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# A SHORT PROOF OF LOCAL REGULARITY OF DISTRIBUTIONAL SOLUTIONS OF POISSON'S EQUATION

GIOVANNI DI FRATTA AND ALBERTO FIORENZA

Abstract. We prove a local regularity result for distributional solutions of the Poisson's equation with $L^p$ data. We use a very short argument based on Weyl's lemma and Riesz-Fréchet representation theorem.

## 1. Introduction

Following the pleasant introduction on the regularity theory of elliptic equations in [15], if $u \in C_0^3(\Omega)$, $\Omega$ being an open, bounded set in $\mathbb{R}^n$, $n \geq 2$, then, using integrations by parts and Schwarz's theorem, and identifying the continuous, compactly supported functions with their corresponding elements in $L^2(\Omega)$,

$$\begin{aligned}\|D^2u\|^2_{L^2(\Omega)} = \int_\Omega \sum_{i,j}^n u_{x_ix_j}u_{x_ix_j}\,dx &= -\int_\Omega \sum_{i,j}^n u_{x_ix_jx_i}u_{x_j}\,dx \\ &= \int_\Omega \sum_{i,j}^n u_{x_ix_i}u_{x_jx_j}\,dx = \|\Delta u\|^2_{L^2(\Omega)}\,.\end{aligned}$$

This means that if $u \in C_0^3(\Omega)$ solves, in the classical sense, the Poisson's equation

$$\Delta u = f\,, \tag{1}$$

then the $L^2$ norm of the datum $f$ controls the $L^2$ norm of *all* second derivatives of $u$. This statement is a typical example of a result in the theory of elliptic regularity, whose main aim is to deduce this kind of results, but under weaker *a priori* hypotheses on the regularity of the solution $u$.

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2010 *Mathematics Subject Classification.* Primary: 35D30; Secondary: 35B65.

**1.1. The notions of weak, very weak, and distributional solutions.** For a given $f \in L^2(\Omega)$, it is natural to study equation (1) in the *weak sense*. This amounts to interpret (1) as equality between elements of the dual space $W^{-1,2}(\Omega)$ of the Sobolev space $W_0^{1,2}(\Omega)$; their images, when tested on every element $v \in W_0^{1,2}(\Omega)$, must coincide. If one looks for functions $u$ in $W_0^{1,2}(\Omega)$ which satisfy (1) in the weak sense (i.e., for *weak solutions*), the requirement is that

$$-\int_\Omega \nabla u \cdot \nabla v \, dx = \int_\Omega f v \, dx \qquad \forall v \in W_0^{1,2}(\Omega)\,. \tag{2}$$

Since $W_0^{1,2}(\Omega)$ is a Hilbert space, by the Riesz-Fréchet representation theorem (see, e.g., [12, p. 118], [3, Theorem 5.5 p. 135]), one gets the existence and uniqueness of the *weak* solution. Note, however, that the weak formulation (2) relies on the apriori assumption that the solution $u$ has first derivatives with the same integrability property of the datum. For such solutions, one can prove the $W^{2,2}_{loc}(\Omega)$ regularity (see, e.g., [15, Theorem 8.2.1]). Also, we recall that under specific assumptions on the regularity of $\Omega$, one can get a better *global* regularity for $u$, while under regularity assumptions on the datum, one can get a better *local* regularity result for the solution (see, e.g., [15, Theorem 8.2.2 and Corollary 8.2.1]).

By (2) one gets the following equivalent equation (the equivalence with (2) immediately follows from a standard density argument), where now the test functions $v$ are in $C_0^\infty(\Omega)$:

$$-\int_\Omega \nabla u \cdot \nabla v \, dx = \int_\Omega f v \, dx \qquad \forall v \in C_0^{\infty}(\Omega)\,. \tag{3}$$

When the problem is in this form, one can look for solutions of equation (1) in the space $W^{1,1}(\Omega)$, because the $L^2(\Omega)$ integrability of the gradient is not needed to give sense to the equation. Regularity results when the datum is in $L^p(\Omega)$, $1 < p < \infty$, are classical, and rely upon the well known Calderón-Zygmund inequality from which one can get the $W^{2,p}_{loc}(\Omega)$ regularity (see, e.g., [15, Theorem 9.2.2, p. 248], or [10, Corollary 9.10, p. 235]). We mention here also the method of difference quotients introduced by Nirenberg (see, e.g., [7, 20], [5, Step 1, p. 121], and [15, Theorem 9.1.2 p.245]).

Equation (3) is a special case of a class of linear equations which can be written in the form

$$-\operatorname{div}(A\nabla u) = f \tag{4}$$

for which it is known ([23]) that even in the case $f \equiv 0$, when $A$ is a matrix function whose entries are locally $L^1$, there exist weak solutions, assumed a priori in $W^{1,1}_{loc}(\Omega)$, which are not in $W^{1,2}_{loc}(\Omega)$. As soon as one assumes that a solution is in $W^{1,2}_{loc}(\Omega)$, much local regularity can be gained by the celebrated De Giorgi's theorem (see e.g. [10, Chap. 8] and references therein).

We recall, in passing, that for $\Omega \subset \mathbb{R}^2$, it is possible to prove an existence and uniqueness theorem for weak solutions of (4) (and even for a nonlinear variant) in a space slightly larger than $W^{1,2}_0(\Omega)$, the so-called grand Sobolev space $W^{1,2)}_0(\Omega)$, when the datum is just in $L^1(\Omega)$ (see [9, Theorem A]). For an excellent survey about solutions of a number of elliptic equations, called very weak because the solutions are assumed a priori in Sobolev spaces with exponents below the natural one, the reader is referred to [11].

One can further weaken the notion of solution, and look for solutions of (1) in the space of regular distributions, that is, among elements of the dual space $\mathcal{D}'(\Omega)$ of $C^\infty_c(\Omega)$ that can be identified with elements of $L^1_{loc}(\Omega)$. In other words, their images, when computed in every element $\varphi \in C^\infty_c(\Omega)$, must coincide:

$$\int_\Omega u\,\Delta\varphi\,dx = \int_\Omega f\varphi\,dx \qquad \forall\varphi \in C^\infty_0(\Omega)\,. \tag{5}$$

Solutions in $L^1_{loc}(\Omega)$ of equation (5) are called *very weak solutions*.

A condition on the datum $f$ ensuring existence and uniqueness in $L^1(\Omega)$ has been found in [4, Lemma 1], namely, if $f$ is in the weighted Lebesgue space where the weight is the distance function from the boundary of $\Omega$, there exists a unique solution $u \in L^1(\Omega)$ such that

$$\|u\|_{L^1(\Omega)} \le \|f \cdot \operatorname{dist}(x, \partial\Omega)\|_{L^1(\Omega)}\,.$$

Differentiability results for very weak solutions are treated in a number of papers, see, e.g., [6, 21] and references therein (see also [8, Theorem 4.2]). However, all such references gain regularity from data in weighted Lebesgue spaces, where the distance to the boundary is involved in the weight and the domain $\Omega$ has itself some regularity assumptions.

When the datum $f$ is identically zero, the masterpiece theorem of regularity for very weak solutions has been proved by Hermann Weyl in [28, pp. 415/6]. It dates back to 1940, well before the introduction of Sobolev spaces [18, 16], and is nowadays referred to as Weyl's Lemma:

**Lemma 1** (H. Weyl, 1940)**.** *Let $\Omega \subset \mathbb{R}^n$ be an open set. Suppose that $u \in L^1_{loc}(\Omega)$ and*

$$\int_\Omega u\,\Delta\varphi dx = 0 \qquad \forall\varphi \in C^\infty_0(\Omega)\,.$$

*Then there exists a unique $\widetilde{u} \in C^\infty(\Omega)$ such that $\Delta\widetilde{u} = 0$ in $\Omega$ and $\widetilde{u} = u$ a.e. in $\Omega$.*

The proof given by Weyl in [28] is elementary and clever. Modern rephrasing of the proof can be found in classical textbooks (see, e.g., [15, Corollary 1.2.1], [5, Theorem 4.7], [24, Appendix, n.2], [27]). A beautiful note devoted entirely on this result and its development is the paper by Strook [26], where Weyl's lemma is stated under the weaker assumption that $u \in \mathcal{D}'(\Omega)$. Indeed, one can go still further, and write (5) (in fact, (1)) in the form

$$\langle u, \Delta\varphi\rangle = \langle f, \varphi\rangle \qquad \forall\varphi \in C_0^\infty(\Omega)\,. \tag{6}$$

Any solution of equation (6), is called a *distributional solution* of the Poisson's equation (1). The statement proved therein is the following (see also [22, 13, 31] for a more general result, valid for a broader class of differential operators).

**Lemma 2** (Weyl's lemma in $\mathcal{D}'(\Omega)$)**.** *Let $\Omega \subset \mathbb{R}^n$ be an open set. Suppose that $u \in \mathcal{D}'(\Omega)$ satisfies $\Delta u = f \in C^\infty(\Omega)$ in the sense of (6). Then $u \in C^\infty(\Omega)$.*

The proof in [26] is very short and elegant. For our purposes, however, it is sufficient the particular case $f \equiv 0$, for which the proof in [26] further simplifies.

**1.2. Contributions of present work.** In this note, we are interested in regularity results for *distributional* solutions of (1), i.e., solutions satisfying (6). We prove a local regularity result for distributional solutions of the Poisson's equation with $L^p$ data. We use a concise argument based on Weyl's lemma and Riesz-Fréchet representation theorem. As a byproduct, we get the following classical result on *very weak* solutions

**Theorem 1.** *If $f \in L^2_{loc}(\Omega)$, then any solution $u \in L^2_{loc}(\Omega)$ of $\Delta u = f$ (i.e., satisfying (5)) belongs to $W^{2,2}_{loc}(\Omega)$.*

Theorem 1, known since 1965 (see [1, Theorem 6.2 p. 58] for a more general result, proved for uniformly elliptic operators with Lipschitz continuous coefficients), is also quoted in the Brezis book [3, Remark 25 p. 306], where it is claimed the delicateness of the proof of interior regularity of very weak solutions, based on estimates for the difference quotient operator (see [1, Def. 3.3 p. 42]). In [2, Section 3 p. 92] the reader can find a modern proof, valid for a wide class of operators, which uses a precise estimate by Hörmander in combination with a spectral representation for hypoelliptic operators. We quote also [29, Theorem 1.3], where for general operators with locally Lipschitz continuous coefficients, in the case $f \equiv 0$, it is shown that very weak solutions in $L^1_{loc}(\Omega)$ are in fact in $W^{2,p}_{loc}(\Omega)$ for every $p \in [1, \infty)$; in [30, Proposition 1.1], the same authors, for general operators having locally Lipschitz continuous coefficients, in the case $f \in L^p_{loc}(\Omega)$, $1 < p < \infty$, get that very weak solutions in $L^1_{loc}(\Omega)$ are in fact in $W^{2,p}_{loc}(\Omega)$.

For other results of regularity for very weak solutions of the Poisson's equation, see, e.g., [17, Section 7.2 p. 223] and [19, Section 4.1 p. 198]. In particular, we mention here that, following Hilbert, one can ask whether a solution, being a distribution, is analytic in the case where the right-hand side $f$ is analytic: the answer is positive for equation (4) when $A$ is analytic, see [14].

## 2. Regularity of very weak solutions of Poisson's equation in the $L^2$-setting

The main ingredient is stated in the following result which, remarkably, is essentially based on Weyl's lemma.

**Lemma 3.** *Let $\Omega \subset \mathbb{R}^n$ be an open set, and let $u \in \mathcal{D}'(\Omega)$. Then*

$$\Delta u \in W^{-1,2}(\Omega) \implies u \in W^{1,2}_{loc}(\Omega)\,. \tag{7}$$

*Proof.* Since $\Delta u \in W^{-1,2}(\Omega)$, by Riesz representation theorem, there exists $v \in W^{1,2}_0(\Omega)$ such that $\Delta v = \Delta u$ in $\mathcal{D}'(\Omega)$. In particular, $\Delta(u-v) = 0$ in $\mathcal{D}'(\Omega)$. By Weyl's lemma (Lemma 2 used with $f \equiv 0$), we know that $u - v \in C^\infty(\Omega)$. Hence $u = (u - v) + v \in C^\infty(\Omega) + W^{1,2}_0(\Omega) \subset W^{1,2}_{loc}(\Omega)$. □

We remark that solutions of Dirichlet problems by Hilbert spaces methods are a classic matter for *weak* solutions, see, e.g., [12, p. 117]. Again, for weak solutions, we quote [25, Lemma 2.1 p.48], where from the assumption of being locally in a Sobolev space, the authors get a better local regularity, still in Sobolev spaces.

**Theorem 2.** *Let $\Omega \subseteq \mathbb{R}^n$ be an open set, and let $u \in \mathcal{D}'(\Omega)$. If $\Delta u \in L^2_{loc}(\Omega)$, then $\nabla u \in W^{1,2}_{loc}(\Omega)$. If, in addition, $u \in L^2_{loc}(\Omega)$, then $u \in W^{2,2}_{loc}(\Omega)$.*

*Proof.* Due to the local character of the result, we can assume $\Delta u \in L^2(\Omega)$. Therefore, it is sufficient to note that if $\Delta u = f$ with $f \in L^2(\Omega)$ then, for any distributional partial derivative of $u$, we have $\Delta(\nabla u) = \nabla f$ with $\nabla f \in W^{-1,2}(\Omega)$. By the previous lemma, we get $\nabla u \in W^{1,2}_{loc}(\Omega)$. Thus, $u \in W^{2,2}_{loc}(\Omega)$ if we assume $u \in L^2_{loc}(\Omega)$. □

## 3. Regularity of Very weak solutions of Poisson's Equation in the $L^p$-setting

We point out that the same argument shows that if $f \in L^p(\Omega)$, $1 < p < \infty$, then $u \in W^{2,p}_{\mathrm{loc}}(\Omega)$. Precisely, the following result holds:

**Theorem 3.** *Let $\Omega \subseteq \mathbb{R}^n$ be an open set, and let $u \in \mathcal{D}'(\Omega)$. If $\Delta u \in L^p_{loc}(\Omega)$, then $\nabla u \in W^{1,p}_{loc}(\Omega)$. If, in addition, $u \in L^p_{loc}(\Omega)$, then $u \in W^{2,p}_{loc}(\Omega)$.*

*Proof.* Indeed (see, e.g., [25, pp. 10-11]), if $1/p + 1/q = 1$ and $F \in W^{-1,q'}(\Omega)$ then there exists a function $u_F \in W^{1,p}(\Omega)$ such that

$$- \int_\Omega \nabla u_F \cdot \nabla \varphi = \langle F, \varphi \rangle \tag{8}$$

for every $\varphi \in W_0^{1,q}(\Omega)$. Note that this can be considered as the $q$-exponent version of the Riesz representation theorem. Now, (8) implies that for any $F \in W^{-1,q'}(\Omega)$ there exists a distribution in $v_F \in W^{1,p}(\Omega)$ such that $\Delta v_F = F$ in $\mathcal{D}'(\Omega)$. After that, assume that $f \in L^p(\Omega)$ and $u \in L^p(\Omega)$ is a distributional solution of (6). Then $\nabla u$ satisfies $\Delta(\nabla u) = \nabla f$ with $\nabla f \in W^{-1,q'}(\Omega)$. Therefore, as in Theorem 3, $\nabla u \in W^{1,p}_{\mathrm{loc}}(\Omega)$, and we conclude. $\square$

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