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OPTIMAL ADDITIVE SCHWARZ PRECONDITIONING FOR ADAPTIVE 2D IGA BOUNDARY ELEMENT METHODS

THOMAS FÜHRER, GREGOR GANTNER, DIRK PRAETORIUS, AND STEFAN SCHIMANKO

ABSTRACT. We define and analyze (local) multilevel diagonal preconditioners for isogeometric boundary elements on locally refined meshes in two dimensions. Hypersingular and weakly-singular integral equations are considered. We prove that the condition number of the preconditioned systems of linear equations is independent of the mesh-size and the refinement level. Therefore, the computational complexity, when using appropriate iterative solvers, is optimal. Our analysis is carried out for closed and open boundaries and numerical examples confirm our theoretical results.

1. INTRODUCTION

In the last decade, the isogeometric analysis (IGA) had a strong impact on the field of scientific computing and numerical analysis. We refer, e.g., to the pioneering work [HCB05] and to [CHB09, BdVBSV14] for an introduction to the field. The basic idea is to utilize the same ansatz functions for approximations as are used for the description of the geometry by some computer aided design (CAD) program. Here, we consider the case, where the geometry is represented by rational splines. For certain problems, where the fundamental solution is known, the boundary element method (BEM) is attractive since CAD programs usually only provide a parametrization of the boundary $\partial\Omega$ and not of the volume Ω itself. Isogeometric BEM (IGABEM) has first been considered for 2D BEM in [PGK⁺09] and for 3D BEM in [SSE⁺13]. We refer to [SBTR12, PTC13, SBLT13, NZW⁺17] for numerical experiments, to [HR10, TM12, MZBF15, DHP16, DHK⁺18, DKSW18] for fast IGABEM based on wavelets, fast multipole, \mathcal{H} -matrices resp. \mathcal{H}^2 -matrices, and to [HAD14, KHZvE17, ACD⁺17, FK18] for some quadrature analysis.

Recently, adaptive IGABEM has been analyzed in [FGP15, FGHP16, FGK⁺18] for rational resp. hierarchical splines in 2D and optimal algebraic convergence rates have been proven in [FGHP17, Sch16] (see also [GPS18]) for rational splines in 2D resp. in [Gan17] for hierarchical splines in 3D. In 2D, the corresponding adaptive algorithms allow for both h -refinement as well as regularity reduction via knot multiplicity increase. Usually, it is assumed that the resulting systems of linear equations are solved exactly. In practice, however, iterative solvers are used and their effectivity hinges on the condition number of the Galerkin

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matrices. It is well-known that the condition number of Galerkin matrices corresponding to the discretization of certain integral operators depend not only on the number of degrees of freedom but also on the ratio h_{\max}/h_{\min} of the largest and smallest element diameter, which can become arbitrarily large on locally refined meshes; see, e.g., [AMT99] for the case of affine boundary elements and lowest-order ansatz functions. Therefore, the construction of optimal preconditioners is a necessity. We say that a preconditioner is optimal, if the condition number of the resulting preconditioned matrices is independent of the mesh-size function h , the number of degrees of freedom and the refinement level.

In this work, we consider simple additive Schwarz methods. The central idea of our local multilevel diagonal preconditioners is to use newly created nodes and old nodes whose multiplicity has changed, to define local diagonal scalings on each refinement level. This allows us to prove optimality of the proposed preconditioner and the computational complexity for applying our preconditioner is linear with respect to the number of degrees of freedom on the finest mesh. In particular, this extends our prior works [FFPS15, FFPS17, FHPS18] on local multilevel diagonal preconditioners for hypersingular integral equations and weakly-singular integral equations for affine geometries in 2D and 3D and lowest-order discretizations. Other results on Schwarz methods for BEM with affine boundaries are found in [Cao02, TS96, TSM97], mainly for uniform mesh-refinements and in [AM03] for some specially local refined meshes. For the higher order case, we refer, e.g., to [Heu96, FMPR15]. Diagonal preconditioners for BEM are covered in [AMT99, GM06]. Another preconditioner technique that leads to uniformly bounded condition numbers is based on the use of integral operators of opposite order. The case of closed boundaries is analyzed in [SW98], whereas open boundaries are treated in the recent works [HJHUT14, HJHUT16, HJHUT17]. The recent work [SvV18] deals with the opposite order operator preconditioning technique in Sobolev spaces of negative order.

To the best of our knowledge, the preconditioning of IGABEM, even on uniform meshes, is still an open problem. For isogeometric finite elements (IGAFEM), a BPX-type preconditioner is analyzed in [BHKS13] on uniform meshes, where the authors consider general pseudodifferential operators of positive order. Recently, a BPX-type preconditioner with local smoothing has been analyzed in [CV17] for locally refined T-meshes. Other multilevel preconditioners for IGAFEM have been studied in [GKT13, ST16, HTZ17, Tak17] for uniform resp. in [HJKZ16] for hierarchical meshes, and domain decomposition methods can be found in [BdVCPS13, BdVPS⁺14, BdVPS⁺17] for uniform meshes.

Model problem. Let Ω be a bounded simply connected Lipschitz domain in \mathbb{R}^2 , with piecewise smooth boundary $\partial\Omega$ and let $\Gamma \subseteq \partial\Omega$ be a connected subset with Lipschitz boundary $\partial\Gamma$. Neumann screen problems on Γ yield the weakly-singular integral equation

$$\mathfrak{W}u(x) := -\frac{\partial}{\partial\nu_x} \int_{\Gamma} \left(\frac{\partial}{\partial\nu_y} G(x, y) \right) u(y) dy = f(x) \quad \text{for all } x \in \Gamma \quad (1.1)$$

with the hypersingular integral operator \mathfrak{W} and some given right-hand side f . Here, ν_x denotes the outer normal unit vector of Ω at some point $x \in \Gamma$, and

$$G(x, y) := -\frac{1}{2\pi} \log|x - y| \quad (1.2)$$

is the fundamental solution of the Laplacian. Similarly, Dirichlet screen problems lead to the weakly-singular integral equation

$$\mathfrak{V}\phi(x) := \int_{\Gamma} G(x, y) \phi(y) dy = g(x) \quad (1.3)$$

with the weakly-singular integral operator \mathfrak{V} and some given right-hand side g .

Outline. The remainder of the work is organized as follows: Section 2 provides the functional analytic setting of the boundary integral operators, the definition of the mesh, B-splines and NURBS together with their basic properties. Auxiliary results that are used in the proof of our main results are stated in Section 3. In Section 4, we define our local multilevel diagonal preconditioner for the hypersingular integral operator on closed and open boundaries and prove its optimality (Theorem 4.1). Then, in Section 5, we extend our local multilevel diagonal preconditioner to the weakly-singular case and give a proof of its optimality (Theorem 5.2). Finally, in Section 6 we restate the abstract results for additive Schwarz operators in matrix formulation (Corollary 6.1). Moreover, numerical examples for closed and open boundaries are presented and some aspects of implementation are discussed.

2. PRELIMINARIES

2.1. Notation. Throughout and without any ambiguity, $|\cdot|$ denotes the absolute value of scalars, the Euclidean norm of vectors in \mathbb{R}^2 , the measure of a set in \mathbb{R} (e.g., the length of an interval), or the arclength of a curve in \mathbb{R}^2 . We write $A \lesssim B$ to abbreviate $A \leq cB$ with some generic constant $c > 0$, which is clear from the context. Moreover, $A \simeq B$ abbreviates $A \lesssim B \lesssim A$. Throughout, mesh-related quantities have the same index, e.g., \mathcal{N}_{\bullet} is the set of nodes of the partition \mathcal{T}_{\bullet} , and h_{\bullet} is the corresponding local mesh-width etc. The analogous notation is used for partitions \mathcal{T}_o resp. \mathcal{T}_{ℓ} etc. We sometimes use $\hat{\cdot}$ to transform notation on the boundary to the parameter domain. The most important symbols are listed in Table 1.

2.2. Sobolev spaces. The usual Lebesgue and Sobolev spaces on Γ are denoted by $L^2(\Gamma) = H^0(\Gamma)$ and $H^1(\Gamma)$. We introduce the corresponding seminorm on any measurable subset $\Gamma_0 \subseteq \Gamma$ via

$$|v|_{H^1(\Gamma_0)} := \|\partial_{\Gamma} v\|_{L^2(\Gamma_0)} \quad \text{for all } v \in H^1(\Gamma), \quad (2.1)$$

with the arclength derivative ∂_{Γ} . We have that

$$\|v\|_{H^1(\Gamma)}^2 = \|v\|_{L^2(\Gamma)}^2 + |v|_{H^1(\Gamma)}^2 \quad \text{for all } v \in H^1(\Gamma), \quad (2.2)$$

Moreover, $\tilde{H}^1(\Gamma)$ is the space of $H^1(\Gamma)$ functions, which have a vanishing trace on the relative boundary $\partial\Gamma$ equipped with the same norm. On Γ , Sobolev spaces of fractional order $0 < \sigma < 1$ are defined by the K -method of interpolation [McL00, Appendix B]: For $0 < \sigma < 1$, we let $H^{\sigma}(\Gamma) := [L^2(\Gamma), H^1(\Gamma)]_{\sigma}$ and $\tilde{H}^{\sigma}(\Gamma) := [L^2(\Gamma), \tilde{H}^1(\Gamma)]_{\sigma}$. We also introduce the Sobolev-Slobodeckij seminorm

$$|v|_{H^{\sigma}(\Gamma_0)} := \left(\int_{\Gamma_0} \int_{\Gamma_0} \frac{|v(x) - v(y)|^2}{|x - y|^{1+2\sigma}} dx dy \right)^{1/2} \quad \text{for all } v \in H^{\sigma}(\Gamma). \quad (2.3)$$

For $0 < \sigma \leq 1$, Sobolev spaces of negative order are defined by duality $H^{-\sigma}(\Gamma) := \tilde{H}^{\sigma}(\Gamma)^*$ and $\tilde{H}^{-\sigma}(\Gamma) := H^{\sigma}(\Gamma)^*$, where duality is understood with respect to the extended $L^2(\Gamma)$ -scalar product $\langle \cdot, \cdot \rangle_{\Gamma}$. In general, there holds the continuous inclusion $\tilde{H}^{\pm\sigma}(\Gamma) \subseteq H^{\pm\sigma}(\Gamma)$

TABLE 1. Important symbols

Name	Description	First appearance	Name	Description	First appearance
$\tilde{B}_{\bullet,i,p}$	B-spline	Section 2.8	$\tilde{R}_{\bullet,i,p}$	NURBS	Section 2.8
$B_{\bullet,i,q}$	B-spline on Γ	Section 2.9	$R_{\bullet,i,p}$	NURBS on Γ	Section 2.9
$\bar{B}_{\bullet,i,p}$	continuous B-spline on Γ	Section 2.9	$\bar{R}_{\bullet,i,p}$	continuous NURBS on Γ	Section 2.9
$\tilde{B}_{\bullet,i,p}^*$	dual B-spline	Section 3.1	$\tilde{R}_{\bullet,i,p}^*$	dual NURBS	Section 3.1
$\text{gen}(\cdot)$	generation fct.	Section 3.3	$t_{\bullet,i}$	knot	Section 2.8
$\tilde{h}_{\bullet,T}$	length in $[a, b]$	Section 2.6	\mathcal{T}_{\bullet}	mesh	Section 2.6
$h_{\bullet,T}$	arclength	Section 2.6	$w_{\bullet,i}$	weight	Section 2.8
$\tilde{h}_{\bullet}(\cdot), h_{\bullet}(\cdot)$	mesh-size functions	Section 2.6	$\tilde{w}(\cdot)$	denominator	Section 2.9
$\bar{h}_{\text{uni}(m)}$	uniform mesh-size in $[a, b]$	Section 3.3	w_{\min}, w_{\max}	bounds for weights	Section 2.9
$\tilde{\mathcal{I}}_{\ell}$	index set	Section 4	\mathcal{W}_{\bullet}	weight vector	Section 2.9
J_{\bullet}	Scott-Zhang operator	Section 3.1	\mathcal{X}_{\bullet}	NURBS space for \mathfrak{W}	Section 2.9
$\tilde{\mathcal{K}}_{\bullet}$	knot vector in $[a, b]$	Section 2.6	$\tilde{\mathcal{X}}_{\ell}$	space of new NURBS	Section 4
\mathcal{K}_{\bullet}	knot vector	Section 2.6	$\mathcal{X}_{\ell,i}$	one-dim. subspace	Section 4
$\tilde{\mathcal{N}}_{\circ\bullet}$	set of new knots	Section 3.1	\mathcal{Y}_{\bullet}	spline space for \mathfrak{B}	Section 5
\mathbb{K}	admissible knot vectors	Section 2.7	$\tilde{\mathcal{Y}}_{\ell}^0$	space of new splines	Section 5
$\text{level}_{\bullet}(\cdot)$	level fct.	Section 3.3	$z_{\bullet,j}$	node	Section 2.6
N_{\bullet}	number of knots	Section 2.6	$\tilde{z}_{\ell,i}$	new knot	Section 4
\mathcal{N}_{\bullet}	set of nodes	Section 2.6	\tilde{r}_{\bullet}	local mesh-ratio in $[a, b]$	Section 2.6
o	1 if $\Gamma = \partial\Omega$, 0 else	Section 2.9	\tilde{r}_{\max}	bound for local mesh-ratio in $[a, b]$	Section 2.7
p	positive polynomial order	Section 2.6	$\Pi_{\text{uni}(m)}$	projection on unif. space	Section 3.3
$\tilde{\mathcal{P}}_L^{\mathfrak{B}}$	Schwarz operator for \mathfrak{B}	Section 5	$\omega_{\bullet}^m(\cdot)$	patch	Section 2.6
$\tilde{\mathcal{P}}_L^{\mathfrak{W}}$	Schwarz operator for \mathfrak{W}	Section 4	$\#\bullet$	multiplicity	Section 2.6
$\text{refine}(\cdot)$	set of refined knot vectors	Section 2.7			

with $\|v\|_{H^{\pm\sigma}(\Gamma)} \lesssim \|v\|_{\tilde{H}^{\pm\sigma}(\Gamma)}$ for all $v \in \tilde{H}^{\pm\sigma}(\Gamma)$. We note that $\tilde{H}^{\pm\sigma}(\Gamma) = H^{\pm\sigma}(\Gamma)$ for $0 < \sigma < 1/2$ with equivalent norms. Moreover, it holds for $\Gamma = \partial\Omega$ that $\tilde{H}^{\pm\sigma}(\partial\Omega) = H^{\pm\sigma}(\partial\Omega)$ even with equal norms for all $0 < \sigma \leq 1$. Finally, the treatment of the closed boundary $\Gamma = \partial\Omega$ requires the definition of $H_0^{\pm\sigma}(\partial\Omega) = \{v \in H^{\pm\sigma}(\partial\Omega) : \langle v, 1 \rangle_{\partial\Omega} = 0\}$ for all $0 \leq \sigma \leq 1$.

Details and equivalent definitions of the Sobolev spaces are, found, e.g., in [McL00, SS11].

2.3. Hypersingular integral equation. For $0 \leq \sigma \leq 1$, the hypersingular integral operator $\mathfrak{W} : \tilde{H}^{\sigma}(\Gamma) \rightarrow H^{\sigma-1}(\Gamma)$ is well-defined, linear, and continuous. Recall that Γ and $\partial\Omega$ are supposed to be connected.

For $\Gamma \subsetneq \partial\Omega$ and $\sigma = 1/2$, $\mathfrak{W} : \tilde{H}^{1/2}(\Gamma) \rightarrow H^{-1/2}(\Gamma)$ is symmetric and elliptic. Hence,

$$\langle\langle u, v \rangle\rangle_{\mathfrak{W}} := \langle \mathfrak{W}u, v \rangle_{\Gamma} \quad \text{for all } u, v \in \tilde{H}^{1/2}(\Gamma), \quad (2.4)$$

defines an equivalent scalar product on $\tilde{H}^{1/2}(\Gamma)$ with corresponding norm $\|\cdot\|_{\mathfrak{W}}$.

For $\Gamma = \partial\Omega$, the operator \mathfrak{W} is symmetric and elliptic up to the constant functions, i.e., $\mathfrak{W} : H_0^{1/2}(\partial\Omega) \rightarrow H_0^{-1/2}(\partial\Omega)$ is elliptic. In particular,

$$\langle\langle u, v \rangle\rangle_{\mathfrak{W}} := \langle \mathfrak{W}u, v \rangle_{\partial\Omega} + \langle u, 1 \rangle_{\partial\Omega} \langle v, 1 \rangle_{\partial\Omega} \quad \text{for all } u, v \in \tilde{H}^{1/2}(\Gamma), \quad (2.5)$$

defines an equivalent scalar product on $H^{1/2}(\partial\Omega) = \tilde{H}^{1/2}(\partial\Omega)$ with norm $\|\cdot\|_{\mathfrak{W}}$.

With this notation and provided that $f \in H_0^{-1/2}(\Gamma)$ in case of $\Gamma = \partial\Omega$, the strong form (1.1) is equivalently stated in variational form: Find $u \in \tilde{H}^{1/2}(\Gamma)$ such that

$$\langle\langle u, v \rangle\rangle_{\mathfrak{W}} = \langle f, v \rangle_{\Gamma} \quad \text{for all } v \in \tilde{H}^{1/2}(\Gamma). \quad (2.6)$$

Therefore, the Lax-Milgram lemma applies and hence (1.1) admits a unique solution $u \in \tilde{H}^{1/2}(\Gamma)$. More details and proofs are found, e.g., in [McL00, SS11, Ste08].

2.4. Weakly-singular integral equation. For $0 \leq \sigma \leq 1$, the weakly-singular integral operator $\mathfrak{W} : \tilde{H}^{\sigma-1}(\Gamma) \rightarrow H^\sigma(\Gamma)$ is well-defined, linear, and continuous. For $\Gamma = \partial\Omega$, we suppose $\text{diam}(\Omega) < 1$.

For $\sigma = 1/2$, $\mathfrak{W} : \tilde{H}^{-1/2}(\Gamma) \rightarrow H^{1/2}(\Gamma)$ is symmetric and elliptic. In particular,

$$\langle\langle \phi, \psi \rangle\rangle_{\mathfrak{W}} := \langle \mathfrak{W}\phi, \psi \rangle_{\Gamma} \quad \text{for all } \phi, \psi \in \tilde{H}^{-1/2}(\Gamma), \quad (2.7)$$

defines an equivalent scalar product on $\tilde{H}^{-1/2}(\Gamma)$ with corresponding norm $\|\cdot\|_{\mathfrak{W}}$. With this notation, the strong form (1.3) with data $g \in H^{1/2}(\Gamma)$ is equivalently stated by

$$\langle\langle \phi, \psi \rangle\rangle_{\mathfrak{W}} = \langle g, \psi \rangle_{\Gamma} \quad \text{for all } \psi \in \tilde{H}^{-1/2}(\Gamma). \quad (2.8)$$

Therefore, the Lax-Milgram lemma applies and hence (1.3) admits a unique solution $\phi \in \tilde{H}^{-1/2}(\Gamma)$. More details and proofs are found, e.g., in [McL00, SS11, Ste08].

2.5. Boundary parametrization. We assume that either $\Gamma = \partial\Omega$ is parametrized by a closed continuous and piecewise continuously differentiable path $\gamma : [a, b] \rightarrow \Gamma$ with $a < b$ such that the restriction $\gamma|_{[a,b]}$ is even bijective, or that $\Gamma \subsetneq \partial\Omega$ is parametrized by a bijective continuous and piecewise continuously differentiable path $\gamma : [a, b] \rightarrow \Gamma$. In the first case, we speak of *closed* $\Gamma = \partial\Omega$, whereas the second case is referred to as *open* $\Gamma \subsetneq \partial\Omega$. For closed $\Gamma = \partial\Omega$, we denote the $(b-a)$ -periodic extension to \mathbb{R} also by γ .

For the left and right derivative of γ , we assume that $\gamma'^{\ell}(t) \neq 0$ for $t \in (a, b]$ and $\gamma'^r(t) \neq 0$ for $t \in [a, b)$. Moreover, we assume that $\gamma'^{\ell}(t) + c\gamma'^r(t) \neq 0$ for all $c > 0$ and $t \in [a, b]$ resp. $t \in (a, b)$. Finally, let $\gamma_{\text{arc}} : [0, |\Gamma|] \rightarrow \Gamma$ denote the arclength parametrization, i.e., $|\gamma'_{\text{arc}}{}^{\ell}(t)| = 1 = |\gamma'_{\text{arc}}{}^r(t)|$, and its periodic extension. Elementary differential geometry yields bi-Lipschitz continuity

$$C_{\Gamma}^{-1} \leq \frac{|\gamma_{\text{arc}}(s) - \gamma_{\text{arc}}(t)|}{|s - t|} \leq C_{\Gamma} \quad \text{for } s, t \in \mathbb{R}, \quad \text{with } \begin{cases} |s - t| \leq \frac{3}{4}|\Gamma|, & \text{for closed } \Gamma, \\ s \neq t \in [0, |\Gamma|], & \text{for open } \Gamma, \end{cases} \quad (2.9)$$

where $C_{\Gamma} > 0$ depends only on Γ . A proof is given in [Gan14, Lemma 2.1] for closed $\Gamma = \partial\Omega$. For open $\Gamma \subsetneq \partial\Omega$, the proof is even simpler.

2.6. Boundary discretization. In the following, we describe the different quantities, which define the discretization.

Nodes $z_{\bullet,j} = \gamma(\hat{z}_{\bullet,j}) \in \mathcal{N}_{\bullet}$ and number of nodes n_{\bullet} . Let $\mathcal{N}_{\bullet} := \{z_{\bullet,j} : j = 1, \dots, n_{\bullet}\}$ and $z_{\bullet,0} := z_{\bullet,n_{\bullet}}$ for closed $\Gamma = \partial\Omega$ resp. $\mathcal{N}_{\bullet} := \{z_{\bullet,j} : j = 0, \dots, n_{\bullet}\}$ for open $\Gamma \subsetneq \partial\Omega$ be a set of nodes. We suppose that $z_{\bullet,j} = \gamma(\hat{z}_{\bullet,j})$ for some $\hat{z}_{\bullet,j} \in [a, b]$ with $a = \hat{z}_{\bullet,0} < \hat{z}_{\bullet,1} < \hat{z}_{\bullet,2} < \dots < \hat{z}_{\bullet,n_{\bullet}} = b$ such that $\gamma|_{[\hat{z}_{\bullet,j-1}, \hat{z}_{\bullet,j}]} \in C^1([\hat{z}_{\bullet,j-1}, \hat{z}_{\bullet,j}])$.

Multiplicity $\#_{\bullet} z_{\bullet,j}$, knot vector \mathcal{K}_{\bullet} and number of knots N_{\bullet} . Let $p \in \mathbb{N}$ be some fixed positive polynomial order. Each interior node $z_{\bullet,j}$ has a multiplicity $\#_{\bullet} z_{\bullet,j} \in \{1, 2, \dots, p\}$ and $\#_{\bullet} z_{\bullet,0} = \#_{\bullet} z_{\bullet,n_{\bullet}} = p + 1$. This induces knots

$$\mathcal{K}_{\bullet} = \left(\underbrace{z_{\bullet,k}, \dots, z_{\bullet,k}}_{\#_{\bullet} z_{\bullet,k} \text{-times}}, \dots, \underbrace{z_{\bullet,n_{\bullet}}, \dots, z_{\bullet,n_{\bullet}}}_{\#_{\bullet} z_{\bullet,n_{\bullet}} \text{-times}} \right), \quad (2.10)$$

with $k = 1$ for $\Gamma = \partial\Omega$ resp. $k = 0$ for $\Gamma \subsetneq \partial\Omega$. We define the number of knots in $\gamma((a, b])$ as

$$N_{\bullet} := \sum_{j=1}^{n_{\bullet}} \#_{\bullet} z_{\bullet,j}. \quad (2.11)$$

Elements $T_{\bullet,j}$, partition \mathcal{T}_{\bullet} . Let $\mathcal{T}_{\bullet} = \{T_{\bullet,1}, \dots, T_{\bullet,n_{\bullet}}\}$ be a partition of Γ into compact and connected segments $T_{\bullet,j} = \gamma(\widehat{T}_{\bullet,j})$ with $\widehat{T}_{\bullet,j} = [\widehat{z}_{\bullet,j-1}, \widehat{z}_{\bullet,j}]$.

Local mesh-sizes $\widehat{h}_{\bullet,T}, h_{\bullet,T}$ and $\widehat{h}_{\bullet}, h_{\bullet}$. For $T \in \mathcal{T}_{\bullet}$, we define $\widehat{h}_{\bullet,T} := |\gamma^{-1}(T)|$ as its length in the parameter domain, and $h_{\bullet,T} := |T|$ as its arclength. We define the local mesh-width functions $\widehat{h}_{\bullet}, h_{\bullet} \in L^{\infty}(\Gamma)$ by $\widehat{h}_{\bullet}|_T = \widehat{h}_{\bullet,T}$ and $h_{\bullet}|_T = h_{\bullet,T}$.

Local mesh-ratio $\widehat{\kappa}_{\bullet}$. We define the local mesh-ratio by

$$\widehat{\kappa}_{\bullet} := \max \{ \widehat{h}_{\bullet,T} / \widehat{h}_{\bullet,T'} : T, T' \in \mathcal{T}_{\bullet} \text{ with } T \cap T' \neq \emptyset \}. \quad (2.12)$$

Patches $\omega_{\bullet}^m(z)$ and $\omega_{\bullet}^m(\Gamma_0)$. For each set $\Gamma_0 \subseteq \Gamma$, we inductively define for $m \in \mathbb{N}_0$

$$\omega_{\bullet}^m(\Gamma_0) := \begin{cases} \Gamma_0 & \text{if } m = 0, \\ \omega_{\bullet}(\Gamma_0) := \bigcup \{ T \in \mathcal{T}_{\bullet} : T \cap \Gamma_0 \neq \emptyset \} & \text{if } m = 1, \\ \omega_{\bullet}(\omega_{\bullet}^{m-1}(\Gamma_0)) & \text{if } m > 1. \end{cases}$$

For points $z \in \Gamma$, we abbreviate $\omega_{\bullet}(z) := \omega_{\bullet}(\{z\})$ and $\omega_{\bullet}^m(z) := \omega_{\bullet}^m(\{z\})$.

2.7. Admissible knot vectors. Throughout, we consider knot vectors \mathcal{K}_{\bullet} as in Section 2.6 with uniformly bounded local mesh-ratio, i.e., we suppose the existence of $\widehat{\kappa}_{\max} \geq 1$ with

$$\widehat{\kappa}_{\bullet} \leq \widehat{\kappa}_{\max}. \quad (2.13)$$

Let \mathcal{K}_{\bullet} and \mathcal{K}_{\circ} be knot vectors (2.13). We say that \mathcal{K}_{\circ} is finer than \mathcal{K}_{\bullet} and write $\mathcal{K}_{\circ} \in \text{refine}(\mathcal{K}_{\bullet})$ if \mathcal{K}_{\bullet} is a subsequence of \mathcal{K}_{\circ} such that \mathcal{K}_{\circ} is obtained from \mathcal{K}_{\bullet} via iterative dyadic bisections in the parameter domain and multiplicity increases. Formally, this means that $\mathcal{N}_{\bullet} \subseteq \mathcal{N}_{\circ}$ with $\#_{\bullet} z \leq \#_{\circ} z$ for all $z \in \mathcal{N}_{\bullet} \cap \mathcal{N}_{\circ}$, and that for all $T \in \mathcal{T}_{\bullet}$ there exists $T' \in \mathcal{T}_{\circ}$ and $j \in \mathbb{N}_0$ with $T \subseteq T'$ and $|\gamma^{-1}(T')| = 2^{-j} |\gamma^{-1}(T)|$. Throughout, we suppose that all considered knot vectors \mathcal{K}_{\bullet} with (2.13) are finer than some fixed initial knot vector \mathcal{K}_0 . We call such a knot vector admissible. The set of all these knot vectors is abbreviated by \mathbb{K} .

2.8. B-splines and NURBS. Throughout this subsection, we consider *knots* $\widehat{\mathcal{K}}_{\bullet} := (t_{\bullet,i})_{i \in \mathbb{Z}}$ on \mathbb{R} with multiplicity $\#_{\bullet} t_{\bullet,i}$, which satisfy that $t_{\bullet,i-1} \leq t_{\bullet,i}$ for $i \in \mathbb{Z}$ and $\lim_{i \rightarrow \pm\infty} t_{\bullet,i} = \pm\infty$. Let $\widehat{\mathcal{N}}_{\bullet} := \{t_{\bullet,i} : i \in \mathbb{Z}\} = \{\widehat{z}_{\bullet,j} : j \in \mathbb{Z}\}$ denote the corresponding set of nodes with $\widehat{z}_{\bullet,j-1} < \widehat{z}_{\bullet,j}$ for $j \in \mathbb{Z}$. For $i \in \mathbb{Z}$, the i -th *B-spline* of degree q is defined inductively by

$$\begin{aligned} \widehat{B}_{\bullet,i,0} &:= \chi_{[t_{\bullet,i-1}, t_{\bullet,i})}, \\ \widehat{B}_{\bullet,i,q} &:= \beta_{\bullet,i-1,q} \widehat{B}_{\bullet,i,q-1} + (1 - \beta_{\bullet,i,q}) \widehat{B}_{\bullet,i+1,q-1} \quad \text{for } q \in \mathbb{N}, \end{aligned} \quad (2.14)$$

where, for $t \in \mathbb{R}$,

$$\beta_{\bullet,i,q}(t) := \begin{cases} \frac{t - t_{\bullet,i}}{t_{\bullet,i+q} - t_{\bullet,i}} & \text{if } t_{\bullet,i} \neq t_{\bullet,i+q}, \\ 0 & \text{if } t_{\bullet,i} = t_{\bullet,i+q}. \end{cases} \quad (2.15)$$

The following lemma collects basic properties of B-splines. Proves are found, e.g., in [dB86].

Lemma 2.1. *For an interval $I = [a, b)$ and $q \in \mathbb{N}_0$, the following assertions (i)–(vii) hold:*

- (i) *The set $\{\widehat{B}_{\bullet,i,q}|_I : i \in \mathbb{Z}, \widehat{B}_{\bullet,i,q}|_I \neq 0\}$ is a basis for the space of all right-continuous $\widehat{\mathcal{N}}_{\bullet}$ -piecewise polynomials of degree lower or equal q on I , which are, at each knot $t_{\bullet,i}$, $q - \#_{\bullet} t_{\bullet,i}$ times continuously differentiable if $q - \#_{\bullet} t_{\bullet,i} \geq 0$.*

- (ii) For $i \in \mathbb{Z}$, $\widehat{B}_{\bullet,i,q}$ vanishes outside the interval $[t_{\bullet,i-1}, t_{\bullet,i+q}]$. It is positive on the open interval $(t_{\bullet,i-1}, t_{\bullet,i+q})$ and a polynomial of degree q on each interval $(\widehat{z}_{j-1}, \widehat{z}_j) \subseteq (t_{\bullet,i-1}, t_{\bullet,i+q})$ for $j \in \mathbb{Z}$.
- (iii) For $i \in \mathbb{Z}$, $\widehat{B}_{\bullet,i,q}$ is completely determined by the $q+2$ knots $t_{\bullet,i-1}, \dots, t_{\bullet,i+q}$, wherefore we also write

$$\widehat{B}(\cdot | t_{\bullet,i-1}, \dots, t_{\bullet,i+q}) := \widehat{B}_{\bullet,i,q} \quad (2.16)$$

- (iv) The B-splines of degree q form a (locally finite) partition of unity, i.e.,

$$\sum_{i \in \mathbb{Z}} \widehat{B}_{\bullet,i,q} = 1 \quad \text{on } \mathbb{R}. \quad (2.17)$$

- (v) For $i \in \mathbb{Z}$ with $t_{\bullet,i-1} < t_{\bullet,i} = \dots = t_{\bullet,i+q} < t_{\bullet,i+q+1}$, it holds that

$$\widehat{B}_{\bullet,i,q}(t_{\bullet,i}-) = 1 \quad \text{and} \quad \widehat{B}_{\bullet,i+1,q}(t_{\bullet,i}) = 1. \quad (2.18)$$

- (vi) Suppose the convention $q/0 := 0$. For $q \geq 1$ and $i \in \mathbb{Z}$, it holds for the right derivative

$$\widehat{B}'_{\bullet,i,q} = \frac{q}{t_{\bullet,i+q-1} - t_{\bullet,i-1}} \widehat{B}_{\bullet,i,q-1} - \frac{q}{t_{\bullet,i+q} - t_{\bullet,i}} \widehat{B}_{\bullet,i+1,q-1}. \quad (2.19)$$

- (vii) Let $t' \in (t_{\ell-1}, t_\ell]$ for some $\ell \in \mathbb{Z}$ and let $\widehat{\mathcal{K}}_\circ$ be the refinement of $\widehat{\mathcal{K}}_\bullet$, obtained by adding t' . Then, for all coefficients $(a_{\bullet,i})_{i \in \mathbb{Z}}$, there exists $(a_{\circ,i})_{i \in \mathbb{Z}}$ such that

$$\sum_{i \in \mathbb{Z}} a_{\bullet,i} \widehat{B}_{\bullet,i,q} = \sum_{i \in \mathbb{Z}} a_{\circ,i} \widehat{B}_{\circ,i,q} \quad (2.20)$$

With the multiplicity $\#_\circ t'$ of t' in the knots $\widehat{\mathcal{K}}_\circ$, the new coefficients can be chosen as

$$a_{\circ,i} = \begin{cases} a_{\bullet,i} & \text{if } i \leq \ell - q + \#_\circ t' - 1, \\ (1 - \beta_{\bullet,i-1,q}(t'))a_{\bullet,i-1} + \beta_{\bullet,i-1,q}(t')a_{\bullet,i} & \text{if } \ell - q + \#_\circ t' \leq i \leq \ell, \\ a_{\bullet,i-1} & \text{if } \ell + 1 \leq i. \end{cases} \quad (2.21)$$

If one assumes $\#_\bullet t_i \leq q+1$ for all $i \in \mathbb{Z}$, these coefficients are unique. Note that these three cases are equivalent to $t_{\bullet,i+q-1} \leq t'$, $t_{\bullet,i-1} < t' < t_{\bullet,i+q-1}$, resp. $t' \leq t_{\bullet,i-1}$. \square

Remark 2.2. Let $j \in \mathbb{Z}$ and $(\delta_{ij})_{i \in \mathbb{Z}}$ be the corresponding Kronecker sequence. Choosing $(a_{\bullet,i})_{i \in \mathbb{Z}} = (\delta_{ij})_{i \in \mathbb{Z}}$ in Lemma 2.1 (vii), one sees that $\widehat{B}_{\bullet,j,q}$ is a linear combination of $\widehat{B}_{\circ,j,q}$ and $\widehat{B}_{\circ,j+1,q}$, where $\widehat{B}_{\bullet,j,q} = \widehat{B}_{\circ,j,q}$ if $j \leq \ell - q + \#_\circ t' - 2$ and $\widehat{B}_{\bullet,j,q} = \widehat{B}_{\circ,j+1,q}$ if $\ell + 1 \leq j$.

In addition to the knots $\widehat{\mathcal{K}}_\bullet = (t_{\bullet,i})_{i \in \mathbb{Z}}$, we consider fixed positive weights $\mathcal{W}_\bullet := (w_{\bullet,i})_{i \in \mathbb{Z}}$ with $w_{\bullet,i} > 0$. For $i \in \mathbb{Z}$ and $q \in \mathbb{N}_0$, we define the i -th NURBS by

$$\widehat{R}_{\bullet,i,q} := \frac{w_{\bullet,i} \widehat{B}_{\bullet,i,q}}{\sum_{k \in \mathbb{Z}} w_{\bullet,k} \widehat{B}_{\bullet,k,q}}. \quad (2.22)$$

Note that the denominator is locally finite and positive.

For any $q \in \mathbb{N}_0$, we define the B-spline space

$$\widehat{\mathcal{S}}^q(\widehat{\mathcal{K}}_\bullet) := \left\{ \sum_{i \in \mathbb{Z}} a_i \widehat{B}_{\bullet,i,q} : a_i \in \mathbb{R} \right\} \quad (2.23)$$

as well as the NURBS space

$$\widehat{\mathcal{S}}^q(\widehat{\mathcal{K}}_\bullet, \mathcal{W}_\bullet) := \left\{ \sum_{i \in \mathbb{Z}} a_i \widehat{R}_{\bullet, i, q} : a_i \in \mathbb{R} \right\} = \frac{\widehat{\mathcal{S}}^q(\widehat{\mathcal{K}}_\bullet)}{\sum_{k \in \mathbb{Z}} w_{\bullet, k} \widehat{B}_{\bullet, k, q}}. \quad (2.24)$$

We define for $0 < \sigma < 1$, any interval I , and $\widehat{v} \in L^2(I)$ the Sobolev-Slobodeckij seminorm $|\widehat{v}|_{H^\sigma(I)}$ as in (2.3) (with Γ_0 and v replaced by I and \widehat{v}).

Lemma 2.3. *Let $q > 0$, $0 < \sigma < 1$, and $K, w_{\min}, w_{\max} > 0$. Suppose that the weights \mathcal{W}_\bullet are bounded by w_{\min} and w_{\max} , i.e., $w_{\min} \leq \inf_{i \in \mathbb{Z}} w_{\bullet, i} \leq \sup_{i \in \mathbb{Z}} w_{\bullet, i} \leq w_{\max}$, and that the local mesh-ratio on \mathbb{R} is bounded by K , i.e., $\sup \left\{ \max \left(\frac{\widehat{z}_j - \widehat{z}_{j-1}}{\widehat{z}_{j-1} - \widehat{z}_{j-2}}, \frac{\widehat{z}_j - \widehat{z}_{j-1}}{\widehat{z}_{j+1} - \widehat{z}_j} \right) : j \in \mathbb{Z} \right\} \leq K$. Then, there exists a constant $C_{\text{scale}} > 0$, which depends only on q, w_{\min}, w_{\max} , and K , such that for all $i \in \mathbb{Z}$ with $|\text{supp} \widehat{R}_{\bullet, i, q}| > 0$, it holds that*

$$|\text{supp} \widehat{R}_{\bullet, i, q}|^{1-2\sigma} \leq C_{\text{scale}} |\widehat{R}_{\bullet, i, q}|_{H^\sigma(\text{supp} \widehat{R}_{\bullet, i, q})}^2. \quad (2.25)$$

Proof. The proof is split into two steps.

Step 1: First, we suppose that $w_i = 1$ for all $i \in \mathbb{Z}$ and hence $\widehat{R}_{\bullet, i, q} = \widehat{B}_{\bullet, i, q}$. The definition of the B-splines implies their invariance with respect to affine transformations of the knots: $\widehat{B}(t|t_0, \dots, t_{q+1}) = \widehat{B}(ct + s|ct_0 + s, \dots, ct_{q+1} + s)$ for all $t_0 \leq \dots \leq t_{q+1}, s, t \in \mathbb{R}$ and $c > 0$.

With the abbreviation $S := \text{supp} \widehat{B}_{\bullet, i, q} = [t_{\bullet, i-1}, t_{\bullet, i+q}]$, it hence holds that

$$\begin{aligned} |\widehat{B}_{\bullet, i, q}|_{H^\sigma(S)}^2 &= \int_S \int_S \frac{|\widehat{B}_{\bullet, i, q}(r) - \widehat{B}_{\bullet, i, q}(s)|^2}{|r - s|^{1+2\sigma}} ds dr \\ &= |S|^{1-2\sigma} \int_0^1 \int_0^1 \frac{|\widehat{B}(r|0, \frac{t_{\bullet, i} - t_{\bullet, i-1}}{|S|}, \dots, \frac{t_{\bullet, i+q} - t_{\bullet, i-1}}{|S|}, 1) - \widehat{B}(s|\dots)|^2}{|r - s|^{1+2\sigma}} ds dr \\ &\geq |S|^{1-2\sigma} \inf_{0 \leq t_1 \leq \dots \leq t_q \leq 1} \int_0^1 \int_0^1 |\widehat{B}(r|0, t_1, \dots, t_q, 1) - \widehat{B}(s|0, t_1, \dots, t_q, 1)|^2 ds dr, \end{aligned} \quad (2.26)$$

where for the last inequality we have used that $|r - s| \leq 1$. We use a compactness argument to conclude the proof. Let $((t_{k,1}, \dots, t_{k,q}))_{k \in \mathbb{N}}$ be a convergent minimizing sequence for the infimum in (2.26). Let $(t_{\infty,1}, \dots, t_{\infty,q})$ be the corresponding limit. With the definition of the B-splines one easily verifies that

$$\widehat{B}(r|0, t_{k,1}, \dots, t_{k,q}, 1) \rightarrow \widehat{B}(r|0, t_{\infty,1}, \dots, t_{\infty,q}, 1) \quad \text{for almost every } r \in \mathbb{R}.$$

The dominated convergence theorem implies that the infimum is attained at $(t_{\infty,1}, \dots, t_{\infty,q})$. Lemma 2.1 (ii) especially implies that $\widehat{B}(\cdot|0, t_{\infty,1}, \dots, t_{\infty,q}, 1)$ is not constant. Therefore the infimum is positive, and we conclude the proof.

Step 2: Recall that $\widehat{R}_{\bullet, i, q} = \frac{w_{\bullet, i} \widehat{B}_{\bullet, i, q}}{\sum_{j=i-q}^{i+q} w_{\bullet, j} \widehat{B}_{\bullet, j, q}}$. As in Step 1, we transform $\text{supp} \widehat{R}_{\bullet, i, q}$ onto the interval $[0, 1]$. Hence, it suffices to prove, with the compact interval $I := [0 - K^q, 1 + K^q]$, that the infimum

$$\inf_{\substack{t_{-q}, \dots, t_{-1}, t_1, \dots, t_q, t_{q+2}, \dots, t_{1+2q} \in I \\ w_{1-q}, \dots, w_{1+q} \in [w_{\min}, w_{\max}]}} \int_0^1 \int_0^1 \left| \frac{w_1 \widehat{B}(r|0, t_1, \dots, t_q, 1)}{\sum_{j=1-q}^{1+q} w_j \widehat{B}(r|t_{j-1}, \dots, t_{j+q})} - \frac{w_1 \widehat{B}(s|0, t_1, \dots, t_q, 1)}{\sum_{j=1-q}^{1+q} w_j \widehat{B}(s|t_{j-1}, \dots, t_{j+q})} \right|^2 ds dr$$

is bigger than 0. This can be proved analogously as before. \square

2.9. Ansatz spaces. Throughout this section, we abbreviate $\gamma|_{[a,b]}^{-1}$ with γ^{-1} if $\Gamma \subsetneq \partial\Omega$ is an open boundary. Additionally to the initial knots $\mathcal{K}_0 \in \mathbb{K}$, suppose that $\mathcal{W}_0 = (w_{0,i})_{i=1-p}^{N_0-p}$ are given initial weights with $w_{0,1-p} = w_{0,N_0-p}$, where $N_0 = |\mathcal{K}_0|$ for closed $\Gamma = \partial\Omega$ resp. $N_0 = |\mathcal{K}_0| - (p+1)$ for open $\Gamma \subsetneq \partial\Omega$. In the weakly-singular case we assume $w_{0,i} = 1$ for $i = 1-p, \dots, N_0-p$. We extend the corresponding knot vector in the parameter domain, $\widehat{\mathcal{K}}_0 = (t_{0,i})_{i=1}^{N_0}$ if $\Gamma = \partial\Omega$ is closed resp. $\widehat{\mathcal{K}}_0 = (t_{0,i})_{i=-p}^{N_0}$ if $\Gamma \subsetneq \partial\Omega$ is open, arbitrarily to $(t_{0,i})_{i \in \mathbb{Z}}$ with $t_{-p} = \dots = t_0 = a$, $t_{0,i} \leq t_{0,i+1}$, $\lim_{i \rightarrow \pm\infty} t_{0,i} = \pm\infty$. For the extended sequence we also write $\widehat{\mathcal{K}}_0$. We define the weight function

$$\widehat{w} := \sum_{k=1-p}^{N_0-p} w_{0,k} \widehat{B}_{0,k,p}|_{[a,b]}. \quad (2.27)$$

Let $\mathcal{K}_\bullet \in \mathbb{K}$ be an admissible knot vector. Outside of the interval $(a, b]$, we extend the corresponding knot sequence $\widehat{\mathcal{K}}_\bullet$ in the parameter domain exactly as before and write again \mathcal{K}_\bullet for the extension as well. This guarantees that $\widehat{\mathcal{K}}_0$ forms a subsequence of $\widehat{\mathcal{K}}_\bullet$. Via knot insertion from $\widehat{\mathcal{K}}_0$ to $\widehat{\mathcal{K}}_\bullet$, Lemma 2.1 (i) proves the existence and uniqueness of weights $\mathcal{W}_\bullet = (w_{\bullet,i})_{i=1-p}^{N_\bullet-p}$ such that

$$\widehat{w} = \sum_{k=1-p}^{N_0-p} w_{0,k} \widehat{B}_{0,k,p}|_{[a,b]} = \sum_{k=1-p}^{N_\bullet-p} w_{\bullet,k} \widehat{B}_{\bullet,k,p}|_{[a,b]}. \quad (2.28)$$

By choosing these weights, we ensure that the denominator of the considered rational splines does not change. Lemma 2.1 (v) states that $\widehat{B}_{\bullet,1-p}(a) = 1 = \widehat{B}_{\bullet,N_\bullet-p}(b-)$, which implies that $w_{\bullet,1-p} = w_{\bullet,N_\bullet-p}$. Further, Lemma 2.1 (iv) and (vii) show that

$$w_{\min} := \min(\mathcal{W}_0) \leq \min(\mathcal{W}_\bullet) \leq \widehat{w} \leq \max(\mathcal{W}_\bullet) \leq \max(\mathcal{W}_0) := w_{\max}. \quad (2.29)$$

In the weakly-singular case there even holds that $w_{\bullet,i} = 1$ for $i = 1-p, \dots, N_\bullet-p$, and $\widehat{w} = 1$. Finally, we extend \mathcal{W}_\bullet arbitrarily to $(w_{\bullet,i})_{i \in \mathbb{Z}}$ with $w_{\bullet,i} > 0$, identify the extension with \mathcal{W}_\bullet and set for the hypersingular case

$$\mathcal{S}^p(\mathcal{K}_\bullet, \mathcal{W}_\bullet) := \{\widehat{V}_\bullet \circ \gamma^{-1} : \widehat{V}_\bullet \in \widehat{\mathcal{S}}^p(\widehat{\mathcal{K}}_\bullet, \mathcal{W}_\bullet)\} \quad (2.30)$$

and for the weakly-singular case

$$\mathcal{S}^{p-1}(\mathcal{K}_\bullet) := \{\widehat{V}_\bullet \circ \gamma^{-1} : \widehat{V}_\bullet \in \widehat{\mathcal{S}}^{p-1}(\widehat{\mathcal{K}}_\bullet)\} \quad (2.31)$$

Lemma 2.1 (iii) shows that the definition does not depend on how the sequences are extended. We define the transformed basis functions

$$R_{\bullet,i,p} := \widehat{R}_{\bullet,i,p} \circ \gamma^{-1} \quad \text{and} \quad B_{\bullet,i,p-1} := \widehat{B}_{\bullet,i,p-1} \circ \gamma^{-1}. \quad (2.32)$$

Later, we will also need the notation $B_{\bullet,i,p}$, which we define analogously.

We introduce the ansatz space for the hypersingular case

$$\mathcal{X}_\bullet := \begin{cases} \{V_\bullet \in \mathcal{S}^p(\mathcal{K}_\bullet, \mathcal{W}_\bullet) : V_\bullet(\gamma(a)) = V_\bullet(\gamma(b-))\} \subset H^{1/2}(\Gamma) & \text{if } \Gamma = \partial\Omega, \\ \{V_\bullet \in \mathcal{S}^p(\mathcal{K}_\bullet, \mathcal{W}_\bullet) : V_\bullet(\gamma(a)) = 0 = V_\bullet(\gamma(b-))\} \subset \widetilde{H}^{1/2}(\Gamma) & \text{if } \Gamma \subsetneq \partial\Omega, \end{cases} \quad (2.33)$$

and for the weakly-singular case

$$\mathcal{Y}_\bullet := \mathcal{S}^{p-1}(\mathcal{K}_\bullet) \subset \widetilde{H}^{-1/2}(\Gamma). \quad (2.34)$$

Note that, in contrast to the hypersingular case, we only allow for non-rational splines in the weakly-singular case. We exploit this restriction in Lemma 5.1 below, which states that $\partial_\Gamma \mathcal{X}_\bullet = \mathcal{Y}_\bullet$. For rational splines, this assertion is in general false. We abbreviate

$$\overline{R}_{\ell,i,p} := \begin{cases} R_{\ell,1-p,p} + R_{\ell,N_\ell-p,p} & \text{for } i = 1 - p, \\ R_{\ell,i,p} & \text{for } i \neq 1 - p. \end{cases} \quad (2.35)$$

We define $\overline{B}_{\ell,i,p}$ analogously. Further, we set

$$o := \begin{cases} 0 & \text{if } \Gamma = \partial\Omega, \\ 1 & \text{if } \Gamma \subsetneq \partial\Omega. \end{cases} \quad (2.36)$$

Lemma 2.1 (i) and (v) show that

$$\mathcal{X}_\bullet = \text{span}\{\overline{R}_{\bullet,i,p} : i = 1 - p + o, \dots, N_\bullet - p - 1\}, \quad (2.37)$$

as well as

$$\mathcal{Y}_\bullet = \text{span}\{B_{\bullet,i,p-1} : i = 1 - (p - 1), \dots, N_\bullet - 1 - (p - 1)\}. \quad (2.38)$$

In both cases, the corresponding sets form a basis of \mathcal{X}_\bullet resp. \mathcal{Y}_\bullet . Note that the spaces \mathcal{X}_\bullet and \mathcal{Y}_\bullet satisfy that $\mathcal{X}_\bullet \subset \widetilde{H}^1(\Gamma)$ and $\mathcal{Y}_\bullet \subset L^2(\Gamma)$. Lemma 2.1 (i) implies nestedness

$$\mathcal{X}_\bullet \subseteq \mathcal{X}_\circ \text{ and } \mathcal{Y}_\bullet \subseteq \mathcal{Y}_\circ \text{ for all } \mathcal{K}_\bullet, \mathcal{K}_\circ \in \mathbb{K} \text{ with } \mathcal{K}_\circ \in \text{refine}(\mathcal{K}_\bullet). \quad (2.39)$$

3. AUXILIARY RESULTS FOR HYPERSINGULAR CASE

3.1. Scott-Zhang-type projection. In this section, we recall a Scott-Zhang-type operator from [GPS18]. Let $\mathcal{K}_\bullet \in \mathbb{K}$. In [BdVBSV14, Section 2.1.5], it is shown that, for $i \in \{1 - p, \dots, N_\bullet - p\}$, there exist dual basis functions $\widehat{B}_{\bullet,i,p}^* \in L^2(a, b)$ such that

$$\text{supp} \widehat{B}_{\bullet,i,p}^* = \text{supp} \widehat{B}_{\bullet,i,p} = [t_{\bullet,i-1}, t_{\bullet,i+p}], \quad (3.1)$$

$$\int_a^b \widehat{B}_{\bullet,i,p}^*(t) \widehat{B}_{\bullet,j,p}(t) dt = \delta_{ij} = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{else,} \end{cases} \quad (3.2)$$

and

$$\|\widehat{B}_{\bullet,i,p}^*\|_{L^2(a,b)} \leq 9^p(2p + 3) |\text{supp} \widehat{B}_{\bullet,i,p}|^{-1/2}. \quad (3.3)$$

Each dual basis function depends only on the knots $t_{\bullet,i-1}, \dots, t_{\bullet,i+p}$. Therefore, we also write

$$\widehat{B}_{\bullet,i,p}^* = \widehat{B}^*(\cdot | t_{\bullet,i-1}, \dots, t_{\bullet,i+p}). \quad (3.4)$$

With the denominator \widehat{w} from (2.28), define

$$\widehat{R}_{\bullet,i,p}^* := \widehat{B}_{\bullet,i,p}^* \widehat{w} / w_{\bullet,i}. \quad (3.5)$$

This immediately proves that

$$\int_a^b \widehat{R}_{\bullet,i,p}^*(t) \widehat{R}_{\bullet,j,p}(t) dt = \delta_{ij} \quad (3.6)$$

and

$$\|\widehat{R}_{\bullet,i,p}^*\|_{L^2(a,b)} \lesssim 9^p(2p + 3) |\text{supp} \widehat{R}_{\bullet,i,p}|^{-1/2}, \quad (3.7)$$

where the hidden constant depends only on w_{\min} and w_{\max} . We define the Scott-Zhang-type operator $J_{\bullet} : L^2(\Gamma) \rightarrow \mathcal{X}_{\bullet}$

$$J_{\bullet} v := \sum_{i=1-p+o}^{N_{\bullet}-p-1} \alpha_{\bullet,i}(v) \bar{R}_{\bullet,i,p} \text{ with } \alpha_{\bullet,i}(v) := \begin{cases} \int_a^b \frac{\widehat{R}_{\bullet,1-p,p}^* + \widehat{R}_{\bullet,N_{\bullet}-p,p}^*}{2} v \circ \gamma dt & \text{if } i = 1-p, \\ \int_a^b \widehat{R}_{\bullet,i,p}^* v \circ \gamma dt & \text{if } i \neq 1-p. \end{cases} \quad (3.8)$$

A similar operator, namely $I_{\bullet} := \sum_{i=1-p}^{N_{\bullet}-p} \left(\int_a^b \widehat{R}_{\bullet,i,p}^* v \circ \gamma dt \right) R_{\bullet,i,p}$, has been analyzed in [BdVBSV14, Section 3.1.2]. However, I_{\bullet} is not applicable here for two reasons: First, for $\Gamma = \partial\Omega$, it does not guarantee that $I_{\bullet} v$ is continuous at $\gamma(a) = \gamma(b)$. Second, for $\Gamma \subsetneq \partial\Omega$, it does not guarantee that $I_{\bullet} v(\gamma(a)) = 0 = I_{\bullet} v(\gamma(b))$.

Lemma 3.1. *Let $\mathcal{K}_{\bullet}, \mathcal{K}_{\circ} \in \mathbb{K}$ with $\mathcal{K}_{\circ} \in \text{refine}(\mathcal{K}_{\bullet})$. Then, each $v \in L^2(\Gamma)$ satisfies that*

$$(J_{\circ} - J_{\bullet})v \in \text{span}\{\bar{R}_{\circ,i,p} : i \in \{1-p+o, \dots, N_{\circ}-p-1\}, \text{supp}\bar{R}_{\circ,i,p} \cap \widetilde{\mathcal{N}}_{\circ \setminus \bullet} \neq \emptyset\}, \quad (3.9)$$

where

$$\widetilde{\mathcal{N}}_{\circ \setminus \bullet} := \mathcal{N}_{\circ} \setminus \mathcal{N}_{\bullet} \cup \{z \in \mathcal{N}_{\circ} \cap \mathcal{N}_{\bullet} : \#_{\circ} z > \#_{\bullet} z\}. \quad (3.10)$$

Proof. We only prove the lemma for closed $\Gamma = \partial\Omega$. For open $\Gamma \subsetneq \partial\Omega$, the proof even simplifies. We split the proof into two steps.

Step 1: We consider the case where $\widehat{\mathcal{K}}_{\circ}$ is obtained from $\widehat{\mathcal{K}}_{\bullet}$ by insertion of a single knot $t' \in [a, b]$ in the parameter domain. Let $b \neq t' \in (t_{\bullet, \ell-1}, t_{\bullet, \ell}]$ with some $\ell \in \{1, \dots, N_{\bullet}-p\}$. Note that $N_{\circ} = N_{\bullet} + 1$. It holds that

$$\begin{aligned} (J_{\circ} - J_{\bullet})v &= \alpha_{\circ,1-p}(v) \bar{R}_{\circ,1-p,p} - \alpha_{\bullet,1-p}(v) \bar{R}_{\bullet,1-p,p} \\ &\quad + \sum_{i=2-p}^{N_{\circ}-p} \left(\int_a^b \widehat{R}_{\circ,i,p}^* v \circ \gamma dt \right) R_{\circ,i,p} - \sum_{i=2-p}^{N_{\bullet}-p-1} \left(\int_a^b \widehat{R}_{\bullet,i,p}^* v \circ \gamma dt \right) R_{\bullet,i,p}. \end{aligned} \quad (3.11)$$

We split the second sum into

$$\begin{aligned} \sum_{i=2-p}^{N_{\circ}-p-1} \left(\int_a^b \widehat{R}_{\bullet,i,p}^* v \circ \gamma dt \right) R_{\bullet,i,p} &= \sum_{i=2-p}^{\ell-p+\#_{\circ} t'-2} \left(\int_a^b \widehat{R}_{\bullet,i,p}^* v \circ \gamma dt \right) R_{\bullet,i,p} \\ &\quad + \sum_{i=\max(2-p, \ell-p+\#_{\circ} t'-1)}^{\ell} \left(\int_a^b \widehat{R}_{\bullet,i,p}^* v \circ \gamma dt \right) R_{\bullet,i,p} + \sum_{i=\ell+1}^{N_{\bullet}-p-1} \left(\int_a^b \widehat{R}_{\bullet,i,p}^* v \circ \gamma dt \right) R_{\bullet,i,p}. \end{aligned}$$

Remark 2.2 and the choice $(a_{\bullet,j})_{j \in \mathbb{Z}} := (w_{\bullet,j})_{j \in \mathbb{Z}}$ in Lemma 2.1 (vii) show the following: first, for $1-p \leq i \leq \ell-p+\#_{\circ} t'-2$, it holds that $\widehat{B}_{\bullet,i,p} = \widehat{B}_{\circ,i,p}$ and $w_{\bullet,i} = w_{\circ,i}$, whence $R_{\bullet,i,p} = R_{\circ,i,p}$; second, for $\ell+1 \leq i \leq N_{\bullet}-p$, it holds that $\widehat{B}_{\bullet,i,p} = \widehat{B}_{\circ,i+1,p}$ and $w_{\bullet,i} = w_{\circ,i+1}$, whence $R_{\bullet,i,p} = R_{\circ,i+1,p}$. Moreover, for $1-p \leq i \leq \ell-p+\#_{\circ} t'-2$, it holds that

$$\widehat{R}_{\bullet,i,p}^* = \widehat{B}^*(\cdot | t_{\bullet,i-1}, \dots, t_{\bullet,i+p}) \widehat{w} / w_{\bullet,i} = \widehat{B}^*(\cdot | t_{\circ,i-1}, \dots, t_{\circ,i+p}) \widehat{w} / w_{\circ,i} = \widehat{R}_{\circ,i,p}^*,$$

and for $\ell+1 \leq i \leq N_{\bullet}-p$ that

$$\widehat{R}_{\bullet,i,p}^* = \widehat{B}^*(\cdot | t_{\bullet,i-1}, \dots, t_{\bullet,i+p}) \widehat{w} / w_{\bullet,i} = \widehat{B}^*(\cdot | t_{\circ,i}, \dots, t_{\circ,i+1+p}) \widehat{w} / w_{\circ,i+1} = \widehat{R}_{\circ,i+1,p}^*.$$

Hence, (3.11) simplifies to

$$(J_\circ - J_\bullet)v = \alpha_{\circ,1-p}(v)\overline{R}_{\circ,1-p,p} - \alpha_{\bullet,1-p}(v)\overline{R}_{\bullet,1-p,p} \quad (3.12)$$

$$+ \sum_{i=\max(2-p,\ell-p+\#_\circ t'-1)}^{\ell+1} \left(\int_a^b \widehat{R}_{\circ,i,p}^* v \circ \gamma dt \right) R_{\circ,i,p} - \sum_{i=\max(2-p,\ell-p+\#_\circ t'-1)}^{\ell} \left(\int_a^b \widehat{R}_{\bullet,i,p}^* v \circ \gamma dt \right) R_{\bullet,i,p}.$$

Remark 2.2 and Lemma 2.1 (ii) imply that

$$\begin{aligned} & \{R_{\circ,i,p} : i = \max(2-p, \ell-p+\#_\circ t'-1), \dots, \ell+1\} \\ & \cup \{R_{\bullet,i,p} : i = \max(2-p, \ell-p+\#_\circ t'-1), \dots, \ell\} \\ & \subseteq \text{span}\{\overline{R}_{\circ,i,p} : i = \max(2-p, \ell-p+\#_\circ t'-1), \dots, \ell+1\} \subseteq \text{span}\{\overline{R}_{\circ,i,p} : \gamma(t') \in \text{supp}\overline{R}_{\circ,i,p}\} \end{aligned} \quad (3.13)$$

We have already seen that $R_{\bullet,N_\bullet-p,p} = R_{\circ,N_\bullet-p+1,p}$ and $R_{\bullet,N_\bullet-p,p}^* = R_{\circ,N_\bullet-p+1,p}^*$ if $N_\bullet - p \geq \ell + 1$, and that $\widehat{R}_{\bullet,1-p,p} = \widehat{R}_{\circ,1-p,p}$ and $\widehat{R}_{\bullet,1-p,p}^* = \widehat{R}_{\circ,1-p,p}^*$ if $3 \leq \ell + \#_\circ t'$. This shows that the first summands in (3.12) cancel each other if $N_\bullet - p \geq \ell + 1$ and $3 \leq \ell + \#_\circ t'$. Otherwise there holds that $\ell = 1$ or $\ell = N_\bullet - p$ and the functions $\overline{R}_{\circ,1-p,p}$, $R_{\circ,2-p,p}$ and $R_{\circ,N_\bullet-p,p}$ are in the last set of (3.13). Since $\overline{R}_{\bullet,1-p,p}$ is a linear combination of these functions, we conclude that

$$(J_\circ - J_\bullet)v \in \text{span}\{\overline{R}_{\circ,i,p} : \gamma(t') \in \text{supp}\overline{R}_{\circ,i,p}\}.$$

Step 2: Let $\mathcal{K}_\circ \in \mathbb{K}$ be arbitrary with $\mathcal{K}_\bullet \leq \mathcal{K}_\circ$ and let $\mathcal{K}_\circ = \mathcal{K}_{(M)}, \mathcal{K}_{(M-1)}, \dots, \mathcal{K}_{(1)}, \mathcal{K}_{(0)} = \mathcal{K}_\bullet$ be a sequence of knot vectors such that each $\mathcal{K}_{(k)}$ is obtained by insertion of one single knot $\gamma(t_{(k)})$ in $\mathcal{K}_{(k-1)}$. Note that these meshes do not necessarily belong to \mathbb{K} , as the $\widehat{\kappa}$ -mesh property (2.13) can be violated. However, the corresponding Scott-Zhang operator $J_{(k)}$ for $\mathcal{K}_{(k)}$ can be defined just as above and Step 1 holds analogously. There holds that

$$(J_\circ - J_\bullet)v = \sum_{k=1}^M (J_{(k)} - J_{(k-1)})v. \quad (3.14)$$

This and Step 1 imply that

$$(J_\circ - J_\bullet)v \in \sum_{k=1}^M \text{span}\{\overline{R}_{(k),i,p} : (\gamma(t_{(k)}) \in \text{supp}\overline{R}_{(k),i,p})\}. \quad (3.15)$$

Remark 2.2 shows that any basis function $\widehat{B}_{(k),i,p}$ with $k < M$ is the linear combination of $\widehat{B}_{(k+1),i,p}$ and $\widehat{B}_{(k+1),i+1,p}$. Moreover, Lemma 2.1 (ii) shows that $\text{supp}\widehat{B}_{(k+1),i,p} \cup \text{supp}\widehat{B}_{(k+1),i+1,p} \subseteq \text{supp}\widehat{B}_{(k),i,p}$. We conclude that

$$\overline{R}_{(k),i,p} \in \text{span}\{\overline{R}_{(k+1),j,p} : \text{supp}\overline{R}_{(k+1),j,p} \subseteq \text{supp}\overline{R}_{(k),i,p}\}.$$

Together with (3.15), this shows that

$$\begin{aligned} (J_\circ - J_\bullet)v & \in \sum_{k=1}^M \text{span}\{\overline{R}_{(M),i,p} : \gamma(t_{(k)}) \in \text{supp}\overline{R}_{(M),i,p}\} \\ & = \text{span}\{\overline{R}_{\circ,i,p} : \text{supp}\overline{R}_{\circ,i,p} \cap \widetilde{\mathcal{N}}_{\circ \setminus \bullet} \neq \emptyset\}, \end{aligned}$$

and concludes the proof. \square

The following proposition is derived in [GPS18]. For closed $\Gamma = \partial\Omega$, the proof is also found in [Sch16, Lemma 3.1].

Proposition 3.2. *For $\mathcal{K}_\bullet \in \mathbb{K}$, the corresponding Scott-Zhang operator J_\bullet satisfies the following properties:*

(i) *Local projection property: For all $v \in L^2(\Gamma)$ and all $T \in \mathcal{T}_\bullet$ it holds that*

$$(J_\bullet v)|_T = v|_T \quad \text{if } v|_{\omega_\bullet^p(T)} \in \mathcal{X}_\bullet|_{\omega_\bullet^p(T)} := \{V_\bullet|_{\omega_\bullet^p(T)} : V_\bullet \in \mathcal{X}_\bullet\}. \quad (3.16)$$

(ii) *Local L^2 -stability: For all $v \in L^2(\Gamma)$ and all $T \in \mathcal{T}_\bullet$, it holds that*

$$\|J_\bullet v\|_{L^2(T)} \leq C_{\text{sz}} \|v\|_{L^2(\omega_\bullet^p(T))}. \quad (3.17)$$

(iii) *Local \tilde{H}^1 -stability: For all $v \in \tilde{H}^1(\Gamma)$ and all $T \in \mathcal{T}_\bullet$, it holds that*

$$|J_\bullet v|_{H^1(T)} \leq C_{\text{sz}} |v|_{H^1(\omega_\bullet^p(T))}. \quad (3.18)$$

(iv) *Local approximation property: For all $v \in \tilde{H}^1(\Gamma)$ and all $T \in \mathcal{T}_\bullet$, it holds that*

$$\|h_\bullet^{-1}(1 - J_\bullet)v\|_{L^2(T)} \leq C_{\text{sz}} |v|_{H^1(\omega_\bullet^p(T))}. \quad (3.19)$$

The constant $C_{\text{sz}} > 0$ depends only on $\hat{\kappa}_{\text{max}}, p, w_{\text{min}}, w_{\text{max}}$, and γ . \square

3.2. Inverse inequalities. In this section, we state some inverse estimates for NURBS from [GPS18], which are well-known for piecewise polynomials [GHS05, AFF⁺15].

Proposition 3.3. *Let $\mathcal{K}_\bullet \in \mathbb{K}$ and $0 \leq \sigma \leq 1$. Then, there hold the inverse inequalities*

$$\|V_\bullet\|_{\tilde{H}^\sigma(\Gamma)} \leq C_{\text{inv}} \|h_\bullet^{-\sigma} V_\bullet\|_{L^2(\Gamma)} \quad \text{for all } V_\bullet \in \mathcal{X}_\bullet, \quad (3.20)$$

and

$$\|h_\bullet^{1-\sigma} \partial_\Gamma V_\bullet\|_{L^2(\Gamma)} \leq C_{\text{inv}} \|V_\bullet\|_{\tilde{H}^\sigma(\Gamma)} \quad \text{for all } V_\bullet \in \mathcal{X}_\bullet. \quad (3.21)$$

The constant $C_{\text{inv}} > 0$ depends only on $\hat{\kappa}_{\text{max}}, p, w_{\text{min}}, w_{\text{max}}, \gamma$, and σ . \square

3.3. Uniform meshes. We consider a sequence $\mathcal{K}_{\text{uni}(m)} \in \mathbb{K}$ of uniform knot vectors: Let $\mathcal{K}_{\text{uni}(0)} := \mathcal{K}_0$ and let $\mathcal{K}_{\text{uni}(m+1)}$ be obtained from $\mathcal{K}_{\text{uni}(m)}$ by uniform refinement, i.e., all elements of $\mathcal{T}_{\text{uni}(m)}$ are bisected in the parameter domain into son elements with half length, where each new knot has multiplicity one. Define $\bar{h}_{\text{uni}(0)} := \max_{T \in \mathcal{T}_0} |\gamma^{-1}(T)|$ as well as

$$\bar{h}_{\text{uni}(m)} := 2^{-m} \bar{h}_{\text{uni}(0)} \quad \text{for each } m \geq 1. \quad (3.22)$$

Note that $\bar{h}_{\text{uni}(m)}$ is equivalent to the usual local mesh-size function on $\mathcal{T}_{\text{uni}(m)}$, i.e., $\bar{h}_{\text{uni}(m)} \simeq |T|$ for all $T \in \mathcal{T}_{\text{uni}(m)}$ and all $m \geq 0$, where the hidden constants depend only on \mathcal{T}_0 and γ . Moreover, let $\mathcal{X}_{\text{uni}(m)}$ denote the associated discrete space with corresponding L^2 -orthogonal projection $\Pi_{\text{uni}(m)} : L^2(\Gamma) \rightarrow \mathcal{X}_{\text{uni}(m)}$. Note that the discrete spaces $\mathcal{X}_{\text{uni}(m)}$ are nested, i.e., $\mathcal{X}_{\text{uni}(m)} \subseteq \mathcal{X}_{\text{uni}(m+1)}$ for all $m \geq 0$.

The next result follows by the approximation property of Proposition 3.2 (iv) and the inverse inequality (3.20) of Proposition 3.3 in combination with [Bor94].

Lemma 3.4. *Let $0 < \sigma < 1$. Then,*

$$\sum_{m=0}^{\infty} \bar{h}_{\text{uni}(m)}^{-2\sigma} \|(1 - \Pi_{\text{uni}(m)})v\|_{L^2(\Gamma)}^2 \leq C_{\text{norm}} \|v\|_{\tilde{H}^\sigma(\Gamma)}^2 \quad \text{for all } v \in \tilde{H}^\sigma(\Gamma), \quad (3.23)$$

where the constant $C_{\text{norm}} > 0$ depends only on $\mathcal{T}_0, \hat{\kappa}_{\text{max}}, p, w_{\text{min}}, w_{\text{max}}, \gamma$, and σ .

Proof. The L^2 -best approximation property of $\Pi_{\text{uni}(m)}$ yields that $\|v - \Pi_{\text{uni}(m)}v\|_{L^2(\Gamma)} \leq \|v - J_{\text{uni}(m)}v\|_{L^2(\Gamma)}$. With Proposition 3.2, we have that

$$\|v - \Pi_{\text{uni}(m)}v\|_{L^2(\Gamma)} \stackrel{(3.19)}{\lesssim} \bar{h}_{\text{uni}(m)}\|v\|_{\tilde{H}^1(\Gamma)} = \bar{h}_{\text{uni}(0)}2^{-m}\|v\|_{\tilde{H}^1(\Gamma)} \quad \text{for all } v \in \tilde{H}^1(\Gamma). \quad (3.24)$$

The approximation property (3.24) (also called Jackson inequality) together with the inverse inequality (3.20) (also called Bernstein inequality) from Proposition 3.3 allow to apply [Bor94, Theorem 1 and Corollary 1] with $X = \tilde{H}^1(\Gamma)$ and $\alpha = 1$. The latter proves that

$$\|v\|_{\tilde{H}^\sigma(\Gamma)}^2 \simeq \bar{h}_{\text{uni}(0)}^{-2\sigma}\|v\|_{L^2(\Gamma)}^2 + \sum_{m=0}^{\infty} \bar{h}_{\text{uni}(m)}^{-2\sigma}\|(1 - \Pi_{\text{uni}(m)})v\|_{L^2(\Gamma)}^2 \quad \text{for all } v \in \tilde{H}^\sigma(\Gamma).$$

This concludes the proof. \square

3.4. Level function. Let $\mathcal{K}_\bullet \in \mathbb{K}$. For given $T \in \mathcal{T}_\bullet$, let $T_0 \in \mathcal{T}_0$ denote its unique ancestor such that $T \subseteq T_0$ and define with the corresponding elements $\hat{T} = \gamma^{-1}(T)$, $\hat{T}_0 = \gamma^{-1}(T_0)$ in the parameter domain the generation of T by

$$\text{gen}(T) := \frac{\log(|\hat{T}|/|\hat{T}_0|)}{\log(1/2)} \in \mathbb{N}_0,$$

i.e., $\text{gen}(T)$ denotes the number of bisections of $T_0 \in \mathcal{T}_0$ in the parameter domain needed to obtain the element $T \in \mathcal{T}_\bullet$. To each node $z \in \mathcal{N}_\bullet$, we associate

$$\text{level}_\bullet(z) := \max \{ \text{gen}(T) : T \in \mathcal{T}_\bullet \text{ and } z \in T \} \quad \text{for all } z \in \mathcal{N}_\bullet. \quad (3.25)$$

The function $\text{level}_\bullet(\cdot)$ provides a link between the mesh \mathcal{T}_\bullet and the sequence of uniformly refined meshes $\mathcal{T}_{\text{uni}(m)}$. A simple proof of the following result is found in [Füh14, Lemma 6.11].

Lemma 3.5. *Let $\mathcal{K}_\bullet \in \mathbb{K}$ and $z \in \mathcal{N}_\bullet$ and $m := \text{level}_\bullet(z)$. Then, it holds that $z \in \mathcal{N}_{\text{uni}(m)}$ and*

$$C_{\text{level}}^{-1}\bar{h}_{\text{uni}(m)} \leq |T| \leq C_{\text{level}}\bar{h}_{\text{uni}(m)} \quad \text{for all } T \in \mathcal{T}_\bullet \text{ with } z \in T. \quad (3.26)$$

The constant $C_{\text{level}} > 0$ depends only on $\mathcal{T}_0, \hat{\kappa}_{\max}$, and γ . \square

4. LOCAL MULTILEVEL DIAGONAL PRECONDITIONER FOR THE HYPERSINGULAR CASE

Throughout this section, let $(\mathcal{K}_\ell)_{\ell \in \mathbb{N}_0}$ be a sequence of refined knot vectors, i.e., $\mathcal{K}_\ell, \mathcal{K}_{\ell+1} \in \mathbb{K}$ with $\mathcal{K}_{\ell+1} \in \text{refine}(\mathcal{K}_\ell)$, and let $L \in \mathbb{N}_0$. We set

$$\tilde{\mathcal{N}}_{0 \setminus -1} := \mathcal{N}_0 \quad \text{and} \quad \omega_{-1}(\cdot) := \omega_0(\cdot). \quad (4.1)$$

For $\ell \in \mathbb{N}_0$, abbreviate the corresponding index set from (3.9)

$$\tilde{\mathcal{I}}_\ell := \{i \in \{1 - p + o, \dots, N_\ell - p - 1\} : \text{supp}\bar{R}_{\ell, i, p} \cap \tilde{\mathcal{N}}_{\ell \setminus \ell-1} \neq \emptyset\}, \quad (4.2)$$

and define the spaces

$$\tilde{\mathcal{X}}_\ell := \text{span}\{\bar{R}_{\ell, i, p} : i \in \tilde{\mathcal{I}}_\ell\} = \sum_{i \in \tilde{\mathcal{I}}_\ell} \mathcal{X}_{\ell, i} \quad \text{with} \quad \mathcal{X}_{\ell, i} := \text{span}\{\bar{R}_{\ell, i, p}\}. \quad (4.3)$$

Note that $\tilde{\mathcal{I}}_0 = \{1 - p + o, \dots, N_0 - p - 1\}$ and $\tilde{\mathcal{X}}_0 = \mathcal{X}_0$. For all $\ell \in \mathbb{N}_0$ and $i \in \tilde{\mathcal{I}}_\ell$, fix a node

$$\tilde{z}_{\ell, i} \in \tilde{\mathcal{N}}_{\ell \setminus \ell-1} \quad \text{with} \quad \tilde{z}_{\ell, i} \in \text{supp}\bar{R}_{\ell, i, p}. \quad (4.4)$$

For $V_L \in \mathcal{X}_L$ and $\ell = 0, 1, \dots, L$, we define (see Lemma 3.1)

$$\tilde{V}_L^\ell := (J_\ell - J_{\ell-1})V_L \in \tilde{\mathcal{X}}_\ell, \quad \text{where } J_{-1} := 0. \quad (4.5)$$

For all $i \in \tilde{\mathcal{I}}_\ell$, we set with the abbreviation $\alpha_{\ell,i}(\tilde{V}_L^\ell)$ from (3.8)

$$V_L^{\ell,i} := \alpha_{\ell,i}(\tilde{V}_L^\ell) \bar{R}_{\ell,i,p}. \quad (4.6)$$

By the duality property (3.6) and the decomposition (4.3), we have the decompositions

$$\tilde{V}_L^\ell = \sum_{i \in \tilde{\mathcal{I}}_\ell} V_L^{\ell,i} \quad \text{and} \quad V_L = \sum_{\ell=0}^L \tilde{V}_L^\ell = \sum_{\ell=0}^L \sum_{i \in \tilde{\mathcal{I}}_\ell} V_L^{\ell,i} \quad (4.7)$$

and hence

$$\mathcal{X}_L = \sum_{\ell=0}^L \sum_{i \in \tilde{\mathcal{I}}_\ell} \mathcal{X}_{\ell,i}. \quad (4.8)$$

With the one-dimensional $\langle \cdot, \cdot \rangle_{\mathfrak{W}}$ -orthogonal projections $\mathcal{P}_{\ell,i}$ onto $\mathcal{X}_{\ell,i}$ defined by

$$\langle \mathcal{P}_{\ell,i} u, V_{\ell,i} \rangle_{\mathfrak{W}} = \langle u, V_{\ell,i} \rangle_{\mathfrak{W}} \quad \text{for all } u \in \tilde{H}^{1/2}(\Gamma), V_{\ell,i} \in \mathcal{X}_{\ell,i}, \quad (4.9)$$

the space decomposition (4.8) gives rise to the additive Schwarz operator

$$\tilde{\mathcal{P}}_L^{\mathfrak{W}} = \sum_{\ell=0}^L \sum_{i \in \tilde{\mathcal{I}}_\ell} \mathcal{P}_{\ell,i}. \quad (4.10)$$

A similar operator for continuous piecewise affine ansatz functions on affine geometries has been investigated in [FFPS17]. Indeed, the proof of the following main result (for the hypersingular case) is essentially inspired by the corresponding proof of [FFPS17, Theorem 1].

Theorem 4.1. *The additive Schwarz operator $\tilde{\mathcal{P}}_L^{\mathfrak{W}} : \tilde{H}^{1/2}(\Gamma) \rightarrow \mathcal{X}_L$ satisfies that*

$$\lambda_{\min}^{\mathfrak{W}} \|V_L\|_{\mathfrak{W}}^2 \leq \langle \tilde{\mathcal{P}}_L^{\mathfrak{W}} V_L, V_L \rangle_{\mathfrak{W}} \leq \lambda_{\max}^{\mathfrak{W}} \|V_L\|_{\mathfrak{W}}^2 \quad \text{for all } V_L \in \mathcal{X}_L, \quad (4.11)$$

where the constants $\lambda_{\min}^{\mathfrak{W}}, \lambda_{\max}^{\mathfrak{W}} > 0$ depend only on $\mathcal{T}_0, \hat{\kappa}_{\max}, p, w_{\min}, w_{\max}$, and γ .

We split the proof into two parts. In Section 4.1, we show the lower bound. The upper bound is proved in Section 4.2.

4.1. Proof of Theorem 4.1 (lower bound). In the remainder of this section, we will show that the decomposition (4.7) of V_L is stable, i.e.,

$$\sum_{\ell=0}^L \sum_{i \in \tilde{\mathcal{I}}_\ell} \|V_L^{\ell,i}\|_{\mathfrak{W}}^2 \lesssim \|V_L\|_{\mathfrak{W}}^2. \quad (4.12)$$

It is well known from additive Schwarz theory [Lio88, Wid89, Zha92, TW05] that this proves the lower bound in Theorem 4.1; see, e.g., [Zha92, Lemma 3.1]. We start with two auxiliary lemmas. In the following, we set $\mathcal{X}_{\text{uni}(m)} := \mathcal{X}_{\text{uni}(0)}$ if $m < 0$.

Lemma 4.2. *Let $\ell \in \mathbb{N}$ and $q \in \mathbb{N}$. There exists a constant $C_1(q) \in \mathbb{N}_0$ such that for all $z \in \mathcal{N}_\ell$ with $m = \text{level}_\ell(z)$, it holds that*

$$\{V|_{\omega_{\ell-1}^q(z)} : V \in \mathcal{X}_{\text{uni}(m-C_1(q))}\} \subseteq \{V|_{\omega_{\ell-1}^q(z)} : V \in \mathcal{X}_{\ell-1}\} \quad (4.13)$$

The constant $C_1(q)$ depends only on $\widehat{\kappa}_{\max}, \gamma$ and q . We abbreviate $C_1 := C_1(2p+1)$.

Proof. We show that

$$\mathcal{N}_{\text{uni}(m-C_1(q))} \cap \omega_{\ell-1}^q(z) \subseteq \mathcal{N}_{\ell-1} \cap \omega_{\ell-1}^q(z). \quad (4.14)$$

Let $\tau \in \mathcal{T}_\ell$ such that $z \in \tau$. Let $T \in \mathcal{T}_{\ell-1}$ be the father element of τ , i.e., $\tau \subseteq T$. We note that $\text{gen}(T) = \text{gen}(\tau)$ or $\text{gen}(T) = \text{gen}(\tau) + 1$ and hence

$$|\text{gen}(\tau) - \text{gen}(T)| \leq 1.$$

Moreover, there exists a constant $C \in \mathbb{N}$, which depends only on $\widehat{\kappa}_{\max}, \gamma$ and q such that

$$|\text{gen}(T) - \text{gen}(T')| \leq C \quad \text{for all } T' \in \mathcal{T}_{\ell-1} \text{ with } T' \subseteq \omega_{\ell-1}^q(z),$$

i.e., the difference in the element generations within some q -th order patch is uniformly bounded. This implies that

$$\text{gen}(\tau) \leq \text{gen}(T') + C + 1 \quad \text{for all } T' \in \mathcal{T}_{\ell-1} \text{ with } T' \subseteq \omega_{\ell-1}^q(z).$$

By definition of $\text{level}_\ell(z)$, we thus infer that $C_1(q) := C + 1 > 0$ yields that

$$m = \text{level}_\ell(z) \leq \min \{ \text{gen}(T') : T' \in \mathcal{T}_{\ell-1} \text{ and } T' \subseteq \omega_{\ell-1}^q(z) \} + C_1(q). \quad (4.15)$$

For $m - C_1(q) \leq 0$, we have that $\mathcal{X}_{\text{uni}(m-C_1(q))} = \mathcal{X}_0$, and the assertion is clear. Therefore, we suppose that $m - C_1(q) \geq 1$. Let $T' \in \mathcal{T}_{\ell-1}$ with $T' \subseteq \omega_{\ell-1}^q(z)$. According to (4.15), it holds that $m - C_1(q) \leq \text{gen}(T')$. Therefore, there exists a father element $Q \in \mathcal{T}_{\text{uni}(m-C_1(q))}$ with $T' \subseteq Q$. Suppose that (4.14) does not hold true. Then there is some $z' \in \mathcal{N}_{\text{uni}(m-C_1(q))} \cap \omega_{\ell-1}^q(z)$, which is not contained in $\mathcal{N}_{\ell-1} \cap \omega_{\ell-1}^q(z)$. Therefore, z' is in the interior of some $T' \in \mathcal{T}_{\ell-1}$ with $T' \subseteq \omega_{\ell-1}^q(z)$ and hence also in the interior of the father $Q \in \mathcal{T}_{\text{uni}(m-C_1(q))}$ of T' . This contradicts $z \in \mathcal{N}_{\text{uni}(m-C_1(q))}$ and concludes the proof of (4.14).

By the definition of $\mathcal{X}_{\text{uni}(m-C_1(q))}$, we even have for the multiplicities that $\#_{\text{uni}(m-C_1(q))} z' \leq \#_{\ell-1} z'$ for $z' \in \mathcal{N}_{\text{uni}(m-C_1(q))} \cap \omega_{\ell-1}^q(z)$. With Lemma 2.1 (i) and the fact that the denominator w in (2.28) is fixed, this proves the assertion. \square

Lemma 4.3. *For each $m \in \mathbb{N}_0$ and $z \in \mathcal{N}_L$, it holds that $|\mathcal{Z}_m(z)| \leq C_2$, where*

$$\mathcal{Z}_m(z) := \{(\ell, i) : \ell \in \{0, \dots, L\}, i \in \widetilde{\mathcal{I}}_\ell, \text{level}_\ell(\widetilde{z}_{\ell,i}) = m, z = \widetilde{z}_{\ell,i}\}. \quad (4.16)$$

The constant $C_2 > 0$ depends only on p .

Proof. As we only use bisection or knot multiplicity increase (with maximal multiplicity p), it holds that $|\{\ell \in \{1, \dots, L\} : z \in \widetilde{\mathcal{N}}_{\ell \setminus \ell-1}\}| \leq p$. This shows that only a bounded number of different ℓ appears in the set of (4.16). For fixed $\ell \in \{0, \dots, L\}$, (4.4) and $\omega_{\ell-1}^{p+1}(z) \subseteq \omega_\ell^{2(p+1)}(z)$ yield that

$$\begin{aligned} \{i \in \widetilde{\mathcal{I}}_\ell : \text{level}_\ell(\widetilde{z}_{\ell,i}) = m, z = \widetilde{z}_{\ell,i}\} &\stackrel{(4.4)}{\subseteq} \{i : \text{supp}(\overline{R}_{\ell,i,p}) \subseteq \omega_{\ell-1}^{p+1}(z)\} \\ &\subseteq \{i : \text{supp}(\overline{R}_{\ell,i,p}) \subseteq \omega_\ell^{2(p+1)}(z)\}. \end{aligned}$$

The cardinality of the last set is bounded by a constant $C_2 > 0$ that depends only on p . \square

Proof of lower bound in (4.11). The proof is split into two steps.

Step 1: We show (4.19). The norm equivalence $\|\cdot\|_{\mathfrak{W}} \simeq \|\cdot\|_{H^{1/2}(\Gamma)}$, the inverse inequality (3.20) for NURBS and $\|h_\ell^{-1/2} R_{\ell,i,p}\|_{L^2(\Gamma)} \lesssim 1$ prove for the functions $V_L^{\ell,i}$ of (4.6) that

$$\|V_L^{\ell,i}\|_{\mathfrak{W}}^2 \lesssim |\alpha_{\ell,i}((J_\ell - J_{\ell-1})V_L)|^2.$$

The Cauchy-Schwarz inequality, and the property (3.7) of the dual basis functions imply that

$$\|V_L^{\ell,i}\|_{\mathfrak{W}}^2 \lesssim |\text{supp}(\bar{R}_{\ell,i,p})|^{-1} \|(J_\ell - J_{\ell-1})V_L\|_{L^2(\text{supp}(\bar{R}_{\ell,i,p}))}^2. \quad (4.17)$$

We abbreviate $m = \text{level}_\ell(\tilde{z}_{\ell,i})$. Proposition 3.2 (i) and Lemma 4.2 together with nestedness $\mathcal{X}_{\ell-1} \subseteq \mathcal{X}_\ell$ imply for almost every $x \in \omega_{\ell-1}^{p+1}(\tilde{z}_{\ell,i})$ that

$$(J_\ell \Pi_{\text{uni}(m-C_1)} V_L)(x) = (\Pi_{\text{uni}(m-C_1)} V_L)(x) = (J_{\ell-1} \Pi_{\text{uni}(m-C_1)} V_L)(x).$$

This together with (4.4) and local L^2 -stability of J_ℓ and $J_{\ell-1}$ (Proposition 3.2 (ii)) shows that

$$\begin{aligned} \|(J_\ell - J_{\ell-1})V_L\|_{L^2(\text{supp}(\bar{R}_{\ell,i,p}))}^2 &= \|(J_\ell - J_{\ell-1})(1 - \Pi_{\text{uni}(m-C_1)})V_L\|_{L^2(\text{supp}(\bar{R}_{\ell,i,p}))}^2 \\ &\leq \|(J_\ell - J_{\ell-1})(1 - \Pi_{\text{uni}(m-C_1)})V_L\|_{L^2(\omega_{\ell-1}^{p+1}(\tilde{z}_{\ell,i}))}^2 \\ &\lesssim \|(1 - \Pi_{\text{uni}(m-C_1)})V_L\|_{L^2(\omega_{\ell-1}^{2p+1}(\tilde{z}_{\ell,i}))}^2. \end{aligned} \quad (4.18)$$

Further, Lemma 3.5 shows that $\bar{h}_{\text{uni}(m)} \simeq |\text{supp}(\bar{R}_{\ell,i,p})|$. Hence, (4.17) and (4.18) prove that

$$\|V_L^{\ell,i}\|_{\mathfrak{W}}^2 \lesssim \bar{h}_{\text{uni}(m)}^{-1} \|(1 - \Pi_{\text{uni}(m-C_1)})V_L\|_{L^2(\omega_{\ell-1}^{2p+1}(\tilde{z}_{\ell,i}))}^2. \quad (4.19)$$

Step 2: We show (4.12), which concludes the lower bound in (4.11). Step 1 gives that

$$\begin{aligned} \sum_{\ell=0}^L \sum_{i \in \tilde{\mathcal{I}}_\ell} \|V_L^{\ell,i}\|_{\mathfrak{W}}^2 &= \sum_{m=0}^{\infty} \sum_{\ell=0}^L \sum_{\substack{i \in \tilde{\mathcal{I}}_\ell \\ \text{level}_\ell(\tilde{z}_{\ell,i})=m}} \|V_L^{\ell,i}\|_{\mathfrak{W}}^2 \\ &\lesssim \sum_{m=0}^{\infty} \sum_{\ell=0}^L \sum_{\substack{i \in \tilde{\mathcal{I}}_\ell \\ \text{level}_\ell(\tilde{z}_{\ell,i})=m}} \bar{h}_{\text{uni}(m)}^{-1} \|(1 - \Pi_{\text{uni}(m-C_1)})V_L\|_{L^2(\omega_{\ell-1}^{2p+1}(\tilde{z}_{\ell,i}))}^2. \end{aligned}$$

There exists a constant $C_3 \in \mathbb{N}$, which depends only on $p, \hat{\kappa}_{\max}, \gamma$, and \mathcal{T}_0 , such that for $z \in \mathcal{N}_\ell$ with $\text{level}_\ell(z) = m$, it holds that

$$\omega_{\ell-1}^{2p+1}(z) \subseteq \omega_{\text{uni}(m)}^{C_3}(z). \quad (4.20)$$

Hence,

$$\begin{aligned} \sum_{\ell=0}^L \sum_{i \in \tilde{\mathcal{I}}_\ell} \|V_L^{i,\ell}\|_{\mathfrak{W}}^2 &\lesssim \sum_{m=0}^{\infty} \sum_{\ell=0}^L \sum_{\substack{i \in \tilde{\mathcal{I}}_\ell \\ \text{level}_\ell(\tilde{z}_{\ell,i})=m}} \bar{h}_{\text{uni}(m)}^{-1} \|(1 - \Pi_{\text{uni}(m-C_1)})V_L\|_{L^2(\omega_{\text{uni}(m)}^{C_3}(\tilde{z}_{\ell,i}))}^2 \\ &\stackrel{(4.16)}{=} \sum_{m=0}^{\infty} \sum_{z \in \mathcal{N}_L} \sum_{(\ell,i) \in \mathcal{Z}_m(z)} \bar{h}_{\text{uni}(m)}^{-1} \|(1 - \Pi_{\text{uni}(m-C_1)})V_L\|_{L^2(\omega_{\text{uni}(m)}^{C_3}(z))}^2. \end{aligned}$$

If $z \in \mathcal{N}_L$ and $(\ell, i) \in \mathcal{Z}_m(z)$, it follows that $z \in \mathcal{N}_\ell$ with $\text{level}_\ell(z) = m$ by definition. Lemma 3.5 implies that $z \in \mathcal{N}_{\text{uni}(m)}$. This and Lemma 4.3 give that

$$\begin{aligned}
& \sum_{m=0}^{\infty} \sum_{z \in \mathcal{N}_L} \sum_{(\ell, i) \in \mathcal{Z}_m(z)} \bar{h}_{\text{uni}(m)}^{-1} \|(1 - \Pi_{\text{uni}(m-C_1)})V_L\|_{L^2(\omega_{\text{uni}(m)}^{C_3}(z))}^2 \\
&= \sum_{m=0}^{\infty} \sum_{z \in \mathcal{N}_L \cap \mathcal{N}_{\text{uni}(m)}} \sum_{(\ell, i) \in \mathcal{Z}_m(z)} \bar{h}_{\text{uni}(m)}^{-1} \|(1 - \Pi_{\text{uni}(m-C_1)})V_L\|_{L^2(\omega_{\text{uni}(m)}^{C_3}(z))}^2 \\
&\stackrel{(4.16)}{\lesssim} \sum_{m=0}^{\infty} \sum_{z \in \mathcal{N}_L \cap \mathcal{N}_{\text{uni}(m)}} \bar{h}_{\text{uni}(m)}^{-1} \|(1 - \Pi_{\text{uni}(m-C_1)})V_L\|_{L^2(\omega_{\text{uni}(m)}^{C_3}(z))}^2 \\
&\leq \sum_{m=0}^{\infty} \sum_{z \in \mathcal{N}_{\text{uni}(m)}} \bar{h}_{\text{uni}(m)}^{-1} \|(1 - \Pi_{\text{uni}(m-C_1)})V_L\|_{L^2(\omega_{\text{uni}(m)}^{C_3}(z))}^2 \\
&\lesssim \sum_{m=0}^{\infty} \bar{h}_{\text{uni}(m)}^{-1} \|(1 - \Pi_{\text{uni}(m-C_1)})V_L\|_{L^2(\Gamma)}^2
\end{aligned}$$

The definition $\Pi_{\text{uni}(m)} = \Pi_{\text{uni}(0)}$ for $m < 0$ yields that

$$\sum_{m=0}^{\infty} \bar{h}_{\text{uni}(m)}^{-1} \|(1 - \Pi_{\text{uni}(m-C_1)})V_L\|_{L^2(\Gamma)}^2 \lesssim \sum_{m=0}^{\infty} \bar{h}_{\text{uni}(m)}^{-1} \|(1 - \Pi_{\text{uni}(m)})V_L\|_{L^2(\Gamma)}^2$$

Combining the latter three estimates, Lemma 3.4 leads us to

$$\sum_{\ell=0}^L \sum_{i \in \tilde{\mathcal{I}}_\ell} \|V_L^{\ell, i}\|_{\mathfrak{W}}^2 \lesssim \sum_{m=0}^{\infty} \bar{h}_{\text{uni}(m)}^{-1} \|(1 - \Pi_{\text{uni}(m)})V_L\|_{L^2(\Gamma)}^2 \stackrel{(3.23)}{\lesssim} \|V_L\|_{H^{1/2}(\Gamma)}^2 \simeq \|V_L\|_{\mathfrak{W}}^2.$$

This proves (4.12) and yields the lower bound in (4.11). \square

4.2. Proof of Theorem 4.1 (upper bound). For $m \in \mathbb{N}_0$, let $\mathcal{K}_{\text{uni}(m,p)} \in \mathbb{K}$ be the knot vector with $\mathcal{T}_{\text{uni}(m,p)} = \mathcal{T}_{\text{uni}(m)}$ and $\#z = p$ for all $z \in \mathcal{N}_{\text{uni}(m,p)} \setminus \{\gamma(a), \gamma(b)\}$. By Lemma 3.5, it holds that $\mathcal{N}_L \subseteq \mathcal{N}_{\text{uni}(M,p)}$, where

$$M := \max_{z \in \mathcal{N}_L} \text{level}_L(z). \tag{4.21}$$

The definition of $\mathcal{X}_{\text{uni}(m,p)}$ yields that $\mathcal{X}_L \subseteq \mathcal{X}_{\text{uni}(M,p)}$. Moreover, we can rewrite the additive Schwarz operator as

$$\tilde{\mathcal{P}}_L^{\mathfrak{W}} = \sum_{m=0}^M \mathcal{Q}_m \quad \text{with} \quad \mathcal{Q}_m := \sum_{\ell=0}^L \sum_{\substack{i \in \tilde{\mathcal{I}}_\ell \\ \text{level}_\ell(\tilde{z}_{\ell, i})=m}} \mathcal{P}_{\ell, i}. \tag{4.22}$$

There holds the following type of strengthened Cauchy-Schwarz inequality.

Lemma 4.4. *For all $0 \leq m \leq M$, $\langle\langle \mathcal{Q}_m(\cdot), (\cdot) \rangle\rangle_{\mathfrak{W}}$ defines a symmetric positive semi-definite bilinear form on $\tilde{H}^{1/2}(\Gamma)$. For $k \in \mathbb{N}_0$, it holds that*

$$\langle\langle \mathcal{Q}_m V_{\text{uni}(k,p)}, V_{\text{uni}(k,p)} \rangle\rangle_{\mathfrak{W}} \leq C_4 2^{-(m-k)} \|V_{\text{uni}(k,p)}\|_{\mathfrak{W}}^2 \quad \text{for all } V_{\text{uni}(k,p)} \in \mathcal{X}_{\text{uni}(k,p)}. \tag{4.23}$$

The constant $C_4 > 0$ depends only on $\mathcal{T}_0, \hat{\kappa}_{\max}, p, w_{\min}, w_{\max}$ and γ .

Proof. Symmetry and positive semi-definiteness follow by the symmetry and positive semi-definiteness of the one-dimensional projectors $\mathcal{P}_{\ell,i}$. To see (4.23), we only consider closed $\Gamma = \partial\Omega$ and split the proof into two steps. For open $\Gamma \subsetneq \partial\Omega$ the proof works analogously.

Step 1: Let $\ell \in \{0, \dots, L\}$ and $i \in \tilde{\mathcal{I}}_\ell$ with $\text{level}_\ell(\tilde{z}_{\ell,i}) = m$. We want to estimate $\langle\langle \mathcal{P}_{\ell,i} V_{\text{uni}(k,p)}, V_{\text{uni}(k,p)} \rangle\rangle_{\mathfrak{W}}$. From the definition (4.9) of $\mathcal{P}_{\ell,i}$, we infer that

$$\langle\langle \mathcal{P}_{\ell,i} V_{\text{uni}(k,p)}, V_{\text{uni}(k,p)} \rangle\rangle_{\mathfrak{W}} = \frac{\langle\langle V_{\text{uni}(k,p)}, \overline{R}_{\ell,i,p} \rangle\rangle_{\mathfrak{W}}^2}{\|\overline{R}_{\ell,i,p}\|_{\mathfrak{W}}^2}. \quad (4.24)$$

Lipschitz continuity of γ gives that $|\widehat{R}_{\ell,i,p}|_{H^{1/2}(\text{supp}(\widehat{R}_{\ell,i,p}))} \lesssim |R_{\ell,i,p}|_{H^{1/2}(\text{supp}(R_{\ell,i,p}))} \lesssim |\overline{R}_{\ell,i,p}|_{H^{1/2}(\Gamma)}$. Hence, Lemma 2.3 with $\sigma = 1/2$ shows that $1 \lesssim |\overline{R}_{\ell,i,p}|_{H^{1/2}(\Gamma)}$. This implies that

$$\|\overline{R}_{\ell,i,p}\|_{L^2(\Gamma)} \lesssim |\text{supp}(\overline{R}_{\ell,i,p})|^{1/2} \lesssim |\text{supp}(\overline{R}_{\ell,i,p})|^{1/2} |\overline{R}_{\ell,i,p}|_{H^{1/2}(\Gamma)} \lesssim |\text{supp}(\overline{R}_{\ell,i,p})|^{1/2} \|\overline{R}_{\ell,i,p}\|_{\mathfrak{W}}.$$

With the Cauchy-Schwarz inequality and $|\text{supp}\overline{R}_{\ell,i,p}| \simeq \overline{h}_{\text{uni}(m)}$ (Lemma 3.5), this gives that

$$\begin{aligned} \langle\langle \mathcal{P}_{\ell,i} V_{\text{uni}(k,p)}, V_{\text{uni}(k,p)} \rangle\rangle_{\mathfrak{W}} &\lesssim \frac{\langle \mathfrak{W} V_{\text{uni}(k,p)}, \overline{R}_{\ell,i,p} \rangle_{\Gamma}^2}{\|\overline{R}_{\ell,i,p}\|_{\mathfrak{W}}^2} + \frac{\langle V_{\text{uni}(k,p)}, 1 \rangle_{\Gamma}^2 \langle \overline{R}_{\ell,i,p}, 1 \rangle_{\Gamma}^2}{\|\overline{R}_{\ell,i,p}\|_{\mathfrak{W}}^2} \\ &\lesssim |\text{supp}(\overline{R}_{\ell,i,p})| (\|\mathfrak{W} V_{\text{uni}(k,p)}\|_{L^2(\text{supp}(\overline{R}_{\ell,i,p}))}^2 + |\text{supp}(\overline{R}_{\ell,i,p})| \|V_{\text{uni}(k,p)}\|_{L^2(\Gamma)}^2) \\ &\lesssim \frac{\overline{h}_{\text{uni}(m)}}{\overline{h}_{\text{uni}(k)}} (\|\overline{h}_{\text{uni}(k)}^{-1/2} \mathfrak{W} V_{\text{uni}(k,p)}\|_{L^2(\text{supp}(\overline{R}_{\ell,i,p}))}^2 + |\text{supp}(\overline{R}_{\ell,i,p})| \|V_{\text{uni}(k,p)}\|_{L^2(\Gamma)}^2) \\ &= 2^{-(m-k)} (\|\overline{h}_{\text{uni}(k)}^{-1/2} \mathfrak{W} V_{\text{uni}(k,p)}\|_{L^2(\text{supp}(\overline{R}_{\ell,i,p}))}^2 + |\text{supp}(\overline{R}_{\ell,i,p})| \|V_{\text{uni}(k,p)}\|_{L^2(\Gamma)}^2). \end{aligned} \quad (4.25)$$

Step 2: We stress that the choice (4.4) of $\tilde{z}_{\ell,i}$ and (4.20) show that

$$\text{supp}(\overline{R}_{\ell,i,p}) \stackrel{(4.4)}{\subseteq} \omega_{\ell-1}^{p+1}(\tilde{z}_{\ell,i}) \subseteq \omega_{\ell-1}^{2p+1}(\tilde{z}_{\ell,i}) \stackrel{(4.20)}{\subseteq} \omega_{\text{uni}(m)}^{C_3}(\tilde{z}_{\ell,i}).$$

Thus, the definition of \mathcal{Q}_m and Step 1 yield that

$$\begin{aligned} \langle\langle \mathcal{Q}_m V_{\text{uni}(k,p)}, V_{\text{uni}(k,p)} \rangle\rangle_{\mathfrak{W}} &= \sum_{\ell=0}^L \sum_{\substack{i \in \tilde{\mathcal{I}}_\ell \\ \text{level}_\ell(\tilde{z}_{\ell,i})=m}} \langle\langle \mathcal{P}_{\ell,i} V_{\text{uni}(k,p)}, V_{\text{uni}(k,p)} \rangle\rangle_{\mathfrak{W}} \\ &\stackrel{(4.25)}{\lesssim} 2^{-(m-k)} \sum_{\ell=0}^L \sum_{\substack{i \in \tilde{\mathcal{I}}_\ell \\ \text{level}_\ell(\tilde{z}_{\ell,i})=m}} \left(\|\overline{h}_{\text{uni}(k)}^{-1/2} \mathfrak{W} V_{\text{uni}(k,p)}\|_{L^2(\omega_{\text{uni}(m)}^{C_3}(\tilde{z}_{\ell,i}))}^2 + |\omega_{\text{uni}(m)}^{C_3}(\tilde{z}_{\ell,i})| \|V_{\text{uni}(k,p)}\|_{L^2(\Gamma)}^2 \right) \\ &\stackrel{(4.16)}{=} 2^{-(m-k)} \sum_{z \in \mathcal{N}_L} \sum_{(\ell,i) \in \mathcal{Z}_m(z)} \left(\|\overline{h}_{\text{uni}(k)}^{-1/2} \mathfrak{W} V_{\text{uni}(k,p)}\|_{L^2(\omega_{\text{uni}(m)}^{C_3}(z))}^2 + |\omega_{\text{uni}(m)}^{C_3}(z)| \|V_{\text{uni}(k,p)}\|_{L^2(\Gamma)}^2 \right). \end{aligned}$$

If $z \in \mathcal{N}_L$ and $(\ell, i) \in \mathcal{Z}_m(z)$, it follows $z \in \mathcal{N}_\ell$ with $\text{level}_\ell(z) = m$. Lemma 3.5 implies that $z \in \mathcal{N}_{\text{uni}(m)}$. Hence, we can replace in the upper sum \mathcal{N}_L by $\mathcal{N}_L \cap \mathcal{N}_{\text{uni}(m)}$. With Lemma 4.3,

we further see that

$$\begin{aligned}
& \sum_{z \in \mathcal{N}_L \cap \mathcal{N}_{\text{uni}(m)}} \sum_{(\ell, i) \in \mathcal{Z}_m(z)} \left(\|\bar{h}_{\text{uni}(k)}^{-1/2} \mathfrak{W} V_{\text{uni}(k,p)}\|_{L^2(\omega_{\text{uni}(m)}^{C_3}(z))}^2 + |\omega_{\text{uni}(m)}^{C_3}(z)| \|V_{\text{uni}(k,p)}\|_{L^2(\Gamma)}^2 \right) \\
& \lesssim \sum_{z \in \mathcal{N}_{\text{uni}(m)}} \left(\|\bar{h}_{\text{uni}(k)}^{-1/2} \mathfrak{W} V_{\text{uni}(k,p)}\|_{L^2(\omega_{\text{uni}(m)}^{C_3}(z))}^2 + |\omega_{\text{uni}(m)}^{C_3}(z)| \|V_{\text{uni}(k,p)}\|_{L^2(\Gamma)}^2 \right) \\
& \lesssim \|\bar{h}_{\text{uni}(k)}^{-1/2} \mathfrak{W} V_{\text{uni}(k,p)}\|_{L^2(\Gamma)}^2 + \|V_{\text{uni}(k,p)}\|_{L^2(\Gamma)}^2.
\end{aligned}$$

Note that boundedness $\mathfrak{W} : H^1(\Gamma) \rightarrow L^2(\Gamma)$ as well as the inverse inequality (3.21) prove that

$$\|\bar{h}_{\text{uni}(k)}^{-1/2} \mathfrak{W} V_{\text{uni}(k,p)}\|_{L^2(\Gamma)}^2 \lesssim \bar{h}_{\text{uni}(k)} \|V_{\text{uni}(k,p)}\|_{H^1(\Gamma)}^2 \lesssim \|V_{\text{uni}(k,p)}\|_{H^{1/2}(\Gamma)}^2.$$

Putting the latter three inequalities together shows that

$$\langle \mathcal{Q}_m V_{\text{uni}(k,p)}, V_{\text{uni}(k,p)} \rangle_{\mathfrak{W}} \lesssim 2^{-(m-k)} (\|V_{\text{uni}(k,p)}\|_{H^{1/2}(\Gamma)}^2 + \|V_{\text{uni}(k,p)}\|_{L^2(\Gamma)}^2) \simeq 2^{-(m-k)} \|V_{\text{uni}(k,p)}\|_{\mathfrak{W}}^2.$$

This finishes the proof. \square

The rest of the proof of the upper bound in Theorem 4.1 follows essentially as in [TS96, Lemma 2.8] and is only given for completeness; see also [FFPS17, Section 4.6].

Proof of upper bound in (4.11). For $k \in \mathbb{N}_0$ let $\mathcal{G}_{\text{uni}(k,p)} : \tilde{H}^{1/2}(\Gamma) \rightarrow \mathcal{X}_{\text{uni}(k,p)}$ denote the Galerkin projection onto $\mathcal{X}_{\text{uni}(k,p)}$ with respect to the scalar product $\langle \cdot, \cdot \rangle_{\mathfrak{W}}$, i.e.,

$$\langle \mathcal{G}_{\text{uni}(k,p)} v, V_{\text{uni}(k,p)} \rangle_{\mathfrak{W}} = \langle v, V_{\text{uni}(k,p)} \rangle_{\mathfrak{W}} \quad \text{for all } v \in \tilde{H}^{1/2}(\Gamma), V_{\text{uni}(k,p)} \in \mathcal{X}_{\text{uni}(k,p)}. \quad (4.26)$$

Note that $\mathcal{G}_{\text{uni}(k,p)}$ is the orthogonal projection onto $\mathcal{X}_{\text{uni}(k,p)}$ with respect to the energy norm $\|\cdot\|_{\mathfrak{W}}$. Moreover, we set $\mathcal{G}_{\text{uni}(-1,p)} := 0$. The proof is split into three steps.

Step 1: Let $V_L \in \mathcal{X}_L \subseteq \mathcal{X}_{\text{uni}(M,p)}$. Lemma 3.5 and the boundedness of the local mesh-ratio by $\hat{\kappa}_{\max}$ yield the existence of a constant $C \in \mathbb{N}_0$, which depends only on $\mathcal{T}_0, \hat{\kappa}_{\max}, \gamma$, and p , such that $\mathcal{N}_\ell \cap \omega_{\ell-1}^{p+2}(z) \subseteq \mathcal{N}_{\text{uni}(m+C,p)}$ for all nodes $z \in \mathcal{N}_\ell$ with $\text{level}_\ell(z) = m$. Lemma 2.1 (i) hence proves that $\mathcal{X}_{\ell,i} \subseteq \mathcal{X}_{\text{uni}(m+C,p)}$ for all $m \in \{0, \dots, M\}, \ell \in \{0, \dots, L\}$, and $i \in \tilde{\mathcal{I}}_\ell$ with $\text{level}_\ell(\tilde{z}_{i,\ell}) = m$. Therefore, the range of \mathcal{Q}_m is a subspace of $\mathcal{X}_{\text{uni}(m+C,p)}$. This shows that

$$\langle \mathcal{Q}_m V_L, V_L \rangle_{\mathfrak{W}} = \langle \mathcal{Q}_m V_L, \mathcal{G}_{\text{uni}(m+C,p)} V_L \rangle_{\mathfrak{W}}. \quad (4.27)$$

Step 2: In Lemma 4.4, we saw that $\langle \mathcal{Q}_m(\cdot), (\cdot) \rangle_{\mathfrak{W}}$ defines a symmetric positive semi-definite bilinear form and hence satisfies a Cauchy-Schwarz inequality. This and (4.27) yield that

$$\begin{aligned}
\langle \mathcal{Q}_m V_L, V_L \rangle_{\mathfrak{W}} &= \langle \mathcal{Q}_m V_L, \mathcal{G}_{\text{uni}(m+C,p)} V_L \rangle_{\mathfrak{W}} = \sum_{k=0}^{m+C} \langle \mathcal{Q}_m V_L, (\mathcal{G}_{\text{uni}(k,p)} - \mathcal{G}_{\text{uni}(k-1,p)}) V_L \rangle_{\mathfrak{W}} \\
&\leq \sum_{k=0}^{m+C} \langle \mathcal{Q}_m V_L, V_L \rangle_{\mathfrak{W}}^{1/2} \langle \mathcal{Q}_m (\mathcal{G}_{\text{uni}(k,p)} - \mathcal{G}_{\text{uni}(k-1,p)}) V_L, (\mathcal{G}_{\text{uni}(k,p)} - \mathcal{G}_{\text{uni}(k-1,p)}) V_L \rangle_{\mathfrak{W}}^{1/2}.
\end{aligned}$$

For the second scalar product, we apply Lemma 4.4 and obtain that

$$\begin{aligned}
& \langle \mathcal{Q}_m (\mathcal{G}_{\text{uni}(k,p)} - \mathcal{G}_{\text{uni}(k-1,p)}) V_L, (\mathcal{G}_{\text{uni}(k,p)} - \mathcal{G}_{\text{uni}(k-1,p)}) V_L \rangle_{\mathfrak{W}} \\
& \lesssim 2^{-(m-k)} \|(\mathcal{G}_{\text{uni}(k,p)} - \mathcal{G}_{\text{uni}(k-1,p)}) V_L\|_{\mathfrak{W}}^2 = 2^{-(m-k)} \langle (\mathcal{G}_{\text{uni}(k,p)} - \mathcal{G}_{\text{uni}(k-1,p)})^2 V_L, V_L \rangle_{\mathfrak{W}}.
\end{aligned}$$

Note that $(\mathcal{G}_{\text{uni}(k,p)} - \mathcal{G}_{\text{uni}(k-1,p)})^2 = \mathcal{G}_{\text{uni}(k,p)} - \mathcal{G}_{\text{uni}(k-1,p)}$, since $\mathcal{G}_{\text{uni}(k,p)} - \mathcal{G}_{\text{uni}(k-1,p)}$ is again an orthogonal projection.

Step 3: With the representation (4.22) of $\tilde{\mathcal{P}}_L^{\mathfrak{W}}$, the two inequalities from Step 2, and the Young inequality, we infer for all $\delta > 0$ that

$$\begin{aligned} \langle\langle \tilde{\mathcal{P}}_L^{\mathfrak{W}} V_L, V_L \rangle\rangle_{\mathfrak{W}} &\stackrel{(4.22)}{=} \sum_{m=0}^M \langle\langle \mathcal{Q}_m V_L, V_L \rangle\rangle_{\mathfrak{W}} \\ &\lesssim \frac{\delta}{2} \sum_{m=0}^M \sum_{k=0}^{m+C} 2^{-(m-k)/2} \langle\langle \mathcal{Q}_m V_L, V_L \rangle\rangle_{\mathfrak{W}} + \frac{\delta^{-1}}{2} \sum_{m=0}^M \sum_{k=0}^{m+C} 2^{-(m-k)/2} \langle\langle (\mathcal{G}_{\text{uni}(k,p)} - \mathcal{G}_{\text{uni}(k-1,p)}) V_L, V_L \rangle\rangle_{\mathfrak{W}}. \end{aligned}$$

We abbreviate $\sum_{k=-C}^{\infty} 2^{-k/2} =: K < \infty$. Changing the summation indices in the second sum, we see with $V_L \in \mathcal{X}_L \subseteq \mathcal{X}_{\text{uni}(M,p)} \subseteq \mathcal{X}_{\text{uni}(M+C,p)}$ that

$$\begin{aligned} \langle\langle \tilde{\mathcal{P}}_L^{\mathfrak{W}} V_L, V_L \rangle\rangle_{\mathfrak{W}} &\lesssim K \frac{\delta}{2} \sum_{m=0}^M \langle\langle \mathcal{Q}_m V_L, V_L \rangle\rangle_{\mathfrak{W}} \\ &\quad + \frac{\delta^{-1}}{2} \sum_{k=0}^{M+C} \sum_{m=\max(k-C,0)}^M 2^{-(m-k)/2} \langle\langle (\mathcal{G}_{\text{uni}(k,p)} - \mathcal{G}_{\text{uni}(k-1,p)}) V_L, V_L \rangle\rangle_{\mathfrak{W}} \\ &\leq K \frac{\delta}{2} \sum_{m=0}^M \langle\langle \mathcal{Q}_m V_L, V_L \rangle\rangle_{\mathfrak{W}} + K \frac{\delta^{-1}}{2} \sum_{k=0}^{M+C} \langle\langle (\mathcal{G}_{\text{uni}(k,p)} - \mathcal{G}_{\text{uni}(k-1,p)}) V_L, V_L \rangle\rangle_{\mathfrak{W}} \\ &= K \frac{\delta}{2} \sum_{m=0}^M \langle\langle \mathcal{Q}_m V_L, V_L \rangle\rangle_{\mathfrak{W}} + K \frac{\delta^{-1}}{2} \langle\langle \mathcal{G}_{\text{uni}(M+C,p)} V_L, V_L \rangle\rangle_{\mathfrak{W}} \\ &\stackrel{(4.22)}{=} K \frac{\delta}{2} \langle\langle \tilde{\mathcal{P}}_L^{\mathfrak{W}} V_L, V_L \rangle\rangle_{\mathfrak{W}} + K \frac{\delta^{-1}}{2} \langle\langle V_L, V_L \rangle\rangle_{\mathfrak{W}}. \end{aligned}$$

Choosing $\delta > 0$ sufficiently small and absorbing the first-term on the right-hand side on the left, we prove the upper bound in (4.11). \square

5. LOCAL MULTILEVEL DIAGONAL PRECONDITIONER FOR THE WEAKLY-SINGULAR CASE

Finally, we generalize the results of the previous sections to the weakly-singular integral equation. The main tool in the following is Maue's formula (see, e.g. [AEF⁺14])

$$\langle \mathfrak{W}u, v \rangle_{\Gamma} = \langle \mathfrak{B} \partial_{\Gamma} u, \partial_{\Gamma} v \rangle_{\Gamma} \quad \text{for all } u, v \in \tilde{H}^1(\Gamma). \quad (5.1)$$

For similar proofs in the case of piecewise constant ansatz functions, we refer to [TS96] (uniform meshes) resp. [FFPS15] (adaptive meshes). Throughout this section, let $(\mathcal{K}_{\ell})_{\ell \in \mathbb{N}_0}$ be a sequence of refined knot vectors, i.e., $\mathcal{K}_{\ell}, \mathcal{K}_{\ell+1} \in \mathbb{K}$ with $\mathcal{K}_{\ell+1} \in \text{refine}(\mathcal{K}_{\ell})$, and let $L \in \mathbb{N}_0$. For each $\Psi_L \in \mathcal{Y}_L$, we consider the unique decomposition

$$\Psi_L = \Psi_L^{00} + \Psi_L^0, \quad \text{where } \Psi_L^{00} := \langle \Psi_L, 1 \rangle_{\Gamma} / |\Gamma| \quad \text{and} \quad \Psi_L^0 := \Psi_L - \Psi_L^{00}. \quad (5.2)$$

Note that

$$\Psi_L^{00} \in \mathcal{Y}^{00} := \text{span}\{1\} \quad \text{and} \quad \Psi_L^0 \in \mathcal{Y}_L^0 := \{\Psi_L \in \mathcal{Y}_L : \langle \Psi_L, 1 \rangle_{\Gamma} = 0\}. \quad (5.3)$$

With hidden constants, which depend only on Γ , it holds that

$$\|\Psi_L^{00}\|_{\mathfrak{Y}} \lesssim \|\Psi_L\|_{\mathfrak{Y}} \quad \text{and} \quad \|\Psi_L^0\|_{\mathfrak{Y}} \lesssim \|\Psi_L\|_{\mathfrak{Y}}. \quad (5.4)$$

Recall the spaces $\mathcal{X}_L, \tilde{\mathcal{X}}_\ell$ and $\mathcal{X}_{\ell,i}$ from Section 4. For $\ell \in \{0, \dots, L\}$, set

$$\tilde{\mathcal{Y}}_\ell^0 := \partial_\Gamma \tilde{\mathcal{X}}_\ell \stackrel{(4.3)}{=} \sum_{i \in \tilde{\mathcal{I}}_\ell} \mathcal{Y}_{\ell,i}^0 \quad \text{with} \quad \mathcal{Y}_{\ell,i}^0 := \partial_\Gamma \mathcal{X}_{\ell,i}. \quad (5.5)$$

Recall that we only consider non-rational splines in the weakly-singular case; see Section 2.9. For rational splines, the following lemma is in general false.

Lemma 5.1. *It holds that*

$$\mathcal{Y}_L^0 = \partial_\Gamma \mathcal{X}_L. \quad (5.6)$$

For closed $\Gamma = \partial\Omega$, we even have that

$$\mathcal{Y}_L^0 = \partial_\Gamma \mathcal{X}_L^0, \quad \text{where} \quad \mathcal{X}_L^0 := \{V_L \in \mathcal{X}_L : \langle V_L, 1 \rangle_\Gamma = 0\}. \quad (5.7)$$

Proof. Since Γ is connected, the kernel of $\partial_\Gamma(\cdot)$ is one-dimensional. Linear algebra yields that $N_L - 1 = \dim \mathcal{X}_L = \dim \mathcal{Y}_L$, $N_L - 2 = \dim \partial_\Gamma \mathcal{X}_L = \dim \mathcal{X}_L^0 = \dim \mathcal{Y}_L^0$ for $\Gamma = \partial\Omega$ resp. $N_L - 1 = \dim \mathcal{Y}_L$, $N_L - 2 = \dim \mathcal{X}_L = \dim \partial_\Gamma \mathcal{X}_L = \dim \mathcal{Y}_L^0$ for $\Gamma \subsetneq \partial\Omega$. It thus remains to prove that $\partial_\Gamma \mathcal{X}_L \subseteq \mathcal{Y}_L^0$. The inclusion $\partial_\Gamma \mathcal{X}_L \subseteq \mathcal{Y}_L$ follows directly from Lemma 2.1 (vi). We stress that for any $v \in \tilde{H}^{1/2}(\Gamma)$, it holds that $\langle \partial_\Gamma v, 1 \rangle_\Gamma = 0$. Thus, any function in $\partial_\Gamma \mathcal{X}_L$ has vanishing integral mean, which concludes the proof. \square

Define the orthogonal projections on \mathcal{Y}^{00} resp. $\mathcal{Y}_{\ell,i}$ via

$$\begin{aligned} \langle \mathcal{P}^{00} \chi, \Psi^{00} \rangle_{\mathfrak{Y}} &= \langle \chi, \Psi^{00} \rangle_{\mathfrak{Y}} \quad \text{for all } \chi \in \tilde{H}^{-1/2}(\Gamma), \Psi^{00} \in \mathcal{Y}^{00}, \\ \langle \mathcal{P}_{\ell,i}^0 \chi, \Psi_{\ell,i}^0 \rangle_{\mathfrak{Y}} &= \langle \chi, \Psi_{\ell,i}^0 \rangle_{\mathfrak{Y}} \quad \text{for all } \chi \in \tilde{H}^{-1/2}(\Gamma), \Psi_{\ell,i}^0 \in \mathcal{Y}_{\ell,i}^0. \end{aligned} \quad (5.8)$$

With Lemma 5.1, we see the decomposition

$$\mathcal{Y}_L = \mathcal{Y}^{00} + \mathcal{Y}_L^0 \stackrel{(5.6)}{=} \mathcal{Y}^{00} + \partial_\Gamma \mathcal{X}_L \stackrel{(4.8)}{=} \mathcal{Y}^{00} + \sum_{\ell=0}^L \tilde{\mathcal{Y}}_\ell^0 \stackrel{(5.5)}{=} \mathcal{Y}^{00} + \sum_{\ell=0}^L \sum_{i \in \tilde{\mathcal{I}}_\ell} \mathcal{Y}_{\ell,i}^0. \quad (5.9)$$

with the corresponding additive Schwarz operator

$$\tilde{\mathcal{P}}_L^{\mathfrak{Y}} := \mathcal{P}^{00} + \sum_{\ell=0}^L \sum_{i \in \tilde{\mathcal{I}}_\ell} \mathcal{P}_{\ell,i}^0. \quad (5.10)$$

Theorem 5.2. *The additive Schwarz operator $\tilde{\mathcal{P}}_L^{\mathfrak{Y}} : \tilde{H}^{-1/2}(\Gamma) \rightarrow \mathcal{Y}_L$ satisfies that*

$$\lambda_{\min}^{\mathfrak{Y}} \|\Psi_L\|_{\mathfrak{Y}}^2 \leq \langle \tilde{\mathcal{P}}_L^{\mathfrak{Y}} \Psi_L, \Psi_L \rangle_{\mathfrak{Y}} \leq \lambda_{\max}^{\mathfrak{Y}} \|\Psi_L\|_{\mathfrak{Y}}^2 \quad \text{for all } \Psi_L \in \mathcal{Y}_L, \quad (5.11)$$

where the constants $\lambda_{\min}^{\mathfrak{Y}}, \lambda_{\max}^{\mathfrak{Y}} > 0$ depend only on $\mathcal{T}_0, \hat{\kappa}_{\max}, p, w_{\min}, w_{\max}$, and γ .

Proof. We only prove the assertion for closed $\Gamma = \partial\Omega$. For open $\Gamma \subsetneq \partial\Omega$, the proof works analogously with $\|\cdot\|_{\mathfrak{W}}^2 = \langle \mathfrak{W}(\cdot), (\cdot) \rangle_\Gamma$.

Step 1: First, we prove the lower bound of (5.11). We have to find a stable decomposition for any $\Psi_L \in \mathcal{Y}_L$. Due to Lemma 5.1, there exists $V_L^0 \in \mathcal{X}_L^0$ with $\partial_\Gamma V_L^0 = \Psi_L^0$. In Section 4.1,

we provided a decomposition $V_L^0 = \sum_{\ell=0}^L \sum_{i \in \tilde{\mathcal{I}}_\ell} V_{\ell,i}$ such that $\sum_{\ell=0}^L \sum_{i \in \tilde{\mathcal{I}}_\ell} \|V_{\ell,i}\|_{\mathfrak{W}}^2 \lesssim \|V_L^0\|_{\mathfrak{W}}^2$. This provides us with a decomposition

$$\Psi_L^0 = \partial_\Gamma V_L^0 = \sum_{\ell=0}^L \sum_{i \in \tilde{\mathcal{I}}_\ell} \Psi_{\ell,i}^0, \quad \text{where} \quad \Psi_{\ell,i}^0 := \partial_\Gamma V_{\ell,i} \in \mathcal{Y}_{\ell,i}^0.$$

Maue's formula (5.1) and $\langle V_L^0, 1 \rangle_\Gamma = 0$ hence show that

$$\sum_{\ell=0}^L \sum_{i \in \tilde{\mathcal{I}}_\ell} \|\Psi_{\ell,i}^0\|_{\mathfrak{W}}^2 = \sum_{\ell=0}^L \sum_{i \in \tilde{\mathcal{I}}_\ell} \langle \mathfrak{W} V_{\ell,i}, V_{\ell,i} \rangle_\Gamma \leq \sum_{\ell=0}^L \sum_{i \in \tilde{\mathcal{I}}_\ell} \|V_{\ell,i}\|_{\mathfrak{W}}^2 \lesssim \|V_L^0\|_{\mathfrak{W}}^2 = \|\Psi_L^0\|_{\mathfrak{W}}^2.$$

With this and (5.4), we finally conclude that

$$\|\Psi_L^{00}\|_{\mathfrak{W}}^2 + \sum_{\ell=0}^L \sum_{i \in \tilde{\mathcal{I}}_\ell} \|\Psi_{\ell,i}^0\|_{\mathfrak{W}}^2 \lesssim \|\Psi_L\|_{\mathfrak{W}}^2.$$

As in Section 4.1, this proves the lower bound.

Step 2: For the upper bound of (5.11), let $\Psi_L = \Psi_L^0 + \Psi_L^{00} \in \mathcal{Y}_L$ with an arbitrary decomposition $\sum_{\ell=0}^L \sum_{i \in \tilde{\mathcal{I}}_\ell} \Psi_{\ell,i}^0 = \Psi_L^0$, where $\Psi_{\ell,i}^0 \in \mathcal{Y}_{\ell,i}^0$. In particular, it holds that $\Psi_{\ell,i}^0 = \alpha_{\ell,i} \partial_\Gamma \bar{B}_{\ell,i,p}$ with some $\alpha_{\ell,i} \in \mathbb{R}$. We define

$$V_L := \sum_{\ell=0}^L \sum_{i \in \tilde{\mathcal{I}}_\ell} V_{\ell,i} \quad \text{with} \quad V_{\ell,i} := \alpha_{\ell,i} \bar{B}_{\ell,i,p}.$$

It is well known from additive Schwarz theory that the existence of a uniform upper bound in Theorem 4.1 is equivalent to $\|V_L\|_{\mathfrak{W}}^2 \lesssim \sum_{\ell=0}^L \sum_{i \in \tilde{\mathcal{I}}_\ell} \|V_{\ell,i}\|_{\mathfrak{W}}^2$ for all decompositions $V_L = \sum_{\ell=0}^L \sum_{i \in \tilde{\mathcal{I}}_\ell} V_{\ell,i}$; see, e.g., [Zha92, Lemma 3.1]. Maue's formula (5.1) yields that

$$\|\Psi_L^0\|_{\mathfrak{W}}^2 = \langle \mathfrak{W} V_L, V_L \rangle_\Gamma \leq \|V_L\|_{\mathfrak{W}}^2 \lesssim \sum_{\ell=0}^L \sum_{i \in \tilde{\mathcal{I}}_\ell} \|V_{\ell,i}\|_{\mathfrak{W}}^2. \quad (5.12)$$

Lemma 2.3 shows that

$$\langle V_{\ell,i}, 1 \rangle_\Gamma^2 \lesssim \alpha_{\ell,i}^2 |\text{supp} V_{\ell,i}|^2 \stackrel{(2.25)}{\lesssim} \alpha_{\ell,i}^2 |\widehat{B}_{\ell,i,p}|_{H^{1/2}(\text{supp} \widehat{B}_{\ell,i,p})}^2 \lesssim \alpha_{\ell,i}^2 |\widehat{B}_{\ell,i,p}|_{H^{1/2}(a,b)}^2. \quad (5.13)$$

Lipschitz continuity of γ proves $|\widehat{B}_{\ell,i,p}|_{H^{1/2}(a,b)}^2 \lesssim |B_{\ell,i,p}|_{H^{1/2}(\Gamma)}^2 \lesssim |\bar{B}_{\ell,i,p}|_{H^{1/2}(\Gamma)}^2$. With the equivalence $|\cdot|_{H^{1/2}(\Gamma)}^2 \simeq \langle \mathfrak{W}(\cdot), (\cdot) \rangle_\Gamma$ on $H^{1/2}(\Gamma)$, (5.13) becomes

$$\langle V_{\ell,i}, 1 \rangle_\Gamma^2 \lesssim \alpha_{\ell,i}^2 |\bar{B}_{\ell,i,p}|_{H^{1/2}(\Gamma)}^2 \simeq \langle \mathfrak{W} V_{\ell,i}, V_{\ell,i} \rangle_\Gamma.$$

By definition of the norm $\|\cdot\|_{\mathfrak{W}}$, we infer $\|V_{\ell,i}\|_{\mathfrak{W}}^2 \lesssim \langle \mathfrak{W} V_{\ell,i}, V_{\ell,i} \rangle_\Gamma$. Hence, (5.12) yields that

$$\|\Psi_L^0\|_{\mathfrak{W}}^2 \stackrel{(5.12)}{\lesssim} \sum_{\ell=0}^L \sum_{i \in \tilde{\mathcal{I}}_\ell} \langle \mathfrak{W} V_{\ell,i}, V_{\ell,i} \rangle_\Gamma \stackrel{(5.1)}{=} \sum_{\ell=0}^L \sum_{i \in \tilde{\mathcal{I}}_\ell} \|\Psi_{\ell,i}^0\|_{\mathfrak{W}}^2.$$

We conclude that

$$\|\Psi_L\|_{\mathfrak{B}}^2 \lesssim \|\Psi_L^{00}\|_{\mathfrak{B}}^2 + \|\Psi_L^0\|_{\mathfrak{B}}^2 \lesssim \|\Psi_L^{00}\|_{\mathfrak{B}}^2 + \sum_{\ell=0}^L \sum_{i \in \tilde{\mathcal{I}}_\ell} \|\Psi_{\ell,i}\|_{\mathfrak{B}}^2.$$

Since $\Psi_L^{00} + \sum_{\ell=0}^L \sum_{i \in \tilde{\mathcal{I}}_\ell} \Psi_{\ell,i} = \Psi_L$ was an arbitrary decomposition, standard additive Schwarz theory proves the upper bound. \square

6. NUMERICAL EXPERIMENTS

In this section, we present a matrix version of Theorem 4.1 and Theorem 5.2. We apply these theorems to define preconditioners for some numerical examples. Throughout this section, let $(\mathcal{K}_\ell)_{\ell \in \mathbb{N}_0}$ be a sequence of refined knot vectors, i.e., $\mathcal{K}_\ell, \mathcal{K}_{\ell+1} \in \mathbb{K}$ with $\mathcal{K}_{\ell+1} \in \mathbf{refine}(\mathcal{K}_\ell)$, and let $L \in \mathbb{N}_0$. For the hypersingular equation (1.1), we allow for arbitrary positive initial weights \mathcal{W}_0 . Whereas, whenever we consider the weakly-singular integral equation (1.3), we suppose that all weights in \mathcal{W}_0 are equal to one, wherefore the denominator satisfies that $w = 1$. The Galerkin approximations $U_\ell \in \mathcal{X}_\ell$ for the hypersingular case resp. $\Phi_\ell \in \mathcal{Y}_\ell$ for the weakly-singular case satisfy that

$$\langle\langle U_\ell, V_\ell \rangle\rangle_{\mathfrak{B}} = \langle f, V_\ell \rangle_\Gamma \quad \text{resp.} \quad \langle\langle \Phi_\ell, \Psi_\ell \rangle\rangle_{\mathfrak{B}} = \langle g, \Psi_\ell \rangle_\Gamma \quad \text{for all } V_\ell \in \mathcal{X}_\ell, \Psi_\ell \in \mathcal{Y}_\ell. \quad (6.1)$$

The discrete solutions U_ℓ, Φ_ℓ are obtained by solving a linear system of equations

$$\mathbf{W}_\ell \mathbf{x}_\ell = \mathbf{f}_\ell \quad \text{resp.} \quad \mathbf{V}_\ell \mathbf{y}_\ell = \mathbf{g}_\ell, \quad (6.2)$$

where

$$U_\ell = \sum_{k=1}^{N_\ell-1-o} (\mathbf{x}_\ell)_k \bar{R}_{\ell,k-p+o,p} \quad \text{resp.} \quad \Phi_\ell = \sum_{k=1}^{N_\ell-1} (\mathbf{y}_\ell)_k \hat{B}_{\ell,k-(p-1),p-1}, \quad (6.3)$$

and

$$\mathbf{W}_\ell = \left(\langle\langle \bar{R}_{\ell,k-p+o,p}, \bar{R}_{\ell,j-p+o,p} \rangle\rangle_{\mathfrak{B}} \right)_{j,k=1}^{N_\ell-1-o}, \quad \mathbf{f}_\ell = \left(\langle f, \bar{R}_{\ell,j-p+o,p} \rangle_\Gamma \right)_{j=1}^{N_\ell-1-o}, \quad (6.4)$$

resp.

$$\mathbf{V}_\ell = \left(\langle\langle \hat{B}_{\ell,k-(p-1),p-1}, \hat{B}_{\ell,j-(p-1),p-1} \rangle\rangle_{\mathfrak{B}} \right)_{j,k=1}^{N_\ell-1}, \quad \mathbf{g}_\ell = \left(\langle g, \hat{B}_{\ell,j-(p-1),p-1} \rangle_\Gamma \right)_{j=1}^{N_\ell-1}. \quad (6.5)$$

For any $L \in \mathbb{N}_0$, we aim to derive preconditioners $(\tilde{\mathbf{S}}_L^{\mathfrak{B}})^{-1}$ resp. $(\tilde{\mathbf{S}}_L^{\mathfrak{B}})^{-1}$ for the Galerkin matrices \mathbf{W}_L resp. \mathbf{V}_L . For their definition, we first have to introduce the following transformation matrices. For $0 \leq \ell \leq L$, let $\mathbf{id}_{\ell \rightarrow L}^{\mathfrak{B}} : \mathcal{X}_\ell \rightarrow \mathcal{X}_L$, $\mathbf{id}_{\tilde{\ell} \rightarrow \ell}^{\mathfrak{B}} : \tilde{\mathcal{X}}_\ell \rightarrow \mathcal{X}_\ell$ and $\mathbf{id}_{\ell \rightarrow L}^{\mathfrak{B}} : \mathcal{Y}_\ell \rightarrow \mathcal{Y}_L$ be the canonical embeddings, i.e., the formal identities, with matrix representations $\mathbf{id}_{\ell \rightarrow L}^{\mathfrak{B}} \in \mathbb{R}^{(N_L-1-o) \times (N_\ell-1-o)}$, $\mathbf{id}_{\tilde{\ell} \rightarrow \ell}^{\mathfrak{B}} \in \mathbb{R}^{(N_\ell-1-o) \times (\#\tilde{\mathcal{I}}_\ell)}$ and $\mathbf{id}_{\ell \rightarrow L}^{\mathfrak{B}} \in \mathbb{R}^{(N_L-1) \times (N_\ell-1)}$. Further, let $\mathbf{id}_{\ell \rightarrow \ell}^{\mathfrak{B}} \in \mathbb{R}^{(N_\ell-1) \times (\#\tilde{\mathcal{I}}_\ell)}$ be the matrix that represents the B-spline derivatives in $\tilde{\mathcal{Y}}_\ell^0$ as B-splines in \mathcal{Y}_ℓ , i.e.,

$$\partial_\Gamma \bar{B}_{\ell,i(k),p} = \sum_{j=1}^{N_\ell-1} (\mathbf{id}_{\tilde{\ell} \rightarrow \ell}^{\mathfrak{B}})_{jk} \hat{B}_{\ell,j-(p-1),p-1} \quad \text{for } k = 1, \dots, \#\tilde{\mathcal{I}}_\ell, \quad (6.6)$$

with the monotonously increasing bijection $i(\cdot) : \{1, \dots, \#\tilde{\mathcal{I}}_\ell\} \rightarrow \tilde{\mathcal{I}}_\ell$. All these matrices can be computed with the help of Lemma 2.1. Finally, let $\mathbf{1} \in \mathbb{R}^{(N_L-1) \times (N_L-1)}$ be the constant one matrix, i.e.,

$$(\mathbf{1})_{jk} = 1 \quad \text{for } j, k = 1, \dots, N_L - 1. \quad (6.7)$$

For any quadratic matrix \mathbf{A} , we define the corresponding diagonal matrix $\text{diag}(\mathbf{A}) = (\mathbf{A}_{jk} \cdot \delta_{jk})_{j,k}$. We consider

$$(\tilde{\mathbf{S}}_L^{\mathfrak{W}})^{-1} := \sum_{\ell=0}^L \mathbf{id}_{\ell \rightarrow L}^{\mathfrak{W}} \mathbf{id}_{\ell \rightarrow \ell}^{\mathfrak{W}} \text{diag}((\mathbf{id}_{\ell \rightarrow \ell}^{\mathfrak{W}})^T \mathbf{W}_\ell \mathbf{id}_{\ell \rightarrow \ell}^{\mathfrak{W}})^{-1} (\mathbf{id}_{\ell \rightarrow \ell}^{\mathfrak{W}})^T (\mathbf{id}_{\ell \rightarrow L}^{\mathfrak{W}})^T, \quad (6.8)$$

resp.

$$(\tilde{\mathbf{S}}_L^{\mathfrak{V}})^{-1} := \langle\langle \mathbf{1}, \mathbf{1} \rangle\rangle_{\mathfrak{V}}^{-1} \mathbf{1} + \sum_{\ell=0}^L \mathbf{id}_{\ell \rightarrow L}^{\mathfrak{V}} \mathbf{id}_{\ell \rightarrow \ell}^{\mathfrak{V}} \text{diag}((\mathbf{id}_{\ell \rightarrow \ell}^{\mathfrak{V}})^T \mathbf{V}_\ell \mathbf{id}_{\ell \rightarrow \ell}^{\mathfrak{V}})^{-1} (\mathbf{id}_{\ell \rightarrow \ell}^{\mathfrak{V}})^T (\mathbf{id}_{\ell \rightarrow L}^{\mathfrak{V}})^T. \quad (6.9)$$

Note that, by the partition of unity property from Lemma 2.1 (iv), there holds that

$$\langle\langle \mathbf{1}, \mathbf{1} \rangle\rangle_{\mathfrak{V}} = \sum_{j,k=1}^{N_L-1} (\mathbf{V}_L)_{jk}. \quad (6.10)$$

Instead of solving $\mathbf{W}_L \mathbf{x}_L = \mathbf{f}_L$ resp. $\mathbf{V}_L \mathbf{y}_L = \mathbf{g}_L$, we consider the preconditioned systems

$$(\tilde{\mathbf{S}}_L^{\mathfrak{W}})^{-1} \mathbf{W}_L \mathbf{x}_L = (\tilde{\mathbf{S}}_L^{\mathfrak{W}})^{-1} \mathbf{f}_L \quad \text{resp.} \quad (\tilde{\mathbf{S}}_L^{\mathfrak{V}})^{-1} \mathbf{V}_L \mathbf{y}_L = (\tilde{\mathbf{S}}_L^{\mathfrak{V}})^{-1} \mathbf{g}_L. \quad (6.11)$$

Elementary manipulations verify that the preconditioned matrices $(\tilde{\mathbf{S}}_L^{\mathfrak{W}})^{-1} \mathbf{W}_L$ resp. $(\tilde{\mathbf{S}}_L^{\mathfrak{V}})^{-1} \mathbf{V}_L$ are just the matrix representations of $\tilde{\mathcal{P}}_L^{\mathfrak{W}}|_{\mathcal{X}_L} : \mathcal{X}_L \rightarrow \mathcal{X}_L$ resp. $\tilde{\mathcal{P}}_L^{\mathfrak{V}}|_{\mathcal{Y}_L} : \mathcal{Y}_L \rightarrow \mathcal{Y}_L$. Theorem 4.1 resp. Theorem 5.2 then immediately prove the next corollary, which states uniform boundedness of the condition number of the preconditioned systems.

For a symmetric and positive definite matrix \mathbf{A} , we denote $\langle \cdot, \cdot \rangle_{\mathbf{A}} := \langle \mathbf{A} \cdot, \cdot \rangle_2$, and by $\|\cdot\|_{\mathbf{A}}$ the corresponding norm resp. induced matrix norm. Here, $\langle \cdot, \cdot \rangle_2$ denotes the Euclidean inner product. The condition number $\text{cond}_{\mathbf{A}}(\mathbf{B})$ of a quadratic matrix \mathbf{B} of same dimension as \mathbf{A} reads

$$\text{cond}_{\mathbf{A}}(\mathbf{B}) := \|\mathbf{B}\|_{\mathbf{A}} \|\mathbf{B}^{-1}\|_{\mathbf{A}}. \quad (6.12)$$

Corollary 6.1. *The matrices $(\tilde{\mathbf{S}}_L^{\mathfrak{W}})^{-1}$, $(\tilde{\mathbf{S}}_L^{\mathfrak{V}})^{-1}$ are symmetric and positive definite with respect to $\langle \cdot, \cdot \rangle_2$, and $\tilde{\mathbf{P}}_L^{\mathfrak{W}} := (\tilde{\mathbf{S}}_L^{\mathfrak{W}})^{-1} \mathbf{W}_L$ resp. $\tilde{\mathbf{P}}_L^{\mathfrak{V}} := (\tilde{\mathbf{S}}_L^{\mathfrak{V}})^{-1} \mathbf{V}_L$ are symmetric and positive definite with respect to $\langle \cdot, \cdot \rangle_{\tilde{\mathfrak{S}}_L^{\mathfrak{W}}}$ resp. $\langle \cdot, \cdot \rangle_{\tilde{\mathfrak{S}}_L^{\mathfrak{V}}}$. Moreover, the minimal and maximal eigenvalues of the matrices $\tilde{\mathbf{P}}_L^{\mathfrak{W}}$ resp. $\tilde{\mathbf{P}}_L^{\mathfrak{V}}$ satisfy that*

$$\lambda_{\min}^{\mathfrak{W}} \leq \lambda_{\min}(\tilde{\mathbf{P}}_L^{\mathfrak{W}}) \leq \lambda_{\max}(\tilde{\mathbf{P}}_L^{\mathfrak{W}}) \leq \lambda_{\max}^{\mathfrak{W}}, \quad (6.13)$$

resp.

$$\lambda_{\min}^{\mathfrak{V}} \leq \lambda_{\min}(\tilde{\mathbf{P}}_L^{\mathfrak{V}}) \leq \lambda_{\max}(\tilde{\mathbf{P}}_L^{\mathfrak{V}}) \leq \lambda_{\max}^{\mathfrak{V}}, \quad (6.14)$$

with the constants $\lambda_{\min}^{\mathfrak{W}}, \lambda_{\max}^{\mathfrak{W}}$ from Theorem 4.1 and $\lambda_{\min}^{\mathfrak{V}}, \lambda_{\max}^{\mathfrak{V}}$ from Theorem 5.2. In particular, the condition number of the additive Schwarz matrices $\tilde{\mathbf{P}}_L^{\mathfrak{W}}$ resp. $\tilde{\mathbf{P}}_L^{\mathfrak{V}}$ is bounded by

$$\text{cond}_{\tilde{\mathfrak{S}}_L^{\mathfrak{W}}}(\tilde{\mathbf{P}}_L^{\mathfrak{W}}) \leq \lambda_{\max}^{\mathfrak{W}} / \lambda_{\min}^{\mathfrak{W}} \quad \text{resp.} \quad \text{cond}_{\tilde{\mathfrak{S}}_L^{\mathfrak{V}}}(\tilde{\mathbf{P}}_L^{\mathfrak{V}}) \leq \lambda_{\max}^{\mathfrak{V}} / \lambda_{\min}^{\mathfrak{V}}. \quad (6.15)$$

Recall that these eigenvalue bounds depend only on \mathcal{T}_0 , $\hat{\kappa}_{\max}$, p , w_{\min} , w_{\max} and γ .

Proof. We only consider the hypersingular case. The weakly-singular case can be treated analogously. Due to (4.8) the operator $\tilde{\mathcal{P}}_L^{\text{2W}}|_{\mathcal{X}_L}$ is positive definite with respect to $\langle\langle \cdot, \cdot \rangle\rangle_{\text{2W}}$. This proves for any $V_L \in \mathcal{X}_L$ with corresponding coefficient vector \mathbf{z}_L that

$$\langle\langle (\tilde{\mathbf{S}}_L^{\text{2W}})^{-1} \mathbf{W}_L \mathbf{z}_L, \mathbf{W}_L \mathbf{z}_L \rangle\rangle_2 = \langle\langle \tilde{\mathcal{P}}_L^{\text{2W}} V_L, V_L \rangle\rangle_{\text{2W}} > 0.$$

Symmetry and positive definiteness of $\tilde{\mathbf{P}}_L^{\text{2W}}$ with respect to $\langle \cdot, \cdot \rangle_{\tilde{\mathcal{S}}_L^{\text{2W}}}$ follow immediately by symmetry and positive definiteness of \mathbf{W}_L . Theorem 4.1 and the fact that $\tilde{\mathbf{P}}_L^{\text{2W}}$ is just the matrix representation of $\tilde{\mathcal{P}}_L^{\text{2W}}|_{\mathcal{X}_L}$ show (6.13). Finally, note that the condition number $\text{cond}_{\tilde{\mathcal{S}}_L^{\text{2W}}}(\tilde{\mathbf{P}}_L^{\text{2W}})$ is just the ratio of the maximal and the minimal eigenvalue of $\tilde{\mathbf{P}}_L^{\text{2W}}$. \square

The corollary can be applied for iterative solution methods such as GMRES [SS86] or CG [Saa03] to solve (6.11). Here, the relative residual of the j -th residual depends only on the condition number $\text{cond}_{\tilde{\mathcal{S}}_L^{\text{2W}}}(\tilde{\mathbf{P}}_L^{\text{2W}})$ resp. $\text{cond}_{\tilde{\mathcal{S}}_L^{\text{2W}}}(\tilde{\mathbf{P}}_L^{\text{2W}})$. Hence, Corollary 6.1 proves that the iterative scheme together with the preconditioners $(\tilde{\mathbf{S}}_L^{\text{2W}})^{-1}$ resp. $(\tilde{\mathbf{S}}_L^{\text{2W}})^{-1}$ is optimal in the following sense: The number of iterations to reduce the relative residual under the tolerance $\epsilon > 0$ is bounded by a constant, which depends only on $\mathcal{T}_0, \hat{\kappa}_{\max}, p, w_{\min}, w_{\max}$ and γ .

Remark 6.2. *The application of the preconditioners $(\tilde{\mathbf{S}}_L^{\text{2W}})^{-1}$ resp. $(\tilde{\mathbf{S}}_L^{\text{2W}})^{-1}$ on a vector \mathbf{z}_L can be done efficiently in $\mathcal{O}(N_L)$ operations. Furthermore, the storage requirements of the preconditioners, i.e., the memory consumption of all the transformation matrices \mathbf{id} and the diagonal matrices $\text{diag}(\cdot)$ in the sum is $\mathcal{O}(N_L)$. This implies the optimal linear complexity of our preconditioners. A detailed description of an algorithm, which implements the matrix-vector multiplication, can be found in our recent work [FFPS17, Algorithm 1] for some local multilevel preconditioner for the hypersingular integral operator on adaptively refined meshes, resp. in [Yse86] for some hierarchical basis preconditioner.*

In the following subsections, we numerically show the optimality of the proposed preconditioners. In all examples, the exact solution is known and singular, wherefore adaptive methods are preferable. To steer the mesh refinement, we apply the following adaptive Algorithm 6.3 proposed in [FGHP16, Algorithm 3.1] for the weakly-singular case resp. in [Sch16, GPS18] for the hypersingular case. We will fix the used refinement indicators $\eta_\ell(z)$ in each experiment separately.

Algorithm 6.3. Input: *Adaptivity parameter $0 < \theta \leq 1$, polynomial order $p \in \mathbb{N}$, initial knots \mathcal{K}_0 , initial weights \mathcal{W}_0 .*

Adaptive loop: *For each $\ell = 0, 1, 2, \dots$ iterate the following steps (i)–(vi):*

- (i) *Compute discrete approximation $U_\ell \in \mathcal{X}_\ell$ in the hypersingular case resp. $\Phi_\ell \in \mathcal{Y}_\ell$ in the weakly-singular case.*
- (ii) *Compute refinement indicators $\eta_\ell(z)$ for all nodes $z \in \mathcal{N}_\ell$.*
- (iii) *Determine a minimal set of nodes $\mathcal{M}_\ell \subseteq \mathcal{N}_\ell$ such that*

$$\theta \eta_\ell^2 \leq \sum_{z \in \mathcal{M}_\ell} \eta_\ell(z)^2. \tag{6.16}$$

- (iv) *If both nodes of an element $T \in \mathcal{T}_\ell$ belong to \mathcal{M}_ℓ , T will be marked.*
- (v) *For all other nodes $z \in \mathcal{M}_\ell$, the multiplicity will be increased if z satisfies that $z \notin \{a, b\}$ and $\#\ell z < p$, otherwise the elements, which contain one of these nodes $z \in \mathcal{M}_\ell$, will be marked.*

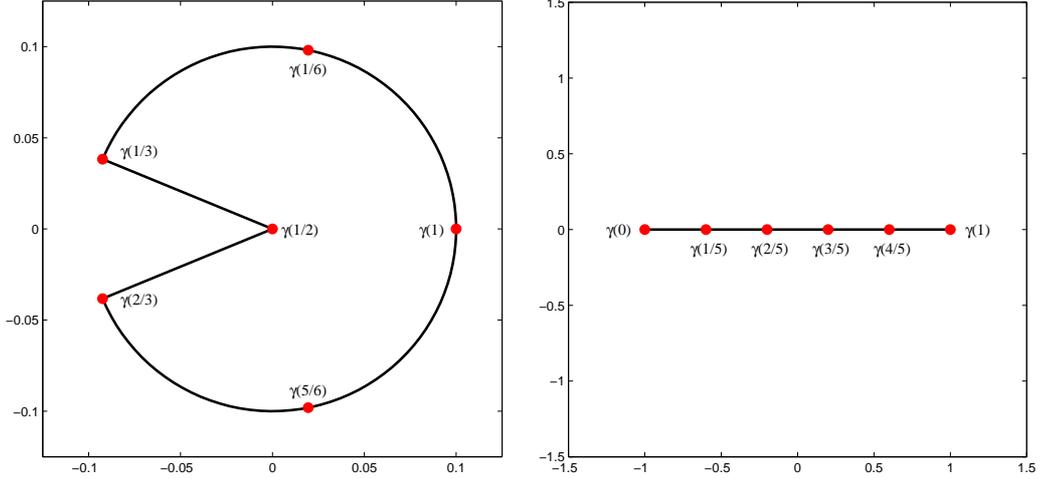


FIGURE 6.1. Geometries and initial nodes for the experiments from Section 6.

- (vi) Refine all marked elements $T \in \mathcal{T}_\ell$ by bisection (insertion of a node with multiplicity one) of the corresponding element $\gamma^{-1}(T)$ in the parameter domain. Use further bisections to guarantee that the new knots $\mathcal{K}_{\ell+1}$ satisfy that

$$\widehat{\kappa}_{\ell+1} \leq 2\widehat{\kappa}_0. \quad (6.17)$$

Output: Approximate solutions U_ℓ resp. Φ_ℓ and error estimators η_ℓ for all $\ell \in \mathbb{N}_0$.

The resulting linear systems are solved by PCG. We compare the preconditioners to simple diagonal preconditioning. In all experiments the initial vector in the PCG-algorithm is set to 0 and the tolerance parameter $\epsilon > 0$ for the relative residual is $\epsilon = 10^{-8}$.

6.1. Adaptive BEM for hypersingular integral equation for Neumann problem on pacman. We consider the boundary $\Gamma = \partial\Omega$ of the pacman geometry

$$\Omega := \left\{ r(\cos(\beta), \sin(\beta)) : 0 \leq r < \frac{1}{10}, \beta \in \left(-\frac{\pi}{2\tau}, \frac{\pi}{2\tau}\right) \right\} \quad (6.18)$$

with $\tau = 4/7$, sketched in Figure 6.1. It can be parametrized by a NURBS curve $\gamma : [0, 1] \rightarrow \Gamma$ of degree two; see [FGHP16, Section 3.2]. With the 2D polar coordinates (r, β) , the function

$$P(x, y) := r^\tau \cos(\tau\beta)$$

satisfies that $-\Delta P = 0$ and has a generic singularity at the origin. With the adjoint double-layer operator \mathfrak{K}' , we define with the normal derivative $\partial_\nu P$

$$f := (1/2 - \mathfrak{K}')\partial_\nu P.$$

Up to an additive constant, there holds $u = P|_\Gamma$, where u is the solution of the corresponding hypersingular integral equation.

For Algorithm 6.3, we choose NURBS of degree two as ansatz space \mathcal{X}_ℓ (i.e., $p = 2$) and the same initial knots \mathcal{K}_0 and weights \mathcal{W}_0 as for the geometry representation.

Due to numerical stability reasons, we replace the right-hand side f in each step by $f_\ell := (1/2 - \mathfrak{K}')\phi_\ell$. Here, ϕ_ℓ is the $L^2(\Gamma)$ -orthogonal projection of $\phi := (\partial_\nu P)$ onto the space of transformed piecewise polynomials of degree p on \mathcal{T}_ℓ , i.e., $\phi_\ell \circ \gamma$ is polynomial

on all $\gamma^{-1}(T)$ with $T \in \mathcal{T}_\ell$. This leads to a perturbed Galerkin approximation U_ℓ^{pert} . To steer the algorithm, we use the $h - h/2$ based error indicators $\eta_\ell(z)^2 := \|\widehat{h}_\ell^{1/2} \partial_\Gamma(U_{\text{fine}(\ell)}^{\text{pert}} - U_\ell^{\text{pert}})\|_{L^2(\omega_\ell(z))}^2 + \|\widehat{h}_\ell^{1/2}(\phi - \phi_\ell)\|_{L^2(\omega_\ell(z))}^2$, where $U_{\text{fine}(\ell)}^{\text{pert}}$ is the perturbed Galerkin approximation in the space $\mathcal{X}_{\text{fine}(\ell)}$ corresponding to the uniformly refined knots $\mathcal{K}_{\text{fine}(\ell)}$. These are generated from \mathcal{K}_ℓ via the refinement steps Algorithm 6.3 (iv)–(vi) with $\mathcal{M}_\ell = \mathcal{N}_\ell$.

In Figure 6.2, we compare the condition numbers of diagonal preconditioning with our proposed additive Schwarz approach. Whereas diagonal preconditioning is suboptimal, we observe optimality for our approach, which numerically verifies our theoretical result in Corollary 6.1. This is also reflected by the number of PCG iterations.

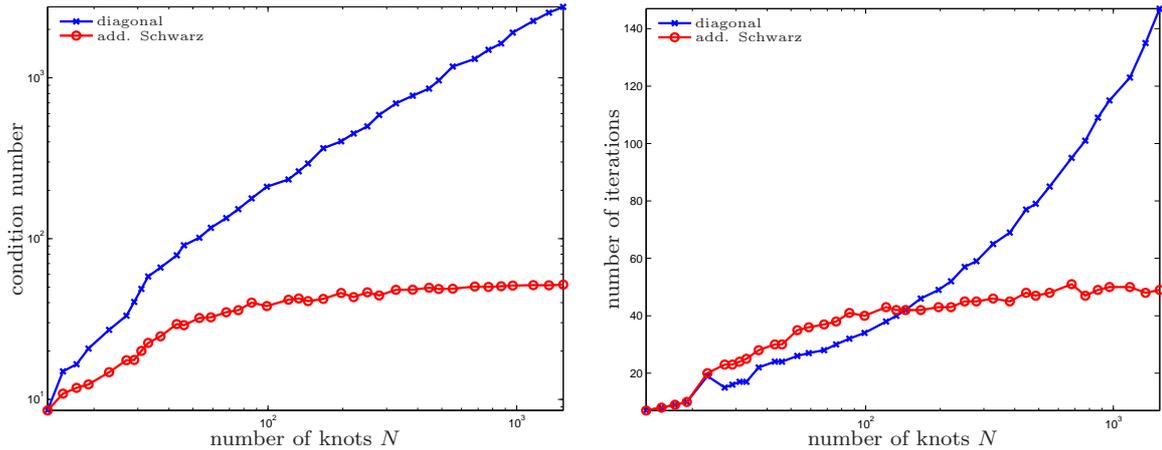


FIGURE 6.2. Condition numbers $\lambda_{\max}/\lambda_{\min}$ of the diagonal and the additive Schwarz preconditioned Galerkin matrices as well as the number of PCG iterations for the hypersingular equation on the pacman from Section 6.1.

6.2. Adaptive BEM for weakly-singular integral equation for Dirichlet problem on pacman. Let Ω and P be as in the previous section. With the double-layer operator \mathfrak{K} and the right-hand side

$$g := (1/2 + \mathfrak{K})P|_\Gamma, \quad (6.19)$$

the solution of the weakly-singular integral equation (1.3) is just the normal derivative of P , i.e., $\phi = \partial_\nu P$. For Algorithm 6.3, we choose splines of degree two as ansatz space \mathcal{Y}_ℓ (i.e., $p = 3$ and all weights are equal to one) and the initial knots \mathcal{K}_0 as for the geometry. To steer the algorithm, we use the weighted-residual error indicators $\eta_\ell(z) := \|\widehat{h}_\ell^{1/2} \partial_\Gamma(g - \mathfrak{B}\Phi_\ell)\|_{L^2(\omega_\ell(z))}$. Figure 6.3 shows a comparison of the diagonal and the additive Schwarz preconditioner.

6.3. Adaptive BEM for hypersingular integral equation on slit. We consider the hypersingular integral equation on the slit $\Gamma = [-1, 1] \times \{0\}$, sketched in Figure 6.1, which is represented as spline curve $\gamma : [0, 1] \rightarrow \Gamma$ of degree one; see [FGHP16, Section 3.4]. For $f := 1$, the exact solution is $u(x, 0) = 2\sqrt{1 - x^2}$. For Algorithm 6.3, we choose splines of degree one as ansatz space \mathcal{X}_ℓ (i.e., $p = 1$ and all weights are equal to one) and the initial knots \mathcal{K}_0 as for the geometry. To steer the algorithm, we use the $h - h/2$ based error indicators $\eta_\ell(z) := \|\widehat{h}_\ell^{1/2} \partial_\Gamma(U_{\text{fine}(\ell)} - U_\ell)\|_{L^2(\omega_\ell(z))}$, where $U_{\text{fine}(\ell)}$ is the Galerkin approximation

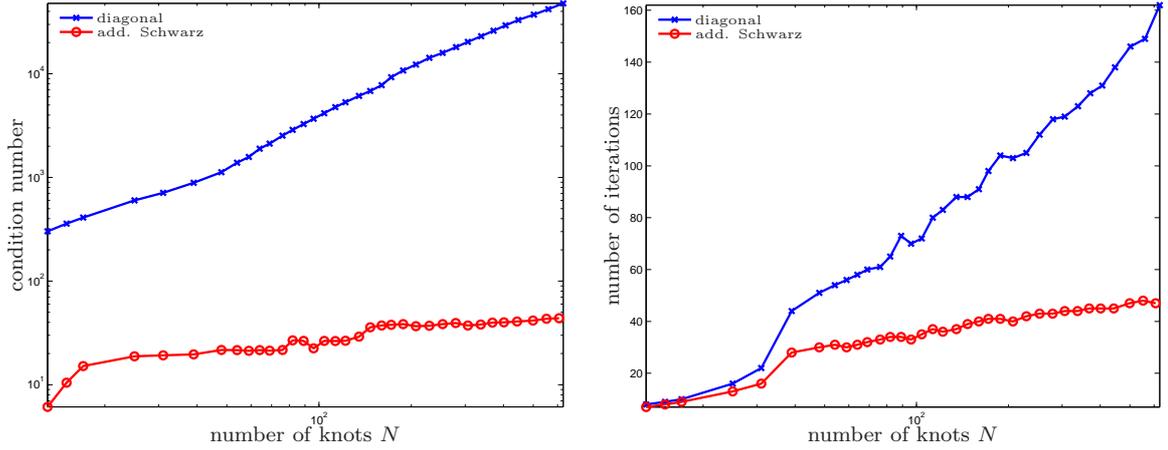


FIGURE 6.3. Condition numbers $\lambda_{\max}/\lambda_{\min}$ of the diagonal and the additive Schwarz preconditioned Galerkin matrices as well as the number of PCG iterations for the weakly-singular equation on the pacman from Section 6.2.

in the space $\mathcal{X}_{\text{fine}(\ell)}$ corresponding to the uniformly refined knots $\mathcal{K}_{\text{fine}(\ell)}$. These are generated from \mathcal{K}_ℓ via the refinement steps Algorithm 6.3 (iv)–(vi) with $\mathcal{M}_\ell = \mathcal{N}_\ell$. Again, we compare diagonal preconditioning and the local multilevel diagonal preconditioner; see Figure 6.4.

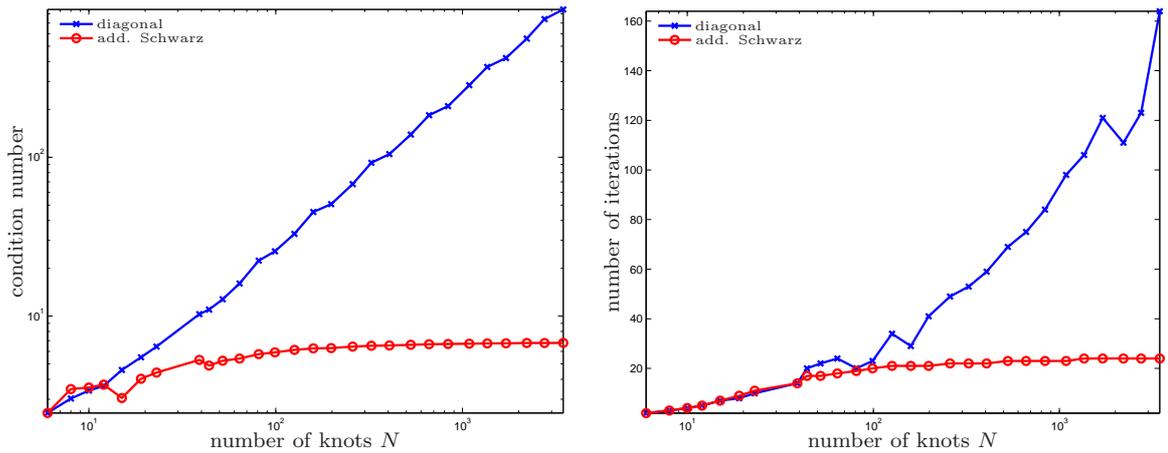


FIGURE 6.4. Condition numbers $\lambda_{\max}/\lambda_{\min}$ of the diagonal and the additive Schwarz preconditioned Galerkin matrices as well as the number of PCG iterations for the hypersingular equation on the slit from Section 6.3.

6.4. Adaptive BEM for weakly-singular integral equation on slit. Let Γ be again the slit $[-1, 1] \times \{0\}$. For the weakly-singular integral equation with $g := -x/2$, the corresponding solution reads $\phi(x, 0) = -x/\sqrt{1-x^2}$. For Algorithm 6.3, we choose splines of degree one as ansatz space \mathcal{Y}_ℓ (i.e., $p = 2$ and all weights are equal to one) and the initial knots \mathcal{K}_0 as for the geometry. To steer the algorithm, we use the weighted-residual error indicators $\eta_\ell(z) := \|\widehat{h}_\ell^{1/2} \partial_\Gamma(g - \mathfrak{B}\Phi_\ell)\|_{L^2(\omega_\ell(z))}$. A comparison of the diagonal and the additive Schwarz preconditioner is found in Figure 6.5.

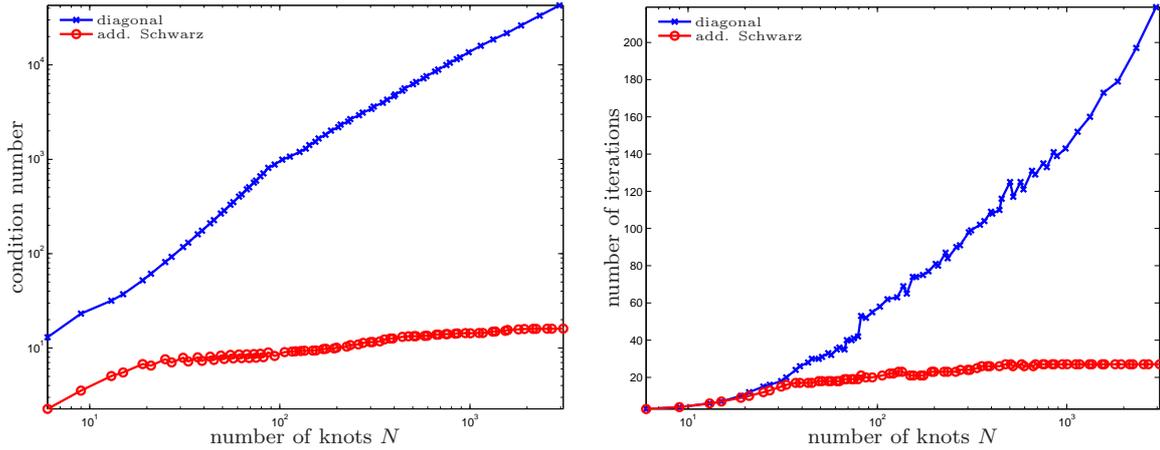


FIGURE 6.5. Condition numbers $\lambda_{\max}/\lambda_{\min}$ of the diagonal and the additive Schwarz preconditioned Galerkin matrices as well as the number of PCG iterations for the weakly-singular equation on the slit from Section 6.4.

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