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M. Ruggeri
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NUMERICAL ANALYSIS OF THE LANDAU–LIFSHITZ–GILBERT EQUATION WITH INERTIAL EFFECTS

Michele Ruggeri

Abstract. We consider the numerical approximation of the inertial Landau–Lifshitz–Gilbert (iLLG) equation, which describes the dynamics of the magnetization in ferromagnetic materials at subpicosecond time scales. We propose and analyze two fully discrete numerical schemes based on two different approaches: The first method is based on a reformulation of the problem as a linear constrained variational formulation for the time derivative of the magnetization. The second method exploits a reformulation of the problem as a first order system in time for the magnetization and the angular momentum. Both schemes are implicit, based on first-order finite elements, and the constructed numerical approximations satisfy the inherent unit-length constraint of iLLG at the vertices of the underlying mesh. For both schemes, we establish a discrete energy law and prove convergence of the approximations towards a weak solution of the problem. Numerical experiments validate the theoretical results and show the applicability of the methods for the simulation of ultrafast magnetic processes.

1. Introduction

1.1. Magnetization dynamics with inertial effects. The understanding of the magnetization dynamics and the capability to perform reliable numerical simulations of magnetic systems play a fundamental role in the design of many technological applications, e.g., hard disk drives. A well-accepted model to describe the magnetization dynamics in ferromagnetic materials is the Landau–Lifshitz–Gilbert equation (LLG), which, in the so-called Gilbert form, is given by

$$\partial_t \mathbf{m} = -\gamma_0 \mathbf{m} \times \mathbf{H}_{\text{eff}}(\mathbf{m}) + \alpha \mathbf{m} \times \partial_t \mathbf{m}.$$  (1)

Here, $\mathbf{m}$ denotes the normalized magnetization (dimensionless and satisfying $|\mathbf{m}| = 1$), the effective field $\mathbf{H}_{\text{eff}}(\mathbf{m})$ (in A/m), up to a negative multiplicative constant, is the functional derivative of the micromagnetic energy $E(\mathbf{m})$ (in J) with respect to the magnetization, while $\gamma_0 > 0$ and $\alpha > 0$ denote the gyromagnetic ratio (in m/(A s)) and the Gilbert damping parameter (dimensionless), respectively. The first term on the right-hand side of (1) describes the precession of the magnetization around the effective field. The second term is a dissipative contribution, which pushes the magnetization towards the effective field. The resulting dynamics is a damped precession, where the magnetization rotates around the effective field while being damped towards it; see Figure 1(a).

![Figure 1](image)

**Figure 1.** (a) Schematic of the magnetization dynamics governed by (1): precession and damping. (b) Schematic of the magnetization dynamics governed by (2): precession, damping, and nutation.

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In 1996, a pioneering experiment showed that, using femtosecond laser excitations, it is possible to manipulate the magnetization of a nickel sample at subpicosecond time scales \[13\]. This discovery gave impulse to several theoretical and experimental studies, which gave rise to the field that nowadays is referred to as \textit{ultrafast magnetism} \[27\].

The standard LLG (1) is not capable to describe the dynamics of the magnetization at such short time scales. Based on the concept of angular momentum in magnetic spin systems, a novel evolution equation has been recently proposed \[16\]. This equation, called \textit{inertial LLG} (iLLG), reads as

\[
\frac{\partial}{\partial t} m = -\gamma_0 m \times H_{\text{eff}}(m) + \alpha m \times \frac{\partial}{\partial t} m + \tau m \times \frac{\partial^2}{\partial t^2} m. 
\]

Here, \(\tau > 0\) denotes the angular momentum relaxation time (in s). In addition to the classical precession and damping contributions, the right-hand side of (2) comprises a third term involving the second time derivative of the magnetization. It has been predicted that the effect of this additional contribution on the magnetization dynamics consists in the appearance of nutation dynamics—superimposed magnetization oscillations occurring at a frequency much higher than the one of the damped precession dynamics; see Figure 1(b). Such inertial magnetization dynamics has been experimentally observed for the first time only very recently \[23\].

1.2. Numerical approximation of LLG and wave map equation. This work is concerned with the numerical analysis of (2). As LLG has some similarities with the harmonic map heat flow into the sphere \[22\]

\[
\frac{\partial}{\partial t} u - \Delta u = |\nabla u|^2 u,
\]

it turns out that iLLG is related to the wave map equation into the sphere \[26\]

\[
\frac{\partial}{\partial t} u - \Delta u = (|\nabla u|^2 - |\frac{\partial}{\partial t} u|^2) u.
\]

The numerical approximation of this class of partial differential equations (PDEs) poses several challenges: nonuniqueness, possible blow-up in time, and low regularity of weak solutions, geometric nonlinearities, a nonconvex pointwise unit-length constraint, intrinsic energy laws as well as (for LLG) the possible coupling with other PDEs, e.g., the Maxwell equations.

In the last twenty years, several numerical integrators have been proposed. Without claiming to be exhaustive and, in particular, restricting ourselves to the methods that are akin to the ones proposed in the present work, we refer to the works \[12, 2, 10, 1\] for LLG and to \[9, 11, 6, 21, 8\] for the wave map equation.

1.3. Contributions and outline of the present work. In this work, we introduce, analyze, and numerically compare two fully discrete numerical schemes for iLLG. For both methods, the spatial discretization is based on first-order finite elements. The first scheme (Algorithm 3.1) is an extension of the tangent plane scheme proposed for (1) in \[1\]. The method is based on an equivalent reformulation of (2) in the tangent space. The unit-length constraint is enforced at the vertices of the underlying mesh by projecting to the sphere the nodal values of the computed approximation at each time-step. The second scheme (Algorithm 3.2) is a novel finite element formulation of the constraint-preserving angular momentum method proposed in \[21\] for the wave map equation (3). Since the resulting method leads to the solution of a nonlinear system of equations per time-step, a linearization based on a convergent constraint-preserving fixed-point iteration—similar to the one considered in \[12\] for the method proposed therein, but requiring a weaker Courant–Friedrichs–Lewy (CFL) condition for convergence—is discussed and analyzed (Algorithm 3.4).

For ease of presentation, in the micromagnetic energy functional (4) below, we consider only the leading-order exchange contribution. The numerical treatment of standard lower-order energy contributions (e.g., magnetocrystalline anisotropy, Zeeman energy, magnetostatic energy, Dzyaloshinskii–Moriya interaction) is well understood; see, e.g., \[11, 25, 17, 20\].

We study the well-posedness and the stability of the proposed schemes, and determine sufficient conditions which guarantee that the algorithms satisfy discrete energy laws resembling the one of the continuous problem (see (10) below). Moreover, we prove that the algorithms generate sequences of finite element solutions that, upon extraction of a subsequence, converge
towards a weak solution of the problem. The proof is constructive and, in particular, establishes also existence of weak solutions of \( (2) \).

The remainder of the work is organized as follows: We conclude this section by collecting some useful notation used throughout the paper. In Section 2, we present the mathematical model under consideration in detail. In Section 3, we introduce the proposed numerical schemes and state the main results of the work. Section 4 is devoted to numerical experiments. Finally, in Section 5, we collect the proofs of the results presented in the paper.

1.4. Notation. We denote by \( \mathbb{N} = \{1, 2, \ldots\} \) the set of natural numbers and set \( \mathbb{N}_0 := \mathbb{N} \cup \{0\} \). We denote the unit sphere by \( S^2 = \{x \in \mathbb{R}^3 : |x| = 1\} \). We denote by \( \{e_1, e_2, e_3\} \) the standard basis of \( \mathbb{R}^3 \). For (spaces of) vector-valued or matrix-valued functions, we use bold letters, e.g., for a generic domain \( \Omega \subset \mathbb{R}^d \) \( (d = 2, 3) \), we denote both \( L^2(\Omega; \mathbb{R}^3) \) and \( L^2(\Omega; \mathbb{R}^{3 \times 3}) \) by \( L^2(\Omega) \). We denote by \( \langle \cdot , \cdot \rangle \) both the scalar product of \( L^2(\Omega) \) and the duality pairing between \( H^1(\Omega) \) and its dual, with the ambiguity being resolved by the arguments. The set of sphere-valued functions in \( H^1(\Omega) \) is denoted by \( H^1(\Omega; S^2) \). We also use the notation \( \lesssim \) to denote smaller than or equal to up to a multiplicative constant, i.e., we write \( A \lesssim B \) if there exists a constant \( c > 0 \), which is clear from the context and always independent of the discretization parameters, such that \( A \leq cB \). Finally, we write \( A \simeq B \) if \( A \lesssim B \) and \( B \lesssim A \) hold simultaneously.

2. Mathematical model

Let \( \Omega \subset \mathbb{R}^d \) \( (d = 2, 3) \) be a bounded Lipschitz domain. The energy of \( m \in H^1(\Omega; S^2) \) is described by the Dirichlet energy functional

\[
\mathcal{E}(m) = \frac{1}{2} \| \nabla m \|_{L^2(\Omega)}^2. \tag{4}
\]

Minimizers \( m \in H^1(\Omega; S^2) \) of \( \mathcal{E} \) satisfy the Euler–Lagrange equations

\[
\begin{align*}
\langle h_{\text{eff}}(m), \phi \rangle &= 0 \quad \text{for all } \phi \in H^1(\Omega) \text{ such that } m \cdot \phi = 0 \text{ a.e. in } \Omega,
\end{align*}
\]

which, in strong form, take the form

\[
\begin{align*}
 m \times h_{\text{eff}}(m) &= 0 \quad \text{in } \Omega, \tag{5a} \\
 \partial_n m &= 0 \quad \text{on } \partial \Omega, \tag{5b}
\end{align*}
\]

where \( n : \partial \Omega \to S^2 \) denote the outward-pointing unit normal vector to \( \partial \Omega \). Here, \( h_{\text{eff}}(m) \) is defined as the opposite of the Gâteaux derivative of the energy, i.e.,

\[
- \langle h_{\text{eff}}(m), \phi \rangle = \langle \frac{\delta \mathcal{E}(m)}{\delta m}, \phi \rangle = \langle \nabla m , \nabla \phi \rangle. \tag{6}
\]

Nonequilibrium magnetization configurations \( m(t) \in H^1(\Omega; S^2) \) evolve according to LLG (see \( 1 \)), which in rescaled form reads as

\[
\partial_t m = -m \times (h_{\text{eff}}(m) - \alpha \partial_t m) \quad \text{in } \Omega, \tag{7}
\]

where \( \alpha > 0 \). Note that stationary solutions to \( (7) \) satisfy \( 5a \). A simple formal computation reveals the orthogonality \( m \cdot \partial_t m = 0 \), from which it follows that the dynamics inherently preserves the unit-length constraint. Moreover, taking the scalar product of \( (7) \) with \( h_{\text{eff}}(m) - \alpha \partial_t m \), one can show that any sufficiently smooth solution of \( (7) \) satisfies the energy law

\[
\frac{d}{dt} \mathcal{E}(m(t)) = -\alpha \| \partial_t m(t) \|_{L^2(\Omega)}^2 \leq 0 \quad \text{for all } t > 0. \tag{8}
\]

Hence, the dynamics is dissipative with the dissipation being modulated by the parameter \( \alpha \).

Inertial effects can be included in the model by adding a term on the right-hand side of \( (7) \) (see \( 2 \)). The resulting equation, iLLG, is given by

\[
\partial_t m = -m \times (h_{\text{eff}}(m) - \alpha \partial_t m - \tau \partial_t^2 m) \quad \text{in } \Omega, \tag{9}
\]

where \( \tau > 0 \). Let \( v := \partial_t m \) and \( w := m \times \partial_t m = m \times v \). Using the jargon of kinematics, we refer to these two quantities as linear velocity and angular momentum, respectively. By
construction, \( m, v, \) and \( w \) are mutually orthogonal. Moreover, since \( |m| = 1 \) and \( m \cdot v = 0 \), it holds that \( |v| = |w| \). The same computation leading to (8) yields the energy law of iLLG:

\[
\frac{d}{dt}\left(\mathcal{E}(m(t)) + \frac{\tau}{2} \|\partial_t m(t)\|^2_{L^2(\Omega)}\right) = -\alpha \|\partial_t m(t)\|^2_{L^2(\Omega)} \leq 0 \quad \text{for all } t > 0.
\]

This motivates the definition of the extended energy functional

\[
\mathcal{J}(m, u) = \mathcal{E}(m) + \frac{T}{2} \|u\|^2_{L^2(\Omega)} \quad \text{for all } m \in H^1(\Omega; S^2) \text{ and } u \in L^2(\Omega).
\]

With this definition, the quantity decaying over time in the dynamics governed by (9) becomes

\[
\text{total energy}
\]

\[
\text{kinetic energy}
\]

\[
\text{potential energy}
\]

The initial boundary value problem considered in this work consists of (9) supplemented with homogeneous Neumann boundary conditions (5b), which make the dynamic problem compatible with the stationary case [5], and suitable initial conditions \( m(0) = m^0 \) and \( \partial_t m(0) = v^0 \).

We conclude this section by presenting the definition of a weak solution of (9), which is obtained by extending [4] Definition 1.2 to the present setting; see also [19].

**Definition 2.1.** Let \( m^0 \in H^1(\Omega; S^2) \) and \( v^0 \in L^2(\Omega) \) such that \( m^0 \cdot v^0 = 0 \) a.e. in \( \Omega \). A vector field \( m : \Omega \times (0, \infty) \to S^2 \) is called a global weak solution of iLLG [4] if \( m \in L^\infty(0, \infty; H^1(\Omega)) \cap W^{1,\infty}(0, \infty; L^2(\Omega)) \) and, for all \( T > 0 \), the following properties are satisfied:

(i) \( m \in H^1(\Omega_T) \), where \( \Omega_T := \Omega \times (0, T) \);

(ii) \( m(t) \to m^0 \) in \( H^1(\Omega) \) and \( \partial_t m(t) \to v^0 \) in \( L^2(\Omega) \) as \( t \to 0 \);

(iii) For all \( \varphi \in C^\infty_c([0, T); H^1(\Omega)) \), it holds that

\[
\int_0^T \langle \partial_t m(t), \varphi(t) \rangle \, dt
\]

\[
= -\int_0^T \langle h_{\text{eff}}(m(t)), \varphi(t) \times m(t) \rangle \, dt + \alpha \int_0^T \langle m(t) \times \partial_t m(t), \varphi(t) \rangle \, dt
\]

\[
- \tau \int_0^T \langle m(t) \times \partial_t m(t), \partial_t \varphi(t) \rangle \, dt - \tau \langle m^0 \times v^0, \varphi(0) \rangle;
\]

(iv) It holds that

\[
\mathcal{J}(m(T), \partial_t m(T)) + \alpha \int_0^T \|\partial_t m(t)\|^2_{L^2(\Omega)} \, dt \leq \mathcal{J}(m^0, v^0).
\]

In Definition 2.1 (12) comes from a variational formulation of [9] in the space-time cylinder \( \Omega_T \), where we integrate by parts in time the inertial contribution in order to lower the requested regularity in time of \( m \). The energy inequality (13) is the weak counterpart of (10).

**Remark 2.2.** The setting discussed in this section can be obtained from the original equations expressed in physical units after a suitable rescaling. Let \( t \) and \( x \) denote the time and spatial variables (measured in \( s \) and \( m \), respectively). First, we perform the change of variables \( t' = \gamma_0 M_s t \) and \( x' = x/\ell_{\text{ex}} \), where \( M_s > 0 \) and \( \ell_{\text{ex}} > 0 \) denote the saturation magnetization (in \( \Lambda/m \)) and the exchange length (in \( m \)) of the material, respectively. Second, we rescale the energy \( \mathcal{E}(m) \) (in J), the effective field \( H_{\text{eff}}(m) \) (in \( \Lambda/m \)), and the angular momentum relaxation time \( \tau \) (in s) in the following way: \( \mathcal{E}'(m) = \mathcal{E}(m)/\mu_0 M_s^2 \ell_{\text{ex}} \), \( h_{\text{eff}}(m) = H_{\text{eff}}(m)/M_s \), \( \tau' = \gamma_0 M_s \tau \). Then, using the chain rule [2], can be rewritten as [4], where all ‘primes’ are omitted from the rescaled quantities in order to simplify the notation.

3. Numerical algorithms and main results

In this section, we introduce two fully discrete algorithms for the numerical approximation of iLLG and we state the corresponding stability and convergence results.
3.1. Preliminaries. For the time discretization, we consider a uniform partition of the positive real axis \((0, \infty)\) with time-step size \(k > 0\), i.e., \(t_i := ik\) for all \(i \in \mathbb{N}_0\). Given a sequence \(\{\phi^t\}_{t \in \mathbb{N}_0}\), for all \(i \in \mathbb{N}_0\), we define \(d_t \phi^{i+1} := (\phi^{i+1} - \phi^i)/k\) and \(\phi^{i+1/2} := (\phi^{i+1} + \phi^i)/2\). Interpreting the sequence \(\{\phi^t\}_{t \in \mathbb{N}_0}\) as a collection of snapshots of a time-dependent function, we consider the time reconstructions \(\phi_k, \phi_k^-, \phi_k^+\) defined, for all \(i \in \mathbb{N}_0\) and \(t \in [t_i, t_{i+1})\), as
\[
\phi_k(t) := \frac{t - t_i}{k} \phi^{i+1} + \frac{t_i + 1 - t}{k} \phi^i, \quad \phi_k^-(t) := \phi^i, \quad \phi_k^+(t) := \phi^{i+1/2}, \quad \text{and} \quad \phi_k^{i+1}(t) := \phi^{i+1}. \tag{14}
\]
Note that \(\phi_k \phi^k(t) = d_t \phi^{i+1}\) for all \(i \in \mathbb{N}_0\) and \(t \in [t_i, t_{i+1})\).

For the spatial discretization, we assume \(\Omega\) to be a polytopal domain with Lipschitz boundary and consider a \(\kappa\)-quasi-uniform family \(\{T_h\}_{h>0}\) of regular tetrahedral meshes of \(\Omega\) \((\kappa \geq 1)\) parametrized by the mesh size \(h = \max_{K \in T_h} \text{diam}(K)\), i.e., \(\kappa^{-1} h \leq |K|^{1/3}\) for all \(K \in T_h\). We denote by \(N_h\) the set of vertices of \(T_h\). For any \(K \in T_h\), let \(\mathcal{P}^1(K)\) be the space of linear polynomials on \(K\). We denote by \(\mathcal{S}^1(T_h)\) the space of piecewise affine and globally continuous functions from \(\Omega\) to \(\mathbb{R}\), i.e., \(\mathcal{S}^1(T_h) = \{v_h \in C^0(\bar{\Omega}) : v_h|_K \in \mathcal{P}^1(K)\} \text{ for all } K \in T_h\). The classical basis for this finite-dimensional linear space is given by the set of the nodal hat functions \(\{\varphi_z\}_{z \in N_h}\), which satisfy \(\varphi_z(z') = \delta_{z, z'}\) for all \(z, z' \in N_h\). Let \(I_h : C^0(\bar{\Omega}) \rightarrow \mathcal{S}^1(T_h)\) denote the nodal interpolant defined by \(I_h[v] = \sum_{z \in N_h} v(z) \varphi_z\) for all \(v \in C^0(\bar{\Omega})\). We denote by \(I_h[\cdot]\) its vector-valued counterpart. We consider the mass-lumped product \(\langle \cdot, \cdot \rangle_h\) defined by
\[
\langle \psi, \phi \rangle_h = \int_{\Omega} I_h[\psi] \cdot \phi \quad \text{for all } \psi, \phi \in C^0(\bar{\Omega}). \tag{15}
\]
Moreover, we define the mapping \(P_h : \mathcal{H}^1(\Omega)^* \rightarrow \mathcal{S}^1(T_h)^3\) by
\[
(P_h u, \phi_h) = \langle u, \phi_h \rangle \quad \text{for all } u \in \mathcal{H}^1(\Omega)^* \text{ and } \phi_h \in \mathcal{S}^1(T_h)^3. \tag{16}
\]
Finally, we say that a mesh satisfies the angle condition if all off-diagonal entries of the so-called stiffness matrix are nonpositive, i.e.,
\[
\langle \nabla \varphi_z, \nabla \varphi_{z'} \rangle \leq 0 \quad \text{for all } z, z' \in N_h \text{ with } z \neq z'. \tag{17}
\]
A sufficient condition for \(\text{(17)}\) to hold in 3D is that the measure of all dihedral angles of all tetrahedra of the mesh is smaller than or equal to \(\pi/2\) [3].

3.2. Numerical algorithms. In the following algorithms, the main identities satisfied by any solution \(m\) of LLG, i.e., \(|m| = 1\) and \(m \cdot \partial_t m = 0\), are imposed only at the vertices of the mesh \(T_h\). To this end, we define the set of admissible discrete magnetizations
\[
\mathcal{M}_h := \{\phi_h \in \mathcal{S}^1(T_h)^3 : |\phi_h(z)| = 1 \text{ for all } z \in N_h\}
\]
and, for \(\psi_h \in \mathcal{S}^1(T_h)^3\), the discrete tangent space of \(\psi_h\)
\[
\mathcal{K}_h(\psi_h) := \{\phi_h \in \mathcal{S}^1(T_h)^3 : \psi_h(z) \cdot \phi_h(z) = 0 \text{ for all } z \in N_h\}. \tag{18}
\]
The discrete counterpart of the functional \(\text{(11)}\) is defined by
\[
J_h(m_h, u_h) = E(m_h) + \frac{\tau}{2} \|u_h\|^2_h \quad \text{for all } m_h, u_h \in \mathcal{S}^1(T_h)^3. \tag{19}
\]
3.2.1. Tangent plane scheme. The first method uses the linear velocity \(v = \partial_t m\) as an auxiliary variable. Using the vector identity
\[
a \times (b \times c) = (a \cdot c)b - (a \cdot b)c \quad \text{for all } a, b, c \in \mathbb{R}^3, \tag{20}
\]
\(\text{together with the properties } |m| = 1 \text{ and } m \cdot v = 0\), iLLG can be formally rewritten as
\[
\tau \partial_t v + v \times m + m \times v = h_{\text{eff}}(m) - |h_{\text{eff}}(m) \cdot m| m - \tau |v|^2 m. \tag{21}
\]
Following the tangent plane paradigm \([2, 10, 11, 23]\), to obtain a numerical scheme for iLLG, we consider a finite element approximation of a mass-lumped variational formulation of \(\text{(20)}\) based on test functions fulfilling the same orthogonality property satisfied by \(v\). This yields a natural linearization of \(\text{(20)}\), as the contributions associated with last two (nonlinear) terms on the right-hand side vanish by orthogonality. More precisely, for all time-steps \(i \in \mathbb{N}_0\), given the current
approximations \( \bm m_h^i \approx \bm m(t_i) \) and \( \bv_h^i \approx \bv(t_i) \), we compute \( \bv_h^{i+1} \approx \bv(t_{i+1}) \) using a discretized version of \((20)\), which is based on the discrete tangent space \( \mathcal K_h(\bm m_h^i) \) introduced in \((18)\) for the spatial discretization and on the implicit Euler method for the temporal discretization. Then, using the available approximations \( \bm m_h^i \) and \( \bv_h^{i+1} \), we obtain \( \bm m_h^{i+1} \approx \bm m(t_{i+1}) \) via a first-order time-stepping, i.e., \( \mathcal I_h\left[ (\bm m_h^i + k\bv_h^{i+1})/(\bm m_h^i + k\bv_h^{i+1}) \right] \), where the nodal projection is employed to ensure that the new approximation belongs to \( \mathcal M_h \). Unlike \([2, 1, 3]\) and as in \([10]\), we use the mass-lumped product \( \langle \cdot, \cdot \rangle_h \), which enhances the efficiency of the scheme without affecting its formal convergence order. The resulting scheme is summarized in the following algorithm.

**Algorithm 3.1** (tangent plane scheme). Input: \( \bm m_h^0 \in \mathcal M_h \) and \( \bv_h^0 \in \mathcal K_h(\bm m_h^0) \). Loop: For all \( i \in \mathbb N_0 \), iterate (i)–(ii):

(i) Compute \( \bv_h^{i+1} \in \mathcal K_h(\bm m_h^i) \) such that, for all \( \phi_h \in \mathcal K_h(\bm m_h^i) \), it holds that

\[
\tau \langle d_t \bv_h^{i+1}, \phi_h \rangle_h + \alpha \langle \bv_h^{i+1}, \phi_h \rangle_h + \langle \bm m_h^i \times \bv_h^{i+1}, \phi_h \rangle_h - k \langle \mathcal P_h \bm h_{\text{eff}}(\bv_h^{i+1}), \phi_h \rangle_h = 0.
\]

(ii) Define \( \bm m_h^{i+1} = \mathcal I_h\left[ (\bm m_h^i + k\bv_h^{i+1})/(\bm m_h^i + k\bv_h^{i+1}) \right] \in \mathcal M_h \).

Output: Sequence of approximations \( \{ (\bm m_h^{i+1}, \bv_h^{i+1}) \}_{i \in \mathbb N_0} \).

Algorithm 3.1 is well-defined: The (nonsymmetric) bilinear form on the left-hand side of \((21)\) in step (i) is elliptic, so that existence and uniqueness of a solution \( \bv_h^{i+1} \in \mathcal K_h(\bm m_h^i) \) are guaranteed by the Lax–Milgram theorem; In the nodal projection appearing in step (ii), the denominator is bounded from below by 1, so that division by zero never occurs.

### 3.2.2. Angular momentum method.

The second method, following \([21]\), uses the angular momentum \( \bm w = \bm m \times \partial_t \bm m \) as an auxiliary variable. On the one hand, it holds that

\[
\bm m \times \bm w = \bm m \times (\bm m \times \partial_t \bm m) = -\partial_t \bm m,
\]

where the second identity follows from \((19)\), together with \(|\bm m| = 1\) and \( \bm m \cdot \partial_t \bm m = 0 \). On the other hand, it holds that

\[
\tau \partial_t \bm m = \tau \bm m \times \partial_t \bm m = \tau \bm m \times \bm h_{\text{eff}}(\bm m) - \alpha \bm m \times \partial_t \bm m + \partial_t \bm m.
\]

We infer the first-order (in time) system

\[
\partial_t \bm m = -\bm m \times \bm w,
\]

\[
\tau \partial_t \bm m = \bm m \times \bm h_{\text{eff}}(\bm m) - \alpha \bm m \times \partial_t \bm m - \bm m \times \bm w.
\]

Our second numerical scheme for iLLG, stated in the following algorithm, consists in a fully discrete mass-lumped variational formulation of this first-order system, where the time discretization is based on the implicit midpoint rule.

**Algorithm 3.2** (nonlinear angular momentum method). Input: \( \bm m_h^0 \in \mathcal M_h \) and \( \bv_h^0 \in \mathcal K_h(\bm m_h^0) \). Initialization: Define \( \bm w_h^0 = \mathcal I_h(\bm m_h^0 \times \bv_h^0) \in \mathcal K_h(\bm m_h^0) \).

Loop: For all \( i \in \mathbb N_0 \), compute \( \bm m_h^{i+1}, \bv_h^{i+1} \) in \( \mathcal M_h \times \mathcal K_h(\bm m_h^{i+1}) \) such that, for all \( (\phi_h, \psi_h) \in S^1(\mathcal T_h)^3 \times S^1(\mathcal T_h)^3 \), it holds that

\[
\langle d_t \bm m_h^{i+1}, \phi_h \rangle_h = -\langle \bm m_h^{i+1/2} \times \bv_h^{i+1/2}, \phi_h \rangle_h,
\]

\[
\tau \langle d_t \bv_h^{i+1}, \psi_h \rangle_h = \langle \bm m_h^{i+1/2} \times \mathcal P_h \bm h_{\text{eff}}(\bm m_h^{i+1/2}), \psi_h \rangle_h - \alpha \langle \bm m_h^{i+1/2} \times d_t \bm m_h^{i+1}, \psi_h \rangle_h - \langle \bm m_h^{i+1/2} \times \bv_h^{i+1/2}, \psi_h \rangle_h.
\]

Output: Sequence of approximations \( \{ (\bm m_h^{i+1}, \bv_h^{i+1}) \}_{i \in \mathbb N_0} \).

In the following proposition, we show that the pointwise constraints \(|\bm m| = 1\) and \( \bm m \cdot \bm w = 0 \) are inherently preserved by Algorithm 3.2 (at the vertices of the mesh). Its proof is deferred to Section 5.1.
Proposition 3.5. Let \( i \in \mathbb{N}_0 \). The approximations generated by Algorithm 3.2 satisfy \( \mathbf{m}^{i+1}_h \in \mathcal{M}_h \) and \( \mathbf{w}^{i+1}_h \in \mathcal{K}_h(\mathbf{m}^{i+1}_h) \).

The computation of \( (\mathbf{m}^{i+1}_h, \mathbf{w}^{i+1}_h) \) satisfying (22) involves the solution of a nonlinear system of equations. An effective implementation requires a linearization.

Let \( \mathbf{u}^i_h := \mathbf{m}^{i+1/2}_h \). Simple algebraic manipulations reveal that \( d_i \mathbf{m}^{i+1}_h = 2(\mathbf{u}^i_h - \mathbf{m}^i_h)/k \) and \( \mathbf{m}^{i+1}_h = 2\mathbf{u}^i_h - \mathbf{m}^i_h \). Similarly, we define \( \mathbf{z}^i_h := \mathbf{w}^{i+1/2}_h \), from which we obtain the identities \( d_i \mathbf{w}^{i+1}_h = 2(\mathbf{z}^i_h - \mathbf{w}^i_h)/k \) and \( \mathbf{w}^{i+1}_h = 2\mathbf{z}^i_h - \mathbf{w}^i_h \). Then, we rewrite (22) with respect to the unknowns \( \mathbf{u}^i_h \) and \( \mathbf{z}^i_h \):

\[
2\langle \mathbf{u}^i_h, \phi_h \rangle_h + k\langle \mathbf{u}^i_h \times \mathbf{z}^i_h, \phi_h \rangle_h = 2\langle \mathbf{m}^i_h, \phi_h \rangle_h, \quad (23a)
\]

\[
2\tau \langle \mathbf{z}^i_h, \psi_h \rangle_h - k\langle \mathbf{u}^i_h \times \mathbf{z}^i_h, \psi_h \rangle_h - 2\alpha \langle \mathbf{w}^i_h \times \mathbf{m}^i_h, \psi_h \rangle_h = 2\tau \langle \mathbf{w}^i_h, \psi_h \rangle_h. \quad (23b)
\]

Starting from this formulation, we introduce a linear fixed-point iteration, which provides an effective implementation of Algorithm 3.2.

Algorithm 3.4 (linearized angular momentum method). Input: \( \mathbf{m}^0_h \in \mathcal{M}_h \) and \( \mathbf{w}^0_h \in \mathcal{K}_h(\mathbf{m}^0_h) \).

Initialization: Define \( \mathbf{w}^0_h = \mathbf{T}_h(\mathbf{m}^0_h \times \mathbf{v}^0_i) \in \mathcal{K}_h(\mathbf{m}^0_h) \).

Loop: For all \( i \in \mathbb{N}_0 \), iterate (i)–(ii):

(i) Let \( \mathbf{u}^0_h = \mathbf{m}^i_h \) and \( \mathbf{z}^0_h = \mathbf{w}^i_h \). For all \( \ell \in \mathbb{N}_0 \), iterate (i-a)–(i-b):

(i-a) Compute \( \mathbf{u}^{i,\ell+1}_h \in S^1(\mathcal{T}_h)^3 \) such that, for all \( \phi_h \in S^1(\mathcal{T}_h)^3 \), it holds that

\[
2\langle \mathbf{u}^{i,\ell+1}_h, \phi_h \rangle_h + k\langle \mathbf{u}^{i,\ell+1}_h \times \mathbf{z}^{i,\ell+1}_h, \phi_h \rangle_h = 2\langle \mathbf{m}_h^{i,\ell+1}, \phi_h \rangle_h. \quad (24)
\]

(i-b) Compute \( \mathbf{z}^{i,\ell+1}_h \in S^1(\mathcal{T}_h)^3 \) such that, for all \( \psi_h \in S^1(\mathcal{T}_h)^3 \), it holds that

\[
2\tau \langle \mathbf{z}^{i,\ell+1}_h, \psi_h \rangle_h + k\langle \mathbf{u}^{i,\ell+1}_h \times \mathbf{z}^{i,\ell+1}_h, \psi_h \rangle_h = k\langle \mathbf{u}^{i,\ell+1}_h \times \mathbf{p}_h^{i,\ell+1}_h, \psi_h \rangle_h + 2\alpha \langle \mathbf{m}_h^{i,\ell+1}, \psi_h \rangle_h + 2\tau \langle \mathbf{w}^i_h, \psi_h \rangle_h; \quad (25)
\]

until

\[
\|\mathbf{u}^{i,\ell+1}_h - \mathbf{u}^{i,\ell}_h\|_h + \|\mathbf{z}^{i,\ell+1}_h - \mathbf{z}^{i,\ell}_h\|_h \leq \varepsilon. \quad (26)
\]

(ii) Let \( \ell^* \in \mathbb{N}_0 \) be the first index for which the stopping criterion (26) is met. Define \( \mathbf{m}^{i+1}_h := 2\mathbf{u}^{i,\ell^*+1}_h - \mathbf{m}^i_h \) and \( \mathbf{w}^{i+1}_h := 2\mathbf{z}^{i,\ell^*+1}_h - \mathbf{w}^i_h \).

Output: Sequence of approximations \( \{ (\mathbf{m}^{i+1}_h, \mathbf{w}^{i+1}_h) \}_{i \in \mathbb{N}_0} \).

The well-posedness and the conservation properties of Algorithm 3.4 are the subject of the following proposition. Its proof is postponed to Section 5.1.

Proposition 3.5. Let \( i \in \mathbb{N}_0 \). Suppose that \( \mathbf{m}^i_h \in \mathcal{M}_h \).

(i) For all \( \ell \in \mathbb{N}_0 \), (24) and (25) admit unique solutions \( \mathbf{u}^{i,\ell+1}_h \) and \( \mathbf{z}^{i,\ell+1}_h \) in \( S^1(\mathcal{T}_h)^3 \). Moreover, it holds that \( \|\mathbf{u}^{i,\ell+1}_h\|_{L^\infty(\Omega)} \leq 1 \).

(ii) There exist \( \delta > 0 \), \( h_0 > 0 \), and \( 0 < \eta < 1 \) such that, if \( k = \delta h \) and \( h < h_0 \), it holds that

\[
\|\mathbf{u}^{i,\ell+2}_h - \mathbf{u}^{i,\ell+1}_h\|_h + \|\mathbf{z}^{i,\ell+2}_h - \mathbf{z}^{i,\ell+1}_h\|_h \leq \eta (\|\mathbf{u}^{i,\ell+1}_h - \mathbf{u}^{i,\ell}_h\|_h + \|\mathbf{z}^{i,\ell+1}_h - \mathbf{z}^{i,\ell}_h\|_h) \quad \text{for all } \ell \in \mathbb{N}_0. \quad (27)
\]

The constants \( \delta, h_0, \eta \) depend only on the mesh constant \( \kappa \) and the problem data.

(iii) Under the assumptions of part (ii), the stopping criterion (26) is met in a finite number of iterations. If \( \ell^* \in \mathbb{N}_0 \) denotes the first index for which (26) holds, the new approximations \( \mathbf{m}^{i+1}_h = 2\mathbf{u}^{i,\ell^*+1}_h - \mathbf{m}^i_h \) and \( \mathbf{w}^{i+1}_h = 2\mathbf{z}^{i,\ell^*+1}_h - \mathbf{w}^i_h \) belong to \( \mathcal{M}_h \) and \( \mathcal{K}_h(\mathbf{m}^{i+1}_h) \), respectively.

Proposition 3.5(ii) shows that, under suitable assumptions, the mapping defining the fixed-point iteration is a contraction. Therefore, under the same assumptions, Banach fixed-point theorem ensures that (22) admits a unique solution so that Algorithm 3.2 is well-posed.
In view of the stability and convergence analysis, we observe that, for all \( i \in \mathbb{N}_0 \), the iterates \((m_h^{i+1}, w_h^{i+1}) \in M_h \times K_h(m^{i+1}_h)\) of Algorithm 3.4 satisfy
\[
(d_t m_h^{i+1} + \phi_h)_h = -(m_h^{i+1/2} \times w_h^{i+1/2} + \phi_h)_h + \langle m_h^{i+1/2} \times r^i_h, \phi_h \rangle_h,
\]
\[
\tau(d_t w_h^{i+1} + \psi_h)_h = \langle m_h^{i+1/2} \times \mathbb{P}_h h_{\text{eff}}(m_h^{i+1/2}), \psi_h \rangle_h - \alpha \langle m_h^{i+1/2} \times d_t m_h^{i+1}, \psi_h \rangle_h
\]
for all \((\phi_h, \psi_h) \in S^1(T_h)^3 \times S^1(T_h)^3\), where \( r_h^i := z_{h}^{i+1/2} - z_{h}^{i} \in S^1(T_h)^3\) satisfies \(\|r_h^i\|_h \leq \varepsilon\).

3.3. Stability and convergence results. In the following proposition, we establish the discrete energy laws satisfied by the algorithms. Its proof is postponed to Section 5.2.

**Proposition 3.6 (Discrete energy law and stability).** Let \( i \in \mathbb{N}_0 \).

(i) Suppose that the angle condition (17) is satisfied. The approximations generated by Algorithm 3.1 satisfy the discrete energy law
\[
J_h(m_h^{i+1}, v_h^{i+1}) + \alpha k \|v_h^{i+1}\|_h^2 + \frac{\tau k^2}{2} \|d_t v_h^{i+1}\|_h^2 + \frac{k^2}{2} \|\nabla v_h^{i+1}\|_{L^2(\Omega)}^2 \leq J_h(m_h^i, v_h^i). \tag{28}
\]

(ii) The approximations generated by Algorithm 3.2 satisfy the discrete energy law
\[
J_h(m_h^{i+1}, w_h^{i+1}) + \alpha k \|d_t m_h^{i+1}\|_h^2 = J_h(m_h^i, w_h^i). \tag{29}
\]

(iii) The approximations generated by Algorithm 3.4 satisfy the discrete energy law
\[
J_h(m_h^{i+1}, w_h^{i+1}) + \alpha k \|d_t m_h^{i+1}\|_h^2 = k \|m_h^{i+1/2} \times r_h^i, \mathbb{P}_h h_{\text{eff}}(m_h^{i+1/2}) - \alpha d_t m_h^{i+1/2} \rangle_h
\]
\[
= J_h(m_h^i, w_h^i). \tag{30}
\]

In the energy law satisfied by Algorithm 3.1 besides the LLG-intrinsic dissipation, we observe the presence of numerical dissipation due to the use of the implicit Euler method; cf. the last two terms on the left-hand side of (28). Algorithm 3.2 fulfills a discrete energy identity, which reflects the fact that the midpoint rule is symplectic. The same identity, apart from an additional term coming from the midpoint dissipation of the nonlinear system, is satisfied by Algorithm 3.4.

From each algorithm, we obtain a sequence of approximations \(\{m_{hk}^i\}_{i \in \mathbb{N}_0}\), which we can use to define the piecewise affine time reconstruction \(m_{hk} : (0, \infty) \rightarrow S^1(T_h)^3\) (denoted by \(m_{hk}^i\) in the case of Algorithm 3.4) as
\[
m_{hk}(t) := \frac{t - t_i}{k} m_h^{i+1} + \frac{t_{i+1} - t_i}{k} m_h^i \quad \text{for all} \quad i \in \mathbb{N}_0 \quad \text{and} \quad t \in [t_i, t_{i+1});
\]
see (14). In the following theorem, we show that, under appropriate assumptions, the sequence \(\{m_{hk}^i\}\) (resp., \(\{m_{hk}^i\}\) for Algorithm 3.4) converges in a suitable sense towards solutions of iLLG as \(h, k \) (and \(\varepsilon\)) go to 0. Its proof is postponed to Section 5.3.

**Theorem 3.7.** Let the approximate initial conditions satisfy
\[
m_h^0 \rightarrow m^0 \quad \text{in} \quad H^1(\Omega) \quad \text{and} \quad v_h^0 \rightarrow v^0 \quad \text{in} \quad L^2(\Omega) \quad \text{as} \quad h \rightarrow 0. \tag{31}
\]

(i) For Algorithm 3.1, assume that the angle condition (17) is satisfied and that \(k = o(h^{3/2})\) as \(h, k \rightarrow 0\). For Algorithm 3.2, assume that the scheme is well-posed. Then, there exist a global weak solution \(m : \Omega \times (0, \infty) \rightarrow S^2\) of iLLG in the sense of Definition 2.1 and a subsequence of \(\{m_{hk}^i\}\) (not relabeled) which converges towards \(m\) as \(h, k \rightarrow 0\). In particular, as \(h, k \rightarrow 0\), it holds that \(m_{hk}^i \rightharpoonup m\) in \(L^\infty(0, \infty; H^1(\Omega; S^2))\) and \(m_{hk}^i|_{\Omega_T} \rightarrow m|_{\Omega_T} \) in \(H^1(\Omega_T)\) for all \(T > 0\).

(ii) For Algorithm 3.4, assume that the scheme is well-posed and that \(\varepsilon = O(h)\) as \(h, \varepsilon \rightarrow 0\). For all \(T > 0\), there exist \(m : \Omega \times (0, T) \rightarrow S^2\), which satisfies the requirements (i)-(iv) of Definition 2.1 and a subsequence of \(\{m_{hk}^i\}\) (not relabeled) such that \(m_{hk}^i|_{\Omega_T} \rightharpoonup m \) in \(L^\infty(0, T; H^1(\Omega; S^2))\) and \(m_{hk}^i|_{\Omega_T} \rightarrow m \) in \(H^1(\Omega_T)\) as \(h, k, \varepsilon \rightarrow 0\).
In order to refer to the limit of the approximations generated by Algorithm 3.4, we have not use the expression ‘global weak solution’. This is a consequence of the fact that, for this algorithm, the boundedness result we are able to show (see Proposition 5.1 below) is not uniform with respect to the (arbitrary but fixed) final time $T > 0$.

To conclude, we summarize the results of our analysis for the proposed algorithms:

- Algorithm 3.1 is unconditionally well-posed, unconditionally stable (under the angle condition (17)), and its convergence towards a global weak solution of iLLG requires the CFL condition $k = o(h^{4/2})$ as $h, k \to 0$.
- Algorithm 3.2 is well-posed if $k = \delta h$ and $h$ is sufficiently small. Assuming its well-posedness, it is unconditionally stable and unconditionally convergent towards a global weak solution of iLLG.
- Algorithm 3.3 is well-posed if $k = \delta h$ and $h$ is sufficiently small. Assuming its well-posedness and choosing a stopping tolerance $\varepsilon$ having the same order of $h$, for all $T > 0$, it is unconditionally stable and unconditionally convergent towards a function $m : \Omega \times (0, T) \to \mathbb{S}^2$ which fulfills the properties (i)–(iv) of Definition 2.1.

4. Numerical results

Before presenting the proof of the results stated in Section 3, we aim to show the effectivity of the proposed algorithms by means of two numerical experiments. For the sake of brevity, in this section, we refer to the tangent plane scheme (Algorithm 3.1) as TPS and to the angular momentum method (Algorithm 3.2 or, more appropriately, its effective realization given in Algorithm 3.3) as AMM. The computation presented in this section were obtained with a MATLAB implementation of the proposed algorithms. All linear systems were solved using the direct solver provided by MATLAB’s backslash operator.

4.1. Finite-time blow-up of weak solutions. In this subsection, we aim to investigate the performance of the algorithms for different choices of the discretization parameters $h$ and $k$ (and $\varepsilon$ for AMM). At the same time, we aim to numerically study for a weak solution $m$ of iLLG the occurrence of a so-called finite-time blow-up, i.e., whether there exists $T^* > 0$ such that

$$\lim_{t \to T^*} \|\nabla m(t)\|_{L^\infty(\Omega)} = \infty.$$  

To this end, we adapt to iLLG the model problem studied in [9, 21] for the wave map equation and in [10] for LLG.

We consider the nondimensional setting presented in Section 2 for the unit square domain $\Omega = (-1/2, 1/2)^2$ in the time interval $(0,2)$. We set $\alpha = \tau = 1$ in [9] and consider the initial conditions $m^0, v^0 : \Omega \to \mathbb{S}^2$ defined as $m^0(x) = (2a(x)x_1, 2a(x)x_1, a(x)^2 - |x|^2)/(a(x)^2 + |x|^2)$ with $a(x) = \max\{0, (1 - 2|x|^2)^4\}$ for all $x = (x_1, x_2) \in \Omega$ and $v^0 \equiv 0$. For the wave map equation [9] and LLG [7], this setting leads to numerical approximations with large gradients, which suggests the occurrence of a finite-time blow-up. For snapshots of numerical approximations which illustrate this phenomenon, we refer to, e.g., [9, Figure 3–4] or [10, Figures 1–2].

First, we investigate the convergence of the fixed-point iteration in AMM analyzed in Proposition 3.4(ii). For $\ell = 5, 6, 7$, we consider a uniform mesh $\mathcal{T}_{h_{\ell}}$ of the unit square consisting of $2^{2\ell+1}$ rectangular triangles. The resulting mesh size is $h_{\ell} = \sqrt{2}^{-\ell}$. A plot of the mesh for $\ell = 5$ is given in Figure 5(a) below. In the stopping criterion (26), in order to better evaluate the convergence of the fixed-point iteration, we use the small tolerance $\varepsilon = 1 \cdot 10^{-12}$. The iteration is terminated either when the stopping criterion (26) is met or when the number of iterations exceeds 1000. We use the time-step size $k_{\ell} = \delta h_{\ell}$ for different values of $0 < \delta < 1$.

In Table [4] we show the average number of fixed-point iterations needed to reach the prescribed tolerance $\varepsilon$ for $\delta = 0.1, 0.2, 0.3, 0.4$. The number of iterations increases as $\delta$ increases and decreases (very slightly) as the mesh size decreases. For all $\ell = 5, 6, 7$, the fixed-point iteration does not converge (within the prescribed maximum number of iterations) if $\delta$ is larger than a threshold value located between 0.46 and 0.47. This behavior is in agreement with the
dependence of the contraction constant on the discretization parameters which can be inferred from the proof of Proposition 3.5(ii), i.e., \( q \approx \delta(1 + h) \) (recall that the tolerance \( \varepsilon \) is fixed).

Next, we compare the performance of TPS and AMM. We consider the uniform mesh \( \mathcal{T}_{h_5} \) (2048 elements and mesh size \( h_5 = 0.0442 \)) and \( k = h_5/10 \). For AMM, we set \( \varepsilon = h_5/10 \) in (26).

In Figure 2, we plot the evolutions of the spatial average of the third magnetization component, i.e., \( \langle m_3(t) \rangle := |\Omega|^{-1} \int_\Omega m_{hk}(t) \cdot e_3 \), the \( W^{1,\infty} \)-seminorm \( \| \nabla m_{hk}(t) \|_{L^\infty(\Omega)} \), and the total discrete energy \( J_h(m_{hk}(t), \partial_t m_{hk}(t)) \) for \( t \in [0, 2] \). We observe that the algorithms capture the same average magnetization dynamics. In particular, at \( t \approx 0.3 \), the approximations attain the largest possible value of the \( W^{1,\infty} \)-seminorm for functions in \( \mathcal{M}_h \) residing in \( \mathcal{T}_{h_5} \), which, for all \( \ell = 5, 6, 7 \), is given by

\[
\max_{\phi_h \in \mathcal{M}_h} \| \nabla \phi_h \|_{L^\infty(\Omega)} = \max_{\phi_h \in \mathcal{M}_h} \max_{T \in \mathcal{T}_{h_5}} \| \nabla \phi_h | T \| = 2/2^{-\ell} = 2^{\ell+1},
\]

which is obtained when the magnetizations of two neighboring vertices point to opposite directions. Indeed, in our case, the magnetization at \((0, 0)\) points to the out-of-plane direction \((1, 0, 0)\), while all surrounding vectors point to the opposite direction. This configuration lasts for some time (see the ‘plateau’ in Figure 2(b)). Then, at \( t \approx 0.8 \), the magnetization at \((0,0)\) is reversed. This gives rise to oscillations of decaying amplitude. Looking at Figure 2(c), we observe that, in agreement with (28), the total energy decays monotonically in the case of TPS. In the case of AMM, the decay is nonmonotone. Note that possible lack of monotonicity is predicted by the energy law of AMM; cf. the (unsigned) third term on the left-hand side (30). Moreover, we see that the curve for TPS is well below the one of AMM. This fact can be related to the numerical dissipation, usually referred to as artificial damping, of the implicit Euler method; cf. the second and the third terms on the left-hand side of (28).

In Figure 3 we show the results obtained repeating the experiment using the same mesh \( \mathcal{T}_{h_5} \), but smaller time-step size \( k = h_5/100 \) and tolerance \( \varepsilon = h_5/100 \). The overall behavior remains the same. However, we see that the numerical dissipation of TPS and the nonmonotonicity of the energy decay of AMM are reduced. This observation confirms the validity of the energy laws established in Proposition 3.6 as the terms responsible for the two above effects can indeed be controlled by time-step size \( k \) and the tolerance \( \varepsilon \), respectively.
Finally, we investigate whether the resolution of the mesh or the use of symmetric meshes have an influence on the detection of the blow-up. In Figure 4(a), we compare the evolution of the $W^{1,\infty}$-seminorm obtained using the uniform meshes $T_{h\ell}$ for $\ell = 5, 6, 7$ ($k = h\ell/100$ and $\varepsilon = h\ell/100$). The appearance of blow-up of the three approximations occurs at the same time ($t \approx 0.3$), but the length of the plateaux, i.e., the duration of the configuration in which the magnetization of the origin has the opposite direction of the surrounding vertices, decreases with the mesh size. Moreover, for all $\ell = 5, 6, 7$, the maximum value attained by the $W^{1,\infty}$-seminorm is always the maximum value (32) allowed by the discrete space.

In Figure 4(b), we compare the evolution of the $W^{1,\infty}$-seminorm computed using $T_{h\ell}$ with the one obtained using an unstructured mesh $T_h$ of comparable number of elements and mesh size; see Figure 5(b). Note that the origin is not a vertex of $T_h$. We observe that a finite-time blow-up at $t \approx 0.3$ occurs also for the approximation residing in the unstructured mesh $T_h$. However, the magnetization configuration with maximum gradient is quickly left (no plateau in the evolution of the $W^{1,\infty}$-seminorm). We believe that the dynamics obtained with $T_h$ is more ‘realistic’, and that the stagnation of the configuration with maximum gradient observed for the uniform meshes is a numerical artifact related to their symmetry.

Since the computations performed with TPS lead to the same conclusions, in order not to overload the plots, in Figure 4 we have shown only the results computed using AMM. It is not clear to the author whether the observed finite-time blow-up also occurs for the weak solution of iLLG towards which the computed approximations converge as $h, k, \varepsilon \to 0$. However, the fact that the phenomenon has been observed for approximations computed using two different schemes and various choices of the discretization parameters seems to provide a clear evidence in this direction.
In order to assess the resulting magnetization dynamics, we analyze the time evolution of the state \( m \) with the gyromagnetic ratio, we consider the value \( \gamma_0 = 2.211 \cdot 10^{9} \) s/Am, while the effective field \( H_{\text{eff}}(m) \) is related to the energy \( E(m) \) via the relation \( \mu_0 M_s H_{\text{eff}}(m) = \frac{\delta E(m)}{\delta m} \).

For the material parameters, we use the values of permalloy (see, e.g., [24]): \( M_s = 8 \cdot 10^{5} \) A/m, \( A = 1.3 \cdot 10^{-11} \) J/m, \( K = 5 \cdot 10^{2} \) J/m², and \( \alpha = 0.023 \). For the angular momentum relaxation time \( \tau \) in [2], we consider the value \( \tau = \alpha \xi \) with \( \xi = 12.3 \) ps; see [23]. As initial conditions, we consider the constant fields \( m^0 \equiv e_1 \) and \( n^0 \equiv 0 \). Note that \( m^0 \) is a global minimum for the energy \( E(m) \) if \( H_{\text{ext}} \equiv 0 \).

The overall simulation time is 30 ps. The experiment consists in perturbing the equilibrium state \( m^0 \equiv e_1 \) with a periodic field and spatially uniform high-frequency pulse field \( H_{\text{ext}}(t) = F(t)M_s e_2 \), where \( F(t) = 0.01 \sin(2\pi ft) \chi_{0 \leq t \leq 2 \cdot 10^{-12}}(t) \) with \( f = 500 \) GHz; see Figure 5(a). In order to assess the resulting magnetization dynamics, we analyze the time evolution of the spatial average of the third magnetization component, i.e., \( \langle m_3 \rangle := |\omega|^{-1} \int_{\omega} m \cdot e_3 \).

For the spatial discretization we consider a triangular mesh of \( \omega \) made of 5998 elements. Its mesh size (3.760 nm) is well below the exchange length of the material \( \ell_{\text{ex}} = \sqrt{2A/(\mu_0 M_s^2)} = 5.686 \) nm. For the time discretization, we consider three different time-step sizes \( (\Delta t = 1, 10, 100) \) fs. In the stopping criterion [25], we use the tolerance \( \varepsilon = 1 \cdot 10^{-6} \).

In Figure 5(b), we compare the magnetization dynamics governed by LLG with the one governed by iLLG. Specifically, we plot the evolution of \( \langle m_3 \rangle \) for both LLG and iLLG. We show the results computed using TPS with \( \Delta t = 1 \) fs. Note that TPS for LLG can be obtained...
from Algorithm 3.1 by omitting the first term on the left-hand side of (21); see [1, 14]. The dynamics induced by the two models are completely different. For LLG, the magnetization reacts to the pulse field and returns straight to the equilibrium state. For iLLG, the deflection from the equilibrium state gives rise to oscillations with approximately the same frequency of the inducing pulse field (500 GHz). Due to damping, the amplitude of the oscillations decays with time and the magnetization regains the initial equilibrium state. This experiment provides a numerical evidence of the inertial nutation dynamics predicted by the model, which has been experimentally observed only very recently [23].

In Figure 6(c), we plot the evolution of \( \langle m_3 \rangle \) computed using TPS with \( \Delta t = 1, 10, 100 \) fs. For larger time-step sizes, we observe a faster decay of the oscillations. This phenomenon is a consequence of the artificial damping of the implicit Euler method used for the time discretization. The observed dependence on \( \Delta t \) reflects the fact that the artificial damping can be controlled by the time-step size; see (28). Finally, in Figure 6(d), we show the same plot for AMM. Unlike TPS, AMM is robust with respect to variations of the time-step size. This reflects the energy conservation properties of the symplectic midpoint rule; see (29)–(30). All the considered time-step sizes are sufficiently small to guarantee the convergence of the fixed-point iteration, which requires 1-2 iterations for \( \Delta t = 1, 10 \) fs and 2-3 iterations for \( \Delta t = 100 \) fs. Note that using larger time-step sizes is not advisable, as they cannot resolve the pulse field and the resulting magnetization dynamics.

This experiment shows the importance of designing a numerical scheme which respects the energy law of the underlying model. This general statement, which holds true for any PDE with a physical background (and, in particular, for LLG), turns out to be a key aspect for iLLG due to the small extent and the ultrafast time scale of the nutation dynamics.
5. Proofs

In this section, we present the proofs of the results stated in Section 3. In view of their later use, we recall some facts: the mass-lumped product \( \langle \cdot, \cdot \rangle_h \) defined in (15) is a scalar product on \( S^1(T_h)^3 \) and the induced norm \( \| \cdot \|_h \) satisfies the norm equivalence
\[
\| \phi_h \|_{L^2(\Omega)} \lesssim \| \phi_h \| \leq \sqrt{d+2} \| \phi_h \|_{L^2(\Omega)} \quad \text{for all } \phi_h \in S^1(T_h)^3.
\]
Moreover, there holds the error estimate
\[
|\langle \phi_h, \psi_h \rangle - \langle \phi_h, \psi_h \rangle| \leq C h^2 \| \nabla \phi_h \|_{L^2(\Omega)} \| \nabla \psi_h \|_{L^2(\Omega)} \quad \text{for all } \phi_h, \psi_h \in S^1(T_h)^3, \tag{33}
\]
where \( C > 0 \) depends only on the shape-regularity of \( T_h \); see [7, Lemma 3.9]. Finally, we recall the following relations between the \( L^p \)-norm of a discrete function and the \( \ell^p \)-norm of the vector collecting its nodal values (see [7, Lemma 3.4]):
\[
\| \phi_h \|_{L^p(\Omega)} \simeq h^d \sum_{z \in N_h} \| \phi_h(z) \|^p \quad \text{and} \quad \| \phi_h \|_{L^\infty(\Omega)} = \max_{z \in N_h} |\phi_h(z)| \quad \text{for all } \phi_h \in S^1(T_h)^3. \tag{34}
\]

5.1. Properties of Algorithm 3.2 and well-posedness of Algorithm 3.4

We start with proving the conservation properties of Algorithm 3.2.

Proof of Proposition 3.3

Note that the definition of the nodal interpolant \( I_h[\cdot] \) yields that
\[
\langle \psi, \phi \rangle_h = \sum_{z \in N_h} \beta_z \psi(z) \cdot \phi(z) \quad \text{for all } \psi, \phi \in C^0(\Omega), \quad \text{with } \beta_z := \int_{\Omega} \varphi_z > 0.
\]

Let \( z \in N_h \). Choosing \( \phi_h = \varphi_z m_h^{i+1/2}(z) \) in (22a), we infer that \(|m_h^{i+1}(z)| = |m_h^i(z)|\). Since \( m_h^0 \in \mathcal{M}_h \) by assumption, we conclude that \( m_h^{i+1} \in \mathcal{M}_h \).

Choosing \( \phi_h = \varphi_z w_h^{i+1/2}(z) \) and \( \psi_h = \varphi_z m_h^{i+1/2}(z) \) in (22b), we obtain the identities
\[
d_t m_h^{i+1}(z) \cdot w_h^{i+1/2}(z) = 0 \quad \text{and} \quad d_t m_h^{i+1}(z) \cdot w_h^{i+1/2}(z) = 0.
\]

Since \( m_h^{0}(z) \cdot w_h^{0}(z) = m_h^{0}(z) \cdot (m_h^{0}(z) \times v_h^{0}(z)) = 0 \), we conclude that \( w_h^{i+1} \in \mathcal{K}_h(m_h^{i+1}) \).

Next, we show that the fixed-point iteration designed for the solution of (23) is well-posed and converges.

Proof of Proposition 3.4

Let \( \ell \in N_0 \). The bilinear forms on the left-hand side of both (24) and (25) are elliptic. Therefore, existence and uniqueness of solutions \( u_h^{i,\ell+1} \) and \( z_h^{i,\ell+1} \) in \( S^1(T_h)^3 \) follow from the Lax–Milgram theorem.

Let \( z \in N_h \) be an arbitrary vertex. Testing (24) with \( \phi_h = \varphi_z u_h^{i,\ell+1}(z) \in S^1(T_h)^3 \), we obtain that \(|u_h^{i,\ell+1}(z)|^2 = u_h^{i,\ell+1}(z) \cdot m_h^{i}(z)|. Hence, \(|u_h^{i,\ell+1}(z)| \leq |m_h^{i}(z)| = 1\). This shows that \(|u_h^{i,\ell+1}|_{L^\infty(\Omega)} \leq 1\) and concludes the proof of part (i).

Let \( u_h^{i,\ell+1} \) and \( u_h^{i,\ell+2} \) (resp., \( z_h^{i,\ell+1} \) and \( z_h^{i,\ell+2} \)) be two successive iterates satisfying (24) (resp., (25)). Taking the difference of the equations satisfied by \( z_h^{i,\ell+2} \) and \( z_h^{i,\ell+1} \) and choosing \( \psi_h = z_h^{i,\ell+2} - z_h^{i,\ell+1} \), we obtain the identity
\[
2\tau \| z_h^{i,\ell+2} - z_h^{i,\ell+1} \|^2_h = \langle u_h^{i,\ell+2} - u_h^{i,\ell+1} \times \mathbb{P}_h u_h^{i,\ell+1} \rangle_h + 2\alpha \langle u_h^{i,\ell+2} - u_h^{i,\ell+1} \times m_h^{i+1}, z_h^{i,\ell+2} - z_h^{i,\ell+1} \rangle_h \tag{35}
\]
Taking the difference of the equations satisfied by $u_h^{i+2}$ and $u_h^{i+1}$ and choosing the test functions $\phi_h = u_h^{i+2} - u_h^{i+1}$ and $\phi_h = z_h^{i+2} - z_h^{i+1}$, we obtain the identities
\[
2\|u_h^{i+2} - u_h^{i+1}\|_h^2 = -k(u_h^{i+1} \times (z_h^{i+1} - z_h^i), u_h^{i+2} - u_h^{i+1})_h, \tag{36}
\]
\[
2(u_h^{i+2} - u_h^{i+1}, z_h^{i+2} - z_h^{i+1})_h + k(u_h^{i+2} \times z_h^{i+1} - u_h^{i+1} \times z_h^{i+2} - z_h^{i+2} - z_h^{i+1})_h = 0. \tag{37}
\]
From (36), since $\|u_h^{i+1}\|_{L^\infty(\Omega)} \leq 1$ from part (i), we deduce that
\[
\|u_h^{i+2} - u_h^{i+1}\|_h \leq \frac{k}{2}\|z_h^{i+1} - z_h^i\|_h. \tag{38}
\]
Combining (35) and (37), we obtain that
\[
2\tau\|z_h^{i+2} - z_h^{i+1}\|_h^2
\]
\[
= k((u_h^{i+2} - u_h^{i+1}) \times P_h h_{\text{eff}}(u_h^{i+2}), z_h^{i+2} - z_h^{i+1})_h \\
+ k(u_h^{i+1} \times (P_h h_{\text{eff}}(u_h^{i+2}) - P_h h_{\text{eff}}(u_h^{i+1})), z_h^{i+2} - z_h^{i+1})_h \\
+ 2\alpha((u_h^{i+2} - u_h^{i+1}) \times m_h^i, z_h^{i+2} - z_h^{i+1})_h \\
- k(u_h^{i+2} \times z_h^{i+2} - u_h^{i+1} \times z_h^{i+1}, z_h^{i+2} - z_h^{i+1})_h \\
+ 2(u_h^{i+2} - u_h^{i+1}, z_h^{i+2} - z_h^{i+1})_h + k(u_h^{i+2} \times z_h^{i+1} - u_h^{i+1} \times z_h^{i+2} - z_h^{i+2} - z_h^{i+1})_h \\
= k((u_h^{i+2} - u_h^{i+1}) \times P_h h_{\text{eff}}(u_h^{i+2}), z_h^{i+2} - z_h^{i+1})_h \\
+ k(u_h^{i+1} \times (P_h h_{\text{eff}}(u_h^{i+2}) - P_h h_{\text{eff}}(u_h^{i+1})), z_h^{i+2} - z_h^{i+1})_h \\
+ 2\alpha((u_h^{i+2} - u_h^{i+1}) \times m_h^i, z_h^{i+2} - z_h^{i+1})_h \\
+ 2(u_h^{i+2} - u_h^{i+1}, z_h^{i+2} - z_h^{i+1})_h + k(u_h^{i+2} \times z_h^{i+1} - u_h^{i+1} \times z_h^{i+2} - z_h^{i+2} - z_h^{i+1})_h.
\]
It follows that
\[
2\tau\|z_h^{i+2} - z_h^{i+1}\|_h^2
\]
\[
\leq (2 + 2\alpha)\|m_h^i\|_{L^\infty(\Omega)} + k\|P_h h_{\text{eff}}(u_h^{i+2})\|_{L^\infty(\Omega)} \|u_h^{i+2} - u_h^{i+1}\|_h \|z_h^{i+2} - z_h^{i+1}\|_h \\
+ k\|u_h^{i+1}\|_{L^\infty(\Omega)} \|P_h h_{\text{eff}}(u_h^{i+2}) - P_h h_{\text{eff}}(u_h^{i+1})\|_h \|z_h^{i+2} - z_h^{i+1}\|_h \\
+ k\|u_h^{i+1}\|_{L^\infty(\Omega)} \|z_h^{i+1}\|_h \|z_h^{i+2} - z_h^{i+1}\|_h \\
+ k\|u_h^{i+1}\|_{L^\infty(\Omega)} \|z_h^{i+2} - z_h^{i+1}\|_h \|z_h^{i+2} - z_h^{i+1}\|_h.
\]
Hence, using the fact that $\|m_h^i\|_{L^\infty(\Omega)} = 1$ and $\|u_h^{i+1}\|_{L^\infty(\Omega)} \leq 1$, together with the estimates
\[
\|P_h h_{\text{eff}}(u_h^{i+2})\|_{L^\infty(\Omega)} \lesssim h^{-2}\|u_h^{i+2}\|_{L^2(\Omega)} \quad \text{and} \quad \|P_h h_{\text{eff}}(u_h^{i+2})\|_h \lesssim h^{-2}\|u_h^{i+2}\|_{L^2(\Omega)}
\]
(-obtained from a global inverse estimate; see, e.g., [12] equations (2.3)–(2.4))), we deduce that
\[
\|z_h^{i+2} - z_h^{i+1}\|_h \lesssim (1 + kh^{-2})\|u_h^{i+2} - u_h^{i+1}\|_h + k\|z_h^{i+1} - z_h^{i+1}\|_h \\
\]
Hence, there exist sufficiently small constants $\delta, h_0 > 0$ and $0 < q < 1$, such that (27) holds if $k = \delta h$ and $h < h_0$. This concludes the proof of part (ii).

By construction, $m_h^{i+1} = 2u_h^{i,E} - m_h^i$ satisfies
\[
\langle d_t m_h^{i+1}, \phi_h \rangle_h = \langle m_h^{i+1/2} \times z_h^{i,E}, \phi_h \rangle_h \quad \text{for all } \phi_h \in S^1(T_h)^3,
\]
while $w_h^{i+1} = 2z_h^{i,E} - w_h^i$ satisfies (22b). The argument used to show Proposition 3.3 shows that $m_h^{i+1} \in M_h$ and $w_h^{i+1} \in K_h(m_h^{i+1})$. This shows part (iii) and concludes the proof. \(\square\)

5.2. Discrete energy laws and stability. We prove the discrete energy laws satisfied by the approximations generated by the algorithms.

**Proof of Proposition 3.4**. Let $i \in \mathbb{N}_0$. We test (21) with $\phi_h = v_h^{i+1} \in K_h(m_h^i)$ and multiply the resulting equation by $k$. We obtain the identity
\[
\tau(v_h^{i+1} - v_h^i, v_h^{i+1}) + h \kappa \|v_h^{i+1}\|^2 + k^2 \|\nabla v_h^{i+1}\|^2_{L^2(\Omega)} = -k\langle \nabla m_h^i, \nabla v_h^{i+1} \rangle_h
\]
Since the angle condition (17) is satisfied, Lemma 3.2 yields that
\[
\|\nabla m_h^{i+1}\|_{L^2(\Omega)} \leq \|\nabla m_h^i + k \nabla v_h^{i+1}\|_{L^2(\Omega)}.
\]
Hence, we obtain that
\[
E(m_h^{i+1}) \leq \frac{1}{2} \|\nabla m_h^i + k \nabla v_h^{i+1}\|^2_{L^2(\Omega)} = E(m_h^i) + k\langle \nabla m_h^i, \nabla v_h^{i+1} \rangle_h + \frac{k^2}{2} \|\nabla v_h^{i+1}\|^2_{L^2(\Omega)}.
\]
Altogether, we obtain that
\[
E(m_h^{i+1}) + \tau(v_h^{i+1} - v_h^i, v_h^{i+1}) + h \kappa \|v_h^{i+1}\|^2 + \frac{k^2}{2} \|\nabla v_h^{i+1}\|^2_{L^2(\Omega)} \leq E(m_h^i).
\]
Applying the vector identity
\[
(a - b) \cdot a = \frac{1}{2} |a|^2 - \frac{1}{2} |b|^2 + \frac{1}{2} |a - b|^2 \quad \text{for all } a, b \in \mathbb{R}^3
\]
to the second term on the left-hand side yields (28). This proves part (i).

To show the stability of Algorithm 3.2, we choose $\phi_h = d_t m_h^{i+1}$ in (22a), $\phi_h = \mathbb{P}_h h_{\text{eff}}(m_h^{i+1/2})$ in (22b), and $\psi_h = w_h^{i+1/2}$ in (22b) to obtain the identities
\[
\|d_t m_h^{i+1}\|^2_h = -\langle m_h^{i+1/2} \times w_h^{i+1/2}, d_t m_h^{i+1} \rangle_h,
\]
\[
\langle d_t m_h^{i+1}, \mathbb{P}_h h_{\text{eff}}(m_h^{i+1/2}) \rangle_h = -\langle m_h^{i+1/2} \times w_h^{i+1/2}, \mathbb{P}_h h_{\text{eff}}(m_h^{i+1/2}) \rangle_h,
\]
\[
\tau\langle d_t w_h^{i+1}, w_h^{i+1/2} \rangle_h = (m_h^{i+1/2} \times \mathbb{P}_h h_{\text{eff}}(m_h^{i+1/2}), w_h^{i+1/2})_h - \alpha \langle m_h^{i+1/2} \times d_t m_h^{i+1}, w_h^{i+1/2} \rangle_h,
\]
respectively. Combining these three equations, we obtain that
\[
\langle d_t m_h^{i+1}, \mathbb{P}_h h_{\text{eff}}(m_h^{i+1/2}) \rangle_h = \tau\langle d_t w_h^{i+1}, w_h^{i+1/2} \rangle_h + \alpha \|d_t m_h^{i+1}\|^2_h.
\]
Since
\[
\mathbb{P}_h h_{\text{eff}}(m_h^{i+1/2}), d_t m_h^{i+1} \rangle_h \leq \frac{1}{k} (E(m_h^{i+1}) - E(m_h^i)),
\]
\[
\langle d_t w_h^{i+1}, w_h^{i+1/2} \rangle_h = \frac{1}{2k} (\|w_h^{i+1}\|^2 - \|w_h^i\|^2),
\]
we obtain (29). This proves part (ii). The proof of (30) from part (iii) can be obtained with the very same argument. \(\square\)
5.3. Convergence results. First, we prove the convergence of the tangent plane scheme.

Proof of Theorem 3.7 for Algorithm 3.1. The proof is largely based on the argument of \[1, 14, 20\]. We start with recalling the estimates

\[
|m_h^{i+1}(z) - m_h^i(z)| \leq k |v_h^{i+1}(z)| \quad \text{and} \quad |m_h^{i+1}(z) - m_h^i(z) - kv_h^{i+1}(z)| \leq \frac{k^2}{2} |v_h^{i+1}(z)|^2, \tag{40}
\]

which hold for all \( i \in \mathbb{N}_0 \) and \( z \in \mathcal{N}_h \); see \[2, 10\].

Let \( j \geq 1 \). Since \( m_j^i \in \mathcal{M}_h \), from (28) and the convergence (31), we deduce the estimate

\[
\|m_j^i\|_{H^1(\Omega)}^2 + \|v_h^i\|_h^2 + k \sum_{i=0}^{j-1} \|v_h^{i+1}\|_h^2 + k^2 \sum_{i=0}^{j-1} \|d_t v_h^{i+1}\|_h^2 + k^2 \sum_{i=0}^{j-1} \|\nabla v_h^{i+1}\|_{L^2(\Omega)}^2 \approx 1. \tag{41}
\]

We infer the uniform boundedness of the sequences \( \{m_{hk}\} \) and \( \{m_{hk}^+\} \) (resp., of \( \{v_{hk}^+\} \)) in \( L^\infty(0, \infty; H^1(\Omega)) \) (resp., in \( L^\infty(0, \infty; L^2(\Omega)) \)). The first inequality in (40) yields that \( \|d_t m_h^i\|_h \leq \|v_h^i\|_h \) from which it follows that \( \{m_{hk}\} \) is uniformly bounded also in \( W^{1,\infty}(0, \infty; L^2(\Omega)) \). Moreover, (41) yields also the convergence \( k\|\nabla v_h^+\|_{L^2(\Omega \times (0,\infty))} \to 0 \) as \( h, k \to 0 \).

Let \( T > 0 \) be arbitrary. With successive extractions of convergent subsequences (not relabeled), one can show that there exists \( m \in L^\infty(0, \infty; H^1(\Omega)) \cap W^{1,\infty}(0, \infty; L^2(\Omega)) \), satisfying \( |m| = 1 \) a.e. in \( \Omega \times (0, \infty) \), such that \( m_{hk} \rightharpoonup m \) in \( L^\infty(0, \infty; H^1(\Omega)) \), \( m_{hk}|_{[t_\tau,t_\tau]} \to m|_{[t_\tau,t_\tau]} \) in \( H^1(\Omega) \), \( \partial_t m_{hk} \rightharpoonup \partial_t m \) in \( L^\infty(0, \infty; L^2(\Omega)) \), and \( m_{hk} \rightharpoonup m \) in \( L^2(\Omega \times (0, \infty)) \) and pointwise almost everywhere in \( \Omega \times (0, \infty) \).

Let \( \varphi \in C^\infty_c([0,T];C(\overline{\Omega})) \) be an arbitrary smooth test function. Let \( N \in \mathbb{N} \) be the smallest integer such that \( T \leq kN = t_N \). Let \( i \in \{0, \ldots, N-1\} \). We choose the test function \( \phi_h = \mathcal{I}_h[m_h^i \times \varphi(t_i)] \in \mathcal{K}_h(m_h^i) \) in (21) to obtain

\[
\begin{align*}
\tau \langle d_t v_h^{i+1}, \mathcal{I}_h[m_h^i \times \varphi(t_i)] \rangle + & \alpha \langle v_h^{i+1}, \mathcal{I}_h[m_h^i \times \varphi(t_i)] \rangle + \langle m_h^i \times v_h^{i+1}, \mathcal{I}_h[m_h^i \times \varphi(t_i)] \rangle_h \\
& - k \langle \mathbb{P}_h \mathcal{I}_h[m_h^i \times \varphi(t_i)] \rangle_h = \langle \mathbb{P}_h \mathcal{I}_h[m_h^i \times \varphi(t_i)] \rangle_h,
\end{align*}
\]

where we extend \( \varphi \) by zero in \( (T, t_N) \). Due to the presence of the mass-lumped scalar product, we can remove the nodal interpolant from the first three terms on the left-hand side without affecting the value of the integrals. Then, multiplying the latter by \( k \), summing over \( i = 0, \ldots, N-1 \), and using (6) and (16), we obtain the identity

\[
\begin{align*}
\tau k \sum_{i=0}^{N-1} \langle d_t v_h^{i+1}, m_h^i \times \varphi(t_i) \rangle_h + & \alpha k \sum_{i=0}^{N-1} \langle v_h^{i+1}, m_h^i \times \varphi(t_i) \rangle_h + k \sum_{i=0}^{N-1} \langle m_h^i \times v_h^{i+1}, m_h^i \times \varphi(t_i) \rangle_h \\
= -k \sum_{i=0}^{N-1} \langle \nabla (m_h^i + k v_h^{i+1}), \nabla \mathcal{I}_h[m_h^i \times \varphi(t_i)] \rangle.
\end{align*}
\]

Next, we rewrite the first term on the left-hand side using the summation by parts formula

\[
\sum_{i=0}^{N-1} (a_{i+1} - a_i) b_i = - \sum_{i=0}^{N-1} a_{i+1} (b_{i+1} - b_i) + a_N b_N - a_0 b_0 \quad \text{for all sequences} \ \{a_i\}_{i=0}^{N}, \ \{b_i\}_{i=0}^{N} \tag{42}
\]
and performing some algebraic manipulations:

\[
\tau k \sum_{i=0}^{N-1} \langle d_i v_h^{i+1}, m_h^i \times \varphi(t_i) \rangle_h \\
= -\tau \sum_{i=0}^{N-1} \langle v_h^{i+1}, m_h^{i+1} \times \varphi(t_{i+1}) - m_h^i \times \varphi(t_i) \rangle_h \\
+ \tau \langle v_h^N, m_h^N \times \varphi(t_N) \rangle_h - \tau \langle v_h^0, m_h^0 \times \varphi(0) \rangle_h \\
= -\tau k \sum_{i=0}^{N-1} \langle v_h^{i+1}, d_i m_h^{i+1} \times \varphi(t_{i+1}) \rangle_h - \tau k \sum_{i=0}^{N-1} \langle v_h^{i+1}, m_h^i \times d_i \varphi(t_{i+1}) \rangle_h \\
+ \tau \langle v_h^N, m_h^N \times \varphi(t_N) \rangle_h - \tau \langle v_h^0, m_h^0 \times \varphi(0) \rangle_h \\
= -\tau k \sum_{i=0}^{N-1} \langle v_h^{i+1}, (d_i m_h^{i+1} - v_h^{i+1}) \times \varphi(t_{i+1}) \rangle_h + \tau k \sum_{i=0}^{N-1} \langle m_h^i \times v_h^{i+1}, d_i \varphi(t_{i+1}) \rangle_h \\
- \tau \langle m_h^N \times v_h^N, \varphi(t_N) \rangle_h + \tau \langle m_h^0 \times v_h^0, \varphi(0) \rangle_h.
\]

Using Hölder inequality, a combination of the second inequality in (40) and the norms equivalence (34), and global inverse estimates (see, e.g., [18, Corollary 1.141]), the first term on the right-hand side can be estimated as

\[
\tau k \sum_{i=0}^{N-1} \langle v_h^{i+1}, (d_i m_h^{i+1} - v_h^{i+1}) \times \varphi(t_{i+1}) \rangle_h \\
\lesssim k \sum_{i=0}^{N-1} \|v_h^{i+1}\|_{L^2(\Omega)} \|d_i m_h^{i+1} - v_h^{i+1}\|_{L^3(\Omega)} \|\varphi(t_{i+1})\|_{L^\infty(\Omega)} \\
\lesssim k^2 \sum_{i=0}^{N-1} \|v_h^{i+1}\|_{L^2(\Omega)}^3 \lesssim k^2 h^{-d/2} \sum_{i=0}^{N-1} \|v_h^{i+1}\|_{L^2(\Omega)}^3 \overset{\text{(31)}}{\sim} k h^{-d/2}.
\]

Altogether, using also the identity

\[
[m_h^i(z) \times v_h^{i+1}(z)] \cdot [m_h^i(z) \times \varphi(z, t_i)] = v_h^{i+1}(z) \cdot \varphi(z, t_i)
\]

(which follows from (19), \(m_h^i \in \mathcal{M}_h\) and \(v_h^{i+1} \in \mathcal{K}_h(m_h^i)\)) and observing that \(\varphi(t_N) = 0\), we thus obtain that

\[
k \sum_{i=0}^{N-1} \langle v_h^{i+1}, \varphi(t_i) \rangle_h \\
= -k \sum_{i=0}^{N-1} \langle \nabla (m_h^i + k v_h^{i+1}), \nabla \varphi_h | m_h^i \times \varphi(t_i) \rangle \rangle + \alpha k \sum_{i=0}^{N-1} \langle m_h^i \times v_h^{i+1}, \varphi(t_i) \rangle_h \rangle \\
- \tau k \sum_{i=0}^{N-1} \langle m_h^i \times v_h^{i+1}, d_i \varphi(t_{i+1}) \rangle_h - \tau \langle m_h^0 \times v_h^0, \varphi(0) \rangle_h + o(1).
\]

Using the approximation properties of the nodal interpolant and estimate (33) (see the argument of [12,11]), we can replace all mass-lumped scalar products by \(L^2\)-products and remove the nodal interpolant from the first term on the left-hand side at the price of an error which goes to zero in the limit. Rewriting the space-time integrals of the resulting equation in terms of the time
reconstructions \([14]\), we obtain that

\[
\int_0^T \langle \nu_h^+(t), \varphi_h^-(t) \rangle \, dt = - \int_0^T \langle \nabla (m_{hh}^-(t) + k \nu_h^+(t)), \nabla (m_{hh}^-(t) \times \varphi_h^-(t)) \rangle \, dt + \alpha \int_0^T \langle m_{hh}^-(t) \times \nu_h^+(t), \varphi_h^-(t) \rangle \, dt - \tau \int_0^T \langle m_{hh}^-(t), \partial_t \varphi_h(t) \rangle \, dt - \tau \langle m_h^0 \times \nu_h^0, \varphi(0) \rangle + o(1).
\]

Using the available convergence results, we can pass to the limit as \(h, k \to 0\) the latter and obtain that each term converges to the corresponding one in \([12]\). By density, it follows that \(m\) satisfies \([12]\) for all \(\varphi \in C_c^\infty([0, T); H^1(\Omega))\). This shows that \(m\) satisfies part (iii) of Definition 2.1.

The proof that \(m\) attains the prescribed initial data \((m^0, v^0)\) continuously in \(H^1(\Omega) \times L^2(\Omega)\) (part (ii) of Definition 2.1) follows the argument of \([9, \text{page 72}]\). The energy inequality \([13]\) (part (iv) of Definition 2.1) can be obtained from the discrete energy law \([28]\) using the available convergence results and standard lower semicontinuity arguments.

Next, we prove the convergence result for the nonlinear angular momentum method.

**Proof of Theorem 3.7 for Algorithm 3.2.** Let \(j \geq 1\). Since \(m_h^j \in \mathcal{M}_h\), from \((29)\) and \((31)\), we deduce the estimate

\[
\|m_h^j\|^2_{H^1(\Omega)} + \|w_h^j\|^2 + k \sum_{i=0}^{j-1} \|d_i m_{i+1}^j\|^2_h \lesssim 1.
\]

We infer the uniform boundedness of the sequences \(\{m_{hh}\}, \{m_{hh}^+\}\) (resp., of \(\{w_{hh}\}\) and \(\{w_{hh}^+\}\)) in \(L^\infty(0, \infty; H^1(\Omega))\) (resp., in \(L^\infty(0, \infty; L^2(\Omega))\)). Testing \((22a)\) with \(\phi_h = d_i m_{i+1}^j\); we infer that \(\|d_i m_{i+1}^j\|_h \leq \|w_{i+1/2}^j\|_h\) for all \(i \in \mathbb{N}_0\), from which it follows that \(m_{hh}\) is uniformly bounded also in \(W^{1,\infty}(0, \infty; L^2(\Omega))\).

Let \(T > 0\) be arbitrary. From the bounds established above, we derive the existence of \(m \in L^\infty(0, \infty; H^1(\Omega)) \cap W^{1,\infty}(0, \infty; L^2(\Omega))\) and \(w \in L^\infty(0, \infty; L^2(\Omega))\) such that, upon extraction of a (nonrelabeled) subsequence, we have that \(m_{hh}, m_{hh}^+, \mu_{hh} \rightharpoonup m\) in \(L^\infty(0, \infty; H^1(\Omega))\), \(\partial_t m_{hh} \rightharpoonup \partial_t m\) in \(L^\infty(0, \infty; L^2(\Omega))\), \(m_{hh}|_{\partial\Omega} \to m|_{\partial\Omega}\) in \(H^1(\Omega_T)\), \(m_{hh}, m_{hh}^+, \mu_{hh} \to m\) in \(L^2(\Omega \times (0, \infty))\) and pointwise almost everywhere in \(\Omega \times (0, \infty)\), as well as \(\mu_{hh}, w_{hh}^+ \rightharpoonup w\) in \(L^\infty(0, \infty; L^2(\Omega))\). Moreover, it holds that \(|m| = 1\) and \(m \cdot w = 0\) a.e. in \(\Omega \times (0, \infty)\).

Let \(\zeta, \varphi \in C_c^\infty([0, T); C(\overline{\Omega}))\) be arbitrary smooth test functions. Let \(N \in \mathbb{N}\) be the smallest integer such that \(T \leq kN = t_N\). Let \(i \in \{0, \ldots, N - 1\}\). We choose the test function \(\phi_h = \mathcal{I}_h(\zeta(t_i)) \in \mathcal{S}^1(\mathcal{T}_h^3)\) in \((22a)\) and \(\psi_h = \mathcal{I}_h(\varphi(t_i)) \in \mathcal{S}^1(\mathcal{T}_h^3)\) in \((22b)\). Multiplying the resulting equation by \(k\), summing over \(i = 0, \ldots, N - 1\), and using \((5)\) and \((16)\) on the term which involves the effective field, we obtain the identities

\[
k \sum_{i=0}^{N-1} \langle d_i m_{i+1}^{i+1}, \zeta(t_i) \rangle_h = -k \sum_{i=0}^{N-1} \langle m_{h}^{i+1/2} \times w_{h}^{i+1/2}, \zeta(t_i) \rangle_h,
\]

\[
t \sum_{i=0}^{N-1} \langle d_i w_{i+1}^{i+1}, \varphi(t_i) \rangle_h = -t \sum_{i=0}^{N-1} \langle \nabla m_{h}^{i+1/2} \times \varphi(t_i) \times m_{h}^{i+1/2} \rangle
\]

\[
= \left\{(N-1) \sum_{i=0}^{N-1} \langle m_{h}^{i+1/2} \times d_i m_{i+1}^{i+1}, \varphi(t_i) \rangle_h - \sum_{i=0}^{N-1} \langle m_{h}^{i+1/2} \times w_{h}^{i+1/2}, \varphi(t_i) \rangle_h \right\}.
\]
To conclude, following [21, Section 3.3], we observe that (44) reveals that Proposition 5.1.

The thresholds Algorithm 3.4 satisfy the inequality

\[ h \times \zeta_n \times \zeta_n = 0 \]

where we extend \( h, \varphi \) by zero in \((T, t_N)\). Next, we rewrite the first term on the left-hand side of both equations using the summation by parts formula [42]:

\[ -k \sum_{i=0}^{N-1} \langle m_h^{i+1}, d_i \zeta_{t_{i+1}} \rangle_h - \langle m_h^0, \zeta_0 \rangle_h = -k \sum_{i=0}^{N-1} \langle m_h^{i+1/2} \times u_h^{i+1/2}, \zeta(t_i) \rangle_h, \]

\[ -\tau k \sum_{i=0}^{N-1} \langle w_h^{i+1}, d_i \varphi(t_{i+1}) \rangle_h - \tau \langle w_h^0, \varphi(0) \rangle_h = -k \sum_{i=0}^{N-1} \langle \nabla m_h^{i+1/2}, \nabla (\varphi(t_i) \times m_h^{i+1/2}) \rangle_h \]

\[ -\alpha k \sum_{i=0}^{N-1} \langle m_h^{i+1/2} \times d_i m_h^{i+1}, \varphi(t_i) \rangle_h - k \sum_{i=0}^{N-1} \langle m_h^{i+1/2} \times w_h^{i+1/2}, \varphi(t_i) \rangle_h. \]

Rewriting the space-time integrals of both equations in terms of the time reconstructions defined in [14], we obtain that

\[ -\int_0^T \langle m_h^{i+1}(t), \partial_t \zeta_k(t) \rangle_h dt - \langle m_h^0, \zeta(0) \rangle_h = -\int_0^T \langle \overline{m}_{hk}(t) \times \overline{w}_{hk}(t), \zeta_k(t) \rangle_h dt, \]

\[ -\tau \int_0^T \langle w_h^{i+1}(t), \partial_t \varphi_k(t) \rangle_h dt - \tau \langle w_h^0, \varphi(0) \rangle_h = -\int_0^T \langle \nabla \overline{m}_{hk}(t), \nabla (\varphi_k(t) \times \overline{m}_{hk}(t)) \rangle dt \]

\[ -\alpha \int_0^T \langle \overline{m}_{hk}(t) \times \partial_t m_{hk}(t), \varphi_k(t) \rangle_h dt - \int_0^T \langle \overline{m}_{hk}(t) \times \overline{w}_{hk}(t), \varphi_k(t) \rangle_h dt. \]

Using the available convergence results and (33), we can proceed as in [12, Section 3] and pass the latter equations to the limit as \( h, k \to 0 \). Rearranging the terms, we obtain that

\[ -\int_0^T \langle m(t), \partial_t \zeta(t) \rangle dt = -\int_0^T \langle m(t) \times w(t), \zeta(t) \rangle dt + \langle m^0, \zeta(0) \rangle, \quad (44) \]

\[ -\int_0^T \langle m(t) \times w(t), \varphi(t) \rangle dt = -\int_0^T \langle \nabla m(t), \nabla (\varphi(t) \times m(t)) \rangle dt \]

\[ +\alpha \int_0^T \langle m(t) \times \partial_t m(t), \varphi(t) \rangle dt - \tau \int_0^T \langle w(t), \partial_t \varphi(t) \rangle dt - \tau \langle m^0 \times v^0, \varphi(0) \rangle. \quad (45) \]

To conclude, following [21, Section 3.3], we observe that (44) reveals that

\[ \partial_t m = -m \times w \quad \text{a.e. in } \Omega \times (0, T). \quad (46) \]

Since \( |m| = 1 \) and \( m \cdot w = 0 \), using (19), it follows that

\[ w = m \times \partial_t m \quad \text{a.e. in } \Omega \times (0, T). \quad (47) \]

Using (46) and (47) in the term on the left-hand side and in the third term on the right-hand side of (45), respectively, we obtain the variational formulation (12). By density, it follows that \( m \) satisfies (12) for all \( \varphi \in C_0^\infty([0, T); H^1(\Omega)) \). This shows that \( m \) satisfies part (iii) of Definition 2.1. The verification of part (ii) and (iv) can be performed using the argument employed for Algorithm 3.2. This proves the desired convergence and concludes the proof. \( \square \)

In the following proposition, we establish the boundedness of the approximations generated by Algorithm 3.3.

**Proposition 5.1.** Let \( j \in \mathbb{N} \). Assume that \( \varepsilon = O(h) \) as \( h, \varepsilon \to 0 \). Then, there exist thresholds \( \varepsilon_0, k_0 > 0 \) such that, if \( h < h_0, k < k_0, \) and \( \varepsilon < \varepsilon_0, \) the approximations generated by Algorithm 3.4 satisfy the inequality

\[ \| \nabla m_h^j \|_{L^2(\Omega)}^2 + \| w_h^j \|_{H^1}^2 + k \sum_{i=0}^{j-1} \| d_i m_h^{i+1} \|_{H^1}^2 \leq C(1 + jk) \exp(jk). \quad (48) \]

The thresholds \( h_0, k_0, \varepsilon_0 > 0 \) and the constant \( C > 0 \) depend only on the mesh constant \( \kappa \) and the problem data.
Proof. Let $i = 0, \ldots, j-1$. Proposition (3.6) (iii) yields (30). Taking the sum over $i = 0, \ldots, j-1$, we obtain the identity

$$
\frac{1}{2} \| \nabla m_h^j \|^2_{L^2(\Omega)} + \frac{\tau}{2} \| w_h^j \|^2_{H^1(\Omega)} + \kappa \sum_{i=0}^{j-1} \| d_t m_h^{i+1} \|^2_{H^1(\Omega)}
$$

$$
= \frac{1}{2} \| \nabla m_h^0 \|^2_{L^2(\Omega)} + \frac{\tau}{2} \| w_h^0 \|^2_{H^1(\Omega)} - \kappa \sum_{i=0}^{j-1} \langle m_h^{i+1/2} \times r_h^i, P_h h_{\text{eff}}(m_h^{i+1/2}) - \alpha d_t m_h^{i+1} \rangle_h.
$$

A straightforward application of the discrete Young inequality yields the inequalities

$$
\sum_{i=0}^{j-1} \| \nabla m_h^{i+1/2} \|^2_{L^2(\Omega)} \leq \frac{j}{2} + \frac{1}{2} \sum_{i=0}^{j-1} \| \nabla m_h^i \|^2_{L^2(\Omega)} + \frac{1}{4} \| \nabla m_h^j \|^2_{L^2(\Omega)},
$$

$$
\| d_t m_h^{i+1} \|_{H^1} \leq \frac{1}{2} + \frac{1}{2} \| d_t m_h^{i+1} \|^2_{H^1(\Omega)}.
$$

Since $\| m_h^{i+1/2} \|_{L^\infty(\Omega)} \leq 1$, $\| r_h^i \| \leq \varepsilon$, and $\| P_h h_{\text{eff}}(m_h^{i+1/2}) \|_h \leq C h^{-1}\| \nabla m_h^{i+1/2} \|_{L^2(\Omega)}$ (where $C > 0$ depends only on the mesh constant $\kappa$), we obtain that

$$
k \sum_{i=0}^{j-1} \langle m_h^{i+1/2} \times r_h^i, P_h h_{\text{eff}}(m_h^{i+1/2}) - \alpha d_t m_h^{i+1} \rangle_h
$$

$$
\leq k \sum_{i=0}^{j-1} \| m_h^{i+1/2} \|_{L^\infty(\Omega)} \| r_h^i \|_h (\| P_h h_{\text{eff}}(m_h^{i+1/2}) \|_h + \alpha \| d_t m_h^{i+1} \|_h)
$$

$$
\leq \varepsilon \sum_{i=0}^{j-1} \left( C h^{-1}\| \nabla m_h^{i+1/2} \|_{L^2(\Omega)} + \alpha \| d_t m_h^{i+1} \|_h \right)
$$

$$
\leq \frac{(Ch^{-1} + \alpha) \varepsilon j k}{2} + C \varepsilon h^{-1} \| \nabla m_h^j \|^2_{L^2(\Omega)} + \frac{C \varepsilon h^{-1}}{2} \sum_{i=0}^{j-1} \| \nabla m_h^i \|^2_{L^2(\Omega)} + \alpha \varepsilon \sum_{i=0}^{j-1} \| d_t m_h^{i+1} \|^2_{H^1(\Omega)}.
$$

Overall, we thus obtain that

$$
\frac{1}{2} \left( 1 - \frac{C \varepsilon h^{-1} k}{2} \right) \| \nabla m_h^j \|^2_{L^2(\Omega)} + \frac{\tau}{2} \| w_h^j \|^2_{H^1(\Omega)} + \frac{\alpha (1 - \varepsilon) k}{2} \sum_{i=0}^{j-1} \| d_t m_h^{i+1} \|^2_{H^1(\Omega)}
$$

$$
\leq \frac{1}{2} \| \nabla m_h^0 \|^2_{L^2(\Omega)} + \frac{\tau}{2} \| w_h^0 \|^2_{H^1(\Omega)} + \frac{(Ch^{-1} + \alpha) \varepsilon j k}{2} + \frac{C \varepsilon h^{-1}}{2} \sum_{i=0}^{j-1} \| \nabla m_h^i \|^2_{L^2(\Omega)} + \alpha \varepsilon \sum_{i=0}^{j-1} \| d_t m_h^{i+1} \|^2_{H^1(\Omega)}.
$$

Owing to the convergence (31) and the assumption $\varepsilon = O(h)$ as $h, \varepsilon \to 0$, there exist constants $A_0, B_0 > 0$ and thresholds $h_0, k_0, \varepsilon_0 > 0$ (independent of the discretization parameters) such that, for all $h < h_0, k < k_0$, and $\varepsilon < \varepsilon_0$, there holds the estimate

$$
\| \nabla m_h^j \|^2_{L^2(\Omega)} + \| w_h^j \|^2_{H^1(\Omega)} + \kappa \sum_{i=0}^{j-1} \| d_t m_h^{i+1} \|^2_{H^1(\Omega)} \leq A_0 (1 + j k) + B_0 k \sum_{i=0}^{j-1} \| \nabla m_h^i \|^2_{L^2(\Omega)}.
$$

Then, an application of the discrete Gronwall lemma yields

$$
\| \nabla m_h^j \|^2_{L^2(\Omega)} + \| w_h^j \|^2_{H^1(\Omega)} + \kappa \sum_{i=0}^{j-1} \| d_t m_h^{i+1} \|^2_{H^1(\Omega)} \leq A_0 (1 + j k) \exp(B_0 j k).
$$

Up to a reformulation of the right-hand side, this yields (48) and concludes the proof. □

The boundedness result established in Proposition [3.1] is the starting point to prove Theorem [3.7] (ii). We omit the presentation of the proof, since this follows line-by-line the argument used to show the convergence of Algorithm [3.4]. We only stress that, in the proof of the variational formulation [12] and the energy inequality [13], the additional contributions arising from
the inexact solution of the nonlinear system are always uniformly bounded by $\varepsilon$ and therefore vanish in the limit.

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REFERENCES


TU WIEN, INSTITUTE OF ANALYSIS AND SCIENTIFIC COMPUTING, WIEDNER HAUPTSTRASSE 8–10, 1040 VIENNA, AUSTRIA

Email address: michele.ruggeri@asc.tuwien.ac.at