On nonsingular BVPs with nonsmooth data.
Part 1: Analytical results

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On nonlinear singular BVPs with nonsmooth data. Part 1: Analytical results

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Abstract

We study boundary value problems for systems of nonlinear ordinary differential equations with a time singularity,

\[ x'(t) = \frac{M(t)}{t} x(t) + \frac{f(t,x(t))}{t}, \quad t \in (0, 1], \quad b(x(0), x(1)) = 0, \]

where \( M : [0, 1] \to \mathbb{R}^{n \times n} \) and \( f : [0, 1] \times \mathbb{R}^n \to \mathbb{R}^n \) are continuous matrix-valued and vector-valued functions, respectively. Moreover, \( b : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n \) is a continuous nonlinear mapping which is specified according to a spectrum of the matrix \( M(0) \). For the case that \( M(0) \) has eigenvalues with nonzero real parts, we prove new results about existence of at least one continuous solution on the closed interval \([0, 1]\) including the singular point, \( t = 0 \). We also formulate sufficient conditions for uniqueness. The theory is illustrated by a numerical simulation based on the collocation method.

Keywords: BVPs, ODEs, time singularity, global existence, uniqueness, fixed point theorems
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1. Motivation

In the present work, we focus on the solvability of the singular nonlinear boundary value problems (BVPs),

\[ x'(t) = \frac{M(t)}{t} x(t) + \frac{f(t,x(t))}{t}, \quad t \in (0, 1], \quad b(x(0), x(1)) = 0 \]

and our main aim is to generalize results from \([8, 9]\), where the existence analysis of linear BVPs with constant, \( M \), and variable, \( M(t) \), coefficient matrix were provided, respectively. Important in this context is to note that the existence and uniqueness results for the analytical solution also answer the question of the well-posedeness of the related BVP. This question has to be resolved before turning to the analysis of any numerical algorithm applied to approximate the analytical solution. It is especially difficult to answer for BVPs with singularities, because in this case only boundary condition of a certain structure guarantee that the analytical problem can be successfully approximated by a numerical method. In this sense, the analytical paper \([9]\) can be seen as a necessary prerequisite for \([10]\) where the convergence of polynomial collocation for the variable coefficient case, \( M(t) \), was studied. Thus, the present article in which the boundary conditions, sufficient for the well-posedeness of the nonlinear BVPs, are precisely specified, is the preparation for the respective convergence analysis. At the end of the paper, we shortly discuss the experimentally observed, quite intriguing, behavior of the collocation schemes applied to solve some nonlinear test examples of the above type. In the next paper, On nonlinear singular BVPs with nonsmooth data. Part 2: Convergence of the collocation schemes, we shall analyze and explain this convergence behavior.
2. Introduction

As already mentioned, the aim for the present work is to analyze the solvability of BVPs for systems of nonlinear ordinary differential equations (ODEs) with a time singularity at the origin,

\[ x'(t) = \frac{M(t)}{t} x(t) + \frac{f(t, x(t))}{t}, \quad t \in (0, 1], \]

\[ b(x(0), x(1)) = 0, \tag{2} \]

where \( f \in C([0, 1] \times \mathbb{R}^n; \mathbb{R}^n), M \in C([0, 1]; \mathbb{R}^{n \times n}), b \in C(\mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}^n), \) and \( n \in \mathbb{N}. \)

**Definition 1.** We say that \( x : [0, 1] \to \mathbb{R}^n \) is a solution of system (1) on \([0, 1]\) if \( x \in C([0, 1]; \mathbb{R}^n) \cap C^1((0, 1]; \mathbb{R}^n) \) and (1) holds for \( t \in (0, 1]. \) We say that \( x : [0, 1] \to \mathbb{R}^n \) is a solution of BVP (1), (2), in case that \( x \) is a solution of system (1) and if it satisfies condition (2).

The boundary conditions in (2) are specified according to a spectrum of the matrix \( M(0). \) Throughout the paper, we assume that eigenvalues of \( M(0) \) have nonzero real parts and discuss the existence of solutions to problem (1), (2) which are continuous on \([0, 1], \) including the singular point \( t = 0. \)

The nonlinear boundary value problems of the form (1), (2) with singularities studied here arise in the modelling of snow avalanche run-up and run-out. The leading-edge model describing the dynamics of dry-flowing avalanches is considered in [24, 30, 31]. In this model, five forces are combined to give the total force governing the avalanche’s dynamics; the driving force, momentum flux, dynamic Coulomb resistive force, turbulent resistive force and passive snow pressure force.

The analysis presented in our paper can be seen as the continuation of the work initiated in [11, 13] and continued in [24, 34], where first and second order linear BVPs with singularities were studied. *Linear* models in these papers are typically of the form

\[ x'(t) = \frac{M(t)}{t} x(t) + f(t), \quad t \in (0, 1], \quad B_0 x(0) + B_1 x(1) = c, \tag{3} \]

where \( f \in C([0, 1]; \mathbb{R}^n), B_0, B_1 \in \mathbb{R}^{n \times n}, c \in \mathbb{R}^n. \) The case of nonsmooth inhomogeneity in (3) written as

\[ x'(t) = \frac{M(t)}{t} x(t) + f(t), \quad t \in (0, 1], \quad B_0 x(0) + B_1 x(1) = c, \tag{4} \]

was investigated in full details in [9], where the attention was focused on the existence, uniqueness and smoothness of solutions. It turned out that it is necessary to prescribe boundary conditions of a certain structure to guarantee the well-posedness of the problem. The form of such conditions depends on spectral properties of the matrix \( M(0). \) The convergence analysis of the collocation method applied to solve (4) can be found in [10], while the analytical results and convergence of collocation applied to (4) with constant coefficient matrix \( M \) are provided in [8].

The first analytical framework for the *nonlinear* singular BVPs of the type

\[ x'(t) = \frac{M(t)}{t} x(t) + f(t, x(t)), \quad t \in (0, 1], \quad B_0 x(0) + B_1 x(1) = c, \tag{5} \]

with \( f \in C([0, 1] \times \mathbb{R}^n; \mathbb{R}^n), \) can be found in [11, 13]. *Both papers do not provide the existence and uniqueness proofs. Only, a local argument is given, based on the assumption that an appropriately smooth solution \( x^* \) exists.*

The main results from [11, 13] state that for the sufficiently smooth problem data, the linearized BVP (at \( x^* \)) is uniquely solvable and stable. Those properties are crucial in the context of any discretization approach aiming at the approximate solution of the BVP (5). Since [11, 13] are mainly focused on the numerical approximation of the problem, this *local analysis* is a precisely tailored prerequisite for the convergence theory of the involved numerical methods. Moreover, it accounts for the fact that in applications of the form (5) multiple solutions may arise.

Following papers, [12, 14, 21, 22, 23, 34, 35], provide further analysis of various numerical methods applied to solve singular BVPs of the first and second order. Here, main focus is on collocation methods, error estimation, grid adaptation and software development.
A different analysis of the nonlinear system of type (1) in the form
\[ tu'(t) = g(t, u(t)), \quad t \in (0, T], \]
was considered in [33], where the local existence and uniqueness of a solution in \( C^w([0, T_1]; \mathbb{R}^n) \), \( T_1 \leq T, m \geq 0 \), without prescribing boundary conditions was shown. This result was obtained under the assumption that \( g \in C^w([0, T] \times \mathbb{R}^n; \mathbb{R}^n) \) and that there exists \( \omega_0 \) such that \( g(0, \omega_0) = 0 \). Moreover, the following assumptions are made on the Jacobian: \( A_0(t, u) = \partial g(t, u)/\partial u \in C^w([0, T] \times \mathbb{R}^n; \mathbb{R}^{\infty\times n}) \) and the eigenvalues \( \lambda_k \) of \( A_0(0, \omega_0) \) have to satisfy the restriction \( m > \text{Re} \lambda_k \). For the case \( m = 0 \), equivalent to considering \( g, A_0 \) and \( u \) only continuous (without continuous higher derivatives), the above condition restricts the analysis to negative real parts of the eigenvalues of \( A_0(0, \omega_0) \), \( 0 > \text{Re} \lambda_k \).

We now shall discuss the novelty and importance of the present paper. From the above discussion, it is clear that there is an extensive literature on singular BVPs. However, global existence and uniqueness results are rare. In this paper we provide the missing global analysis for problem (1), (2).

- We discuss the global existence and uniqueness of continuous solutions of the nonlinear singular system (1). In this setting, we provide a complete study of the problem where \( M(0) \) has arbitrary eigenvalues with nonzero real parts and general Jordan canonical form. We point out that our theory precisely describes initial, terminal or boundary conditions, respectively, which are necessary and sufficient for the related IVP, TVP, and BVP to have a solution in \( C[0, 1] \).

- The global results in Theorems 15, 20 and 26 are, to our knowledge, the first such existence results in the literature. They even provide a new insight for the problem (5) where a time singularity occurs only in the linear term because previous papers dealing with (5) only give local arguments based on an assumption that an appropriately smooth solution exists.

- The comparison of the present paper with [33] is only possible in the special case when we look at Lemma 14 in our paper and Theorem 3 in [33]. Both results deal with the local existence and uniqueness of a solution to (1) for \( M(0) \) with eigenvalues whose real parts are negative. However, in [33] more smoothness on the problem data is required to show an analogous statement.

- The now available global theory is an important preparation for the numerical analysis and the following computational treatment, since it provides an essential structural information on the underlying BVP, namely its well-posedness. This property is indispensable for any analytical problem which is subject to numerical treatment. It simply states that the unique solution of the analytical problem depends continuously on the problem data. This, for instance, enables to estimate the effects of the modelling errors and round-off errors, in case that they can be interpreted as small perturbations in the inhomogeneity.

- Moreover, the precise knowledge of initial/terminal/boundary conditions, which are necessary and sufficient for the related IVP/TVP/BVP to be well-posed can also be utilized in applications. It enables the user to verify if the prescribed boundary conditions are correctly stated and in case that they are missing, he knows how to correctly close the system. The resulting IVPs/TVPs/BVPs are then suitable for the numerical treatment.

Another strong motivation for the present analysis is the important class of regular problems posed on the infinite interval \([0, \infty)\) which can be related to the singular problems of type (1), (2) posed on \((0, 1]\). Originally, such problems often have the form,
\[ x'(s) = N(s)x(s) + g(s, x(s)), \quad s \in [0, \infty), \quad b(x(0), x(\infty)) = 0, \]
with continuous \( N, g \) and \( b \). There are numerous applications of this type, where \( s \) takes a role of time, and there are several possibilities to solve problems (6) numerically.

The oldest method is to solve the problem on a finite interval \([0, L]\), where \( L \) is appropriately large and replace the boundary conditions \( b(x(0), x(\infty)) = 0 \) by \( b(x(0), x(L)) = 0 \), see [18]. The disadvantages of this approach are severe. It is often necessary to solve the problem for different values of \( L \) in order to find out the accuracy of the approximation.
Moreover, it may be necessary to solve the problem on very large intervals to guarantee a reasonable quality of the numerical solution, cf. [28], where $L = O(10^3)$. To overcome these difficulties, another development was proposed, where the asymptotically correct boundary conditions were derived and imposed at the right boundary $L$ [27, 29]. This method is by no means straightforward and results in highly nonlinear conditions.

Another idea is to use a coordinate transformation and solve the so-called free boundary formulation, see [16]. The main idea of this method is as follows. Let us consider the BVP,

$$x''(s) = g(t, x(s), x'(s)), \quad s \in [0, \infty), \quad x(0) = x_0, \quad x(\infty) = x_\infty.$$  

Then, its free boundary formulation reads:

$$x''_s(t) = g(t, x_s(t), x'_s(t)), \quad t \in [0, t_\epsilon], \quad x_s(0) = x_0, \quad x_s(t_\epsilon) = x_\infty, \quad x'_s(t_\epsilon) = \epsilon,$$

where $t_\epsilon$ is an unknown parameter and $\epsilon \to 0$. Note that this method can be applied to ODEs of at least order two provided that the first derivative of $x$ tends monotonically to zero at infinity, $\lim_{t \to \infty} x'(t) = 0$. Moreover, usually, we have to solve a series of BVPs for different values of $\epsilon$.

In view of difficulties of approaches based on truncation of the interval of integration, another idea seems to be more promising. It is based on a transformation of the interval $s \in [0, \infty)$ to the finite domain $t \in (0, 1]$. The advantages of working on a finite small interval $[0, 1]$ while discretizing the analytical problem are evident, however with this transformation, we usually introduce a singularity at $t = 0$ and the problem data becomes nonsmooth. For the transformation $t = e^{-s}$, system in (6) takes the form (1) and this means that the resulting singularity is of the first kind, where the power of $t$ in the denominator of the right-hand side is one. Such singularity can in general, be handled more efficiently, when compared to a singularity of the second kind, where the power of $t$ in the denominator of the right-hand side in (1) is larger than one. As an example, we point to a boundary value problem from a theory for the explosive crystallization of thin amorphous layers on a substrate [6, 25, 26]. Here, the aim is to compute the crystallization rate and the temperature distribution of a crystallization front propagating through a thin layer of amorphous material on a substrate. The original problem posed on a semi-infinite interval $s \in [0, \infty)$ when transformed to $t \in [0, 1]$ using $t = e^{s}$, results in a boundary value problem (1) for a system of two nonlinear equations. Using the standard transformation [6, 21], $t = 1 - 1/ \sqrt[3]{1 + s}$, introduces a numerically less advantageous singularity of the second kind.

Motivated by the above discussion, we focus our attention on the singular nonlinear system (1),

$$x'(t) = \frac{M(t)}{t} x(t) + \frac{f(t, x(t))}{t}, \quad t \in (0, 1].$$

Our aim is to specify the boundary conditions in (2) depending on the spectrum of $M(0)$. We first deal with the case of only negative real parts of eigenvalues of $M(0)$ and investigate the associated IVP. Then, we study the case of only positive real parts of eigenvalues of $M(0)$ and analyze a terminal value problem (TVP). Finally, a BVP is studied in the case of both positive and negative real parts of eigenvalues of $M(0)$. In all three cases we intend to prove the existence of solutions which are at least continuous.

The Banach Fixed Point Theorem seems to be very helpful in dealing with difficulties caused by the singularity at $t = 0$. It provides both, the existence and uniqueness of a solution of (1) subject to suitable initial, terminal or boundary conditions. However, certain restrictions have to be imposed on the length of the interval where the unique solvability is guaranteed. This means that we first investigate system (1) on a sufficiently small interval $[0, \delta]$, $0 < \delta \leq 1$. The form of initial, terminal or boundary conditions that specify the unique continuous solution on $[0, \delta]$ depends on the spectral properties of the constant matrix $M(0)$. Then, having the existence and uniqueness on $[0, \delta]$, we investigate a corresponding IVP, TVP or BVP on $[0, 1]$ by means of the Leray-Schauder alternative.

The plan of the paper is organized as follows. Preliminaries are introduced in Section 3. Results for the linear case with a constant coefficient matrix are summarized in Section 4. The main Section 5 is devoted to the analysis of the nonlinear problem.
We first deal with the case of only negative real parts of eigenvalues of $M(0)$. It turns out that the unique continuous solution on $[0, \delta]$ is determined by the following structure of initial condition:

$$M(0)x(0) + f(0, x(0)) = 0. \tag{7}$$

Conditions sufficient for the solvability of the IVP (1), (7) on $[0, 1]$ are presented in Section 5.1, together with the discussion of the unique solvability of the IVP (1), (7). Note that the form of the initial condition (7) follows from the requirement that the solution $x$ of (1) is continuous on the closed interval $[0, 1]$ including the singular point $t = 0$. Hence, we can interpret (7) as the necessary condition for (1) to be the well-posed.

The case where all eigenvalues of the matrix $M(0)$ have positive real parts is considered Section 5.2. In this situation, each solution of equation (1) on $[0, 1]$ satisfies condition (7). Consequently, there is no uniquely solvable IVP and therefore, we have to study the system (1) subject to a terminal condition which is chosen for simplicity as

$$x(1) = c, \quad c \in \mathbb{R}^n. \tag{8}$$

In Section 5.2, conditions for the existence and uniqueness of solutions to the TVP (1), (8) on $[0, 1]$ are formulated. Finally, in Section 5.3, we admit the matrix $M(0)$ to have a mixed spectrum without zero and purely imaginary eigenvalues. Based on the results of Section 5.1 and Section 5.2 for $M(0)$ with only negative and only positive real parts of eigenvalues, respectively, system (1) equipped with the following boundary conditions is studied:

$$NM(0)x(0) + Nf(0, x(0)) = 0, \quad Px(1) = Pc, \quad c \in \mathbb{R}^n, \tag{9}$$

where $N$ and $P$ are appropriately defined projection matrices. We provide existence and uniqueness results for the BVP (1), (9). In Section 6, we numerically simulate three model problems covering the three different spectra of the matrix $M(0)$ analyzed in the paper. Finally, conclusions can be found in Section 8.

3. Preliminaries

Throughout the paper we use $\mathbb{R}^n$ and $\mathbb{C}^n$ to denote the $n$-dimensional vector space of real-valued and complex-valued vectors $x$, respectively, equipped with the maximum norm, $\|x\| = \max\{|x_i|; 1 \leq i \leq n\}$. Similarly, we denote by $\mathbb{R}^{m\times n}$ and $\mathbb{C}^{m\times n}$ the space of real-valued and complex-valued $m \times n$ matrices $B$, respectively, with the norm defined as $\|B\| = \max\{|\sum_{j=1}^{m} |b_{ij}|; 1 \leq i \leq n\}$. Moreover, $I \in \mathbb{R}^{m\times n}$ is the identity matrix and $\Theta \in \mathbb{R}^{m\times n}$ is the zero matrix. Let $\mathcal{F} \subset [0, 1]$, then $C(\mathcal{F}; \mathbb{R}^n)$ is the space of the real-valued functions which are continuous on $\mathcal{F}$, while $C(\mathcal{F}; (0, \infty))$ and $C(\mathcal{F}; [0, \infty))$ are the spaces of continuous positive and nonnegative functions, respectively. We equip the space $C([0, \delta]; \mathbb{R}^n)$, $\delta \in (0, 1]$, with the norm $\|x\|_\delta := \max\{|x(t)|; t \in [0, \delta]\}$ and the space $C((\delta, 1]; \mathbb{R}^n)$, $\delta \in (0, 1)$ with the norm $\|x\|_\delta := \max\{|x(t)|; t \in [\delta, 1]\}$. Finally, $C^1(\mathcal{F}; \mathbb{R}^n)$ is the space of real-valued vector functions which are continuously differentiable on $\mathcal{F}$, $C(\mathcal{F} \times \mathbb{R}^n; \mathbb{R}^n)$ and $C(\mathcal{F}; \mathbb{R}^{m\times n})$ are the spaces of real-valued vector functions and matrix functions, continuous on the particular sets.

The existence of solutions (on the whole interval $[0, 1]$) of the IVP (1), (7), and the TVP (1), (8) is shown using the Leray-Schauder alternative [15] and by the special form of the Bihari inequality [5, 17], formulated in the following lemmas.

**Lemma 2** (Bihari). Let $\delta \in (0, 1)$ and $u \in C([\delta, 1]; [0, \infty))$. Let $w \in C((0, \infty); (0, \infty))$ be nondecreasing, and

$$\int_0^\infty \frac{1}{w(s)} \, ds = \infty.$$

If $u$ satisfies the integral inequality

$$u(t) \leq B_1 + B_2 \int_\delta^t w(u(s)) \, ds, \quad t \in [\delta, 1],$$

where $B_1, B_2$ are positive constants, then

$$u(t) \leq G^{-1}(G(B_1) + B_2(t - \delta)), \quad t \in [\delta, 1],$$

where $G$ is a function defined by

$$G(s) := \int_0^s \frac{1}{w(s)} \, ds.$$
where \( G \) is defined as

\[
G(x) = \int_0^x \frac{1}{w(s)} \, ds, \quad x \in [0, \infty),
\]

and \( G^{-1} \) is the inverse of \( G \).

Note, that if we apply Lemma 2 with \( u(t) \) replaced by \( u(1 + \delta - t) \), we obtain a useful modification of the Bihari inequality.

**Lemma 3.** Let \( \delta, w \) and \( G \) be as in Lemma 2. If \( u \) satisfies the integral inequality

\[
u(t) \leq B_1 + B_2 \int_t^1 w(u(s)) \, ds, \quad t \in [\delta, 1],
\]

where \( B_1, B_2 \) are positive constants, then

\[
u(t) \leq G^{-1}(G(B_1) + B_2(1 - t)), \quad t \in [\delta, 1].
\]

Another important tool used in the proofs is the Leray-Schauder alternative, cf. [15, Cor. 8.1].

**Lemma 4** (Leray-Schauder alternative). Let \( Y \) be a Banach space and \( K : Y \to Y \) a completely continuous operator. Then the following alternative holds:

Either \( x = \lambda Kx \) has a solution for every \( \lambda \in [0, 1] \) or the set \( S = \{ x \in Y : x = \lambda Kx \text{ for some } \lambda \in (0, 1) \} \) is unbounded.

### 4. Linear systems with constant coefficient matrix \( A \)

In this section, we collect the most important results for the linear case with a constant coefficient matrix \( A \). We use these prerequisites for the investigation of the nonlinear system (1) in Section 5.

We first consider the linear homogeneous system

\[
x'(t) = \frac{A}{t} x(t), \quad t \in (0, 1],
\]

with a regular matrix \( A \in \mathbb{R}^{n \times n} \). If \( \Phi \) is the fundamental solution matrix of system (10) satisfying \( \Phi(1) = I \), then \( \Phi \) has the form

\[
\Phi(t) = t^A = \exp(A \ln t) = \sum_{k=0}^{\infty} \frac{A^k(\ln t)^k}{k!}, \quad t \in (0, 1].
\]

Let \( A \) have the eigenvalues \( \lambda_1, \ldots, \lambda_m \in \mathbb{C} \), \( m \leq n \) and let us denote by \( J \) the Jordan canonical form of \( A \). Moreover, let \( n_1, \ldots, n_m \) be the dimensions of the Jordan boxes \( J_1, \ldots, J_m \) corresponding to the (not necessarily different) eigenvalues \( \lambda_1, \ldots, \lambda_m \). Let \( E \in \mathbb{C}^{n \times n} \) be the associated matrix of generalized eigenvectors of \( A \), that is, the matrix transforming \( A \) to its canonical form. Then,

\[
A = EJ\!E^{-1}, \quad r^A = Et^A\!E^{-1}, \quad J = \text{diag}(J_1, \ldots, J_m), \quad t^J = \text{diag}(t^{n_1}, \ldots, t^{n_m}).
\]

The basic properties of the fundamental solution matrix are

\[
Ar^A = r^A A, \quad 1^A = I, \quad t^A \frac{1}{t} = t^{A-1}, \quad t \in (0, 1].
\]

From (10), we see that the matrices \( r^A \) and \( t^{-A} \) satisfy

\[
(r^A)' = \frac{A}{t} r^A, \quad (t^{-A})' = -\frac{A}{t} t^{-A} = -Ar^{-A-1}, \quad t \in (0, 1].
\]
Choose \( k \in \{1, \ldots, m\} \) and consider the eigenvalue \( \lambda_k = \sigma_k + i\rho_k \). Let us denote by \( \Lambda_k(t) \) the following \( n_k \times n_k \) matrix:

\[
\Lambda_k(t) = \begin{pmatrix}
1 & \ln t & \frac{(\ln t)^2}{2} & \cdots & \frac{(\ln t)^{n_k-1}}{(n_k-1)!} \\
0 & 1 & \ln t & \cdots & \frac{(\ln t)^{n_k-2}}{(n_k-2)!} \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & \cdots & 1 \\
\end{pmatrix}, \quad t \in (0, 1].
\]

Then, we have \( r^h = r^h \Lambda_k(t) \), for \( t \in (0, 1] \), and hence

\[
r^{-h} = \left( \frac{1}{r} \right)^h = \left( \frac{1}{r} \right)^h \Lambda_k \left( \frac{1}{r} \right) = r^{-\sigma_k - i\rho_k} \Lambda_k \left( \frac{1}{r} \right), \quad t \in (0, 1].
\]

Since \( \ln (1/r) = -\ln r \geq 0 \) for \( t \in (0, 1] \), we obtain

\[
|\Lambda_k(t)| = \left| \Lambda_k \left( \frac{1}{r} \right) \right| = \sum_{j=0}^{n_k-1} \frac{(-\ln r)^j}{j!}, \quad t \in (0, 1], \tag{14}
\]

\[
|r^h| = r^{|\sigma_k|} |\Lambda_k(t)|, \quad |r^{-h}| = r^{-|\sigma_k|} |\Lambda_k(t)|, \quad t \in (0, 1].
\]

Clearly, from

\[
\lim_{t \to 0^+} r^s (\ln r)^j = 0 \quad \text{for} \quad \alpha > 0, \quad j \in \mathbb{N},
\]

it follows

\[
\sigma_k < 0 \Rightarrow \lim_{t \to 0^+} |r^{-h}| = 0 \quad \text{and} \quad \sigma_k > 0 \Rightarrow \lim_{t \to 0^+} |r^h| = 0. \tag{16}
\]

**Eigenvalues of \( A \) with negative real parts.**

Assume that all eigenvalues \( \lambda_k \) of \( A \) have negative real parts

\[
\lambda_k = \sigma_k + i\rho_k, \quad \sigma_k < 0, \quad k = 1, \ldots, m.
\]

Then, by (11) and (16), we have

\[
\lim_{t \to 0^+} r^A = \Theta. \tag{17}
\]

Integration of (13) over \([\tau, t] \subset (0, 1]\) gives

\[
r^{-A} - r^{-A} = -A \int_\tau^t s^{-A-l} \, ds.
\]

Letting \( \tau \to 0^+ \) and using (17), we obtain

\[
\int_0^t s^{-A-l} \, ds = -A^{-1} r^{-A}, \quad t \in (0, 1],
\]

and it follows by (12)

\[
\int_0^1 s^{-A-l} \, ds = -A^{-1}. \tag{18}
\]

The next lemma is a special case of Lemma 3 and Lemma 4 from [8]. Since its proof is short in this setting, we present it here for a reader’s convenience.

**Lemma 5.** Assume that all eigenvalues \( \lambda_k \) of \( A \) have negative real parts. Then,

\[
\lim_{t \to 0^+} \int_0^t |s^{-J_k-l}| \, ds = 0, \quad \int_0^1 |s^{-J_k-l}| \, ds = \sum_{j=0}^{n_k-1} \frac{1}{(-\sigma_k)^{j+1}}, \quad k = 1, \ldots, m.
\]

where \( I_k \in \mathbb{R}^{n_k \times n_k} \) is the identity matrix.
For $k = 1, \ldots, m$, consider

$$\psi_k(s) = \sum_{j=0}^{n-1} \sum_{p=0}^{j} \frac{s^{(-\ln s)^p}}{p!(-\sigma_k)^{j+1}}, \quad s \in (0, 1].$$  \hspace{1cm} (20)$$

We first show

$$\int_0^t |s^{-j-1}| \, ds = \psi_k(t), \quad t \in (0, 1].$$  \hspace{1cm} (21)$$

For $s = 1$, it follows from (20) that

$$\psi_k(1) = \sum_{j=0}^{n-1} \frac{1}{(-\sigma_k)^{j+1}},$$

and by (15) and (20), we obtain

$$\lim_{x \to 0} \psi_k(x) = 0.$$  

Using (12), (14) and repeated integration by parts, we conclude for $[\tau, t] \subset (0, 1)$

$$\int_{\tau}^t |s^{-j-1}| \, ds = \int_{\tau}^t |s^{1-\psi_k}| \cdot \Lambda_k \left( \frac{1}{s} \right) \, ds = \int_{\tau}^t s^{1-\psi_k} \sum_{j=0}^{n-1} \frac{(-\ln s)^j}{j!} \, ds = \psi_k(t) - \psi_k(\tau).$$

Finally, let $\tau$ tend to zero. Then, (21) holds and this completes the proof.

Next corollary is a direct consequence of Lemma 5 and (11).

**Corollary 6.** Assume that all eigenvalues of $A$ have negative real parts. Then,

$$\int_0^1 |s^{-j-1}| \, ds = \max_{1 \leq k \leq m} \frac{1}{(-\sigma_k)^{j+1}}, \quad \int_0^1 |s^{-\sigma_k-1}| \, ds \leq |E| \cdot |E^{-1}| \cdot \max_{1 \leq k \leq m} \frac{1}{(-\sigma_k)^{j+1}}.$$  \hspace{1cm} (22)$$

**Lemma 7.** Assume that all eigenvalues of $A$ have negative real parts and consider $\delta \in (0, 1]$ and $h \in C([0, \delta]; \mathbb{R}^n)$. Then,

$$\lim_{\ell \to 0^+} \int_0^1 \! s^{-\sigma_k-1} h(s) \, ds = -A^{-1}h(0).$$  \hspace{1cm} (23)$$

**Proof.** Let $\ell \in \mathbb{N}$ and let us define

$$u_\ell(t) = \int_0^t \! s^{-\sigma_k-1} h(s) \, ds, \quad t \in [0, \delta].$$

Clearly, $u_\ell \in C([0, \delta]; \mathbb{R}^n)$ for $\ell \in \mathbb{N}$. We now show that

$$\lim_{\ell \to \infty} \left| u_\ell(t) - \int_0^t \! s^{-\sigma_k-1} h(s) \, ds \right| = 0 \quad \text{uniformly on} \quad [0, \delta].$$  \hspace{1cm} (24)$$

For $t \in [0, \delta]$, we have by (21)

$$\left| \int_0^t \! s^{-\sigma_k-1} h(s) \, ds - \int_0^t \! s^{-\sigma_k-1} h(s) \, ds \right| = \left| \int_0^t \! s^{-\sigma_k-1} h(s) \, ds \right| \leq \int_0^t \! |s^{-\sigma_k-1}| \cdot |h|_0 \, ds \leq |E| \cdot |E^{-1}| \max_{1 \leq k \leq m} \left\{ \psi_k \left( \frac{1}{\ell} \right) \right\} \cdot |h|_0.$$

Moreover, (19) implies that $\lim_{\ell \to \infty} \psi_k(1/\ell) = 0$ for $k = 1, \ldots, m$, and hence, (24) follows. Consequently, the function

$$u_\infty(t) = \int_0^t \! s^{-\sigma_k-1} h(s) \, ds$$

is continuous on $[0, \delta]$ and therefore, due to (18),

$$\lim_{\ell \to 0^+} u_\infty(t) = u_\infty(0) = \int_0^1 \! s^{-\sigma_k-1} h(s) \, ds = -A^{-1}h(0).$$  \hspace{1cm} \square
**Eigenvalues of A with positive real parts.**

Assume that all eigenvalues \( \lambda_i \) of \( A \) have positive real parts

\[
\lambda_k = \sigma_k + i\rho_k, \quad \sigma_k > 0, \quad k = 1, \ldots, m.
\]

Then, by (11) and (16), we have

\[
\lim_{t \to 0^+} r^k = 0.
\]  \hspace{1cm} (25)

Integration of (13) over \([t, \delta] \subset (0, 1]\) gives

\[
\delta^A - t^A = -A \int_t^\delta s^{A-1} ds.
\]

We now multiply the last equation by \(-A^k\) and use (12) to obtain

\[
\int_t^\delta \left( \frac{t}{s} \right)^A \frac{1}{s} ds = A^{-1} \left( I - \left( \frac{t}{\delta} \right)^A \right), \quad t \in (0, \delta],
\]  \hspace{1cm} (26)

and, by (25),

\[
\lim_{t \to 0^+} \int_t^\delta \left( \frac{t}{s} \right)^A \frac{1}{s} ds = A^{-1}.
\]

Next lemma follows from [8, Lemma 7] and [9, Lemma 3.4].

**Lemma 8.** Assume that all eigenvalues of \( A \) have positive real parts. Then, there exists a constant \( c > 0 \) such that

\[
\int_t^\delta \left| \left( \frac{t}{s} \right)^A \right| \frac{1}{s} ds \leq c, \quad t \in (0, 1],
\]  \hspace{1cm} (27)

\[
\lim_{t \to 0^+} \int_t^\delta \left| \left( \frac{t}{s} \right)^A \right| \frac{1}{s} ds = 0 \quad \text{for each } \delta \in (0, 1].
\]  \hspace{1cm} (28)

**Lemma 9.** Assume that all eigenvalues of \( A \) have positive real parts. Let \( \delta \in (0, 1]. \) Then, for \( t \in (0, \delta] \)

\[
\int_t^\delta \left( \frac{t}{s} \right)^{\delta A} \frac{1}{s} ds \leq \sum_{j=0}^{n_k-1} \frac{1}{\delta \sigma_k}, \quad k = 1, \ldots, m.
\]  \hspace{1cm} (29)

**Proof.** Choose \( k \in \{0, \ldots, m\}. \) Then, by (12), (14), we can write for \( t \in (0, \delta] \)

\[
\int_t^\delta \left( \frac{t}{s} \right)^{\delta A} \frac{1}{s} ds = \int_t^\delta \left( \frac{t}{s} \right)^\sigma_k \frac{1}{s} \sum_{j=0}^{n_k-1} \left( -\ln \left( \frac{t}{s} \right) \right)^j ds = \sum_{j=0}^{n_k-1} h_j(t),
\]  \hspace{1cm} (30)

where

\[
h_j(t) = t^{\sigma_k} \int_t^\delta s^{-\delta A - 1} \left( -\ln \left( \frac{t}{s} \right) \right)^j ds, \quad j = 0, \ldots, n_k - 1.
\]  \hspace{1cm} (31)

Hence,

\[
h_0(t) = t^{\sigma_k} \int_t^\delta s^{-\delta A - 1} ds \leq \frac{1}{\delta \sigma_k}, \quad t \in (0, \delta].
\]  \hspace{1cm} (32)

Integration by parts yields, for \( t \in (0, \delta] \) and \( j \in \{1, \ldots, n_k\}, \)

\[
h_j(t) = t^{\sigma_k} \int_t^\delta \left( \ln \left( \frac{t}{s} \right) \right)^j \frac{\sigma_k - \delta A}{\sigma_k} + \frac{j}{\sigma_k} \int_t^\delta s^{-\delta A - 1} \left( -\ln \left( \frac{t}{s} \right) \right)^{j-1} ds,
\]

and, due to (31) and the fact that the first term is nonpositive, we deduce

\[
h_j(t) \leq \frac{1}{\delta \sigma_k} - h_{j-1}(t), \quad t \in (0, \delta].
\]  \hspace{1cm} (33)

Consequently, (29) follows from (30), (32), and (33). \( \square \)
As a direct consequence of Lemma 9 and (11), we have the following result.

**Corollary 10.** Assume that all eigenvalues of $A$ have positive real parts. Then, for $\delta \in (0,1]$ and $t \in (0,\delta]$

\[
\int_0^t \left( \frac{1}{s} \right)^{\frac{1}{n}} \frac{1}{s} \, ds \leq \max_{1 \leq k \leq m} \sum_{j=0}^{n-1} \frac{1}{\sigma_k^{j+1}}, \quad \int_0^t \left( \frac{1}{s} \right)^{\frac{1}{n}} \frac{1}{s} \, ds \leq |E| \cdot |E^{-1}| \cdot \max_{1 \leq k \leq m} \sum_{j=0}^{n-1} \frac{1}{\sigma_k^{j+1}}.
\]  

(34)

**Lemma 11.** Assume that all eigenvalues of $A$ have positive real parts and consider $\delta \in (0,1]$ and $h \in C([0,\delta];\mathbb{R}^n)$.

Then,

\[
\lim_{t \to \delta} \int_0^t \left( \frac{1}{s} \right)^{\frac{1}{n}} \frac{1}{s} h(s) \, ds = A^{-1}h(0).
\]  

(35)

**Proof.** Choose $\varepsilon > 0$. Then there exists $\delta_1 \in (0,\delta]$ such that

\[
|h(s) - h(0)| < \frac{\varepsilon}{2c}, \quad s \in (0,\delta_1),
\]  

(36)

where the constant $c$ is from (27). The continuity of $h$ on $[0,\delta]$ provides a constant $c_1 > 0$ with

\[
|h(s) - h(0)| \leq c_1, \quad s \in [0,\delta].
\]  

(37)

According to (28) in Lemma 8, there exists $\delta_2 > 0$ such that

\[
\int_{\delta_1}^{\delta_2} \left( \frac{1}{s} \right)^{\frac{1}{n}} \frac{1}{s} \, ds < \frac{\varepsilon}{2c_1}, \quad t \in (0,\delta_2).
\]  

(38)

For $t \in (0,\delta]$, let us introduce

\[I_1(t) := \int_t^{\delta_2} \left( \frac{1}{s} \right)^{\frac{1}{n}} \frac{1}{s} |h(s) - h(0)| \, ds, \quad I_2(t) := \left| \int_t^{\delta_2} \left( \frac{1}{s} \right)^{\frac{1}{n}} - \frac{1}{s} h(0) \, ds - A^{-1}h(0) \right|.
\]

Let $\delta_0 = \min\{\delta_1,\delta_2\}$. By (27), (36)-(38), if $t \in (0,\delta_0)$, then

\[I_1(t) \leq \int_t^{\delta_1} \left( \frac{1}{s} \right)^{\frac{1}{n}} \frac{1}{s} |h(s) - h(0)| \, ds + \int_{\delta_1}^{\delta_2} \left( \frac{1}{s} \right)^{\frac{1}{n}} \frac{1}{s} |h(s) - h(0)| \, ds < \frac{\varepsilon}{2c_1} + c_1 \frac{\varepsilon}{2c_1} = \varepsilon.
\]

Consequently, $\lim_{t \to \delta} I_1(t) = 0$. Moreover, by (26),

\[I_2(t) \leq |A^{-1} \left( \frac{f(t)}{\delta} \right) - A^{-1}| |h(0)| \leq |A^{-1}||\left( \frac{f(t)}{\delta} \right) ||h(0)||.
\]

Finally, by (25) we have $\lim_{t \to \delta} I_2(t) = 0$ and the statement (35) follows from the following inequality letting $t \to 0^+$

\[\left| \int_t^{\delta_2} \left( \frac{1}{s} \right)^{\frac{1}{n}} \frac{1}{s} h(s) \, ds - A^{-1}h(0) \right| \leq I_1(t) + I_2(t), \quad t \in (0,\delta].
\]

\[\square\]

5. Existence and uniqueness results for the nonlinear problem

In this section we investigate the original nonlinear system (1),

\[x'(t) = \frac{M(t)}{t}x(t) + \frac{f(t,x(t))}{t}, \quad t \in (0,1],
\]

From now on, we assume that the eigenvalues $\lambda_k$ of $M(0)$ satisfy

\[\lambda_k = \sigma_k + ip_k, \quad \sigma_k \neq 0, \quad k = 1, \ldots, m.
\]

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According to (11), we have for $A = M(0)$
\[ M(0) = EJ^{-1}, \quad t^M(0) = Et^J E^{-1}, \quad J = \text{diag}(J_1, \ldots, J_m), \quad t^J = \text{diag}(t^{J_1}, \ldots, t^{J_m}), \]

(39)

where $E \in \mathbb{C}^{n \times n}$ is the matrix of generalized eigenvectors associated with $M(0)$. In what follows, we denote
\[ g(t, x) := (M(t) - M(0))x + f(t, x), \quad t \in [0, 1], \quad x \in \mathbb{R}^n, \]

(40)

\[ \beta := |E| \cdot |E^{-1}| \cdot \max_{1 \leq k \leq n} \frac{1}{\|\sigma_k\|^2}, \]

(41)

where $n_k$ is the dimension of the Jordan box $J_k$, $k = 1, \ldots, m$. Depending on the spectrum of $M(0)$, we shall distinguish between IVPs, TVPs, and two-point BVPs for system (1).

5.1. Eigenvalues of $M(0)$ with negative real parts – IVPs

Assume that the eigenvalues $\lambda_k$ of $M(0)$ satisfy
\[ \lambda_k = \sigma_k + ip_k, \quad \sigma_k < 0, \quad k = 1, \ldots, m, \]

and consider system (1) subject to the nonlinear initial condition (7),
\[ M(0)x(0) + f(0, x(0)) = 0. \]

As a first step in the existence proof of at least one solution of the IVP (1), (7) on the interval $[0, 1]$, we show the existence of a unique solution of the IVP (1), (7) on an interval $[0, \delta] \subset [0, 1]$ for a sufficiently small $\delta$.

On the basis of Lemma 7, where $h(\tau) = (M(\tau) - M(0))x(\tau) + f(\tau, x(\tau))$ and $x \in C([0, \delta]; \mathbb{R}^n)$, we define an operator $\mathcal{H} : C([0, \delta]; \mathbb{R}^n) \rightarrow C([0, \delta]; \mathbb{R}^n)$ by
\[ (\mathcal{H}x)(t) = \int_0^1 s^{-M(0)-1}((M(st) - M(0))x(st) + f(st, x(st)))\, ds, \quad t \in [0, \delta], \]

(42)

and note that (18) and (23) imply
\[ (\mathcal{H}x)(0) = (-M(0))^{-1}f(0, x(0)) = \lim_{t \to 0^+} (\mathcal{H}x)(t). \]

(43)

Lemma 12. Assume that all eigenvalues $\lambda_k$ of $M(0)$ have negative real parts and consider $\delta \in (0, 1]$. A mapping $x \in C([0, \delta]; \mathbb{R}^n)$ is a fixed point of the operator $\mathcal{H}$ if and only if $x$ is a solution of the IVP (1), (7) on $[0, \delta]$, or equivalently, $x \in C([0, \delta]; \mathbb{R}^n) \cap C^1((0, \delta]; \mathbb{R}^n)$ satisfies equation (1) on $(0, \delta]$ and the initial condition (7).

Proof. We use notation (40): $g(t, x) := (M(t) - M(0))x + f(t, x)$, for $t \in [0, 1], x \in \mathbb{R}^n$, and carry out the substitution $\tau = st$ in the right-hand side of (42). Then, for $t \in [0, \delta]$
\[ \int_0^1 s^{-M(0)-1}g(st, x(st))\, ds = t^M(0) \int_0^\tau t^{-M(0)-1}g(\tau, x(\tau))\, d\tau. \]

(44)

1. Assume that $x$ is a fixed point of $\mathcal{H}$. Then due to the above substitution,
\[ x(t) = t^M(0) \int_0^\tau t^{-M(0)-1}g(\tau, x(\tau))\, d\tau, \]

and by the basic properties (12) and (13) of the fundamental solution matrix $t^M(0)$,
\[ x'(t) = M(0)t^{-M(0)-1} \int_0^\tau t^{-M(0)-1}g(\tau, x(\tau))\, d\tau + t^M(0)t^{-M(0)-1}g(t, x(t)) = \frac{M(t)}{t} x(t) + \frac{f(t, x(t))}{t}, \quad t \in (0, \delta]. \]
Therefore, by (43), $x$ is continuous at $t = 0$ and satisfies condition (7). So, $x$ is a solution of the IVP (1), (7) on $[0, \delta]$.

2. Let $x$ be a solution of system (1) on $[0, \delta]$. According to Definition 1, $x$ is continuous on $[0, \delta]$. Multiplying (1) by $t^{-M(0)}$, we obtain

\[
(t^{-M(0)}x(t))' = t^{-M(0)}x(t) - M(0)t^{-M(0)}x(t) = t^{-M(0)}g(t, x(t)), \quad t \in (0, \delta].
\]

Integrating (45) over $[t, \tau] \subset (0, \delta)$, we have

\[
t^{-M(0)}x(t) - t^{-M(0)}x(\tau) = \int_{t}^{\tau} \left(s^{-M(0)}x(s)\right)' \, ds = \int_{t}^{\tau} s^{-M(0)}x(s) - M(0)s^{-M(0)}x(s) \, ds = \int_{t}^{\tau} s^{-M(0)}g(s, x(s)) \, ds.
\]

Letting $\tau \to 0+$ and using (17), we get

\[
t^{-M(0)}x(t) = \int_{0}^{t} s^{-M(0)}g(s, x(s)) \, ds, \quad t \in (0, \delta].
\]

Consequently by (44),

\[
x(t) = \int_{0}^{t} s^{-M(0)}g(st, x(st)) \, ds, \quad t \in (0, \delta],
\]

and by (18)

\[
x(0) = \int_{0}^{1} s^{-M(0)}g(0, x(0)) \, ds = -(M(0)^{-1}g(0, x(0)) = -(M(0)^{-1}f(0, x(0)).
\]

Therefore, $x$ is a fixed point of $\mathcal{H}$, and the result follows.

\[\square\]

**Remark 13.** In the case that all eigenvalues of $M(0)$ have negative real parts, any solution $x$ of system (1) on $[0, \delta]$, $\delta \in (0, 1]$, satisfies the initial condition (7). This follows from the second part of the proof in Lemma 12 and clarifies the form of a correctly posed initial condition (7).

**Lemma 14.** Assume that all eigenvalues $\lambda_k$ of $M(0)$ have negative real parts and let $\beta$ in (41) be given. Assume that $a \in (0, 1)$ and $L \in (0, 1/\beta)$ exist such that

\[
|f(t, x) - f(t, y)| \leq L|x - y|, \quad t \in [0, a], \quad x, y \in \mathbb{R}^n.
\]

Then, there exists $\delta \in (0, a]$ such that the IVP (1), (7) has a unique solution $u$ on $[0, \delta]$.

**Proof.** Since $M$ is continuous on $[0, 1]$, there exists $\delta \in [0, a]$ such that

\[
|M(t) - M(0)| \leq \frac{1}{2} \left(\frac{1}{\beta} - L\right), \quad t \in [0, \delta].
\]

Therefore, by (46), for $t \in [0, \delta]$ and $x, y \in \mathbb{R}^n$,

\[
|(M(t) - M(0))(x - y) + f(t, x) - f(t, y)| \leq L_1|x - y|, \quad L_1 = \frac{1}{2} \left(\frac{1}{\beta} + L\right).
\]

Let $C([0, \delta]; \mathbb{R}^n)$ be a Banach space equipped with the norm $\|x\|_\infty = \max\{|x(t)| : t \in [0, \delta]\}$ and let $\mathcal{H}$ be given by (42). By Lemma 12, a mapping $u \in C([0, \delta]; \mathbb{R}^n)$ is a solution of the IVP (1), (7) on $[0, \delta]$ if and only if $u$ is a fixed point of the operator $\mathcal{H}$. We show that $\mathcal{H}$ is a contraction. Let $x, y \in C([0, \delta]; \mathbb{R}^n)$ and $g$ be given by (40). Then, by (22) with $A = M(0)$, (41) and (47), we have for $t \in [0, \delta]$,

\[
|(\mathcal{H}x)(t) - (\mathcal{H}y)(t)| = \left|\int_{0}^{t} s^{-M(0)}(g(st, x(st)) - g(st, y(st))) \, ds\right| \leq L_1 \int_{0}^{t} |s^{-M(0)}| |x(st) - y(st)| \, ds \leq L_1 \|x - y\|_\infty,
\]

and hence, $\|\mathcal{H}x - \mathcal{H}y\|_\infty \leq L_1 \|x - y\|_\infty$. Assumption $L_1 \beta < 1$ implies $L_1 \beta < 1$ due to (47) and therefore, $\mathcal{H}$ is a contraction. Hence, the IVP (1), (7) has a unique solution $u$ on $[0, \delta]$.

\[\square\]
We are now ready to show the first main result. To this end, applying Lemma 2 and Lemma 4, we construct a
continuation of the solution \( u \) from Lemma 14 onto the interval \([\delta, 1]\).

**Theorem 15.** Assume that all eigenvalues \( \lambda_k \) of \( M(0) \) have only negative real parts and let \( \beta \) be specified by (41). Let \( a \in (0, 1) \) and \( L \in (0, 1/\beta) \) exist such that \( f \) satisfies the Lipschitz condition (46). Finally, assume that there exists \( W > 0 \) such that

\[
|f(t, x)| \leq W + \omega(|x|), \quad t \in [a, 1], \quad x \in \mathbb{R}^n,
\]

where \( \omega \in C([0, \infty); (0, \infty)) \) is nondecreasing, \( \omega(s) \geq s \) for \( s \in [0, \infty) \), and

\[
\int_0^\infty ds \omega(s) = \infty.
\]

Then, the IVP (1), (7) has at least one solution on \([0, 1]\).

**Proof.** By the Lipschitz condition (46), we can see that

\[
|f(t, x)| \leq |f(t, 0)| + L|x|, \quad t \in [0, a], \quad x \in \mathbb{R}^n.
\]

Denote \( Q_1 := W + \max(|f(t, 0)| : t \in [0, a]) \). Then (48) and (49) yield

\[
|f(t, x)| \leq Q_1 + L|x| + \omega(|x|), \quad t \in [0, 1], \quad x \in \mathbb{R}^n.
\]

Denote \( Q_2 := L + 1 + \max(|M(t) - M(0)| : t \in [0, 1]) \), and consider \( g \) specified in (40). Then (50) yields

\[
|g(t, x)| \leq Q_1 + Q_2\omega(|x|), \quad t \in [0, 1], \quad x \in \mathbb{R}^n.
\]

By Lemma 14, there exists \( \delta \in (0, a) \) such that the IVP (1), (7) has a unique solution \( u \) on \([0, \delta]\). We now prove that there exists at least one solution \( v \) of system (1) which is a \( C^1 \)-continuation of \( u \) onto \([\delta, 1]\). To this end, for \( b \in \mathbb{R}^n \), we discuss system (1) on \([\delta, 1]\) together with the initial condition

\[
x(\delta) = b.
\]

Let \( C([\delta, 1]; \mathbb{R}^n) \) be equipped with the norm \( ||x||_\infty = \max||x(t)|| : t \in [\delta, 1] \) and let \( \Phi \) be the fundamental solution matrix of the system

\[
x'(t) = \frac{M(0)}{t}x(t), \quad t \in [\delta, 1],
\]

with \( \Phi(1) = I \). Then,

\[
\Phi'(t) = \frac{M(0)}{t}\Phi(t), \quad \Phi(t) = t^{M(0)}, \quad t \in [\delta, 1],
\]

and \( \Phi, \Phi^{-1} \in C([\delta, 1]; \mathbb{R}^{n\times n}) \). So we can write

\[
||\Phi||_\infty = \max||\Phi(t)|| : t \in [\delta, 1], \quad ||\Phi^{-1}||_\infty = \max||\Phi^{-1}(t)|| : t \in [\delta, 1].
\]

It is clear that a mapping \( v \) is a solution of the IVP (1), (52) on \([\delta, 1]\) if and only if \( v \) is a solution of the integral equation

\[
x(t) = \Phi(t)\Phi^{-1}(\delta)b + \Phi(t)\int_\delta^t \Phi^{-1}(s)\frac{g(s, x(s))}{s} \, ds
\]

in \( C([\delta, 1]; \mathbb{R}^n) \). Hence, we define an operator \( \mathcal{K} : C([\delta, 1]; \mathbb{R}^n) \to C([\delta, 1]; \mathbb{R}^n) \) as

\[
(\mathcal{K}x)(t) = \Phi(t)\Phi^{-1}(\delta)b + \Phi(t)\int_\delta^t \Phi^{-1}(s)\frac{g(s, x(s))}{s} \, ds.
\]

In order to show that \( \mathcal{K} \) admits a fixed point \( v \), we apply Lemma 4. Due to the continuity of the vector function \( g \) and the matrix functions \( \Phi, \Phi^{-1} \), it is not difficult to verify that \( \mathcal{K} \) is continuous and \( \mathcal{K} \) maps bounded sets into bounded
sets. Next, let \( \Omega \subset C([\delta, 1]; \mathbb{R}^n) \) be bounded and denote \( \mu_* := \sup \{ ||x||_* : x \in \Omega \} \). Then, for \( \delta \leq t_1 < t_2 \leq 1 \) and \( x \in \Omega \), we have

\[
|\langle Kx \rangle(t_2) - \langle Kx \rangle(t_1) \rangle \leq |\Phi(t_2) - \Phi(t_1)| ||\Phi^{-1}(\delta)|| |b| + |\Phi(t_2) - \Phi(t_1)| \int_\delta^{t_2} |\Phi^{-1}(s)| \frac{|g(s, x(s))|}{s} ds
\]

\[
+ |\Phi(t_2)| \int_\delta^{t_1} |\Phi^{-1}(s)| \frac{|g(s, x(s))|}{s} ds \leq |\Phi(t_2) - \Phi(t_1)| ||\Phi^{-1}||_* \left( |b| + \frac{r_2}{\delta} \right) + ||\Phi||_* ||\Phi^{-1}||_* \frac{r_2}{\delta} (t_2 - t_1),
\]

where \( r_2 = \sup \{ |g(t, x)| : t \in [\delta, 1], |x| \leq \mu_* \} \). Hence the set \( \{ Kx : x \in \Omega \} \) is equicontinuous on \([\delta, 1] \). Consequently, \( K \) is completely continuous.

Finally, we prove that the set \( S = \{ x \in C([\delta, 1]; \mathbb{R}^n) : x = \lambda Kx \text{ for some } \lambda \in (0, 1) \} \) is bounded in \( C([\delta, 1]; \mathbb{R}^n) \). Let \( x \in C([\delta, 1]; \mathbb{R}^n) \) satisfy \( x = \lambda Kx \) for some \( \lambda \in (0, 1) \). Then, by (51) and (54),

\[
|x(t)| \leq |\langle Kx \rangle(t)| \leq ||\Phi||_* ||\Phi^{-1}||_* \left( |b| + \int_\delta^t \frac{Q_1 + Q_2 \omega(|x(s)|)}{s} ds \right) \leq K_1 + K_2 \int_\delta^t \omega(|x(s)|) ds, \; t \in [\delta, 1],
\]

where

\[
K_1 = ||\Phi||_* ||\Phi^{-1}||_* \left( |b| + \frac{Q_1}{\delta} (1 - \delta) \right), \quad K_2 = ||\Phi||_* ||\Phi^{-1}||_* \frac{Q_2}{\delta}.
\]

Let us define

\[
G(v) := \int_0^v \frac{d\xi}{\omega(\xi)}, \; v \geq 0.
\]

Combining the inequality

\[
|x(t)| \leq K_1 + K_2 \int_\delta^t \omega(|x(s)|) ds, \; t \in [\delta, 1],
\]

with Lemma 2, we obtain

\[
|x(t)| \leq G^{-1}(G(K_1) + K_2(1 - \delta)), \; t \in [\delta, 1],
\]

where \( G^{-1} \) is the inverse of the function \( G \). Hence, the set \( S \) is bounded in \( C([\delta, 1]; \mathbb{R}^n) \). Consequently, by Lemma 4, the IVP (1), (52) has a solution \( v \) on \([\delta, 1] \). Due to (1) and (52), with \( b = u(\delta) \), we have \( u(\delta) = v(\delta) \) and \( u'(\delta) = v'(\delta) \) which means that \( v \) is a \( C^1 \)-continuation of \( u \) onto \([\delta, 1] \). As a result, the mapping

\[
x(t) = \begin{cases} u(t), & t \in [0, \delta], \\ v(t), & t \in [\delta, 1], \end{cases}
\]

is a solution of the IVP (1), (7). \( \square \)

**Corollary 16.** Let Lipschitz condition (46) hold on \([0, 1] \), which means that \( a = 1 \) in Theorem 15. Then, the IVP (1), (7) has a unique solution on \([0, 1] \).

**Proof.** The existence of a unique solution \( u \) of the IVP (1), (7) on some interval \([0, \delta], \delta \in (0, 1) \), follows from Lemma 14. Since the Lipschitz condition holds on the whole interval \([0, 1] \), the solution \( u \) can be uniquely extended as a solution of (1) onto \([0, 1] \). \( \square \)

### 5.2. Eigenvalues of \( M(0) \) with positive real parts – TVPs

Assume that all eigenvalues \( \lambda_k \) of \( M(0) \) have positive real parts,

\[
\lambda_k = \sigma_k + ip_k, \quad \sigma_k > 0, \quad k = 1, \ldots, m,
\]

and let us investigate system (1) with the terminal condition (8),

\[
x(1) = c, \quad c \in \mathbb{R}^n.
\]
In order to prove the existence of at least one solution of the TVP (1), (8) on \([0,1]\), we first show the existence of a unique solution \(x\) of system (1) on \([0,\delta]\), for a sufficiently small \(\delta \in (0,1]\), such that \(x\) satisfies the terminal condition

\[
x(\delta) = b, \quad b \in \mathbb{R}^n.
\]

(55)

Consider \(g\) specified in (40), that is, \(g(t,x) = (M(t) - M(0))x + f(t,x)\), for \(t \in [0,1], x \in \mathbb{R}^n\). Due to Lemma 11, where \(h(s) = g(s,x(s))\) and \(x \in C([0,\delta]; \mathbb{R}^n)\), we define an operator \(\mathcal{H} : C([0,\delta]; \mathbb{R}^n) \to C([0,\delta]; \mathbb{R}^n)\) by

\[
(\mathcal{H}x)(t) = \left\{
\begin{array}{ll}
\left(\frac{t}{s}\right)^{M(0)} b - \int_{s}^{t} \left(\frac{r}{s}\right)^{M(0)} g(s,x(s)) \, ds, & \text{if } t \in (0,\delta], \\
\lim_{t \to 0^+}(\mathcal{H}x)(t) = -(M(0))^{-1} f(0,x(0)), & \text{if } t = 0.
\end{array}
\right.
\]

(56)

**Lemma 17.** Assume that the eigenvalues \(\lambda_k\) of \(M(0)\) have positive real parts and let \(\delta \in (0,1]\). A mapping \(x \in C([0,\delta]; \mathbb{R}^n)\) is a fixed point of the operator \(\mathcal{H}\) if and only if \(x\) is a solution of the TVP (1), (55) on \([0,\delta]\), that is,

\[
x \in C([0,\delta]; \mathbb{R}^n) \cap C^1((0,\delta]; \mathbb{R}^n)\]

satisfies equation (1) on \((0,\delta]\) and the terminal condition (55).

**Proof.** 1. Let \(x\) be a fixed point of \(\mathcal{H}\). Then \(x(\delta) = b\), and by basic properties (12) of the fundamental matrix \(e^{M(0)t}\),

\[
x'(t) = M(0)e^{M(0)(t-s)}b - M(0)e^{M(0)(s-t)}g(s,x(s)) \, ds + \frac{g(t,x(t))}{t} = M(0)x(t) + \frac{f(t,x(t))}{t}, \quad t \in (0,\delta].
\]

Let \(x\) be a solution of the TVP (1), (55). Then multiplying (1) by \(e^{-M(0)t}\) and substituting \(g\) by (40), we see that \(x\) satisfies (45) which integrated over \([t,\delta] \subset (0,\delta]\) yields

\[
x(t) = \left(\frac{t}{\delta}\right)^{M(0)} b - \int_{t}^{\delta} \left(\frac{r}{s}\right)^{M(0)} \frac{1}{s} g(s,x(s)) \, ds, \quad t \in (0,\delta].
\]

Since \(x\) is continuous on \([0,\delta]\), it follows from (25) and (35),

\[
x(0) = \lim_{t \to 0^+} x(t) = -(M(0))^{-1} g(0,x(0)) = -(M(0))^{-1} f(0,x(0)).
\]

(57)

According to (56), \(x\) is a fixed point of \(\mathcal{H}\).

**Remark 18.** In the case of only positive real parts of the eigenvalues of \(M(0)\) each solution \(x\) of (1) on \([0,\delta]\), \(\delta \in (0,1]\), satisfies the initial condition (7). To see this, we multiply (1) by \(e^{-M(0)t}\) and integrate over \([t,\delta] \subset (0,\delta]\). Therefore, \(x\) satisfies

\[
x(t) = \left(\frac{t}{\delta}\right)^{M(0)} x(\delta) - \int_{t}^{\delta} \left(\frac{r}{s}\right)^{M(0)} \frac{1}{s} g(s,x(s)) \, ds, \quad t \in (0,\delta].
\]

Thus, \(x(0) = -(M(0))^{-1} f(0,x(0))\), see (57) in the last part of the proof of Lemma 17.

**Lemma 19.** Assume that the eigenvalues \(\lambda_k\) of \(M(0)\) have positive real parts and let \(\beta\) be given via (41). Moreover, let there exist \(a \in (0,1)\) and \(L \in (0,1/\beta)\) such that the Lipschitz condition (46) holds. Then, there exists \(\delta \in (0,a]\) such that for each \(b \in \mathbb{R}^n\) the TVP (1), (55) has a unique solution on \([0,\delta]\). In addition, this solution satisfies the initial condition (7).

**Proof.** The arguments are similar to those from the proof of Lemma 14. We take the space \(C([0,\delta]; \mathbb{R}^n)\), where \(\delta \in (0,a]\) is such that (47) holds and define an operator \(\mathcal{H}\) by (56). In order to prove that \(\mathcal{H}\) is a contraction, we consider \(x, y \in C([0,\delta]; \mathbb{R}^n)\) and \(g\) from (40). By (34) with \(A = M(0)\) and (47) we obtain,

\[
||\mathcal{H}x(t) - \mathcal{H}y(t)|| = \int_{t}^{\delta} \left(\frac{r}{s}\right)^{M(0)} \frac{1}{s} g(s,x(s)) - g(s,y(s)) \, ds \leq L_1 \int_{t}^{\delta} \left(\frac{r}{s}\right)^{M(0)} \frac{1}{s} |x(s) - y(s)| \, ds \leq L_1 \beta ||x - y||,
\]

for \(t \in (0,\delta]\), and

\[
||\mathcal{H}x(0) - \mathcal{H}y(0)|| = \lim_{t \to 0^+} ||\mathcal{H}x(t) - \mathcal{H}y(t)|| \leq L_1 \beta ||x - y||.
\]

Hence, \(||\mathcal{H}x - \mathcal{H}y|| \leq L_1 \beta ||x - y||\). The assumption \(L_\beta < 1\) implies \(L_1 \beta < 1\) due to (47). Therefore, \(\mathcal{H}\) is a contraction and consequently, the TVP (1), (55) has a unique solution on \([0,\delta]\). By Remark 18, this solution satisfies (7).
We can now show our second main result characterizing the existence of a solution of the TVP (1), (8) on [0, 1].

**Theorem 20.** Assume that the eigenvalues $\lambda_k$ of $M(0)$ have positive real parts and let $\beta$ be specified in (41). Let there exist $a \in (0, 1)$ and $L \in (0, 1/\beta)$ such that $f$ satisfies the Lipschitz condition (46). Finally, assume that there exists $W > 0$ such that the function $f$ satisfies (48). Then, for each $c \in \mathbb{R}^n$, the TVP (1), (8) has at least one solution on [0, 1]. Moreover, this solution satisfies the initial condition (7).

**Proof.** We argue similar to the proof of Theorem 15 and find $Q_1, Q_2 \in (0, \infty)$ such that the function $g$ specified in (40) satisfies (51). By Lemma 19 there exists $\delta \in (0, a]$ such that for each $b \in \mathbb{R}^n$ the TVP (1), (55) has a unique solution on $[0, \delta]$. We first prove that for such $\delta$ and each $c \in \mathbb{R}^n$, there exists at least one solution $v$ of the TVP (1), (8) on the interval $[\delta, 1]$. Let $\Phi$ be the fundamental solution matrix given in (53). Then, $\Phi, \Phi^{-1} \in C([\delta, 1]; \mathbb{R}^{n \times n})$ and we can use (54). Substituting $g$ into (1), we see that a mapping $v$ is a solution of the TVP (1), (8) on $[\delta, 1]$ if and only if $v$ is a solution of the integral equation

$$x(t) = \Phi(t)c - \Phi(t) \int_{\delta}^{t} \Phi^{-1}(s) \frac{g(s, x(s))}{s} \, ds$$

in $C([\delta, 1]; \mathbb{R}^n)$. Hence, we define an operator $\mathcal{K} : C([\delta, 1]; \mathbb{R}^n) \rightarrow C([\delta, 1]; \mathbb{R}^n)$ as

$$(\mathcal{K}x)(t) = \Phi(t)c - \Phi(t) \int_{\delta}^{t} \Phi^{-1}(s) \frac{g(s, x(s))}{s} \, ds.$$  

(58)

Using the same arguments as in the proof of Theorem 15, we conclude that $\mathcal{K}$ is completely continuous. Finally, assume that $x \in C([\delta, 1]; \mathbb{R}^n)$ satisfies $x = \lambda\mathcal{K}x$ for some $\lambda \in (0, 1)$. Then, by (51) and (54),

$$|x(t)| \leq \|\mathcal{K}x(t)\| \leq \|\Phi\| \left( |c| + \|\Phi^{-1}\| \int_{\delta}^{1} \left| \frac{Q_1 + Q_2\omega(|x(s)|)}{s} \right| \, ds \right) \leq K_1 + K_2 \int_{\delta}^{1} \omega(|x(s)|) \, ds, \quad t \in [\delta, 1],$$

where

$$K_1 = \|\Phi\| \left( |c| + \|\Phi^{-1}\| \frac{Q_1}{\delta} (1 - \delta) \right), \quad K_2 = \|\Phi\| \|\Phi^{-1}\| \frac{Q_2}{\delta}.$$

Let us define

$$\mathcal{G}(v) := \int_{\delta}^{1} \frac{d\xi}{\omega(\xi)}, \quad v \geq 0.$$

Combining the inequality

$$|x(t)| \leq K_1 + K_2 \int_{\delta}^{1} \omega(|x(s)|) \, ds, \quad t \in [\delta, 1],$$

with Lemma 3, we obtain

$$|x(t)| \leq \mathcal{G}^{-1}(\mathcal{G}(K_1) + K_2(1 - \delta)), \quad t \in [\delta, 1],$$

where $\mathcal{G}^{-1}$ is the inverse of function $\mathcal{G}$. Hence, the set $\mathcal{S} = \{x \in C([\delta, 1]; \mathbb{R}^n) : x = \lambda\mathcal{K}x$ for some $\lambda \in (0, 1)\}$ is bounded in $C([\delta, 1]; \mathbb{R}^n)$ and we can apply the Leray-Schauder alternative from Lemma 4. Consequently, the TVP (1), (8) has a solution $v$ on $[\delta, 1]$.

Let us choose $b = v(\delta)$. Then, it follows from Lemma 19 that a unique solution of problem (1), (55) on $[0, \delta]$ exists such that $u(\delta) = v(\delta)$. Due to (1) we have $u'(\delta) = v'(\delta)$ which means that $v$ is a $C^1$-continuation of $u$ on $[\delta, 1]$. As a result, the mapping

$$x(t) = \begin{cases} u(t), & t \in [0, \delta], \\ v(t), & t \in [\delta, 1], \end{cases}$$

is a solution of the TVP (1), (8) on $[0, 1]$. In addition, by Lemma 19, $u$ satisfies (7). □

**Corollary 21.** Let the Lipschitz condition (46) hold on $[0, 1]$ which means that $a = 1$ in Theorem 20. Then, for each $c \in \mathbb{R}^n$, the TVP (1), (8) has a unique solution on $[0, 1]$. Moreover, this solution satisfies the initial condition (7).

**Proof.** Choose $c \in \mathbb{R}^n$ and let $\delta$ be as in Lemma 19. Then, the TVP (1), (8) has a unique solution $v$ on $[\delta, 1]$ due to the Lipschitz condition (46) which is now satisfied for $t \in [0, 1]$. Also, for $b := v(\delta)$, Lemma 19 provides a unique solution $u$ of the TVP (1), (55) on $[0, \delta]$ with $u(\delta) = v(\delta)$. Due to (1), $u'(\delta) = v'(\delta)$, and thus, $x$ is a solution of the TVP (1), (8) and satisfies (7), as in the final part of the proof of Theorem 20. The uniqueness of the solution $x$ follows from the uniqueness of $u$ and $v$. □
5.3. Mixed spectrum of $M(0)$ – BVPs

Consider the matrix function $M$ from system (1) and assume that the eigenvalues $\lambda_k$ of $M(0)$ have nonzero real parts. Let matrices $N, P \in \mathbb{R}^{n \times n}$ represent projections onto the subspaces $X_-, X_+ \subset \mathbb{R}^n$, where $X_-$ is associated with the eigenvalues of $M(0)$ having negative real parts, $X_+$ is associated with the eigenvalues of $M(0)$ having positive real parts, and

$$N : \mathbb{R}^n \to X_-, \quad P : \mathbb{R}^n \to X_+, \quad X_- \oplus X_+ = \mathbb{R}^n, \quad Nx + Px = x, \quad x \in \mathbb{R}^n. \quad (59)$$

Due to (39),

$$N = E\tilde{N}E^{-1}, \quad P = E\tilde{P}E^{-1}. \quad (60)$$

Here $\tilde{N} \in \mathbb{R}^{n \times n}$ is the diagonal matrix with ones at the positions where the matrix $J$, see (39), has the eigenvalues with negative real parts and zero entries elsewhere. Similarly, $\tilde{P} \in \mathbb{R}^{n \times n}$ is the diagonal matrix with ones at the positions corresponding the positions of the eigenvalues with positive real parts in the matrix $J$ and zero entries elsewhere. Then,

$$|\tilde{N}x + \tilde{P}y| = \max[|\tilde{N}x|, |\tilde{P}y|], \quad x, y \in \mathbb{R}^n. \quad (61)$$

Remark 22. By [8, Lemma 18], we know that the matrices $N$ and $P$ commute with the matrices $M(0)$ and $t^{M(0)}$. For the proof see [32, Lemma 19].

According to the previous sections, the structure of the correctly posed boundary conditions depends on the spectrum of the matrix $M(0)$. In Section 5.1 the case of only negative real parts of eigenvalues of $M(0)$ was discussed and the IVP (1), (7) was studied. Section 5.2 was concerned with the case of only positive real parts of eigenvalues of $M(0)$ and the TVP (1), (8) with arbitrary $c \in \mathbb{R}^n$ was of interest. On the basis of these results we now investigate system (1) with a mixed spectrum of $M(0)$ subject to the boundary conditions (9) of the form

$$NM(0)x(0) + Nf(0, x(0)) = 0, \quad Px(1) = Pc, \quad c \in \mathbb{R}^n. \quad (62)$$

As in Sections 5.1 and 5.2, we first discuss the unique solvability of an auxiliary BVP on an interval $[0, \delta]$ for a sufficiently small $\delta$. Hence, for $b \in \mathbb{R}^n$ and $\delta \in (0, 1]$, we introduce the boundary conditions

$$NM(0)x(0) + Nf(0, x(0)) = 0, \quad Px(\delta) = Pb, \quad (63)$$

and define the operators $N_b : C([0, \delta]; \mathbb{R}^n) \to C([0, \delta]; X_-), \quad P_b : C([0, \delta]; \mathbb{R}^n) \to C([0, \delta]; X_+)$

$$(N_b \phi)(t) = Nx(t), \quad (P_b \phi)(t) = Px(t), \quad t \in [0, \delta]. \quad (64)$$

Again, consider $g$ given via (40): $g(t, x) = (M(t) - M(0))x + f(t, x)$, for $t \in [0, 1], x \in \mathbb{R}^n$, and define an operator $H_1 : C([0, \delta]; \mathbb{R}^n) \to C([0, \delta]; \mathbb{R}^n)$ by (42) and an operator $H_2 : C([0, \delta]; \mathbb{R}^n) \to C([0, \delta]; \mathbb{R}^n)$ by (56). Consequently, $H_1$ and $H_2$ read, note that $g(0, x) = f(0, x),

$$(H_1 \phi)(t) = \int_0^1 s^{M(0)-1} g(st, x(st)) \, ds, \quad t \in [0, \delta], \quad (65)$$

$$(H_2 \phi)(t) = \begin{cases} \left( {\dot{z} \over z} \right)^{M(0)} b - \int_0^\delta \left( {\dot{z} \over z} \right)^{M(0)} {g(s, x(s)) \over z} \, ds, & t \in (0, \delta], \\ -(M(0))^{-1}g(0, x(0)), & t = 0. \end{cases} \quad (66)$$

respectively. Consequently, $N \circ H_1 + P \circ H_2 : C([0, \delta]; \mathbb{R}^n) \to C([0, \delta]; \mathbb{R}^n)$.

Lemma 23. Assume that the eigenvalues $\lambda_k$ of $M(0)$ have nonzero real parts and let $\delta \in (0, 1]$. Then, a mapping $x \in C([0, \delta]; \mathbb{R}^n)$ is a fixed point of the operator $N \circ H_1 + P \circ H_2$ if and only if $x$ is a solution of the BVP (1), (62) on $[0, \delta], \text{ that is, } x \in C([0, \delta]; \mathbb{R}^n) \cap C^1((0, \delta]; \mathbb{R}^n)$ satisfies equation (1) on $(0, \delta)$ and the boundary conditions (62).

Proof. 1. Assume that $x$ is a fixed point of $N \circ H_1 + P \circ H_2$. Since

$$x = (N \circ H_1 + P \circ H_2)x = N_0 H_1x + P \circ H_2x,$$
and by (59), \( N_\delta x + P_\delta x = x \), we can see that \( x \) satisfies
\[
N_\delta x = N_\delta H_1 x, \quad P_\delta x = P_\delta H_2 x. \tag{66}
\]
Due to (44), (63) and (64), for \( t \in [0, \delta] \),
\[
(N_\delta x)(t) = N \left( \int_0^t s^{-M(0) - I} g(st, x(st)) \, ds \right) = N \left( \int_0^t \tau^{-M(0) - I} g(\tau, x(\tau)) \, d\tau \right),
\]
and as in Part 1 of the proof of Lemma 12 having in mind Remark 22, we conclude
\[
(N_\delta x)'(t) = N \left( \frac{M(t)}{t} x(t) + \frac{f(t, x(t))}{t} \right), \quad t \in (0, \delta).
\]
The condition \( N x(0) = N(-M(0))^{-1} f(0, x(0)) \) follows from (43) and (64). As in Part 1 of the proof of Lemma 17, we deduce from (65) and Remark 22 that
\[
(P_\delta x)'(t) = P \left( \frac{M(t)}{t} x(t) + \frac{f(t, x(t))}{t} \right), \quad t \in (0, \delta),
\]
and \( P x(\delta) = P b \). Consequently, \( x \) is a solution of the BVP (1), (62) on \([0, \delta]\).

2. Let \( x \) be a solution of system (1) on \([0, \delta]\). Now, we follow the arguments from the Part 2 of the proof of Lemma 12, integrate (45) over \([t, \tau] \subset (0, \delta]\) and obtain
\[
(N_\delta x)(t) = N \left( \int_0^\tau s^{-M(0) - I} x(s) + \int_0^\tau s^{-M(0) - I} g(s, x(s)) \, ds \right).
\]
Letting \( \tau \to 0^+ \) and using (17), (44), and (64), we derive the first equality in (66). Now, assume in addition that \( x \) satisfies the terminal condition (55) and integrate (45) over \([t, \delta] \subset (0, \delta]\) as in the Part 2 of the proof of Lemma 17. Then, we have
\[
(P_\delta x)(t) = P \left( \left( \frac{1}{\delta} \right)^{M(0)} b - \int_t^\delta \left( \frac{1}{s} \right)^{M(0)} \frac{1}{s} g(s, x(s)) \, ds \right), \quad t \in (0, \delta).
\]
Finally, it follows from (25) and (35),
\[
(P_\delta x)(0) = \lim_{t \to 0^+} (P_\delta x)(t) = P \left( -(M(0))^{-1} g(0, x(0)) \right),
\]
this means that the second equality in (66) holds. Hence, \( x \) is a fixed point of \( N_\delta H_1 + P_\delta H_2 \).

**Remark 24.** Let the eigenvalues \( \lambda_k \) of \( M(0) \) have nonzero real parts and let \( x \) be a solution of system (1) on \([0, \delta]\), \( \delta \in (0, 1] \). Then \( x \) satisfies (7). To show this statement, we use the arguments from the second part of the proof of Lemma 12 and obtain
\[
N x(t) = N \int_0^\delta s^{-M(0) - I} g(st, x(st)) \, ds, \quad t \in (0, \delta), \quad N x(0) = N(-M(0))^{-1} f(0, x(0)).
\]
As in Remark 18, we have
\[
P x(t) = P \left( \left( \frac{1}{\delta} \right)^{M(0)} x(\delta) - \int_0^\delta \left( \frac{1}{s} \right)^{M(0)} \frac{1}{s} g(s, x(s)) \, ds \right), \quad t \in (0, \delta).
\]
As in the last part of the proof of Lemma 17, we conclude \( P x(0) = P(-M(0))^{-1} f(0, x(0)) \) and the condition (7) follows.

We now proceed with the proof of the existence of a unique solution of the BVP (1), (62) on \([0, \delta]\) for a sufficiently small \( \delta > 0 \).
Consequently, using (22), (34), and (41), we obtain

Step 1. Define an operator $K$ initial condition

Theorem 25. Assume that all eigenvalues $\lambda_k$ of $M(0)$ have nonzero real parts and let $\beta$ be specified in (41). Let there exist $a \in (0, 1)$ and $L \in (0, 1/\beta)$ such that the Lipschitz condition (46) holds. Then, there exists $\delta \in (0, a]$ such that for each $b \in \mathbb{R}^n$ the BVP (1), (62) has a unique solution on $[0, \delta]$. In addition, this solution satisfies the initial condition (7).

Proof. As in the proof of Lemma 14, we choose the space $C([0, \delta]; \mathbb{R}^n)$, where $\delta \in (0, a]$ is such that (47) holds. Choose $b \in \mathbb{R}^n$ and consider the operators $N_b$ and $P_b$ given in (63) and the operators $\mathcal{H}_1$ and $\mathcal{H}_2$ defined in (64) and (65). By Theorem 23, in order to prove that a unique solution of the BVP (1), (62) on $[0, \delta]$ exists, it is sufficient to show that the operator $N_b \mathcal{H}_1 + P_b \mathcal{H}_2$ is a contraction on $C([0, \delta]; \mathbb{R}^n)$. Using (39), (40), (47), (60), and (61), we can deduce for $t \in [0, \delta]$ and $x, y \in C([0, \delta]; \mathbb{R}^n)$,

$$\| (N_b \mathcal{H}_1 + P_b \mathcal{H}_2)x(t) - (N_b \mathcal{H}_1 + P_b \mathcal{H}_2)y(t) \| \leq E \left( \int_0^t \left| s^{-\lambda} - \frac{1}{s} \right| \| |E||E|^{-1} \| |x - y\|_{\delta}. \right.$$ 

Consequently, using (22), (34), and (41), we obtain

$$\| (N_b \mathcal{H}_1 + P_b \mathcal{H}_2)x - (N_b \mathcal{H}_1 + P_b \mathcal{H}_2)y \|_{\delta} \leq \max_{0 \leq k \leq m} \sum_{j=0}^{m-1} \frac{1}{|\sigma_k|^{k+1}} |E||E|^{-1} \| |x - y\|_{\delta} = \beta L_1 \| |x - y\|_{\delta}. \right.$$ 

By (47), the assumption $\beta L_1 < 1$ implies $\beta L_1 < 1$ and hence, $N_b \mathcal{H}_1 + P_b \mathcal{H}_2$ is a contraction on $C([0, \delta]; \mathbb{R}^n)$. Therefore, the BVP (1), (55) has a unique solution on $[0, \delta]$. By Remark 24, this solution satisfies (7).

In order to move from the BVP (1), (62) on $[0, \delta]$ to the BVP (1), (9) on $[0, 1]$, we require the additional assumption,

$$P f(t, x) = P f(t, x), \quad t \in [\delta, 1], \quad x \in \mathbb{R}^n. \quad (67)$$ 

Theorem 26. Assume that the eigenvalues $\lambda_k$ of $M(0)$ have nonzero real parts and let $\beta$ be specified in (41). Let there exist $a \in (0, 1)$ and $L \in (0, 1/\beta)$ such that the Lipschitz condition (46) holds. Finally, assume that $f$ satisfies (48) and condition (67). Then, for each $c \in \mathbb{R}^n$ the BVP (1), (9) has at least one solution on $[0, 1]$. This solution satisfies the initial condition (7).

Proof. By Theorem 25 there exists $\delta \in (0, a]$ such that for each $b \in \mathbb{R}^n$ the BVP (1), (62) has a unique solution on $[0, \delta]$. Consider such $\delta$ and $\Phi$ from (53). As in the beginning of Section 5.3, take the subspaces $X_\delta$ and $X_\epsilon$ associated with the eigenvalues of $M(0)$ and the projections $N : \mathbb{R}^n \to X_\delta$ and $P : \mathbb{R}^n \to X_\epsilon$. Operators $N_b$ and $P_b$ are given in (63). Similarly, introduce the operators $N^* : C([\delta, 1]; \mathbb{R}^n) \to C([\delta, 1]; \mathbb{R}_X)$, $P^* : C([\delta, 1]; \mathbb{R}^n) \to C([\delta, 1]; X_\epsilon)$

$$(N^* x)(t) = N x(t), \quad (P^* x)(t) = P x(t), \quad t \in [\delta, 1].$$ 

Step 1. Define an operator $K_2 : C([\delta, 1]; \mathbb{R}^n) \to C([\delta, 1]; \mathbb{R}^n)$ by (58), or equivalently,

$$(K_2 x)(t) = \Phi(t)c - \Phi(t) \int_{t_\delta}^t \Phi^1(s) g(x(s)) \frac{d s}{s}. \quad (68)$$ 

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Applying the arguments from the proof of Theorem 20 to the operator $P^*K_2$, we see that there exists its fixed point $w \in C([\delta, 1]; X_+).$ That is, $P^*K_2w = w$ and we have

$$w(t) = P\left(\Phi(t)c - \Phi(t)\int_{t}^{1} \Phi^{-1}(s) \frac{g(s, w(s))}{s} \, ds\right), \quad t \in [\delta, 1].$$

Differentiation now yields

$$w'(t) = P\left(\Phi'(t)c - \Phi'(t)\int_{t}^{1} \Phi^{-1}(s) \frac{g(s, w(s))}{s} \, ds\right) + \frac{Pg(t, w(t))}{t}, \quad t \in [\delta, 1],$$

and using (40) and (53), we see that $w \in C([\delta, 1]; X_+)$ is a solution of the TVP on $[\delta, 1],

$$w'(t) = \frac{M(t)}{t}w(t) + \frac{Pf(t, w(t))}{t}, \quad t \in [\delta, 1], \quad w(1) = Pc.$$  \hspace{1cm} (68)

Step 2. Consider the BVP (1), (62) on $[0, \delta]$ and choose $Pb = w(\delta).$ Then, by Theorem 25 there exists a unique solution $u$ of the BVP (1), (62) on $[0, \delta]$ and

$$Pu(\delta) = w(\delta).$$  \hspace{1cm} (69)

In addition, $u$ satisfies the initial condition (7).  

Step 3. Define an operator $K_1 : C([\delta, 1]; \mathbb{R}^n) \rightarrow C([\delta, 1]; \mathbb{R}^n)$ by

$$(K_1, x)(t) = \Phi(t)\Phi^{-1}(\delta)u(\delta) + \Phi(t)\int_{\delta}^{t} \Phi^{-1}(s) \frac{g(s, w(s) + x(s))}{s} \, ds,$$

where $w \in C([\delta, 1]; X_+)$ is from Step 1 and $u(\delta) \in \mathbb{R}^n$ is from Step 2. Applying the arguments from the proof of Theorem 15 to the operator $N^*K_1$, we see that there exists its fixed point $z \in C([\delta, 1]; X_-).$ Hence, $N^*K_1z = z$ and we have

$$z(t) = N\left(\Phi(t)\Phi^{-1}(\delta)u(\delta) + \Phi(t)\int_{\delta}^{t} \Phi^{-1}(s) \frac{g(s, w(s) + x(s))}{s} \, ds\right), \quad t \in [\delta, 1].$$  \hspace{1cm} (70)

Differentiating (70), we obtain for $t \in [\delta, 1]

$$z'(t) = N\left(\Phi'(t)\Phi^{-1}(\delta)u(\delta) + \Phi'(t)\int_{\delta}^{t} \Phi^{-1}(s) \frac{g(s, w(s) + x(s))}{s} \, ds\right) + \frac{Ng(t, z(t) + z(t))}{t}.$$ 

Using (53), (40), and (70), we can see that $z \in C([\delta, 1]; X_-)$ is a solution of the IVP

$$z'(t) = \frac{M(t)}{t}z(t) + \frac{Nf(t, w(t) + z(t))}{t}, \quad t \in [\delta, 1], \quad z(\delta) = Nu(\delta),$$  \hspace{1cm} (71)

on $[\delta, 1].$ Let $v := w + z.$ Then, by (59), (69), and (71)

$$u(\delta) = Pu(\delta) + Nu(\delta) = w(\delta) + z(\delta) = v(\delta).$$  \hspace{1cm} (72)

Moreover, $v$ belongs to $C([\delta, 1]; \mathbb{R}^n),

$$Pv(t) = w(t), \quad Nv(t) = z(t), \quad t \in [\delta, 1],$$  \hspace{1cm} (73)

and by (67),

$$Pf(t, v(t)) = Pf(t, Pv(t)) = Pf(t, w(t)), \quad t \in [\delta, 1].$$  \hspace{1cm} (74)

Therefore the equation in (71) can be written in the form

$$z'(t) = \frac{M(t)}{t}Nv(t) + \frac{Nf(t, v(t))}{t}, \quad t \in [\delta, 1].$$
and, by (74), the equation in (68) can be rewritten as
\[ w'(t) = \frac{M(t)}{t} P v(t) + \frac{P f(t, v(t))}{t}, \quad t \in [\delta, 1]. \]

Adding the last two equations to each other results in
\[ v'(t) = \frac{M(t)}{t} v(t) + \frac{f(t, v(t))}{t}, \quad t \in [\delta, 1]. \]  
(75)

Therefore, \( v \) is a solution of equation (1) on \([\delta, 1]\). Due to (1), (72), and (75), we have \( u'(\delta) = v'(\delta) \) and so, \( v \) is a \( C^1 \)-continuation of \( u \) on \([\delta, 1]\). As a result, the mapping
\[ x(t) = \begin{cases} u(t), & t \in [0, \delta], \\ v(t), & t \in [\delta, 1], \end{cases} \]
is a solution of equation (1) on \([0, 1]\). Since the condition in (68) can be due to (73) expressed as \( P v(1) = P c \) and \( u \) satisfies (7) according to Step 2, \( x \) satisfies (9) as well as (7).

**Corollary 27.** Let the Lipschitz condition (46) hold on the whole interval \([0, 1]\). This means that in Theorem 26, \( a = 1 \). Then, for each \( c \in \mathbb{R}^n \) the BVP (1), (9) has a unique solution on \([0, 1]\). This solution satisfies the initial condition (7).

**Proof.** Let \( \delta \) be from Theorem 25 and choose \( c \in \mathbb{R}^n \). Since the Lipschitz condition (46) is satisfied for \( t \in [\delta, 1] \), there exists a unique solution \( w \) of the TVP (68) on \([\delta, 1]\) and \( w \in C([\delta, 1]; X_\cdot) \). By Theorem 25, there exists a unique solution \( u \) of the BVP (1), (62) on \([0, \delta]\) and \( P u(\delta) = w(\delta) \). Similarly, since the Lipschitz condition (46) is satisfied for \( t \in [\delta, 1] \), there exists a unique solution \( z \) of the IVP (71) on \([\delta, 1]\) and \( z \in C([\delta, 1]; X_\cdot) \). Finally,
\[ x(t) = \begin{cases} u(t), & t \in [0, \delta], \\ w(t) + z(t), & t \in [\delta, 1], \end{cases} \]
is a solution \( x \) of the BVP (1), (9) on \([0, 1]\). The uniqueness of the solution \( x \) follows since \( u, w, \) and \( z \) are unique.

**Remark 28.** Theorem 26 and Corollary 27 hold also when we replace assumption (67) by
\[ N f(t, x) = N f(t, Nx), \quad t \in [0, \delta], \quad x \in \mathbb{R}^n. \]  
(76)

6. **Numerical simulation of model problems**

We use the open domain MATLAB code \texttt{bvpsuite2.0} [2] to solve the model problems designed in Section 6, see [21] for \texttt{bvpsuite1.1}, the original version of the code. The basic solver of the code is based on polynomial collocation, a widely used and well-studied standard solution method for two-point BVPs [1]. Moreover, for singular problems, many popular discretization methods like finite differences, Runge–Kutta and multistep methods show order reductions, thus making computations inefficient and prohibiting asymptotically correct error estimation and reliable mesh adaptation. Therefore, in our code development, we have chosen collocation as a high-order, robust, general-purpose numerical method. The above codes are new versions of the general purpose MATLAB code \texttt{sbvp}, cf. [3], which has already been successfully applied to a variety of problems, see for example [4, 7, 19, 20, 21]. The code is designed to solve implicit systems of differential equations whose order may vary\(^1\). Here, as an example, we consider the problem
\[ F(t, y^{(4)}(t), y^{(3)}(t), y^{(2)}(t), y'(t), y(t)) = 0, \quad 0 < t \leq 1, \]  
(77)
\[ b(y^{(3)}(0), y''(0), y'(0), y(0), y^{(3)}(1), y''(1), y'(1), y(1)) = 0. \]  
(78)
\[^1\text{The order can also be zero, which means that algebraic constrains which do not involve derivatives are also admitted.}\]
Problem (77), (78) may also include unknown parameters to be calculated along with the solution \( y \). In this case, appropriately many additional boundary conditions in (78) need to be prescribed. For this reason the problem at hand is in scope of bvp-suite and we can run the code with no further pre-handling.

The numerical approximation defined by collocation is computed as follows: On a grid
\[
\Delta := \{ \tau_i : i = 0, \ldots, L \}, \quad 0 = \tau_0 < \tau_1 \cdots < \tau_L = 1,
\]
we approximate each component of the analytical solution by a piecewise defined collocating function
\[
p(s) := p_i(s), \quad s \in [\tau_i, \tau_{i+1}], \quad i = 0, \ldots, L - 1,
\]
where we require \( p \in C^{r-1}[0, 1] \) if the order of the underlying differential equation (or the highest derivative of the solution component) is \( q \). Here, \( p_i \) are polynomials of maximal degree \( k - 1 + q \) which satisfy the system (77) in the collocation points,
\[
(t_{ij} = \tau_i + p_j(\tau_{i+1} - \tau_i), \quad i = 0, \ldots, L - 1, \quad j = 1, \ldots, k), \quad 0 < \rho_1 < \cdots < \rho_k < 1.
\]
The associated boundary conditions (78) are also prescribed for \( p \). Classical theory [1] predicts that the convergence order is at least \( O(h^k) \), where \( h \) is the maximal stepsize, \( h := \max|\tau_{i+1} - \tau_i| \). The same could be shown for the first order problems with a singularity of the first kind. Quite often, even the superconvergence order, in case of Gaussian points \( O(h^{2k}) \), can be observed in practice. To make the computations more efficient, an adaptive mesh selection strategy based on an a posteriori estimate for the global error of the collocation solution has been implemented.

The general form of the examples used as an illustration for the theory is
\[
x'(t) = M(t) x(t) + \mu \frac{f(t, x(t))}{t}, \quad t \in (0, 1],
\]
where \( \mu > 0 \) is a given parameter. Moreover,
\[
M(t) = \begin{pmatrix} A_1 + m_{11}(t) & m_{12}(t) \\ m_{21}(t) & A_2 + m_{22}(t) \end{pmatrix}, \quad M(0) = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}, \quad m_{ij} \in C[0, 1], \quad i, j = 1, 2.
\]
We assume that \( A_1, A_2 \in \mathbb{R}, A_1 \neq 0, A_2 \neq 0, \) and define \( \beta := \max\{|1/|A_1|, 1/|A_2|\} \). The function \( f \) is chosen to have the following form:
\[
f(t, x) = p(t) + H(t) r(t, x), \quad (79)
\]
where \( p \in C([0, 1]; \mathbb{R}^2), H \in C([0, 1]; \mathbb{R}^{2 \times 2}), \) and \( r \in C([0, 1] \times \mathbb{R}; \mathbb{R}^2) \) are constructed in such a way that the function \( \mu f \) satisfies the Lipschitz condition (46) and the growth condition (48).

6.1. Example 1

The first example has two versions. The eigenvalues of \( M(0) \), \( \lambda_1 = -1, \lambda_2 = -2, \) are negative and therefore, we formulate these problems as IVPs and relate Examples 1.1 and 1.2 to Corollary 16 and Theorem 15, respectively.

Example 1.1: Here, we consider
\[
x'(t) = \frac{M(t)}{t} x(t) + \frac{1}{3} \frac{f(t, x(t))}{t}, \quad t \in (0, 1], \quad M(0)x(0) + \frac{1}{3} f(0, x(0)) = 0,
\]
where
\[
M(t) = \begin{pmatrix} -1 + (\exp(t) - 1) & \exp(-t) - 1 \\ \exp(2t) - 1 & -2 + (\exp(-2t) - 1) \end{pmatrix}, \quad M(0) = \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix}, \quad \beta = \max\left\{ 1, \frac{1}{2} \right\} = 1.
\]
The function \( f \) has the form specified in (79) with
\[
p(t) = \begin{pmatrix} t^2/2 \\ -1 \end{pmatrix}, \quad r(t, x) = \begin{pmatrix} \sin(x_1 + x_2) \\ \cos(x_1 + x_2) \end{pmatrix}, \quad H(t) = \begin{pmatrix} t & 0 \\ \sin(t) & \cos(t) \end{pmatrix}.
\]
Since \( \max_{t \in [0,1]} |H(t)| = \sqrt{2} \) and \( |r(t, x) - r(t, y)| \leq 2|x - y| \), for \( t \in [0,1], x, y \in \mathbb{R}^2 \), we conclude
\[
\left| \frac{1}{3} f(t, x) - \frac{1}{3} f(t, y) \right| \leq \frac{2 \sqrt{2}}{3} |x - y|, \quad t \in [0,1], \quad x, y \in \mathbb{R}^2.
\]
Hence, \( 1/3f \) satisfies the Lipschitz condition (46) on \([0,1]\) and by Corollary 16, the IVP (80) has a unique solution on \([0,1]\), see Figure 1, left.

**Example 1.2:** In the second version of Example 1, we change the value of \( \mu \) to \( \mu = \frac{1}{2} \) and obtain
\[
x'(t) = \frac{M(t)}{t} x(t) + \frac{5}{2} f(t, x(t)), \quad t \in (0,1], \quad M(0)x(0) + \frac{5}{2} f(0, x(0)) = 0.
\]
(81)
The matrices \( M(t) \) and \( H(t) \) as well as the vector \( r(t) \) remain unchanged and \( \beta = 1 \). In \( f \) we now have
\[
r(t, x) = a_1(t) \left( \frac{\sin(x_1 + x_2)}{\cos(x_1 + x_2)} \right) + a_2(t) \left( \frac{x_1 \log(1 + |x_1 + x_2|)}{x_2 \log(1 + |x_1 + x_2|)} \right),
\]
where
\[
a_1(t) = 0.5(t - 0.5)^2, \quad a_2(t) = \begin{cases} 0, & t < 0.5, \\ -(t - 0.5)^2, & t \geq 0.5. \end{cases}
\]
We choose \( a = 0.5 \). Then \( \max_{t \in [0,0.5]} |H(t)| < \sqrt{2} \) and \( \max_{t \in [0,0.5]} |r_1(t)| = 1/8 \). Thus, \( |r(t, x) - r(t, y)| \leq 1/4|x - y| \), for \( t \in [0,0.5], x, y \in \mathbb{R}^2 \). Consequently, \( 5/2f \) satisfies the Lipschitz condition (46) on \([0,0.5]\). In addition \( 5/2f \) satisfies the growth condition (48) on \([0.5,1]\) with \( w(s) = (1 + 2s) \ln(1 + 2s) + 1, s \in [0,\infty) \). Now, from Theorem 15 it follows that the IVP (81) has a solution on \([0,1]\), see Figure 1, right.

**6.2. Example 2**
The second example is also given in two versions. Now, the eigenvalues of \( M(0) \), \( \lambda_1 = 1, \lambda_2 = 2 \), are positive and Examples 2.1 and 2.2 take the form of TVPs. Again, we relate Examples 2.1 and 2.2 to Corollary 21 and Theorem 20, respectively.
Example 2.1: The structure of the problem and the data functions are similar to those in Example 1.1. We study the TVP
\[ x'(t) = \frac{M(t)}{t} x(t) + \frac{1}{3} \frac{f(t, x(t))}{t}, \quad t \in (0, 1], \quad x(1) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \]
\[ M(t) = \begin{pmatrix} 1 + \exp(t) - 1 & \exp(-t) - 1 \\ \exp(2t) - 1 & 2 + \exp(-2t) - 1 \end{pmatrix}, \quad M(0) = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}, \quad \beta = \max \left\{ 1, \frac{1}{2} \right\} = 1. \]

Here \( f \) is given by (79), where \( p, r, H \) are from Example 1.1. The solvability follows from Corollary 21, cf. Figure 2.

Example 2.2: In the second version of Example 2, we change the value of \( \mu \) to \( \mu = 5/2 \) and obtain
\[ x'(t) = \frac{M(t)}{t} x(t) + \frac{5}{2} \frac{f(t, x(t))}{t}, \quad t \in (0, 1], \quad x(1) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \] (82)

The matrix \( M(t) \) remains unchanged and \( \beta = 1 \). Function \( f \) is here determined by \( p(t), H(t) \) and \( r(t, x) \) from Example 2.2. By Theorem 20, a solution of the TVP (82) exists, cf. Figure 3.
6.3. Example 3
We finally address the case of general BVPs. This means that the eigenvalues of \( M(0) \) have different signs, \( \lambda_1 = -1 \) is negative and \( \lambda_2 = 2 \) is positive. Consequently, the boundary conditions are two-point conditions involving both boundaries \( t = 0 \) and \( t = 1 \). The corresponding theoretical results for Examples 3.1 and 3.2 are formulated in Corollary 27 and Theorem 26, respectively.

**Example 3.1:** The ODE system and the data functions but \( H(t) \), are identical to those in Example 1.1. However, we consider a two-point BVP of the form

\[
x'(t) = \frac{M(t)}{t} x(t) + \frac{1}{3} \frac{f(t, x(t))}{t}, \quad t \in (0, 1], \quad -x_1(0) + \frac{1}{3} f_1(0, x(0)) = 0, \quad x_2(1) = 1.
\]

(83)

The function \( f \) has again the form (79) with

\[
p(t) = \begin{pmatrix} r^2 / 2 \\ -1 \end{pmatrix}, \quad r(t, x) = \begin{pmatrix} \sin(x_2) \\ \cos(x_1 + x_2) \end{pmatrix}, \quad H(t) = \begin{pmatrix} \sin(t) & \cos(t) \end{pmatrix}.
\]

Since \( M(0) \) coincides with its Jordan canonical form \( J \), we have by (60), \( N = \tilde{N}, P = \tilde{P} \), and

\[
\tilde{N} M(0) x(0) = \begin{pmatrix} \lambda_1 x_1(0) \\ 0 \end{pmatrix}, \quad \tilde{N} \frac{1}{3} f(0, x(0)) = \begin{pmatrix} \frac{1}{3} f_1(0, x(0)) \\ 0 \end{pmatrix}, \quad \tilde{P} c(1) = \begin{pmatrix} 0 \\ x_2(1) \end{pmatrix}.
\]

Also note that function \( f \) satisfies (67). Thus, the correctly posed boundary conditions read:

\[
\tilde{N} M(0) x(0) + \tilde{N} \frac{1}{3} f(0, x(0)) = 0, \quad \tilde{P} c(1) = \tilde{P} c, \quad c \in \mathbb{R}^2.
\]

For \( c = (1, 1)^T \) this amounts to

\[
\lambda_1 x_1(0) + \frac{1}{3} f_1(0, x(0)) = 0, \quad x_2(1) = 1 \iff -x(0) + \frac{1}{3} f_1(0, x(0)) = 0, \quad x_2(1) = 1,
\]

cf. (83). From Corollary 27, we conclude that the solution to the BVP (83) exists and is unique. The solution of the BVP (83) can be found in Figure 4, left.

![Figure 4: Example 3.1 (left): Numerical solution obtained from bvpsuite2.0 with four Gaussian collocation points. Mesh adaptation has been used with tolerances Tol_0 = Tol_1 = 10^{-12}. The number of subintervals in the initial mesh was 200, in the final mesh 643. The almost equidistant final mesh, where every third point is shown, is also included. Example 3.2 (right): Numerical solution obtained from bvpsuite2.0 with four Gaussian collocation points. Mesh adaptation has not been used. The number of equidistant subintervals in the final mesh was 400. The error estimate indicates a maximal global relative error of 3.7 \cdot 10^{-7}. Again the data functions are nonsmooth, cf. \( \alpha_2 \).](image)

**Example 3.2:** In the final problem, we change the value of \( \mu \) to \( \mu = 5/2 \) and obtain

\[
x'(t) = \frac{M(t)}{t} x(t) + \frac{5}{2} \frac{f(t, x(t))}{t}, \quad t \in (0, 1], \quad -x_1(0) + \frac{5}{2} f_1(0, x(0)) = 0, \quad x_2(1) = 1.
\]

(84)

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The matrix \( M(t) \) as well as \( p(t) \) remain unchanged and \( \beta = 1 \). The functions \( H \) and \( r \) specifying function \( f \) read

\[
H(t) = \begin{pmatrix} t \\ \sin(t) \\ \cos(t) \end{pmatrix}, \quad r(t, x) = \alpha_1(t) \left( \frac{\sin(x_1)}{\cos(x_1 + x_2)} \right) + \alpha_2(t) \left( \frac{x_1 \log(1 + |x_1 + x_2|)}{x_2 \log(1 + |x_1 + x_2|)} \right),
\]

where \( \alpha_1 \) and \( \alpha_2 \) remains unchanged from Example 1.2. As before in Example 3.1, we can derive from (60) the correctly posed boundary conditions and see that (76) holds. The only alteration is the factor in front of \( f \), cf. (84). Finally, the existence of a solution to the BVP (84) follows from Theorem 26 and Remark 28, cf. Figure 4, right.

7. Future work: convergence of the collocation

We regard the convergence proof for the polynomial collocation as our next goal. As a first step, for the Examples 1.1, 1.2, and 1.3, we computed the convergence orders for the collocation schemes based on collocation at \( k = 2, 3, 4, 5 \) collocation points, Gaussian points and equidistantly spaced inner collocation points. The step sizes of coherently refined meshes varied from \( h_{\text{start}} = 1/2 \) to \( h_{\text{final}} \approx 1/1024 \). To study the errors, we computed the reference solution on a mesh with 2048 points using the collocation scheme based on 7 Gaussian points.

We monitor two types of errors, the maximal global error taken in the mesh points and the maximal global error computed ‘uniformly’ in \( t \). Let us assume that the numerical solution \( Y_h \) has been computed using the step size \( h \) and the mesh containing \( N + 1 \) mesh points. Moreover, let \( Y \) be the reference solution. Then, the maximal error taken over the mesh points is

\[
||Y_h - Y||_\Delta := \max_{0 \leq j \leq N} |Y_h(t_j) - Y(t_j)|,
\]

where \( | \cdot | \) denotes the maximum vector norm. To obtain the maximal error ‘uniformly’ in \( t \), we introduce 2048 equidistant points \( t_j, 0 \leq j \leq 2048 \) between 0 and 1 and compute

\[
||Y_h - Y||_u := \max_{0 \leq j \leq 2048} |Y_h(t_j) - Y(t_j)|.
\]

The order of convergence and the error constant \( c \) are estimated using two consecutive meshes with step sizes \( h \) and \( h/2 \). From the ansatz \( ||Y_h - Y|| \approx ch^p \) for \( h \to 0 \), we have

\[
||Y_h - Y|| = ch^p, \quad ||Y_{h/2} - Y|| = c \left( \frac{h}{2} \right)^p \Rightarrow p = \ln \left( \frac{||Y_h - Y||}{||Y_{h/2} - Y||} \right) \ln(2).
\]

Having \( p \) we calculate the error constant from \( c = ||Y_{h/2} - Y|| / \left( \frac{1}{2} \right)^p \). For these errors, following observations could be made:

1. For the IVP, Example 1.1, where the eigenvalues of \( M(0) \) are negative, we observe for the equidistant collocation points, the uniform convergence order \( O(h^k) \) for even \( k \) and \( O(h^{k+1}) \) for odd \( k \). For Gaussian collocation points, the maximal error in the mesh points shows the small superconvergence order \( O(h^{k+1}) \) for \( k = 2 \), see Table 1. Typically for the singular ODEs, the full superconvergence order \( O(h^2) \) cannot be observed [12].

2. For the TVP in Example 2.1 and the BVP in Example 3.1, there are small positive eigenvalues in the spectrum of \( M(0) \), \( \lambda_1 = 1, \lambda_2 = 2 \) and \( \lambda = 2 \), respectively. In this case, the uniform convergence order (equidistant collocation points) and the convergence order for the error in the mesh points (Gaussian collocation points) is \( O(h^2) \) for all values of \( k \) which means a clear order reduction, cf. Tables 2 and 3.

However, when the positive eigenvalue becomes larger, the above order reduction is not observed, see Table 4.

The convergence behavior of the collocation will stay in focus of our future research. Especially, it will be subject to further tests. Moreover, as next, we will analyze the situation and try to provide the full convergence theory of the collocation applied to the present class of nonlinear singular ODEs. Technically, we intend to rely on the approach proposed in [23], as we did to obtain the convergence results for the linear case with the variable coefficient matrix, cf. (4). The respective convergence analysis in the context of (4) can be found in [10]. It turns out that the existence and uniqueness analysis of the underlying analytical problem in [9] is an important prerequisite for studying the properties
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of the collocation scheme. With other words, not only [23] but also [9] constitute the basis for [10]. This close relationship between [9] and [10] can be seen in the proofs of [10, Theorems 5.2,7,2]. There, an auxiliary analytical perturbed IVP and BVP, respectively, is constructed, whose solution, a piecewise polynomial function \( e(t) \), differs from the true global error only by higher order terms and the inhomogeneities of these auxiliary problems contain defects whose leading terms show the proper powers of \( h \). In order to estimates \( e(t) \) and thus the global error of the scheme, analytical results derived in [9] are used. Analytical results from the present paper will be used for the convergence analysis of the collocation scheme in the nonlinear case. We will apply the way of reasoning proposed in [23], but certainly, a very careful step by step checking of the arguments given there is necessary. This is the aim for our next paper *On nonlinear singular BVP with nonsmooth data. Part 2: Convergence of the collocation schemes*.

8. Conclusions

In the present paper, we study existence and uniqueness of solutions to the nonlinear systems of ODEs with the time singularity (1). It turns out that the eigenvalues of the matrix \( M(0) \) decide about the kind and structure of boundary conditions which are necessary for the problem to be well-posed and for the solution to be at least in \( C[0,1] \). We assume that \( M(0) \) has only eigenvalues with nonzero real parts and distinguish between three possible cases:

**Case 1.** Let all eigenvalues of \( M(0) \) have negative real parts, then the correctly posed BVP has the form of an IVP,

\[
x'(t) = \frac{M(t)}{t} x(t) + \frac{f(t,x(t))}{t}, \quad t \in (0,1], \quad M(0)x(0) + f(0,x(0)) = 0.
\]

**Case 2.** Let all eigenvalues of \( M(0) \) have positive real parts, then the correctly posed BVP has the form of a TVP,

\[
x'(t) = \frac{M(t)}{t} x(t) + \frac{f(t,x(t))}{t}, \quad t \in (0,1], \quad x(1) = c \in \mathbb{R}^n
\]

whose solution also satisfies condition \( M(0)x(0) + f(0,x(0)) = 0 \).

**Case 3.** Let \( M(0) \) have both, eigenvalues with negative and eigenvalues with positive real parts. Then, we have to formulate the problem as a two-point BVP,

\[
x'(t) = \frac{M(t)}{t} x(t) + \frac{f(t,x(t))}{t}, \quad t \in (0,1], \quad NM(0)x(0) + Nf(0,x(0)) = 0, \quad \begin{aligned} Px(1) &= Pc, \\
&= c \in \mathbb{R}^n.
\end{aligned}
\]

As before, the solution \( x \) satisfies \( M(0)x(0) + f(0,x(0)) = 0 \). Here, \( N \) and \( P \) are suitably defined projections.

Under appropriate assumptions on the problem data, we can show existence and uniqueness of solutions in Cases 1 to 3. Theoretical findings are illustrated by numerical simulation of model problems carried out in MATLAB with the software package *bvpsuite2.0* [2].

Acknowledgements

The authors Jana Burkotová, Irena Rachůnková and Svatoslav Staněk gratefully acknowledge support received from the grant No. 14-06958S of the Grant Agency of the Czech Republic.
