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ADAPTIVE BEM WITH OPTIMAL CONVERGENCE RATES FOR THE HELMHOLTZ EQUATION
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Abstract. We analyze an adaptive boundary element method for the weakly-singular and hypersingular integral equations for the 2D and 3D Helmholtz problem. The proposed adaptive algorithm is steered by a residual error estimator and does not rely on any a priori information that the underlying meshes are sufficiently fine. We prove convergence of the error estimator with optimal algebraic rates, independently of the (coarse) initial mesh. As a technical contribution, we prove certain local inverse-type estimates for the boundary integral operators associated with the Helmholtz equation.

1. Introduction

Adaptive boundary element methods (ABEMs) with (dis)continuous piecewise polynomials for second order elliptic problems are well understood if the boundary integral operator is strongly elliptic. In particular for the Laplace equation and lowest order boundary elements, optimal algebraic rates of convergence have been proved in [FFK14, FFK15, FKMP13] for polyhedral boundaries and in [Gan15] for smooth boundaries. An abstract framework is also found in [CFPP14]. With the recent work [AFF17], these results can also be extended to piecewise smooth boundaries.

In recent years, isogeometric analysis has lead to a variety of works proving optimal rates for ABEM using spline basis functions; see, e.g., [FGP15, FGHP16, FGHP17] for the Laplace problem in two dimension as well as [Gan17] for a generalization to second-order linear elliptic PDEs in three dimensions.

On the other hand, boundary element methods (BEMs) for the Helmholtz equation are very popular and used in many applications; see, e.g., [CWGLS12, CK83] for an overview of techniques in acoustic scattering. To our knowledge, there are no results concerning optimal convergence of ABEM for indefinite problems, even for sufficiently fine initial meshes. With this paper, we fill this gap in the theory.

In this work, we generalize existing results concerning optimal convergence of ABEM for the Laplace equation to the Helmholtz equation. To this end, let \( \Omega \subset \mathbb{R}^d \) with \( d = 2, 3 \) be a bounded Lipschitz domain with boundary \( \partial \Omega \supseteq \Gamma \). We consider adaptive BEM for the Dirichlet or Neumann boundary value problem for the Helmholtz equation, i.e.,

\[
-\Delta u - k^2 u = 0 \quad \text{in} \quad \Omega \quad \text{subject to either} \quad u = g \quad \text{on} \quad \Gamma \quad \text{or} \quad \partial_n u = \phi \quad \text{on} \quad \Gamma.
\]

where \( k \in \mathbb{R} \) denotes the wavenumber. Independently, whether a direct or an indirect approach is used, the Dirichlet boundary value problems leads to the following weakly-singular integral equation. Suppose that \( k^2 \) is not an eigenvalue of the interior Dirichlet Problem (IDP). Given a right-hand side \( f \in H^{1/2}(\Gamma) \), find \( \phi \in H^{-1/2}(\Gamma) \) such that

\[
V_k \phi = f \quad \text{on} \quad \Gamma,
\]

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where \( V_k \) denotes the simple-layer operator associated with the Helmholtz equation. For \( k = 0 \), \( V_k \) coincides with the simple-layer operator of the Laplace equation. In this case, we refer to [FKMP13, FFK14, Tso13], where optimal algebraic convergence rates for the weakly-singular integral equation for the Laplace operator are shown.

In this paper, we focus on the case \( k \neq 0 \). We build on the abstract framework developed in [BHP17] and propose an adaptive algorithm (Algorithm 7) for the numerical solution of problem (2). In the algorithm, the local mesh-refinement is guided by the weighted-residual \textit{a posteriori} error estimator, and the classical adaptive loop is complemented by an additional step that performs a uniform mesh-refinement if the Galerkin formulation of (2) does not admit a unique solution. Thus, our algorithm does not require any \textit{a priori} information on whether the initial mesh (or any locally refined mesh generated by the algorithm) is sufficiently fine.

The main result of this work is Theorem 10. It states that Algorithm 7 generates a convergent sequence of discrete solutions and, moreover, the generated sequence of \textit{a posteriori} error estimators converges linearly with an optimal algebraic rate. Theorem 10 is the first result that proves optimal convergence rates for adaptive BEM for the Helmholtz equation. In addition to that, we emphasize that our adaptive algorithm effects the optimal rate for any given, possibly coarse, initial mesh. Although the presentation focuses on the weakly-singular equation in (2), the adaptive algorithm and the main result of Theorem 10 extend immediately to the hypersingular integral equation corresponding to the Neumann boundary value problem in (1).

The proof of Theorem 10 relies on the abstract framework developed in [BHP17] for compactly perturbed elliptic problems and requires verification of the so-called \textit{axioms of adaptivity} [CFPP14] for the weighted-residual error estimator. In this work, we verify these by employing novel inverse-type estimates for the underlying boundary integral operators. These estimates exploit potential decompositions from [Mel12] and generalize existing results for the Laplacian \((k = 0)\) to the case of an arbitrary wavenumber \( k \geq 0 \).

**Outline.** This work and its main results are structured as follows: Section 2 recaps the functional analytic framework and introduces the involved integral operators as well as the Galerkin discretization by piecewise polynomials. In Section 3, we prove inverse-type estimates in the style of [AFF17] for the Helmholtz operators. The exact adaptive algorithm and the \textit{a posteriori} weighted-residual error estimator are given in Section 4. The main result (Theorem 10) of this work is given in Section 5. Further, Section 6 comments on the extension of the analysis to the hypersingular equation. In the last Section 7, we underpin our theoretical findings with some numerical experiments. A rigorous proof of the essential estimator properties and Theorem 10 is given in the Appendix.

Throughout all statements, the dependencies of all constants are given. In proofs, we may abbreviate the notation by use of the symbol \( \lesssim \) which indicates \( \leq \) up to some multiplicative constant which is clear from the context. Analogously, \( \gtrsim \) indicates \( \geq \) up to a multiplicative constant. The symbol \( \simeq \) states that both estimates \( \lesssim \) and \( \gtrsim \) hold.

2. Preliminaries

Let \( \Omega \subset \mathbb{R}^d \) with \( d = 2, 3 \) be a bounded Lipschitz domain with piecewise \( C^\infty \)-boundary \( \partial \Omega \) and exterior normal vector \( \mathbf{n}(y) \) for every \( y \in \partial \Omega \); see [SS11, Definition 2.2.10]. Let \( \Omega^\ext := \mathbb{R}^d \setminus \overline{\Omega} \) denote the corresponding exterior domain. We suppose that \( \Gamma = \partial \Omega \)
or $\emptyset \neq \Gamma \subset \partial \Omega$ is a relative open set which stems from a Lipschitz dissection $\partial \Omega = \Gamma \cup \partial \Gamma \cup (\partial \Omega \setminus \Gamma)$; see [McL00, p. 99].

2.1. Sobolev spaces. For $s \in \{-1/2, 0, 1/2\}$, the Sobolev spaces $H^{1/2+s}(\partial \Omega)$ are defined as in [McL00, p. 100] via Bessel-potentials and the Lipschitz parametrization of $\partial \Omega$. Let $\langle \cdot, \cdot \rangle$ denote the duality pairing which extends the $L^2(\partial \Omega)$-scalar product. For $s \in \{-1/2, 0, 1/2\}$, the negative-order Sobolev spaces are defined by duality $H^{-1(1/2+s)}(\partial \Omega) := H^{1/2+s}(\partial \Omega)'$.

If $\Gamma \subset \neq \partial \Omega$, let $E_{0, \Gamma}$ denote the extension operator which extends a function on $\Gamma$ to $\partial \Omega$ by zero. Then, the spaces $H^{1/2+s}(\Gamma)$ and $\tilde{H}^{1/2+s}(\Gamma)$ are defined as in [McL00] by

$$H^{1/2+s}(\Gamma) := \{v|_\Gamma : v \in H^{1/2+s}(\partial \Omega)\}, \quad \|v\|_{H^{1/2+s}(\Gamma)} := \inf \{\|w\|_{H^{1/2+s}(\partial \Omega)} : w|_\Gamma = v\},$$

$$\tilde{H}^{1/2+s}(\Gamma) := \{v : E_{0, \Gamma}v \in H^{1/2+s}(\partial \Omega)\}, \quad \|v\|_{\tilde{H}^{1/2+s}(\Gamma)} := \|E_{0, \Gamma}v\|_{H^{1/2+s}(\partial \Omega)}.$$

For $s = 1/2$, we have the following equivalences

$$\|u\|_{H^{1}(\Omega)}^2 \simeq \|u\|_{L^2(\Gamma)}^2 + \|\nabla u\|_{L^2(\partial \Omega)}^2$$

as well as

$$\|v\|_{\tilde{H}^{1/2}(\Gamma)}^2 \simeq \|v\|_{L^2(\Gamma)}^2 + \|\nabla v\|_{L^2(\partial \Omega)}^2;$$

see, e.g. [AFF+17, Facts 2.1]. For $s \in \{-1/2, 0, 1/2\}$, the corresponding negative-order spaces are obtained by duality

$$\tilde{H}^{-1(1/2+s)}(\Gamma) := H^{1/2+s}(\Gamma)' \quad \text{and} \quad H^{-1(1/2+s)}(\Gamma) := \tilde{H}^{1/2+s}(\Gamma).$$

We emphasize that, for all $\psi \in L^2(\Gamma)$, it holds that $E_{0, \Gamma}\psi \in H^{-1/2}(\Omega)$ as well as

$$\|\psi\|_{\tilde{H}^{-1/2}(\Gamma)} = \|E_{0, \Gamma}\psi\|_{H^{-1/2}(\partial \Omega)}.$$ We note the continuous inclusions

$$\tilde{H}^{1(1/2+s)}(\Gamma) \subseteq H^{1(1/2+s)}(\Gamma) \quad \text{and} \quad \tilde{H}^{1(1/2+s)}(\partial \Omega) = H^{1(1/2+s)}(\partial \Omega).$$

We make the following convention: If $\Gamma \subset \neq \partial \Omega$, and it is clear from the context, we identify any $v \in \tilde{H}^{1/2+s}(\Gamma)$ with its extension $E_{0, \Gamma}v \in H^{1/2+s}(\partial \Omega)$. Further, the operators $\tilde{V}_k, V_k, K'_k$ are often applied to functions in $L^2(\Gamma)$, resp. $\tilde{K}_k, K_k, W_k$ are applied to functions in $H^{1/2}(\Gamma)$. To ease notation, for $\psi \in L^2(\Gamma)$ and $v \in \tilde{H}^{1/2}(\Gamma)$, we implicitly extend by zero, e.g., we write $V_k\psi$ instead of $V_k(E_{0, \Gamma}\psi)$ and $K_kv$ instead of $K_k(E_{0, \Gamma}\psi)$.

2.2. Trace operators. We denote by $\gamma^{\text{int}}_0 : H^1(\Omega) \to H^{1/2}(\partial \Omega)$ the interior trace operator. For $u \in H^1_\Delta := \{u \in H^1(\Omega) : -\Delta u \in L^2(\Omega)\}$, we define the interior conormal derivative operator via Green’s first identity as

$$\gamma^{\text{int}}_1 : H^1_\Delta(\Omega) \to H^{-1/2}(\partial \Omega), \quad \langle \gamma^{\text{int}}_1u , \gamma^{\text{int}}_0v \rangle_{\partial \Omega} := \langle \nabla u , \nabla v \rangle_{\Omega} - \langle -\Delta u , v \rangle_{\Omega} \quad \forall v \in H^1(\Omega).$$

To define the exterior counterparts $\gamma^{\text{ext}}_0$ and $\gamma^{\text{ext}}_1$, let $U \subset \mathbb{R}^d$ be a bounded Lipschitz domain such that $\overline{\Omega} \subset U \subset \mathbb{R}^d$. Then, the exterior trace operator $\gamma^{\text{ext}}_0 : H^1(U \setminus \overline{\Omega}) \to H^{1/2}(\partial \Omega)$ is defined analogously as restriction to $\partial \Omega$. The exterior conormal derivative operator $\gamma^{\text{ext}}_1 : H^1_\Delta(U \setminus \overline{\Omega}) \to H^{-1/2}(\partial \Omega)$ is defined by $\langle \gamma^{\text{ext}}_1u , \gamma^{\text{ext}}_0v \rangle_{\partial \Omega} := \langle \nabla u , \nabla v \rangle_{U \setminus \Omega} - \langle -\Delta u , v \rangle_{U \setminus \Omega}$ for all $v \in H^1(U \setminus \overline{\Omega})$ with $\gamma^{\text{ext}}_0v = 0$ on $\partial U$. If a function $u$ admits interior and exterior trace, resp., interior and exterior conormal derivative, we define the jump

$$[\gamma_1 u] := \gamma^{\text{ext}}_1 u - \gamma^{\text{int}}_1 u \quad \text{resp.} \quad [u] = \gamma^{\text{ext}}_0 u - \gamma^{\text{int}}_0 u.$$ We denote the surface gradient by $\nabla_\Gamma(\cdot)$. 

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2.3. Layer potentials and boundary integral operators. Let $k$ denote the wavenumber of the Helmholtz equation. For $k > 0$, the Helmholtz kernel is given by
\[ G_k(x, y) = \frac{e^{ik|x-y|}}{4\pi|x-y|} \quad \text{for } d = 3, \]
where $H_0^{(1)}$ is the first-kind Hankel function of order zero. For $k < 0$, we define $G_k := \overline{G_{-k}}$ and if $k = 0$, we employ the fundamental solution of the Laplace operator
\[ G_0(x, y) = -\frac{1}{2\pi} \log |x-y| \quad \text{for } d = 2 \quad \text{resp.} \quad G_0(x, y) = \frac{1}{4\pi|x-y|} \quad \text{for } d = 3. \]
For all $(4)$
\[ (\mathcal{V}_k \phi)(x) := \int_{\partial \Omega} G_k(x, y) \phi(y) \, dy \quad \text{and} \quad (\overline{\mathcal{K}_k} \phi)(x) := \int_{\partial \Omega} \partial_n(y) G_k(x, y) \phi(y) \, dy \]
which give rise to corresponding bounded linear operators $\overline{\mathcal{V}}_k \in L(H^{-1/2}(\partial \Omega); H^1(U))$ and $\overline{\mathcal{K}}_k \in L(H^{1/2}(\partial \Omega); H^1(U \setminus \partial \Omega))$.

The simple-layer potential induces the simple-layer operator
\[ V_k := \gamma_0^\text{int} \overline{\mathcal{V}}_k : H^{-1/2+s}(\partial \Omega) \to H^{1/2+s}(\partial \Omega) \]
for $-1/2 < s \leq 1/2$; see, e.g., Theorem 2 in the case of $s = 1/2$. For $k = 0$, $V_0$ is even a well-defined isomorphism for $-1/2 < s \leq 1/2$, and elliptic as well as symmetric for $s = 0$.

For $k \neq 0$, the simple-layer operator $V_k$ is invertible, if and only if $k^2$ is not an eigenvalue of the interior Dirichlet problem (IDP) for the Laplace operator, i.e., it holds that
\[ (\text{IDP}) \quad \forall u \in H^1(\Omega) \left( -\Delta u = k^2 u \quad \text{with} \quad \gamma_0^\text{int} u = 0 \implies u = 0 \quad \text{in } \Omega \right); \]
see, e.g., [SS11, Theorem 3.9.1]. Throughout, we assume that $k^2$ satisfies (IDP).

The double-layer potential induces the double-layer operators
\[ K_k^\sigma := \gamma_0^\sigma \overline{\mathcal{K}}_k : H^{1/2+s}(\partial \Omega) \to H^{1/2+s}(\partial \Omega) \]
with $\sigma \in \{ \text{int, ext} \}$ and $-1/2 < s \leq 1/2$; see Theorem 2 for $s = 1/2$. Combining the two operators, we define $K_k := \frac{1}{2}(K_k^\text{int} + K_k^\text{ext}) : H^{1/2+s}(\partial \Omega) \to H^{1/2+s}(\partial \Omega)$.

We define the adjoint double-layer operator $K_k^\dagger : H^{-1/2+s}(\partial \Omega) \to H^{-1/2+s}(\partial \Omega)$ by $K_k^\dagger := -\frac{1}{2} \text{Id} + \gamma_1^\text{int} \overline{\mathcal{V}}_k$. Further, the hypersingular operator is given by $W_k := -\gamma_1^\text{int} \overline{\mathcal{K}}_k : H^{1/2+s}(\partial \Omega) \to H^{-1/2+s}(\partial \Omega)$ for $k = 0$, the operators $K_0, K_0'$, as well as $W_0$ are even well defined for $s = \pm 1/2$; see [SS11, Remark 3.1.18].

2.4. Admissible triangulations. Let $T_{\text{ref}}$ denote the reference element defined by
\[ T_{\text{ref}} = (0, 1) \quad \text{for } d = 2 \quad \text{resp.} \quad T_{\text{ref}} = \text{conv}\{ (0, 0), (1, 0), (0, 1) \} \quad \text{for } d = 3, \]
i.e., $T_{\text{ref}}$ is the open unit interval for $d = 2$ and the Kuhn simplex for $d = 3$. A set $T_\ast$ is a regular triangulation of $\Gamma$, if the following conditions (a)–(d) hold:
\begin{enumerate}
  \item Each $T \in T_\ast$ is a relative open subset of $\Gamma$, and there exists a bijective element map $g_T \in C^\infty(T_{\text{ref}}, T)$ such that $g_T(T_{\text{ref}}) = T$.
  \item The union of all elements cover $\Gamma$, i.e., $\Gamma = \bigcup_{T \in T_\ast} T$.
  \item For all $T, T' \in T_\ast$, the intersection $T \cap T'$ is either empty, or a joint node ($d \geq 2$), or a joint facet ($d = 3$).
\end{enumerate}
(d) In the case of \( d = 3 \), there holds the following: If \( \overline{T} \cap \overline{T}' \) is a facet, there exist facets \( f, f' \subseteq \partial T_{\text{ref}} \) of \( T_{\text{ref}} \) such that \( \overline{T} \cap \overline{T}' = g_T(f) = g_T(f') \), and the composition \( g_T^{-1} \circ g_T : f' \to f \) is even affine.

The element patch of \( T \in \mathcal{T}_* \), is given by

\[
\omega_*(T) := \text{interior} \left( \bigcup_{\overline{T} \subseteq T, \overline{T} \cap \overline{T} \neq \emptyset} \overline{T} \right).
\]

For a set of elements \( \mathcal{U} \subseteq \mathcal{T}_* \), let \( \omega_*(\mathcal{U}) := \{ T' \in \mathcal{T}_* : \forall T \in \mathcal{U} : T' \subseteq \omega_*(T) \} \). Define the local mesh-size function \( h_* \in L^\infty(\Omega) \) by \( h_*(T) := |T|^{1/(d-1)} \) for all \( T \in \mathcal{T}_* \).

To introduce shape regularity, let \( G_T(x) := D g_T(x)^T D g_T(x) \in \mathbb{R}^{(d-1) \times (d-1)} \) be the symmetric Gramian matrix of \( g_T \) and \( \lambda_{\min}(G_T(x)) \) and \( \lambda_{\max}(G_T(x)) \) its extremal eigenvalues. A regular triangulation \( \mathcal{T}_* \) is \( \gamma \)-shape regular triangulation, if the following holds:

- For all \( T \in \mathcal{T}_* \), the corresponding element maps \( g_T(\cdot) \) satisfy that

\[
\sigma(T) := \sup_{x \in T_{\text{ref}}} \left( \frac{h_*(T)^2}{\lambda_{\min}(G_T(x))} + \frac{\lambda_{\max}(G_T(x))}{h_*(T)^2} \right) \leq \gamma.
\]

- If \( d = 2 \), it additionally holds that

\[
\tilde{\sigma}(\mathcal{T}_*) := \max_{T, T' \in \mathcal{T}_*} \left( \frac{|T|}{|T'|} \right) \leq \gamma.
\]

Note that the Gramian matrix \( G_T(x) \) is symmetric and positive definite. This implies that \( 0 \leq \lambda_{\min}(G_T) \leq \lambda_{\max}(G_T) \) and hence, \( \sigma(T) \geq 0 \). The additional assumption for \( d = 2 \) ensures that the mesh-size of neighboring elements remains comparable.

Throughout, we assume that \( \mathcal{T}_* \) is a \( \gamma \)-shape regular triangulation. The next lemma recaps some important properties of \( \gamma \)-shape regular meshes; see [AFF+17, Lemma 2.6].

**Lemma 1.** There exists a constant \( C > 0 \) that depends only on \( \gamma \) and the Lipschitz character of \( \partial \Omega \), such that the following assertions (a)–(d) hold:

(a) For all \( T, T' \in \mathcal{T}_* \) such that \( \overline{T} \cap \overline{T}' \neq \emptyset \), it holds that \( h_*(T) \leq Ch_*(T') \).

(b) The number of elements in an element patch is bounded by \( C \).

(c) For all \( T \in \mathcal{T}_* \) and all elements \( T', T'' \subseteq \omega_*(T) \), there exists a sequence \( T' = T_1, \ldots, T_n = T'' \) with \( T_i \subseteq \omega_*(T) \) for all \( 1 \leq i \leq n \) such that \( \overline{T}_i \cap \overline{T}_{i+1} \) is a joint facet of \( T_i \) and \( T_{i+1} \) (for \( d = 3 \)), resp., a joint node (for \( d = 2 \)).

(d) There exists a constant \( C_{\text{shape}} > 0 \) which depends only on \( \gamma \), such that

\[
\max_{T \in \mathcal{T}_*} \left( \frac{\text{diam}(T)}{h_*(T)} \right) \leq C_{\text{shape}} \quad \text{with} \quad \text{diam}(T) := \sup_{x, y \in T} |x - y|.
\]

2.5. Discrete spaces. Let \( \mathcal{T}_* \) be a regular triangulation of \( \Gamma \). For a fixed polynomial degree \( p \geq 0 \), we define the space of (discontinuous) \( \mathcal{T}_* \)-piecewise polynomials by

\[
\mathcal{P}^p(\mathcal{T}_*) := \{ \Phi_* \in L^\infty(\Gamma) : \forall T \in \mathcal{T}_*, \Phi_* \circ g_T \text{ is a polynomial of degree } \leq p \}.
\]

Further, let \( S^p(\mathcal{T}_*) := \mathcal{P}^p(\mathcal{T}_*) \cap H^1(\Gamma) \) resp. \( \tilde{S}^p(\mathcal{T}_*) := \mathcal{P}^p(\mathcal{T}_*) \cap H^1(\Gamma) \) be the space of continuous piecewise polynomials. Note the following (compact) inclusions

\[
\mathcal{P}^p(\mathcal{T}_*) \subset L^2(\Gamma) \subset \tilde{H}^{-1/2}(\Gamma) \quad \text{and} \quad \tilde{S}^p(\mathcal{T}_*) \subset \tilde{H}^1(\Gamma) \subset \tilde{H}^{1/2}(\Gamma).
\]
In the case of $\Gamma = \partial \Omega$, there holds $\tilde{S}^p(\mathcal{T}_*) = S^p(\mathcal{T}_*)$ and $S^p(\mathcal{T}_*) \subset H^1(\Gamma)$. Throughout this paper, we use the following convention: All quantities which are associated with a triangulation $\mathcal{T}_*$, have the same index, e.g., $h_* \in \mathcal{P}^0(\mathcal{T}_*)$ is the local mesh size function or $\Phi_*$ will denote the discrete solution in $\mathcal{P}^p(\mathcal{T}_*)$.

3. Inverse estimate

The main result of this section is the following inverse-type estimate which generalizes [FKMP13, Theorem 3.1] and [AFF+17, Theorem 3.1] from $k = 0$ to general $k \geq 0$.

**Theorem 2.** The simple-layer and the double-layer operator satisfy

\begin{equation}
V_k \in L(L^2(\Gamma), H^1(\Gamma)) \quad \text{resp.} \quad K_k \in L(\tilde{H}^1(\Gamma), H^1(\Gamma)).
\end{equation}

Additionally, let $\mathcal{T}_*$ be a $\gamma$-shape regular triangulation of $\Gamma$. Then, there exists a constant $C_{\text{inv}} > 0$ which depends only on $\Gamma$, $\Omega$, and $\gamma$, such that for all $k \geq 0$, it holds that

\begin{align}
C_{\text{inv}}^{-1} \| h_*^{1/2} \nabla V_k \psi \|_{L^2(\Gamma)} &\leq (1 + k^3) \| \psi \|_{\tilde{H}^{-1/2}(\Gamma)} + \| h_*^{1/2} \nabla \psi \|_{L^2(\Gamma)}, \\
C_{\text{inv}}^{-1} \| h_*^{1/2} \nabla K_k \psi \|_{L^2(\Gamma)} &\leq (1 + k^3) \| \psi \|_{\tilde{H}^{-1/2}(\Gamma)} + \| h_*^{1/2} \nabla \psi \|_{L^2(\Gamma)}, \\
C_{\text{inv}}^{-1} \| h_*^{1/2} K_k \psi \|_{L^2(\Gamma)} &\leq (1 + k^3) \| \psi \|_{\tilde{H}^{-1/2}(\Gamma)} + \| h_*^{1/2} \nabla \psi \|_{L^2(\Gamma)}, \\
C_{\text{inv}}^{-1} \| h_*^{1/2} W_k \psi \|_{L^2(\Gamma)} &\leq (1 + k^3) \| \psi \|_{\tilde{H}^{-1/2}(\Gamma)} + \| h_*^{1/2} \nabla \psi \|_{L^2(\Gamma)},
\end{align}

for all $\psi \in L^2(\Gamma)$ and $v \in \tilde{H}^1(\Gamma)$. Furthermore, there exists $\tilde{C}_{\text{inv}} > 0$ which depends only on $\Omega$, $\Gamma$, and $p$, such that

\begin{align}
\| h_*^{1/2} \nabla V_k \Psi_* \|_{L^2(\Gamma)} + \| h_*^{1/2} K_k \Psi_* \|_{L^2(\Gamma)} &\leq \tilde{C}_{\text{inv}}(1 + k^3) \| \Psi_* \|_{\tilde{H}^{-1/2}(\Gamma)}, \\
\| h_*^{1/2} \nabla K_k V_* \|_{L^2(\Gamma)} + \| h_*^{1/2} W_k V_* \|_{L^2(\Gamma)} &\leq \tilde{C}_{\text{inv}}(1 + k^3) \| V_* \|_{\tilde{H}^{1/2}(\Gamma)},
\end{align}

for all $\Psi_* \in \mathcal{P}^p(\mathcal{T}_*)$ and $V_* \in \tilde{S}^{p+1}(\mathcal{T}_*)$. In particular, the constants $C_{\text{inv}}, \tilde{C}_{\text{inv}}$ are independent of the wavenumber $k \geq 0$.

The proof of Theorem 2 is based on the decomposition of the layer potentials into a singular part, which consists of the layer potentials $\tilde{V}_0$, resp. $\tilde{K}_0$, of the Laplacian and two smoothing operators $\tilde{S}$ and $\tilde{A}$. For the decomposition, we employ the following notation

\[ |\nabla^n \psi(x)|^2 := \sum_{\alpha \in \mathbb{N}^d} \frac{n!}{\alpha!} |D^n \psi(x)|^2 \quad \text{with} \quad \alpha! := \alpha_1! \cdot \alpha_2! \cdot \ldots \cdot \alpha_d! \quad \text{and} \quad |\nabla^0 \psi(x)|^2 := |\psi(x)|^2. \]

Lemma 3 provides such a decomposition for the simple-layer potential, while Lemma 4 states a similar result for the double-layer potential.

**Lemma 3** ([Mel12, Theorem 5.1.1]). Let $R > 0$ with $\overline{\Omega} \subseteq B_R := \{x \in \mathbb{R}^d : |x| < R\}$. Let $0 < \rho < 1$. Then, it holds that

\begin{equation}
\tilde{V}_k = \tilde{V}_0 + \tilde{S}_{V,k} + \tilde{A}_{V,k},
\end{equation}

with linear potential operators $\tilde{S}_{V,k} : H^{-1/2+s}(\partial \Omega) \to H^{3+s}(B_R)$ and $\tilde{A}_{V,k} : H^{-1/2+s}(\partial \Omega) \to H^{3+s}(B_R) \cap C^\infty(B_R)$ for all $-1/2 < s < 1/2$. Moreover, there exist positive constants

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For bounded Lipschitz domains \( \Omega \subset \mathbb{R}^d \) and \( s \in \mathbb{N}_0 \) as well as \( s' \in (0, 1) \), we require the Besov space \( B^{s', s}_{2, \infty}(\Omega) := (H^s(\Omega), H^{s+1}(\Omega))_{s', \infty} \).

**Lemma 4** ([Mel12, Theorem 5.2]). Let \( R > 0 \) with \( \overline{\Omega} \subseteq_B B_R := \{ x \in \mathbb{R}^d : |x| < R \} \). It holds that

\[
\tilde{K}_k = \tilde{K}_0 + \tilde{S}_{K,k} + \tilde{A}_{K,k},
\]

with linear potential operators \( \tilde{S}_{K,k} : L^2(\partial \Omega) \to B^{5/2}_{2, \infty}(B_R) \) as well as \( \tilde{A}_{K,k} : L^2(\partial \Omega) \to B^{5/2}_{2, \infty}(B_R) \cap C^\infty(B_R) \). Moreover, there exist constants \( C_1^K, C_2^K, C_3^K > 0 \), such that

\[
\| \tilde{S}_{K,k} \|_{L^2(\partial \Omega)} \leq C_1^K k \| v \|_{L^2(\partial \Omega)},
\]

\[
\| \nabla^n \tilde{A}_{K,k} \|_{L^2(B_R)} \leq C_2^K k^{n+1} \| \tilde{K}_0^\prime v \|_{L^2(B_R)} \leq C_3^K k^{n+1} \| v \|_{L^2(\partial \Omega)} \text{ for all } n \in \mathbb{N}_0.
\]

The constants \( C_1^K, C_2^K, C_3^K \) depend only on \( R, \Omega \), but not on the wavenumber \( k \). \( \square \)

**Proof of Theorem 2.** Let \( k > 0 \) and \( R > 0 \) with \( B_R \supseteq \overline{\Omega} \). For convenience of the reader, we split the proof into several steps.

**Step 1** (Proof of (8) for \( V_k \)): Let \( \psi \in L^2(\Gamma) \) and recall that \( \| \psi \|_{\overline{H}^{-1/2}(\partial \Omega)} = \| \psi \|_{H^{-1/2}(\partial \Omega)} \), where we identify \( \psi \) with its extension \( E_{0,\Gamma} \psi \). With Lemma 3 and the definition of \( V_k := \gamma_0^{\text{int}} \tilde{V}_k \), we decompose \( V_k = V_0 + S_{V,k} + A_{V,k} \), where

\[
S_{V,k} := \gamma_0^{\text{int}} \tilde{S}_{V,k} \quad \text{and} \quad A_{V,k} := \gamma_0^{\text{int}} \tilde{A}_{V,k}.
\]

For all \( 1/2 < s' \leq 3 + s \leq 3 + 1/2 \), equation (16) implies that

\[
\| \tilde{S}_{V,k} \psi \|_{H^{s'}(B_R)} \lesssim \rho^2 (\rho k^{-1})^{1+s-s'} \| \psi \|_{\overline{H}^{-1/2+s'}(\partial \Omega)}.
\]

For \( s' = 2 \) and \( s = 0 \), this reveals \( \tilde{S}_{V,k} \psi \in H^2(B_R) \). Further, stability of \( \gamma_0^{\text{int}} \) yields that

\[
\| S_{V,k} \psi \|_{H^1(\Gamma)} \lesssim \| S_{V,k} \psi \|_{H^1(\partial \Omega)} \leq \| \tilde{S}_{V,k} \psi \|_{H^{3/2}(B_R)} \lesssim \| \tilde{S}_{V,k} \psi \|_{H^2(\Gamma)} \lesssim \rho k \| \psi \|_{\overline{H}^{-1/2}(\Gamma)}.
\]

Next, note that equation (17) proves that \( A_{V,k} \psi \in H^2(B_R) \). With the (compact) embedding \( H^{-1/2}(\partial \Omega) \subset H^{-1}(\partial \Omega) \) with \( \| \cdot \|_{H^{-1}(\partial \Omega)} \lesssim \| \cdot \|_{H^{-1/2}(\partial \Omega)} \), this yields that

\[
\| A_{V,k} \psi \|_{H^2(B_R)} \lesssim (k + k^2 + k^3) \| \psi \|_{H^{-1}(\partial \Omega)} \lesssim (1 + k^3) \| \psi \|_{\overline{H}^{-1/2}(\Gamma)}.
\]

Similarly to (22), continuity of the trace operator proves that

\[
\| A_{V,k} \psi \|_{H^1(\Gamma)} \leq \| A_{V,k} \psi \|_{H^1(\partial \Omega)} \lesssim \| \tilde{A}_{V,k} \psi \|_{H^2(\Gamma)} \lesssim (1 + k^3) \| \psi \|_{\overline{H}^{-1/2}(\Gamma)}.
\]
Combining the estimates (22) and (24) with the (compact) embedding $L^2(\Gamma) \subset \tilde{H}^{-1/2}(\Gamma)$, we see that $A_{V,k}, S_{V,k} \in L(L^2(\Gamma), H^1(\Gamma))$. With $V_0 \in L(L^2(\Gamma), H^1(\Gamma))$, we conclude that $V_k = V_0 + S_{V,k} + A_{V,k} \in L(L^2(\Gamma), H^1(\Gamma))$.

**Step 2 (Proof of equation (9)):** Recall that $V_k = V_0 + S_{V,k} + A_{V,k}$. This decomposition directly yields that

$$\|h_1^{1/2}\nabla_{\Gamma} V_k \psi\|_{L^2(\Gamma)} \leq \|h_0^{1/2}\nabla_{\Gamma} V_0 \psi\|_{L^2(\Gamma)} + \|h_1^{1/2}\nabla_{\Gamma} S_{V,k} \psi\|_{L^2(\Gamma)} + \|h_1^{1/2}\nabla_{\Gamma} A_{V,k} \psi\|_{L^2(\Gamma)}.$$  

We treat each term on the right-hand side separately. [AFF+17, Theorem 3.1] yields that

$$\|h_0^{1/2}\nabla_{\Gamma} V_0 \psi\|_{L^2(\Gamma)} \lesssim \|\psi\|_{H^{-1/2}(\Gamma)} + \|h_1^{1/2} \psi\|_{L^2(\Gamma)}.$$

Next, $\|h_\bullet \|_{L^\infty(\Gamma)} \lesssim \text{diam}(\Omega) \lesssim 1$ and equation (22) imply that

$$\|h_1^{1/2}\nabla_{\Gamma} S_{V,k} \psi\|_{L^2(\Gamma)} \lesssim \|S_{V,k} \psi\|_{H^1(\Gamma)} \lesssim k \|\psi\|_{H^{-1/2}(\Gamma)}.$$

Finally, we use equation (24) to estimate the last term on the right hand side of (25) by

$$\|h_1^{1/2}\nabla_{\Gamma} A_{V,k} \psi\|_{L^2(\Gamma)} \lesssim \|A_{V,k} \psi\|_{H^1(\Gamma)} \lesssim (1 + k^3) \|\psi\|_{H^{-1/2}(\Gamma)}.$$

Combining the latter four estimates, we prove that

$$\|h_1^{1/2}\nabla_{\Gamma} V_k \psi\|_{L^2(\Gamma)} \lesssim (1 + k^3) \|\psi\|_{H^{-1/2}(\Gamma)} + \|h_1^{1/2} \psi\|_{L^2(\Gamma)}.$$

This concludes the proof of (9).

**Step 3 (Proof of equation (11)):** Recall the definition of the adjoint double-layer operator. With $K'_k = -\frac{1}{2}\text{Id} + \gamma_1^{\text{int}} V_k = K_0' + \gamma_1^{\text{int}} \tilde{S}_{V,k} + \gamma_1^{\text{int}} \tilde{A}_{V,k}$, this implies that

$$\|h_1^{1/2} K'_k \psi\|_{L^2(\Gamma)} \leq \|h_0^{1/2} K'_0 \psi\|_{L^2(\Gamma)} + \|h_1^{1/2} \gamma_1^{\text{int}} \tilde{S}_{V,k} \psi\|_{L^2(\Gamma)} + \|h_1^{1/2} \gamma_1^{\text{int}} \tilde{A}_{V,k} \psi\|_{L^2(\Gamma)}.$$  

Again, we treat each term on the right-hand side separately. First, [AFF+17, Theorem 3.1] yields that

$$\|h_1^{1/2} K'_0 \psi\|_{L^2(\Gamma)} \lesssim \|\psi\|_{H^{-1/2}(\Gamma)} + \|h_1^{1/2} \psi\|_{L^2(\Gamma)}.$$

Recall from Step 1 that $\tilde{S}_{V,k} \varphi, \tilde{A}_{V,k} \varphi \in H^2(B_R)$. Therefore, [SS11, Remark 2.7.5] implies that $\gamma_1^{\text{int}} \tilde{S}_{V,k} \varphi, \gamma_1^{\text{int}} \tilde{A}_{V,k} \varphi \in H^{1/2}(\partial \Omega)$. With $\|h_\bullet \|_{L^\infty(\Gamma)} \lesssim \text{diam}(\Omega) \lesssim 1$, the (compact) embedding $H^{1/2}(\partial \Omega) \subset L^2(\partial \Omega)$ and stability ([SS11, Remark 2.7.5]) of the conormal derivative yield that

$$\|h_1^{1/2} \gamma_1^{\text{int}} \tilde{S}_{V,k} \psi\|_{L^2(\Gamma)} \lesssim \|\gamma_1^{\text{int}} \tilde{S}_{V,k} \psi\|_{H^{1/2}(\partial \Omega)} \lesssim \|\tilde{S}_{V,k} \psi\|_{L^2(\Gamma)} \lesssim k \|\psi\|_{H^{-1/2}(\Gamma)}.$$  

Third, we argue as before and prove that

$$\|h_1^{1/2} \gamma_1^{\text{int}} \tilde{A}_{V,k} \psi\|_{L^2(\Gamma)} \lesssim \|\gamma_1^{\text{int}} \tilde{A}_{V,k} \psi\|_{H^{1/2}(\partial \Omega)} \lesssim \|\tilde{A}_{V,k} \psi\|_{H^2(B_R)} \lesssim (1 + k^3) \|\psi\|_{H^{-1/2}(\Gamma)}.$$  

Combining the right-hand sides of all estimates, we obtain that

$$\|h_1^{1/2} K'_k \psi\|_{L^2(\Gamma)} \lesssim (1 + k^3) \|\psi\|_{H^{-1/2}(\Gamma)} + \|h_1^{1/2} \psi\|_{L^2(\Gamma)},$$

and conclude the proof of (11).

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Step 4 (Proof of (8) for $K_k$): Let $v \in \tilde{H}^1(\Gamma)$. Analogously to Step 1, Lemma 4 yields that $K_k = K_0 + S_{k,k} + A_{k,k}$, where $S_{k,k} := \gamma_0 \int S_{k,k}$ and $A_{k,k} := \gamma_0 \int A_{k,k}$.

For $-\infty < s < \infty$, 0 < $q < \infty$, and 0 < $r, t \leq \infty$, there holds the continuous embedding $B^s_{q,r}(B_R) \subset B^s_{q,t}(B_R)$; see, e.g., [Tri92, Section 2.32]. This implies that $B^5_{2,\infty}(B_R) \subset B^2_{2,2}(B_R) = H^2(B_R)$ with $\| \cdot \|_{H^2(B_R)} \lesssim \| \cdot \|_{B^5_{2,\infty}(B_R)}$. Analogously to (22), continuity of the interior trace operator $\gamma_0$ and inequality (19) reveal that

$$\|S_{k,k}v\|_{H^1(\Gamma)} \lesssim \|S_{k,k}v\|_{H^2(B_R)} \lesssim \|\tilde{S}_{k,k}v\|_{B^5_{2,\infty}(B_R)} \lesssim k \|v\|_{L^2(\Omega)} = k \|v\|_{L^2(\Gamma)}.$$ (26)

The operator $A_{k,k}$ is treated analogously to Step 1 and hence satisfies that

$$\|A_{k,k}v\|_{H^1(\Gamma)} \lesssim \|\tilde{A}_{k,k}v\|_{H^3/2(B_R)} \lesssim \|\tilde{A}_{k,k}v\|_{H^3/2(\Gamma)} \lesssim (1 + k^3) \|v\|_{L^2(\Gamma)}.$$ (27)

Then, the estimates (26) and (27) prove that $S_{k,k}$, $A_{k,k} \in L(L^2(\Gamma), H^1(\Gamma))$. With $K_0 \in L(\tilde{H}^1(\Gamma), H^1(\Gamma))$, we conclude that $K_k = K_0 + S_{k,k} + A_{k,k} \in L(H^1(\Gamma), H^1(\Gamma))$.

Step 5 (Proof of equation (10)): Let $v \in \tilde{H}^1(\Gamma)$. Analogously to Step 2, the decomposition $K_k = K_0 + S_{k,k} + A_{k,k}$ implies that

$$\|h^{1/2}_* \nabla_{\Gamma} K_kv\|_{L^2(\Gamma)} \lesssim \|h^{1/2}_* \nabla_{\Gamma} K_0v\|_{L^2(\Gamma)} + \|h^{1/2}_* \nabla_{\Gamma} S_{k,k}v\|_{L^2(\Gamma)} + \|h^{1/2}_* \nabla_{\Gamma} A_{k,k}v\|_{L^2(\Gamma)}.$$ We proceed as before. First, [AFF+17, Theorem 3.1] yields that

$$\|h^{1/2}_* \nabla_{\Gamma} K_0v\|_{L^2(\Gamma)} \lesssim \|v\|_{\tilde{H}^{1/2}(\Gamma)} + \|h^{1/2}_* \nabla_{\Gamma} v\|_{L^2(\Gamma)}.$$ Second, $\|h^{*}_{L^\infty}\| \lesssim \text{diam}(\Omega) \lesssim 1$ and equation (26) imply that

$$\|h^{1/2}_* \nabla_{\Gamma} S_{k,k}v\|_{L^2(\Gamma)} \lesssim \|S_{k,k}v\|_{H^1(\Gamma)} \lesssim k \|v\|_{L^2(\Gamma)}.$$ Third, we use equation (27) to see that

$$\|h^{1/2}_* \nabla_{\Gamma} A_{k,k}v\|_{L^2(\Gamma)} \lesssim \|A_{k,k}v\|_{H^1(\Gamma)} \lesssim (1 + k^3) \|v\|_{L^2(\Gamma)}.$$ Combining the latter estimates, we obtain that

$$\|h^{1/2}_* \nabla_{\Gamma} K_kv\|_{L^2(\Gamma)} \lesssim \|v\|_{\tilde{H}^{1/2}(\Gamma)} + (1 + k^3) \|v\|_{L^2(\Gamma)} + \|h^{1/2}_* \nabla_{\Gamma} v\|_{L^2(\Gamma)}$$ \lesssim (1 + k^3) \|v\|_{\tilde{H}^{1/2}(\Gamma)} + \|h^{1/2}_* \nabla_{\Gamma} v\|_{L^2(\Gamma)}.$$ This concludes the proof of (10).

Step 6 (Proof of equation (12)): Recall the definition of $W_k$. With $\tilde{K}_k = \tilde{K}_0 + \tilde{S}_{k,k} + \tilde{A}_{k,k}$ there holds $W_k = -\gamma_1 \int \tilde{K}_k = W_0 - \gamma_1 \int \tilde{S}_{k,k} - \gamma_1 \int \tilde{A}_{k,k}$ and hence

$$\|h^{1/2}_* W_kv\|_{L^2(\Gamma)} \leq \|h^{1/2}_* W_0v\|_{L^2(\Gamma)} + \|h^{1/2}_* \gamma_1 \int \tilde{S}_{k,k}v\|_{L^2(\Gamma)} + \|h^{1/2}_* \gamma_1 \int \tilde{A}_{k,k}v\|_{L^2(\Gamma)}.$$ We proceed as before. First, [AFF+17, Theorem 3.1] yields that

$$\|h^{1/2}_* W_0v\|_{L^2(\Gamma)} \lesssim \|v\|_{\tilde{H}^{1/2}(\Gamma)} + \|h^{1/2}_* \nabla_{\Gamma} v\|_{L^2(\Gamma)}.$$
Recall from Step 4 that \( \bar{S}_{K,k}v, \bar{A}_{K,k}v \in H^2(B_R) \) and hence \( \gamma_1^{\text{int}} \bar{S}_{K,k}v, \gamma_1^{\text{int}} \bar{A}_{K,k}v \in H^{1/2}(\partial \Omega) \). As in Step 3, stability of \( \gamma_1^{\text{int}} \) gives

\[
\| h_1^{1/2} \gamma_1^{\text{int}} \bar{S}_{K,k}v \|_{L^2(\Gamma)} \lesssim \| \bar{S}_{K,k}v \|_{H^2(B_R)} \lesssim k \| v \|_{L^2(\Gamma)},
\]

and

\[
\| h_1^{1/2} \gamma_1^{\text{int}} \bar{A}_{K,k}v \|_{L^2(\Gamma)} \lesssim \| \bar{A}_{K,k}v \|_{H^2(B_R)} \lesssim (1 + k^2) \| v \|_{L^2(\Gamma)}.
\]

Combining the latter four estimates, we conclude the proof of (12).

**Step 7 (Proof of equations (13)–(14)):** According to [GHS05, Geo08] or [AFF+17, Lemma A.1], there hold the following inverse estimates

\[
\| h_1^{1/2} (p + 1)^{-1} \Psi \|_{L^2(\Gamma)} \lesssim \| \Psi \|_{H^{-1/2}(\Gamma)} \quad \text{for all } \Psi \in P^p(\mathcal{T}_\varepsilon),
\]

and

\[
\| h_1^{1/2} (p + 1)^{-1} \nabla \chi \|_{L^2(\Gamma)} \lesssim \| \chi \|_{H^{1/2}(\Gamma)} \quad \text{for all } \chi \in \bar{S}^p(\mathcal{T}_\varepsilon),
\]

where \( p \) is the fixed polynomial degree. The hidden constant depends only on \( \partial \Omega, \Gamma \), and the shape regularity of \( \mathcal{T}_\varepsilon \). Applying (28)–(29) to the right-hand sides of equations (9) and (12), we conclude (13). Using (28)–(29) to estimate the right-hand sides of (10) and (12), we reveal (14). This concludes the proof.

\[ \square \]

4. Adaptive Algorithm

In this section, we introduce the adaptive algorithm as well as a suitable a posteriori error estimator. We show that our ABEM setting fits in the abstract framework of [BHP17, Section 2], where an adaptive algorithm for compactly perturbed elliptic problems is analyzed and optimal algebraic convergence rates are proved.

**4.1. Framework.** We consider the model problem (2) in the following functional analytic framework. For each admissible triangulation \( \mathcal{T}_\varepsilon \), we consider \( \mathcal{T}_\varepsilon \)-piecewise polynomial ansatz and test spaces \( P^p(\mathcal{T}_\varepsilon) \). On Lipschitz boundaries \( \partial \Omega \), the operator \( C_k := V_k - V_0 : H^{-1/2}(\partial \Omega) \rightarrow H^{1/2}(\partial \Omega) \) is compact; see, e.g., [SS11, Lemma 3.9.8] or [Ste08a, Section 6.9]. This implies compactness of \( C_k : \tilde{H}^{-1/2}(\Gamma) \rightarrow H^{1/2}(\Gamma) \). Therefore, the model problem (2) can equivalently be reformulated as follows: Given \( f \in H^{1/2}(\Gamma) \), find \( \phi \in \tilde{H}^{-1/2}(\Gamma) \) such that

\[
(V_0 + C_k) \phi = f \quad \text{on } \Gamma.
\]

The weak formulation of (30) thus seeks \( \phi \in \tilde{H}^{-1/2}(\Gamma) \) such that

\[
\langle V_0 \phi, \psi \rangle + \langle C_k \phi, \psi \rangle = \langle f, \psi \rangle \quad \text{for all } \psi \in \tilde{H}^{-1/2}(\Gamma).
\]

Recall that \( V_0 \) is an elliptic and symmetric isomorphism. Hence,

\[
a(\chi, \psi) := \langle V_0 \chi, \psi \rangle \quad \text{for all } \chi, \psi \in \tilde{H}^{-1/2}(\Gamma),
\]
defines a scalar product that induces an equivalent energy norm \( \| \psi \| := a(\psi, \psi)^{1/2} \simeq \| \psi \|_{\tilde{H}^{-1/2}(\Gamma)} \) on \( \tilde{H}^{-1/2}(\Gamma) \). The Galerkin discretization seeks \( \Phi_\varepsilon \in P^p(\mathcal{T}_\varepsilon) \) such that

\[
a(\Phi_\varepsilon, \Psi_\varepsilon) + \langle C_k \Phi_\varepsilon, \Psi_\varepsilon \rangle = \langle f, \Psi_\varepsilon \rangle \quad \text{for all } \Psi_\varepsilon \in P^p(\mathcal{T}_\varepsilon).
\]

Existence of solutions to the (32) can be guaranteed by the following proposition which is applied for \( H = \tilde{H}^{-1/2}(\Gamma) \) and \( \mathcal{X}_\varepsilon = P^p(\mathcal{T}_\varepsilon) \); see [SS11, Theorem 4.2.9] or [BHP17, July 31, 2018 10].
Proposition 5. Let $H$ be a separable Hilbert space and let $(X_{\ell})_{\ell \in \mathbb{N}_0}$ be a dense sequence of discrete subspaces $X_{\ell} \subset H$, i.e., $\min_{\Psi \in X_{\ell}} \|\psi - \Psi\|_H \to 0$ as $\ell \to \infty$ for all $\psi \in H$. Let $a(\cdot, \cdot)$ be a hermitian continuous, and elliptic sesquilinear form on $H$. Moreover, let $C : H \to H^*$ be a compact operator and $f \in H^*$. Consider the following variational formulation: Find $\phi \in H$ such that

\begin{equation}
 b(\phi, \psi) := a(\phi, \psi) + \langle C\phi, \psi \rangle = \langle f, \psi \rangle \quad \text{for all } \psi \in H.
\end{equation}

Suppose well-posedness of (33), i.e., for all $\phi \in H$, it holds that

\begin{equation}
 \phi = 0 \iff \forall \psi \in H \ b(\phi, \psi) = 0.
\end{equation}

Then, there exists some index $\ell_* \in \mathbb{N}$ such that for all discrete subspaces $X_{\ell_*} \subseteq H$ with $X_{\ell_*} \supseteq X_{\ell}$, the following holds: There exists $\beta > 0$, which depends only on $X_{\ell_*}$, such that the discrete inf-sup constant on $X_{\ell_*}$ is uniformly bounded from below, i.e.,

\begin{equation}
 0 < \beta \leq \beta_* := \inf_{\Phi_* \in X_{\ell_*} \setminus \{0\}} \sup_{\Psi_* \in X_{\ell_*} \setminus \{0\}} \frac{|b(\Phi_*, \Psi_*)|}{\|\Phi_*\|_H \|\Psi_*\|_H}.
\end{equation}

In particular, the discrete formulation (32) admits a unique solution $\Phi_* \in X_{\ell_*}$ and there exists $C > 0$, which depends only on $b(\cdot, \cdot)$ and $\beta$, but not on $X_{\ell_*}$, such that

\begin{equation}
 \|\phi - \Phi_*\|_H \leq C \min_{\Psi_* \in X_{\ell_*}} \|\phi - \Psi_*\|_H,
\end{equation}

i.e., uniform validity of the Céa lemma. If the spaces $X_{\ell}$ are nested, i.e., $X_{\ell} \subseteq X_{\ell+1}$ for all $\ell \in \mathbb{N}_0$, the latter guarantees convergence $\|\phi - \Phi_\ell\|_H \to 0$ as $\ell \to \infty$. \hfill $\Box$

4.2. Mesh-refinement. From now on, suppose that $T_0$ is a given $\gamma$-shape regular triangulation of $\Gamma$. For mesh-refinement, we consider 2D newest vertex bisection (NVB) for $d = 3$ (see e.g., [Ste08b]), or extended 1D bisection (EB) from [AFF+13] for $d = 2$. Given a $\gamma$-shape regular triangulation $T_0$ and a set of marked elements $M_* \subseteq T_0$, the call $T_* = \text{refine}(T_0, M_*)$ returns for both refinement strategies the coarsest refinement $T_0$ of $T_0$ such that all $T \in M_*$ have been refined, i.e.,

- $M_* \subseteq T_0 \setminus T_0$,
- the number of elements $\#T_0$ is minimal amongst all other refinements $T'$ of $T_0$.

Furthermore, we write $T_\ell \in \text{refine}(T_0)$ if $T_\ell$ is obtained by a finite number of refinement steps, i.e., there exists $n \in \mathbb{N}_0$ as well as a finite sequence $T^{(0)}, \ldots, T^{(n)}$ of triangulations and corresponding sets $M^{(j)} \subseteq T^{(j)}$ such that

- $T_0 = T^{(0)}$,
- $T^{(j+1)} = \text{refine}(T^{(j)}, M^{(j)})$ for all $j = 0, \ldots, n-1$,
- $T_n = T^{(n)}$.

In particular, $T_* \in \text{refine}(T_0)$. To abbreviate notation, we let $T := \text{refine}(T_0)$ be the set of all possible triangulations which can be obtained from the initial triangulation $T_0$.

Both refinement strategies guarantee uniform $\gamma$-shape regularity of all $T_* \in T$, where $\gamma$ depends only on $T_0$. Hence, Lemma 1 applies for any triangulation $T_* \in T$. Moreover, for all $T \in T_*$, it holds that $T = \bigcup\{T' \in T_* : T' \subseteq T\}$. In the following, we recall further properties of these mesh-refinement strategies, which are exploited below.
First, refining an element results in at least 2 and at most $C_{\text{son}}$ sons, where $C_{\text{son}} = 2$ for EB and $C_{\text{son}} = 4$ for NVB; see e.g., [KPP13] for NVB and [AFF+13, Section 3] for EB. In particular, it holds that

$$\#(T_0 \setminus T_\circ) + \#T_\circ \leq \#T_0 \quad \text{for all } T_0 \in \mathbb{T} \text{ and all } T_\circ \in \text{refine}(T_0).$$

Second, refinement of an element yields a contraction of the local mesh-size function. Even though the proof is found, e.g., in [Gan17], we include it for the sake of completeness.

**Lemma 6.** There exist $0 < q_{\text{mesh}} < 1$, such that for all $T_\circ, T_0 \in \mathbb{T}$ with $T_0 \in \text{refine}(T_\circ)$, it holds that $h_0|_T \leq q_{\text{mesh}} h_0|_T$ on all $T \in T_0 \setminus T_\circ$.

**Proof.** We argue by contradiction. To this end, let $(T_\circ_n)_{n \in \mathbb{N}}, (T_0^n)_{n \in \mathbb{N}} \subset \mathbb{T}$ be sequences of refinements with $T_0^n \in \text{refine}(T_\circ^n)$ and elements $T_\circ^n \in T_0^n \setminus T_\circ^n$ as well as $T_0^n \in T_\circ^n \setminus T_\circ^n$ such that

$$T_0^n \subseteq T_\circ^n \quad \text{as well as} \quad \frac{|T_\circ^n|}{|T_0^n|} \to 1 \quad \text{as } n \to \infty.$$

This implies that $|T_\circ^n \setminus T_0^n| / |T_\circ^n| \to 0$ as $n \to \infty$. Further, for all $n \in \mathbb{N}$ there exists $T \in T_0^n$ such that $T_\circ^n \not\subseteq T_\circ^n \subseteq T$. We obtain a corresponding sequence $\tilde{T}_\circ^n \subseteq \tilde{T}_\circ^n \subseteq T_{\text{ref}}$ with $g_T(\tilde{T}_\circ^n) = T_\circ^n$ as well as $g_T(T_\circ^n) = T_0^n$. Since bisection is done at first on the reference element, it holds that $|\tilde{T}_\circ^n| \leq |\tilde{T}_\circ^n| / 2$ for all $n \in \mathbb{N}_0$. Then, $\gamma$-shape regularity implies that $|\det G_T(x)| \simeq (h_0(T))^2/d = |T|^2$ for all $x \in T_{\text{ref}}$. This reveals the contradiction

$$\frac{1}{2} \leq \frac{|\tilde{T}_\circ^n \setminus \tilde{T}_0^n|}{|T_\circ^n|} \simeq \frac{\int_{\tilde{T}_\circ^n \setminus \tilde{T}_0^n} |\det G_T(t)|^{1/2} dt}{\int_{\tilde{T}_\circ^n} |\det G_T(t)|^{1/2} dt} = \frac{|T_\circ^n \setminus T_0^n|}{|T_\circ^n|} \to 0 \quad n \to \infty$$

and this concludes the proof. \square

Third, for a sequence $(T_\ell)_{\ell \in \mathbb{N}_0}$ with $T_\ell = \text{refine}(T_{\ell-1}, \mathcal{M}_{\ell-1})$ for arbitrary $\mathcal{M}_{\ell-1} \subseteq \mathcal{T}_{\ell-1}$, EB and NVB satisfy the mesh-closure estimate

$$\#T_\ell - \#T_0 \leq C_{\text{mesh}} \sum_{j=0}^{\ell-1} \#\mathcal{M}_j \quad \text{for all } \ell \in \mathbb{N},$$

where the constant $C_{\text{mesh}} \geq 1$ depends only on the initial mesh $T_0$. In particular, (38) guarantees that the number of additional refinements of elements, in order to avoid hanging nodes and preserve conformity (NVB) or to preserve $\gamma$-shape regularity (EB), does not dominate the number of marked elements. For newest vertex bisection, the mesh-closure estimate has first been proved for $d = 2$ in [BDD04] and later for $d \geq 2$ in [Ste08b]. While both works require an additional admissibility assumption on $T_0$, [KPP13] proved that this condition is redundant for $d = 2$. For EB, (38) is proved in [AFF+13, Theorem 2.3].

Finally, we recall the overlay-estimate; For all $T \in \mathbb{T}$ as well as $T_\circ, T_0 \in \text{refine}(T)$ there exists a common refinement $T_\circ + T_0 \in \text{refine}(T_\circ) \cap \text{refine}(T_0) \subseteq \text{refine}(T)$, such that

$$\#(T_0 + T_\circ) \leq \#T_0 + \#T_\circ - \#T.$$

For NVB, the proof is found in [CKNS08, Ste07]. For EB, the proof is trivial; see [AFF+13].

**4.3. Residual a posteriori error estimator.** Let $T_\circ \in \text{refine}(T_0)$. Suppose that $f \in H^1(\Gamma)$ and that the solution $\Phi_\circ \in \mathcal{P}^0(T_\circ)$ of (32) exists. Recall that $V_k : L^2(\Gamma) \to \mathbb{R}$.

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$H^1(\Gamma)$. Therefore, we can compute for all $T \in \mathcal{T}_*$ the local refinement indicators $\eta_\bullet(T) := \|h^{1/2}_\bullet \nabla (V_k \Phi_\bullet - f)\|_{L^2(T)} \geq 0$ as well as the corresponding \textit{a posteriori} error estimator

$$\eta_\bullet := \eta_\bullet(\mathcal{T}_*) \text{ with } \eta_\bullet(\mathcal{U}_*) := \left( \sum_{T \in \mathcal{U}_*} \eta_\bullet(T) \right)^{1/2} \text{ for all } \mathcal{U}_* \subseteq \mathcal{T}_*.$$  

For $\mathcal{U}_* \subseteq \mathcal{T}_*$, define

$$\bigcup \mathcal{U}_* := \{ x \in \Gamma : \exists T \in \mathcal{T}_*, x \in T \}.$$ 

It holds that $\eta_\bullet(\mathcal{U}_*) = \|h^{1/2} \nabla (V_k \Phi_\bullet - f)\|_{L^2(\bigcup \mathcal{U}_*)}$. The error estimator (40) has first been proposed for \textit{a posteriori} BEM error control for the weakly-singular integral equation in 2D in [CS95, Car96] and later in 3D in [CMS01].

### 4.4. Adaptive algorithm.

Based on the error estimator $\eta_\bullet$ we consider the following algorithm, where the expanded making strategy in Step(iv)–(v) goes back to [BHP17].

**Algorithm 7.** \textbf{INPUT:} Parameters $0 < \theta \leq 1$ and $C_{\text{mark}} \geq 1$ as well as initial triangulation $\mathcal{T}_0$ with $\Phi_{-1} := 0 \in \mathcal{P}^0(\mathcal{T}_0)$ and $\eta_{-1} := 1$.

**Adaptive loop:** For all $\ell = 0, 1, 2, \ldots$, iterate the following Steps (i)–(vi):

(i) **If:** (32) does not admit a unique solution in $\mathcal{P}^p(\mathcal{T}_\ell)$:
- Define $\Phi_\ell := \Phi_{\ell-1} \in \mathcal{P}^p(\mathcal{T}_0)$ and $\eta_\ell := \eta_{\ell-1}$.
- Let $\mathcal{T}_{\ell+1} := \text{refine}(\mathcal{T}_\ell, \mathcal{T}_\ell)$ be the uniform refinement of $\mathcal{T}_\ell$,
- Increase $\ell \to \ell + 1$ and continue with Step (i).

(ii) **Else:** compute the unique solution $\Phi_\ell \in \mathcal{P}^p(\mathcal{T}_\ell)$ to (32).

(iii) Compute the corresponding indicators $\eta_\ell(T)$ for all $T \in \mathcal{T}_\ell$.

(iv) Determine a set $\mathcal{M}_\ell^p \subseteq \mathcal{T}_\ell$ of up to the multiplicative factor $C_{\text{mark}}$ minimal cardinality such that $\theta h^{2}_\ell \leq \eta_\ell(\mathcal{M}_\ell^p)^2$.

(v) Find $\mathcal{M}_\ell^p \subseteq \mathcal{T}_\ell$ such that $\# \mathcal{M}_\ell^p = \# \mathcal{M}_\ell^p$ as well as $h_\ell(T) \geq h_\ell(T')$ for all $T \in \mathcal{M}_\ell^p$ and $T' \in \mathcal{T}_\ell \setminus \mathcal{M}_\ell^p$. Define $\mathcal{M}_\ell := \mathcal{M}_\ell^p \cup \mathcal{M}_\ell^p$.

(vi) Generate $\mathcal{T}_{\ell+1} := \text{refine}(\mathcal{T}_\ell, \mathcal{M}_\ell)$, increase $\ell \to \ell + 1$, and continue with Step (i).

**OUTPUT:** Sequences of successively refined triangulations $\mathcal{T}_\ell$, discrete solutions $\Phi_\ell$, and corresponding estimators $\eta_\ell$.

**Remark 8.** • Apart from Step (i) and Step (v), Algorithm 7 is the usual adaptive loop based on the Dörfler marking strategy [Dör96] in Step (iv) as used, e.g., in [CKNS08, FFP14, CFPP14] as well as [FFK+14, FK15, FKMP13].

• While $C_{\text{mark}} = 1$ requires to sort the indicators and hence leads to log-linear effort, Stevenson [Ste07] showed that $C_{\text{mark}} = 2$ allows to determine $\mathcal{M}_\ell$ in linear complexity.

• Step (v) of Algorithm 7 is called expanded Dörfler marking and ensures $\|h_\ell\|_{L^\infty(\Gamma)} \to \infty$ as $\ell \to \infty$, see [BHP17, Proposition 16]. In particular, Step (v) implies $\bigcup_{\ell \in \mathbb{N}} \mathcal{P}^p(\mathcal{T}_\ell) = \overline{\mathcal{H}^{1/2}(\Gamma)}$. This guarantees definiteness and hence well-posedness of (31) on the discrete limit space, i.e., [BHP17, Axiom (A5)] is satisfied.

### 4.5. Properties of the error estimator.

The proof of convergence with optimal algebraic rates for the adaptive scheme relies on the following essential properties of the \textit{a posteriori} error estimator. These, so-called \textit{axioms of adaptivity} are found in [BHP17,
Lemma 9. There exist $C_{\text{stb}}, C_{\text{red}}, C_{\text{rel}}, C_{\text{rel}} > 0$ and 0 < $q_{\text{red}} < 1$ such that for all $\mathcal{T}_o \in \mathbb{T}$ and all $\mathcal{T}_o \in \text{refine}(\mathcal{T}_o)$ the following implication holds: Provided that the discrete solutions $\Phi_o \in \mathcal{P}^p(\mathcal{T}_o)$ and $\Phi_o \in \mathcal{P}^p(\mathcal{T}_o)$ exist, there holds the following (i)–(iv):

(i) **Stability on non-refined element domains**

\begin{equation}
|\eta_o(\mathcal{T}_o \cap \mathcal{T}_o) - \eta_o(\mathcal{T}_o \cap \mathcal{T}_o)| \leq C_{\text{stb}} \|\Phi_o - \Phi_o\|_{\bar{H}^{-1/2}(\Gamma)},
\end{equation}

(ii) **Reduction on refined element domains**

\begin{equation}
\eta_o(\mathcal{T}_o \setminus \mathcal{T}_o)^2 \leq q_{\text{red}} \eta_o(\mathcal{T}_o \setminus \mathcal{T}_o)^2 + C_{\text{red}} \|\Phi_o - \Phi_o\|_{\bar{H}^{-1/2}(\Gamma)}^2.
\end{equation}

(iii) **Discrete reliability**

\begin{equation}
\|\Phi_o - \Phi_o\|_{\bar{H}^{-1/2}(\Gamma)} \leq C_{\text{rel}} \beta_o^{-1} \eta_o(\mathcal{R}_{\text{opt}}),
\end{equation}

where $\beta_o$ is the discrete inf-sup constant from (35) on $\mathcal{P}^p(\mathcal{T}_o)$ and $\mathcal{R}_{\text{opt}} := \omega_o(\mathcal{T}_o \setminus \mathcal{T}_o) \subseteq \mathcal{T}_o$. In particular, it holds that $\mathcal{T}_o \setminus \mathcal{T}_o \subseteq \mathcal{R}_{\text{opt}}$ as well as $\#\mathcal{R}_{\text{opt}} \leq C_{\text{rel}} \#(\mathcal{T}_o \setminus \mathcal{T}_o)$.

(iv) **Reliability**

\[ \|\phi - \Phi_o\|_{\bar{H}^{-1/2}(\Gamma)} \leq C_{\text{rel}} \eta_o. \]

The involved constants $C_{\text{stb}}, C_{\text{red}}, C_{\text{rel}}, C_{\text{rel}}, q_{\text{red}} > 0$ depend only on the given data, the polynomial degree $p$, the initial mesh $\mathcal{T}_0$, and $\gamma$-shape regularity.

5. Optimal Convergence

In this section, we prove linear convergence as well as optimal algebraic convergence rates for the sequence of a posteriori error estimators, generated by Algorithm 7. According to Section 4, the error estimator as well as the mesh-refinement strategy satisfy all assumptions needed in order to apply the abstract framework from [BHP17, Section 2].

5.1. Approximation class. For $N \in \mathbb{N}$, we define the set of all refinements which have at most $N$ elements more than a given mesh $\mathcal{T}$, i.e.,

\[ \mathcal{T}_N(\mathcal{T}) := \{ \mathcal{T}_o \in \text{refine}(\mathcal{T}) : \#\mathcal{T}_o - \#\mathcal{T} \leq N \} \]

and the unique solution $\Phi_o \in \mathcal{P}^p(\mathcal{T}_o)$ to (32) exists.

If $\mathcal{T}_N(\mathcal{T}) = \emptyset$ for some $N \in \mathbb{N}$, then we set $\min_{\mathcal{T}_o \in \mathcal{T}_N(\mathcal{T})} \eta_o = 0$. For any $s > 0$, we define the abstract approximation class

\begin{equation}
\|\phi\|_{\Lambda_s(\mathcal{T})} := \sup_{N \in \mathbb{N}_0} \left( (N + 1)^s \min_{\mathcal{T}_o \in \mathcal{T}_N(\mathcal{T})} \eta_o \right),
\end{equation}

where $\eta_o$ denotes the estimator corresponding to the optimal mesh $\mathcal{T}_o \in \mathcal{T}_N$. To abbreviate notation, we define $\mathcal{T}_N := \mathcal{T}_N(\mathcal{T}_0)$ and $\|\phi\|_{\Lambda_s} := \|\phi\|_{\Lambda_s(\mathcal{T}_0)}$. That means that, if
In particular, this implies convergence (45) in Appendix B and improves \cite{BHP17}.

Moreover, there exist \(\eta\) for the sake of completeness, a rigorous proof of Theorem 10 is given (46) in general, the sequence of corresponding optimal spaces \(P^n(T)\) is not necessarily nested.

5.2. Optimal convergence rates. The next theorem is the main result of this work. It proves that Algorithm 7 does not only lead to convergence as well as linear convergence of the sequence of solutions, but also guarantees optimal algebraic convergence rates for the sequence of a posteriori error estimators.

**Theorem 10.** Let \((T)\in\mathbb{N}_0\) and \((\eta)\in\mathbb{N}_0\) be the sequences of meshes and corresponding estimators produced by Algorithm 7. Let \(0 < \theta \leq 1\). Then, there exist constants \(0 < q_{\text{lin}} < 1\) and \(C_{\text{lin}} > 0\) as well as \(\ell_{\text{lin}} \in \mathbb{N}_0\) such that Algorithm 7 guarantees that

\[
\eta_{\ell+n} \leq C_{\text{lin}} q_{\text{lin}}^n \eta_{\ell} \quad \text{for all } \ell, n \in \mathbb{N} \text{ with } \ell \geq \ell_{\text{lin}}.
\]

In particular, this implies convergence (47)

\[
\lim_{\ell \to \infty} \|\phi - \Phi_{\ell}\|_{H^{-1/2}(\Gamma)} = \lim_{\ell \to \infty} \eta_{\ell} = 0.
\]

Moreover, there exist \(C_{\text{Cea}} \in \mathbb{N}_0\) and \(C_{\ell} \geq 1\) with \(\lim_{\ell \to \infty} C_{\ell} = 1\) such that the sequence of discrete solutions \(\Phi_{\ell} \in \mathcal{P}^p(T)\) satisfies

\[
\|\phi - \Phi_{\ell}\| \leq C_{\ell} \min_{\psi_{\ell} \in \mathcal{P}^p(T)} \|\phi - \psi_{\ell}\| \quad \text{for all } \ell \geq C_{\text{Cea}}.
\]

Moreover, there exists \(\hat{\beta} > 0\), \(\ell_{\text{opt}} > 0\), as well as \(\theta_{\text{opt}} := (1 + C_{\text{stb}}^2 C_{\text{rel}}^2 \hat{\beta})^{-1}\), such that for all \(0 < \theta < \theta_{\text{opt}}\) and all \(s > 0\), it holds that

\[
\|\phi\|_{L^s} < \infty \iff \exists C_{\text{opt}} > 0 \quad \forall \ell \geq \ell_{\text{opt}} \quad \eta_{\ell} \leq C_{\text{opt}} (\#T_{\ell} - \#T_0 + 1)^{-s}.
\]

The constant \(C_{\text{opt}}\) depends only on \(C_{\text{son}}, \varphi_{\text{son}}, T_0, \theta, s\), and on the constants in Lemma 9.

The proof follows ideas of \cite[Section 4.3]{BHP17}, where we exploit the estimator properties of Lemma 9. For the sake of completeness, a rigorous proof of Theorem 10 is given in Appendix B and improves \cite{BHP17}.

**Remark 11.** For the presentation, we focus on the model problem (2) for some indirect BEM. In the case of a direct boundary element approach, the model problem reads

\[
V_k \phi = (K_k + \frac{1}{2} \text{Id}) g \quad \text{on } \Gamma,
\]

where \(g \in H^{1/2}(\Gamma)\) is the given Dirichlet data and \(\phi = \partial_n u \in H^{-1/2}(\partial \Omega)\) is the sought normal derivative of the solution \(u \in H^1(\Omega)\) of the (equivalent) boundary value problem

\[-\Delta u - k^2 u = 0 \quad \text{in } \Omega \quad \text{subject to } \quad u = g \quad \text{on } \Gamma.
\]

The implementation of the right-hand side requires to approximate \(g \approx G_{\bullet} \in S^{p+1}(T)\).

Suitable approximations \(G_{\bullet} = I_{\bullet} g\) together with some local data oscillations which control the approximation error \(\|g - G_{\bullet}\|_{H^{1/2}(\partial \Omega)}\), are discussed and analyzed for the Laplace problem in \cite{FFK+14}. Provided that \(g \in H^1(\partial \Omega)\), it is shown that the adaptive algorithm then still leads to optimal convergence behavior. Together with the present analysis, the results of \cite{FFK+14} transfer immediately to the direct boundary element approach (49).
6. Hypersingular Integral Equation

In this section, we briefly comment on the extension of our analysis to the hypersingular integral equation. In case of the Laplace equation \( k = 0 \), a proof of optimal algebraic convergence rates is found in [FFK+15, Tso13]. Throughout this section, we additionally suppose that \( \partial \Omega \) is connected. The model problem reads as follows: Given \( f \in H^{-1/2}(\partial \Omega) := \{ \phi \in H^{-1/2}(\partial \Omega) : \langle \phi, 1 \rangle = 0 \} \) and the hypersingular operator \( W_k := -\gamma_1^{\text{int}} K_k \), find \( u \in H^{1/2}(\partial \Omega) := \{ v \in H^{1/2}(\partial \Omega) : \langle 1, v \rangle = 0 \} \) such that

\[
W_k u = f \quad \text{on } \partial \Omega.
\]

The proof of convergence as well as optimal convergence rates for the related adaptive boundary element method follows similar to the one for the weakly singular integral equation. Therefore, we focus only on the differences and highlight the necessary modifications. For the Laplace case \( k = 0 \), we also refer to [FFK+15].

6.1. Framework. The operator \( W_0 \) is symmetric and positive semi-definite on \( H^{1/2}(\partial \Omega) \):

\[
\langle W_0 v, w \rangle = \langle W_0 w, v \rangle \quad \text{and} \quad \langle W_0 v, v \rangle \geq 0 \quad \text{for all } v, w \in H^{1/2}(\partial \Omega).
\]

Since \( \partial \Omega \) is connected, the kernel of \( W_0 \) are the constant functions, and the bilinear form \( \langle W_0(\cdot), \cdot \rangle \) provides a scalar product on \( H^{1/2}(\partial \Omega) \). Hence, this can be expanded to

\[
a(u, v) := \langle W_0 v, w \rangle + \langle 1, v \rangle \langle 1, w \rangle \quad \text{for all } v, w \in H^{1/2}(\partial \Omega),
\]

which is a scalar product on \( H^{1/2}(\partial \Omega) \). According to the Rellich compactness theorem, there holds the norm equivalence \( \| v \|_1^2 := a(v, v) \simeq \| v \|_{H^{1/2}(\partial \Omega)}^2 \) for all \( v \in H^{1/2}(\partial \Omega) \).

For \( k \neq 0 \), it is well-known that the hypersingular integral operator \( W_k \) is invertible, if and only if \( k^2 \) is not an eigenvalue of the interior Neumann problem (see [Ste13, Proposition 2.5]), i.e., it holds that

\[
\text{(INP)} \quad \forall u \in H^1(\Omega) \quad \left( \Delta u = k^2 u \text{ with } \gamma_1^{\text{int}} u = 0 \text{ and } \int u \, dx = 0 \Longrightarrow u = 0 \text{ in } \Omega \right);
\]

To ensure solvability, we assume throughout that \( k^2 \) satisfies (INP). On Lipschitz boundaries \( \partial \Omega \), the operator \( \tilde{C}_{W_k} := W_k - W_0 : H^{1/2}(\partial \Omega) \to H^{-1/2}(\partial \Omega) \) is compact; see [SS11, Lemma 3.9.8]. Define \( \langle C_{W_k} v, w \rangle := \langle \tilde{C}_{W_k} v, w \rangle - \langle 1, v \rangle \langle 1, w \rangle \) for all \( v, w \in H^{1/2}(\partial \Omega) \).

Note that, \( C_{W_k} : H^{1/2}(\partial \Omega) \to H^{-1/2}(\partial \Omega) \) is uniquely defined and compact. Reformulation of (50) yields the following equivalent formulation: Given \( f \in H^{-1/2}(\partial \Omega) \), find \( u \in H^{1/2}(\partial \Omega) \) such that

\[
a(u, v) + \langle C_{W_k} u, v \rangle = \langle f, v \rangle \quad \text{for all } v \in H^{1/2}(\partial \Omega).
\]

The corresponding discrete formulation of (51) reads as: Find \( U_\bullet \in S^0(T_\bullet) \) such that

\[
a(U_\bullet, V_\bullet) + \langle C_{W_k} U_\bullet, V_\bullet \rangle = \langle f, V_\bullet \rangle \quad \text{for all } V_\bullet \in S^0(T_\bullet).
\]

Then, the weak formulation (51) together with its Galerkin formulation (52) fits in the abstract framework of Proposition 5. Hence, existence and uniqueness of solutions of (52) are guaranteed in the sense of Proposition 5.

6.2. Error estimator. Analogously to \( H^{k\pm 1/2}(\partial \Omega) \), we define \( L^2_k(\partial \Omega) := \{ \phi \in L^2(\partial \Omega) : \int_{\Gamma} f \, ds = 0 \} \). Let \( T_\bullet \in T := \text{refine}(T_0) \) be a triangulation such that the
corresponding discrete solution \( U_\bullet \in S^p(\mathcal{T}_\bullet) \) of (52) exists. Suppose that \( f \in L^2_2(\partial \Omega) \). Then, the local contributions of the weighted-residual error estimator for the hypersingular integral equation are defined by

\[
\eta_\bullet(T) := \| h^{1/2}_\bullet(f - W_k U_\bullet) \|_{L^2(T)} \quad \text{for all } T \in \mathcal{T}_\bullet.
\]

The proofs of the estimator properties in Lemma 9 (i.e., stability on non-refined domains, reduction on refined element domains, discrete reliability as well as reliability) are similar to the ones for the weakly-singular case and can be found in [FFK+15, Proposition 3.5]. The main difference is the use of the inverse inequality (12) instead of (9).

6.3. Optimal convergence rates. We apply Algorithm 7 to the model problem (51). Recall that the error estimator (53) satisfies Lemma 9 and model problem (51) fits in the abstract setting of [BHP17]. Verbatim argumentation as for the weakly-singular case proves Theorem 10 for the hypersingular integral equation. For details, see [Hab18].

Similarly to Remark 11, one may consider a direct formulation for the Neumann boundary-value problem. In this case, the model problem reads as follows: Given Neumann data \( \phi \in H^{-1/2}(\partial \Omega) \), find \( u \in H^{1/2}(\partial \Omega) \) such that

\[
W_k u = (\frac{1}{2}\text{Id} - K'_k) \phi \quad \text{on } \partial \Omega.
\]

In practice, the implementation of the right-hand side requires to approximate \( \phi \approx \Phi_\bullet \in P^{p-1}(\mathcal{T}_\bullet) \). Provided that \( \phi \in L^2(\partial \Omega) \), a suitable approximation \( \Phi_\bullet := \Pi_\bullet \phi \) is given by the \( L^2 \)-orthogonal projection onto \( P^{p-1}(\mathcal{T}_\bullet) \). The local data oscillations which control the additional approximation error \( \| \phi - \Phi_\bullet \|_{H^{-1/2}(\Gamma)} \) are discussed and analyzed in [FFK+15] for the Laplace problem. There, it is shown that the adaptive algorithm then still leads to optimal convergence behavior. Together with the present analysis, the results of [FFK+15] transfer immediately to the direct boundary element approach (54).

7. Numerical Experiments

In this section, we present some numerical experiments for the 3D Helmholtz equation that underpin the theoretical findings of this work. We use lowest-order BEM and consider \( X_\bullet = \mathcal{P}^0(\mathcal{T}_\bullet) \) for the weakly-singular integral equation and \( X_\bullet = S^1(\mathcal{T}_\bullet) \) for the hypersingular equation. The numerical computations were done with help of BEM++, which is an open-source Galerkin boundary element library. We refer to [SBA+15, GBB+15, vWGBA15] for details on BEM++.

We consider sound-soft (exterior Dirichlet) and sound-hard (exterior Neumann) acoustic scattering problems in \( \mathbb{R}^3 \setminus \Omega \), where \( \Omega \subset \mathbb{R}^3 \) denotes the scatterer. Let \( a \in \mathbb{R}^3 \) with \( |a| = 1 \) denote the directional vector of the incident wave. Then, the incident (plane-) wave is given by \( u^{\text{inc}} = \exp(ika \cdot x) \). Let \( u^{\text{scat}} \) be the scattered field and the resulting total field is defined by \( u^{\text{tot}} = u^{\text{inc}} + u^{\text{scat}} \).

For the sake of simplicity, we restrict the numerical examples to an indirect approach, in which the solution is in the form of a layer potential with some unknown density. For the sound-soft scattering problem, we obtain: Find \( u^{\text{scat}} = V_k(\phi) \) such that

\[
V_k \phi = g \quad \text{subject to} \quad g = -u^{\text{inc}} \quad \text{on } \partial \Omega.
\]
The indirect approach for the sound-hard reads: Find \( u^{\text{scat}} = \tilde{K}_k(\phi) \) such that
\[
W_k \phi = g \quad \text{subject to} \quad g = -\partial_n u^{\text{inc}} \quad \text{on } \partial \Omega.
\]

7.1. Sound-soft scattering on a L-shaped domain (non-convex case). As first numerical example, we consider a so called L-shaped domain in \((x,y)\)-direction and expand it on the \(z\)-axis up to \([-1,1]\) (Figure 1). We compare two directions of the incident wave. One with \( a = (-1/\sqrt{2},1/\sqrt{2},0)^T \) (Figure 2, left) hitting the scatterer on the non-convex part vs. \( a = (1/\sqrt{2},-1/\sqrt{2},0)^T \) hitting the convex part of \( \Omega \) (Figure 2, right).

7.1.1. Non-convex case. First, we comment on the non-convex case. Figure 3 (left) shows the convergence rate of \( \eta^2 \) for \( k = 1 \) and different marking strategies. We compare uniform refinement to standard Dörfler marking (i.e., with Algorithm 7 without Step (v)) as well as to the expanded Dörfler marking (Algorithm 7), both using \( \theta = 0.4 \). The experiments show that uniform mesh-refinement leads to a suboptimal rate of \( \mathcal{O}(N^{-2/3}) \) for \( \eta^2 \), while adaptive refinement with Algorithm 7 leads to the improved rate \( \mathcal{O}(N^{-\delta}) \) with \( \delta = 1.075 \), independently of the actual marking. Empirically, the results generated by employing the standard Dörfler marking are of no difference compared to the results generated by employing the expanded Dörfler marking. The same observation is made for all computations (not displayed).

Figure 3 (right) compares uniform vs. adaptive refinement for fixed \( \theta = 0.4 \) but various \( k \in \{1,2,4,8,16\} \). As expected, the preasymptotic phase increases with \( k \), but adaptive mesh-refinement asymptotically regains improved convergence rates for every \( k \). For \( k = 8 \) and \( k = 16 \), the last mesh of the preasymptotic phase is marked with a black symbol. Table 1 displays the number of elements per wavelength, when asymptotic convergence behavior kicks in.

<table>
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<tr>
<th>( k )</th>
<th>#( \mathcal{T}_* )</th>
<th>max ( \text{el. per } \lambda )</th>
<th>min ( \text{el. per } \lambda )</th>
<th>#( \mathcal{T}_* )</th>
<th>max ( \text{el. per } \lambda )</th>
<th>min ( \text{el. per } \lambda )</th>
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<td>0.125</td>
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<td>0.125</td>
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<td>0.5</td>
<td>1.57</td>
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<tr>
<td>8-uniform</td>
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<td>0.25</td>
</tr>
<tr>
<td>8-adaptive</td>
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<td>0.5</td>
<td>0.5</td>
<td>1.57</td>
<td>126</td>
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<td>0.5</td>
<td>3.14</td>
<td>56</td>
<td>0.5</td>
</tr>
</tbody>
</table>

**Table 1.** Ex. 7.1.1 and Ex. 7.1.2. Number of elements per wavelength (el. per \( \lambda \)), on the surface-part hit by the incoming wave, for the last mesh of the preasymptotic phase. The corresponding meshes are marked by black symbols in Figure 3 and Figure 6. Here max and min denote the maximal and minimal diameter in \((x,y)\)-direction. For adaptive meshes, the number of elements per \( \lambda \) is computed with the maximal diameter. For \( k \leq 4 \) the asymptotic phase starts with \( \mathcal{T}_0 \). For \( k = 16 \) the coarsest uniform refinement with 6 elements per wavelength has 1792 elements.

Figure 5 compares the convergence of the estimator for different values of the marking parameter \( \theta \in \{0.2,0.4,0.6,0.8\} \) as well as for uniform mesh-refinement. Again, uniform mesh-refinement leads to a suboptimal rate of convergence for the error estimator,
while adaptive refinement with Algorithm 7 regains the improved rate of convergence, independently of the actual choice of the marking parameter. Although Theorem 10 predicts optimal convergence rates only for small marking parameters \(0 < \theta < \theta_{opt} := (1 + C^2_{stab}/C^2_{rel}/\hat{\beta})^{-1}\), we observe that Algorithm 7 is stable with respect to \(\theta\), where we tested \(\theta \in \{0.2, 0.4, 0.6, 0.8\}\). In Figure 4, one can see some of the obtained adaptive meshes \(T_\ell\) with \(\ell = 4, 8, 12\). The mesh-refinement is focused around the facets and edges hit by the incoming wave, while all facets in the shadow essentially remain coarse.

Figure 7 illustrates the condition number of the arising linear system in (32). As expected, the condition number grows with progressing mesh-adaptation, but stays bounded in the first couple of steps. This indicates that the linear system in the discrete formulation (32) allows for a unique solution for every \(\ell \in \mathbb{N}\). Hence, Algorithm 7 never enforced uniform mesh-refinement in Step (i).

7.1.2. Convex case. In the second case, the scatterer is hit on the convex part of the domain (Figure 2, right), we compute very similar results as in the non-convex case. As shown in Figure 6, the expanded as well as the standard Dörfler marking both lead to improved rates of \(O(N^{-1.06})\) for the \(\eta^2\), while uniform refinement leads only to \(O(N^{-2/3})\). The rate of convergence is independent of the wavelength \(k > 0\), but increasing \(k\) leads to a longer preasymptotic phase. Figure 8 shows the triangulation \(T_{16}\). Again, the mesh-refinement is focused around the facets and edges hit by the incoming wave. All facets in the shadow essentially remain coarse.

7.2. Sound-hard scattering on a L-shaped domain. For the second example, we consider sound-hard scattering on an L-shaped domain from Figure 1. The direction of the incident wave is given by \(a = (-1/\sqrt{2}, 1/\sqrt{2}, 0)^T\), hitting the scatterer on the non-convex part; see Figure 9 (left).

Figure 10 (left) compares uniform vs. adaptive mesh-refinement for fixed \(k = 1\) and various \(\theta = \{0.2, 0.4, 0.6, 0.8\}\). Algorithm 7 leads to the improved rate \(O(N^{-1})\) for \(\eta^2\), while uniform mesh-refinement leads to a reduced rate \(O(N^{-2/3})\). Figure 11 shows the
Figure 2. Ex. 7.1.1 and Ex. 7.1.2: Total field $u^{\text{tot}}$ at the plane $z = 0$ for different directions of $u^{\text{inc}}$ with $k = 8$. The incident wave hits the scatterer on the non-convex part (left) and on the convex part (right).

Figure 3. Ex. 7.1.1: Convergence of $\eta^2_{\ell}$ for standard Dörfler marking vs. expanded Dörfler and uniform refinement with $k = 1$ (left). Expanded Dörfler marking (squares) vs. uniform refinement (circles) for different values of $k > 0$ (right). Both plots are computed with $\theta = 0.4$. The black symbols mark last meshes of the preasymptotic phase for $k = 8$ and $k = 16$.

The adaptive rate for various $k \in \{1, 2, 4, 8, 16\}$ and fixed $\theta = 0.2$ (left) as well as $\theta = 0.4$ (right). A higher wavenumber $k$ just influences the invoked constants and the length of the preasymptotic phase, but does not effect the rate of convergence. For $k = 16$, we admit that the computed range is not sufficient to observe a better rate of convergence for the adaptive scheme. Finally, Figure 10 (right) plots the condition number of the
Figure 4. Ex. 7.1.1: Triangulations $T_4$, $T_8$, and $T_{12}$ with 208,766 and 2332 elements. The color indicates the element contribution of the error estimator $\eta_\ell(T)^2$ for all $T \in T_\ell$.

Figure 5. Ex. 7.1.1: Convergence of $\eta^2_\ell$ for different values of $\theta \in \{0.2, 0.4, 0.6, 0.8\}$ as well as for uniform refinement. Both plots use expanded Dörfler marking with $k = 1$ (left) and $k = 16$ (right).

arising linear system of the discrete formulation (52). Similar to sound-soft scattering, the condition number indicates that the linear system admits a unique solution for every $\ell \in \mathbb{N}$ and hence Algorithm 7 never enforced uniform mesh-refinement in Step (i).

Appendix A. Proof of Lemma 9.

Proof of Lemma 9 (i). Let $T_\bullet, T_\circ \in \mathbb{T}$ such that $T_\circ \in \text{refine}(T_\bullet)$ and the corresponding discrete solutions $\Phi_\bullet \in \mathcal{P}^p(T_\bullet)$ and $\Phi_\circ \in \mathcal{P}^p(T_\circ)$ exist. For all non-refined elements $T \in T_\bullet \cap T_\circ$, it holds that $h_\bullet(T) = h_\circ(T)$. Let $\mathcal{U} := T_\bullet \cap T_\circ$. Together with the inverse
triangle inequality and the inverse estimate (13), we obtain that
\[
|\eta(U) - \eta(\mathcal{U})| = \left| \| h^{1/2} \nabla_{\Gamma} (V_k \Phi - f) \|_{L^2(\mathcal{U} \setminus \mathcal{C}^\star)} - \| h^{1/2} \nabla_{\Gamma} V_k (\Phi - f) \|_{L^2(\mathcal{U} \setminus \mathcal{C}^\star)} \right|
\leq \| h^{1/2} \nabla_{\Gamma} V_k (\Phi - \Phi_0) \|_{L^2(\Gamma)} \leq C_{\text{inv}} (1 + k^3) \| \Phi - \Phi_0 \|_{H^{-1/2}(\Gamma)}.
\]

**Figure 6.** Ex. 7.1.2: Convergence of $\eta^2$ for standard Dörfler marking vs. expanded Dörfler with $\theta = 0.4$ and uniform refinement (left). Expanded Dörfler marking (squares) vs. uniform refinement (circles) for different values of $k > 0$ (right).

**Figure 7.** Ex. 7.1.1: Condition number of the arising linear system. The condition number for the first 20 adaptive steps for non-convex case (left) and the convex case (right).
This concludes (41) with $C_{\text{stab}} := (1 + k^3) \tilde{C}_\text{inv}$. \hfill ∎

**Proof of Lemma 9 (ii).** Let $\mathcal{T}_\bullet, \mathcal{T}_o \in \mathcal{T}$ such that $\mathcal{T}_o \in \text{refine}(\mathcal{T}_\bullet)$ and the corresponding discrete solutions $\Phi_\bullet \in \mathcal{P}^p(\mathcal{T}_\bullet)$ and $\Phi_o \in \mathcal{P}^p(\mathcal{T}_o)$ exist. For all $T \in \mathcal{T}_o \setminus \mathcal{T}_\bullet$, reduction of the local mesh size implies that $h_o|_T \leq q_{\text{mesh}} h_\bullet|_T$. Using the Young inequality with arbitrary $\delta > 0$, we estimate

$$\eta_o(\mathcal{T}_o \setminus \mathcal{T}_\bullet)^2 = \sum_{T \in \mathcal{T}_o \setminus \mathcal{T}_\bullet} \|h_o^{1/2} \nabla \Gamma (V_k \Phi_o - f)\|_{L^2(T)}^2 \leq \sum_{T \in \mathcal{T}_o \setminus \mathcal{T}_\bullet} \left( \|h_o^{1/2} \nabla \Gamma (V_k \Phi_\bullet - f)\|_{L^2(T)} + \|h_o^{1/2} \nabla \Gamma V_k (\Phi_o - \Phi_\bullet)\|_{L^2(T)} \right)^2 \leq \sum_{T \in \mathcal{T}_o \setminus \mathcal{T}_\bullet} \left( (1 + \delta) q_{\text{mesh}} \|h_o^{1/2} \nabla \Gamma (V_k \Phi_\bullet - f)\|_{L^2(T)}^2 + (1 + \delta^{-1}) \|h_o^{1/2} \nabla \Gamma V_k (\Phi_o - \Phi_\bullet)\|^2_{L^2(T)} \right).$$
Figure 10. Ex. 7.2: Convergence of $\eta^2_k$ for different values of $\theta \in \{0.2, 0.4, 0.6, 0.8\}$ as well as uniform refinement (left). The plot uses expanded Dörfler marking with $k = 1$. Condition number of the linear system in (52) (right).

Figure 11. Ex. 7.2: Convergence of $\eta^2_k$ for expanded Dörfler (squares) vs. uniform refinement (circles) for different values of $k \in \{1, 2, 4, 8, 16\}$. The computations use $\theta = 0.2$ (left) as well as $\theta = 0.4$ (right).
Next, the inverse inequality (13) yields that
\[
\eta_\ast(T_0 \setminus T_\ast)^2 \leq (1 + \delta) \eta_{\text{mesh}} \eta_\ast(T_0 \setminus T_\ast)^2 + (1 + \delta^{-1}) (1 + k^3)^2 \tilde{C}_{\text{inv}}^2 \|\Phi_\ast - \Phi_0\|_{\tilde{H}^{-1/2}(\Gamma)}^2.
\]
Choosing \(\delta > 0\) sufficiently small such that \(q_{\text{red}} := (1 + \delta) q_{\text{mesh}} < 1\), we conclude (42) with \(C_{\text{red}} = (1 + \delta^{-1}) (1 + k^3)^2 \tilde{C}_{\text{inv}}^2\).

**Proof of Lemma 9 (iii).** We follow the arguments from [FKMP13, Theorem 5.3] for the case of \(k = 0\). Recall that notation of Proposition 33. Then, existence and uniqueness of \(\Phi_0 \in \mathcal{P}^p(T_0)\) is equivalent to \(\beta_0 > 0\). The discrete inf–sup condition (35) for \(\mathcal{X}_0 := \mathcal{P}^p(T_0)\) and \(W_0 := \Phi_0 - \Phi_\ast\) reads as

\[
(57) \quad \beta_0 \|\Phi_0 - \Phi_\ast\|_{\tilde{H}^{-1/2}(\Gamma)} \leq \sup_{\Psi_0 \in \mathcal{X}_0 \setminus \{0\}} \frac{\langle V_k(\Phi_0 - \Phi_\ast), \Psi_0 \rangle}{\|\Psi_0\|_{\tilde{H}^{-1/2}(\Gamma)}}.
\]

Let \(\mathcal{N}_\ast\) denote the set of nodes corresponding to a triangulation \(T_\ast\). Let \(\rho_z \in \mathcal{S}^1(T_\ast)\) denote the hat function associated with a node \(z \in \mathcal{N}_\ast\). Further, let \(\mathcal{N}_\ast^R := \mathcal{N}_\ast \cap (\bigcup (T_\ast \setminus T_\ast))\) be the set of all nodes which belong to the refined elements. Define \(\mathcal{R}_{\ast,0} := \omega_\ast(T_\ast \setminus T_0)\) and \(\mathcal{Q}_\ast := \mathcal{R}_{\ast,0} \setminus (T_\ast \setminus T_0)\). These definitions give rise to disjoint decompositions

\[
\mathcal{R}_{\ast,0} = (T_\ast \setminus T_0) \cup \mathcal{Q}_\ast \quad \text{and} \quad \mathcal{T}_\ast = (T_\ast \setminus \mathcal{R}_{\ast,0}) \cup (T_\ast \setminus T_0) \cup \mathcal{Q}_\ast.
\]

Define \(\chi := \sum_{z \in \mathcal{N}_\ast^R} \rho_z\). Then, \(\chi \in \mathcal{S}^1(T_\ast)\) satisfies \(\text{supp}(\chi) = \bigcup \mathcal{R}_{\ast,0}\) and \(\chi|_{\bigcup (T_\ast \setminus T_0)} \equiv 1\).

We define the operator \(\pi_\ast : \mathcal{P}^p(T_0) \rightarrow \mathcal{P}^p(T_\ast)\) by

\[
\pi_\ast(\Psi_0) := \begin{cases} 0 & \text{on } \bigcup (T_\ast \setminus T_0), \\ \Psi_0 & \text{elsewhere}. \end{cases}
\]

For any \(\Psi_0 \in \mathcal{P}^p(T_0)\) and \(\Psi_\ast \in \mathcal{P}^p(T_\ast)\), the Galerkin orthogonality yields that

\[
(58) \quad \langle V_k(\Phi_0 - \Phi_\ast), \Psi_0 \rangle = \langle f - V_k \Phi_\ast, \Psi_0 \rangle = \langle f - V_k \Phi_\ast, \Psi_0 - \Psi_\ast \rangle.
\]

Choose \(\Psi_\ast := \pi_\ast(\Psi_0) \in \mathcal{P}^p(T_\ast)\) and note that \(\text{supp}((1 - \pi_\ast) \Psi_0) \subseteq \bigcup (T_\ast \setminus T_0)\). Using (58), we derive that

\[
\langle V_k(\Phi_0 - \Phi_\ast), \Psi_0 \rangle = \langle f - V_k \Phi_\ast, (1 - \pi_\ast) \Psi_0 \rangle = \left\langle \sum_{z \in \mathcal{N}_\ast^R} \rho_z (f - V_k \Phi_\ast), (1 - \pi_\ast) \Psi_0 \right\rangle
\]

\[
= \left\langle \sum_{z \in \mathcal{N}_\ast^R} \rho_z (f - V_k \Phi_\ast), \Psi_0 \right\rangle - \left\langle \sum_{z \in \mathcal{N}_\ast^R} \rho_z (f - V_k \Phi_\ast), \Psi_0 \right\rangle.
\]

Since \(\mathcal{Q}_\ast \subset T_\ast \cap T_0\), we obtain that \(h_\ast(T) = h_\ast(T)\) for all \(T \in \mathcal{Q}_\ast\). We estimate

\[
|\langle V_k(\Phi_0 - \Phi_\ast), \Psi_0 \rangle| \leq \left\| \sum_{z \in \mathcal{N}_\ast^R} \rho_z (f - V_k \Phi_\ast) \right\|_{H^{1/2}(\Gamma)} \|\Psi_0\|_{\tilde{H}^{-1/2}(\Gamma)}
\]

\[
+ \left\| h_\ast^{-1/2} \sum_{z \in \mathcal{N}_\ast^R} \rho_z (f - V_k \Phi_\ast) \right\|_{L^2(\Gamma)} \|h_\ast^{1/2} \Psi_0\|_{L^2(\Gamma)}.
\]
Applying the inverse estimate (28) to the right-hand side, we see that
\[
\langle V_k(\Phi_o - \Phi_*), \Psi_o \rangle \lesssim \left( \left\| \sum_{z \in N^R_o} \rho_z (f - V_k \Phi_*) \right\|_{H^{1/2}(\Gamma)}^2 + \left\| h_o^{-1/2} \sum_{z \in N^R_o} \rho_z (f - V_k \Phi_*) \right\|_{L^2(\Gamma)} \right) \| \Psi_o \|_{\tilde{H}^{-1/2}(\Gamma)}.
\]

The terms in the parentheses are estimated as in [CMS01]. The sole difference is that compared to [CMS01, Theorem 3.2] only hat functions associated with nodes \( z \in N^R_o \) are involved. Hence, the upper bound affects only \( \bigcup R_{*,o} \subset \Gamma \) and reads
\[
(59) \quad \langle V_k(\Phi_o - \Phi_*), \Psi_o \rangle \lesssim h_o^{-1/2} \nabla_{\Gamma} (f - V_k \Phi_*) \| L^2(\bigcup R_{*,o}) \| \Psi_o \|_{\tilde{H}^{-1/2}(\Gamma)}.
\]

Altogether, the combination of (57)–(59) proves that
\[
\| \Phi_o - \Phi_* \|_{\tilde{H}^{-1/2}(\Gamma)} \leq \frac{1}{\beta_o} \sup_{\Psi_o \in X_o} \frac{\langle V_k(\Phi_o - \Phi_*), \Psi_o \rangle}{\| \Psi_o \|_{\tilde{H}^{-1/2}(\Gamma)}} \lesssim \beta_o^{-1} h_o^{-1/2} \nabla_{\Gamma} (f - V_k \Phi_*) \| L^2(\bigcup R_{*,o}).
\]

This concludes the proof.

**Proof of Lemma 9 (iv).** Let \( \varepsilon > 0 \). Since uniform mesh-refinement yields convergence, we may choose \( T_o \in \text{refine}(T_*) \) such that \( \| \phi - \Phi_o \|_{\tilde{H}^{-1/2}(\Gamma)} \leq \varepsilon \). Lemma 9 (iii) hence proves that
\[
\| \phi - \Phi_* \|_{\tilde{H}^{-1/2}(\Gamma)} \leq \| \phi - \Phi_o \|_{\tilde{H}^{-1/2}(\Gamma)} + \| \Phi_* - \Phi_o \|_{\tilde{H}^{-1/2}(\Gamma)} \leq \varepsilon + C_{rel} \eta_*.
\]

For \( \varepsilon \to 0 \), this concludes the proof. \( \square \)

**Appendix B. Proof of Theorem 10**

The following Section gives a rigorous proof of Theorem 10. In doing so, we fill a gap in the proof of (48) in the abstract setting of [BHP17].

**B.1. Proof of (45)–(47).** To see convergence (46) and, in particular, linear convergence (45) of Algorithm 7, note that Step(v) of Algorithm 7 implies \( \bigcup_{l \geq 0} X_l = \tilde{H}^{-1/2}(\Gamma) \) and hence the well-posedness [BHP17, (A5)] of the Galerkin formulation on the “discrete limit space”. Moreover, Section 4.5 proves [BHP17, (A1)–(A4)]. Additionally, recall that \( a(\cdot, \cdot) := \langle V_0(\cdot), \cdot \rangle_{\Gamma} \) induces an equivalent energy norm \( \| \cdot \| \) on \( \tilde{H}^{-1/2}(\Gamma) \). Then, using \( \mathcal{H} := \tilde{H}^{-1/2}(\Gamma) \) in [BHP17, Proposition 11] and [BHP17, Theorem 19, Theorem 20], we immediately derive (45)–(47).

**B.2. Proof of (48).** First, recall [BHP17, Lemma 21], which recaps some important properties of the mesh refinement.

**Lemma 12.** There exist \( m \in \mathbb{N} \) and \( \gamma > 0 \) such that the \( m \)-times uniform refinement \( T_0 \) of \( T_0 \) satisfies the following properties (a)–(d):

(a) For all \( T_0 \in \text{refine}(T_0) \), the discrete inf–sup constant is bounded from below by \( \gamma_* \geq \gamma > 0 \). In particular, there exists a unique Galerkin solution \( \Phi_* \in \mathcal{P}^p(T_*) \) to (2).
Lemma 13 (optimality of Dörfler marking). Suppose the assumptions of Theorem 10. Then, for all \(0 < \theta < \theta_{\text{opt}}\), there exists some \(0 < \gamma_{\text{opt}} < 1\) such that for all \(T_i \in \text{refine}(T_0)\) and all \(T_0 \in \text{refine}(T_{\star})\), it holds that

\[
\eta_0 \leq \gamma_{\text{opt}} \eta_{\star} \quad \Longrightarrow \quad \theta \eta_{\star}^2 \leq \eta_{\star}(R_{0,\star})^2,
\]

where \(R_{0,\star}\) is the enlarged set of refined elements from (3) in Lemma 9. \(\square\)

Lemma 14. Suppose the assumptions of Theorem 10. Then, there exist constants \(C_1, C_2 > 0\) such that for all \(\ell \geq \ell_3\) and all \(s > 0\), there exists \(R_\ell \subseteq T_\ell\) such that the following holds: \(\|\phi\|_{A_s(T_{\ell_3})} < \infty\), then it holds that

\[
\#R_\ell \leq C_1 \left(C_2 \|\phi\|_{A_s(T_{\ell_3})}\right)^{1/s} \eta_{\ell}^{-1/s}
\]

as well as the Dörfler marking criterion

\[
\theta \eta_{\ell}^2 \leq \eta_{\ell}(R_{\ell})^2.
\]

The constant \(C_2\) depends only on \(\theta, \gamma_0\), and on the constants in Lemma 9. The constant \(C_1\) depends additionally on \(#T_{\ell_3}\) and \(T_0\). \(\square\)

Proof of (48). The implication “\(\iff\)” in (48) follows by definition of the approximation class, cf. [CFPP14, Proposition 4.15]. Hence, we focus on the converse implication “\(\implies\)”.

We suppose that \(\|\phi\|_{A_s} < \infty\). With \(\ell_3\) being the constant from Lemma 12, define \(\ell_{\text{opt}} := \max\{\ell_1, \ell_3\}\). Then, [BHP17, Lemma 23] implies that \(\|\phi\|_{A_s(T_{\ell_{\text{opt}}})} < \infty\). Further, let \(M_\ell\) denote the set of marked elements in the \(\ell\)-th step of Algorithm 7. For \(\ell \geq \ell_{\text{opt}}\), Lemma 14 provides a set \(R_\ell \subseteq T_\ell\) with (61)–(62). According to the minimality of \(M_\ell\) (cf. Step(iv) and Step(v) of Algorithm 7), we obtain that

\[
\#M_\ell \leq \#R_\ell \lesssim \|\phi\|_{A_s(T_{\ell_{\text{opt}}})}^{1/s} \eta_{\ell}^{-1/s} \quad \text{for all } \ell \geq \ell_{\text{opt}}.
\]

Linear convergence (45) yields that \(\eta_{\ell} \lesssim q_{\text{lin}}^{\ell-j} \eta_j\) for all \(\ell_{\text{opt}} \leq j \leq \ell\) and hence

\[
\eta_{\ell}^{-1/s} \lesssim q_{\text{lin}}^{(\ell-j)/s} \eta_j^{-1/s} \quad \text{for all } \ell_{\text{opt}} \leq j \leq \ell.
\]

The mesh-closure estimate (38) yields that

\[
\#T_\ell - \#T_0 + 1 \leq C_{\text{mesh}} \sum_{j=0}^{\ell-1} \#M_j.
\]
Together with $C := \max_{j=0,\ldots,\ell_{opt}} \#M_j$, we obtain that

\[ (66) \quad \sum_{j=0}^{\ell-1} \#M_j = \sum_{j=0}^{\ell_{opt}} \#M_j + \sum_{j=\ell_{opt}}^{\ell-1} \#M_j \leq (\ell_{opt} C + 1) \sum_{j=0}^{\ell-1} \#M_j. \]

Combining (63)–(66) as well as $0 < q := q_{\text{lin}}^{1/s} < 1$, we exploit the geometric series and reveal that

\[ \#T_\ell - \#T_0 + 1 \lesssim \sum_{j=0}^{\ell_{opt}} \#M_j \lesssim \sum_{j=0}^{\ell_{opt}} \#M_j \lesssim \|\phi\|_{A_{\delta}(T_{\ell_{opt}})}^{1/s} \sum_{j=\ell_{opt}}^{\ell-1} \eta_j^{-1/s} \]

\[ \lesssim \|\phi\|_{A_{\delta}(T_{\ell_{opt}})}^{1/s} \eta^{1/s}_{\text{lin}} \sum_{j=\ell_{opt}}^{\ell-1} q_{\text{lin}}^{(\ell-j)/s} \lesssim \|\phi\|_{A_{\delta}(T_{\ell_{opt}})}^{1/s} \eta^{1/s}_{\text{lin}}. \]

Rearranging the terms, we conclude that $\eta_{\ell} \leq C_{\text{opt}} \big( \#T_\ell - \#T_0 + 1 \big)^{-s}$. The constant $C_{\text{opt}} > 0$ is given by

\[ C_{\text{opt}} = C_{\text{lin}} C_2 \|\phi\|_{A_{\delta}(T_{\ell_{opt}})} \left( \frac{2C_{\text{mark}}C_{\text{mesh}}C_1 (\ell_4 C + 1)}{1 - q_{\text{lin}}^{1/s}} \right)^s, \]

where $C_1, C_2$ are the constants from Lemma 14. This concludes the proof. \hfill \Box

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