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A. Sokolova and H. Woracek

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Institute for Analysis and Scientific Computing
Vienna University of Technology
Wiedner Hauptstraße 8–10
1040 Wien, Austria

E-Mail: admin@asc.tuwien.ac.at
WWW: http://www.asc.tuwien.ac.at
FAX: +43-1-58801-10196

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Proper Semirings and Proper Convex Functors

Ana Sokolova[1] and Harald Woracek[2]

[1] University of Salzburg, Austria
ana.sokolova@cs.uni-salzburg.at
[2] TU Vienna, Austria
harald.woracek@tuwien.ac.at

Abstract. Esik and Maletti introduced the notion of a proper semiring and proved that some important (classes of) semirings – Noetherian semirings, natural numbers – are proper. Properness matters as the equivalence problem for weighted automata over proper and finitely and effectively presented semirings is decidable. Milius generalised the notion of properness from a semiring to a functor. As a consequence, a semiring is proper if and only if its associated "cubic functor" is proper. Moreover, properness of a functor renders soundness and completeness proofs of axiomatizations of equivalent behaviour.
In this paper we provide a method for proving properness of functors, and instantiate it to cover both the known cases and several novel ones: (1) properness of the semirings of positive rationals and positive reals, via properness of the corresponding cubic functors; and (2) properness of two functors on (positive) convex algebras. The latter functors are important for axiomatizing trace equivalence of probabilistic transition systems. Our proofs rely on results that stretch all the way back to Hilbert and Minkowski.

Keywords: proper semirings, proper functors, coalgebra, weighted automata, probabilistic transition systems

1 Introduction

In this paper we deal with algebraic categories and deterministic weighted automata functors on them. Such categories are the target of generalized determinization [25, 26, 12] and enable coalgebraic modelling beyond sets. For example, non-deterministic automata, weighted, or probabilistic ones are coalgebraically modelled over the categories of join-semilattices, semimodules for a semiring, and convex sets, respectively. Moreover, expressions for axiomatizing behavior semantics often live in algebraic categories.

In order to prove completeness of such axiomatizations, the common approach [24, 5, 26] is to prove finality of a certain object in a category of coalgebras over an algebraic category. Proofs are significantly simplified if it suffices to verify finality only w.r.t. coalgebras carried by free finitely generated algebras, as those are the coalgebras that result from generalized determinization.

In recent work, Milius [18] proposed the notion of a proper functor on an algebraic category that provides a sufficient condition for this purpose. This notion is an extension of the notion of a proper semiring introduced by Esik and Maletti [9]: A semiring is proper if and only if its "cubic" functor is proper. A cubic functor is a functor $\mathbb{S} \times (-)^A$ where A is a finite alphabet and $\mathbb{S}$ is a free algebra with a single generator in the algebraic category. Cubic functors model deterministic weighted automata which are models of determinizations of non-deterministic and probabilistic transition systems.

Properness is the property that for any two states that are behaviourally equivalent in coalgebras with free finitely generated carriers, there is a zig-zag of homomorphisms (called a chain of simulations in the original works on weighted automata and proper semirings) that identifies the two states, whose nodes are all carried by free finitely generated algebras.

Even though the notion of properness is relatively new for a semiring and very new for a functor, results on properness of semirings can be found in more distant literature as well. Here is a brief history, to the best of our knowledge:

- The Boolean semiring was proven to be proper in [4].
- Finite commutative ordered semirings were proven to be proper in [8, Theorem 5.1]. Interestingly, the proof provides a zig-zag with at most seven intermediate nodes.
- Any euclidean domain and any skew field were proven proper in [2, Theorem 3]. In each case the zig-zag has two intermediate nodes.
- The semiring of natural numbers $\mathbb{N}$, the Boolean semiring $\mathbb{B}$, the ring of integers $\mathbb{Z}$ and any skew field were proven proper in [3, Theorem 1]. Here, all zig-zag were spans, i.e., had a single intermediate node with outgoing arrows.
- Noetherian semirings were proven proper in [9, Theorem 4.2], commutative rings also in [9, Corollary 4.4], and finite semirings as well in [9, Corollary 4.5], all with a zig-zag being a span. Moreover, the tropical semiring is not proper, as proven in [9, Theorem 5.4].

Having properness of a semiring, together with the property of the semiring being finitely and effectively presentable, yields decidability of the equivalence problem (decidability of trace equivalence) for weighted automata.

In this paper, motivated by the wish to prove properness of a certain functor $\widehat{F}$ on convex algebras used for axiomatizing trace semantics of probabilistic systems in [26], as well as by the open questions stated in [18, Example 3.19], we provide a framework for proving properness. We instantiate this framework on known cases like Noetherian semirings and $\mathbb{N}$ (with a zig-zag that is a span), and further prove new results of properness:

- The semirings $\mathbb{Q}_+$ and $\mathbb{R}_+$ of positive rationals and reals, respectively, are proper. The shape of the zig-zag is a span as well.
- The functor $[0,1] \times (-)^A$ on `PCA` is proper, again the zig-zag being a span.
- The functor $\widehat{F}$ on `PCA` is proper. This proof is the most involved, and interestingly, provides the only case where the zig-zag is not a span: it contains three intermediate nodes of which the middle one forms a span.

Our framework requires a proof of so-called *extension* and *reduction lemma* in each case. While the extension lemma is a generic result that covers all cubic functors of interest, the reduction lemma is in all cases a nontrivial property intrinsic to the algebras under consideration. For the semiring of natural numbers it is a consequence of a result that we trace back to Hilbert; for the case of convex algebra $[0,1]$ the result is due to Minkowski. In case of $\widehat{F}$, we use Kakutani's set-valued fixpoint theorem.

It is an interesting question for future work whether these new properness results may lead to new complete axiomatizations of expressions for certain weighted automata.

The organization of the rest of the paper is as follows. In Section 2 we give some basic definitions and introduce the semirings, the categories, and the functors of interest. Section 3 provides the general framework as well as proofs of propernes of the cubic functors. Section 4–Section 6 lead us to properness of $\widehat{F}$ on $\mathtt{PCA}$. For space reasons, we present the ideas of proofs and constructions in the main paper and refer all detailed proofs to the appendix.

2 Proper functors

We start with a brief introduction of the basic notions from algebra and coalgebra needed in the rest of the paper, as well as the important definition of proper functors [18]. We refer the interested reader to [23, 13, 11] for more details. We assume basic knowledge of category theory, see e.g. [16] or Appendix A.

Let $\mathtt{C}$ be a category and F a $\mathtt{C}$-endofunctor. The category $\mathsf{Coalg}(F)$ of F-*coalgebras* is the category having as objects pairs (X, c) where X is an object of $\mathtt{C}$ and c is a $\mathtt{C}$-morphism from X to FX, and as morphisms $f\colon (X, c) \to (Y, d)$ those $\mathtt{C}$-morphisms from X to Y that make the diagram on the right commute.

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "c"'] & Y \arrow[d, "d"] \\ FX \arrow[r, "Ff"] & FY \end{tikzcd}$$

All base categories $\mathtt{C}$ in this paper will be *algebraic categories*, i.e., categories $\mathtt{Set}^T$ of Eilenberg-Moore algebras of a finitary monad [3] in $\mathtt{Set}$. Hence, all base categories are concrete with forgetful functor that is identity on morphisms.

In such categories behavioural equivalence [15, 28, 27] can be defined as follows. Let (X, c) and (Y, d) be F-coalgebras and let $x \in X$ and $y \in Y$. Then x and y are *behaviourally equivalent*, and we write $x \sim y$, if there exists an F-coalgebra (Z, e) and $\mathsf{Coalg}(F)$-morphisms $f\colon (X, c) \to (Z, e)$, $g\colon (Y, d) \to (Z, e)$, with $f(x) = g(y)$.

$$(X, c) \xrightarrow{\ f\ } \underset{f(x)=g(y)}{(Z, e)} \xleftarrow{\ g\ } (Y, d)$$

If there exists a final coalgebra in $\mathsf{Coalg}(F)$, and all categories considered in this paper will have this property, then two elements are behaviourally equivalent if

[3] The notions of monads and algebraic categories are central to this paper. We recall them in Appendix A to make the paper accessible to all readers.

and only if they have the same image in the final. If we have a *zig-zag diagram* in $\mathsf{Coalg}(F)$

$$\begin{tikzcd}
(X,c) \arrow[dr, "f_1"] & & (Z_2,e_2) \arrow[dl, "f_2"'] \arrow[dr, "f_3"] & & \cdots \arrow[dl, "f_4"'] \arrow[dr, "f_{2n-1}"] & & (Y,d) \arrow[dl, "f_{2n}"'] \\
& (Z_1,e_1) & & (Z_3,e_1) & & (Z_{2n-1},e_1) &
\end{tikzcd} \tag{1}$$

which relates x with y in the sense that there exist elements $z_{2k} \in Z_{2k}$, $k = 1, \ldots, n-1$, with (setting $z_0 = x$ and $z_{2n} = y$)

$$f_{2k}(z_{2k}) = f_{2k-1}(z_{2k-2}), \quad k = 1, \ldots, n,$$

then $x \sim y$.

We now recall the notion of a proper functor, introduced by Milius [18] which is central to this paper. It is very helpful for establishing completeness of regular expressions calculi, cf. [18, Corollary 3.17].

Definition 1. Let $T\colon \mathtt{Set} \to \mathtt{Set}$ be a finitary monad with unit η and multiplication μ. A $\mathtt{Set}^T$-endofunctor F is *proper*, if the following statement holds.

For each pair (TB_1, c_1) and (TB_2, c_2) of F-coalgebras with B_1 and B_2 finite sets, and each two elements $b_1 \in B_1$ and $b_2 \in B_2$ with $\eta_{B_1}(b_1) \sim \eta_{B_2}(b_2)$, there exists a zig-zag (1) in $\mathsf{Coalg}(F)$ which relates $\eta_{B_1}(b_1)$ with $\eta_{B_2}(b_2)$, and whose nodes (Z_j, e_j) all have free and finitely generated carrier.

This notion generalizes the notion of a proper semiring introduced by Esik and Maletti in [9, Definition 3.2], cf. [18, Remark 3.10].

Remark 2. In the definition of properness the condition that intermediate nodes have free *and* finitely generated carrier is necessary for nodes with incoming arrows (the nodes Z_{2k-1} in (1)). For the intermediate nodes with outgoing arrows (Z_{2k} in (1)), it is enough to require that their carrier is finitely generated. This follows since every F-coalgebra with finitely generated carrier is the image under an F-coalgebra morphism of an F-coalgebra with free and finitely generated carrier.

Moreover, note that zig-zag's which start (or end) with incoming arrows instead of outgoing ones, can also be allowed since a zig-zag of this form can be turned into one of the form (1) by appending identity maps.

Some concrete monads and functors

We deal with the following base categories.

- The category $\mathbb{S}$-$\mathtt{SMOD}$ of semimodules over a semiring $\mathbb{S}$ induced by the monad $T_{\mathbb{S}}$ of finitely supported maps into $\mathbb{S}$, see, e.g., [17, Example 4.2.5].
- The category $\mathtt{PCA}$ of positively convex algebras induced by the monad of finitely supported subprobability distributions, see, e.g., [6, 7] and [20].

For $n \in \mathbb{N}$, the free algebra with n generators in $\mathbb{S}$-$\mathtt{SMOD}$ is the direct product $\mathbb{S}^n$, and in $\mathtt{PCA}$ it is the n-simplex $\Delta^n = \{(\xi_1, \ldots, \xi_n) \mid \xi_j \geq 0, \sum_{j=1}^n \xi_j \leq 1\}$.

Concerning semimodule-categories, we mainly deal with the semirings $\mathbb{N}$, $\mathbb{Q}_+$, and $\mathbb{R}_+$, and their ring completions $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{R}$. For these semirings the categories of $\mathbb{S}$-semimodules are

- $\mathtt{CMON}$ of commutative monoids for $\mathbb{N}$,
- $\mathtt{AB}$ of abelian groups for $\mathbb{Z}$,
- $\mathtt{CONE}$ of convex cones for $\mathbb{R}_+$,
- $\mathbb{Q}$-$\mathtt{VEC}$ and $\mathbb{R}$-$\mathtt{VEC}$ of vector spaces over the field of rational and real numbers, respectively, for $\mathbb{Q}$ and $\mathbb{R}$.

We consider the following functors, where A is a fixed finite alphabet.

- The *cubic functor* $F_{\mathbb{S}}$ on $\mathbb{S}$-$\mathtt{SMOD}$. By this we mean the functor acting as

$$F_{\mathbb{S}}X = \mathbb{S} \times X^A \text{ for } X \text{ object of } \mathbb{S}\text{-}\mathtt{SMOD},$$
$$F_{\mathbb{S}}f = \mathrm{id}_{\mathbb{S}} \times (f \circ -) \text{ for } f\colon X \to Y \text{ morphism of } \mathbb{S}\text{-}\mathtt{SMOD}.$$

  We chose the name since $F_{\mathbb{S}}$ assigns to objects X a full direct product, i.e., a full cube. The underlying $\mathtt{Set}$ functors of cubic functors are also sometimes called deterministic-automata functors, see e.g. [12], as their coalgebras are deterministic weighted automata with output in the semiring.
- The *cubic functor* $F_{[0,1]}$ on $\mathtt{PCA}$. By this we mean the functor $F_{[0,1]}X = [0,1] \times X^A$ and $F_{[0,1]}f = \mathrm{id}_{[0,1]} \times (f \circ -)$.
- A *subcubic convex functor* $\widehat{F}$ on $\mathtt{PCA}$ whose action will be introduced in Definition 11 [4]. The name origins from the fact that $\widehat{F}X$ is a certain convex subset of $F_{[0,1]}X$ and that $\widehat{F}f = (F_{[0,1]}f)|_{\widehat{F}X}$ for $f\colon X \to Y$.

All functors are liftings of $\mathtt{Set}$-endofunctors. In particular, they preserve surjective algebra homomorphisms, which is a property needed to apply the work of Milius, cf. [18, Assumptions 3.1].

Remark 3. We can now formulate precisely the connection between proper semirings and proper functors mentioned after Definition 1. A semiring $\mathbb{S}$ is proper in the sense of [9], if and only if for every finite input alphabet A the cubic functor $F_{\mathbb{S}}$ on $\mathbb{S}$-$\mathtt{SMOD}$ is proper.

We shall interchangeably think of direct products as sets of functions or as sets of tuples. Taking the viewpoint of tuples, the definition of $F_{\mathbb{S}}f$ reads as

$$(F_{\mathbb{S}}f)\big((o, (x_a)_{a\in A})\big) = \big(o, (f(x_a))_{a\in A}\big), \quad o \in \mathbb{S},\ x_a \in X \text{ for } a \in A.$$

A coalgebra structure $c\colon X \to F_{\mathbb{S}}X$ writes as

$$c(x) = \big(c_{\mathrm{o}}(x), (c_a(x))_{a\in A}\big), \quad x \in X,$$

[4] This functor was denoted $\hat{G}$ in [26] where it was first studied in the context of axiomatization of trace semantics.

and we use $c_{\mathrm{o}} : X \to \mathbb{S}$ and $c_a : X \to X$ as generic notation for the components of the map c. More generally, we define $c_w \colon X \to X$ for any word $w \in A^*$ inductively as $c_\varepsilon = \mathrm{id}_X$ and $c_{wa} = c_a \circ c_w,\ w \in A^*, a \in A$.

The map from a coalgebra (X, c) into the final $F_{\mathbb{S}}$-coalgebra, the *trace map*, is then given as $\mathrm{tr}_c(x) = \big((c_{\mathrm{o}} \circ c_w)(x)\big)_{w \in A^*}$ for $x \in X$. Behaviour equivalence for cubic functors is the kernel of the trace map.

3 Properness of cubic functors

Our proofs of properness in this section and in Section 6 below start from the following idea. Let $\mathbb{S}$ be a semiring, and assume we are given two $F_{\mathbb{S}}$-coalgebras which have free finitely generated carrier, say $(\mathbb{S}^{n_1}, c_1)$ and $(\mathbb{S}^{n_2}, c_2)$. Moreover, assume $x_1 \in \mathbb{S}^{n_1}$ and $x_2 \in \mathbb{S}^{n_2}$ are two elements having the same trace. Let $d_j \colon \mathbb{S}^{n_1} \times \mathbb{S}^{n_2} \to F_{\mathbb{S}}(\mathbb{S}^{n_1} \times \mathbb{S}^{n_2})$ be given by

$$d_j(y_1, y_2) = \Big(c_{j\,\mathrm{o}}(y_j), ((c_{1\,a}(y_1), c_{2\,a}(y_2)))_{a \in A}\Big).$$

Denoting by $\pi_j \colon \mathbb{S}^{n_1} \times \mathbb{S}^{n_2} \to \mathbb{S}^{n_j}$ the canonical projections, both sides of the following diagram separately commute.

$$\begin{tikzcd}
\mathbb{S}^{n_1} \arrow[d, "c_1"'] & \mathbb{S}^{n_1} \times \mathbb{S}^{n_2} \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] \arrow[d, bend right, "d_1"'] \arrow[d, bend left, "d_2"] \arrow[d, phantom, "\neq"] & \mathbb{S}^{n_2} \arrow[d, "c_2"'] \\
F_{\mathbb{S}}\mathbb{S}^{n_1} & F_{\mathbb{S}}(\mathbb{S}^{n_1} \times \mathbb{S}^{n_2}) \arrow[l, "F_{\mathbb{S}}\pi_1"'] \arrow[r, "F_{\mathbb{S}}\pi_2"] & F_{\mathbb{S}}\mathbb{S}^{n_2}
\end{tikzcd}$$

However, in general the maps d_1 and d_2 do not coincide.

The next lemma contains a simple observation: there exists a subsemimodule Z of $\mathbb{S}^{n_1} \times \mathbb{S}^{n_2}$, such that the restrictions of d_1 and d_2 to Z coincide and turn Z into an $F_{\mathbb{S}}$-coalgebra.

Lemma 4. *Let Z be the subsemimodule of $\mathbb{S}^{n_1} \times \mathbb{S}^{n_2}$ generated by the pairs $(c_{1\,w}(x_1), c_{2\,w}(x_2))$ for $w \in A^*$. Then $d_1|_Z = d_2|_Z$ and $d_j(Z) \subseteq F_{\mathbb{S}}(Z)$.*

The significance of Lemma 4 in the present context is that it leads to the diagram (we denote $d = d_j|_Z$)

$$\begin{tikzcd}
 & \mathbb{S}^{n_1} \times \mathbb{S}^{n_2} \arrow[d, phantom, "\cup\!|"] & \\
\mathbb{S}^{n_1} \arrow[d, "c_1"'] & Z \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] \arrow[d, "d"] & \mathbb{S}^{n_2} \arrow[d, "c_2"'] \\
F_{\mathbb{S}}\mathbb{S}^{n_1} & F_{\mathbb{S}}Z \arrow[l, "F_{\mathbb{S}}\pi_1"'] \arrow[r, "F_{\mathbb{S}}\pi_2"] \arrow[d, phantom, "|\!\cap"] & F_{\mathbb{S}}\mathbb{S}^{n_2} \\
 & \mathbb{S} \times (\mathbb{S}^{n_1} \times \mathbb{S}^{n_2})^A &
\end{tikzcd}$$

In other words, it leads to the zig-zag in $\mathsf{Coalg}(F_{\mathbb{S}})$

$$(\mathbb{S}^{n_1}, c_1) \xleftarrow{\pi_1} (Z, d) \xrightarrow{\pi_2} (\mathbb{S}^{n_2}, c_2) \tag{2}$$

This zig-zag relates x_1 with x_2 since $(x_1, x_2) \in Z$.

As a corollary we reobtain the result [9, Theorem 4.2] of Esik and Maletti establishing properness of certain cubic functors.

Corollary 5 (Esik–Maletti 2010). *Every Noetherian semiring is proper.*

Our first main result is Theorem 6 below, where we show properness of the cubic functors $F_{\mathbb{S}}$ on $\mathbb{S}$-`SMOD`, for $\mathbb{S}$ being one of the semirings $\mathbb{N}$, $\mathbb{Q}_+$, $\mathbb{R}_+$, and of the cubic functor $F_{[0,1]}$ on `PCA`. The case of $F_{\mathbb{N}}$ is known from [3, Theorem 4] [5], the case of $F_{[0,1]}$ is stated as an open problem in [18, Example 3.19].

Theorem 6. *The cubic functors $F_{\mathbb{N}}$, $F_{\mathbb{Q}_+}$, $F_{\mathbb{R}_+}$, and $F_{[0,1]}$ are proper.*

In fact, for any two coalgebras with free finitely generated carrier and any two elements having the same trace, a zig-zag with free and finitely generated nodes relating those elements can be found, which is a span (has a single intermediate node with outgoing arrows).

The proof proceeds via relating to the Noetherian case. It always follows the same scheme, which we now outline. Observe that the ring completion of each of $\mathbb{N}$, $\mathbb{Q}_+$, $\mathbb{R}_+$, is Noetherian (for the last two it actually is a field), and that $[0, 1]$ is the positive part of the unit ball in $\mathbb{R}$.

Step 1. The extension lemma: We use an extension of scalars process to pass from the given category `C` to an associated category $\mathbb{E}$-`MOD` with a Noetherian ring $\mathbb{E}$. This is a general categorical argument.

To unify notation, we agree that $\mathbb{S}$ may also take the value $[0, 1]$, and that $T_{[0,1]}$ is the monad of finitely supported subprobability distributions giving rise to the category `PCA`.

$\mathbb{S}$	$\mathbb{N}$	$\mathbb{Q}_+$	$\mathbb{R}_+$	$[0,1]$
`C`	$\mathbb{N}$-`SMOD` (`CMON`)	$\mathbb{Q}_+$-`SMOD`	$\mathbb{R}_+$-`SMOD` (`CONE`)	`PCA`
$\mathbb{E}$-`MOD`	$\mathbb{Z}$-`MOD` (`AB`)	$\mathbb{Q}$-`MOD` ($\mathbb{Q}$-`VEC`)	$\mathbb{R}$-`MOD` ($\mathbb{R}$-`VEC`)	$\mathbb{R}$-`MOD` ($\mathbb{R}$-`VEC`)

For the formulation of the extension lemma, recall that the starting category `C` is the Eilenberg-Moore category of the monad $T_{\mathbb{S}}$ and the target category $\mathbb{E}$-`MOD` is the Eilenberg-Moore category of $T_{\mathbb{E}}$. We write $\eta_{\mathbb{S}}$ and $\mu_{\mathbb{S}}$ for the unit and multiplication of $T_{\mathbb{S}}$ and analogously for $T_{\mathbb{E}}$. We have $T_{\mathbb{S}} \leq T_{\mathbb{E}}$, via the inclusion $\iota\colon T_{\mathbb{S}} \Rightarrow T_{\mathbb{E}}$ given by $\iota_X(u) = u$, as $\eta_{\mathbb{E}} = \iota \circ \eta_{\mathbb{S}}$ and $\mu_{\mathbb{E}} \circ \iota\iota = \iota \circ \mu_{\mathbb{S}}$ where $\iota\iota \overset{\text{def}}{=} T_{\mathbb{E}}\iota \circ \iota \overset{\text{nat.}}{=} \iota \circ T_{\mathbb{S}}\iota$.

[5] In [3] only a sketch of the proof is given, cf. [3, §3.3]. In this sketch one important point is not mentioned. Using the terminology of [3, §3.3]: it could a priori be possible that the size of the vectors in G and the size of G both oscillate.

Definition 7. Let $(X, \alpha_X) \in \mathtt{Set}^{T_\mathbb{S}}$ and $(Y, \alpha_Y) \in \mathtt{Set}^{T_\mathbb{E}}$ where $T_\mathbb{S}$ and $T_\mathbb{E}$ are monads with $T_\mathbb{S} \leq T_\mathbb{E}$ via $\iota\colon T_\mathbb{S} \Rightarrow T_\mathbb{E}$. A $\mathtt{Set}$-arrow $h\colon X \to Y$ is a $T_\mathbb{S} \leq T_\mathbb{E}$-homomorphism from (X, α_X) to (Y, α_Y) if and only if the following diagram commutes (in $\mathtt{Set}$)

$$\begin{tikzcd} T_\mathbb{S}X \arrow[r, "\iota h"] \arrow[d, "\alpha_X"'] & T_\mathbb{E}Y \arrow[d, "\alpha_Y"] \\ X \arrow[r, "h"] & Y \end{tikzcd}$$

where ιh denotes the map $\iota h \overset{\text{def}}{=} T_\mathbb{E}h \circ \iota_X \overset{\text{nat.}}{=} \iota_Y \circ T_\mathbb{S}h$.

Now we can formulate the extension lemma.

Proposition 8 (Extension Lemma). *For every $F_\mathbb{S}$-coalgebra $T_\mathbb{S}B \xrightarrow{c} F_\mathbb{S}(T_\mathbb{S}B)$ with free finitely generated carrier $T_\mathbb{S}B$ for a finite set B, there exists an $F_\mathbb{E}$-coalgebra $T_\mathbb{E}B \xrightarrow{\tilde{c}} F_\mathbb{E}(T_\mathbb{E}B)$ with free finitely generated carrier $T_\mathbb{E}B$ such that*

$$\begin{tikzcd} T_\mathbb{S}B \arrow[r, "\iota"] \arrow[d, "c"'] & T_\mathbb{E}B \arrow[d, "\tilde{c}"] \\ F_\mathbb{S}(T_\mathbb{S}B) \arrow[r, "\iota \times \iota^A"] & F_\mathbb{E}(T_\mathbb{E}B) \end{tikzcd}$$

where the horizontal arrows (ι and $\iota \times \iota^A$) are $T_\mathbb{S} \leq T_\mathbb{E}$-homomorphisms, and moreover they both amount to inclusion.

Step 2. The basic diagram: Let $n_1, n_2 \in \mathbb{N}$, let B_j be the n_j-element set consisting of the canonical basis vectors of $\mathbb{E}^{n_j}$, and set $X_j = T_\mathbb{S}B_j$. Assume we are given $F_\mathbb{S}$-coalgebras (X_1, c_1) and (X_2, c_2), and elements $x_j \in X_j$ with $\mathrm{tr}_{c_1} x_1 = \mathrm{tr}_{c_2} x_2$.

The extension lemma provides $F_\mathbb{E}$-coalgebras $(\mathbb{E}^{n_j}, \tilde{c}_j)$ with $\tilde{c}_j|_{X_j} = c_j$. Clearly, $\mathrm{tr}_{\tilde{c}_1} x_1 = \mathrm{tr}_{\tilde{c}_2} x_2$. Using the zig-zag diagram (2) in $\mathsf{Coalg}(F_\mathbb{E})$ and appending inclusion maps, we obtain what we call the *basic diagram.* In this diagram all solid arrows are arrows in $\mathbb{E}$-$\mathtt{MOD}$, and all dotted arrows are arrows in $\mathtt{C}$. The horizontal dotted arrows denote the inclusion maps, and π_j are the restrictions to Z of the canonical projections.

$$\begin{tikzcd} & & \mathbb{E}^{n_1} \times \mathbb{E}^{n_2} \arrow[d, phantom, "\cup|"] & & \\ X_1 \arrow[r, dotted] \arrow[d, dotted, "c_1"'] & \mathbb{E}^{n_1} \arrow[d, "\tilde{c}_1"'] & Z \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] \arrow[d, "d"] & \mathbb{E}^{n_2} \arrow[d, "\tilde{c}_2"'] & X_2 \arrow[l, dotted] \arrow[d, dotted, "c_2"] \\ F_\mathbb{S}X_1 \arrow[r, dotted] & F_\mathbb{E}\mathbb{E}^{n_1} & F_\mathbb{E}Z \arrow[l, "F_\mathbb{E}\pi_1"'] \arrow[r, "F_\mathbb{E}\pi_2"] \arrow[d, phantom, "|\cap"] & F_\mathbb{E}\mathbb{E}^{n_2} & F_\mathbb{S}X_2 \arrow[l, dotted] \\ & & \mathbb{E} \times (\mathbb{E}^{n_1} \times \mathbb{E}^{n_2})^A & & \end{tikzcd}$$

Commutativity of this diagram yields $d\big(\pi_j^{-1}(X_j)\big) \subseteq (F_\mathbb{E}\pi_j)^{-1}\big(F_\mathbb{S}X_j\big)$ for $j = 1, 2$. Now we observe the following properties of cubic functors.

Lemma 9. *We have $F_{\mathbb{E}}X \cap F_{\mathbb{S}}Y = F_{\mathbb{S}}(X \cap Y)$. Moreover, if $Y_j \subseteq X_j$, then $(F_{\mathbb{E}}\pi_1)^{-1}(F_{\mathbb{S}}Y_1) \cap (F_{\mathbb{E}}\pi_2)^{-1}(F_{\mathbb{S}}Y_2) = F_{\mathbb{S}}(Y_1 \times Y_2)$.*

Using this, yields

$$\begin{aligned} d\big(Z \cap (X_1 \times X_2)\big) &\subseteq F_{\mathbb{E}}Z \cap (F_{\mathbb{E}}\pi_1)^{-1}\big(F_{\mathbb{S}}X_1\big) \cap (F_{\mathbb{E}}\pi_2)^{-1}\big(F_{\mathbb{S}}X_2\big) \\ &= F_{\mathbb{E}}Z \cap F_{\mathbb{S}}(X_1 \times X_2) = F_{\mathbb{S}}\big(Z \cap (X_1 \times X_2)\big). \end{aligned}$$

This shows that $Z \cap (X_1 \times X_2)$ becomes an $F_{\mathbb{S}}$-coalgebra with the restriction $d|_{Z\cap(X_1\times X_2)}$. Again referring to the basic diagram, we have the following zig-zag in $\mathsf{Coalg}(F_{\mathbb{S}})$ (to shorten notation, denote the restrictions of d, π_1, π_2 to $Z \cap (X_1 \times X_2)$ again as d, π_1, π_2):

$$(X_1, c_1) \xleftarrow{\pi_1} \big(Z \cap (X_1 \times X_2), d\big) \xrightarrow{\pi_2} (X_2, c_2) \tag{3}$$

This zig-zag relates x_1 with x_2 since $(x_1, x_2) \in Z \cap (X_1 \times X_2)$.

Step 3. The reduction lemma: In view of the zig-zag (3), the proof of Theorem 6 can be completed by showing that $Z \cap (X_1 \times X_2)$ is finitely generated as an algebra in $\mathtt{C}$. Since Z is a submodule of the finitely generated module $\mathbb{E}^{n_1} \times \mathbb{E}^{n_2}$ over the Noetherian ring $\mathbb{E}$, it is finitely generated as an $\mathbb{E}$-module. The task thus is to show that being finitely generated is preserved when reducing scalars.

This is done by what we call the *reduction lemma*. Contrasting the extension lemma, the reduction lemma is not a general categorical fact, and requires specific proof in each situation.

Proposition 10 (Reduction Lemma). *Let $n_1, n_2 \in \mathbb{N}$, let B_j be the set consisting of the n_j canonical basis vectors of $\mathbb{E}^{n_j}$, and set $X_j = T_{\mathbb{S}}B_j$. Moreover, let Z be an $\mathbb{E}$-submodule of $\mathbb{E}^{n_1} \times \mathbb{E}^{n_2}$. Then $Z \cap (X_1 \times X_2)$ is finitely generated as an algebra in $\mathtt{C}$.*

4 A subcubic convex functor

Recall the following definition from [26, p.309].

Definition 11. We introduce a functor $\widehat{F}\colon \mathtt{PCA} \to \mathtt{PCA}$.

1. Let X be a $\mathtt{PCA}$. Then

$$\begin{aligned} \widehat{F}X = \Big\{(o, \phi) \in [0,1] \times X^A \mid & \\ & \exists\, n_a \in \mathbb{N}.\ \exists\, p_{a,j} \in [0,1], x_{a,j} \in X \text{ for } j = 1, \ldots, n_a, a \in A. \\ & o + \sum_{a\in A}\sum_{j=1}^{n_a} p_{a,j} \leq 1,\ \phi(a) = \sum_{j=1}^{n_a} p_{a,j}x_{a,j}\Big\}. \end{aligned}$$

2. Let X, Y be $\mathtt{PCA}$'s, and $f\colon X \to Y$ a convex map. Then $\widehat{F}f\colon \widehat{F}X \to \widehat{F}Y$ is the map $\widehat{F}f = \mathrm{id}_{[0,1]} \times (f \circ -)$.

For every X we have $\widehat{F}X \subseteq F_{[0,1]}X$, and for every $f\colon X \to Y$ we have $\widehat{F}f = (F_{[0,1]}f)|_{\widehat{F}X}$. For this reason, we think of $\widehat{F}$ as a *subcubic functor*.

The definition of $\widehat{F}$ can be simplified.

Lemma 12. *Let X be a* `PCA`*, then*

$$\widehat{F}X = \Big\{(o,f) \in [0,1] \times X^A \mid \exists\, p_a \in [0,1], x_a \in X \text{ for } a \in A. \\ o + \sum_{a\in A} p_a \leq 1,\ f(a) = p_a x_a \Big\}.$$

From this representation it is obvious that $\widehat{F}$ is monotone in the sense that

- If $X_1 \subseteq X_2$, then $\widehat{F}X_1 \subseteq \widehat{F}X_2$.
- If $f_1\colon X_1 \to Y_1, f_2\colon X_2 \to Y_2$ with $X_1 \subseteq X_2, Y_1 \subseteq Y_2$ and $f_2|_{X_1} = f_1$, then $\widehat{F}f_2|_{\widehat{F}X_1} = \widehat{F}f_1$.

Note that $\widehat{F}$ is not compatible with direct products.

For a `PCA` X whose carrier is a compact subset of a euclidean space, $\widehat{F}X$ can be described with help of a geometric notion, namely using the Minkowksi functional of X. Before we can state this fact, we have to make a brief digression to explain this notion and its properties.

Definition 13. Let $X \subseteq \mathbb{R}^n$ be a `PCA`. The *Minkowski functional* of X is the map $\mu_X\colon \mathbb{R}^n \to [0,\infty]$ defined as

$$\mu_X(x) = \begin{cases} \inf\{t > 0 \mid x \in tX\}\,, & x \in \bigcup_{t>0} tX, \\ \infty & \text{, otherwise.} \end{cases}$$

Minkowski functionals, sometimes also called *gauge*, are a central and exhaustively studied notion in convex geometry, see, e.g., [22, p.34] or [21, p.28].

We list some basic properties whose proof can be found in the mentioned textbooks.

1. $\mu_X(px) = p\mu_X(x)$ for $x \in \mathbb{R}^n, p \geq 0$,
2. $\mu_X(x+y) \leq \mu_X(x) + \mu_X(y)$ for $x, y \in \mathbb{R}^n$,
3. $\mu_{X\cap Y}(x) = \max\{\mu_X(x), \mu_Y(x)\}$ for $x \in \mathbb{R}^n$.
4. If X is bounded, then $\mu_X(x) = 0$ if and only if $x = 0$.

The set X can almost be recovered from μ_X.

5. $\{x \in \mathbb{R}^n \mid \mu_X(x) < 1\} \subseteq X \subseteq \{x \in \mathbb{R}^n \mid \mu_X(x) \leq 1\}$.
6. If X is closed, equality holds in the second inclusion of 5.
7. Let X, Y be closed. Then $X \subseteq Y$ if and only if $\mu_X \geq \mu_Y$.

Example 14. As two simple examples, consider the n-simplex $\Delta^n \subseteq \mathbb{R}^n$ and a convex cone $C \subseteq \mathbb{R}^n$. Then (here $\geq$ denotes the product order on $\mathbb{R}^n$)

$$\mu_{\Delta^n}(x) = \begin{cases} \sum_{j=1}^n \xi_j\,, & x = (\xi_1, \ldots, \xi_n) \geq 0, \\ \infty & \text{, otherwise.} \end{cases} \qquad \mu_C(x) = \begin{cases} 0\ , & x \in C, \\ \infty\,, & \text{otherwise.} \end{cases}$$

Observe that $\Delta^n = \{x \in \mathbb{R}^n \mid \mu_{\Delta^n}(x) \leq 1\}$.

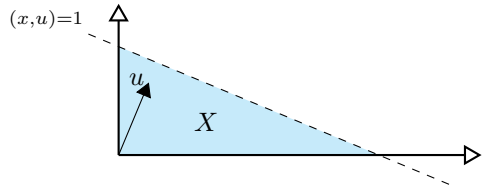
Another illustrative example is given by general pyramids in a euclidean space. This example will play an important role later on.

Example 15. For $u \in \mathbb{R}^n$ consider the set

$$X = \big\{x \in \mathbb{R}^n \mid x \geq 0 \text{ and } (x,u) \leq 1\big\},$$

where $(\cdot,\cdot)$ denotes the euclidean scalar product on $\mathbb{R}^n$. The set X is intersection of the cone $\mathbb{R}^n_+$ with the half-space given by the inequality $(x,u) \leq 1$, hence it is convex and contains 0. Thus X is a `PCA`.

Let us first assume that u is strictly positive, i.e., $u \geq 0$ and no component of u equals zero. Then X is a pyramid (in 2-dimensional space, a triangle).



The n-simplex Δ^n is of course a particular pyramid. It is obtained using the vector $u = (1,\ldots,1)$.

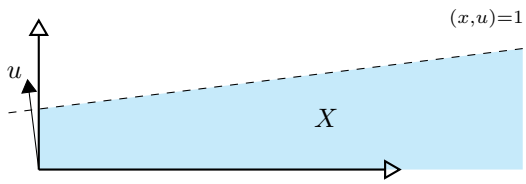
The Minkowski functional of the pyramid X associated with u is

$$\mu_X(x) = \begin{cases} (x,u)\,, & x \geq 0, \\ \infty \quad\;\; , & \text{otherwise.} \end{cases}$$

Write $u = \sum_{j=1}^n \alpha_j e_j$, where e_j is the j-th canonical basis vector, and set $y_j = \frac{1}{\alpha_j} e_j$. Clearly, $\{y_1,\ldots,y_n\}$ is linearly independent. Each vector $x = \sum_{j=1}^n \xi_j e_j$ can be written as $x = \sum_{j=1}^n (\xi_j \alpha_j) y_j$, and this is a subconvex combination if and only if $\xi_j \geq 0$ and $\sum_{j=1}^n \xi_j \alpha_j \leq 1$, i.e., if and only if $x \in X$. Thus X is generated by $\{y_1,\ldots,y_n\}$ as a `PCA`.

The linear map given by the diagonal matrix made up of the α_j's induces a bijection of X onto Δ^n, and maps the y_j's to the corner points of Δ^n. Hence, X is free with basis $\{y_1,\ldots,y_n\}$.

If u is not strictly positive, the situation changes drastically. Then X is unbounded, in particular, not finitely generated as a `PCA`.



Now we return to the functor $\widehat{F}$.

Lemma 16. *Let $X \subseteq \mathbb{R}^n$ be a* PCA*, and assume that X is compact. Then*

$$\widehat{F}X = \Big\{(o,\phi) \in \mathbb{R} \times (\mathbb{R}^n)^A \mid \ o \geq 0,\ o + \sum_{a \in A} \mu_X(\phi(a)) \leq 1\Big\}.$$

In the following we use the elementary fact that every convex map has a linear extension.

Lemma 17. *Let V_1, V_2 be vector spaces, let $X \subseteq V_1$ be a* PCA*, and let $c\colon X \to V_2$ be a convex map. Then c has a linear extension $\tilde{c}\colon V_1 \to V_2$. If $\operatorname{span} X = V_1$, this extension is unique.*

Rescaling in this representation of $\widehat{F}X$ leads to a characterisation of $\widehat{F}$-coalgebra maps. We give a slightly more general statement; for the just said, use $X = Y$.

Corollary 18. *Let $X, Y \subseteq \mathbb{R}^n$ be* PCA*'s, and assume that X and Y are compact. Further, let $c\colon X \to \mathbb{R}_+ \times (\mathbb{R}^n)^A$ be a convex map, and let $\tilde{c}\colon \mathbb{R}^n \to \mathbb{R} \times (\mathbb{R}^n)^A$ be a linear extension of c.*

Then $c(X) \subseteq \widehat{F}Y$, if and only if

$$\tilde{c}_{\mathrm{o}}(x) + \sum_{a \in A} \mu_Y(\tilde{c}_a(x)) \leq \mu_X(x), \quad x \in \mathbb{R}^n. \tag{4}$$

5 An extension theorem for $\widehat{F}$-coalgebras

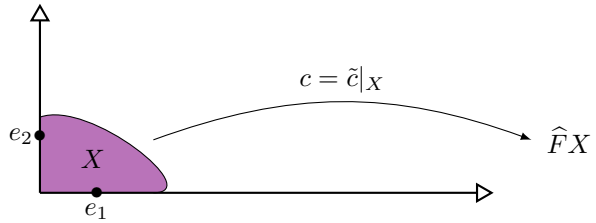
In this section we establish an extension theorem for $\widehat{F}$-coalgebras. It states that an $\widehat{F}$-coalgebra, whose carrier has a particular geometric form, can, under a mild additional condition, be embedded into an $\widehat{F}$-coalgebra whose carrier is free and finitely generated.

Theorem 19. *Let (X, c) be an $\widehat{F}$-coalgebra whose carrier X is a compact subset of a euclidean space $\mathbb{R}^n$ with $\Delta^n \subseteq X \subseteq \mathbb{R}^n_+$. Assume that the output map c_{o} does not vanish on invariant coordinate hyperplanes in the sense that (e_j denotes again the j-th canonical basis vector in $\mathbb{R}^n$)*

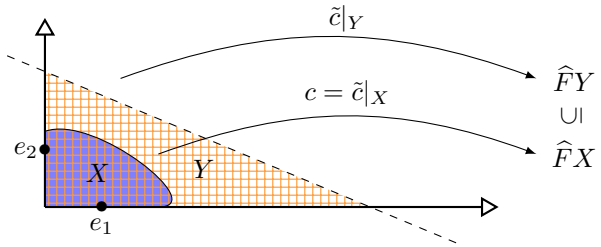
$$\begin{aligned} &\nexists\, I \subseteq \{1, \ldots, n\}. \\ &I \neq \emptyset, \quad c_{\mathrm{o}}(e_j) = 0, j \in I, \quad c_a(e_j) \subseteq \operatorname{span}\{e_i \mid i \in I\}, a \in A, j \in I. \end{aligned} \tag{5}$$

Then there exists an $\widehat{F}$-coalgebra (Y, d), such that $X \subseteq Y \subseteq \mathbb{R}^n_+$, the inclusion map $\iota\colon X \to Y$ is a $\mathsf{Coalg}(\widehat{F})$-morphism, and Y is the subconvex hull of n linearly independent vectors (in particular, Y is free with n generators).

The idea of the proof can be explained by geometric intuition. Say, we have an $\widehat{F}$-coalgebra (X, c) of the stated form, and let $\tilde{c}\colon \mathbb{R}^n \to \mathbb{R} \times (\mathbb{R}^n)^A$ be the linear extension of c to all of $\mathbb{R}^n$, cf. Lemma 17.



Remembering that pyramids are free and finitely generated, we will be done if we find a pyramid $Y \supseteq X$ which is mapped into $\widehat{F}Y$ by $\tilde{c}$:



This task can be reformulated as follows: For each pyramid Y_1 containing X let $P(Y_1)$ be the set of all pyramids Y_2 containing X, such that $\tilde{c}(Y_2) \subseteq \widehat{F}Y_1$. If we find Y with $Y \in P(Y)$, we are done.

Existence of Y can be established by applying a fixed point principle for set-valued maps. The result sufficient for our present level of generality is Kakutani's generalisation [14, Corollary] of Brouwers fixed point theorem.

6 Properness of $\widehat{F}$

In this section we give the second main result of the paper.

Theorem 20. *The functor $\widehat{F}$ is proper.*

In fact, for each two given coalgebras with free finitely generated carrier and each two elements having the same trace, a zig-zag with free and finitely generated nodes relating those elements can be found, which has three intermediate nodes with the middle one forming a span.

We try to follow the proof scheme familiar from the cubic case. Assume we are given two $\widehat{F}$-coalgebras with free finitely generated carrier, say (Δ^{n_1}, c_1) and (Δ^{n_2}, c_2), and elements $x_1 \in \Delta^{n_1}$ and $x_2 \in \Delta^{n_2}$ having the same trace. Since $\widehat{F}\Delta^{n_j} \subseteq \mathbb{R} \times (\mathbb{R}^{n_j})^A$ we can apply Lemma 17 and obtain $F_{\mathbb{R}}$-coalgebras $(\mathbb{R}^{n_j}, \tilde{c}_j)$

with $\tilde{c}_j|_{\Delta^{n_j}} = c_j$. This leads to the basic diagram:

$$\begin{tikzcd}
 & & \mathbb{R}^{n_1}\times\mathbb{R}^{n_2} \arrow[d, phantom, "\cup|"] & & \\
\Delta^{n_1} \arrow[r, dotted] \arrow[d, dotted, "c_1"'] & \mathbb{R}^{n_1} \arrow[d, "\tilde{c}_1"'] & Z \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] \arrow[d, "d"] & \mathbb{R}^{n_2} \arrow[d, "\tilde{c}_2"'] & \Delta^{n_2} \arrow[l, dotted] \arrow[d, dotted, "c_2"] \\
\widehat{F}\Delta^{n_1} \arrow[r, dotted] & F_{\mathbb{R}}\mathbb{R}^{n_1} & F_{\mathbb{R}}Z \arrow[l, "F_{\mathbb{R}}\pi_1"'] \arrow[r, "F_{\mathbb{R}}\pi_2"] & F_{\mathbb{R}}\mathbb{R}^{n_2} & \widehat{F}\Delta^{n_2} \arrow[l, dotted] \\
 & & \mathbb{R}\times(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})^A \arrow[u, phantom, "|\cap"] & &
\end{tikzcd}$$

At this point the line of argument known from the cubic case breaks: it is *not* granted that $Z \cap (\Delta^{n_1} \times \Delta^{n_2})$ becomes an $\widehat{F}$-coalgebra with the restriction of d.

The substitute for $Z \cap (\Delta^{n_1} \times \Delta^{n_2})$ suitable for proceeding one step further is given by the following lemma, where we tacitly identify $\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ with $\mathbb{R}^{n_1+n_2}$.

Lemma 21. *We have* $d(Z \cap 2\Delta^{n_1+n_2}) \subseteq \widehat{F}(Z \cap 2\Delta^{n_1+n_2})$.

This shows that $Z \cap 2\Delta^{n_1+n_2}$ becomes an $\widehat{F}$-coalgebra with the restriction of d. Still, we cannot return to the usual line of argument: it is *not* granted that $\pi_j(Z \cap 2\Delta^{n_1+n_2}) \subseteq \Delta^{n_j}$. This forces us to introduce additional nodes to produce a zig-zag in $\mathsf{Coalg}(\widehat{F})$. These additional nodes are given by the following lemma. There $\mathrm{co}(-)$ denotes the convex hull.

Lemma 22. *Set* $Y_j = \mathrm{co}(\Delta^{n_j} \cup \pi_j(Z \cap 2\Delta^{n_1+n_2}))$. *Then* $\tilde{c}_j(Y_j) \subseteq \widehat{F}Y_j$.

This shows that Y_j becomes an $\widehat{F}$-coalgebra with the restriction of $\tilde{c}_j$. We are led to a zig-zag in $\mathsf{Coalg}(\widehat{F})$:

$$(\Delta^{n_1}, c_1) \xrightarrow{\subseteq} (Y_1, \tilde{c}_1) \xleftarrow{\pi_1} \big(Z \cap 2\Delta^{n_1+n_2}, d\big) \xrightarrow{\pi_2} (Y_2, \tilde{c}_2) \xleftarrow{\supseteq} (\Delta^{n_2}, c_2)$$

This zig-zag relates x_1 and x_2 since $(x_1, x_2) \in Z \cap 2\Delta^{n_1+n_2}$.

By Minkowski's Theorem, see Appendix B Theorem 31, the middle node has finitely generated carrier. The two nodes with incoming arrows are, as convex hulls of two finitely generated `PCA`'s, of course also finitely generated. But in general they will not be free (and this is essential, remember Remark 2). Now Theorem 19 comes into play.

Lemma 23. *Assume that each of* (Δ^{n_1}, c_1) *and* (Δ^{n_2}, c_2) *satisfies the following condition:*

$$\begin{aligned}
&\nexists\, I \subseteq \{1, \ldots, n\}. \\
&I \neq \emptyset,\ c_{j_o}(e_k) = 0, k \in I,\ c_{j_a}(e_k) \subseteq \mathrm{co}(\{e_i \mid i \in I\} \cup \{0\}), a \in A, k \in I.
\end{aligned} \tag{6}$$

Then there exist free finitely generated `PCA`*'s* U_j *with* $Y_j \subseteq U_j \subseteq \mathbb{R}_+^{n_j}$ *which satisfy* $\tilde{c}_j(U_j) \subseteq \widehat{F}U_j$.

This shows that U_j, under the additional assumption (6) on (Δ^{n_j}, c_j), becomes an $\widehat{F}$-coalgebra with the restriction of $\tilde{c}_j$. Thus we have a zig-zag in $\mathsf{Coalg}(\widehat{F})$ relating x_1 and x_2 whose nodes with incoming arrows are free and finitely generated, and whose node with outgoing arrows is finitely generated:

$$\begin{tikzcd}
(\Delta^{n_1}, c_1) \arrow[r, dotted, "\subseteq"] \arrow[dr] & (Y_1, \tilde{c}_1) \arrow[d, dotted, "\cap"] & (Z \cap 2\Delta^{n_1+n_2}, d) \arrow[l, dotted, "\pi_1"'] \arrow[r, dotted, "\pi_2"] \arrow[dl] \arrow[dr] & (Y_2, \tilde{c}_2) \arrow[d, dotted, "\cap"] & (\Delta^{n_2}, c_2) \arrow[l, dotted, "\supseteq"'] \arrow[dl] \\
& (U_1, \tilde{c}_1) & & (U_2, \tilde{c}_2) &
\end{tikzcd}$$

Removing the additional assumption on (Δ^{n_j}, c_j) is a easy.

Lemma 24. *Let (Δ^n, c) be an $\widehat{F}$-coalgebra. Assume that I is a nonempty subset of $\{1, \dots, n\}$ with*

$$c_{\mathrm{o}}(e_k) = 0,\ k \in I \quad \text{and} \quad c_a(e_k) \in \mathrm{co}\big(\{e_i \mid i \in I\} \cup \{0\}\big),\ a \in A, k \in I. \tag{7}$$

Let X be the free PCA *with basis $\{e_k \mid k \in \{1, \dots, n\} \setminus I\}$, and let $f\colon \Delta^n \to X$ be the* PCA*-morphism with*

$$f(e_k) = \begin{cases} 0\ , & k \in I, \\ e_k\,, & k \notin I. \end{cases}$$

Further, let $g\colon X \to [0,1] \times X^A$ be the PCA*-morphism with*

$$g(e_k) = \Big(c_{\mathrm{o}}(e_k), \big(f(c_a(e_k))\big)_{a \in A}\Big), \quad k \in \{1, \dots, n\} \setminus I.$$

Then (X, g) is an $\widehat{F}$-coalgebra, and f is an $\widehat{F}$-coalgebra morphism of (Δ^n, c) onto (X, g).

Corollary 25. *Let (Δ^n, c) be an $\widehat{F}$-coalgebra. Then there exists $k \leq n$, an $\widehat{F}$-coalgebra (Δ^k, g), such that (Δ^k, g) satisfies the assumption in Lemma 24 and that there exists an $\widehat{F}$-coalgebra map f of (Δ^n, c) onto (Δ^k, g).*

The proof of Theorem 20 is now finished by putting together what we showed so far. Starting with $\widehat{F}$-coalgebras (Δ^{n_j}, c_j) without any additional assumptions, and elements $x_j \in \Delta^{n_j}$ having the same trace, we first reduce by means of Corollary 25 and then apply Lemma 23. This gives a zig-zag as required:

$$\begin{tikzcd}
(\Delta^{n_1}, c_1) \arrow[d, dotted, "\psi_1"'] \arrow[dr] & & (Z \cap 2\Delta^{k_1+k_2}, d) \arrow[dl] \arrow[dr] & & (\Delta^{n_2}, c_2) \arrow[dl] \arrow[d, dotted, "\psi_2"] \\
(\Delta^{k_1}, g_1) \arrow[r, dotted] & (U_1, \tilde{g}_1) & & (U_2, \tilde{g}_2) & (\Delta^{k_2}, g_2) \arrow[l, dotted]
\end{tikzcd}$$

and completes the proof of properness of $\widehat{F}$.

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A. Category theory basics

We start by recalling the basic notions of category, functor and natural transformation, so that all of the results in the paper are accessible also to non-experts.

A category $\mathtt{C}$ is a collection of objects and a collection of arrows (or morphisms) from one object to another. For every object $X \in \mathtt{C}$, there is an identity arrow $\mathrm{id}_X \colon X \to X$. For any three objects $X, Y, Z \in \mathtt{C}$, given two arrows $f \colon X \to Y$ and $g \colon Y \to Z$, there exists an arrow $g \circ f \colon X \to Z$. Arrow composition is associative and id_X is neutral w.r.t. composition. The standard example is $\mathtt{Set}$, the category of sets and functions.

A functor F from a category $\mathtt{C}$ to a category $\mathtt{D}$, notation $F \colon \mathtt{C} \to \mathtt{D}$, assigns to every object $X \in \mathtt{C}$, an object $FX \in \mathtt{D}$, and to every arrow $f \colon X \to Y$ in $\mathtt{C}$ an arrow $Ff \colon FX \to FY$ in $\mathtt{D}$ such that identity arrows and composition are preserved.

A category $\mathtt{C}$ is concrete, if it admits a canonical forgetful functor $\mathcal{U} \colon \mathtt{C} \to \mathtt{Set}$. By a forgetful functor we mean a functor that is identity on arrows. Intuitively, a concrete category has objects that are sets with some additional structure, e.g. algebras, and morphisms that are particular kind of functions. All categories that we consider are algebraic and hence concrete.

Let $F \colon \mathtt{C} \to \mathtt{D}$ and $G \colon \mathtt{C} \to \mathtt{D}$ be two functors. A natural transformation $\sigma \colon F \Rightarrow G$ is a family of arrows $\sigma_X \colon FX \to GX$ in $\mathtt{D}$ such that the diagram on the right commutes for all arrows $f \colon X \to Y$.

$$\begin{tikzcd} FX \arrow[r, "Ff"] \arrow[d, "\sigma_X"] & FY \arrow[d, "\sigma_Y"] \\ GX \arrow[r, "Gf"] & GY \end{tikzcd}$$

A.1. Monads and Algebras

A monad is a functor $T \colon \mathtt{C} \to \mathtt{C}$ together with two natural transformations: a unit $\eta \colon \mathrm{id}_{\mathtt{C}} \Rightarrow T$ and multiplication $\mu \colon T^2 \Rightarrow T$. These are required to make the following diagrams commute, for $X \in \mathtt{C}$.

$$\begin{tikzcd} TX \arrow[r, "\eta_{TX}"] \arrow[dr, equal] & T^2X \arrow[d, "\mu_X"] & TX \arrow[l, "T\eta_X"'] \arrow[dl, equal] \\ & TX & \end{tikzcd} \qquad \begin{tikzcd} T^3X \arrow[r, "\mu_{TX}"] \arrow[d, "T\mu_X"'] & T^2X \arrow[d, "\mu_X"] \\ T^2X \arrow[r, "\mu_X"'] & TX \end{tikzcd}$$

Given two monads S, T with units and multiplications η^S, η^T and μ^S, μ^T, respectively, and a natural transformation $\iota \colon S \Rightarrow T$, we say S is a *submonad* of T along ι, and write $S \leq T$, if $\eta^T = \sigma \circ \eta^S$ and $\iota \circ \mu^S = \mu^T \circ \iota\iota$ where $\iota\iota \overset{\text{def}}{=} T_{\mathbb{E}}\iota \circ \iota \overset{\text{nat.}}{=} \iota \circ T_{\mathbb{S}}\iota$.

We briefly describe some examples of monads on $\mathtt{Set}$.

– The finitely supported subprobability distribution monad $\mathcal{D}$ is defined, for a set X and a function $f \colon X \to Y$, as

$$\mathcal{D}X \;=\; \{\varphi \colon X \to [0,1] \mid \sum_{x \in X} \varphi(x) \leq 1,\ \mathrm{supp}(\varphi) \text{ is finite}\}$$

and

$$\mathcal{D}f(\varphi)(y) \quad = \sum_{x \in f^{-1}(\{y\})} \varphi(x).$$

Here and below $\mathrm{supp}(\varphi) = \{x \in X \mid \varphi(x) \neq 0\}$. The unit of $\mathcal{D}$ is given by a Dirac distribution $\eta_X(x) = \delta_x = (x \mapsto 1)$ for $x \in X$ and the multiplication by $\mu_X(\Phi)(x) = \sum_{\varphi \in \mathrm{supp}(\Phi)} \Phi(\varphi) \cdot \varphi(x)$ for $\Phi \in \mathcal{D}\mathcal{D}X$.

- For a semiring $\mathbb{S}$ the $\mathbb{S}$-valuations monad $T_{\mathbb{S}}$ is defined as $T_{\mathbb{S}}X = \{\varphi \colon X \to \mathbb{S} \mid \mathrm{supp}(\varphi) \text{ is finite}\}$ and on functions $f \colon X \to Y$ we have $T_{\mathbb{S}}f(\varphi)(y) = \sum_{x \in f^{-1}(\{y\})} \varphi(x)$. Its unit is given by $\eta_X(x) = (x \mapsto 1)$ and multiplication by $\mu_X(\Phi)(x) = \sum_{\varphi \in \mathrm{supp}\,\Phi} \Phi(\varphi) \cdot \varphi(x)$ for $\Phi \in T_{\mathbb{S}}T_{\mathbb{S}}X$.
- To illustrate the connection between $\mathcal{D}$ and $T_{\mathbb{S}}$, consider yet another monad: For a semiring $\mathbb{S}$, and a (suitable) subset $S \subseteq \mathbb{S}$, the $(\mathbb{S}, S)$-valuations monad $T_{\mathbb{S},S}$ is defined as follows. On objects it acts like

$$T_{\mathbb{S},S}X = \{\varphi \colon X \to \mathbb{S} \mid \mathrm{supp}(\varphi) \text{ is finite and } \sum_{x \in X} \varphi(x) \in S\}$$

on functions it acts like $T_{\mathbb{S}}$. The unit and multiplication are defined as in $T_{\mathbb{S}}$. Note that $\mathcal{D} = T_{\mathbb{R}_+,[0,1]}$.

With a monad T on a category $\mathtt{C}$ one associates the Eilenberg-Moore category $\mathtt{Set}^T$ of Eilenberg-Moore algebras. Objects of $\mathtt{Set}^T$ are pairs $\mathbb{A} = (A, \alpha)$ of an object $A \in \mathtt{C}$ and an arrow $\alpha \colon TA \to A$, making the first two diagrams below commute.

$$\begin{tikzcd} A \arrow[r, "\eta_A"] \arrow[dr, equal] & TA \arrow[d, "\alpha"] \\ & A \end{tikzcd} \qquad \begin{tikzcd} T^2A \arrow[r, "T\alpha"] \arrow[d, "\mu_A"'] & TA \arrow[d, "\alpha"] \\ TA \arrow[r, "\alpha"'] & A \end{tikzcd} \qquad \begin{tikzcd} TA \arrow[r, "Th"] \arrow[d, "\alpha"'] & TB \arrow[d, "\beta"] \\ A \arrow[r, "h"'] & B \end{tikzcd}$$

A homomorphism from an algebra $\mathbb{A} = (A, \alpha)$ to an algebra $\mathbb{B} = (B, b)$ is a map $h \colon A \to B$ in $\mathtt{C}$ between the underlying objects making the diagram above on the right commute.

A free Eilenberg-Moore algebra for a monad T generated by X is (TX, μ_X) and we will often denote it simply by TX. A free finitely generated Eilenberg-Moore algebra for T is an algebra TX with X a finite set. The diagram in the middle thus says that the map α is a homomorphism from TA to $\mathbb{A}$.

Indeed, TX is free in the algebraic sense as for any map $f \colon X \to \mathbb{A}$ there is a unique (Kleisli) extension that is an algebra homomorphism from TX to $\mathbb{A}$. This extension is given by $f^{\#} = \alpha \circ Tf$.

B. Proof details for properness of cubic functors

Proof (of Lemma 4). Since $\mathrm{tr}_{c_1} x_1 = \mathrm{tr}_{c_2} x_2$, we have

$$c_{1_{\mathrm{o}}}(c_{1_w}(x_1)) = [\mathrm{tr}_{c_1} x_1](w) = [\mathrm{tr}_{c_2} x_2](w) = c_{2_{\mathrm{o}}}(c_{2_w}(x_2)), \quad w \in A^*,$$

and therefore $d_1|_Z = d_2|_Z$. Moreover,

$$c_{j\,a}(c_{j\,w}(x_j)) = c_{j\,wa}(x_j), \quad w \in A^*,$$

and therefore $d_j(Z) \subseteq \mathbb{S} \times Z^A$. □

Proof (of Corollary 5). Remembering Remark 3, we have to show that the functor $F_{\mathbb{S}}$ is proper. We have the zig-zag (2), and the $\mathbb{S}$-semimodule Z is, as a subsemimodule of the finitely generated $\mathbb{S}$-semimodule $\mathbb{S}^{n_1} \times \mathbb{S}^{n_2}$, itself finitely generated. □

B.1. Proof of the extension lemma

The proof of the extension lemma follows directly from the following two abstract properties.

Lemma 26. *Assume $T_{\mathbb{S}} \leq T_{\mathbb{E}}$ via $\iota\colon T_{\mathbb{S}} \Rightarrow T_{\mathbb{E}}$ and let X be a finite set. Let $\mathbb{Y} \in \mathtt{Set}^{T_{\mathbb{S}}}$ and $\mathbb{Z} \in \mathtt{Set}^{T_{\mathbb{E}}}$ and assume we are given an arrow $a_Y\colon T_{\mathbb{S}}X \to \mathbb{Y}$ in $\mathtt{Set}^{T_{\mathbb{S}}}$ and a $T_{\mathbb{S}} \leq T_{\mathbb{E}}$-homomorphism $h\colon \mathbb{Y} \to \mathbb{Z}$. Then there exists an arrow $a_Z\colon T_{\mathbb{E}}X \to \mathbb{Z}$ in $\mathtt{Set}^{T_{\mathbb{E}}}$ making the following diagram commute.*

$$\begin{tikzcd} T_{\mathbb{S}}X \arrow[r, "\iota"] \arrow[d, "a_Y"'] & T_{\mathbb{E}}X \arrow[d, "a_Z"] \\ \mathbb{Y} \arrow[r, "h"] & \mathbb{Z} \end{tikzcd}$$

Proof. Consider the map $h \circ a_Y \circ \eta_{\mathbb{S},X}\colon X \to Z$. Let $a_Z = (h \circ a_Y \circ \eta_{\mathbb{S},X})^{\#_{\mathbb{E}}}$.

For any monad T with unit η, an Eilenberg-Moore algebra $\mathbb{A} = (A, \alpha)$, and a set X, the unique (Kleisli) extension $(f \circ \eta)^{\#}\colon TX \to \mathbb{A}$ in $\mathtt{Set}^T$ for a map $f\colon TX \to \mathbb{A}$ satisfies $(f \circ \eta)^{\#} = f$. Indeed, we have, using that f is an algebra homomorphism from the free algebra TX to $\mathbb{A}$, and the monad laws:

$$(f \circ \eta)^{\#} = \alpha \circ T(f \circ \eta) = \alpha \circ Tf \circ T\eta = f \circ \mu \circ T\eta = f.$$

Furthermore, for any map $f\colon X \to \mathbb{A}$ we have $f^{\#} \circ \eta = f$ since $f^{\#} \circ \eta = \alpha \circ Tf \circ \eta = \alpha \circ \eta \circ f$ by naturality of η and the Eilenberg-Moore law. Hence, we have

$$\begin{aligned} a_Z \circ \iota_X &= (h \circ a_Y \circ \eta_{\mathbb{S},X})^{\#_{\mathbb{E}}} \circ \iota_X \\ &= ((h \circ a_Y \circ \eta_{\mathbb{S},X})^{\#_{\mathbb{E}}} \circ \iota_X \circ \eta_{\mathbb{S},X})^{\#_{\mathbb{S}}} \\ &\overset{(*)}{=} (h \circ a_Y \circ \eta_{\mathbb{S},X})^{\#_{\mathbb{S}}} \\ &= h \circ a_Y \end{aligned}$$

and the equation marked with $(*)$ holds because $T_{\mathbb{S}} \leq T_{\mathbb{E}}$ via ι and so

$$(h \circ a_Y \circ \eta_{\mathbb{S},X})^{\#_{\mathbb{E}}} \circ \iota_X \circ \eta_{\mathbb{S},X} = (h \circ a_Y \circ \eta_{\mathbb{S},X})^{\#_{\mathbb{E}}} \circ \eta_{\mathbb{E},X} = h \circ a_Y \circ \eta_{\mathbb{S},X}.$$

□

Lemma 27. *The map $\iota \times \iota^A$ is a $T_{\mathbb{S}} \leq T_{\mathbb{E}}$-homomorphism from $F_{\mathbb{S}}(T_{\mathbb{S}}X)$ to $F_{\mathbb{E}}(T_{\mathbb{E}}X)$.*

Proof. Recall that $F_{\mathbb{S}}(T_{\mathbb{S}}X) = \mathbb{S} \times (T_{\mathbb{S}}X)^A = T_{\mathbb{S}}1 \times (T_{\mathbb{S}}X)^A$ and in the same way $F_{\mathbb{E}}(T_{\mathbb{E}}X) = T_{\mathbb{E}}1 \times (T_{\mathbb{E}}X)^A$, hence both algebras are finite products of free finitely generated algebras. We will prove a more general property. Given two $T_{\mathbb{S}} \leq T_{\mathbb{E}}$-homomorphisms $h_1 \colon T_{\mathbb{S}}X \to T_{\mathbb{E}}X$ and $h_2 \colon T_{\mathbb{S}}Y \to T_{\mathbb{E}}Y$, their product $h_1 \times h_2$ is a $T_{\mathbb{S}} \leq T_{\mathbb{E}}$-homomorphism as well from $T_{\mathbb{S}}X \times T_{\mathbb{S}}Y$ to $T_{\mathbb{E}}X \times T_{\mathbb{E}}Y$. We have

$$\begin{aligned}
&(h_1 \times h_2) \circ (\mu_{\mathbb{S},X} \times \mu_{\mathbb{S},X}) \circ \langle T_{\mathbb{S}}\pi_1, T_{\mathbb{S}}\pi_2 \rangle \\
&\qquad = (\mu_{\mathbb{E},X} \times \mu_{\mathbb{E},X}) \circ (\iota h_1 \times \iota h_2) \circ \langle T_{\mathbb{S}}\pi_1, T_{\mathbb{S}}\pi_2 \rangle \\
&\qquad = (\mu_{\mathbb{E},X} \times \mu_{\mathbb{E},X}) \circ \langle T_{\mathbb{E}}\pi_1, T_{\mathbb{E}}\pi_2 \rangle \circ \iota(h_1 \times h_2)
\end{aligned}$$

where the first equation holds by assumption since h_1 and h_2 are $T_{\mathbb{S}} \leq T_{\mathbb{E}}$-homomorphisms, and the second since $\pi_1 \circ (h_1 \times h_2) = h_1 \circ \pi_1$ and $\pi_2 \circ (h_1 \times h_2) = h_2 \circ \pi_2$, by the general property $(*)$ below, and by properties of products and pairings. Moreover $(\mu_{\mathbb{S}} \times \mu_{\mathbb{S}}) \circ \langle T_{\mathbb{S}}\pi_1, T_{\mathbb{S}}\pi_2 \rangle$ and $(\mu_{\mathbb{E}} \times \mu_{\mathbb{E}}) \circ \langle T_{\mathbb{E}}\pi_1, T_{\mathbb{E}}\pi_2 \rangle$ are the algebra structures of the corresponding products of the two free algebras.
$(*)$: Assume maps a, b, f, g such that the left diagram below commutes in `Set`. Then the right one commutes as well.

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "a"'] & B \arrow[d, "b"] \\
C \arrow[r, "g"] & D
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
T_{\mathbb{S}}A \arrow[r, "\iota f"] \arrow[d, "T_{\mathbb{S}}a"'] & T_{\mathbb{E}}B \arrow[d, "T_{\mathbb{E}}b"] \\
T_{\mathbb{S}}C \arrow[r, "\iota g"] & T_{\mathbb{E}}D
\end{tikzcd}$$

This is indeed the case because

$$\begin{aligned}
T_{\mathbb{E}}b \circ \iota f &\overset{\text{def.}}{=} T_{\mathbb{E}}b \circ \iota \circ T_{\mathbb{S}}f \\
&\overset{\text{nat.}}{=} \iota \circ T_{\mathbb{S}}b \circ T_{\mathbb{S}}f \\
&\overset{\text{hyp.}}{=} \iota \circ T_{\mathbb{S}}g \circ T_{\mathbb{S}}a \\
&\overset{\text{def.}}{=} \iota g \circ T_{\mathbb{S}}a.
\end{aligned}$$

Finally, we remark that, since $T_{\mathbb{S}} \leq T_{\mathbb{E}}$ via ι, ι_X is a $T_{\mathbb{S}} \leq T_{\mathbb{E}}$-homomorphism from the free algebra $T_{\mathbb{S}}X$ to the free algebra $T_{\mathbb{E}}X$, which completes the proof. □

Proof (of Lemma 9). Since $\mathbb{S} \subseteq \mathbb{E}$, we have

$$F_{\mathbb{E}}X \cap F_{\mathbb{S}}Y = (\mathbb{E} \times X^A) \cap (\mathbb{S} \times Y^A) = \mathbb{S} \times (X \cap Y)^A = F_{\mathbb{S}}(X \cap Y).$$

Assume now that $Y_j \subseteq X_j$. We have

$$(F_{\mathbb{E}}\pi_1)^{-1}(F_{\mathbb{S}}Y_1) = \{(o, ((x_{1a}, x_{2a}))_{a \in A}) \in \mathbb{E} \times (X_1 \times X_2)^A \mid o \in \mathbb{S}, x_{1a} \in Y_1\},$$

and the analogous formula for $(F_{\mathbb{E}}\pi_2)^{-1}(F_{\mathbb{S}}Y_2)$. This shows that the intersection of these two inverse images is equal to $\mathbb{S} \times (Y_1 \times Y_2)^A$. □

B.2. Proof of the reduction lemma

➢ Reducing from `AB` to `CMON`

The reduction lemma for passing from abelian groups to commutative monoids arises from a classical result of algebra. Namely, it is a corollary of the following theorem due to D.Hilbert, cf. [10, Theorem II] see also [1, Theorem 1.1].

Theorem 28 (Hilbert 1890). *Let W be a $n \times m$-matrix with integer entries, and let X be the commutative monoid*

$$X = \{x \in \mathbb{Z}^n \mid x \cdot W \geq 0\}.$$

Then X is finitely generated as a commutative monoid.

The reduction lemma for passing from `AB` to `CMON` is a corollary Since every finitely generated abelian group is also finitely generated as a commutative monoid, we obtain a somewhat stronger variant.

Lemma 29. *Let Z be a finitely generated abelian group, let $m \in \mathbb{N}$, and let $\varphi\colon Z \to \mathbb{Z}^m$ be a group homomorphism. Then $\varphi^{-1}(\mathbb{N}^m)$ is finitely generated as a commutative monoid.*

Proof. Write Z, up to an isomorphism, as a direct sum of cyclic abelian groups

$$Z = \mathbb{Z}^k \oplus \Big[\bigoplus_{j=1}^{n} \mathbb{Z}/a_j\mathbb{Z}\Big] \tag{8}$$

with $a_j \geq 2$. Since φ maps into the torsionfree group $\mathbb{Z}^m$, we must have

$$\varphi\Big(\bigoplus_{j=1}^{n} \mathbb{Z}/a_j\mathbb{Z}\Big) = \{0\}.$$

Hence, an element $x \in Z$ satisfies $\varphi(x) \geq 0$, if and only if $\varphi(x_0) \geq 0$ where $x = x_0 + x_1$ is the decomposition of x according to the direct sum (8). The action of the map $\psi = \varphi|_{\mathbb{Z}^k} : \mathbb{Z}^k \to \mathbb{Z}^m$ is described as multiplication of $x_0 = (\xi_1, \ldots, \xi_k)$ with some $k \times m$-matrix W having integer coefficients. Thus

$$\psi^{-1}(\mathbb{N}^m) = \{x_0 \in \mathbb{Z}^k \mid x_0 \cdot W \geq 0\},$$

and by Hilbert's Theorem $\psi^{-1}(\mathbb{N}^m)$ is finitely generated as a commutative monoid.

The set $\bigoplus_{j=1}^{n} \mathbb{Z}/a_j\mathbb{Z}$ also has a finite set of generators as a monoid, for example the residue classes $1/a_j\mathbb{Z}$, $j = 1, \ldots, n$. Together we see that $\varphi^{-1}(\mathbb{N}^m)$ has a finite set of generators as a commutative monoid. □

➢ Reducing from $\mathbb{Q}$-VEC to $\mathbb{Q}_+$-MOD

The reduction lemma for passing from vector spaces over $\mathbb{Q}$ to $\mathbb{Q}_+$-semimodules is a corollary of the one passing from AB to CMON. Thus we have the corresponding stronger variant also in this case.

Lemma 30. *Let Z be a finite dimensional $\mathbb{Q}$-vector space, let $m \in \mathbb{N}$, and let $\varphi\colon Z \to \mathbb{Q}^m$ be $\mathbb{Q}$-linear. Then $\varphi^{-1}(\mathbb{Q}_+^n)$ is finitely generated as a $\mathbb{Q}_+$-semimodule.*

Proof. Let $\{u_1, \dots, u_k\}$ be a set of generators of Z as a $\mathbb{Q}$-vector space. Write

$$\varphi(u_j) = \Big(\frac{a_{j,1}}{b_{j,1}}, \dots, \frac{a_{j,m}}{b_{j,m}}\Big), \quad j = 1, \dots, k,$$

with $a_{j,i} \in \mathbb{Z}$ and $b_{j,i} \in \mathbb{N} \setminus \{0\}$. Set $b = \prod_{j=1}^k \prod_{i=1}^m b_{j,i}$, then $\varphi(bu_j) \in \mathbb{Z}^m$, $j = 1, \dots, k$.

Let $Z' \subseteq Z$ be the $\mathbb{Z}$-submodule generated by $\{bu_1, \dots, bu_k\}$, and set $\psi = \varphi|_{Z'}$. Then ψ is a $\mathbb{Z}$-linear map of Z' into $\mathbb{Z}^m$. By Lemma 29, $\psi^{-1}(\mathbb{N}^m)$ is finitely generated as $\mathbb{N}$-semimodule, say by $\{v_1, \dots, v_l\} \subseteq Z'$.

Given $x \in \varphi^{-1}(\mathbb{Q}_+^m)$, choose $\nu_1, \dots, \nu_k \in \mathbb{Q}$ with $x = \sum_{j=1}^k \nu_j u_j$. Write $\nu_j = \frac{\alpha_j}{\beta_j}$ with $\alpha_j \in \mathbb{Z}$ and $\beta_j \in \mathbb{N} \setminus \{0\}$, and set $\beta = \prod_{j=1}^k \beta_j$. Then

$$\beta b \cdot x = \sum_{j=1}^k (\beta \nu_j) \cdot bu_j \in Z',$$

and

$$\psi(\beta b \cdot x) = \varphi(\beta b \cdot x) = \beta b \cdot \varphi(x) \in \mathbb{Q}_+^m \cap \mathbb{Z}^m = \mathbb{N}^m.$$

Thus $\beta b \cdot x$ is an $\mathbb{N}$-linear combination of the elements $v_1, \dots, v_l$, and hence x is a $\mathbb{Q}_+$-linear combination of these elements. This shows that $\varphi^{-1}(\mathbb{Q}_+^m)$ is generated by $\{v_1, \dots, v_l\}$ as a $\mathbb{Q}_+$-semimodule. □

➢ Reducing from $\mathbb{R}$-VEC to CONE

The reduction lemma for passing from vector spaces over $\mathbb{R}$ to convex cones arises from a different source than the previously studied. Namely, it is a corollary of the below classical theorem of H.Minkowski, cf. [19] see also [21, Theorem 19.1].

Recall that a convex subset X of $\mathbb{R}^n$ is called *polyhedral*, if it is a finite intersection of half-spaces, i.e., if there exist $l \in \mathbb{N}$, $u_1, \dots, u_l \in \mathbb{R}^n$, and $\nu_1, \dots, \nu_l \in \mathbb{R}$, such that

$$X = \big\{x \in \mathbb{R}^n \mid (x, u_j) \le \nu_j, j = 1, \dots, l\big\},$$

where $(\cdot, \cdot)$ denotes the euclidean scalar product on $\mathbb{R}^n$. On the other hand, X is said to be *generated by points $a_1, \dots, a_{l_1}$ and directions $b_1, \dots, b_{l_2}$*, if

$$X = \Big\{\sum_{j=1}^{l_1} \alpha_j a_j + \sum_{j=1}^{l_2} \beta_j b_j \mid \alpha_j \in [0,1], \sum_{j=1}^{l_1} \alpha_j = 1,\ \beta_j \ge 0, j = 1, \dots, l_2\Big\}.$$

Note that a convex set generated by some points and directions is bounded, if and only if no (nonzero) directions are present. Further, a convex set is a cone, if and only if it allows a representation where only directions occur.

Theorem 31 (Minkowski 1896). *Let X be a convex subset of $\mathbb{R}^n$. Then X is polyhedral, if and only if X is generated by a finite set of points and directions.*

The relevance of Minkowski's Theorem in the present context is that it shows that the intersection of two finitely generated sets is finitely generated (since the intersection of two polyhedral sets is obviously polyhedral).

The reduction lemma for passing from $\mathbb{R}$-`VEC` to `CONE` is an immediate corollary. Since every finite dimensional $\mathbb{R}$-vector space is also finitely generated as a convex cone, we have the corresponding stronger version.

Lemma 32. *Let Z be a finite dimensional $\mathbb{R}$-vector space, let $m \in \mathbb{N}$, and let $\varphi\colon Z \to \mathbb{R}^m$ be $\mathbb{R}$-linear. Then $\varphi^{-1}(\mathbb{R}^n_+)$ is finitely generated as a convex cone.*

Proof.

Step 1: The image $\varphi(Z)$ is a linear subspace of $\mathbb{R}^n$, in particular, poyhedral. The positive cone $\mathbb{R}^n_+$ is obviously also polyhedral. We conclude that the convex cone $\varphi(Z) \cap \mathbb{R}^n_+$ is generated by some finite set of directions.

Step 2: The kernel $\varphi^{-1}(\{0\})$ is, as a linear subspace of the finite dimensional vector space Z, itself finite dimensional (generated, say, by $\{u_1, \ldots, u_k\}$). Thus it is also finitely generated as a convex cone (in fact, $\{\pm u_1, \ldots, \pm u_k\}$ is a set of generators).

Choose a finite set of directions $\{a_1, \ldots, a_l\}$ generating $\varphi(Z) \cap \mathbb{R}^n_+$ as a convex cone, and choose $v_j \in Z$ with $\varphi(v_j) = a_j$, $j = 1, \ldots, l$. Then $\{\pm u_1, \ldots, \pm u_k\} \cup \{v_1, \ldots, v_l\}$ generates $\varphi^{-1}(Z)$ as a convex cone.

□

➢ Reducing from $\mathbb{R}$-`VEC` to `PCA`

The reduction lemma for passing from vector spaces over $\mathbb{R}$ to positively convex algebras is again a corollary of Theorem 31. However, in a sense the situation is more complicated. One, the corresponding strong version fails; in fact, no (nonzero) $\mathbb{R}$-vector space is finitely generated as a `PCA`. Two, unlike in categories of semimodules, the direct product $T_{[0,1]}B_1 \times T_{[0,1]}B_2$ does not coincide with $T_{[0,1]}(B_1 \dot\cup B_2)$.

Lemma 33. *Let $n_1, n_2 \in \mathbb{N}$, and let Z be a linear subspace of $\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$. Then $Z \cap (\Delta^{n_1} \times \Delta^{n_2})$ is finitely generated as a positively convex algebra.*

Proof. Obviously, Z and $\Delta^{n_1} \times \Delta^{n_2}$ are both polyhedral. We conclude that $Z \cap (\Delta^{n_1} \times \Delta^{n_2})$ is generated by a finite set of points and directions. Since it is bounded, no direction can occur, and it is thus finitely generated as a `PCA`. □

C. Self-contained proof of Lemma 29

We provide a short and self-contained proof of the named reduction lemma. It proceeds via an argument very specific for $\mathbb{N}$; the essential ingredient is that the order of $\mathbb{N}$ is total and satisfies the descending chain condition. Note that the following argument also proves Hilbert's Theorem.

First, a common fact about the product order on $\mathbb{N}^m$ (we provide an explicit proof since we cannot appoint a reference).

Lemma 34. *Let $m \in \mathbb{N}$, and let $M \subseteq \mathbb{N}^m$ be a set of pairwise incomparable elements. Then M is finite.*

Proof. Assume that M is infinite, and choose a sequence $(a_n)_{n\in\mathbb{N}}$ of different elements of M. Write $a_n = (\alpha_{n,1}, \dots, \alpha_{n,m})$. We construct, in m steps, a subsequence $(b_n)_{n\in\mathbb{N}}$ of $(a_n)_{n\in\mathbb{N}}$ with the property that (we write $b_n = (\beta_{n,1}, \dots, \beta_{n,m})$)

$$\forall k \in \{1, \dots, m\}.\ L_k = \sup_{n\in\mathbb{N}} \beta_{n,k} < \infty \ \vee\ \beta_{0,k} < \beta_{1,k} < \beta_{2,k} < \cdots \tag{9}$$

In the first step, extract a subsequence of $(a_n)_{n\in\mathbb{N}}$ according to the behaviour of the sequence of first components $(\alpha_{n,1})_{n\in\mathbb{N}}$. If $\sup_{n\in\mathbb{N}} \alpha_{n,1} < \infty$, take the whole sequence $(a_n)_{n\in\mathbb{N}}$ as the subsequence. If $\sup_{n\in\mathbb{N}} \alpha_{n,1} = \infty$, take a subsequence $(a_{n_j})_{j\in\mathbb{N}}$ with

$$\alpha_{n_0,1} < \alpha_{n_1,1} < \alpha_{n_2,1} < \cdots .$$

Repeating this step, always starting from the currently chosen subsequence, we succesively extract subsequences which after l steps satisfy the property (9) for the components up to l.

Denote

$$I_1 = \big\{k \in \{1, \dots, m\} \mid \sup_{n\in\mathbb{N}} \beta_{n,k} < \infty\big\}, \quad I_2 = \big\{k \in \{1, \dots, m\} \mid \sup_{n\in\mathbb{N}} \beta_{n,k} = \infty\big\}$$

The map $n \mapsto (\beta_{n,k})_{k\in I_1}$ maps $\mathbb{N}$ into the finite set $\prod_{k\in I_1}\{0, \dots, L_k\}$, and hence is not injective. Choose $n_1 < n_2$ with $\beta_{n_1,k} = \beta_{n_2,k}$, $k \in I_1$. Since $\beta_{n_1,k} < \beta_{n_2,k}$, $k \in I_2$, we obtain $b_{n_1} \leq b_{n_2}$. However, by our choice of the elements a_n, $b_{n_1} \neq b_{n_2}$. Thus M contains a pair of different but comparable elements. □

Proof (of Lemma 29). If $\varphi^{-1}(\mathbb{N}^m) = \{0\}$, there is nothing to prove. Hence, assume that $\varphi^{-1}(\mathbb{N}^m) \neq \{0\}$.

Step 1: We settle the case that $Z \subseteq \mathbb{Z}^m$ and φ is the inclusion map. Let M be the set of minimal elements of $(Z \cap \mathbb{N}^m) \setminus \{0\}$. From the descending chain condition we obtain

$$\forall x \in (Z \cap \mathbb{N}^m) \setminus \{0\}.\ \exists\, y \in M.\ y \leq x$$

By Lemma 34, M is finite, say $M = \{a_1, \dots, a_l\}$. Now we show that M generates Z as commutative monoid. Let $x \in Z$, and assume that $x - \sum_{j=1}^{l} \alpha_j a_j \neq 0$ for all $\alpha_j \in \mathbb{N}$. By the descending chain condition, the set of all elements of this

form contains a minimial element, say, $x - \sum_{j=1}^{l} \tilde{\alpha}_j a_j$. Choose $y \in M$ with $y \leq x - \sum_{j=1}^{l} \tilde{\alpha}_j a_j$. Since $y \neq 0$, we have $x - \sum_{j=1}^{l} \tilde{\alpha}_j a_j - y < x - \sum_{j=1}^{l} \tilde{\alpha}_j a_j$ and we reached a contradiction.

Step 2: The kernel $\varphi^{-1}(\{0\})$ is, as a subgroup of the finitely generated abelian group Z, itself finitely generated (remember here that $\mathbb{Z}$ is a Noetherian ring). Let $\{u_1, \ldots, u_k\}$ be a set of generators of $\varphi^{-1}(\{0\})$ as abelian group. Then $\{\pm u_1, \ldots, \pm u_k\}$ is a set of generators of $\varphi^{-1}(\{0\})$ as a commutative monoid.

By Step 1 we find $\{a_1, \ldots, a_l\} \subseteq \mathbb{Z}^m$ generating $\varphi(Z) \cap \mathbb{N}^m$ as a commutative monoid. Choose $v_j \in Z$ with $\varphi(v_j) = a_j$, $j = 1, \ldots, l$. Then we find, for each $x \in Z$, a linear combination of the v_j's with nonnegative integer coefficients such that

$$\varphi\Big(x - \sum_{j=1}^{l} \nu_j v_j\Big) = 0.$$

Hence, $\{\pm u_1, \ldots, \pm u_k\} \cup \{v_1, \ldots, v_l\}$ generates $\varphi^{-1}(Z)$ as commutative monoid.

□

D. Properties of $\widehat{F}$

Proof (of Lemma 12). Here the inclusion "$\supseteq$" is obvious. For the reverse inclusion, let $(o, \phi) \in \widehat{F}X$ and choose $p_{a,j}$ and $x_{a,j}$ according to Definition 11. Set $p_a = \sum_{j=1}^{n_a} p_{a,j}$. If $p_a = 0$, set $x_a = 0$. If $p_a > 0$, set $x_a = \sum_{j=1}^{n} \frac{p_{a,j}}{p_a} x_{a,j}$. Then $x_a \in X$ and $f(a) = \sum_{j=1}^{n_a} p_{a,j} x_{a,j} = p_a x_a$. □

Proof (of Lemma 16). Let $(o, \phi) \in \widehat{F}X$, and choose $p_a \in [0,1]$ and $x_a \in X$ as in Lemma 12. Then $\mu_X(\phi(a)) = p_a \mu_X(x_a) \leq p_a$, and hence $o + \sum_{a \in A} \mu_X(\phi(a)) \leq 1$. Further, $o \in [0,1]$, in particular $o \geq 0$.

Conversely, assume that $o \geq 0$ and $o + \sum_{a \in A} \mu_X(\phi(a)) \leq 1$. Let $a \in A$. Set $p_a = \mu_X(\phi(a))$, then $p_a \in [0,1]$ since $\sum_{a \in A} p_a \leq 1$. To define x_a consider first the case that $\mu_X(\phi(a)) = 0$. In this case $\phi(a) = 0$ since X is bounded, and we set $x_a = 0$. If $\mu_X(\phi(a)) > 0$, set $x_a = \frac{1}{\mu_X(\phi(a))} \phi(a)$. Since X is closed, we have $x_a \in X$. In both cases, we obtained a representation $\phi(a) = p_a x_a$ with $p_a \in [0,1]$ and $x_a \in X$. Clearly, $o + \sum_{a \in A} p_a \leq 1$, and we conclude that $(o, \phi) \in \widehat{F}X$. □

Proof (of Lemma 17). We build the extension in three stages.

① We extend c to the cone generated by X: Set $C = \bigcup_{t>0} tX$, and define $c_1 \colon C \to V_2$ by the following procedure. Given $x \in C$, choose $t > 0$ with $x \in tX$, and set $c_1(x) = t \cdot c\big(\frac{1}{t} x\big)$. By this procedure the map c_1 is indeed well-defined. Assume $x \in tX \cap sX$ where w.l.o.g. $s \leq t$. Then $\frac{1}{t} x = \frac{s}{t} \cdot \frac{1}{s} x$. Since $\frac{s}{t} \leq 1$, it follows that $c(\frac{1}{t} x) = \frac{s}{t} c(\frac{1}{s} x)$. Let us check that c_1 is cone-morphism, i.e., that

$$c_1(x + y) = c_1(x) + c_2(y),\ x, y \in C, \qquad c_1(px) = p c_1(x),\ x \in C, p \geq 0.$$

Given $x, y \in C$, choose $t > 0$ such that $x, y, x+y \in tX$. Observe here that C is a union of an increasing family of sets. Then

$$\begin{aligned} c_1(x+y) &= 2t \cdot c\Big(\frac{1}{2t}(x+y)\Big) = 2t \cdot c\Big(\frac{1}{2} \cdot \frac{1}{t}x + \frac{1}{2} \cdot \frac{1}{t}y\Big) \\ &= 2t \cdot \Big[\frac{1}{2}c\big(\frac{1}{t}x\big) + \frac{1}{2}c\big(\frac{1}{t}y\big)\Big] = 2t \cdot \Big[\frac{1}{2}c\big(\frac{1}{t}x\big) + \frac{1}{2}c\big(\frac{1}{t}y\big)\Big] \\ &= t \cdot c\Big(\frac{1}{t}x\Big) + t \cdot c\Big(\frac{1}{t}y\Big) = c_1(x) + c_2(y) \end{aligned}$$

Given $x \in C$ and $p > 0$, choose $t > 0$ with $x \in tX$. Then $px \in (pt)X$, and we obtain

$$c_1(px) = pt \cdot c\Big(\frac{1}{pt}(px)\Big) = pt \cdot c\Big(\frac{1}{t}x\Big) = pt \cdot \frac{1}{t}c_1(x) = pc_1(x).$$

For $p = 0$, the required equality is trivial. Finally, observe that c_1 extends c, since for $x \in X$ we can choose $t = 1$ in the definition of c_1.

② We extend c_1 to the linear subspace generated by C: Since C is a cone, we have $\operatorname{span} C = C - C$. We define $c_2\colon \operatorname{span} C \to V_2$ by the following procedure. Given $x \in \operatorname{span} C$, choose $a_+, a_- \in C$ with $x = a_+ - a_-$, and $c_2(x) = c_1(a_+) - c_2(a_-)$. By this procedure the map c_2 is indeed well-defined. Assume $x = a_+ - a_- = b_+ - b_-$. Then $a_+ + b_- = b_+ + a_-$, and we obtain

$$c_1(a_+) + c_1(b_-) = c_1(a_+ + b_-) = c_1(b_+ + a_-) = c_1(b_+) + c_1(a_-),$$

which yields $c_1(a_+) - c_2(a_-) = c_1(b_+) - c_2(b_-)$. Let us check that c_2 is linear. Given $x, y \in \operatorname{span} C$, choose representations $x = a_+ - a_-$, $y = b_+ - b_-$. Then $x + y = (a_+ + b_+) - (a_- + b_-)$, and we obtain

$$\begin{aligned} c_2(x+y) &= c_1(a_+ + b_+) - c_1(a_- + b_-) = \big[c_1(a_+) + c_1(b_+)\big] - \big[c_1(a_-) + c_1(b_-)\big] \\ &= \big[c_1(a_+) - c_1(a_-)\big] + \big[c_1(b_+) - c_1(b_-)\big] = c_2(x) + c_2(y). \end{aligned}$$

Given $x \in \operatorname{span} C$ and $p \in \mathbb{R}$, choose a representation $x = a_+ - a_-$ and distinguish cases according to the sign of p. If $p > 0$, we have the representation $px = pa_+ - pa_-$ and hence

$$\begin{aligned} c_2(px) &= c_1(pa_+) - c_1(pa_-) = pc_1(a_+) - pc_1(a_-) \\ &= p\big[c_1(a_+) - c_1(a_-)\big] = pc_2(x). \end{aligned}$$

If $p < 0$, we have the representation $px = (-p)a_- - (-p)a_+$ and hence

$$\begin{aligned} c_2(px) &= c_1((-p)a_-) - c_1((-p)a_+) = (-p)c_1(a_-) - (-p)c_1(a_+) \\ &= p\big[c_1(a_+) - c_1(a_-)\big] = pc_2(x). \end{aligned}$$

For $p = 0$, the required equality is trivial. Finally, observe that c_2 extends c_1, since for $x \in C$ we can choose the representation $x = x - 0$ in the definition of c_2.

③ We extend c_2 to V_1: By linear algebra a linear map given on a subspace can be extended to a linear map on the whole space.

The uniqueness statement is clear. □

Proof (of Corollary 18). First assume that (4) holds. Let $x \in X$. Then $\mu_X(x) \leq 1$, and we obtain

$$c_{\mathrm{o}}(x) + \sum_{a \in A} \mu_Y(c_a(x)) = \tilde{c}_{\mathrm{o}}(x) + \sum_{a \in A} \mu_Y(\tilde{c}_a(x)) \leq \mu_X(x) \leq 1.$$

Further, $c_{\mathrm{o}}(x) \geq 0$ by assumption. Now Lemma 16 gives $c(x) \in \widehat{F}Y$.

Conversely, assume $c(X) \subseteq \widehat{F}Y$, and let $x \in \mathbb{R}^n$ be given. If $\mu_X(x) = \infty$, the relation (4) trivially holds. If $\mu_X(x) = 0$, then $x = 0$ since X is bounded. Hence, the left side of (4) equals 0, and again (4) holds. Assume that $\mu_X(x) \in (0, \infty)$. Since X is closed, we have $\mu_X(x)^{-1}x \in X$, and hence $c(\mu_X(x)^{-1}x) \in \widehat{F}Y$. From Lemma 16, we get the estimate

$$\begin{aligned}\tilde{c}_{\mathrm{o}}(x) + \sum_{a \in A} \mu_Y(\tilde{c}_a(x)) &= \mu_X(x)\Big(\tilde{c}_{\mathrm{o}}\big(\frac{1}{\mu_X(x)}x\big) + \sum_{a \in A} \mu_Y\big(\tilde{c}_a(\frac{1}{\mu_X(x)}x)\big)\Big) \\ &= \mu_X(x)\Big(c_{\mathrm{o}}\big(\frac{1}{\mu_X(x)}x\big) + \sum_{a \in A} \mu_Y\big(c_a(\frac{1}{\mu_X(x)}x)\big)\Big) \leq \mu_X(x).\end{aligned}$$

□

E. Proof details of the Extension Theorem

Recall Kakutani's theorem [14, Corollary].

Theorem 35 (Kakutani 1941). *Let $M \subseteq \mathbb{R}^n$ and $P\colon M \to \mathcal{P}(M)$. Assume*

1. *M is nonempty, compact, and convex,*
2. *for each $x \in M$, the set $P(x)$ is nonempty, closed, and convex,*
3. *the map P has closed graph in the sense that, whenver $x_n \in M$, $x_n \to x$, and $y_n \in P(x_n)$, $y_n \to y$, it follows that $y \in P(x)$.*

Then there exists $x \in M$ with $x \in P(x)$.

Note that P having closed graph implies that $P(x)$ is closed for all x. To see this, let $y_n \in P(x)$, $y_n \to y$, and use the constant sequence $x_n = x$ in the closed graph property.

In the proof of Theorem 19 we shall, as in Example 15, identify a pyramid Y with the appropriately scaled normal vector u of its inclined side. Then, for two pyramids Y_1 and Y_2 with corresponding normal vectors u_1 and u_2, the requirement that $X \subseteq Y_j$ becomes $(x, u_j) \leq \mu_X(x)$, $x \geq 0$, and the requirement $\tilde{c}(Y_2) \subseteq \widehat{F}Y_1$ becomes $\tilde{c}_{\mathrm{o}}(x) + \sum_{a \in A}(\tilde{c}_a(x), u_1) \leq (x, u_2)$, $x \geq 0$, cf. Corollary 18.

Proof (of Theorem 19). Let M be the set

$$M = \big\{u \in \mathbb{R}^n \mid u \geq 0 \text{ and } (x,u) \leq \mu_X(x), x \geq 0\big\}.$$

We have to include vectors u with possibly vanishing components into M to ensure closedness. It will be a step in the proof to show that a fixed point must be strictly positive.

Let $P\colon M \to \mathcal{P}(M)$ be the map

$$P(u) = \big\{v \in M \mid \tilde{c}_{\mathrm{o}}(x) + \sum_{a\in A}(\tilde{c}_a(x), u) \leq (x,v), x \geq 0\big\}.$$

Here we again denote by $\tilde{c}\colon \mathbb{R}^n \to \mathbb{R} \times (\mathbb{R}^n)^A$ the linear extension of c. Observe that $\tilde{c}(x) \geq 0$ for all $x \geq 0$, since $\Delta^n \subseteq X$ and $c(x) \geq 0$ for $x \in X$.

It is easy to check that M and P satisfy the hypothesis of Kakutani's Theorem, the crucial point being that $P(u) \neq \emptyset$.

① M is nonempty: We have $0 \in M$.

② M is compact: To show that M is closed let $u_n \in M$ with $u_n \to u$. Since $u_n \geq 0$ also $u \geq 0$, and for each fixed $x \geq 0$ continuity of the scalar product yields $(x,u) = \lim_{n\to\infty}(x,u_n) \leq \mu_X(x)$. Further, M is bounded since $(e_j,u) \leq \mu_X(e_j) \leq 1$, $j = 1,\dots,n$, by our assumption that $\Delta^n \subseteq X$, and hence $u \in [0,1]^n$.

③ M is convex: Let $u_1, u_2 \in M$ and $p \in [0,1]$. First, clearly, $pu_1 + (1-p)u_2 \geq 0$. Second, for each $x \geq 0$,

$$\begin{aligned}(x, pu_1 + (1-p)u_2) &= p(x,u_1) + (1-p)(x,u_2)\\ &\leq p\mu_X(x) + (1-p)\mu_X(x) = \mu_X(x).\end{aligned}$$

④ $P(u)$ is nonempty: Let $u \in M$ be given. The map $x \mapsto \tilde{c}_{\mathrm{o}}(x) + \sum_{a\in A}(\tilde{c}_a(x),u)$ is a linear functional on $\mathbb{R}^n$. Thus we find $v \in \mathbb{R}^n$ representing it as $x \mapsto (x,v)$. Since $e_j \in X$, we have

$$(e_j, v) = \tilde{c}_{\mathrm{o}}(e_j) + \sum_{a\in A}(\tilde{c}_a(e_j), u) \geq 0.$$

Further, using that $u \in M$ and $\tilde{c}(X) \subseteq \widehat{F}X$, we obtain that for each $x \geq 0$

$$(x,v) = \tilde{c}_{\mathrm{o}}(x) + \sum_{a\in A}(\tilde{c}_a(x),u) \leq \tilde{c}_{\mathrm{o}}(x) + \sum_{a\in A}\mu_X(\tilde{c}_a(x)) \leq \mu_X(x).$$

Together, we see that $v \in M$. By its definition, therefore, $v \in P(u)$.

⑤ $P(u)$ is convex: Let $v_1, v_2 \in P(u)$ and $p \in [0,1]$. First, since M is convex, $pv_1 + (1-p)v_2$ belongs to M. Second, for each $x \geq 0$,

$$\begin{aligned}(x, pv_1 + (1-p)v_2) &= p(x,v_1) + (1-p)(x,v_2)\\ &\geq p\Big(\tilde{c}_{\mathrm{o}}(x) + \sum_{a\in A}(\tilde{c}_a(x),u)\Big) + (1-p)\Big(\tilde{c}_{\mathrm{o}}(x) + \sum_{a\in A}(\tilde{c}_a(x),u)\Big)\\ &= \tilde{c}_{\mathrm{o}}(x) + \sum_{a\in A}(\tilde{c}_a(x),u).\end{aligned}$$

⑥ P has closed graph: Let $u_n \in M$, $u_n \to u$, and $v_n \in P(u_n)$, $v_n \to v$. Then $u, v \in M$ since M is closed. Now fix $x \geq 0$. Continuity of the scalar product allows to pass to the limit in the relation

$$\tilde{c}_{\mathrm{o}}(x) + \sum_{a \in A} (\tilde{c}_a(x), v_n) \leq (x, u_n),$$

which holds for all $n \in \mathbb{N}$. This yields $\tilde{c}_{\mathrm{o}}(x) + \sum_{a \in A} (\tilde{c}_a(x), v) \leq (x, u)$.

Having verified all necessary hypothesis, Theorem 35 can be applied and furnishes us with $u \in M$ satisfying $u \in P(u)$, explicitly, $u \in \mathbb{R}^n$ with

$$u \geq 0, \quad (x, u) \leq \mu_X(x), x \geq 0, \quad \tilde{c}_{\mathrm{o}}(x) + \sum_{a \in A} (\tilde{c}_a(x), u) \leq (x, u), x \geq 0. \tag{10}$$

Set $Y = \{x \geq 0 \mid (x, u) \leq 1\}$. Then Y is a `PCA`, and by definition contained in $\mathbb{R}^n_+$. It contains X since $u \in M$, and since $u \in P(u)$ we have $\tilde{c}(Y) \subseteq \widehat{F}Y$. Thus $d = \tilde{c}|_Y$ turns Y into an $\widehat{F}$-coalgebra, and since $c = \tilde{c}|_X = (\tilde{c}|_Y)|_X = d|_X$, the inclusion map $\iota \colon X \to Y$ is an $\widehat{F}$-morphism.

It remains to show that Y is generated by n linearly independent vectors. Remembering again Example 15, this is equivalent to u being strictly positiv. Let $I = \{j \in \{1, \ldots, n\} \mid (e_j, u) = 0\}$. For each $j \in I$ the last relation in (10) implies that $c_{\mathrm{o}}(e_j) = 0$ and $(c_a(e_j), u) = 0$, $a \in A$. Since $u \geq 0$ and $c_a(e_j) \geq 0$, we conclude that the vector $c_a(e_j)$ can have nonzero components only in those coordinates where u has zero component. In other words, $c_a(e_j) \in \operatorname{span}\{e_i \mid i \in I\}$. Now (5) gives $I = \emptyset$. □

F. Proof details for properness of $\widehat{F}$

Proof (of Lemma 21). We denote $v_j = (1, \ldots, 1) \in \mathbb{R}^{n_j}$. By Example 15

$$\mu_{\Delta^{n_j}}(x_j) = (x_j, v_j), \; x_j \in \mathbb{R}^{n_j}_+.$$

Since (Δ^{n_j}, c_j) is an $\widehat{F}$-coalgebra, Corollary 18 yields

$$\tilde{c}_{j_{\mathrm{o}}}(x_j) + \sum_{a \in A} (\tilde{c}_{j_a}(x_j), v_j) \leq (x_j, v_j), \quad x_j \in \mathbb{R}^{n_j}_+, j = 1, 2.$$

Summing up these two inequalities yields that for $x_1 \in \mathbb{R}^{n_1}_+$ and $x_2 \in \mathbb{R}^{n_2}_+$

$$\Big[\tilde{c}_{1_{\mathrm{o}}}(x_1) + \tilde{c}_{2_{\mathrm{o}}}(x_2)\Big] + \sum_{a \in A} \Big[(\tilde{c}_{1_a}(x_1), v_1) + (\tilde{c}_{2_a}(x_2), v_2)\Big] \leq (x_1, v_1) + (x_2, v_2). \tag{11}$$

The definition of the map d in the basic diagram ensures that for $(x_1, x_2) \in Z$

$$d_{\mathrm{o}}((x_1, x_2)) = \tilde{c}_{1_{\mathrm{o}}}(x_1) = \tilde{c}_{2_{\mathrm{o}}}(x_2), \quad d_a((x_1, x_2)) = \big(\tilde{c}_{1_a}(x_2), \tilde{c}_{2_a}(x_2)\big).$$

Set $v = \frac{1}{2}(1, \dots, 1) \in \mathbb{R}^{n_1+n_2}$. Plugging the above into (11) and dividing by 2 yields

$$d_{\mathrm{o}}((x_1, x_2)) + \sum_{a \in A} \big(d_a((x_1, x_2)), v\big) \leq \big((x_1, x_2), v\big), \quad (x_1, x_2) \in Z \cap \mathbb{R}_+^{n_1+n_2}.$$

By Example 14 and Example 15, we have

$$\mu_{Z \cap 2\Delta^{n_1+n_2}}(x) = \max\big\{\mu_Z(x), \mu_{2\Delta^{n_1+n_2}}(x)\big\} = \begin{cases} (x, v)\,, & x \in Z \cap \mathbb{R}_+^{n_1+n_2}, \\ \infty \quad , & \text{otherwise.} \end{cases}$$

From Lemma 16 we now obtain $d(Z \cap 2\Delta^{n_1+n_2}) \subseteq \widehat{F}(Z \cap 2\Delta^{n_1+n_2})$. □

Proof (of Lemma 22). Using the basic diagram, we obtain

$$\begin{aligned} &\tilde{c}_j(\Delta^{n_j}) \subseteq \widehat{F}\Delta^{n_j} \subseteq \widehat{F}Y_j, \\ &\tilde{c}_j(\pi_j(Z \cap 2\Delta^{n_1+n_2})) \subseteq \widehat{F}(\pi_j(Z \cap 2\Delta^{n_1+n_2})) \subseteq \widehat{F}Y_j. \end{aligned}$$

Since $\tilde{c}_j$ is linear, in particular convex, and $\widehat{F}Y_j$ is convex, it follows that

$$\tilde{c}_j\big(\operatorname{co}(\Delta^{n_j} \cup \pi_j(Z \cap 2\Delta^{n_1+n_2}))\big) \subseteq \widehat{F}Y_j.$$

□

Proof (Lemma 23). We check that the `PCA` Y_j satisfies the hypothesis of Theorem 19. By its definition $\Delta^{n_j} \subseteq Y_j \subseteq \mathbb{R}_+^{n_j}$. Since Y_j is finitely generated, it is a compact subset of $\mathbb{R}^{n_j}$. Finally, since the coalgebra structure on Y_j is an extension of the one on Δ^{n_j}, the present assumption (6) implies that the condition (5) of Theorem 19 is satisfied. Note here that $\Delta^{n_j} \cap \operatorname{span}\{e_i \mid i \in I\} = \operatorname{co}(\{e_i \mid i \in I\} \cup \{0\})$.

Applying Theorem 19 we obtain extensions U_j as required. □

Proof (of Lemma 24). We show that the diagram

$$\begin{tikzcd} \Delta^n \arrow[r, "f"] \arrow[d, "c"'] & X \arrow[d, "g"] \\ \widehat{F}\Delta^n \arrow[r, "\mathrm{id} \times (f \circ -)"'] & {[0,1] \times X^A} \end{tikzcd}$$

commutes. First, for $k \notin I$, we have $((\mathrm{id} \times (f \circ -)) \circ c)(e_k) = (g \circ f)(e_k)$ by the definition of g. Second, consider $k \in I$. Then $(g \circ f)(e_k) = 0$ since $f(e_k) = 0$. By (7), also $((\mathrm{id} \times (f \circ -)) \circ c)(e_k) = 0$.

Since $\widehat{F}f$ maps $\widehat{F}\Delta^n$ into $\widehat{F}X$, we have $g(X) \subseteq \widehat{F}X$. This says that X indeed becomes an $\widehat{F}$-coalgebra with g. Revisiting the above diagram shows that f is an $\widehat{F}$-coalgebra morphism. □

Proof (of Corollary 25). Applying Lemma 24 repeatedly, we obtain after finitely many steps an $\widehat{F}$-coalgebra (Δ^k, g) such that no nonempty subset $I \subseteq \{1, \ldots, k\}$ with (7) exists for (Δ^k, g), and that we have an $\widehat{F}$-coalgebra morphism $f\colon (\Delta^n, c) \to (\Delta^k, g)$. Note here that in each application of the lemma the number of generators decreases. $\square$