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# AN ESTIMATE OF THE BLOW-UP OF LEBESGUE NORMS IN THE NON-TEMPERED CASE

GIOVANNI DI FRATTA, ALBERTO FIORENZA, AND VALERIY SLASTIKOV

Abstract. We prove that if $p > 1$ and $\psi : ]0, p-1] \to ]0, \infty[$ is just nondecreasing and differentiable (hence not necessarily $\Delta_2$), then for every $f$ Lebesgue measurable function on $(0, 1)$

$$(*) \qquad \sup_{0<\varepsilon<p-1} \psi(\varepsilon)\|f\|_{L^{p-\varepsilon}(0,1)} \lesssim \sup_{0<t<1} S_\psi(t)\|f^*\|_{L^p(t,1)}\,,$$

where $f^*$ denotes the decreasing rearrangement of $f$ and $S_\psi$ is defined, for $\varepsilon \in ]0, p-1[$, through

$$\begin{cases} S_\psi(\nu(\varepsilon)) &= \displaystyle\sup_{\varepsilon<\tau<p-1} \psi\left(\frac{p-1}{\tau}\varepsilon\right)\left[\frac{d}{d\tau}(\tau\psi^p(\tau))\right]^{\frac{1}{p}} \\ \nu(\varepsilon) &= c_\psi \displaystyle\int_0^\varepsilon e^{-\frac{p-1}{\sigma}} \inf_{\sigma<\lambda<p-1}\left[\frac{\psi\left(\frac{(p-1)\sigma}{\lambda}\right)}{\psi(p-1)}\right]^{\frac{p^2}{\lambda}} \frac{d\sigma}{\sigma^2}\,, \end{cases}$$

where $c_\psi$ is the normalizing constant chosen so that $\nu((p-1)-) = 1$. If $\psi$ is in a class of functions satisfying the $\Delta_2$ condition, essentially characterized by the so-called $\nabla'$ condition, then inequality $(*)$ is sharp, in the sense that both sides are equivalent. Estimate $(*)$ generalizes an inequality of the type obtained by the second author with Farroni and Giova in [6] under the growth condition $\psi \in \Delta_2$.

## 1. Introduction

This paper is a follow-up of [1,5,6,8], where, by different methods and overlapping special cases, the blow-up, as $\varepsilon \to 0$, of the function norm

$$F : \varepsilon \in ]0, p-1[ \;\mapsto\; F(\varepsilon) = \|f\|_{L^{p-\varepsilon}(\Omega)} \in ]0, \infty[$$

has been studied. Here, $1 < p < \infty$ and $f$ is a (Lebesgue) measurable function $f$ over $\Omega \subset \mathbb{R}^n$ ($n \geqslant 1$, $|\Omega| = 1$) such that

$$f \in \bigcap_{0<\varepsilon<p-1} L^{p-\varepsilon}(\Omega)\,.$$

We consider a nondecreasing and differentiable function $\psi : ]0, p-1] \to ]0, \infty[$, and we prove the existence of a function $S_\psi$ (which also depends on $p$) such that for every Lebesgue

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measurable function $f$ on $(0,1)$ there holds

$$\sup_{0<\varepsilon<p-1} \psi(\varepsilon)\|f\|_{L^{p-\varepsilon}(0,1)} \lesssim \sup_{0<t<1} S_\psi(t)\|f^*\|_{L^p(t,1)}\,, \tag{1.1}$$

where $f^*$ denotes the decreasing rearrangement of $f$.

Symbols $\lesssim$ and $\approx$ above (and analogously the symbol $\gtrsim$) mean, as usual, inequalities up to not influent constants (in our case, depending possibly only on $p$ and $\psi$).

The estimate (1.1) is optimal in the following precise sense: if $\psi$ is in the class of functions essentially characterized by the so-called $\nabla'$ condition (see Section 3), then both sides of (1.1) are equivalent. In fact, when $\psi$ is in this class, we prove that the following equivalence holds (see Proposition 3.1):

$$S_\psi(t) \approx \psi\left(\frac{p-1}{1-\log t}\right)\,. \tag{1.2}$$

It is known from [6] that if (1.2) holds, the inequality in (1.1) is an equivalence and, moreover, that such equivalence holds if and only if $\psi$ satisfies the growth condition $\psi \in \Delta_2 \cap L^\infty$. Precisely, the next result holds.

**Theorem 1.1** (cf. Theorem 1.1 in [6])**.** *If $1 < p < \infty$, and $\psi : ]0, p-1] \to ]0,\infty[$ is nondecreasing, then for all Lebesgue measurable functions $f$ on $(0,1)$*

$$\left(\sup_{0<\varepsilon<p-1} \psi(\varepsilon)\|f\|_{L^{p-\varepsilon}(0,1)} \approx \sup_{0<t<1} \psi\left(\frac{p-1}{1-\log t}\right)\|f^*\|_{L^p(t,1)}\right) \quad \Longleftrightarrow \quad \psi \in \Delta_2 \cap L^\infty\,. \tag{1.3}$$

We recall that since $\psi$ is, in particular, nondecreasing and continuous, the growth condition $\psi \in \Delta_2$ implies that $\psi(2\cdot) \approx \psi(\cdot)$, which means that $\psi$ is *tempered* (see [9]).

When no hypotheses on the growth of $\psi$ are assumed, given the characterization in Theorem 1.1, it is natural to ask whether or not there exists a function $S_\psi$ such that (1.1) holds and such that it is equivalent to the optimal expression in (1.2), at least in the case when $\psi$ is of power type (see Proposition 3.1). The question is of some importance in the study of generalized grand Lebesgue spaces and is stated in a recent work by Astashkin and Milman where the authors collect some open problems in the area of Extrapolation Theory (cf. Problem 20 in [2]).

The interest of the problem stems from the premise that the left-hand side of (1.1) is a Banach function norm introduced in [3] and investigated in [4] (see [7] for the more detailed history). The difficulty of the problem lies in the fact that the optimality obtained in [6] requires an expression for $S_\psi$ to be *essentially different* comparing to the case $\psi \in \Delta_2$.

We emphasize that, in (1.3), a precise choice of $S_\psi$ is involved. In fact, in our setting, one can think about (1.3) as claiming that if $\psi(\cdot) \in \Delta_2 \cap L^\infty$, then the optimal map $\psi \mapsto S_\psi$ for which the equivalence is achieved, can be chosen to be pointwise related to $\psi$ and even linear in $\psi$. However, a careful analysis of the estimates in [5] and [6], suggests that this is no more the case under weaker hypotheses on $\psi$. This is the motivation behind the

more complex expression of $S_\psi$ which now appears as a nonlinear and nonlocal function of $\psi$.

The ultimate natural goal would be to obtain an *optimal* expression — if it exists — of the type of the right-hand side of (1.1) valid for *all* possible choices of increasing $\psi$. This goal is not reached in this paper. However (see Theorem 2.1 below), our result moves towards the right direction: we find an explicit expression of a function $S_\psi$ satisfying (1.1) which gives back the optimal result in (1.3) at least in the case $\psi$ of power type. Our argument uses a careful variant of the techniques originated in [5] and generalized in [6].

We stress that an important feature of the expression for $S_\psi$ is to become smaller as soon as $\psi$ and $\psi'$ become smaller: without this property, (1.1) holds trivially setting $S_\psi \equiv 1$, although then the upper bound in (1.1) would be uninformative in the most interesting case, because it would be infinity when $f \in L^{p-\varepsilon}(0,1)$ for $0 < \varepsilon < p-1$, and $f \notin L^p(0,1)$.

The paper is organized as follows. In Section 2 we state and prove our main result. In Section 3 we first discuss typical growth conditions on $\psi$ and then prove that the estimate (1.1) is optimal when $\psi$ is in the class of functions essentially characterized by the $\nabla'$ condition.

## 2. The main result

**Theorem 2.1.** *If $p > 1$ and $\psi : ]0, p-1]] \to ]0, \infty[$ is nondecreasing and differentiable, then (1.1) holds for every $f$ Lebesgue measurable function on $(0,1)$, with $S_\psi$ defined, for $\varepsilon \in ]0, p-1[$, through*

$$\begin{cases} S_\psi(\nu(\varepsilon)) &= \displaystyle\sup_{\varepsilon<\tau<p-1} \psi\left(\frac{p-1}{\tau}\varepsilon\right)\left[\frac{d}{d\tau}(\tau\psi^p(\tau))\right]^{\frac{1}{p}} \\ \nu(\varepsilon) &= c_\psi \displaystyle\int_0^\varepsilon e^{-\frac{p-1}{\sigma}} \inf_{\sigma<\lambda<p-1}\left[\frac{\psi\left(\frac{(p-1)\sigma}{\lambda}\right)}{\psi(p-1)}\right]^{\frac{p^2}{\lambda}} \frac{d\sigma}{\sigma^2}, \end{cases}$$

*where $c_\psi$ is the normalizing constant chosen so that $\nu((p-1)-) = 1$.*

**Remark 2.1.** It is possible to show that tempered functions, i.e., functions satisfying the $\Delta_2$ condition, are *bounded from below* (near the origin) by some (and therefore *any* sufficiently large) positive power function — see the discussion before the statement of Proposition 3.1. However, in order to control the blow-up of the $L^p$ norm, one is mainly interested in "weights" $\psi^p$ which are very flat around the origin, i.e., basically convex. In this case, we can show that $S_\psi$ is a nondecreasing function, and therefore it optimally controls around the origin the blow-up of $\|f^*\|_{L^p(t,1)}$.

The proof goes as follows: setting $\tau(\varepsilon,\lambda) := \lambda + \frac{\varepsilon}{p-1}(1-\lambda)$, $0<\lambda<1$, we can rewrite $S^p_\psi$ under the form

$$S^p_\psi(\nu(\varepsilon)) = \sup_{0<\lambda<1} \psi^p\left(\frac{\varepsilon}{\tau(\varepsilon,\lambda)}\right) \cdot \frac{d}{d\tau}(\tau\psi^p(\tau))\Big|_{\tau:=(p-1)\tau(\varepsilon,\lambda)} .$$

Note that, for any $0<\lambda<1$, both $\tau(\varepsilon,\lambda)$ and $\varepsilon/\tau(\varepsilon,\lambda)$ are increasing functions of $\varepsilon$. Also we have

$$\frac{d}{d\tau}(\tau\psi^p(\tau))\Big|_{\tau:=(p-1)\tau(\varepsilon,\lambda)} = \Big[\psi^p((p-1)\tau(\varepsilon,\lambda)) + (p-1)\tau(\varepsilon,\lambda)(\psi^p)'((p-1)\tau(\varepsilon,\lambda))\Big] .$$

Therefore, if $\psi^p$ is convex, we have that the function

$$\begin{aligned}\psi^p\left(\frac{\varepsilon}{\tau(\varepsilon,\lambda)}\right) &\frac{d}{d\tau}(\tau\psi^p(\tau))\Big|_{\tau:=(p-1)\tau(\varepsilon,\lambda)} \\ &= \psi^p\left(\frac{\varepsilon}{\tau(\varepsilon,\lambda)}\right)\Big[\psi^p((p-1)\tau(\varepsilon,\lambda)) + (p-1)\tau(\varepsilon,\lambda)(\psi^p)'((p-1)\tau(\varepsilon,\lambda))\Big]\end{aligned}$$

is nondecreasing in $\varepsilon$ because is the product of nondecreasing functions of $\varepsilon$. Our assertion follows, because suprema of nondecreasing functions are nondecreasing functions.

*Proof of Theorem 2.1.* We split the proof in four steps.

**Step 1.** Proceeding as in [6], we split left-hand side of (1.1) as follows:

$$\begin{aligned}\sup_{0<\varepsilon<p-1} \psi(\varepsilon)\|f\|_{L^{p-\varepsilon}(0,1)} &= \sup_{0<\varepsilon<p-1} \psi(\varepsilon)\left[\int_0^1 |f(x)|^{p-\varepsilon}dx\right]^{\frac{1}{p-\varepsilon}} \\ &= \sup_{0<\varepsilon<p-1} \psi(\varepsilon)\left[\int_0^1 f^*(s)^{p-\varepsilon}ds\right]^{\frac{1}{p-\varepsilon}} \\ &= \sup_{0<\varepsilon<p-1} \psi(\varepsilon)\left[\int_0^{\nu(\varepsilon)} f^*(s)^{p-\varepsilon}ds + \int_{\nu(\varepsilon)}^1 f^*(s)^{p-\varepsilon}ds\right]^{\frac{1}{p-\varepsilon}} \qquad (2.1)\end{aligned}$$

To shorten notation, it is convenient to set

$$I_1(\varepsilon) := \psi(\varepsilon)\left(\int_0^{\nu(\varepsilon)} f^*(s)^{p-\varepsilon}ds\right)^{\frac{1}{p-\varepsilon}}, \qquad I_2(\varepsilon) := \psi(\varepsilon)\left(\int_{\nu(\varepsilon)}^1 f^*(s)^{p-\varepsilon}ds\right)^{\frac{1}{p-\varepsilon}}. \qquad (2.2)$$

From (2.1) we immediately infer the following upper bound:

$$\sup_{0<\varepsilon<p-1} \psi(\varepsilon)\|f\|_{L^{p-\varepsilon}(0,1)} \leqslant \sup_{0<\varepsilon<p-1} I_1(\varepsilon) + \sup_{0<\varepsilon<p-1} I_2(\varepsilon) . \qquad (2.3)$$

Next, we estimate $I_1$ and $I_2$ separately. We start by estimating $I_2$.

**Step 2** (Estimate for $I_2$). Setting $\tau = p-1$ in the supremum defining $S_\psi(\nu(\varepsilon))$, we get that $S_\psi(\nu(\varepsilon)) \gtrsim \psi(\varepsilon)$, up to a constant factor that does not depend on $\varepsilon$. Precisely, we have

$$S_\psi(\nu(\varepsilon)) \geqslant \left(\frac{d}{d\tau}(\tau\psi^p(\tau))_{|\tau=p-1}\right)^{\frac{1}{p}} \psi(\varepsilon) \geqslant k_2^{-1}\psi(\varepsilon)\,, \tag{2.4}$$

with $k_2^{-1} := \left(\frac{d}{d\tau}(\tau\psi^p(\tau))_{|\tau=p-1}\right)^{\frac{1}{p}}$. Note that $k_2 > 0$ because it consists of the sum of two positive terms, one of which is $\psi^p(p-1) > 0$. Next, by Hölder's inequality, we have

$$\begin{aligned}
\psi(\varepsilon)\left(\int_{\nu(\varepsilon)}^1 f^*(x)^{p-\varepsilon}\,dx\right)^{\frac{1}{p-\varepsilon}} &\leqslant \psi(\varepsilon)\left(\int_{\nu(\varepsilon)}^1 dx\right)^{\frac{\varepsilon}{p(p-\varepsilon)}}\left(\int_{\nu(\varepsilon)}^1 f^*(x)^p\,dx\right)^{\frac{1}{p}} \\
&\leqslant k_2(1-\nu(\varepsilon))^{\frac{\varepsilon}{p(p-\varepsilon)}} S_\psi(\nu(\varepsilon))\left(\int_{\nu(\varepsilon)}^1 f^*(x)^p\,dx\right)^{\frac{1}{p}} \\
&\leqslant k_2(1-\nu(\varepsilon))^{\frac{\varepsilon}{p(p-\varepsilon)}} \sup_{0<t<1} S_\psi(t)\|f^*\|_{L^p(t,1)} \\
&\leqslant k_2 \sup_{0<t<1} S_\psi(t)\|f^*\|_{L^p(t,1)}\,.
\end{aligned} \tag{2.5}$$

Overall, we infer the estimate

$$\sup_{0<\varepsilon<p-1} I_2(\varepsilon) \leqslant k_2 \sup_{0<t<1} S_\psi(t)\|f^*\|_{L^p(t,1)}\,. \tag{2.6}$$

**Step 3** (Estimate for $I_1$). To estimate $I_1$ we need some properties of the integral function $\Psi$ defined by

$$\Psi(x) := \int_0^x S_\psi^p(\nu(\sigma))d\sigma\,, \qquad x \in ]0, p-1[\,. \tag{2.7}$$

Note that, thanks to (2.4), $\Psi$ is positive and increasing. We prove the following result.

**Lemma 2.2.** *For every $0 < \varepsilon < p-1$, the following estimate holds:*

$$\int_0^{\nu(\varepsilon)} \Psi(\nu^{-1}(s))f^*(s)^p ds \leqslant \varepsilon \sup_{0<t<1} S_\psi^p(t)\|f^*\|^p_{L^p(t,1)}\,. \tag{2.8}$$

*Moreover, for every $0 < \varepsilon < p-1$ and every $t > 1$, we have*

$$\varepsilon\psi^p(\varepsilon) \leqslant t\left[\psi\left(\frac{p-1}{t}\right)\right]^{-p} \Psi\left(\frac{\varepsilon}{t}\right)\,. \tag{2.9}$$

*Proof.* For any $0 < \sigma < \varepsilon$, recalling that $\nu$ is increasing and that $0 < \nu(\varepsilon) < 1$, we have

$$S_\psi^p(\nu(\sigma))\int_{\nu(\sigma)}^{\nu(\varepsilon)} f^*(s)^p ds \leqslant \sup_{0<\tau<p-1} S_\psi^p(\nu(\tau))\int_{\nu(\tau)}^1 f^*(s)^p ds = \sup_{0<t<1} S_\psi^p(t)\|f^*\|^p_{L^p(t,1)}\,.$$

Integrating both sides with respect to $\sigma$, we obtain

$$\int_0^{\varepsilon} S_\psi^p(\nu(\sigma)) \int_{\nu(\sigma)}^{\nu(\varepsilon)} f^*(s)^p ds d\sigma \leqslant \varepsilon \sup_{0<t<1} S_\psi^p(t) \|f^*\|_{L^p(t,1)}^p \quad \forall \varepsilon \in ]0, p-1[\,. \tag{2.10}$$

Changing the order of integration, we get

$$\int_0^{\nu(\varepsilon)} f^*(s)^p \int_0^{\nu^{-1}(s)} S_\psi^p(\nu(\sigma)) d\sigma\, ds \leqslant \varepsilon \sup_{0<t<1} S_\psi^p(t) \|f^*\|_{L^p(t,1)}^p \quad \forall \varepsilon \in ]0, p-1[\,, \tag{2.11}$$

from which (2.8) follows.

Now we prove (2.9). For any $t > 1$, we have that $\sigma < \sigma t < p-1$ whenever $0 < \sigma < \varepsilon/t$ and $0 < \varepsilon < p-1$. Therefore,

$$\begin{aligned} t\psi^{-p}\left(\frac{p-1}{t}\right)\Psi\left(\frac{\varepsilon}{t}\right) &= t\psi^{-p}\left(\frac{p-1}{t}\right)\int_0^{\varepsilon/t} \sup_{\sigma<\tau<p-1}\left[\psi^p\left(\frac{p-1}{\tau}\sigma\right)\frac{d}{d\tau}(\tau\psi^p(\tau))\right] d\sigma \\ &\overset{(\text{choice } \tau=\sigma t)}{\geqslant} t\int_0^{\varepsilon/t} \frac{d}{d\tau}(\tau\psi^p(\tau))_{|\tau=\sigma t} d\sigma \\ &= \int_0^{\varepsilon} \frac{d}{d\tau}(\tau\psi^p(\tau))_{|\tau=\sigma} d\sigma \\ &= \varepsilon[\psi(\varepsilon)]^p\,. \end{aligned} \tag{2.12}$$

This completes the proof. □

Now we can estimate $I_1$. By Hölder's inequality and (2.8), we have, for every $0 < \varepsilon < p-1$,

$$\begin{aligned} I_1(\varepsilon) &= \psi(\varepsilon)\left(\int_0^{\nu(\varepsilon)} f^*(s)^{p-\varepsilon} ds\right)^{\frac{1}{p-\varepsilon}} \\ &= \psi(\varepsilon)\left(\int_0^{\nu(\varepsilon)} \left[\Psi(\nu^{-1}(s))\right]^{-\frac{p-\varepsilon}{p}} \left[\Psi(\nu^{-1}(s)) f^*(s)^p\right]^{\frac{p-\varepsilon}{p}} ds\right)^{\frac{1}{p-\varepsilon}} \\ &\leqslant \psi(\varepsilon)\left[\int_0^{\nu(\varepsilon)} \left[\Psi(\nu^{-1}(s))\right]^{-\frac{p-\varepsilon}{\varepsilon}} ds\right]^{\frac{\varepsilon}{p(p-\varepsilon)}} \left[\int_0^{\nu(\varepsilon)} \Psi(\nu^{-1}(s)) f^*(s)^p ds\right]^{\frac{1}{p}} \\ &\overset{(2.8)}{\leqslant} \varepsilon^{\frac{1}{p}}\psi(\varepsilon)\left[\int_0^{\nu(\varepsilon)} \left[\Psi(\nu^{-1}(s))\right]^{-\frac{p-\varepsilon}{\varepsilon}} ds\right]^{\frac{\varepsilon}{p(p-\varepsilon)}} \sup_{0<t<1} S_\psi(t)\|f^*\|_{L^p(t,1)} \\ &=: C(\varepsilon)\cdot \sup_{0<t<1} S_\psi(t)\|f^*\|_{L^p(t,1)}\,, \end{aligned} \tag{2.13}$$

where coefficient $C(\varepsilon)$ is defined by

$$C(\varepsilon) := \varepsilon^{\frac{1}{p}}\psi(\varepsilon)\left[\int_0^{\nu(\varepsilon)}\left[\Psi(\nu^{-1}(s))\right]^{-\frac{p-\varepsilon}{\varepsilon}}ds\right]^{\frac{\varepsilon}{p(p-\varepsilon)}}. \tag{2.14}$$

The coefficient $C(\varepsilon)$ can be estimated as follows (recall that $\nu(0+)=0$):

$$\begin{aligned}
C(\varepsilon) &\overset{s=\nu(\varepsilon/t)}{=} \varepsilon^{\frac{1}{p}}\psi(\varepsilon)\left[\int_1^\infty\left[\Psi\left(\frac{\varepsilon}{t}\right)\right]^{-\frac{p-\varepsilon}{\varepsilon}}\left[-\frac{d}{dt}\left(\nu\left(\frac{\varepsilon}{t}\right)\right)\right]dt\right]^{\frac{\varepsilon}{p(p-\varepsilon)}}\\
&= \varepsilon^{\frac{1}{p}}\psi(\varepsilon)\left[\int_1^\infty\left[\frac{t}{\psi^p\left(\frac{p-1}{t}\right)}\Psi\left(\frac{\varepsilon}{t}\right)\frac{\psi^p\left(\frac{p-1}{t}\right)}{t}\right]^{-\frac{p-\varepsilon}{\varepsilon}}\left[-\frac{d}{dt}\left(\nu\left(\frac{\varepsilon}{t}\right)\right)\right]dt\right]^{\frac{\varepsilon}{p(p-\varepsilon)}}\\
&\overset{(2.9)}{\leqslant} \varepsilon^{\frac{1}{p}}\psi(\varepsilon)\left[\int_1^\infty\left[\varepsilon\psi^p(\varepsilon)\frac{\psi^p\left(\frac{p-1}{t}\right)}{t}\right]^{-\frac{p-\varepsilon}{\varepsilon}}\left[-\frac{d}{dt}\left(\nu\left(\frac{\varepsilon}{t}\right)\right)\right]dt\right]^{\frac{\varepsilon}{p(p-\varepsilon)}}\\
&= \left[\int_1^\infty\left[\frac{t}{\psi^p\left(\frac{p-1}{t}\right)}\right]^{\frac{p-\varepsilon}{\varepsilon}}\left[-\frac{d}{dt}\left(\nu\left(\frac{\varepsilon}{t}\right)\right)\right]dt\right]^{\frac{\varepsilon}{p(p-\varepsilon)}}.
\end{aligned} \tag{2.15}$$

Recalling that for every $x\in]0,p-1[$

$$\nu(x) = c_\psi\int_0^x e^{-\frac{p-1}{\sigma}}\inf_{\sigma<\lambda<p-1}\left[\frac{\psi\left(\frac{(p-1)\sigma}{\lambda}\right)}{\psi(p-1)}\right]^{\frac{p^2}{\lambda}}\frac{d\sigma}{\sigma^2},$$

we obtain that

$$\nu\left(\frac{\varepsilon}{t}\right) = c_\psi\int_0^{\frac{\varepsilon}{t}} e^{-\frac{p-1}{\sigma}}\inf_{\sigma<\lambda<p-1}\left[\frac{\psi\left(\frac{(p-1)\sigma}{\lambda}\right)}{\psi(p-1)}\right]^{\frac{p^2}{\lambda}}\frac{d\sigma}{\sigma^2}.$$

Therefore, we can estimate the derivative of $\nu\left(\frac{\varepsilon}{t}\right)$ as follows:

$$-\frac{d}{dt}\left(\nu\left(\frac{\varepsilon}{t}\right)\right) = c_\psi e^{-\frac{p-1}{\varepsilon}t}\inf_{\varepsilon/t<\lambda<p-1}\left[\frac{\psi\left(\frac{(p-1)\varepsilon}{t\lambda}\right)}{\psi(p-1)}\right]^{\frac{p^2}{\lambda}}\frac{t^2}{\varepsilon^2}\frac{\varepsilon}{t^2} \overset{\lambda=\varepsilon}{\leqslant} \frac{c_\psi}{\varepsilon}e^{-\frac{p-1}{\varepsilon}t}\left[\frac{\psi\left(\frac{p-1}{t}\right)}{\psi(p-1)}\right]^{\frac{p^2}{\varepsilon}}$$

from which the following bound on the last integrand in (2.15) follows:

$$\left[\frac{t}{\psi^p\left(\frac{p-1}{t}\right)}\right]^{\frac{p-\varepsilon}{\varepsilon}}\left[-\frac{d}{dt}\left(\nu\left(\frac{\varepsilon}{t}\right)\right)\right] \leqslant \frac{c_\psi}{\varepsilon}\left(\frac{t}{\psi^p\left(\frac{p-1}{t}\right)}\right)^{\frac{p-\varepsilon}{\varepsilon}}e^{-\frac{p-1}{\varepsilon}t}\left[\frac{\psi\left(\frac{p-1}{t}\right)}{\psi(p-1)}\right]^{\frac{p^2}{\varepsilon}}. \tag{2.16}$$

The right-hand side, which can be rearranged as follows

$$\frac{c_\psi}{\varepsilon} t^{\frac{p-\varepsilon}{\varepsilon}} e^{-\frac{p-1}{\varepsilon}t} \left( \frac{1}{\psi^p\left(\frac{p-1}{t}\right)} \right)^{\frac{p-\varepsilon}{\varepsilon}} \left[ \frac{\psi\left(\frac{p-1}{t}\right)}{\psi(p-1)} \right]^{\frac{p^2}{\varepsilon}},$$

is dominated by

$$\begin{aligned} A_\varepsilon(t) &:= \frac{c_\psi \psi^p (p-1)}{\varepsilon} t^{\frac{p-\varepsilon}{\varepsilon}} e^{-\frac{p-1}{\varepsilon}t} \frac{1}{\left[\psi\left(\frac{p-1}{t}\right)\right]^{\frac{p^2}{\varepsilon}}} \left[ \frac{\psi\left(\frac{p-1}{t}\right)}{\psi(p-1)} \right]^{\frac{p^2}{\varepsilon}} \\ &= \hat{c}_{\psi,p} \frac{1}{\varepsilon} t^{\frac{p-\varepsilon}{\varepsilon}} e^{-\frac{p-1}{\varepsilon}t} \left[ \frac{1}{\psi(p-1)} \right]^{\frac{p^2}{\varepsilon}}, \end{aligned} \tag{2.17}$$

where $\hat{c}_{\psi,p} := c_\psi \psi^p (p-1)$ is a constant not depending on $\varepsilon$. Combining (2.16) and (2.17), and then integrating, we get that

$$\begin{aligned} &\left[ \int_1^\infty \left[ \frac{t}{\left[\psi\left(\frac{p-1}{t}\right)\right]^p} \right]^{\frac{p-\varepsilon}{\varepsilon}} \left[ -\frac{d}{dt}\left(\nu\left(\frac{\varepsilon}{t}\right)\right) \right] dt \right]^{\frac{\varepsilon}{p(p-\varepsilon)}} \\ &\qquad \leqslant \left[ \hat{c}_{\psi,p} \frac{1}{\varepsilon} \left[ \frac{1}{\psi(p-1)} \right]^{\frac{p^2}{\varepsilon}} \int_1^\infty t^{\frac{p-\varepsilon}{\varepsilon}} e^{-\frac{p-1}{\varepsilon}t} dt \right]^{\frac{\varepsilon}{p(p-\varepsilon)}}. \end{aligned} \tag{2.18}$$

By the previous relation (2.18) we infer that

$$k_1 := \sup_{0<\varepsilon<p-1} C(\varepsilon) < \infty, \tag{2.19}$$

with $k_1$ depending only on $p$ and $\psi$. Indeed, the only non-trivial term to investigate is

$$\varepsilon \in ]0, p-1[ \to \left[ \int_1^\infty t^{\frac{p-\varepsilon}{\varepsilon}} e^{-\frac{p-1}{\varepsilon}t} dt \right]^{\frac{\varepsilon}{p(p-\varepsilon)}}.$$

But this term has already been proven to be bounded, by a constant independent of $\varepsilon$, at the end of the proof of Theorem 3.1 in [5] or in (2.12) in [6], through an asymptotic expansion of the incomplete Euler's gamma function. Overall, combining estimates (2.13) and (2.18) we obtain that

$$I_1(\varepsilon) \leqslant k_1 \sup_{0<t<1} S_\psi(t) \|f^*\|_{L^p(t,1)}. \tag{2.20}$$

**Step 4** (Combining the estimates). Using (2.20) and (2.6), we can estimate (2.3) to obtain

$$\sup_{0<\varepsilon<p-1} \psi(\varepsilon) \|f\|_{L^{p-\varepsilon}(0,1)} \leqslant (k_1 + k_2) \sup_{0<t<1} S_\psi(t) \|f^*\|_{L^p(t,1)}.$$

This completes the proof. □

## 3. The optimality result

In the introduction of the paper, we stated that if $\psi$ is in a class of functions satisfying the $\Delta_2$ condition, *essentially* characterized by the so-called $\nabla'$ condition, then inequality (1.1) is sharp, in the sense that both sides are equivalent. This section is devoted to clarifying this optimality claim.

Let $p > 1$ and $\psi : ]0, p-1] \to ]0, \infty[$ be nondecreasing. In [6] the notion of $\Delta_2$ condition, which is classical in the Orlicz space theory for convex functions (see, e.g., [10, Def. 1 p. 22]) has been used for functions defined on intervals of the type $]0, p-1]$, in the form

$$\psi \in \Delta_2 \quad \Leftrightarrow \quad \psi(2\varepsilon) \leqslant c\psi(\varepsilon) \tag{3.1}$$

for some $c > 1$ and for every $\varepsilon$ sufficiently small. In our framework, we will need this notion of condition $\Delta_2$, along with the following two remarks.

- First, functions satisfying the $\Delta_2$ condition are *greater* than some positive power near the origin. The idea of the proof can be found in [10, Corollary 5 p. 26]. For completeness we give here the details. Incidentally, unlike the one in [10], the following argument does not make use of any convexity or concavity assumptions. Let $n_0 \in \mathbb{N}$ be such that $2^{-n_0} < p-1$ and set

$$q := \sup_{m \in \mathbb{N}} \frac{m \log_2 c}{n_0 + m - 1} = \log_2 c\,, \tag{3.2}$$

where $c > 1$ is from the $\Delta_2$ condition in (3.1). For every $0 < x < 2^{-n_0}$, denoting by $m$ the integer such that

$$2^{-n_0-m} \leqslant x < 2^{-n_0-m+1}\,, \tag{3.3}$$

we have

$$\begin{aligned}
\psi(x) &\geqslant \psi(2^{-n_0-m}) \geqslant c^{-m}\psi(2^{-n_0}) = \psi(2^{-n_0})2^{\log_2 c^{-m}} \\
&= \psi(2^{-n_0})2^{-m\log_2 c} = \psi(2^{-n_0})(2^{-n_0-m+1})^{\frac{m\log_2 c}{n_0+m-1}} \\
&\geqslant \psi(2^{-n_0})(2^{-n_0-m+1})^q \\
&> \psi(2^{-n_0})x^q\,.
\end{aligned} \tag{3.4}$$

This concludes the argument.

- Second, we observe that if

$$\sup_{\varepsilon \in ]0,p-1]} \frac{\varepsilon\psi'(\varepsilon)}{\psi(\varepsilon)} < \infty\,, \tag{3.5}$$

then $\psi \in \Delta_2$. Indeed, (3.5) implies that for some $q > 0$ the function $\psi(t)/t^q$ is nonincreasing, and the comparison of its values in $2t$ and $t$ gives immediately the $\Delta_2$ condition.

The next proposition states our second main result and concerns the optimality of the estimate in (1.1). The claim borrows another growth condition well-known in the theory of Orlicz spaces: namely, the $\nabla'$ condition (see, e.g., [10, Def. 7 p. 28]). We set

$$\psi \in \nabla' \quad \Leftrightarrow \quad \exists\, b > 0 \,:\, \psi(x)\psi(y) \leqslant \psi(bxy) \text{ for } x, y, bxy \in ]0, p-1]. \tag{3.6}$$

**Proposition 3.1.** *Let $p > 1$ and $\psi : ]0, p-1] \to ]0, \infty[$ be nondecreasing and differentiable. If $\psi \in \nabla'$ satisfies (3.5), then, for every $t \in ]0, 1]$, there holds*

$$S_\psi(t) = \sup_{\nu^{-1}(t)<\tau<p-1} \psi\left(\frac{p-1}{\tau}\nu^{-1}(t)\right)\left[\frac{d}{d\tau}(\tau\psi^p(\tau))\right]^{\frac{1}{p}} \approx \psi\left(\frac{p-1}{1-\log t}\right)$$

*Proof.* We split the proof in two steps.

**Step 1.** First, we show that

$$S_\psi(t) \gtrsim \psi\left(\frac{p-1}{1-\log t}\right) \quad \forall t \in ]0, 1]\,. \tag{3.7}$$

In the proof of Theorem 2.1 we already remarked that (see (2.4)) $S_\psi(\nu(\varepsilon)) \geqslant k_2^{-1}\psi(\varepsilon)$, and therefore, setting $\nu(\varepsilon) = t$, we have $S_\psi(t) \geqslant k_2^{-1}\psi(\nu^{-1}(t))$. Since

$$\inf_{\sigma<\lambda<p-1}\left[\frac{\psi\left(\frac{(p-1)\sigma}{\lambda}\right)}{\psi(p-1)}\right]^{\frac{p^2}{\lambda}} \leqslant 1\,,$$

we have that for any $0 < x < p-1$

$$\begin{aligned}
\nu(x) &= c_\psi \int_0^x e^{-\frac{p-1}{\sigma}} \inf_{\sigma<\lambda<p-1}\left[\frac{\psi\left(\frac{(p-1)\sigma}{\lambda}\right)}{\psi(p-1)}\right]^{\frac{p^2}{\lambda}} \frac{d\sigma}{\sigma^2} \\
&\leqslant c_\psi \int_0^x e^{-\frac{p-1}{\sigma}} \frac{d\sigma}{\sigma^2} = c_\psi \frac{1}{p-1} e^{-\frac{p-1}{x}} \\
&\leqslant c_\psi \frac{p}{p-1} e^{-\frac{p-1}{x}}\,.
\end{aligned} \tag{3.8}$$

Therefore, since $0 < \nu(x) < 1$, we have $-|\log \nu(x)| \leqslant \left|\log\left(\frac{pc_\psi}{p-1}\right)\right| - \frac{p-1}{x}$, from which we infer that

$$\frac{1}{x} \leqslant \frac{|\log \nu(x)| + \left|\log\left(\frac{pc_\psi}{p-1}\right)\right|}{p-1} \leqslant \kappa_p \frac{1 + |\log \nu(x)|}{p-1}$$

with $\kappa_p := \left(1 + \left|\log\left(\frac{pc_\psi}{p-1}\right)\right|\right)$. Overall, we obtain the following lower bound on $\nu^{-1}$:

$$p - 1 \geqslant \nu^{-1}(t) \geqslant \kappa_p^{-1} \frac{p-1}{1 + |\log t|}\,, \qquad \forall t \in ]0,1]\ . \tag{3.9}$$

But $S_\psi(t) \geqslant k_2^{-1}\psi(\nu^{-1}(t))$, and therefore, taking into account that $\psi$ is nondecreasing, from (3.9) we infer that

$$S_\psi(t) \geqslant \frac{1}{k_2}\psi(\nu^{-1}(t)) \geqslant \frac{1}{k_2}\psi\left(\frac{p-1}{\kappa_p(1+|\log t|)}\right)\ .$$

Finally, by the $\Delta_2$ condition (precisely, as a consequence of (3.5)), we know that $\psi(\varepsilon) \leqslant c^m\psi\left(2^{-m}\varepsilon\right)$ for every $m \in \mathbb{N}$. Hence

$$S_\psi(t) \geqslant \frac{1}{k_2}\psi\left(\frac{p-1}{\kappa_p(1+|\log t|)}\right) \geqslant \frac{1}{k_2 c^{m_p}}\psi\left(\frac{p-1}{1+|\log t|}\right)$$

with $m_p \in \mathbb{N}$ such that $\kappa_p^{-1} > 2^{-m_p}$. This completes the proof of the estimate (3.7).

**Step 2.** Now, we show that

$$S_\psi(t) \lesssim \psi\left(\frac{p-1}{1-\log t}\right) \quad \forall t \in ]0,1]\ . \tag{3.10}$$

By (3.4), the property $\psi \in \Delta_2$ entrains that $\psi(\varepsilon) \geqslant c\varepsilon^q$ for some $c > 0$ and some $q > 0$. Hence, taking into account the assumption (3.5) we are allowed to set

$$q_* := \max\left\{q, \sup_{\varepsilon \in ]0,p-1]} \frac{\varepsilon\psi'(\varepsilon)}{\psi(\varepsilon)}\right\} < \infty\,. \tag{3.11}$$

Then, we proceed to estimate the infimum that appears in the definition of $\nu$. We begin showing the existence of a constant $0 < k < 1$ such that for all $0 < \sigma < p-1$ there holds

$$\inf_{\sigma<\lambda<p-1}\left[\frac{\psi\left(\frac{(p-1)\sigma}{\lambda}\right)}{\psi(p-1)}\right]^{\frac{p^2}{\lambda}} \geqslant k^{\frac{p-1}{\sigma}}\,, \tag{3.12}$$

where the constant $0 < k < 1$ can be expressed under the form

$$k := c_*^{p^2/(p-1)}\left[\inf_{0<\tau<1}\tau^\tau\right]^{\frac{q_* p^2}{p-1}}\,, \qquad c_* := \min\left\{\frac{c}{\psi(p-1)}(p-1)^{q_*}, 1\right\}\ . \tag{3.13}$$

Indeed, classical estimates give the following chain of inequalities:

$$\begin{aligned}
\inf_{\sigma<\lambda<p-1}\left[\frac{\psi\left(\frac{(p-1)\sigma}{\lambda}\right)}{\psi(p-1)}\right]^{\frac{p^2}{\lambda}} &\geqslant \inf_{\sigma<\lambda<p-1}\left[\frac{c(p-1)^{q_*}}{\psi(p-1)}\left(\frac{\sigma}{\lambda}\right)^{q_*}\right]^{\frac{p^2}{\lambda}} \\
&\geqslant \inf_{\sigma<\lambda<p-1} c_*^{p^2/\lambda}\left[\left(\frac{\sigma}{\lambda}\right)^{q_*}\right]^{\frac{p^2}{\lambda}} \\
&\geqslant c_*^{p^2/(p-1)} \inf_{\sigma<\lambda<p-1}\left[\left(\frac{\sigma}{\lambda}\right)^{q_*}\right]^{\frac{p^2}{\lambda}} \\
&= c_*^{p^2/(p-1)} \inf_{\sigma<\lambda<p-1}\left[\left(\frac{\sigma}{\lambda}\right)^{\frac{\sigma}{\lambda}}\right]^{\frac{q_* p^2}{\sigma}} \qquad (3.14)
\end{aligned}$$

Taking into account that $0<\sigma/\lambda<1$ we get

$$\begin{aligned}
(3.14) &= c_*^{p^2/(p-1)}\left[\inf_{\sigma<\lambda<p-1}\left(\frac{\sigma}{\lambda}\right)^{\frac{\sigma}{\lambda}}\right]^{\frac{q_* p^2}{\sigma}} \\
&\geqslant c_*^{p^2/(p-1)}\left[\inf_{0<\tau<1}\tau^{\tau}\right]^{\frac{q_* p^2}{\sigma}} \\
&= c_*^{p^2/(p-1)}\left(\left[\inf_{0<\tau<1}\tau^{\tau}\right]^{\frac{q_* p^2}{p-1}}\right)^{\frac{p-1}{\sigma}} \\
&\geqslant k^{\frac{p-1}{\sigma}} \qquad (3.15)
\end{aligned}$$

This concludes the proof of (3.12). After that, we can estimate $\nu$ as follows:

$$\begin{aligned}
\nu(x) &= c_\psi \int_0^x e^{-\frac{p-1}{\sigma}} \inf_{\sigma<\lambda<p-1}\left[\frac{\psi\left(\frac{(p-1)\sigma}{\lambda}\right)}{\psi(p-1)}\right]^{\frac{p^2}{\lambda}} \frac{d\sigma}{\sigma^2} \\
&\geqslant c_\psi \int_0^x e^{-\frac{p-1}{\sigma}} k^{\frac{p-1}{\sigma}} \frac{d\sigma}{\sigma^2} \\
&= c_\psi \int_0^x E^{-\frac{p-1}{\sigma}} \frac{d\sigma}{\sigma^2} \\
&= \frac{c_\psi}{(p-1)\log E} E^{-\frac{p-1}{x}} \\
&\geqslant a E^{-\frac{p-1}{x}} \qquad (3.16)
\end{aligned}$$

with $a := \min\left\{\frac{c_\psi}{(p-1)\log E}, 1\right\}$ and $E := e/k > e$ because of $k < 1$. Taking the logarithm of both sides of the previous relation, we obtain that for every $x \in ]0, p-1[$

$$-|\log_E \nu(x)| \geqslant -|\log_E a| - \frac{p-1}{x}.$$

This last expression can be rewritten under the form

$$x|\log \nu(x)| \leqslant (p-1)\log E + x|\log a|$$

from which the next upper bound follows

$$x(1 + |\log \nu(x)|) \leqslant (p-1)\log E + (p-1)(1 + |\log a|)\,.$$

Overall, we get that for every $x \in ]0, p-1[$ there holds

$$x \leqslant \frac{c_*(p-1)}{1 + |\log \nu(x)|}\,, \tag{3.17}$$

with $c_* := \log E + 1 + \log a$. Since $\nu$ is an increasing function we get that

$$\nu^{-1}(t) \leqslant c_* \frac{p-1}{1 + |\log t|}\,, \qquad \forall 0 < t < 1. \tag{3.18}$$

Finally, by (3.5) and the $\nabla'$ condition (3.6) we have that

$$\begin{aligned}
S_\psi(\nu(\varepsilon)) &= \sup_{\varepsilon<\tau<p-1} \psi\left(\frac{p-1}{\tau}\varepsilon\right)\left[\frac{d}{d\tau}(\tau\psi^p(\tau))\right]^{\frac{1}{p}} \\
&= \sup_{\varepsilon<\tau<p-1} \psi\left(\frac{p-1}{\tau}\varepsilon\right)\left[\psi^p(\tau) + p\psi^{p-1}(\tau)\tau\psi'(\tau)\right]^{\frac{1}{p}} \\
&\leqslant \sup_{\varepsilon<\tau<p-1} \psi\left(\frac{p-1}{\tau}\varepsilon\right)\left[\psi^p(\tau) + p\psi^{p-1}(\tau)q_*\psi(\tau)\right]^{\frac{1}{p}} \\
&= [1 + pq_*]^{\frac{1}{p}} \sup_{\varepsilon<\tau<p-1} \psi\left(\frac{p-1}{\tau}\varepsilon\right)\psi(\tau) \\
&\leqslant [1 + pq_*]^{\frac{1}{p}} \psi\left(b(p-1)\varepsilon\right) \qquad (3.19)\\
&\lesssim \psi(\varepsilon)\,, \qquad (3.20)
\end{aligned}$$

where in the last $\lesssim$ we used the $\Delta_2$ condition. This completes the proof of (3.10) and, therefore, also the proof of the main statement. □

**Remark 3.1.** If $\psi$ is of power type, i.e. if $c_1\varepsilon^q \leqslant \psi(\varepsilon) \leqslant c_2\varepsilon^q$ for some positive $c_1, c_2, q$, then $\psi \in \nabla'$ and (3.5) is satisfied. Therefore in this case the estimate (1.1) proved in our main result Theorem 2.1 is optimal.

**Corollary 3.2.** *Let $p > 1$ and $\psi : ]0, p-1] \to ]0, \infty[$ be nondecreasing and differentiable. If $\psi \in \nabla'$ and (3.5) holds, then*

$$\sup_{0<\varepsilon<p-1} \psi(\varepsilon)\|f\|_{L^{p-\varepsilon}(0,1)} \approx \sup_{0<t<1} S_\psi(t)\|f^*\|_{L^p(t,1)} \,. \tag{3.21}$$

*Proof.* By Proposition 3.1 we have $S_\psi(t) \approx \psi\left(\frac{p-1}{1-\log t}\right)$, and, since the assumption entrains that $\psi \in \Delta_2$, from [6] we get that the equivalence holds. □

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