathbb{R}^d$. (We omit the details here and refer to the proof of Proposition 3.6 above for a very similar argument.) This along with (6.48)(i) and (6.56) yields

$$\tilde{u} = (b_{i+1} - b_i)\chi_{\{x_d \ge \alpha_i\}} + S_iM_jx + d_i \qquad \text{on } D_i \cap \Omega. \tag{6.57}$$

Then, in view of (6.55) and the fact that $j_i = j_{i+1}$ for $i \in \mathcal{I}_u$, we check that $b_{i+1} - b_i = \tilde{b}_{i+1} - \tilde{b}_i$ for all $i \in \mathcal{I}_u$. Therefore, by (6.54) there exist $\{s_j\}_j \subset \mathbb{R}^d$ such that

$$\tilde{u} = \sum\nolimits_j (SM_j x + s_j)\chi_{P_j} + \sum\nolimits_{i=1}^{I+1} b_i \chi_{B_i}. \tag{6.58}$$

We define $\bar{u} = u + \sum_j (SM_j x + s_j)\chi_{P_j}$. We observe that $u - \bar{u} \in \mathcal{T}(y, \mathcal{P})$, and by (6.42) and (6.58) we note that

$$\bar{u} = \tilde{u} + u'. \tag{6.59}$$

In view of (6.53), (6.56), and (6.59), we find that

$$\lim_{\varepsilon\to 0} \varepsilon^{-1}\big(y^\varepsilon - (R^\varepsilon M_j^\varepsilon\, x + t_j^\varepsilon)\big) = \tilde{u} + \lim_{\varepsilon\to 0}\varepsilon^{-1}(y^\varepsilon - \tilde{y}^\varepsilon) = \tilde{u} + u' = \bar{u}$$

in measure on P_j for every j. In other words, by (3.14) this means

$$u^\varepsilon \to \bar{u} \quad \text{in measure in } \Omega, \tag{6.60}$$

where $\{u^\varepsilon\}_\varepsilon$ is defined in (3.18) with respect to $\{y^\varepsilon\}_\varepsilon$ and the quadruples $\{(R^\varepsilon, \mathcal{P}^\varepsilon, \mathcal{M}^\varepsilon, \mathcal{T}^\varepsilon)\}_\varepsilon$. Now, we see that $(y, \mathcal{P}, \bar{u})$ is an admissible limit for the quadruples $\{(R^\varepsilon, \mathcal{P}^\varepsilon, \mathcal{M}^\varepsilon, \mathcal{T}^\varepsilon)\}_\varepsilon$. Indeed, all properties apart from (3.12), (3.15), and (3.19)–(3.20) follow from the corresponding properties of $\{\tilde{y}^\varepsilon\}_\varepsilon$. For (3.12) and (3.15) we additionally take (6.53) and $u' \in C^\infty(\overline{\Omega}; \mathbb{R}^d)$ into account, and for (3.19) we use (6.60). Finally, to see (3.20), we use $\nabla\tilde{u}^\varepsilon \to \nabla\tilde{u}$ in $L^2_{\rm loc}(\Omega; \mathbb{M}^{d\times d})$, where $\tilde{u}^\varepsilon$ is defined in (3.18) with respect to $\{\tilde{y}^\varepsilon\}_\varepsilon$, and $\nabla\bar{u} = \nabla\tilde{u} + \nabla u$ by (6.42) and (6.59), as well as $\nabla u^\varepsilon = \nabla\tilde{u}^\varepsilon + \nabla u$ by (3.18), (6.42), and (6.53). Thus, $y^\varepsilon \to (y, \bar{u}, \mathcal{P})$ in the sense of Definition 3.5. As $u - \bar{u} \in \mathcal{T}(y, \mathcal{P})$, by Proposition 3.6(iii) we then also find $y^\varepsilon \to (y, u, \mathcal{P})$, as desired. This concludes this step of the proof.

Step 6: Convergence of the energies. The goal of this step is to prove $\limsup_{\varepsilon\to 0} \mathcal{E}_\varepsilon(y^\varepsilon) \le \mathcal{E}_0^{\mathcal{A}}(y, u, \mathcal{P})$. To this end, fix $\delta, \theta > 0$. Recalling the construction of the local recovery sequences in Step 3, it is not restrictive to suppose that

$$\mathcal{L}^d\Big(\bigcup\nolimits_{i=1}^{I} D_i\Big) \le \theta^2 \tag{6.61}$$

by choosing the constant $h > 0$ sufficiently small, see before equation (6.44). In view of (6.50), we see that we essentially need to estimate the difference of $\mathcal{E}_\varepsilon(y^\varepsilon)$ and $\mathcal{E}_\varepsilon(\tilde{y}^\varepsilon)$.

First, we note that $\varepsilon|\nabla u| \le \delta$ for ε small enough since $\nabla u \in C^\infty(\overline{\Omega}; \mathbb{M}^{d\times d})$. Define $\Omega_\varepsilon = \{x \in \Omega\colon \operatorname{dist}(\nabla y^\varepsilon, SO(d)\{A, B\}) < \delta\}$. By (3.1), Lemma 6.3, (6.42), (6.53), and a quadratic expansion we calculate

$$\mathcal{E}_\varepsilon(y^\varepsilon) \le \mathcal{E}_\varepsilon(\tilde{y}^\varepsilon) + \frac{C_\delta}{\varepsilon}\int_\Omega \sqrt{W(\nabla\tilde{y}^\varepsilon)}|\nabla u|\,\mathrm{d}x + \int_{\Omega_\varepsilon} \frac{1}{2} D^2 W(\nabla\tilde{y}^\varepsilon)\nabla u : \nabla u\,\mathrm{d}x + \rho_\delta \int_{\Omega_\varepsilon} |\nabla u|^2\,\mathrm{d}x + \gamma_\varepsilon, \tag{6.62}$$

where ρ_δ and C_δ are the constants from Lemma 6.3, and γ_ε is defined by

$$\gamma_\varepsilon := \varepsilon^3 \int_\Omega 2\nabla^2\tilde{y}^\varepsilon : \nabla^2 u\,\mathrm{d}x + \varepsilon^4\int_\Omega |\nabla^2 u|^2\,\mathrm{d}x + \bar{\eta}^2_{\varepsilon,d}\sum\nolimits_{1\le\min\{i,j\}<d}\int_\Omega \Big(2\varepsilon\partial^2_{ij}\tilde{y}^\varepsilon\,\partial^2_{ij}u + \varepsilon^2|\partial^2_{ij}u|^2\Big)\,\mathrm{d}x.$$

As $\mathcal{E}_\varepsilon(\tilde{y}^\varepsilon) \le C$ by (6.50) and $\nabla u \in C^\infty(\overline{\Omega}; \mathbb{M}^{d\times d})$, the fact that $\lim_{\varepsilon\to 0}\varepsilon\bar{\eta}_{\varepsilon,d} = 0$ (see (3.3)) along with Young's inequality shows that $\lim_{\varepsilon\to 0}\gamma_\varepsilon = 0$. (More precisely, for the third term we use an estimate of the form $\bar{\eta}^2_{\varepsilon,d}\varepsilon\partial^2_{ij}\tilde{y}^\varepsilon\,\partial^2_{ij}u \le \frac{\bar{\eta}^2_{\varepsilon,d}}{2\lambda_\varepsilon^2}|\partial^2_{ij}\tilde{y}^\varepsilon|^2 + \frac{1}{2}\varepsilon^2\bar{\eta}^2_{\varepsilon,d}\lambda_\varepsilon^2|\partial^2_{ij}u|^2$ for a sequence $\{\lambda_\varepsilon\}_\varepsilon$ such that $\lambda_\varepsilon \to \infty$ and $\lambda_\varepsilon\varepsilon\bar{\eta}_{\varepsilon,d} \to 0$.) Moreover, for the second term in (6.62) we compute by Hölder's inequality, (6.49), H3., and (6.61)

$$\begin{aligned}\frac{1}{\varepsilon}\int_\Omega \sqrt{W(\nabla\tilde{y}^\varepsilon)}|\nabla u|\,\mathrm{d}x &= \frac{1}{\varepsilon}\int_{\bigcup_i D_i} \sqrt{W(\nabla\tilde{y}^\varepsilon)}|\nabla u|\,\mathrm{d}x \le \frac{1}{\varepsilon}\Big(\int_\Omega W(\nabla\tilde{y}^\varepsilon)\,\mathrm{d}x\Big)^{1/2}\|\nabla u\|_{L^2(\bigcup_i D_i)}\\ &\le C\|\nabla u\|_{L^\infty(\Omega)}\Big(\mathcal{L}^d\big(\bigcup\nolimits_i D_i\big)\Big)^{1/2} \le C\theta,\end{aligned} \tag{6.63}$$

where in the penultimate step we have also used the fact that $\int_\Omega W(\nabla\tilde{y}^\varepsilon)\,\mathrm{d}x \le C\varepsilon^2$ by (6.50). Then, from (6.47), (6.62), (6.63), $\gamma_\varepsilon \to 0$, the regularity of W, and the dominated convergence theorem we obtain

$$\limsup_{\varepsilon\to 0}\mathcal{E}_\varepsilon(y^\varepsilon) \le \limsup_{\varepsilon\to 0}\mathcal{E}_\varepsilon(\tilde{y}^\varepsilon) + \int_\Omega \mathcal{Q}_{\rm lin}(\nabla y(x), \nabla u(x))\,\mathrm{d}x + CC_\delta\theta + \rho_\delta\|\nabla u\|^2_{L^2(\Omega)},$$

where $\mathcal{Q}_{\rm lin}$ is defined in (2.11). In view of (6.50), this yields

$$\limsup_{\varepsilon\to 0}\mathcal{E}_\varepsilon(y^\varepsilon) \le \mathcal{E}_0^{\mathcal{A}}(y,u,\mathcal{P}) + CC_\delta\theta + \rho_\delta\|\nabla u\|^2_{L^2(\Omega)}.$$

The limsup inequality now follows by first letting $\theta \to 0$, then $\delta \to 0$, and by recalling the comment in (6.40).

Step 7: Proof of (6.52). To conclude the proof, it remains to show the technical property (6.52). We observe that it suffices to prove the estimates

$$\begin{aligned}
&\text{(i)} \quad J_u \setminus J_{\nabla y} \subset \big(J_{\tilde{u}} \cup \big(\textstyle\bigcup_j \partial\tilde{P}_j \cap \Omega\big)\big) \setminus J_{\nabla y},\\
&\text{(ii)} \quad \textstyle\bigcup_j (\partial P_j \cap \Omega) \setminus J_{\nabla y} \subset \bigcup_j (\partial\tilde{P}_j \cap \Omega) \setminus J_{\nabla y},\\
&\text{(iii)} \quad \textstyle\bigcup_j (\partial P_j \cap \Omega) \setminus J_{\nabla y} \supset \bigcup_j (\partial\tilde{P}_j \cap \Omega) \setminus J_{\nabla y}.
\end{aligned} \tag{6.64}$$

In fact, (6.64)(ii),(iii) along with Definition 3.9 show (6.52)(i). By (6.64)(i) and Definition 3.9 we obtain one inclusion in (6.52)(ii). The other one then follows from (6.51).

Let us now show (6.64) by contradiction. First, if (6.64)(i) were wrong, we would find a cylindrical set $\alpha_i e_d + D_{\omega_i,l}$ for $i \in \mathcal{I}_\mathcal{P} \cup \mathcal{I}_u$ (see (6.43)) and $l > 0$ sufficiently small and some component $\tilde{P}_j$ of $\tilde{\mathcal{P}}$ such that $(\alpha_i e_d + D_{\omega_i,l}) \cap \Omega \subset \tilde{P}_j$ and $(\alpha_i e_d + D_{\omega_i,l}) \cap J_{\tilde{u}} = \emptyset$. By Theorem 3.3 applied for $\{\tilde{y}^\varepsilon\}_\varepsilon$, we then get (see also (6.56))

$$\varepsilon^{-1}\big(\tilde{y}^\varepsilon - (R^\varepsilon M x + t_j^\varepsilon)\big) \to \tilde{u} \qquad \text{in measure on } (\alpha_i e_d + D_{\omega_i,l}) \cap \tilde{P}_j, \tag{6.65}$$

where $R^\varepsilon \to \mathrm{Id}$, $\{t_j^\varepsilon\}_\varepsilon \subset \mathbb{R}^d$, and M such that $\nabla y \equiv M$ on $\tilde{P}_j$. In view of the fact that $(\alpha_i e_d + D_{\omega_i,l}) \cap J_{\tilde{u}} = \emptyset$, we obtain a contradiction to (6.48)(i),(ii). On the other hand, if (6.64)(ii) were wrong, we would find $i \in \mathcal{I}_\mathcal{P}$ such that (6.65) holds. But then (6.65) and the fact that $\tilde{u}$ is finite a.e. contradict (6.48)(ii).

Finally, suppose that (6.64)(iii) were wrong. Then, there would exist a cylindrical set $D := \alpha e_d + D_{\omega,l}$ which intersects two components $\tilde{P}_{j_1}$ and $\tilde{P}_{j_2}$, but not $\bigcup_{i\in\mathcal{I}_\mathcal{P}}(\omega_i \times \{\alpha_i\})$, i.e., there exists P_j such that $D \cap \Omega \subset P_j$. Similarly to (6.65), we find sequences $\{t_{j_1}^\varepsilon\}_\varepsilon, \{t_{j_2}^\varepsilon\}_\varepsilon \subset \mathbb{R}^d$ from the sequence $\{\mathcal{T}^\varepsilon\}_\varepsilon$ given in Theorem 3.3 such that

$$\varepsilon^{-1}\big(\tilde{y}^\varepsilon - (R^\varepsilon M x + t_{j_k}^\varepsilon)\big) \to \tilde{u} \qquad \text{in measure on } D \cap \tilde{P}_{j_k} \text{ for } k = 1, 2, \tag{6.66}$$

where M is such that $\nabla y \equiv M$ on P_j. On the other hand, we find a sequence of isometries $\{I^\varepsilon\}_\varepsilon$ converging to the identity as $\varepsilon \to 0$ such that $\varepsilon^{-1}(\tilde{y}^\varepsilon - I^\varepsilon \circ y)$ converges to a finite value a.e. on $\Omega \cap D$ due to (6.48)–(6.49), where we exploit that D does not intersect $\bigcup_{i\in\mathcal{I}_\mathcal{P}}(\omega_i \times \{\alpha_i\})$. This along with (6.66) shows $\limsup_{\varepsilon\to 0}|(t_{j_1}^\varepsilon - t_{j_2}^\varepsilon)/\varepsilon| < +\infty$. This, however, contradicts (3.17). This argument concludes the proof of (6.64), and thus we have completed the proof of (6.52). ☐

We conclude this subsection by showing that W satisfies the estimates in Lemma 6.3.

Proof of Lemma 6.3. Fix $0 < \delta \le \delta_W/2$. We start with (i). By a Taylor expansion, by assumption H5., and the fact that D^2W is uniformly continuous on $\overline{\mathcal{V}_\delta}$ we find that for any $F \in \mathcal{V}_\delta$, and $G \in B_\delta(0)$ there holds

$$W(F+G) \le W(F) + DW(F) : G + \frac{1}{2}D^2W(F)\,G : G + \rho_\delta|G|^2,$$

where $\rho_\delta \to 0$ as $\delta \to 0$. Letting $R_F \in SO(d)\{A,B\}$ be such that $|R_F - F| = \mathrm{dist}(F, SO(d)\{A,B\})$, assumptions H3. and H4., together with the fact that DW is Lipschitz on $\mathcal{V}_\delta$ and $DW(R_F) = 0$, give

$$|DW(F)| \le |DW(R_F)| + C|F - R_F| = C\mathrm{dist}(F, SO(d)\{A,B\}) \le (C/\sqrt{c_1})\sqrt{W(F)}$$

for a constant C only depending on W. By the Cauchy-Schwarz inequality this concludes the proof of (i). To prove (ii), we exploit H7. to find for $F \in \mathbb{M}^{d\times d}$ and $G \in B_\delta(0)$ that

$$W(F+G) \le W(F) + c_3(1+2|F|+\delta)|G|.$$

For $F \in \mathbb{M}^{d\times d} \setminus \mathcal{V}_\delta$ one has $\max\{\delta,\, 1+2|F|\} \le C_\delta \mathrm{dist}(F, SO(d)\{A,B\})$ for a sufficiently large constant depending on δ. The desired estimate follows then again from H4. □

6.3. **Properties of the double-profile energy.** In this subsection we analyze the double-profile energy functional introduced in (6.5) and address its relation to K and $K^M_{\rm dp}$. In particular, we prove Proposition 6.2. We start by stating the results of this subsection.

Proposition 6.6 (Properties of the double-profile energy function)**.** *The functions $\mathcal{F}^M_{\rm dp}$, $M \in \{A,B\}$, satisfy for all $h>0$ and all open, bounded sets $\omega \subset \mathbb{R}^{d-1}$ with $\mathcal{H}^{d-1}(\partial\omega) = 0$:*

(i) $\mathcal{F}^M_{\rm dp}(\alpha\omega;\alpha h) \ge \alpha^{d-1}\mathcal{F}^M_{\rm dp}(\omega;h)$ *for all* $0<\alpha<1$.
(ii) $\mathcal{F}^M_{\rm dp}(\omega;h) = \mathcal{H}^{d-1}(\omega)\,\mathcal{F}^M_{\rm dp}(Q';h)$, *where* $Q' := (-\frac{1}{2},\frac{1}{2})^{d-1}$.
(iii) $\mathcal{F}^M_{\rm dp}(\omega;h) = \mathcal{F}^M_{\rm dp}(\omega;1)$.

We now address the relationship between the optimal-profile and double-profile energies.

Proposition 6.7 (Relation between K and $K^M_{\rm dp}$)**.** *There holds $K^M_{\rm dp} \ge \mathcal{F}^M_{\rm dp}(Q',1) \ge 2K$ for $M\in\{A,B\}$, where $Q' = (-\frac{1}{2},\frac{1}{2})^{d-1}$, and K, $K^M_{\rm dp}$ are defined in (3.4) and (3.26), respectively.*

Finally, if $2K = K^M_{\rm dp}$ for $M \in \{A,B\}$, in the definition (3.26) one can replace cubes by general Lipschitz domains, and the formula holds for every $h>0$ and general $\{w_\varepsilon\}_\varepsilon \in \mathcal{W}_d$.

Proposition 6.8 (Characterization of $K^M_{\rm dp}$)**.** *Let $M\in\{A,B\}$, and suppose that the constant $K^M_{\rm dp}$ defined in (3.26) satisfies $K^M_{\rm dp} = 2K$. Then there holds*

$$\inf\Big\{\limsup_{\varepsilon\to0}\mathcal{E}_\varepsilon(y^\varepsilon, D_{\omega,h})\colon\ \frac{y^\varepsilon - Mx}{w_\varepsilon} \to y^M_{\rm dp} \text{ in measure in } D_{\omega,h} \text{ as } \varepsilon\to0\Big\} = K^M_{\rm dp}\,\mathcal{H}^{d-1}(\omega), \tag{6.67}$$

for every Lipschitz domain $\omega\subset\mathbb{R}^{d-1}$, $h>0$, and $\{w_\varepsilon\}_\varepsilon \in \mathcal{W}_d$.

We point out that Propositions 6.6 and 6.7 directly imply Proposition 6.2. Proposition 6.8 will be instrumental in Subsection 6.4 below for the proof of Proposition 6.5. We prove it here as it completes the characterization of the relation between $K^M_{\rm dp}$, $M\in\{A,B\}$, and the double-profile energy functions. We now proceed with the proofs of Propositions 6.6, 6.7, and 6.8. As a preparation, we start with a standard rescaling argument which we will use several times.

Remark 6.9. For a configuration $y \in H^2(\alpha D_{\omega,h};\mathbb{R}^d)$ and $0<\alpha<1$, we define $\bar y \in H^2(D_{\omega,h};\mathbb{R}^d)$ by $\bar y(x) = y(\alpha x)/\alpha$. We observe that $\nabla \bar y(x) = \nabla y(\alpha x)$ and $\nabla^2\bar y(x) = \alpha\nabla^2 y(\alpha x)$ for all $x \in D_{\omega,h}$. Since $\{\bar\eta_{\varepsilon,d}\}_\varepsilon$ is increasing as $\varepsilon\to0$ (see (3.3)), we get $\bar\eta^2_{\sqrt{\alpha}\varepsilon,d} \ge \alpha\bar\eta^2_{\varepsilon,d}$. Thus, we obtain by (3.1)–(3.2)

$$\begin{aligned}\mathcal{E}_{\sqrt{\alpha}\varepsilon}(y,\alpha D_{\omega,h}) &\ge \frac{1}{\alpha\varepsilon^2}\int_{\alpha D_{\omega,h}} W(\nabla y)\,\mathrm{d}x + \alpha\varepsilon^2\int_{\alpha D_{\omega,h}}|\nabla^2 y|^2\,\mathrm{d}x + \alpha\bar\eta^2_{\varepsilon,d}\int_{\alpha D_{\omega,h}}(|\nabla^2 y|^2 - |\partial^2_{dd}y|^2)\,\mathrm{d}x\\ &= \frac{\alpha^{d-1}}{\varepsilon^2}\int_{D_{\omega,h}} W(\nabla\bar y)\,\mathrm{d}x + \alpha^{d-1}\varepsilon^2\int_{D_{\omega,h}}|\nabla^2\bar y|^2\,\mathrm{d}x + \alpha^{d-1}\bar\eta^2_{\varepsilon,d}\int_{D_{\omega,h}}(|\nabla^2\bar y|^2 - |\partial^2_{dd}\bar y|^2)\,\mathrm{d}x\\ &= \alpha^{d-1}\mathcal{E}_\varepsilon(\bar y, D_{\omega,h}).\end{aligned} \tag{6.68}$$

Proof of Proposition 6.6. We prove (i). Let $0<\alpha<1$. By (6.5), for a given $\delta>0$, we find sequences $\{\varepsilon_i\}_i$ with $\varepsilon_i\to0$, $\{w_i\}_i \in \mathcal{W}_d$, $u\in\mathcal{U}_{\rm dp}(\alpha D_{\omega,h})$, and $\{y^i\}_i \subset H^2(\alpha D_{\omega,h};\mathbb{R}^d)$ with $w_i^{-1}(y^i - Mx)\to u$ in measure in $\alpha D_{\omega,h}$ such that

$$\liminf_{i\to\infty}\mathcal{E}_{\sqrt{\alpha}\varepsilon_i}(y^i,\alpha D_{\omega,h}) \le \mathcal{F}^M_{\rm dp}(\alpha\omega;\alpha h) + \delta. \tag{6.69}$$

Let $\{\bar y^i\}_i \subset H^2(D_{\omega,h};\mathbb{R}^d)$ be the rescaled functions defined before (6.68). Note that $\alpha w_i^{-1}(\bar y^i - Mx) = w_i^{-1}(y^i(\alpha x) - M(\alpha x)) \to \alpha \bar u$ in measure in $D_{\omega,h}$, where $\bar u(x) = u(\alpha x)/\alpha$ for $x \in D_{\omega,h}$. Then the definition of $\mathcal{F}^M_{\rm dp}$, (6.68), and (6.69) imply

$$\delta + \mathcal{F}^M_{\rm dp}(\alpha\omega;\alpha h) \geq \liminf_{i\to\infty} \mathcal{E}_{\sqrt{\alpha}\varepsilon_i}(y^i, \alpha D_{\omega,h}) \geq \alpha^{d-1} \liminf_{i\to\infty} \mathcal{E}_{\varepsilon_i}(\bar y^i, D_{\omega,h}) \geq \alpha^{d-1}\mathcal{F}^M_{\rm dp}(\omega; h).$$

Since $\delta > 0$ was arbitrary, (i) follows.

The proof of (ii) and (iii) is exactly as in [32, Propostion 4.6] and we refer therein for details. (See also [23, Lemma 4.3] for similar arguments.) ☐

We now move to the proof of Proposition 6.7. We first state two technical lemmas. Recall the definition of y_0^+ and y_0^- below (2.5).

Lemma 6.10 (Lower energy bound)**.** *Let $\{\varepsilon_i\}_i$ be an infinitesimal sequence, and $\{\tau_i\}_i \subset \mathbb{R}$ be a bounded sequence with $\varepsilon_i/\sqrt{\tau_i} \to 0$. Let $\omega \subset \mathbb{R}^{d-1}$ be a bounded Lipschitz domain. Suppose that there exists a sequence $\{v^i\}_i$ with $v^i \in H^2(D_{\omega,\tau_i};\mathbb{R}^d)$, and*

$$\tau_i^{-1}\|\nabla v^i - \nabla y_0^+\|^2_{L^2(D_{\omega,\tau_i})} \to 0. \tag{6.70}$$

Then

$$\liminf_{i\to\infty} \mathcal{E}_{\varepsilon_i}(v^i, D_{\omega,\tau_i}) \geq K\mathcal{H}^{d-1}(\omega), \tag{6.71}$$

where K is the constant from (3.4).

Lemma 6.11 (Zooming to two interfaces)**.** *Let $\{\varepsilon_i\}_i$ be an infinitesimal sequence. Let $\mathcal{Q}' \subset \mathbb{R}^{d-1}$ be a cube and let $h > 0$. Let $M \in \{A, B\}$. For every $i \in \mathbb{N}$, let $y^i \in H^2(D_{\mathcal{Q}',h};\mathbb{R}^d)$ with $\mathcal{E}_{\varepsilon_i}(y^i, D_{\mathcal{Q}',h}) \leq C_0 < +\infty$, let $\{\tau_i\}_i \in \mathcal{W}_d$, let $u \in \mathcal{U}_{\rm dp}(D_{\mathcal{Q}',h})$, and assume that*

$$\frac{y^i - Mx}{\tau_i} \to u \text{ in measure in } D_{\mathcal{Q}',h} \text{ as } i \to \infty. \tag{6.72}$$

Then, there exist $\mu > 0$, sequences $\{\alpha_i^1\}_i, \{\alpha_i^2\}_i \subset \mathbb{R}$ such that $D_i^j := \alpha_i^j e_d + D_{\mathcal{Q}',\mu\tau_i}$, $j = 1, 2$, satisfy $D_i^1, D_i^2 \subset D_{\mathcal{Q}',h}$ and $D_i^1 \cap D_i^2 = \emptyset$, and there exists a sequence of isometries $\{I_i\}_i$ such that the maps $v^i \in H^2(D_i^1 \cup D_i^2;\mathbb{R}^d)$, defined by

$$v^i(x) = I_i \circ y^i(x) \quad \text{for every } x \in D_i^1 \cup D_i^2, \tag{6.73}$$

satisfy, up to a subsequence, for $j = 1, 2$ that

$$\min\Big\{\tau_i^{-1}\|\nabla v^i(\cdot + \alpha_i^j e_d) - \nabla y_0^+\|^2_{L^2(D_{\mathcal{Q}',\mu\tau_i})},\ \tau_i^{-1}\|\nabla v^i(\cdot + \alpha_i^j e_d) - \nabla y_0^-\|^2_{L^2(D_{\mathcal{Q}',\mu\tau_i})}\Big\} \to 0. \tag{6.74}$$

The lemma states that one finds two cylindrical sets with height $\mu\tau_i$ such that each 'contains an interface', i.e., asymptotically a big portion of $D_i^j \cap \{x_d \geq \alpha_i^j\}$ and $D_i^j \cap \{x_d \leq \alpha_i^j\}$, respectively, is contained in the A and B-phase region, respectively, cf. Figure 7.

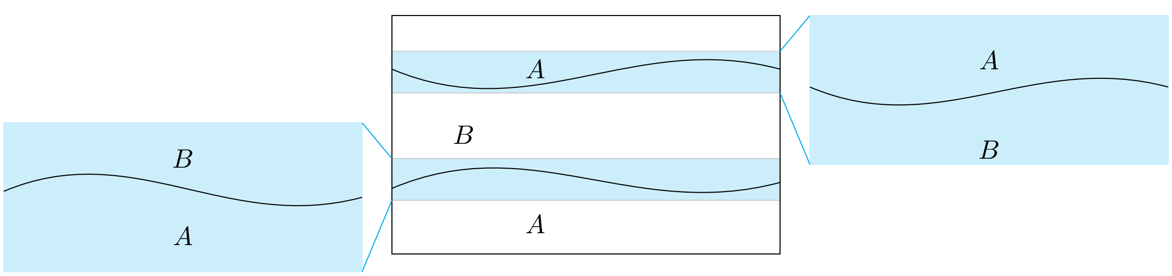

FIGURE 7. By 'zooming in' one can identify two regions in which phase transitions occur: the interfaces between the A and B-phase regions become asymptotically flat as $i \to \infty$.

Loosely speaking, the result shows that, under assumption (6.72), there are at least two interfaces and the interfaces between the A and B-phase regions become asymptotically flat, where the nonflatness is asymptotically small compared to the sequence $\{\tau_i\}_i$. An analogous result for a single interface between the A and B-phase region has been derived in [32, Lemma 4.9].

We postpone the proofs of these two lemmas and proceed with the proof of Proposition 6.7.

Proof of Proposition 6.7. Let $M \in \{A, B\}$. First, the inequality $K^M_{\rm dp} \geq \mathcal{F}^M_{\rm dp}(Q', 1)$ follows immediately from the definitions in (3.26) and (6.5). We now show $\mathcal{F}^M_{\rm dp}(Q', 1) \geq 2K$. We again let $Q = (-\frac{1}{2}, \frac{1}{2})^d$. Given $\delta > 0$, we choose sequences $\{\varepsilon_i\}_i$, $\{w_i\}_i \in \mathcal{W}_d$, $u \in \mathcal{U}_{\rm dp}(Q)$, and $\{y^i\}_i \subset H^2(Q; \mathbb{R}^d)$ such that $w_i^{-1}(y^i - Mx) \to u$ in measure in Q, and

$$\limsup_{i\to\infty} \mathcal{E}_{\varepsilon_i}(y^i, Q) \leq \mathcal{F}^M_{\rm dp}(Q', \tfrac{1}{2}) + \delta = \mathcal{F}^M_{\rm dp}(Q', 1) + \delta, \tag{6.75}$$

where the last step follows from Proposition 6.6(iii). By Lemma 6.11 applied for $\mathcal{Q}' = Q'$, $h = 1/2$, and $\tau_i = w_i$ we find $\mu > 0$ and pairwise disjoint sets $D^j_i := \alpha^j_i e_d + D_{Q', \mu w_i}$, $j = 1, 2$, with $D^1_i, D^2_i \subset Q$, and isometries $\{I_i\}_i$ such that the maps $v^i \in H^2(D^1_i \cup D^2_i; \mathbb{R}^d)$, defined by $v^i(x) = I_i \circ y^i(x)$ for $x \in D^1_i \cup D^2_i$ satisfy (6.74) (after extraction of a subsequence). Possibly after a transformation of the form $x \mapsto -v^i(-x)$, we may suppose that $w_i^{-1}\|\nabla v^i(\cdot + \alpha^j_i e_d) - \nabla y^+_0\|^2_{L^2(D_{Q',\mu w_i})} \to 0$ for $j = 1, 2$. Then H2. and Lemma 6.10 for $\tau_i = w_i$ (note that $\varepsilon_i/\sqrt{\tau_i} \to 0$ by (3.24)) imply

$$\liminf_{i\to\infty} \mathcal{E}_{\varepsilon_i}(y^i, Q) \geq \sum\nolimits_{j=1,2} \liminf_{i\to\infty} \mathcal{E}_{\varepsilon_i}(v^i(\cdot + \alpha^j_i e_d), D_{Q',\mu w_i}) \geq 2K.$$

This along with (6.75) and the fact that $\delta > 0$ was arbitrary concludes the proof. □

We continue with the proofs of Lemma 6.10 and Lemma 6.11.

Proof of Lemma 6.10. First, suppose that $\tau_i \geq h > 0$ for all $i \in \mathbb{N}$ for some $h > 0$. Then, up to translations we have $v^i \to y^+_0$ in $L^1(D_{\omega,h}; \mathbb{R}^d)$, and we immediately get

$$\liminf_{i\to\infty} \mathcal{E}_{\varepsilon_i}(v^i, D_{\omega,\tau_i}) \geq \liminf_{i\to\infty} \mathcal{E}_{\varepsilon_i}(v^i, D_{\omega,h}) \geq \mathcal{F}(\omega; h)$$

by (6.3). The result now follows from Proposition 6.1.

We can therefore concentrate on the case $\tau_i \to 0$. We prove the statement first for $\omega = \mathcal{Q}'$, where $\mathcal{Q}' \subset \mathbb{R}^{d-1}$ is a cube. For notational convenience we set $\gamma_i := \tau_i^{-1}$. We define $y^i \in H^2(D_{\gamma_i\mathcal{Q}',1}; \mathbb{R}^d)$ by $y^i(x) = v^i(\tau_i x)/\tau_i$. By using (6.68) with $\alpha_i = \tau_i$, we get

$$\mathcal{E}_{\varepsilon_i}(v^i, D_{\mathcal{Q}',\tau_i}) = \mathcal{E}_{\sqrt{\tau_i}\sqrt{\gamma_i}\varepsilon_i}(v^i, D_{\mathcal{Q}',\tau_i}) \geq \tau_i^{d-1}\mathcal{E}_{\sqrt{\gamma_i}\varepsilon_i}(y^i, D_{\gamma_i\mathcal{Q}',1}). \tag{6.76}$$

Let $\delta > 0$. We can (almost) cover $D_{\gamma_i\mathcal{Q}',1}$ by $\lfloor\gamma_i\rfloor^{d-1}$ pairwise disjoint translated copies of $D_{\mathcal{Q}',1}$. This implies that we can find $z_i \in \mathbb{R}^{d-1} \times \{0\}$ such that, by a classical De Giorgi argument (see the explanation at the beginning of the proof of [26, Lemma 4.3] for details on this technique), for $i \in \mathbb{N}$ sufficiently large we get by (6.76) and a change of variables that

$$\begin{aligned}
&\text{(i)}\ \ \mathcal{E}_{\sqrt{\gamma_i}\varepsilon_i}(y^i, z_i + D_{\mathcal{Q}',1}) \leq \frac{(1+\delta)}{\lfloor\gamma_i\rfloor^{d-1}}\,\mathcal{E}_{\sqrt{\gamma_i}\varepsilon_i}(y^i, D_{\gamma_i\mathcal{Q}',1}) \leq \frac{(1+\delta)}{(\lfloor\gamma_i\rfloor\tau_i)^{d-1}}\,\mathcal{E}_{\varepsilon_i}(v^i, D_{\mathcal{Q}',\tau_i}),\\
&\text{(ii)}\ \ \|\nabla y^i - \nabla y^+_0\|^2_{L^2(z_i + D_{\mathcal{Q}',1})} \leq \frac{C}{\delta}\tau_i^{d-1}\|\nabla y^i - \nabla y^+_0\|^2_{L^2(D_{\gamma_i\mathcal{Q}',1})} = \frac{C}{\delta\tau_i}\|\nabla v^i - \nabla y^+_0\|^2_{L^2(D_{\mathcal{Q}',\tau_i})}.
\end{aligned} \tag{6.77}$$

Since $\tau_i \to 0$, there holds $\tau_i\lfloor\gamma_i\rfloor \to 1$. This along with (6.77)(i) yields

$$\liminf_{i\to\infty} \mathcal{E}_{\sqrt{\gamma_i}\varepsilon_i}(y^i, z_i + D_{\mathcal{Q}',1}) \leq (1+\delta)\,\liminf_{i\to\infty} \mathcal{E}_{\varepsilon_i}(v^i, D_{\mathcal{Q}',\tau_i}). \tag{6.78}$$

Moreover, by (6.70) (with $\omega = \mathcal{Q}'$) and (6.77)(ii) we obtain $\|\nabla y^i - \nabla y^+_0\|^2_{L^2(z_i + D_{\mathcal{Q}',1})} \to 0$. Since $\sqrt{\gamma_i}\varepsilon_i \to 0$ by assumption on $\{\tau_i\}_i$, (6.3), (6.78), and the translational invariance of $\mathcal{E}_\varepsilon$ imply

$$\mathcal{F}(\mathcal{Q}', 1) \leq \liminf_{i\to\infty} \mathcal{E}_{\sqrt{\gamma_i}\varepsilon_i}(y^i, z_i + D_{\mathcal{Q}',1}) \leq (1+\delta)\,\liminf_{i\to\infty} \mathcal{E}_{\varepsilon_i}(v^i, D_{\mathcal{Q}',\tau_i}).$$

Since $\delta > 0$ was arbitrary, in view of Proposition 6.1, the statement follows for $\omega = \mathcal{Q}'$.

Now we consider a general bounded Lipschitz domain $\omega \subset \mathbb{R}^{d-1}$. Given $\delta > 0$, we can choose pairwise disjoint cubes $\mathcal{Q}'_j \subset \omega$, $j = 1, \ldots, N$, contained in ω such that $\mathcal{H}^{d-1}(\omega \setminus \bigcup_{j=1}^N \mathcal{Q}'_j) \leq \delta$. Then by applying (6.71) on each cube $\mathcal{Q}'_j$ we get

$$\liminf_{i\to\infty} \mathcal{E}_{\varepsilon_i}(v^i, D_{\omega,\tau_i}) \geq \sum\nolimits_{j=1}^N \liminf_{i\to\infty} \mathcal{E}_{\varepsilon_i}(v^i, D_{\mathcal{Q}'_j,\tau_i}) \geq K \sum\nolimits_{j=1}^N \mathcal{H}^{d-1}(\mathcal{Q}'_j) \geq K\big(\mathcal{H}^{d-1}(\omega) - \delta\big).$$

Since $\delta > 0$ was arbitrary, (6.71) holds. □

Proof of Lemma 6.11. We prove the result only in the case $M = A$. The case $M = B$ is the same, up to a different notational realization. The proof is similar to the one of [32, Lemma 4.9] where the problem with one interface only has been addressed.

Step 1: Subdivision into phases. As $\{\tau_i\}_i \in \mathcal{W}_d$, see (3.24), and $\alpha(d) = 1/(2d)$, we can choose $\lambda_i = \varepsilon_i^{1+1/(4d)} \subset (0, 1/4)$ such that

$$\tau_i^{-1}\lambda_i \to 0, \qquad \varepsilon_i^{-2+\alpha(d)}\tau_i\,\lambda_i^{(d-1)/d} \to \infty. \tag{6.79}$$

We use Proposition 4.1 for $y^i \in H^2(D_{\mathcal{Q}',h};\mathbb{R}^d)$ to find a corresponding set T_i with properties (4.1). Recall that T_i corresponds to the A-phase regions and $D_{\mathcal{Q}',h} \setminus T_i$ to the B-phase regions of the function y^i. Let

$$\begin{aligned} \mathcal{T}^i_A &= \big\{t \in (-h,h)\colon\ \mathcal{H}^{d-1}((\mathcal{Q}' \times \{t\}) \cap T_i) \geq (1-\lambda_i)\mathcal{H}^{d-1}(\mathcal{Q}')\big\}, \\ \mathcal{T}^i_B &= \big\{t \in (-h,h)\colon\ \mathcal{H}^{d-1}((\mathcal{Q}' \times \{t\}) \setminus T_i) \geq (1-\lambda_i)\mathcal{H}^{d-1}(\mathcal{Q}')\big\}. \end{aligned} \tag{6.80}$$

Define the indicator function $\psi_i\colon (-h,h) \to \{A,B\}$ by $\psi_i(t) = A$ if $\sup\{t' \leq t, t' \in \mathcal{T}^i_A \cup \mathcal{T}^i_B\} \in \overline{\mathcal{T}^i_A}$ and $\psi_i(t) = B$ else. We get that

$$\mathcal{H}^1\big((-h,h) \setminus (\mathcal{T}^i_A \cup \mathcal{T}^i_B)\big) \leq cC_0\varepsilon_i^{2-\alpha(d)}\,\lambda_i^{\frac{1-d}{d}}\big(\mathcal{H}^{d-1}(\mathcal{Q}')\big)^{\frac{2-d}{d-1}}, \tag{6.81}$$

and that the function ψ_i jumps at most

$$N_i \leq 2cC_0\,(\mathcal{H}^{d-1}(\mathcal{Q}'))^{-1} + 1 \tag{6.82}$$

times, where $c > 0$ is the constant from Proposition 4.1, and $C_0 > 0$ is such that $\mathcal{E}_{\varepsilon_i}(y^i, D_{\mathcal{Q}',h}) \leq C_0$ for all $i \in \mathbb{N}$. We point out that the above estimates are obtained by performing analogous arguments to the ones in the proof of [32, Lemma 4.9; (4.39)-(4.43)]. The expert reader can thus skip the remaining part of this step and move directly to Step 2. To keep the presentation self-contained, we include here a short proof of (6.81) and (6.82).

For i sufficiently large (i.e., λ_i small), the relative isoperimetric inequality on $\mathcal{Q}' \times \{t\}$ in dimension $d-1$, cf. [35, Theorem 2, Section 5.6.2], shows that

$$\mathcal{H}^{d-2}\big((\mathcal{Q}' \times \{t\}) \cap \partial^* T_i\big) \leq \lambda_i^{\frac{d-1}{d}}(\mathcal{H}^{d-1}(\mathcal{Q}'))^{\frac{d-2}{d-1}} \qquad \Rightarrow \qquad t \in \mathcal{T}^i_A \cup \mathcal{T}^i_B. \tag{6.83}$$

Indeed, by the relative isoperimetric inequality we get

$$\min\big\{\mathcal{H}^{d-1}\big((\mathcal{Q}' \times \{t\}) \cap T_i\big), \mathcal{H}^{d-1}\big((\mathcal{Q}' \times \{t\}) \setminus T_i\big)\big\} \leq C\big(\lambda_i^{\frac{d-1}{d}}(\mathcal{H}^{d-1}(\mathcal{Q}'))^{\frac{d-2}{d-1}}\big)^{\frac{d-1}{d-2}} \leq \lambda_i\mathcal{H}^{d-1}(\mathcal{Q}')$$

for i large enough, where we used $(d-1)^2/(d(d-2)) > 1$. (For $d = 2$, the term after the first inequality has to be interpreted as zero.) This gives (6.83). Thus, by (4.1)(iii), (6.83), and $\mathcal{E}_{\varepsilon_i}(y^i, D_{\mathcal{Q}',h}) \leq C_0$ we obtain (6.81).

To prove (6.82), we use the coarea formula to get for $\mathcal{H}^1$-a.e. $t_A \in \mathcal{T}^i_A$, $t_B \in \mathcal{T}^i_B$

$$\begin{aligned} \mathcal{H}^{d-1}\big(\partial^* T_i \cap (\mathcal{Q}' \times (t_A,t_B))\big) &\geq \int_{\partial^* T_i \cap (\mathcal{Q}' \times (t_A,t_B))} |\langle \nu_{T_i}, e_d\rangle|\,\mathrm{d}\mathcal{H}^{d-1} \\ &= \int_{\Pi_d} \mathcal{H}^0\big((z + (t_A,t_B)e_d) \cap \partial^* T_i \cap (\mathcal{Q}' \times (t_A,t_B))\big)\,\mathrm{d}\mathcal{H}^{d-1}(z), \end{aligned}$$

where $\Pi_d := \mathbb{R}^{d-1} \times \{0\}$, and ν_{T_i} denotes the outer unit normal to T_i. In view of (6.80) and $\lambda_i \leq \frac{1}{4}$, we get

$$\int_{\Pi_d} \mathcal{H}^0\big((z + (t_A,t_B)e_d) \cap \partial^* T_i \cap (\mathcal{Q}' \times (t_A,t_B))\big)\,\mathrm{d}\mathcal{H}^{d-1}(z) \geq \frac{1}{2}\mathcal{H}^{d-1}(\mathcal{Q}').$$

Property (6.82) follows then by (4.1)(i).

Step 2: Rigidity estimates. Theorem 3.2 and Proposition 4.1 yield rotations $R_i \in SO(d)$ such that

$$\|\nabla y^i - R_i A\|_{L^2(D_{\mathcal{Q}',h}\cap T_i)} + \|\nabla y^i - R_i B\|_{L^2(D_{\mathcal{Q}',h}\setminus T_i)} \le C\varepsilon_i, \tag{6.84}$$

where C depends on the uniform energy bound C_0 and on $D_{\mathcal{Q}',h}$. (Note that the estimate holds in the entire set $D_{\mathcal{Q}',h}$ since it is a paraxial cuboid.) For later purposes, we estimate integrals on sets $D = \alpha e_d + D_{\mathcal{Q}',\sigma} \subset D_{\mathcal{Q}',h}$ for $\alpha \in \mathbb{R}$ and $\sigma > 0$. Let $L \ge \sqrt{d}$ sufficiently large such that $\mathrm{dist}(F, SO(d)\{A,B\}) \ge |F - RM|/2$ for all $F \in \mathbb{M}^{d\times d}$ with $|F| \ge L$, $R \in SO(d)$, and $M \in \{A,B\}$. We now show that for every $q \in \{1,2\}$ there holds

$$\begin{aligned}
&\text{(i)} \quad \int_D |R_i^T \nabla y^i - A|^q \,\mathrm{d}x \le C\big(\mathcal{L}^d(D)\big)^{1-q/2}\varepsilon_i^q + (2L)^q \mathcal{L}^d(D\setminus T_i),\\
&\text{(ii)} \quad \int_D |R_i^T \nabla y^i - B|^q \,\mathrm{d}x \le C\big(\mathcal{L}^d(D)\big)^{1-q/2}\varepsilon_i^q + (2L)^q \mathcal{L}^d(D\cap T_i).
\end{aligned} \tag{6.85}$$

To see this, define $E_i = D \cap \{|\nabla y^i| \le L\}$. First, by using H4. we observe that

$$\|\nabla y^i - R_i A\|^2_{L^2(D\setminus E_i)} + \|\nabla y^i - R_i B\|^2_{L^2(D\setminus E_i)} \le C\int_D W(\nabla y^i)\,\mathrm{d}x \le C\varepsilon_i^2, \tag{6.86}$$

where C depends on c_1 and C_0. For the integral on E_i, we calculate

$$\begin{aligned}
\int_{E_i} |R_i^T\nabla y^i - A|^q\,\mathrm{d}x &= \int_{E_i\cap T_i} |\nabla y^i - R_i|^q\,\mathrm{d}x + \int_{E_i\setminus T_i} |\nabla y^i - R_i|^q\,\mathrm{d}x\\
&\le \big(\mathcal{L}^d(D)\big)^{1-q/2}\Big(\int_{D\cap T_i} |\nabla y^i - R_i|^2\,\mathrm{d}x\Big)^{q/2} + (2L)^q\mathcal{L}^d(D\setminus T_i)
\end{aligned}$$

for $q \in \{1,2\}$, where in the second step we have used Hölder's inequality. This along with (6.84), (6.86), and Hölder's inequality shows (6.85)(i). In a similar fashion, one can show (6.85)(ii).

Step 3: Asymptotic behavior of phases. We now use (6.85) to show the properties

$$\text{(i)}\ \liminf_{i\to\infty}\frac{1}{\tau_i}\mathcal{H}^1\Big(\mathcal{T}_B^i \cap (-\tfrac{h}{2},\tfrac{h}{2})\Big) > 0 \quad \text{and} \quad \text{(ii)}\ \lim_{i\to\infty}\mathcal{H}^1\Big(\mathcal{T}_B^i \cap \big((-h,h)\setminus(-\tfrac{h}{2},\tfrac{h}{2})\big)\Big) \to 0. \tag{6.87}$$

Suppose by contradiction that (6.87)(i) were false. Let $D^\sigma := D_{\mathcal{Q}',\sigma}$ for $0 < \sigma < \frac{h}{2}$. Then by (6.79)–(6.81) we get (for a subsequence, not relabeled)

$$\frac{1}{\tau_i}\mathcal{L}^d(D^\sigma\setminus T_i) \le \frac{1}{\tau_i}\mathcal{H}^{d-1}(\mathcal{Q}')\Big(\lambda_i\mathcal{H}^1\big((-\sigma,\sigma)\cap\mathcal{T}_A^i\big) + \mathcal{H}^1\big((-\sigma,\sigma)\cap\mathcal{T}_B^i\big) + \mathcal{H}^1\big((-\sigma,\sigma)\setminus(\mathcal{T}_A^i\cup\mathcal{T}_B^i)\big)\Big) \to 0. \tag{6.88}$$

By (6.85)(i) for $q=1$ and the fact that $\limsup_{i\to\infty}(\varepsilon_i/\tau_i) < +\infty$, see (3.24), this implies

$$\limsup_{i\to\infty}\frac{1}{\tau_i}\int_{D^\sigma}|R_i^T\nabla y^i - A|\,\mathrm{d}x \le C\big(2\sigma\mathcal{H}^{d-1}(\mathcal{Q}')\big)^{1/2}\limsup_{i\to\infty}(\varepsilon_i/\tau_i) \le c_\sigma$$

for a constant c_σ with $c_\sigma \to 0$ as $\sigma \to 0$. By Poincare's inequality and a BV compactness result, we find $\{b_i\}_i \subset \mathbb{R}^d$ such that the sequence

$$f_i^\sigma(x) := \tau_i^{-1}(y^i - (R_i\,x + b_i)) \quad \text{for } x \in D^\sigma$$

converges weakly* in BV to some $f^\sigma \in BV(D^\sigma;\mathbb{R}^d)$ with $|Df^\sigma|(D^\sigma) \le c_\sigma$. In view of (6.72), it is not hard to check that $f^\sigma(x) = u(x) + Sx + b$ for some $S \in \mathbb{M}^{d\times d}_{\mathrm{skew}}$ and $b \in \mathbb{R}^d$. On the other hand, by (6.4), for σ sufficiently small we find $c_\sigma < |D^ju|(\mathcal{Q}'\times\{0\})$, where D^ju denotes the jump part of the distributional derivative. This contradicts the fact that $|D^ju|(D^\sigma) = |D^jf^\sigma|(D^\sigma) \le c_\sigma$.

Now suppose by contradiction that (6.87)(ii) were false. In view of (6.82), by passing to a subsequence, we find $h > \sigma > 0$ and $\alpha \in (-h+\sigma, h-\sigma)$ such that $\mathcal{H}^1((\alpha-\sigma,\alpha+\sigma)\cap\mathcal{T}_A^i) = 0$ for all i sufficiently large. Define $D := \alpha e_d + D_{\mathcal{Q}',\sigma}$. Repeating the argument in (6.88), in particular using (6.79)–(6.81), we find $\tau_i^{-1}\mathcal{L}^d(D\cap T_i) \to 0$. Then, by (6.85)(ii) and the fact that $\limsup_{i\to\infty}(\varepsilon_i/\tau_i) < +\infty$ we get that

$$\limsup_{i\to\infty}\frac{1}{\tau_i}\int_D |R_i^T\nabla y^i - B|\,\mathrm{d}x < +\infty.$$

By Poincare's inequality and a BV compactness result, we find $\{b_i\}_i \subset \mathbb{R}^d$ such that the sequence $f_i(x) := \tau_i^{-1}(y^i - (R_i\, B\, x + b_i))$ for $x \in D$ converges pointwise a.e. to some $f \in BV(D;\mathbb{R}^d)$ (up to passing to a subsequence). By (6.72), this implies that $\tau_i^{-1}((R_i\, B - A)x + b_i)$ converges a.e. on D to a finite limit. This, however, is impossible, and therefore (6.87)(ii) holds.

Step 4: Definition of cylindrical sets. In the following, we denote by $s^i_1 < s^i_2 \ldots < \ldots < s^i_{N_i}$ the jump points of the function ψ_i defined below (6.80). Let $\mathcal{J}_i = \{0 \le j \le N_i \colon (s^i_j, s^i_{j+1}) \cap \mathcal{T}^i_A = \emptyset\}$, where we set $s^i_0 = -h$ and $s^i_{N_i+1} = h$. Note that for $j \in \mathcal{J}_i \setminus \{0\}$ there holds $(s^i_{j-1}, s^i_j) \cap \mathcal{T}^i_B = \emptyset$. Recalling (6.82), up to passing to a subsequence, we can assume that $\mathcal{J}_i$ and N_i are independent of i, which we denote by $\mathcal{J}$ and N, respectively, for simplicity. Moreover, we can suppose that $\{s^i_j\}_i$ converges for all $1 \le j \le N$. In view of (6.87)(i), possibly by selecting a further subsequence, we find an index $k \in \mathcal{J}$ and a constant $\bar{c} > 0$ independently of i such that $s^i_k, s^i_{k+1} \in (-\frac{h}{2}, \frac{h}{2})$ and

$$s^i_{k+1} - s^i_k \ge \bar{c}\,\tau_i. \tag{6.89}$$

We now show that there exist $1 \le j_1 \le k$ and $k+1 \le j_2 \le N$, as well as $\mu_1, \mu_2 > 0$ such that

$$\begin{aligned}
&\text{(i)} \quad \lim_{i\to\infty} \tau_i^{-1}\mathcal{H}^1\big((s^i_{j_1} - \mu_1\tau_i, s^i_{j_1}) \cap \mathcal{T}^i_B\big) = 0, \qquad \lim_{i\to\infty} \tau_i^{-1}\mathcal{H}^1\big((s^i_{j_1}, s^i_{j_1} + \mu_1\tau_i) \cap \mathcal{T}^i_A\big) = 0,\\
&\text{(ii)} \quad \lim_{i\to\infty} \tau_i^{-1}\mathcal{H}^1\big((s^i_{j_2} - \mu_2\tau_i, s^i_{j_2}) \cap \mathcal{T}^i_A\big) = 0, \qquad \lim_{i\to\infty} \tau_i^{-1}\mathcal{H}^1\big((s^i_{j_2}, s^i_{j_2} + \mu_2\tau_i) \cap \mathcal{T}^i_B\big) = 0.
\end{aligned} \tag{6.90}$$

Indeed, choose $j_1 \in \mathcal{J}$, $j_1 \le k$, as the largest index such that $\liminf_{i\to\infty} \tau_i^{-1}(s^i_{j_1} - s^i_{j_1-1}) > 0$ and set

$$\mu_1 := \min\big\{\liminf_{i\to\infty} \tau_i^{-1}(s^i_{j_1} - s^i_{j_1-1}), \bar{c}/2\big\} > 0,$$

where $\bar{c}$ is the constant from (6.89). Note that such an index exists by (6.79), (6.81), (6.87)(ii), and the fact that $(s^i_{j-1}, s^i_j) \cap \mathcal{T}^i_B = \emptyset$ for each $j \in \mathcal{J} \setminus \{0\}$ by the definition of $\mathcal{J}$. This immediately implies the first part of (6.90)(i). The second part of (6.90)(i) follows from the fact that $\liminf_{i\to\infty} \tau_i^{-1}(s^i_j - s^i_{j-1}) = 0$ for all $j \in \mathcal{J}$ with $j_1 < j \le k$, $(s^i_j, s^i_{j+1}) \cap \mathcal{T}^i_A = \emptyset$ for $j \in \mathcal{J}$, (6.89), and the fact that $\mu_1 \le \bar{c}/2$. The index $j_2 \ge k+1$, $j_2 \notin \mathcal{J}$, and $\mu_2 \in (0, \bar{c}/2]$ in (6.90)(ii) can be chosen in a similar fashion: let $j_2 \ge k+1$, $j_2 \notin \mathcal{J}$, be the smallest index such that $\liminf_{i\to\infty} \tau_i^{-1}(s^i_{j_2+1} - s^i_{j_2}) > 0$ and let $\mu_2 = \min\{\liminf_{i\to\infty} \tau_i^{-1}(s^i_{j_2+1} - s^i_{j_2}), \bar{c}/2\}$.

We define $\mu = \min\{\mu_1, \mu_2\}$, $\alpha^1_i = s^i_{j_1}$, and $\alpha^2_i = s^i_{j_2}$. Then, the sets $D^1_i := \alpha^1_i e_d + D_{\mathcal{Q}', \mu\tau_i}$ and $D^2_i := \alpha^2_i e_d + D_{\mathcal{Q}', \mu\tau_i}$ satisfy $D^1_i \cap D^2_i = \emptyset$ by (6.89) and the fact that $\mu \le \bar{c}/2$. Moreover, there holds

$$\begin{aligned}
&\text{(i)} \quad \tau_i^{-1}\big(\mathcal{L}^d(D^1_i \cap \{x_d \le \alpha^1_i\} \setminus T_i) + \mathcal{L}^d(D^1_i \cap \{x_d \ge \alpha^1_i\} \cap T_i)\big) \to 0,\\
&\text{(ii)} \quad \tau_i^{-1}\big(\mathcal{L}^d(D^2_i \cap \{x_d \le \alpha^2_i\} \cap T_i) + \mathcal{L}^d(D^2_i \cap \{x_d \ge \alpha^2_i\} \setminus T_i)\big) \to 0
\end{aligned} \tag{6.91}$$

as $i \to \infty$. Indeed, e.g., for the first term in (6.91)(i), we compute by (6.79)–(6.81) and (6.90)(i) that

$$\begin{aligned}
&\tau_i^{-1}\mathcal{L}^d(\{x \in D^1_i \colon\ x_d \le \alpha^1_i\} \setminus T_i)\\
&\quad \le \tau_i^{-1}\mathcal{H}^{d-1}(\mathcal{Q}')\Big(\mathcal{H}^1\big((-h,h) \setminus (\mathcal{T}^i_A \cup \mathcal{T}^i_B)\big) + \mathcal{H}^1\big((s^i_{j_1} - \mu_1\tau_i, s^i_{j_1}) \cap \mathcal{T}^i_B\big) + \mu\tau_i\,\lambda_i\Big) \to 0
\end{aligned}$$

as $i \to \infty$. The other three terms can be treated in a similar fashion.

Step 5: Proof of (6.74). We define v^i as in (6.73) for isometries I_i whose derivative is given by R_i^T. To see (6.74), we apply (6.85)(i) for $q = 2$ on $D = D^1_i \cap \{x_d \le \alpha^1_i\}$ and $D = D^2_i \cap \{x_d \ge \alpha^2_i\}$, as well as (6.85)(ii) for $q = 2$ on $D = D^1_i \cap \{x_d \ge \alpha^1_i\}$ and $D = D^2_i \cap \{x_d \le \alpha^2_i\}$. This along with (6.91) and $\tau_i^{-1}\varepsilon_i^2 \to 0$ (see (3.24)) shows the desired estimate. This concludes the proof. □

We conclude this subsection with the proof of Proposition 6.8.

Proof of Proposition 6.8. Let $M \in \{A, B\}$. First, it is clear that the left hand side in (6.67) is not smaller than $\mathcal{F}^M_{\rm dp}(\omega, h)$, see (6.5). We also note by Proposition 6.2 that

$$\mathcal{F}^M_{\rm dp}(\omega, h) \ge 2K\,\mathcal{H}^{d-1}(\omega) = K^M_{\rm dp}\,\mathcal{H}^{d-1}(\omega), \tag{6.92}$$

where in the last step we used the assumption $K^M_{\rm dp} = 2K$. To prove the reverse inequality, we argue by contradiction: if the statement were false, there would exist $\delta > 0$, a Lipschitz domain $\omega \subset \mathbb{R}^{d-1}$, $h > 0$, and a sequence $\{w_\varepsilon\}_\varepsilon \in \mathcal{W}_d$ such that

$$\inf\left\{\limsup_{\varepsilon\to 0}\mathcal{E}_\varepsilon(y^\varepsilon; D_{\omega,h}):\ \frac{y^\varepsilon - Mx}{w_\varepsilon} \to y^M_{\rm dp} \text{ in measure in } D_{\omega,h} \text{ as } \varepsilon \to 0\right\} \geq (K^M_{\rm dp} + 2\delta)\mathcal{H}^{d-1}(\omega). \tag{6.93}$$

Up to translations of ω, we can select a cube $\mathcal{Q}' \subset \mathbb{R}^{d-1}$ containing both ω and $Q' = (-\frac{1}{2}, \frac{1}{2})^{d-1}$ such that $\alpha\mathcal{Q}' = Q'$ for some $0 < \alpha < 1$. In view of (3.26), we can find a sequence of functions $\{y^\varepsilon\}_\varepsilon \subset H^2(D_{Q',\alpha h};\mathbb{R}^d)$ such that $(w_\varepsilon\alpha)^{-1}(y^\varepsilon - Mx) \to y^M_{\rm dp}$ in measure in $D_{Q',\alpha h}$ and

$$\limsup_{\varepsilon\to 0}\mathcal{E}_{\sqrt{\alpha}\varepsilon}(y^\varepsilon, D_{Q',\alpha h}) \leq K^M_{\rm dp} + \delta\alpha^{d-1}\mathcal{H}^{d-1}(\omega). \tag{6.94}$$

Then the functions $\{\bar{y}^\varepsilon\}_\varepsilon \subset H^2(D_{\mathcal{Q}',h};\mathbb{R}^d)$ defined by $\bar{y}^\varepsilon(x) = y^\varepsilon(\alpha x)/\alpha$ satisfy $w_\varepsilon^{-1}(\bar{y}^\varepsilon - Mx) = (w_\varepsilon\alpha)^{-1}(y^\varepsilon(\alpha x) - M(\alpha x)) \to y^M_{\rm dp}$ in measure in $D_{\mathcal{Q}',h}$. In particular, as $D_{\omega,h} \subset D_{\mathcal{Q}',h}$, by (6.93) we find an infinitesimal sequence $\{\varepsilon_i\}_i$ such that

$$\liminf_{i\to\infty}\mathcal{E}_{\varepsilon_i}(\bar{y}^{\varepsilon_i}, D_{\omega,h}) \geq (K^M_{\rm dp} + 2\delta)\,\mathcal{H}^{d-1}(\omega). \tag{6.95}$$

Then, using (6.5), (6.68), (6.92), and (6.95), we derive

$$\begin{aligned}
\liminf_{i\to\infty}\alpha^{1-d}\mathcal{E}_{\sqrt{\alpha}\varepsilon_i}(y^{\varepsilon_i}, D_{Q',\alpha h}) &\geq \liminf_{i\to\infty}\mathcal{E}_{\varepsilon_i}(\bar{y}^{\varepsilon_i}, D_{\mathcal{Q}',h})\\
&\geq \liminf_{i\to\infty}\mathcal{E}_{\varepsilon_i}(\bar{y}^{\varepsilon_i}, D_{\mathcal{Q}',h}\setminus D_{\omega,h}) + \liminf_{i\to\infty}\mathcal{E}_{\varepsilon_i}(\bar{y}^{\varepsilon_i}, D_{\omega,h})\\
&\geq \mathcal{F}^M_{\rm dp}(\mathcal{Q}'\setminus\omega; h) + (K^M_{\rm dp} + 2\delta)\mathcal{H}^{d-1}(\omega)\\
&\geq K^M_{\rm dp}\,\mathcal{H}^{d-1}(\mathcal{Q}'\setminus\omega) + (K^M_{\rm dp} + 2\delta)\mathcal{H}^{d-1}(\omega) = \alpha^{1-d}K^M_{\rm dp} + 2\delta\mathcal{H}^{d-1}(\omega).
\end{aligned}$$

In the last step, we used $\alpha\mathcal{Q}' = Q'$. This estimate, however, contradicts (6.94). □

6.4. **Construction of local recovery sequences.** This subsection is devoted to the proofs of Propositions 6.4 and 6.5, namely to the construction of local recovery sequences performing single and double phase transitions, respectively, in an energetically optimal way. The crucial point is that the sequences coincide with isometries far from the interfaces as this allows to 'glue together' different sequences, as done in the proof of Theorem 3.14. We begin with the proof of Proposition 6.4.

Proof of Proposition 6.4. The result has been proved in [32, Proposition 4.7] in the special case in which $\Omega = D_{\omega',h}$. We briefly explain how to obtain the result for strictly star-shaped sets Ω and cylindrical sets $D_{\omega',h}$ such that $(\partial\omega' \times (-h,h)) \cap \Omega = \emptyset$. Choose $\omega \subset \mathbb{R}^{d-1}$ such that $\omega \times \{0\} = (\omega' \times \{0\}) \cap \Omega$. As Ω is strictly star-shaped, we can find sequences $\{h_i\}_i, \{\alpha_i\}_i \subset \mathbb{R}$, with $h_i \to 0$ and $\alpha_i \to 0$ as $i \to \infty$, and a sequence of decreasing Lipschitz sets $\{\omega_i\}_i$ with $\omega \subset\subset \omega_i \subset\subset \omega'$ for all $i \in \mathbb{N}$ and

$$\mathcal{H}^{d-1}(\omega_i) \leq \mathcal{H}^{d-1}(\omega) + 1/i \tag{6.96}$$

such that $\alpha_i e_d + D_{\omega_i,h_i} \subset D_{\omega',h}$, and $(\partial\omega_i \times (-h_i + \alpha_i, \alpha_i + h_i)) \cap \Omega = \emptyset$.

We apply [32, Proposition 4.7] on $D_i := \alpha_i e_d + D_{\omega_i,h_i}$, to obtain a recovery sequence $v^{\pm,i}_\varepsilon \subset H^2(D_i;\mathbb{R}^d)$ and isometries $\{I^{\pm,i}_{1,\varepsilon}\}_\varepsilon$, $\{I^{\pm,i}_{2,\varepsilon}\}_\varepsilon$ such that (6.33) holds for D_i in place of $D_{\omega',h} \cap \Omega$ and for $y^\pm_0(\cdot - \alpha_i e_d)$ in place of $y^\pm_0$, and (6.35) holds for h_i in place of h, up to a translation by $\alpha_i e_d$. Moreover, instead of (6.34) we get

$$\lim_{\varepsilon\to 0}\mathcal{E}_\varepsilon(v^{\pm,i}_\varepsilon, D_i) = K\,\mathcal{H}^{d-1}(\omega_i). \tag{6.97}$$

In view of (6.35) for $v^{\pm,i}_\varepsilon$ and the fact that $(\partial\omega_i \times (-h_i + \alpha_i, \alpha_i + h_i)) \cap \Omega = \emptyset$, we can extend $v^{\pm,i}_\varepsilon$ to an H^2-function on $D_{\omega',h} \cap \Omega$ by setting $v^{\pm,i}_\varepsilon = I^{\pm,i}_{1,\varepsilon} \circ y^\pm_0$ on $\{\alpha_i + 3h_i/4 \leq x_d < h\}$ and $v^{\pm,i}_\varepsilon = I^{\pm,i}_{2,\varepsilon} \circ y^\pm_0$ on $\{-h < x_d \leq \alpha_i - 3h_i/4\}$, respectively. Note that the extensions (not relabeled) still satisfy (6.33) (for $y^\pm_0(\cdot - \alpha_i e_d)$ in place of $y^\pm_0$). Now we obtain a sequence satisfying (6.33)–(6.35) by choosing a suitable diagonal sequence in $\{v^{\pm,i}_\varepsilon\}_{\varepsilon,i}$ as $\varepsilon \to 0$ and $i \to \infty$ via Attouch's diagonalization lemma [9, Lemma 1.15 and Corollary 1.16], and by taking (6.96)–(6.97) into account. □

The remaining part of this subsection is devoted to the proof of Proposition 6.5. The argument hinges upon applying some careful transformations to maps locally attaining the double-profile energy in Proposition 6.8, so that the modified maps satisfy (6.37). As a first step, we show that the energy of optimal sequences concentrates near the interface. We recall the definitions of $\mathcal{W}_d$ and $y^M_{\rm dp}$ in (3.24) and (3.25), respectively.

Lemma 6.12 (Concentration of the energy near the interface)**.** *Let $h > \tau > 0$, and let $\omega \subset \mathbb{R}^{d-1}$ be a bounded Lipschitz domain. Let $M \in \{A, B\}$ and suppose that $K^M_{\rm dp} = 2K$. Let $\{\varepsilon_i\}_i$ be an infinitesimal sequence and let $\{w_{\varepsilon_i}\}_i \in \mathcal{W}_d$. Then, there exists $\{y^{\varepsilon_i}\}_i \subset H^2(D_{\omega,h};\mathbb{R}^d)$ satisfying $\lim_{i\to\infty} \|y^{\varepsilon_i} - Mx\|_{H^1(D_{\omega,h})} = 0$, and, as $i \to \infty$, we have*

$$\mathcal{E}_{\varepsilon_i}(y^{\varepsilon_i}, D_{\omega,h}) \to 2K\mathcal{H}^{d-1}(\omega), \quad \mathcal{E}_{\varepsilon_i}(y^{\varepsilon_i}, D_{\omega,h} \setminus D_{\omega,\tau}) \to 0, \quad \frac{y^{\varepsilon_i} - Mx}{w_{\varepsilon_i}} \to y^M_{\rm dp} \quad \textit{in measure in } D_{\omega,h}.$$

Proof. First, by Proposition 6.8, $K^M_{\rm dp} = 2K$, and a standard diagonal argument we find a sequence $\{y^{\varepsilon_i}\}_i \subset H^2(D_{\omega,h};\mathbb{R}^d)$ with

$$\limsup_{i\to\infty} \mathcal{E}_{\varepsilon_i}(y^{\varepsilon_i}, D_{\omega,h}) \leq 2K\mathcal{H}^{d-1}(\omega), \qquad \frac{y^{\varepsilon_i} - Mx}{w_{\varepsilon_i}} \to y^M_{\rm dp} \quad \text{in measure in } D_{\omega,h}.$$

By (6.5) and Proposition 6.2, we also get $\liminf_{i\to\infty} \mathcal{E}_{\varepsilon_i}(y^{\varepsilon_i}, D_{\omega,\tau}) \geq 2K\mathcal{H}^{d-1}(\omega)$. This in turn implies $\mathcal{E}_{\varepsilon_i}(y^{\varepsilon_i}, D_{\omega,h} \setminus D_{\omega,\tau}) \to 0$ and $\mathcal{E}_{\varepsilon_i}(y^{\varepsilon_i}, D_{\omega,h}) \to 2K\mathcal{H}^{d-1}(\omega)$. The convergence in measure to $y^M_{\rm dp}$ along with $w_{\varepsilon_i} \to 0$ implies that $y^{\varepsilon_i} \to Mx$ in measure on $D_{\omega,h}$. Then, by Lemma 2.1 we deduce $\lim_{i\to\infty} \|y^{\varepsilon_i} - Mx\|_{H^1(D_{\omega,h})} = 0$. □

Motivated by Lemma 6.12, for $0 < \tau \leq h/4$ we introduce the notion of *ε-closeness of y to Mx*, defined as

$$\delta^M_\varepsilon(y;\omega,h,\tau) := \mathcal{E}_\varepsilon(y, D_{\omega,h} \setminus D_{\omega,\tau}) + (\mathcal{L}^d(D_{\omega,4\tau}))^{-1}\|\nabla y - M\|^2_{L^2(D_{\omega,4\tau})}, \tag{6.98}$$

for $M \in \{A, B\}$. In the following, we will use that, for given $\omega \subset \mathbb{R}^{d-1}$, $0 < \tau \leq h/4$, and $\{\varepsilon_i\}_i$ converging to zero, there exists a sequence $\{y^{\varepsilon_i}\}_i \subset H^2(D_{\omega,h};\mathbb{R}^d)$ of deformations attaining asymptotically the double-profile energy $K^M_{\rm dp} = 2K$ such that

$$\delta^M_{\varepsilon_i}(y^{\varepsilon_i};\omega,h,\tau) \to 0 \qquad \text{as } i \to \infty.$$

Owing to the quantitative rigidity estimate in Theorem 3.2, it is possible to find $(d-1)$-dimensional slices on which the energy of y and the L^2-distance of ∇y from suitable rotations of $M \in \{A, B\}$ can be quantified in terms of $\delta^M_\varepsilon(y;\omega,h,\tau)$. Recall $\kappa = |A - B|$, and c_1 in H4. In addition, define

$$p_d := \begin{cases} 2 & \text{if } d = 2 \\ 2(d-1)/d & \text{if } d > 2. \end{cases}$$

Proposition 6.13 (Properties of $(d-1)$-dimensional slices)**.** *Let $d \in \mathbb{N}$, $d \geq 2$, and let $M \in \{A, B\}$. Let $h > 0$, $0 < \tau \leq h/4$, and let $\omega, \hat\omega \subset \mathbb{R}^{d-1}$ be bounded Lipschitz domains such that $\omega \subset\subset \hat\omega$. Then there exist $\varepsilon_0 = \varepsilon_0(\omega, \hat\omega, h, \kappa, c_1, \tau) \in (0, 1)$ and $C = C(\omega, \hat\omega, h, \kappa, c_1) > 0$ with the following properties:*
For all $0 < \varepsilon \leq \varepsilon_0$ and for each $y \in H^2(D_{\hat\omega,h};\mathbb{R}^d)$ with $\delta^M_\varepsilon(y;\hat\omega,h,\tau) \leq (\kappa/64)^2$ we can find two rotations $R^+, R^- \in SO(d)$ and two constants $s^+ \in (\tau, 2\tau)$, $s^- \in (-2\tau, -\tau)$ such that

(i) $\displaystyle\int_{\Gamma^+} |\nabla y - R^+M|^p \,\mathrm{d}\mathcal{H}^{d-1} + \int_{\Gamma^-} |\nabla y - R^-M|^p \,\mathrm{d}\mathcal{H}^{d-1} \leq \frac{C}{\tau}(\delta^M_\varepsilon(y;\hat\omega,h,\tau))^{p/2}\,\varepsilon^p$ *for all* $1 \leq p \leq p_d$,

(ii) $\|\nabla y - M\|^2_{L^2(s^+e_d + D_{\omega,\varepsilon^2})} + \|\nabla y - M\|^2_{L^2(s^-e_d + D_{\omega,\varepsilon^2})} \leq C\varepsilon^2\delta^M_\varepsilon(y;\hat\omega,h,\tau)$,

(iii) $\displaystyle\varepsilon^2\int_{\Gamma^+\cup\Gamma^-} |\nabla^2 y|^2 \,\mathrm{d}\mathcal{H}^{d-1} + \bar\eta^2_{\varepsilon,d}\int_{\Gamma^+\cup\Gamma^-} (|\nabla^2 y|^2 - |\partial^2_{dd} y|^2) \,\mathrm{d}\mathcal{H}^{d-1} \leq \frac{C}{\tau}\delta^M_\varepsilon(y;\hat\omega,h,\tau)$,

(iv) $\displaystyle\mathcal{E}_\varepsilon\big(y, s^+e_d + D_{\omega,\varepsilon^2}\big) + \mathcal{E}_\varepsilon\big(y, s^-e_d + D_{\omega,\varepsilon^2}\big) \leq \frac{C\varepsilon^2}{\tau}\delta^M_\varepsilon(y;\hat\omega,h,\tau)$,

(v) $|R^+ - \mathrm{Id}|^2 + |R^- - \mathrm{Id}|^2 \leq C\delta^M_\varepsilon(y;\hat\omega,h,\tau)$, *where we set* $\Gamma^\pm = \omega \times \{s^\pm\}$ *for brevity.*

Proof. The statement has been proven in [32, Proposition 4.12] in the case in which the bound on $\delta_\varepsilon^M(y;\omega,h,\tau)$ is replaced by a smallness assumption on

$$\delta_\varepsilon(y;\omega,h,\tau) := \mathcal{E}_\varepsilon(y, D_{\omega,h} \setminus D_{\omega,\tau}) + (\mathcal{L}^d(D_{\omega,4\tau}))^{-1}\|\nabla y - \nabla y_0^+\|^2_{L^2(D_{\omega,4\tau})}, \tag{6.99}$$

where y_0^+ is the map defined right after (2.5) (see also [32, Subsection 4.5]). Since the identifications of $R^\pm$ and $s^\pm$ are completely independent from each other (see also [32, Remark 4.21]), the proof of Proposition 6.13 follows by analogous arguments. □

Remark 6.14 (Integrability exponent)**.** Note that the results in [32] are proved using the most general formulation of the quantitative rigidity estimate in [32, Theorem 3.1], thus allowing for different integrability exponents p, as well as for a smaller penalization $\eta_{\varepsilon,d} < \bar\eta_{\varepsilon,d}$ (see (3.3)). The proposition is stated in its generality in order to ease the reference to [32]. Under suitable simplifications (see [32, Remark 4.17]), analogous estimates hold for $p = 2$.

The following lemma deals with the transition between a $(d-1)$-dimensional slice and a rigid movement. Recall the definition of c_2 in H6.

Lemma 6.15 (Transition to a rigid movement)**.** *Let $d \in \mathbb{N}$, $d \geq 2$, and let $M \in \{A,B\}$. Let $h,\tau,\varepsilon > 0$ and $\omega \subset\subset \hat\omega \subset \mathbb{R}^{d-1}$ satisfy the assumptions of Proposition 6.13. Assume that the elastic energy density W satisfies assumptions H1.–H4. and H6. Let $y \in H^2(D_{\hat\omega,h};\mathbb{R}^d)$ with $\delta_\varepsilon^M(y;\hat\omega,h,\tau) \leq (\kappa/64)^2$ and let $R^+, R^- \in SO(d)$, $s^+ \in (\tau, 2\tau)$, $s^- \in (-2\tau,-\tau)$ be the associated rotations and constants provided by Proposition 6.13. Then there exist a map $y_+^M \in H^2(\omega\times(0,\infty);\mathbb{R}^d)$ and a constant $b_+^M \in \mathbb{R}^d$ such that*

$$\begin{aligned}
&\text{(i)}\ \ y_+^M = y \ \text{ on } \ \omega\times(0,s^+), \qquad y_+^M(x) = R^+Mx + b_+^M \quad \text{for all } x \in \omega\times(s^++\tau,\infty),\\
&\text{(ii)}\ \ \|\nabla y_+^M - R^+M\|^2_{L^2(\omega\times(s^+,\infty))} \leq C\varepsilon^2\delta_\varepsilon^M(y;\hat\omega,h,\tau),\\
&\text{(iii)}\ \ \mathcal{E}_\varepsilon(y_+^M, \omega\times(s^+,\infty)) \leq C\delta_\varepsilon^M(y;\hat\omega,h,\tau)
\end{aligned} \tag{6.100}$$

where $C = C(\omega,\hat\omega,h,\tau,\kappa,c_1,c_2) > 0$. Analogously, there exist a map $y_-^M \in H^2(\omega\times(-\infty,0);\mathbb{R}^d)$ and a constant $b_-^M \in \mathbb{R}^d$ for which (6.100) holds with s^-, and R^- in place of s^+, and R^+, respectively.

Proof. The result follows directly by [32, Lemma 4.20]. Indeed, in [32, Lemma 4.20] an analogous result is proven in the case in which the ε-closeness δ_ε^M is replaced by the quantity defined in (6.99). The thesis follows by observing that the constructions around the slices s^+ and s^- are independent (see also [32, Remark 4.21]). □

After these preparations, we are now in a position to exhibit local recovery sequences performing a double phase transition in an energetically optimal way.

Proof of Proposition 6.5. We will prove the result only in the special case that $\Omega = D_{\omega',h}$. In fact, to treat the general case of strictly star-shaped sets Ω and cylindrical sets $D_{\omega',h}$ with $(\partial\omega'\times(-h,h))\cap\Omega = \emptyset$ one can apply the diagonal argument explained in the proof of Proposition 6.4 in a similar fashion and therefore we omit the details. For simplicity, we will write ω in place of ω' in the following.

Let $M \in \{A,B\}$, let $h > 0$, let $\omega \subset \mathbb{R}^{d-1}$ be a bounded Lipschitz domain, and let $\{w_\varepsilon\}_\varepsilon \in \mathcal{W}_d$. Fix $\rho > 0$ and choose a Lipschitz domain $\tilde\omega$ such that $\omega \subset\subset \tilde\omega$, with $\mathcal{H}^{d-1}(\tilde\omega\setminus\omega) \leq \rho$. We first observe that by Lemma 6.12 there exists a sequence $\{y^\varepsilon\}_\varepsilon \subset H^2(D_{\tilde\omega,h};\mathbb{R}^d)$ such that

$$\lim_{\varepsilon\to 0}\|y^\varepsilon - Mx\|_{H^1(D_{\tilde\omega,h})} = 0, \qquad \frac{y^\varepsilon - Mx}{w_\varepsilon} \to y^M_{\mathrm{dp}} \quad \text{in measure on } D_{\tilde\omega,h}, \tag{6.101}$$

where y^M_{dp} is the function defined in (3.25), as well as

$$\lim_{\varepsilon\to 0}\mathcal{E}_\varepsilon(y^\varepsilon, D_{\tilde\omega,h}) = 2K\mathcal{H}^{d-1}(\tilde\omega), \qquad \lim_{\varepsilon\to 0}\mathcal{E}_\varepsilon(y^\varepsilon, D_{\tilde\omega,h}\setminus D_{\tilde\omega,h/16}) = 0. \tag{6.102}$$

In view of Lemma 6.12, the existence of a sequence $\{y^{\varepsilon_i}\}_i$ satisfying (6.101)–(6.102) is guaranteed for every $\{\varepsilon_i\}_i$ with $\varepsilon_i \to 0$. Hence, in what follows, for notational simplicity we directly work with the continuous parameter ε.

Fix $\tau = h/8$. By (6.98) and (6.101)–(6.102) we find that $\delta^M_\varepsilon(y^\varepsilon;\tilde\omega,h,\tau) \to 0$ as $\varepsilon \to 0$. Without loss of generality we can assume that $\varepsilon < \varepsilon_0$ (see Proposition 6.13) and $\delta^M_\varepsilon(y^\varepsilon;\tilde\omega,h,\tau) \le (\kappa/64)^2$. Applying Proposition 6.13 to $\{y^\varepsilon\}_\varepsilon$ for $\hat\omega = \tilde\omega$, we find sequences of rotations $\{R^+_\varepsilon\}_\varepsilon$, $\{R^-_\varepsilon\}_\varepsilon \subset SO(d)$, and of slices $\{s^+_\varepsilon\}_\varepsilon \subset (\tau, 2\tau)$, and $\{s^-_\varepsilon\}_\varepsilon \subset (-2\tau,-\tau)$. Let now $\{y^M_{\varepsilon,\pm}\}_\varepsilon$ be the maps provided by Lemma 6.15. We define $v^M_\varepsilon \in H^2(D_{\omega,h};\mathbb{R}^d)$ by

$$v^M_\varepsilon(x) := \begin{cases} y^M_{\varepsilon,+} & \text{if } x_d \ge s^+_\varepsilon, \\ y^\varepsilon & \text{if } s^-_\varepsilon \le x_d \le s^+_\varepsilon, \\ y^M_{\varepsilon,-} & \text{if } x_d \le s^-_\varepsilon, \end{cases} \tag{6.103}$$

for every $x \in D_{\omega,h}$. We proceed by checking that $\{v^M_\varepsilon\}_\varepsilon$ satisfies (6.36)–(6.37). First, since $|s^\pm_\varepsilon| \le 2\tau$ and $\tau = h/8$, by Lemma 6.15 we have that $v^M_\varepsilon = I^M_{1,\varepsilon} \circ Mx$ and $v^M_\varepsilon = I^M_{2,\varepsilon} \circ Mx$ for $x_d \ge 3h/8$ and $x_d \le -3h/8$, respectively, for two suitable sequences of isometries $\{I^M_{1,\varepsilon}\}_\varepsilon, \{I^M_{2,\varepsilon}\}_\varepsilon$. This yields the second part of (6.37). For brevity, we define the sets $F^+_{\omega,h} = \omega \times (h/16, h)$ and $F^-_{\omega,h} = \omega \times (-h,-h/16)$. A key step will be to show that for $\varepsilon \to 0$

$$w_\varepsilon^{-1}(v^M_\varepsilon - Mx) \to y^M_{\rm dp} \quad \text{in measure on } F^-_{\omega,h} \cup F^+_{\omega,h}. \tag{6.104}$$

This along with (6.101) and the fact that $v^M_\varepsilon = y^\varepsilon$ on $D_{\omega,h/8}$ then shows (6.36). Moreover, note that (6.104) also implies that the isometries $\{I^M_{1,\varepsilon}\}_\varepsilon$ and $\{I^M_{2,\varepsilon}\}_\varepsilon$ converge to the identity as $\varepsilon \to 0$.

Let us now show (6.104). We only show the result on $F^+_{\omega,h}$ as the argument on $F^-_{\omega,h}$ is analogous. Moreover, it clearly suffices to prove the property for any subsequence as then convergence holds for the whole sequence by Urysohn's property. First, we note that $\mathcal{E}_\varepsilon(v^M_\varepsilon, F^+_{\omega,h}) \to 0$ as $\varepsilon \to 0$ by Lemma 6.15(iii), (6.102), (6.103), and the fact that $\delta^M_\varepsilon(y^\varepsilon;\tilde\omega,h,\tau) \to 0$. Then, applying the compactness result and the lower bound for $\Omega = F^+_{\omega,h}$ (see Theorem 3.3 and Theorem 3.13) we find a subsequence (not relabeled) and $(y,u,\mathcal{P}) \in \mathcal{A}$ such that $v^M_\varepsilon \to (y,u,\mathcal{P})$ and $\mathcal{E}^{\mathcal{A}}_0(y,u,\mathcal{P}) = 0$, where here the limiting energy $\mathcal{E}^{\mathcal{A}}_0$ defined in (3.23) has to be understood with respect to the set $F^+_{\omega,h}$.

In view of (3.23) and $\mathcal{E}^{\mathcal{A}}_0(y,u,\mathcal{P}) = 0$, we find that $\mathcal{P}$ is trivial, consisting just of the component $F^+_{\omega,h}$. Moreover, ∇y is constant, and then $\nabla y = M$ by (3.15), (6.101), and the fact that $v^M_\varepsilon = y^\varepsilon$ on $G^+_{\omega,h} := \omega \times (h/16, h/8)$. (Recall that $s^+_\varepsilon \ge \tau = h/8$.) As $\mathcal{E}^{\mathcal{A}}_0(y,u,\mathcal{P}) = 0$ and $F \mapsto \mathcal{Q}_{\rm lin}(M, FM)$ is positive definite on $\mathbb{M}^{d\times d}_{\rm sym}$ (see (2.12)) we also get that u is affine on $F^+_{\omega,h}$ and has the form $u(x) = SMx + s$ for each $x \in F^+_{\omega,h}$, where $S \in \mathbb{M}^{d\times d}_{\rm skew}$ and $s \in \mathbb{R}^d$. Moreover, in view of (3.18)–(3.19), we find $\{t^\varepsilon\}_\varepsilon \subset \mathbb{R}^d$ and $\{\bar R^\varepsilon\}_\varepsilon \subset SO(d)$ such that

$$\varepsilon^{-1}\big(v^M_\varepsilon - (\bar R^\varepsilon Mx + t^\varepsilon)\big) \to u \quad \text{in measure in } F^+_{\omega,h}. \tag{6.105}$$

On the other hand, by (6.101) and the fact that $v^M_\varepsilon = y^\varepsilon$ on $G^+_{\omega,h} = \omega \times (h/16, h/8)$ we have

$$w_\varepsilon^{-1}(v^M_\varepsilon - Mx) \to y^M_{\rm dp} \quad \text{in measure in } G^+_{\omega,h}. \tag{6.106}$$

Passing to another subsequence (not relabeled) we can assume that $\lambda := \lim_{\varepsilon\to 0} \varepsilon/w_\varepsilon$ exists, cf. (3.24). By multiplying (6.105) with $\varepsilon/w_\varepsilon$ and by subtracting (6.106) we get

$$w_\varepsilon^{-1}\big(Mx - (\bar R^\varepsilon Mx + t^\varepsilon)\big) \to \lambda u - y^M_{\rm dp} \quad \text{in measure in } G^+_{\omega,h}.$$

As the mappings in the left-hand side, as well as u and $y^M_{\rm dp}$ are affine, this convergence holds also on the larger set $F^+_{\omega,h}$. This along with (6.105) yields

$$w_\varepsilon^{-1}(v^M_\varepsilon - Mx) \to \lambda u - (\lambda u - y^M_{\rm dp}) = y^M_{\rm dp} \quad \text{in measure on } F^+_{\omega,h}.$$

This concludes the proof of (6.104). To conclude, it remains to show the asymptotic behavior of the energies in (6.37). Using (6.5), (6.36), and Proposition 6.2, it follows that $\liminf_{\varepsilon\to 0} \mathcal{E}_\varepsilon(v^M_\varepsilon, D_{\omega,h}) \ge$

$2K\mathcal{H}^{d-1}(\omega)$. To prove the opposite inequality, we observe that by (6.103) and Lemma 6.15(iii) there holds

$$\begin{aligned}\mathcal{E}_\varepsilon\big(v_\varepsilon^M, D_{\omega,h}\big) &\le \mathcal{E}_\varepsilon\big(y_{\varepsilon,+}^M, \omega\times(s_\varepsilon^+, h)\big) + \mathcal{E}_\varepsilon\big(y_{\varepsilon,-}^M, \omega\times(-h, s_\varepsilon^-)\big) + \mathcal{E}_\varepsilon\big(y^\varepsilon, \omega\times(s_\varepsilon^-, s_\varepsilon^+)\big)\\ &\le C\delta_\varepsilon^M(y^\varepsilon, \tilde\omega, h, \tau) + \mathcal{E}_\varepsilon(y^\varepsilon, D_{\tilde\omega,h}).\end{aligned}$$

Thus, by (6.102), the fact that $\delta_\varepsilon^M(y^\varepsilon;\tilde\omega,h,\tau)\to 0$, and $\mathcal{H}^{d-1}(\tilde\omega\setminus\omega)\le\rho$, we have

$$\limsup_{\varepsilon\to 0}\mathcal{E}_\varepsilon(v_\varepsilon^M, D_{\omega,h}) \le 2K\mathcal{H}^{d-1}(\tilde\omega) \le 2K\mathcal{H}^{d-1}(\omega) + 2K\rho.$$

The convergence in (6.37) follows then by the arbitrariness of ρ and by a diagonal argument. □

6.5. **One-dimensional profiles and compatibility condition.** In this subsection we assume that the density W satisfies (3.27). We will show that in this case optimal profiles for single transitions are one-dimensional in a sense to make precise below. Moreover, we show that the compatibility condition $K^A_{\rm dp} = K^B_{\rm dp} = 2K$ holds. Let us start by discussing a model case for (3.27), see (3.28). Suppose that W is of the form

$$W(F) = \phi\big(\mathrm{dist}(F, SO(d)A), \mathrm{dist}(F, SO(d)B)\big) \qquad \text{for all } F\in\mathbb{M}^{d\times d},$$

where $\phi\colon ([0,\infty))^2\to[0,\infty)$ is a smooth function with $c_1(\min\{t_1,t_2\})^2 \le \phi(t_1,t_2)\le c_2(\min\{t_1,t_2\})^2$ for all $t_1,t_2\in[0,\infty)$, and is increasing in both entries. Then, we can check that H1.–H6. hold. Moreover, also H7. is satisfied if ϕ fulfills a corresponding local Lipschitz condition. We can also confirm (3.27). Indeed, for each $F\in\mathbb{M}^{d\times d}$, by H3., the monotonicity assumptions on ϕ, and the triangle inequality we compute

$$\begin{aligned}W(F) &= \phi\big(\mathrm{dist}(F,SO(d)A), \mathrm{dist}(F,SO(d)B)\big) = \phi\Big(\min_{R\in SO(d)}|F-RA|, \min_{R\in SO(d)}|F-RB|\Big)\\ &\ge \phi\Big(\min_{R\in SO(d)}|Fe_d - RAe_d|, \min_{R\in SO(d)}|Fe_d-RBe_d|\Big) \ge \phi\Big(\big||Fe_d|-|Ae_d|\big|, \big||Fe_d|-|Be_d|\big|\Big)\\ &= \phi\Big(\big||Fe_d|-1\big|, \big||Fe_d|-(1+\kappa)\big|\Big) = \phi\Big(\big|\mathrm{Id}+(|Fe_d|-1)e_{dd}-A\big|, \big|\mathrm{Id}+(|Fe_d|-1)e_{dd}-B\big|\Big)\\ &\ge \phi\Big(\mathrm{dist}\big(\mathrm{Id}+(|Fe_d|-1)e_{dd}, SO(d)A\big), \mathrm{dist}\big(\mathrm{Id}+(|Fe_d|-1)e_{dd}, SO(d)B\big)\Big)\\ &= W\big(\mathrm{Id}+(|Fe_d|-1)e_{dd}\big).\end{aligned} \tag{6.107}$$

We now check that under condition (3.27) optimal profiles for single transitions are one-dimensional.

Lemma 6.16 (One-dimensional profiles)**.** *Under condition* (3.27)*, there holds*

$$K = \inf\Big\{\liminf_{\varepsilon\to 0}\mathcal{E}_\varepsilon(y^\varepsilon, Q)\colon\ y^\varepsilon(x) = (x', \psi^\varepsilon(x_d)) \text{ for } x=(x',x_d)\in Q, \quad \lim_{\varepsilon\to 0}\|y^\varepsilon - y_0^+\|_{L^1(Q)} = 0\Big\}, \tag{6.108}$$

where K is defined in (3.4).

Proof. We denote the right-hand side of (6.108) by K_{1d}. Clearly, we get $K_{1d}\ge K$. To see the reverse inequality, by a standard diagonal argument, we choose a sequence $\{y^\varepsilon\}_\varepsilon\subset H^2(Q;\mathbb{R}^d)$ with $\lim_{\varepsilon\to 0}\|y^\varepsilon - y_0^+\|_{L^1(Q)} = 0$ and

$$\liminf_{\varepsilon\to 0}\mathcal{E}_\varepsilon(y^\varepsilon, Q) = K.$$

Then, by Fatou's lemma, and by Lemma 2.1, we can find $x'\in(-\frac12,\frac12)^{d-1}$ such that

$$\liminf_{\varepsilon\to 0}\int_{-\frac12}^{\frac12}\Big(\frac{1}{\varepsilon^2}W(\nabla y^\varepsilon(x',t)) + \varepsilon^2|\nabla^2 y^\varepsilon(x',t)|^2 + \bar\eta^2_{\varepsilon,d}(|\nabla^2 y^\varepsilon(x',t)|^2 - |\partial^2_{dd}y^\varepsilon(x',t)|^2)\Big)\,\mathrm{d}t \le K \tag{6.109}$$

as well as

$$\lim_{\varepsilon\to 0}\Big(\int_{-\frac12}^0|\nabla y^\varepsilon(x',t)-B|^2\,\mathrm{d}t + \int_0^{\frac12}|\nabla y^\varepsilon(x',t)-A|^2\,\mathrm{d}t\Big) = 0. \tag{6.110}$$

We let $\tau^\varepsilon := \partial_d y^\varepsilon(x',\cdot) \in H^1((-\frac{1}{2},\frac{1}{2});\mathbb{R}^d)$ and we choose the unique function $\psi^\varepsilon\colon (-\frac{1}{2},\frac{1}{2}) \to \mathbb{R}$ with $\psi^\varepsilon(0) = 0$ and $(\psi^\varepsilon)' = |\tau^\varepsilon|$. Then, we define the sequence $\{v^\varepsilon\}_\varepsilon \subset H^2(Q;\mathbb{R}^d)$ by $v^\varepsilon(x',x_d) = (x',\psi^\varepsilon(x_d))$ for $(x',x_d) \in Q$. We observe that

$$\nabla v^\varepsilon(x) = \sum\nolimits_{i=1}^{d-1} e_{ii} + |\tau^\varepsilon(x_d)| e_{dd}. \tag{6.111}$$

We note that $\{v^\varepsilon\}_\varepsilon$ is an admissible sequence in the definition of K_{1d}. Indeed, by H3., (6.110), (6.111), and the triangle inequality we find

$$\begin{aligned}\int_Q |\nabla v^\varepsilon - \nabla y_0^+|^2\,\mathrm{d}x &= \int_{Q\cap\{x_d\le 0\}} |\partial_d v^\varepsilon - Be_d|^2\,\mathrm{d}t + \int_{Q\cap\{x_d\ge 0\}} |\partial_d v^\varepsilon - Ae_d|^2\,\mathrm{d}t \\ &= \int_{-\frac{1}{2}}^0 \big||\partial_d y^\varepsilon(x',t)| - |Be_d|\big|^2\,\mathrm{d}t + \int_0^{\frac{1}{2}} \big||\partial_d y^\varepsilon(x',t)| - |Ae_d|\big|^2\,\mathrm{d}t \\ &\le \int_{-\frac{1}{2}}^0 \big|(\nabla y^\varepsilon(x',t) - B)e_d\big|^2\,\mathrm{d}t + \int_0^{\frac{1}{2}} \big|(\nabla y^\varepsilon(x',t) - A)e_d\big|^2\,\mathrm{d}t \to 0,\end{aligned}$$

and therefore also $v^\varepsilon \to y_0^+$ in $L^1(Q;\mathbb{R}^d)$ since $v^\varepsilon(0) = 0$ for all ε. Consequently, in view of (3.27), (6.108), (6.109), (6.111), and the fact that $\frac{\mathrm{d}}{\mathrm{d}t}|\tau^\varepsilon|(t) \le |\partial_{dd} y^\varepsilon(x',t)|$ for $t \in (-\frac{1}{2},\frac{1}{2})$ we get

$$\begin{aligned}K_{1d} &\le \liminf_{\varepsilon\to 0} \mathcal{E}_\varepsilon(v^\varepsilon, Q) = \liminf_{\varepsilon\to 0} \int_Q \Big(\frac{1}{\varepsilon^2} W(\nabla v^\varepsilon) + \varepsilon^2 |\partial^2_{dd} v^\varepsilon|^2\Big)\,\mathrm{d}x \\ &\le \liminf_{\varepsilon\to 0} \int_{-\frac{1}{2}}^{\frac{1}{2}} \Big(\frac{1}{\varepsilon^2} W\Big(\mathrm{Id} + (|\nabla y^\varepsilon(x',t)e_d| - 1)e_{dd}\Big) + \varepsilon^2 \Big|\frac{\mathrm{d}}{\mathrm{d}t}|\tau^\varepsilon|(t)\Big|^2\Big)\,\mathrm{d}t \\ &\le \liminf_{\varepsilon\to 0} \int_{-\frac{1}{2}}^{\frac{1}{2}} \Big(\frac{1}{\varepsilon^2} W\big(\nabla y^\varepsilon(x',t)\big) + \varepsilon^2 |\partial^2_{dd} y^\varepsilon(x',t)|^2\Big)\,\mathrm{d}t \le K.\end{aligned}$$

This concludes the proof. $\square$

We point out that without an additional assumption, such as (3.27), optimal profiles for single transitions are in general not one-dimensional, see [26, Remark 6.2] for an example in a linearized setting. We are now in a position to prove Proposition 3.16.

Proof of Proposition 3.16. We start with a consequence of Lemma 6.16. Define $\tilde{W}\colon \mathbb{R} \to \mathbb{R}$ by $\tilde{W}(t) = W(\mathrm{Id} + (t-1)e_{dd})$ for $t \in \mathbb{R}$. Note that $\tilde{W}$ is a two-well potential with $\tilde{W}(t) = 0$ if and only if $t \in \{1, 1+\kappa\}$, see H3. In view of (3.1) and (6.108), we find

$$K = \inf\Big\{\liminf_{\varepsilon\to 0} \int_{-\frac{1}{2}}^{\frac{1}{2}} \Big(\frac{1}{\varepsilon^2}\tilde{W}\big(\psi'_\varepsilon\big) + \varepsilon^2 |\psi''_\varepsilon|^2\Big)\,\mathrm{d}t\colon\ \psi^\varepsilon \in H^2((-\tfrac{1}{2},\tfrac{1}{2});\mathbb{R}),\quad \lim_{\varepsilon\to 0} \|\psi^\varepsilon - \tilde{y}_0^+\|_{L^2(-\frac{1}{2},\frac{1}{2})} = 0\Big\},$$

where $\tilde{y}_0^+(t) := t\chi_{\{t>0\}} + (1+\kappa)t\chi_{\{t<0\}}$ for $t \in (-\frac{1}{2},\frac{1}{2})$. By a cut-off argument one can further show that (see e.g. [23, Proof of Proposition 5.3] for details)

$$\begin{aligned}K = \inf\Big\{\liminf_{\varepsilon\to 0} \int_{-\frac{1}{2}}^{\frac{1}{2}} \Big(\frac{1}{\varepsilon^2}\tilde{W}\big(\psi'_\varepsilon\big) + \varepsilon^2 |\psi''_\varepsilon|^2\Big)\,\mathrm{d}t\colon\ \psi^\varepsilon \in H^2((-\tfrac{1}{2},\tfrac{1}{2});\mathbb{R}),\ \lim_{\varepsilon\to 0} \|\psi^\varepsilon - \tilde{y}_0^+\|_{L^2(-\frac{1}{2},\frac{1}{2})} = 0, \\ \psi'_\varepsilon(t) = 1+\kappa \text{ near } t = -\tfrac{1}{2},\ \psi'_\varepsilon(t) = 1 \text{ near } t = \tfrac{1}{2}\Big\}.\end{aligned} \tag{6.112}$$

We now start with the proof. We prove the result only for $M = A$. The arguments for $M = B$ are similar up to a different notational realization. Let $Q' = (-\frac{1}{2},\frac{1}{2})^{d-1}$. Fix $\delta > 0$. In view of (3.26), we choose $h > 0$ and $\{w_\varepsilon\}_\varepsilon \in \mathcal{W}_d$ such that

$$K^A_{\mathrm{dp}} - \delta \le \inf\Big\{\limsup_{\varepsilon\to 0} \mathcal{E}_\varepsilon\big(y^\varepsilon, D_{Q',h}\big)\colon\ w_\varepsilon^{-1}(y^\varepsilon - x) \to y^A_{\mathrm{dp}} \text{ in measure in } D_{Q',h}\Big\}, \tag{6.113}$$

where we recall the notations in (3.25) and (6.2). We start by observing that it suffices to show that there exists a sequence $\{z_\varepsilon\}_\varepsilon \subset H^2((-h,h);\mathbb{R})$ such that

$$\begin{aligned} &\text{(i)}\quad w_\varepsilon^{-1}\big(z_\varepsilon - \mathrm{id}\big) \to \chi_{\{t>0\}} \qquad \text{in measure in } (-h,h),\\ &\text{(ii)}\quad \limsup_{\varepsilon\to 0} \int_{-h}^{h} \Big(\frac{1}{\varepsilon^2}\tilde{W}\big(z_\varepsilon'\big) + \varepsilon^2|z_\varepsilon''|^2\Big)\,\mathrm{d}t \le 2K+\delta. \end{aligned} \tag{6.114}$$

In fact, then the sequence $y^\varepsilon \in H^2(D_{Q',h};\mathbb{R}^d)$ defined by $y^\varepsilon(x',x_d) = (x', z_\varepsilon(x_d))$ clearly satisfies

$$w_\varepsilon^{-1}(y^\varepsilon - x) \to y^A_{\mathrm{dp}} \qquad \text{in measure in } D_{Q',h}$$

by (6.114)(i). Therefore, it is an admissible sequence in (6.113), and thus $\limsup_{\varepsilon\to 0}\mathcal{E}_\varepsilon(y^\varepsilon, D_{Q',h}) \ge K^A_{\mathrm{dp}} - \delta$. By (3.1), (6.114)(ii), and the definition of $\tilde{W}$, we also have $\limsup_{\varepsilon\to 0}\mathcal{E}_\varepsilon(y^\varepsilon, D_{Q',h}) \le 2K+\delta$. Thus, $K^A_{\mathrm{dp}} - \delta \le 2K+\delta$ and therefore $K^A_{\mathrm{dp}} \le 2K$ holds by passing to $\delta\to 0$. The other inequality $K^A_{\mathrm{dp}} \ge 2K$ follows from Proposition 3.15.

We now construct a sequence $\{z_\varepsilon\}_\varepsilon \subset H^2((-h,h);\mathbb{R})$ satisfying (6.114). Given $\delta>0$, we use (6.112) to find $\varepsilon_0>0$ and a function $\psi \in H^2((-\frac12,\frac12);\mathbb{R})$ such that

$$\int_{-\frac12}^{\frac12} \Big(\frac{1}{\varepsilon_0^2}\tilde{W}\big(\psi'\big) + \varepsilon_0^2|\psi''|^2\Big)\,\mathrm{d}t \le K + \delta/2, \tag{6.115}$$

as well as $\psi'(t) = 1+\kappa$ near $t=-\frac12$ and $\psi'(t) = 1$ near $t=\frac12$. Let $\varepsilon>0$ sufficiently small and let $\epsilon := \varepsilon/\varepsilon_0$ for brevity. We define $z_\varepsilon \in H^2((-h,h);\mathbb{R})$ as the continuous function with $z_\varepsilon(0)=0$ and the derivative

$$z_\varepsilon'(t) := \begin{cases} 1 & \text{if } t\in(-h,-\epsilon^2)\\ \psi'\big(\frac{1}{\epsilon^2}(-\frac12\epsilon^2 - t)\big) & \text{if } t\in(-\epsilon^2,0)\\ 1+\kappa & \text{if } t\in(0,w^\kappa_\varepsilon)\\ \psi'\big(\frac{1}{\epsilon^2}(t - w^\kappa_\varepsilon - \frac12\epsilon^2)\big) & \text{if } t\in(w^\kappa_\varepsilon, w^\kappa_\varepsilon+\epsilon^2)\\ 1 & \text{if } t\in(w^\kappa_\varepsilon+\epsilon^2, h) \end{cases}$$

for $t\in(-h,h)$, where for shorthand we write $w^\kappa_\varepsilon = w_\varepsilon/\kappa$. Indeed, we note that z_ε' is continuous since ψ' is constant near $t=-\frac12$ and $t=\frac12$. By using $\tilde{W}(t)=0$ for $t\in\{1,1+\kappa\}$ and (6.115), it is not hard to check that

$$\begin{aligned} \int_{-h}^{h}\Big(\frac{1}{\varepsilon^2}\tilde{W}\big(z_\varepsilon'\big) + \varepsilon^2|z_\varepsilon''|^2\Big)\,\mathrm{d}t &= 2\int_{-\epsilon^2/2}^{\epsilon^2/2}\Big(\frac{1}{\varepsilon^2}\tilde{W}\big(\psi'(t/\epsilon^2)\big) + \frac{\varepsilon^2}{\epsilon^4}\big|\psi''(t/\epsilon^2)\big|^2\Big)\,\mathrm{d}t\\ &= 2\int_{-\frac12}^{\frac12}\Big(\frac{1}{\varepsilon_0^2}\tilde{W}\big(\psi'(s)\big) + \varepsilon_0^2|\psi''(s)|^2\Big)\,\mathrm{d}s \le 2K+\delta, \end{aligned}$$

where in the second step we used a change of variables and $\epsilon = \varepsilon/\varepsilon_0$. This shows (6.114)(ii). We now prove (6.114)(i). As by a scaling argument we have

$$\|z_\varepsilon'\|_{L^1((-\epsilon^2,0))} + \|z_\varepsilon'\|_{L^1((w^\kappa_\varepsilon, w^\kappa_\varepsilon+\epsilon^2))} \le 2\epsilon^2\int_{-\frac12}^{\frac12}|\psi'|\,\mathrm{d}t \le C\varepsilon^2,$$

we get that

$$\|z_\varepsilon' - \tilde{z}_\varepsilon'\|_{L^1((-h,h))} \le C\varepsilon^2,$$

where $\tilde{z}_\varepsilon$ denotes the continuous piecewise affine function with $\tilde{z}_\varepsilon(0)=0$, $\tilde{z}_\varepsilon' = 1$ on $(-h,0)\cup(w^\kappa_\varepsilon,h)$, and $\tilde{z}_\varepsilon' = 1+\kappa$ on $(0,w^\kappa_\varepsilon)$. By Poincaré's inequality and $z_\varepsilon(0) = \tilde{z}_\varepsilon(0) = 0$ this also yields

$$\|z_\varepsilon - \tilde{z}_\varepsilon\|_{L^1((-h,h))} \le C\varepsilon^2. \tag{6.116}$$

Since $w_\varepsilon\to 0$ as $\varepsilon\to 0$ and $w^\kappa_\varepsilon = w_\varepsilon/\kappa$, it is easy to check that $w_\varepsilon^{-1}(\tilde{z}_\varepsilon - \mathrm{id}) \to \chi_{\{t>0\}}$ in measure in $(-h,h)$. This along with (6.116) and the fact that $\varepsilon^2/w_\varepsilon \to 0$ as $\varepsilon\to 0$ (see (3.24)) implies (6.114)(i). This concludes the proof. $\square$

Appendix A. SBV functions and Caccioppoli partitions

Let $d \in \mathbb{N}$, and let $\Omega \subset \mathbb{R}^d$ be a bounded open set. In the whole paper we use standard notations for the space $BV(\Omega)$. We refer the reader to [7] for definitions and main properties. We discuss here only some basic properties of special functions of bounded variation (SBV) and Caccioppoli partitions.

Special functions of bounded variation. Let $m \in \mathbb{N}$. We say that a function $u \in BV(\Omega;\mathbb{R}^m)$ is a *special function of bounded variation*, i.e., $u \in SBV(\Omega;\mathbb{R}^m)$, if the Cantor part of its gradient (see [7, Definition 3.91]) satisfies

$$D^c u = 0.$$

In particular, for every $u \in SBV(\Omega;\mathbb{R}^m)$ there holds

$$Du = \nabla u \, \mathcal{L}^d + (u^+ - u^-) \otimes \nu_u \mathcal{H}^{d-1} \lfloor J_u,$$

where ∇u is the approximate differential, u^+ and u^- are the approximate one-sided limits, J_u is the jump set of u, and ν_u is the normal to J_u (see [7, Chapter 3]).

A function $u \in L^1_{\mathrm{loc}}(\Omega;\mathbb{R}^m)$ (namely $u \in L^1(K;\mathbb{R}^m)$ for every compact set $K \subset \Omega$) is a *special function of locally bounded variation*, i.e., $u \in SBV_{\mathrm{loc}}(\Omega;\mathbb{R}^m)$, if $u \in SBV(O;\mathbb{R}^m)$ for every open set $O \subset\subset \Omega$.

We stress that $SBV(\Omega;\mathbb{R}^m)$ is a proper subset of $BV(\Omega;\mathbb{R}^m)$, see [7, Corollary 4.3]. The set $SBV^2(\Omega;\mathbb{R}^m)$ is defined as the collection of maps $u \in SBV(\Omega;\mathbb{R}^m)$ such that $\nabla u \in L^2(\Omega;\mathbb{R}^{m\times d})$ and $\mathcal{H}^{d-1}(J_u) < +\infty$.

Sets of finite perimeter and Caccioppoli partitions. For every $\mathcal{L}^d$-measurable set $E \subset \mathbb{R}^d$ and every $t \in [0,1]$, we denote by E^t the set of points of E having density t, namely

$$E^t := \big\{x \in E \colon \lim_{\rho\to 0} \mathcal{L}^d(E \cap B_\rho(x))/\mathcal{L}^d(B_\rho(x)) = t\big\}.$$

The *essential boundary of* E, denoted by $\partial^* E$, is defined as $\partial^* E := \mathbb{R}^d \setminus (E^0 \cup E^1)$. We say that E has finite perimeter if $\mathcal{H}^{d-1}(\partial^* E) < +\infty$. We refer the reader to [7, Subsections 3.3 and 3.5] for basic properties. Moreover, a partition $\mathcal{P} = \{P_j\}_j$ of Ω is called a *Caccioppoli partition* if

$$\sum\nolimits_j \mathcal{H}^{d-1}(\partial^* P_j) < +\infty.$$

We say that a partition is *ordered* if $\mathcal{L}^d(P_i) \geq \mathcal{L}^d(P_j)$ for $i \leq j$, and recall that every Caccioppoli partition of a bounded domain induces an ordered one just by a permutation of the indices.

We say that a set of finite perimeter E is *indecomposable* if it cannot be written as $E^1 \cup E^2$ with $E^1 \cap E^2 = \emptyset$, $\mathcal{L}^d(E^1), \mathcal{L}^d(E^2) > 0$ and $\mathcal{H}^{d-1}(\partial^* E) = \mathcal{H}^{d-1}(\partial^* E^1) + \mathcal{H}^{d-1}(\partial^* E^2)$. Note that this notion generalizes the concept of connectedness to sets of finite perimeter. By [8, Theorem 1] for each set of finite perimeter E there exists a unique finite or countable family of pairwise disjoint indecomposable sets $\{E_i\}_i$ such that $\mathcal{H}^{d-1}(\partial^* E) = \sum_i \mathcal{H}^{d-1}(\partial^* E_i)$. The set $\{E_i\}_i$ are called the *connected components* of E.

We conclude this section by stating a compactness result for ordered Caccioppoli partitions (see [7, Theorem 4.19, Remark 4.20]).

Theorem A.1 (Compactness for Caccioppoli partitions)**.** *Let $\Omega \subset \mathbb{R}^d$ be a bounded open set with Lipschitz boundary. Let $\mathcal{P}_i = \{P_{j,i}\}_j$, $i \in \mathbb{N}$, be a sequence of ordered Caccioppoli partitions of Ω with*

$$\sup\nolimits_{i\in\mathbb{N}} \Big\{ \sum\nolimits_j \mathcal{H}^{d-1}(\partial^* P_{j,i}) \Big\} < +\infty.$$

Then, there exists a Caccioppoli partition $\mathcal{P} = \{P_j\}_j$ and a not relabeled subsequence such that $P_{j,i} \to P_j$ in measure for all $j \in \mathbb{N}$ as $i \to \infty$.

In the proofs, we also sometimes use the fact that $P_{i,j} \to P_j$ in measure for all $j \in \mathbb{N}$ is equivalent to $\sum_j \mathcal{L}^d(P_{i,j} \triangle P_j) \to 0$.

Acknowledgements

We would like to thank Roberto Alicandro, Giuliano Lazzaroni, and Ben Schweizer for interesting discussions on the topics of this paper. The support by the Alexander von Humboldt Foundation is gratefully acknowledged. This work was supported by the DFG projects FR 4083/1-1, FR 4083/3-1, and by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) under Germany's Excellence Strategy EXC 2044 -390685587, Mathematics Münster: Dynamics–Geometry–Structure. This work has been funded by the Vienna Science and Technology Fund (WWTF) through Project MA14-009, as well as by the Austrian Science Fund (FWF) projects F65, V 662 N32, and I 4052 N32, and by BMBWF through the OeAD-WTZ project CZ04/2019.

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Institute of Analysis and Scientific Computing, Vienna University of Technology, Wiedner Hauptstrasse 8-10, 1040 Vienna, Austria

E-mail address: `elisa.davoli@tuwien.ac.at`

Institute for Computational and Applied Mathematics, University of Münster, Einsteinstrasse 62, D-48149 Münster, Germany

E-mail address: `manuel.friedrich@uni-muenster.de`