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AN ESTIMATE OF THE BLOW-UP OF LEBESGUE NORMS IN THE NON-TEMPERED CASE

GIOVANNI DI FRATTA, ALBERTO FIORENZA, AND VALERIY SLASTIKOV

Abstract. We prove that if \( p > 1 \) and \( \psi : ]0, p - 1[ \to ]0, \infty[ \) is just nondecreasing and differentiable (hence not necessarily \( \Delta_2 \)), then for every \( f \) Lebesgue measurable function on \( (0, 1) \)

\[
(*) \quad \sup_{0 < \varepsilon < p - 1} \psi(\varepsilon) \|f\|_{L^p-\varepsilon(0,1)} \lesssim \sup_{0 < t < 1} S_\psi(t) \|f^*\|_{L^p(t,1)},
\]

where \( f^* \) denotes the decreasing rearrangement of \( f \) and \( S_\psi \) is defined, for \( \varepsilon \in ]0, p - 1[ \), through

\[
S_\psi(\nu(\varepsilon)) = \sup_{\varepsilon < \tau < p - 1} \psi \left( \frac{p - 1}{\tau} \varepsilon \right) \left[ \frac{d}{d\tau} (\tau \psi^p(\tau)) \right]^{\frac{1}{p}},
\]

\[
\nu(\varepsilon) = c_\psi \int_0^\varepsilon e^{-\frac{\psi}{\psi(p-1)}} \inf_{\sigma < \lambda < p - 1} \left[ \frac{\psi \left( \frac{(p-1)\sigma}{\lambda} \right)}{\psi(p-1)} \right] \frac{d\sigma}{\sigma^2},
\]

where \( c_\psi \) is the normalizing constant chosen so that \( \nu((p-1)-1) = 1 \). If \( \psi \) is in a class of functions satisfying the \( \Delta_2 \) condition, essentially characterized by the so-called \( \nabla' \) condition, then inequality (\( * \)) is sharp, in the sense that both sides are equivalent.

Estimate (\( * \)) generalizes an inequality of the type obtained by the second author with Farroni and Giova in [6] under the growth condition \( \psi \in \Delta_2 \).

1. Introduction

This paper is a follow-up of [1, 5, 6, 8], where, by different methods and overlapping special cases, the blow-up, as \( \varepsilon \to 0 \), of the function norm

\[
F : \varepsilon \in ]0, p - 1[ \mapsto F(\varepsilon) = \|f\|_{L^p-\varepsilon(\Omega)} \in ]0, \infty[ ,
\]

has been studied. Here, \( 1 < p < \infty \) and \( f \) is a (Lebesgue) measurable function \( f \) over \( \Omega \subset \mathbb{R}^n (n \geq 1, |\Omega| = 1) \) such that

\[
f \in \bigcap_{0 < \varepsilon < p - 1} L^{p-\varepsilon}(\Omega).
\]

We consider a nondecreasing and differentiable function \( \psi : ]0, p - 1[ \to ]0, \infty[ , \) and we prove the existence of a function \( S_\psi \) (which also depends on \( p \)) such that for every Lebesgue

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measurable function \( f \) on \((0, 1)\) there holds
\[
\sup_{0 < \varepsilon < p - 1} \psi(\varepsilon)\|f\|_{L^p - \varepsilon(0, 1)} \lesssim \sup_{0 < t < 1} S_{\psi}(t)\|f^*\|_{L^p(t, 1)},
\]
where \( f^* \) denotes the decreasing rearrangement of \( f \).

Symbols \( \lesssim \) and \( \approx \) above (and analogously the symbol \( \gtrsim \)) mean, as usual, inequalities up to not influent constants (in our case, depending possibly only on \( p \) and \( \psi \)).

The estimate (1.1) is optimal in the following precise sense: if \( \psi \) is in the class of functions essentially characterized by the so-called \( \nabla' \) condition (see Section 3), then both sides of (1.1) are equivalent. In fact, when \( \psi \) is in this class, we prove that the following equivalence holds (see Proposition 3.1):
\[
S_{\psi}(t) \approx \psi\left(\frac{p - 1}{1 - \log t}\right).
\]

It is known from [6] that if (1.2) holds, the inequality in (1.1) is an equivalence and, moreover, that such equivalence holds if and only if \( \psi \) satisfies the growth condition \( \psi \in \Delta_2 \cap L^\infty \). Precisely, the next result holds.

**Theorem 1.1** (cf. Theorem 1.1 in [6]). If \( 1 < p < \infty \), and \( \psi : ]0, p - 1] \to ]0, \infty[ \) is nondecreasing, then for all Lebesgue measurable functions \( f \) on \((0, 1)\)
\[
\left( \sup_{0 < \varepsilon < p - 1} \psi(\varepsilon)\|f\|_{L^p - \varepsilon(0, 1)} \approx \sup_{0 < t < 1} \psi\left(\frac{p - 1}{1 - \log t}\right)\|f^*\|_{L^p(t, 1)} \right) \iff \psi \in \Delta_2 \cap L^\infty.
\]

We recall that since \( \psi \) is, in particular, nondecreasing and continuous, the growth condition \( \psi \in \Delta_2 \) implies that \( \psi(2\cdot) \approx \psi(\cdot) \), which means that \( \psi \) is *tempered* (see [9]).

When no hypotheses on the growth of \( \psi \) are assumed, given the characterization in Theorem 1.1, it is natural to ask whether or not there exists a function \( S_{\psi} \) such that (1.1) holds and such that it is equivalent to the optimal expression in (1.2), at least in the case when \( \psi \) is of power type (see Proposition 3.1). The question is of some importance in the study of generalized grand Lebesgue spaces and is stated in a recent work by Astashkin and Milman where the authors collect some open problems in the area of Extrapolation Theory (cf. Problem 20 in [2]).

The interest of the problem stems from the premise that the left-hand side of (1.1) is a Banach function norm introduced in [3] and investigated in [4] (see [7] for the more detailed history). The difficulty of the problem lies in the fact that the optimality obtained in [6] requires an expression for \( S_{\psi} \) to be *essentially different* comparing to the case \( \psi \in \Delta_2 \).

We emphasize that, in (1.3), a precise choice of \( S_{\psi} \) is involved. In fact, in our setting, one can think about (1.3) as claiming that if \( \psi(\cdot) \in \Delta_2 \cap L^\infty \), then the optimal map \( \psi \mapsto S_{\psi} \) for which the equivalence is achieved, can be chosen to be pointwise related to \( \psi \) and even linear in \( \psi \). However, a careful analysis of the estimates in [5] and [6], suggests that this is no more the case under weaker hypotheses on \( \psi \). This is the motivation behind the
more complex expression of $S_\psi$ which now appears as a nonlinear and nonlocal function of $\psi$.

The ultimate natural goal would be to obtain an optimal expression — if it exists — of the type of the right-hand side of (1.1) valid for all possible choices of increasing $\psi$. This goal is not reached in this paper. However (see Theorem 2.1 below), our result moves towards the right direction: we find an explicit expression of a function $S_\psi$ satisfying (1.1) which gives back the optimal result in (1.3) at least in the case $\psi$ of power type. Our argument uses a careful variant of the techniques originated in [5] and generalized in [6].

We stress that an important feature of the expression for $S_\psi$ is to become smaller as soon as $\psi$ and $\psi'$ become smaller: without this property, (1.1) holds trivially setting $S_\psi \equiv 1$, although then the upper bound in (1.1) would be uninformative in the most interesting case, because it would be infinity when $f \in L^{p-\varepsilon}(0,1)$ for $0 < \varepsilon < p-1$, and $f \notin L^p(0,1)$.

The paper is organized as follows. In Section 2 we state and prove our main result. In Section 3 we first discuss typical growth conditions on $\psi$ and then prove that the estimate (1.1) is optimal when $\psi$ is in the class of functions essentially characterized by the $\nabla'$ condition.

## 2. The main result

**Theorem 2.1.** If $p > 1$ and $\psi : [0, p-1] \to [0, \infty]$ is nondecreasing and differentiable, then (1.1) holds for every $f$ Lebesgue measurable function on $(0,1)$, with $S_\psi$ defined, for $\varepsilon \in ]0, p-1[$, through

\[
\begin{align*}
S_\psi(\nu(\varepsilon)) &= \sup_{\varepsilon < \tau < p-1} \psi \left( \frac{p-1}{\tau} \varepsilon \right) \left[ \frac{d}{d\tau} (\tau \psi^p(\tau)) \right]^{\frac{1}{p}} \\
\nu(\varepsilon) &= c_\psi \int_{0}^{\varepsilon} e^{-\frac{p-1}{\sigma}} \inf_{\sigma < \lambda < p-1} \left[ \frac{\psi \left( \frac{(p-1)\sigma}{\lambda} \right)}{\psi(p-1)} \right] \frac{d\sigma}{\sigma^2} ,
\end{align*}
\]

where $c_\psi$ is the normalizing constant chosen so that $\nu((p-1)-) = 1$.

**Remark 2.1.** It is possible to show that tempered functions, i.e., functions satisfying the $\Delta_2$ condition, are bounded from below (near the origin) by some (and therefore any sufficiently large) positive power function — see the discussion before the statement of Proposition 3.1. However, in order to control the blow-up of the $L^p$ norm, one is mainly interested in “weights” $\psi^p$ which are very flat around the origin, i.e., basically convex. In this case, we can show that $S_\psi$ is a nondecreasing function, and therefore it optimally controls around the origin the blow-up of $\|f^*\|_{L^p(t,1)}$. 

The proof goes as follows: setting $\tau(\varepsilon, \lambda) := \lambda + \frac{\varepsilon}{p-1}(1-\lambda)$, $0 < \lambda < 1$, we can rewrite $S^p_\psi$ under the form
\[
S^p_\psi(\nu(\varepsilon)) = \sup_{0 < \lambda < 1} \psi^p \left( \frac{\varepsilon}{\tau(\varepsilon, \lambda)} \right) \cdot \left. \frac{d}{d\tau}(\tau \psi^p(\tau)) \right|_{\tau := (p-1)\tau(\varepsilon, \lambda)}.
\]
Note that, for any $0 < \lambda < 1$, both $\tau(\varepsilon, \lambda)$ and $\varepsilon/\tau(\varepsilon, \lambda)$ are increasing functions of $\varepsilon$. Also we have
\[
\left. \frac{d}{d\tau}(\tau \psi^p(\tau)) \right|_{\tau := (p-1)\tau(\varepsilon, \lambda)} = \left[ \psi^p((p-1)\tau(\varepsilon, \lambda)) + (p-1)\tau(\varepsilon, \lambda)(\psi^p)'((p-1)\tau(\varepsilon, \lambda)) \right].
\]
Therefore, if $\psi^p$ is convex, we have that the function
\[
\psi^p \left( \frac{\varepsilon}{\tau(\varepsilon, \lambda)} \right) \left. \frac{d}{d\tau}(\tau \psi^p(\tau)) \right|_{\tau := (p-1)\tau(\varepsilon, \lambda)}
= \psi^p \left( \frac{\varepsilon}{\tau(\varepsilon, \lambda)} \right) \left[ \psi^p((p-1)\tau(\varepsilon, \lambda)) + (p-1)\tau(\varepsilon, \lambda)(\psi^p)'((p-1)\tau(\varepsilon, \lambda)) \right]
\]
is nondecreasing in $\varepsilon$ because is the product of nondecreasing functions of $\varepsilon$. Our assertion follows, because suprema of nondecreasing functions are nondecreasing functions.

**Proof of Theorem 2.1.** We split the proof in four steps.

**Step 1.** Proceeding as in [6], we split left-hand side of (1.1) as follows:
\[
\sup_{0 < \varepsilon < p-1} \psi(\varepsilon) \|f\|_{L^{p-\varepsilon}(0,1)} = \sup_{0 < \varepsilon < p-1} \psi(\varepsilon) \left[ \int_0^1 |f(x)|^{p-\varepsilon} dx \right]^{\frac{1}{p-\varepsilon}}
= \sup_{0 < \varepsilon < p-1} \psi(\varepsilon) \left[ \int_0^1 f^*(s)^{p-\varepsilon} ds \right]^{\frac{1}{p-\varepsilon}}
= \sup_{0 < \varepsilon < p-1} \psi(\varepsilon) \left[ \int_0^{\nu(\varepsilon)} f^*(s)^{p-\varepsilon} ds + \int_{\nu(\varepsilon)}^1 f^*(s)^{p-\varepsilon} ds \right]^{\frac{1}{p-\varepsilon}}.
\tag{2.1}
\]
To shorten notation, it is convenient to set
\[
I_1(\varepsilon) := \psi(\varepsilon) \left( \int_0^{\nu(\varepsilon)} f^*(s)^{p-\varepsilon} ds \right)^{\frac{1}{p-\varepsilon}}, \quad I_2(\varepsilon) := \psi(\varepsilon) \left( \int_{\nu(\varepsilon)}^1 f^*(s)^{p-\varepsilon} ds \right)^{\frac{1}{p-\varepsilon}}.
\tag{2.2}
\]
From (2.1) we immediately infer the following upper bound:
\[
\sup_{0 < \varepsilon < p-1} \psi(\varepsilon) \|f\|_{L^{p-\varepsilon}(0,1)} \leq \sup_{0 < \varepsilon < p-1} I_1(\varepsilon) + \sup_{0 < \varepsilon < p-1} I_2(\varepsilon).
\tag{2.3}
\]
Next, we estimate $I_1$ and $I_2$ separately. We start by estimating $I_2$. 
Step 2 (Estimate for $I_2$). Setting $\tau = p - 1$ in the supremum defining $S_\psi(\nu(\varepsilon))$, we get that $S_\psi(\nu(\varepsilon)) \gtrsim \psi(\varepsilon)$, up to a constant factor that does not depend on $\varepsilon$. Precisely, we have

$$S_\psi(\nu(\varepsilon)) \geq \left(\frac{d}{d\tau}(\tau \psi^p(\tau))_{|\tau=p-1}\right)^{\frac{1}{p}} \psi(\varepsilon) \geq k_2^{-1} \psi(\varepsilon), \quad (2.4)$$

with $k_2^{-1} := \left(\frac{d}{d\tau}(\tau \psi^p(\tau))_{|\tau=p-1}\right)^{\frac{1}{p}}$. Note that $k_2 > 0$ because it consists of the sum of two positive terms, one of which is $\psi^p(p - 1) > 0$. Next, by Hölder’s inequality, we have

$$\psi(\varepsilon) \left(\int_{\nu(\varepsilon)}^{1} f^*(x)^{p-\varepsilon} \, dx\right)^{\frac{1}{p-\varepsilon}} \leq \psi(\varepsilon) \left(\int_{\nu(\varepsilon)}^{1} dx\right)^{\frac{\varepsilon}{p(p-\varepsilon)}} \left(\int_{\nu(\varepsilon)}^{1} f^*(x)^p \, dx\right)^{\frac{1}{p}}$$

$$\leq k_2 (1 - \nu(\varepsilon))^{\frac{\varepsilon}{p(p-\varepsilon)}} S_\psi(\nu(\varepsilon)) \left(\int_{\nu(\varepsilon)}^{1} f^*(x)^p \, dx\right)^{\frac{1}{p}}$$

$$\leq k_2 (1 - \nu(\varepsilon))^{\frac{\varepsilon}{p(p-\varepsilon)}} \sup_{0 < t < 1} S_\psi(t) \|f^*\|_{L^p(t,1)}^p$$

$$\leq k_2 \sup_{0 < t < 1} S_\psi(t) \|f^*\|_{L^p(t,1)}^p. \quad (2.5)$$

Overall, we infer the estimate

$$\sup_{0 < \varepsilon < p - 1} I_2(\varepsilon) \leq k_2 \sup_{0 < t < 1} S_\psi(t) \|f^*\|_{L^p(t,1)}^p. \quad (2.6)$$

Step 3 (Estimate for $I_1$). To estimate $I_1$ we need some properties of the integral function $\Psi$ defined by

$$\Psi(x) := \int_0^x S_\psi^p(\nu(\sigma)) \, d\sigma, \quad x \in ]0, p - 1[. \quad (2.7)$$

Note that, thanks to $(2.4)$, $\Psi$ is positive and increasing. We prove the following result.

Lemma 2.2. For every $0 < \varepsilon < p - 1$, the following estimate holds:

$$\int_0^{\nu(\varepsilon)} \Psi(\nu^{-1}(s)) f^*(s)^p \, ds \leq \varepsilon \sup_{0 < t < 1} S_\psi^p(t) \|f^*\|_{L^p(t,1)}^p. \quad (2.8)$$

Moreover, for every $0 < \varepsilon < p - 1$ and every $t > 1$, we have

$$\varepsilon \psi^p(\varepsilon) \leq t \left[\psi \left(\frac{p - 1}{t}\right)\right]^{-p} \Psi \left(\frac{\varepsilon}{t}\right). \quad (2.9)$$

Proof. For any $0 < \sigma < \varepsilon$, recalling that $\nu$ is increasing and that $0 < \nu(\varepsilon) < 1$, we have

$$S_\psi^p(\nu(\sigma)) \int_{\nu(\sigma)}^{\nu(\varepsilon)} f^*(s)^p \, ds \leq \sup_{0 < \tau < p - 1} S_\psi^p(\nu(\tau)) \int_{\nu(\tau)}^{1} f^*(s)^p \, ds = \sup_{0 < t < 1} S_\psi^p(t) \|f^*\|_{L^p(t,1)}^p. \quad (2.9)$$
Integrating both sides with respect to $\sigma$, we obtain
\[
\int_0^\varepsilon S^p_\psi(\nu(\sigma)) \int_{\nu(\sigma)}^{\nu(\varepsilon)} f^*(s)^p ds d\sigma \leq \varepsilon \sup_{0 < t < 1} S^p_\psi(t) \| f^* \|_{L^p(t,1)}^p \quad \forall \varepsilon \in ]0, p - 1[. \tag{2.10}
\]
Changing the order of integration, we get
\[
\int_{\nu(\sigma)}^{\nu(\varepsilon)} S^p_\psi(\nu(\sigma)) d\sigma ds \leq \varepsilon \sup_{0 < t < 1} S^p_\psi(t) \| f^* \|_{L^p(t,1)}^p \quad \forall \varepsilon \in ]0, p - 1[, \tag{2.11}
\]
from which (2.8) follows.

Now we prove (2.9). For any $t > 1$, we have that $s < \sigma < \frac{p - 1}{p} - 1$ whenever $0 < \sigma < \varepsilon/t$ and $0 < \varepsilon < p - 1$. Therefore,
\[
t \psi^{-p} \left( \frac{p - 1}{t} \right) \Psi \left( \frac{\varepsilon}{t} \right) = t \psi^{-p} \left( \frac{p - 1}{t} \right) \int_0^{\varepsilon/t} \sup_{\sigma < \tau < \frac{p - 1}{p} - 1} \left[ \psi^p \left( \frac{p - 1}{\tau} \sigma \right) \frac{d}{d\tau}(\tau \psi^p(\tau)) \right] d\sigma \]
\[
(\text{choice } \tau = \sigma) \geq t \int_0^{\varepsilon/t} \frac{d}{d\tau}(\tau \psi^p(\tau))|_{\tau = \sigma} d\sigma \]
\[
= \int_0^{\varepsilon} \frac{d}{d\tau}(\tau \psi^p(\tau))|_{\tau = \sigma} d\sigma \]
\[
= \varepsilon [\psi(\varepsilon)]^p. \tag{2.12}
\]
This completes the proof. \hspace{1cm} \square

Now we can estimate $I_1$. By Hölder’s inequality and (2.8), we have, for every $0 < \varepsilon < p - 1$,
\[
I_1(\varepsilon) = \psi(\varepsilon) \left( \int_0^{\nu(\varepsilon)} f^*(s)^{p - \varepsilon} ds \right)^{\frac{1}{p - \varepsilon}}
\]
\[
= \psi(\varepsilon) \left( \int_0^{\nu(\varepsilon)} \left[ \Psi(\nu^{-1}(s)) \right]^{-\frac{p - \varepsilon}{p}} \left[ \Psi(\nu^{-1}(s)) f^*(s)^p \right]^{\frac{p - \varepsilon}{p}} ds \right)^{\frac{1}{p - \varepsilon}}
\]
\[
\leq \psi(\varepsilon) \left[ \int_0^{\nu(\varepsilon)} \left[ \Psi(\nu^{-1}(s)) \right]^{-\frac{p - \varepsilon}{\varepsilon}} ds \right]^{\frac{\varepsilon}{p(p - \varepsilon)}} \left[ \int_0^{\nu(\varepsilon)} \Psi(\nu^{-1}(s)) f^*(s)^p ds \right]^{\frac{1}{p}}
\]
\[
\leq \varepsilon^{\frac{1}{p - \varepsilon}} \psi(\varepsilon) \left[ \int_0^{\nu(\varepsilon)} \left[ \Psi(\nu^{-1}(s)) \right]^{-\frac{p - \varepsilon}{\varepsilon}} ds \right]^{\frac{\varepsilon}{p(p - \varepsilon)}} \sup_{0 < t < 1} S_\psi(t) \| f^* \|_{L^p(t,1)}
\]
\[
=: C(\varepsilon) \sup_{0 < t < 1} S_\psi(t) \| f^* \|_{L^p(t,1)}, \tag{2.13}
\]
where coefficient $C(\varepsilon)$ is defined by

$$C(\varepsilon) := \varepsilon^{\frac{1}{p^*}} \psi(\varepsilon) \left[ \int_0^{\nu(\varepsilon)} \left[ \Psi(\nu^{-1}(s)) \right]^{\frac{p-1}{p}} ds \right]^{\frac{\varepsilon}{p(p-1)}}.$$  

(2.14)

The coefficient $C(\varepsilon)$ can be estimated as follows (recall that $\nu(0^+) = 0$):

$$C(\varepsilon) \overset{s=\nu(\varepsilon/t)}{=} \varepsilon^{\frac{1}{p^*}} \psi(\varepsilon) \left[ \int_1^{\infty} \left[ \frac{t}{\psi^p \left( \frac{p-1}{t} \right)} \Psi \left( \frac{\varepsilon}{t} \right) \right]^{\frac{p-1}{p}} \left[ -\frac{d}{dt} \left( \nu \left( \frac{\varepsilon}{t} \right) \right) \right]^{\frac{\varepsilon}{p(p-1)}} dt \right]^{\frac{\varepsilon}{p(p-1)}}$$

$$\overset{(2.9)}{=} \varepsilon^{\frac{1}{p^*}} \psi(\varepsilon) \left[ \int_1^{\infty} \left[ \varepsilon \psi^p(\varepsilon) \frac{p-1}{p} \right]^{\frac{p-1}{p}} \left[ -\frac{d}{dt} \left( \nu \left( \frac{\varepsilon}{t} \right) \right) \right]^{\frac{\varepsilon}{p(p-1)}} dt \right]^{\frac{\varepsilon}{p(p-1)}}$$

$$= \left[ \int_1^{\infty} \left[ \frac{t}{\psi^p \left( \frac{p-1}{t} \right)} \right]^{\frac{p-1}{p}} \left[ -\frac{d}{dt} \left( \nu \left( \frac{\varepsilon}{t} \right) \right) \right]^{\frac{\varepsilon}{p(p-1)}} dt \right]^{\frac{\varepsilon}{p(p-1)}}. \quad (2.15)$$

Recalling that for every $x \in ]0, p - 1[$

$$\nu(x) = c_\psi \int_0^x e^{-\frac{p-1}{\sigma}} \inf_{\sigma < \lambda < p-1} \left[ \frac{\psi \left( \frac{(p-1)\sigma}{\lambda} \right)}{\psi(p-1)} \right]^{\frac{\sigma^2}{\lambda}} d\sigma \frac{\sigma^2}{\sigma^2},$$

we obtain that

$$\nu \left( \frac{\varepsilon}{t} \right) = c_\psi \int_0^{\frac{\varepsilon}{t}} e^{-\frac{p-1}{\sigma}} \inf_{\sigma < \lambda < p-1} \left[ \frac{\psi \left( \frac{(p-1)\sigma}{\lambda} \right)}{\psi(p-1)} \right]^{\frac{\sigma^2}{\lambda}} d\sigma \frac{\sigma^2}{\sigma^2}.$$  

Therefore, we can estimate the derivative of $\nu \left( \frac{\varepsilon}{t} \right)$ as follows:

$$-\frac{d}{dt} \left( \nu \left( \frac{\varepsilon}{t} \right) \right) = c_\psi e^{-\frac{p-1}{\varepsilon}} \inf_{\varepsilon < \lambda < p-1} \left[ \frac{\psi \left( \frac{(p-1)\varepsilon}{\lambda} \right)}{\psi(p-1)} \right]^{\frac{\varepsilon^2}{\lambda}} \frac{\varepsilon^2 t^{-2}}{\varepsilon^2 t^{-2}} \leq c_\psi e^{-\frac{p-1}{\varepsilon}} \left[ \frac{\psi \left( \frac{(p-1)\varepsilon}{\lambda} \right)}{\psi(p-1)} \right]^{\frac{\varepsilon^2}{\lambda}} \frac{\varepsilon^2 t^{-2}}{\varepsilon^2 t^{-2}}$$

from which the following bound on the last integrand in (2.15) follows:

$$\left[ \frac{t}{\psi^p \left( \frac{p-1}{t} \right)} \right]^{\frac{p-1}{p}} \left[ -\frac{d}{dt} \left( \nu \left( \frac{\varepsilon}{t} \right) \right) \right] \leq c_\psi \frac{e^{-\frac{p-1}{\varepsilon}}}{\psi^p \left( \frac{p-1}{t} \right)} \left[ \frac{\psi \left( \frac{(p-1)\varepsilon}{\lambda} \right)}{\psi(p-1)} \right]^{\frac{\varepsilon^2}{\lambda}}. \quad (2.16)$$
The right-hand side, which can be rearranged as follows

\[
\frac{c_\psi}{\varepsilon} t^{\frac{p-\varepsilon}{p}} e^{-\frac{p-1}{\varepsilon} t} \left( \frac{1}{\psi(p-1)} \right)^{\frac{p-\varepsilon}{p}} \left[ \frac{\psi\left(\frac{p}{t}\right)}{\psi(p-1)} \right]^{\frac{p-\varepsilon}{p}} \frac{\varepsilon^2}{\varepsilon} t^{\frac{p-\varepsilon}{p}},
\]

is dominated by

\[
A_\varepsilon(t) := \frac{c_\psi \psi^p(p-1)}{\varepsilon} t^{\frac{p-\varepsilon}{p}} e^{-\frac{p-1}{\varepsilon} t} \left[ \frac{1}{\psi(p-1)} \right]^{\frac{p-\varepsilon}{p}} \left[ \frac{\psi\left(\frac{p}{t}\right)}{\psi(p-1)} \right]^{\frac{p-\varepsilon}{p}} \frac{\varepsilon^2}{\varepsilon},
\]

where \( \hat{c}_{\psi,p} := c_\psi \psi^p(p-1) \) is a constant not depending on \( \varepsilon \). Combining (2.16) and (2.17), and then integrating, we get that

\[
\left[ \int_1^\infty \left[ \frac{t}{\psi\left(\frac{p}{t}\right)} \right]^{\frac{p-\varepsilon}{p}} \left[ - \frac{d}{dt} \left( \frac{\varepsilon}{t} \right) \right] dt \right]^{\varepsilon \frac{p}{\varepsilon} \frac{1}{\psi(p-1)}} \leq \left( \hat{c}_{\psi,p} \frac{1}{\varepsilon} \right)^{\frac{p-\varepsilon}{p}} \frac{\varepsilon^2}{\varepsilon} \int_1^\infty t^{\frac{p-\varepsilon}{p}} e^{-\frac{p-1}{\varepsilon} t} dt.
\]

(2.18)

By the previous relation (2.18) we infer that

\[
k_1 := \sup_{0<\varepsilon<p-1} C(\varepsilon) < \infty,
\]

with \( k_1 \) depending only on \( p \) and \( \psi \). Indeed, the only non-trivial term to investigate is

\[
\varepsilon \in ]0, p-1[ \rightarrow \left[ \int_1^\infty t^{\frac{p-\varepsilon}{p}} e^{-\frac{p-1}{\varepsilon} t} dt \right]^{\frac{p^2}{\varepsilon(p-\varepsilon)}}.
\]

But this term has already been proven to be bounded, by a constant independent of \( \varepsilon \), at the end of the proof of Theorem 3.1 in [5] or in (2.12) in [6], through an asymptotic expansion of the incomplete Euler’s gamma function. Overall, combining estimates (2.13) and (2.18) we obtain that

\[
I_1(\varepsilon) \leq k_1 \sup_{0<t<1} S_\psi(t) \| f^* \|_{L^p(t,1)}.
\]

(2.20)

**Step 4** (Combining the estimates). Using (2.20) and (2.6), we can estimate (2.3) to obtain

\[
\sup_{0<\varepsilon<p-1} \psi(\varepsilon) \| f \|_{L^{p-\varepsilon}(0,1)} \leq (k_1 + k_2) \sup_{0<t<1} S_\psi(t) \| f^* \|_{L^p(t,1)}.
\]

This completes the proof.
3. The optimality result

In the introduction of the paper, we stated that if \( \psi \) is in a class of functions satisfying the \( \Delta_2 \) condition, essentially characterized by the so-called \( \nabla' \) condition, then inequality (1.1) is sharp, in the sense that both sides are equivalent. This section is devoted to clarifying this optimality claim.

Let \( p > 1 \) and \( \psi : [0, p - 1] \to ]0, \infty[ \) be nondecreasing. In [6] the notion of \( \Delta_2 \) condition, which is classical in the Orlicz space theory for convex functions (see, e.g., [10, Def. 1 p. 22]) has been used for functions defined on intervals of the type \( [0, p - 1] \), in the form

\[
\psi \in \Delta_2 \iff \psi(2\varepsilon) \leq c\psi(\varepsilon)
\]

for some \( c > 1 \) and for every \( \varepsilon \) sufficiently small. In our framework, we will need this notion of condition \( \Delta_2 \), along with the following two remarks.

- First, functions satisfying the \( \Delta_2 \) condition are greater than some positive power near the origin. The idea of the proof can be found in [10, Corollary 5 p. 26]. For completeness we give here the details. Incidentally, unlike the one in [10], the following argument does not make use of any convexity or concavity assumptions.

Let \( n_0 \in \mathbb{N} \) be such that \( 2^{-n_0} < p - 1 \) and set

\[
q := \sup_{m \in \mathbb{N}} \frac{m \log_2 c}{n_0 + m - 1} = \log_2 c,
\]

where \( c > 1 \) is from the \( \Delta_2 \) condition in (3.1). For every \( 0 < x < 2^{-n_0} \), denoting by \( m \) the integer such that

\[
2^{-n_0 - m} \leq x < 2^{-n_0 - m + 1},
\]

we have

\[
\psi(x) \geq \psi(2^{-n_0 - m}) \geq c^{-m} \psi(2^{-n_0}) = \psi(2^{-n_0}) 2^{\log_2 c}^{-m} = \psi(2^{-n_0}) 2^{-m \log_2 c} = \psi(2^{-n_0}) (2^{-n_0 - m + 1})^{m \log_2 c} \geq \psi(2^{-n_0}) (2^{-n_0 - m + 1})^q > \psi(2^{-n_0}) x^q.
\]

This concludes the argument.

- Second, we observe that if

\[
\sup_{\varepsilon \in [0, p - 1]} \frac{\varepsilon \psi'(\varepsilon)}{\psi(\varepsilon)} < \infty,
\]

then \( \psi \in \Delta_2 \). Indeed, (3.5) implies that for some \( q > 0 \) the function \( \psi(t)/t^q \) is nonincreasing, and the comparison of its values in \( 2t \) and \( t \) gives immediately the \( \Delta_2 \) condition.
The next proposition states our second main result and concerns the optimality of
the estimate in (1.1). The claim borrows another growth condition well-known in
the theory of Orlicz spaces: namely, the $\nabla'$ condition (see, e.g., [10, Def. 7 p. 28]). We set
\[
\psi \in \nabla' \iff \exists b > 0 : \psi(x)\psi(y) \leq \psi(bxy) \text{ for } x, y, bxy \in ]0, p-1[. \tag{3.6}
\]

**Proposition 3.1.** Let $p > 1$ and $\psi : ]0, p-1[ \to ]0, \infty[$ be nondecreasing and differentiable. If $\psi \in \nabla'$ satisfies (3.5), then, for every $t \in ]0, 1[$, there holds
\[
S_\psi(t) = \sup_{\nu^{-1}(t) < \tau < p-1} \psi \left( \frac{p-1}{\tau} \nu^{-1}(t) \right) \left[ \frac{d}{d\tau} (\tau \psi^p(\tau)) \right] \frac{1}{p} \approx \psi \left( \frac{p-1}{1 - \log t} \right)
\]

**Proof.** We split the proof in two steps.

**Step 1.** First, we show that
\[
S_\psi(t) \gtrsim \psi \left( \frac{p-1}{1 - \log t} \right) \forall t \in ]0, 1[. \tag{3.7}
\]

In the proof of Theorem 2.1 we already remarked that (see (2.4)) $S_\psi(\nu(\epsilon)) \geq k_2^{-1} \psi(\epsilon)$, and therefore, setting $\nu(\epsilon) = t$, we have $S_\psi(t) \geq k_2^{-1} \psi(\nu^{-1}(t))$. Since
\[
\inf_{\sigma < \lambda < p-1} \left[ \frac{\psi \left( \frac{(p-1)\sigma}{\lambda} \right)}{\psi(p-1)} \right] \leq 1,
\]
we have that for any $0 < x < p-1$
\[
\nu(x) = c_\psi \int_0^x e^{-\frac{x-1}{\sigma}} \inf_{\sigma < \lambda < p-1} \left[ \frac{\psi \left( \frac{(p-1)\sigma}{\lambda} \right)}{\psi(p-1)} \right] \frac{d\sigma}{\sigma^2}
\leq c_\psi \int_0^x e^{-\frac{x-1}{\sigma}} \frac{d\sigma}{\sigma^2} = c_\psi \frac{1}{p-1} e^{-\frac{x-1}{x}}
\leq c_\psi \frac{p}{p-1} e^{-\frac{x-1}{x}}. \tag{3.8}
\]

Therefore, since $0 < \nu(x) < 1$, we have $-|\log \nu(x)| \leq |\log \left( \frac{pc_\psi}{p-1} \right)| - \frac{p-1}{x}$, from which we infer that
\[
\frac{1}{x} \leq \frac{|\log \nu(x)| + |\log \left( \frac{pc_\psi}{p-1} \right)|}{p-1} \leq \kappa_p \frac{1 + |\log \nu(x)|}{p-1}
\]
with \( \kappa_p := \left( 1 + \left| \log \left( \frac{\mu_p}{p-1} \right) \right| \right) \). Overall, we obtain the following lower bound on \( \nu^{-1} \):

\[
p - 1 \geq \nu^{-1}(t) \geq \kappa_p^{-1} \frac{p-1}{1 + |\log t|}, \quad \forall t \in [0, 1].
\]  

But \( S_\psi(t) \geq \kappa_p^{-1} \psi(\nu^{-1}(t)) \), and therefore, taking into account that \( \psi \) is nondecreasing, from (3.9) we infer that

\[
S_\psi(t) \geq \frac{1}{k_2} \psi(\nu^{-1}(t)) \geq \frac{1}{k_2} \psi \left( \frac{p-1}{\kappa_p(1 + |\log t|)} \right).
\]

Finally, by the \( \Delta_2 \) condition (precisely, as a consequence of (3.5)), we know that \( \psi(\varepsilon) \leq c^m \psi(2^{-m} \varepsilon) \) for every \( m \in \mathbb{N} \). Hence

\[
S_\psi(t) \geq \frac{1}{k_2} \psi \left( \frac{p-1}{\kappa_p(1 + |\log t|)} \right) \geq \frac{1}{k_2 c^m} \psi \left( \frac{p-1}{1 + |\log t|} \right)
\]

with \( m_p \in \mathbb{N} \) such that \( \kappa_p^{-1} > 2^{-m_p} \). This completes the proof of the estimate (3.7).

**Step 2.** Now, we show that

\[
S_\psi(t) \lesssim \psi \left( \frac{p-1}{1 - \log t} \right) \quad \forall t \in [0, 1].
\]  

By (3.4), the property \( \psi \in \Delta_2 \) entrains that \( \psi(\varepsilon) \geq c \varepsilon^q \) for some \( c > 0 \) and some \( q > 0 \). Hence, taking into account the assumption (3.5) we are allowed to set

\[
q_* := \max \left\{ q, \sup_{\varepsilon \in [0, p-1]} \varepsilon \psi'(\varepsilon) \right\} < \infty.
\]  

Then, we proceed to estimate the infimum that appears in the definition of \( \nu \). We begin showing the existence of a constant \( 0 < k < 1 \) such that for all \( 0 < \sigma < p - 1 \) there holds

\[
\inf_{\sigma < \lambda < p-1} \left[ \psi \left( \frac{(p-1)\sigma}{\lambda} \right) \right]^\frac{p^2}{\lambda} \geq k^{\frac{p-1}{\sigma}},
\]  

where the constant \( 0 < k < 1 \) can be expressed under the form

\[
k := c_*^{\frac{p^2}{(p-1)}} \left[ \inf_{0 < \tau < 1} \tau^r \right]^{\frac{q_* p^2}{p-1}}, \quad c_* := \min \left\{ \frac{c}{\psi(p-1)(p-1)^{q_*}}, 1 \right\}.
\]
Indeed, classical estimates give the following chain of inequalities:

\[
\inf_{\sigma<\lambda<p^{-1}} \left[ \frac{\psi \left( \frac{(p-1)\sigma}{\lambda} \right)}{\psi(p-1)} \right] \geq \inf_{\sigma<\lambda<p^{-1}} \left[ \frac{c(p-1)^q \psi \left( \frac{(p-1)\sigma}{\lambda} \right)}{\psi(p-1)} \right] \geq \inf_{\sigma<\lambda<p^{-1}} \frac{c^p}{\lambda} \left[ \left( \frac{\sigma}{\lambda} \right) \right]^{q^*} \frac{x^2}{\lambda}
\]

\[
\geq c^p \inf_{\sigma<\lambda<p^{-1}} \left[ \left( \frac{\sigma}{\lambda} \right) \right]^{q^*} \frac{x^2}{\lambda}
\]

\[
= c^p \inf_{\sigma<\lambda<p^{-1}} \left[ \left( \frac{\sigma}{\lambda} \right) \right]^{q^*} \frac{x^2}{\lambda}
\]

(3.14)

Taking into account that \(0 < \sigma/\lambda < 1\) we get

\[
(3.14) = c^p \inf_{0<\tau<1} \left[ \left( \frac{\tau}{\lambda} \right) \right]^{q^*} \frac{x^2}{\lambda}
\]

\[
\geq k \frac{x^p}{\lambda}
\]

(3.15)

This concludes the proof of (3.12). After that, we can estimate \(\nu\) as follows:

\[
\nu(x) = c_\psi \int_0^x e^{-\frac{p-1}{\sigma} \frac{x^2}{\lambda}} \inf_{\sigma<\lambda<p^{-1}} \left[ \frac{\psi \left( \frac{(p-1)\sigma}{\lambda} \right)}{\psi(p-1)} \right] \frac{x^2}{\lambda} d\sigma \frac{d\sigma}{\sigma^2}
\]

\[
\geq c_\psi \int_0^x e^{-\frac{p-1}{\sigma} \frac{x^2}{\lambda}} k \frac{p-1}{\sigma} \frac{d\sigma}{\sigma^2}
\]

\[
= c_\psi \int_0^x E^{-\frac{p-1}{\sigma}} \frac{d\sigma}{\sigma^2}
\]

\[
= \frac{c_\psi}{(p-1) \log E} E^{-\frac{p-1}{x}}
\]

\[
\geq a E^{-\frac{p-1}{x}}
\]

(3.16)
with \( a := \min \left\{ \frac{c_1}{(p-1) \log E}, 1 \right\} \) and \( E := e/k > e \) because of \( k < 1 \). Taking the logarithm of both sides of the previous relation, we obtain that for every \( x \in ]0, p-1[ \)
\[
-|\log_E \nu(x)| \geq -|\log_E a| - \frac{p-1}{x}.
\]
This last expression can be rewritten under the form
\[
x|\log \nu(x)| \leq (p-1) \log E + x|\log a|
\]
from which the next upper bound follows
\[
x(1 + |\log \nu(x)|) \leq (p-1) \log E + (p-1)(1 + |\log a|).
\]
Overall, we get that for every \( x \in ]0, p-1[ \) there holds
\[
x \leq \frac{c_*(p-1)}{1 + |\log \nu(x)|}, \tag{3.17}
\]
with \( c_* := \log E + 1 + \log a \). Since \( \nu \) is an increasing function we get that
\[
\nu^{-1}(t) \leq c_* \frac{p-1}{1 + |\log t|}, \quad \forall 0 < t < 1. \tag{3.18}
\]
Finally, by (3.5) and the \( \nabla' \) condition (3.6) we have that
\[
S_{\psi}(\nu(\varepsilon)) = \sup_{\varepsilon < \tau < p-1} \psi \left( \frac{p-1}{\tau} \varepsilon \right) \left[ \frac{d}{d\tau} (\tau \psi^p(\tau)) \right]^\frac{1}{p} \\
= \sup_{\varepsilon < \tau < p-1} \psi \left( \frac{p-1}{\tau} \varepsilon \right) [\psi^p(\tau) + p\psi^{p-1}(\tau)\tau\psi'(\tau)]^\frac{1}{p} \\
\leq \sup_{\varepsilon < \tau < p-1} \psi \left( \frac{p-1}{\tau} \varepsilon \right) [\psi^p(\tau) + pq\psi^{p-1}(\tau)q_*\psi(\tau)]^\frac{1}{p} \\
= [1 + pq_*]^\frac{1}{p} \sup_{\varepsilon < \tau < p-1} \psi \left( \frac{p-1}{\tau} \varepsilon \right) \psi(\tau) \\
\leq [1 + pq_*]^\frac{1}{p} \psi(b(p-1)\varepsilon) \tag{3.19} \\
\lesssim \psi(\varepsilon), \tag{3.20}
\]
where in the last \( \lesssim \) we used the \( \Delta_2 \) condition. This completes the proof of (3.10) and, therefore, also the proof of the main statement. \[\square\]

**Remark 3.1.** If \( \psi \) is of power type, i.e. if \( c_1\varepsilon^q \leq \psi(\varepsilon) \leq c_2\varepsilon^q \) for some positive \( c_1, c_2, q \), then \( \psi \in \nabla' \) and (3.5) is satisfied. Therefore in this case the estimate (1.1) proved in our main result Theorem 2.1 is optimal.
Corollary 3.2. Let $p > 1$ and $\psi : ]0, p - 1[ \to ]0, \infty[ $ be nondecreasing and differentiable. If $\psi \in \nabla'$ and (3.5) holds, then
\[
\sup_{0 < \epsilon < p - 1} \psi(\epsilon) \| f \|_{L^{p - \epsilon}(0,1)} \approx \sup_{0 < t < 1} S_{\psi}(t) \| f^* \|_{L^p(t,1)}.
\] (3.21)

Proof. By Proposition 3.1 we have $S_{\psi}(t) \approx \psi \left( \frac{p - 1}{1 - \log t} \right)$, and, since the assumption entrains that $\psi \in \Delta_2$, from [6] we get that the equivalence holds.

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