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ADAPTIVE IGAFEM WITH OPTIMAL CONVERGENCE RATES: HIERARCHICAL B-SPLINES

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Abstract. We consider an adaptive algorithm for finite element methods for the isogeometric analysis (IGAFEM) of elliptic (possibly non-symmetric) second-order partial differential equations in arbitrary space dimension \( d \geq 2 \). We employ hierarchical B-splines of arbitrary degree and different order of smoothness. We propose a refinement strategy to generate a sequence of locally refined meshes and corresponding discrete solutions. Adaptivity is driven by some weighted residual \textit{a posteriori} error estimator. We prove linear convergence of the error estimator (resp. the sum of energy error plus data oscillations) with optimal algebraic rates. Numerical experiments underpin the theoretical findings.

1. Introduction

1.1. Adaptivity in isogeometric analysis. The central idea of isogeometric analysis (IGA) is to use the same ansatz functions for the discretization of the partial differential equation (PDE) as for the representation of the problem geometry in computer aided design (CAD); see [HCB05, CHB09, BBdVC+06]. The CAD standard for spline representation in a multivariate setting relies on tensor-product B-splines. However, to allow for adaptive refinement, several extensions of the B-spline model have recently emerged, e.g., analysis-suitable T-splines [SLSH12, BdVBSV13], hierarchical splines [VGJS11, GJS12, KVvdZvB14], or LR-splines [DLP13, JKD14]; see also [JRK15, HKMP16] for a comparison of these approaches. All these concepts have been studied via numerical experiments. However, so far there exists only little literature concerning the thorough mathematical analysis of adaptive isogeometric finite element methods (IGAFEM). Indeed, we are only aware of the works [BG16a] which investigates an estimator reduction of an IGA FEM with certain hierarchical splines introduced in [BG15], as well as [BG16b] which investigates linear convergence of an IGA FEM with truncated hierarchical B-splines introduced in [GJS12]. In the continuation of the latter work [BG16b], [BGMP16] studied the corresponding mesh-refinement strategy together with some refinement-related properties for the proof of optimal convergence. However, the mathematical proof that the adaptive strategy of [BG16b] leads to optimal convergence rates, is still missing in the literature. Moreover, in this case one cannot use hierarchical B-splines for the implementation, but has to use truncated hierarchical B-splines instead. It is important to note that the procedure of truncation requires a specific construction that entails complicated supports of the basis functions, which are in general not even connected, and their use may produce an overhead with an adaptive strategy that cannot be neglected. For standard FEM with globally continuous piecewise polynomials, adaptivity is well understood; see, e.g., [Dör96, MNS00, BDD04, Ste07, CKNS08, FFP14, CFPP14] for milestones

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on convergence and optimal convergence rates. In the frame of isogeometric boundary element methods (IGABEM), we also mention our recent work [FGHP16b] which shows linear convergence with optimal rates for some adaptive isogeometric boundary element method in 2D from [FGP15, FGHP16a], where it is however sufficient to use the span of univariate non-uniform rational B-splines (NURBS).

1.2. Model problem. On the bounded Lipschitz-domain \( \Omega \subset \mathbb{R}^d \) with \( d \geq 2 \), we consider a general second-order linear elliptic PDE in divergence form with homogenous Dirichlet boundary conditions

\[
Lu := -\text{div}(A \nabla u) + b \cdot \nabla u + cu = f \quad \text{in } \Omega, \\
u = 0 \quad \text{on } \partial \Omega. \tag{1.1}
\]

We pose the following regularity assumptions on the coefficients: \( A(x) \in \mathbb{R}^{d \times d}_{\text{sym}} \) is a symmetric matrix with \( A \in W^{1,\infty}(\Omega) \). The vector \( b(x) \in \mathbb{R}^d \) and the scalar \( c(x) \in \mathbb{R} \) satisfy \( b, c \in L^\infty(\Omega) \). We interpret \( L \) in its weak form and define the corresponding bilinear form

\[
\langle w, v \rangle_L := \int_{\Omega} A(x) \nabla w(x) \cdot \nabla v(x) + b(x) \cdot \nabla w(x)v(x) + c(x)w(x)v(x) \, dx. \tag{1.2}
\]

The bilinear form is continuous, i.e., it holds that

\[
\langle w, v \rangle_L \leq (\| A \|_{L^\infty(\Omega)} + \| b \|_{L^\infty(\Omega)} + \| c \|_{L^\infty(\Omega)}) \| w \|_{H^1(\Omega)} \| v \|_{H^1(\Omega)} \quad \text{for all } v, w \in H^1(\Omega). \tag{1.3}
\]

Additionally, we suppose ellipticity of \( \langle \cdot, \cdot \rangle_L \) on \( H^1_0(\Omega) \), i.e.,

\[
\langle v, v \rangle_L \geq C_{\text{ell}} \| v \|_{H^1(\Omega)}^2 \quad \text{for all } v \in H^1_0(\Omega). \tag{1.4}
\]

Note that (1.4) is for instance satisfied if \( A(x) \) is uniformly positive definite and if \( b \in H(\text{div}, \Omega) \) with \( -\frac{1}{2} \text{div} b(x) + c(x) \geq 0 \) almost everywhere in \( \Omega \).

Overall, the boundary value problem (1.1) fits into the setting of the Lax-Milgram theorem and therefore admits a unique solution \( u \in H^1_0(\Omega) \) to the weak formulation

\[
\langle u, v \rangle_L = \int_{\Omega} fv \, dx \quad \text{for all } v \in H^1_0(\Omega). \tag{1.5}
\]

Finally, we note that the additional regularity \( A \in W^{1,\infty}(\Omega) \) (instead of \( A \in L^\infty(\Omega) \)) is only required for the well-posedness of the residual a posteriori error estimator, see Section 2.

1.3. Outline & Contributions. The remainder of this work is roughly organized as follows: Section 2 provides an abstract framework for adaptive mesh-refinement for conforming FEM for the model problem (1.1). Its main result is Theorem 2.4 which states optimal convergence behavior of some standard adaptive algorithm. Section 3 considers conforming FEM based on hierarchical splines. Its main result is Theorem 3.8 which states that hierarchical splines fit into the framework of Section 2. The proofs of Theorem 2.4 and Theorem 3.8 are given in Section 4 and Section 5, respectively. Three numerical experiments in Section 6 underpin the optimal convergence behavior of adaptive IGAFEM with hierarchical splines.

In more detail, the contribution of Section 2 can be paraphrased as follows: We formulate the standard adaptive strategy (Algorithm 2.3) driven by some residual a posteriori error estimator (2.9) in the frame of conforming FEM. We formulate three assumptions (M1)–(M3) for the underlying meshes (Section 2.1), five assumptions (R1)–(R5) on the mesh-refinement (Section 2.2), and six assumptions (S1)–(S6) on the FEM spaces (Section 2.3). First, these
assumptions are sufficient to guarantee that the error estimator $\eta$ associated with the FEM solution $U_\bullet \in X_\bullet \subset H^1_0(\Omega)$ is efficient and reliable, i.e., there exist $C_{\text{eff}}, C_{\text{rel}} > 0$ such that

$$C_{\text{eff}}^{-1} \eta_\bullet \leq \inf_{V_\bullet \in X_\bullet} \left( \|u - V_\bullet\|_{H^1(\Omega)} + \text{osc}_\bullet(V_\bullet) \right) \leq \|u - U_\bullet\|_{H^1(\Omega)} + \text{osc}_\bullet(U_\bullet) \leq C_{\text{rel}} \eta_\bullet. \quad (1.6)$$

where $\text{osc}_\bullet(\cdot)$ denotes certain data oscillation terms. Second, Theorem 2.4 states that Algorithm 2.3 leads to linear convergence with optimal rates in the spirit of [CKNS08, CFPP14]: Let $\eta_\ell$ denote the error estimator in the $\ell$-th step of the adaptive algorithm. Then, there exist $C > 0$ and $0 < q < 1$ such that

$$\eta_{\ell+n} \leq C q^n \eta_\ell \quad \text{for all } \ell, n \in \mathbb{N}_0. \quad (1.7)$$

Moreover, for sufficiently small marking parameters in Algorithm 2.3, the estimator (resp. the so-called total error $\|u - U_\ell\|_{H^1(\Omega)} + \text{osc}_\ell(U_\ell)$; see (1.6)) decays even with the optimal algebraic convergence rate in the sense of certain nonlinear approximation classes (Section 2.7).

In explicit terms, we identify sufficient conditions of the underlying meshes, the local FEM spaces, as well as the employed (local) mesh-refinement rule which guarantee that the related residual \textit{a posteriori} error estimator satisfies the \textit{axioms of adaptivity} from [CFPP14], so that linear convergence with optimal rates for the standard adaptive algorithm follows. While we exploit this framework only for IGAFEM with hierarchical splines, we believe that it might also serve as a promising starting point to analyze different technologies for adaptive IGAFEM like (analysis-suitable) $T$-splines or LR-splines, as well as for other conforming discretizations like the virtual element method (VEM) from [BdVBC+13].

Section 3 recalls the definition of hierarchical splines from [VGJS11], derives the canonical basis of the hierarchical spline space $X_\bullet \subset H^1_0(\Omega)$ with Dirichlet boundary conditions (Section 3.3), formulates an admissibility criterion (3.20) for hierarchical meshes (Section 3.4), and introduces some local mesh-refinement rule for admissible hierarchical meshes (Section 3.5). One crucial observation is that the new mesh-refinement strategy for hierarchical meshes (Algorithm 3.5) guarantees that the number of (truncated) hierarchical B-splines on each element as well as the number of active elements contained in the support of each (truncated) hierarchical B-spline is uniformly bounded (Proposition 3.3). If one uses the strategy of [BG16b, BGMP16] instead, this property is not satisfied for hierarchical B-splines, but only for truncated hierarchical B-splines. In general, the latter have a smaller, but also more complicated and not necessarily connected support.

The main result of Section 3 and the entire work is Theorem 3.8 which states that hierarchical splines together with the proposed local mesh-refinement strategy satisfy all assumptions of Section 2, so that Theorem 2.4 applies. In particular, our work goes beyond [BG16b] in two respects: While [BG16b] only proves linear convergence of the adaptive algorithm, we give the first proof of optimal convergence rates for IGAFEM. Moreover, [BG16b] adapts the analysis of [CKNS08] and is hence restricted to symmetric problems (i.e., $b = 0$ and $c \geq 0$ in (1.1)). Our analysis exploits the framework of [CFPP14] together with some recent ideas from [FFP14, BHP17] and also covers the non-symmetric problem (1.1). Technical contributions of general interest include the following: We prove that a hierarchical mesh is admissible if and only if it can be obtained by the mesh-refinement strategy of Algorithm 3.5 (Proposition 5.2). Moreover, admissible meshes also allow a simpler computation of truncated hierarchical B-splines in the sense that truncation simplifies considerably (Proposition 5.6). Together with some ideas from [SM16], we use this observation to define
a Scott-Zhang-type projector \( J_{\bullet} : L^2(\Omega) \to X_{\bullet} \) which is locally \( L^2 \)- and \( H^1 \)-stable and has a first-order approximation property (Section 5.10).

1.4. General notation. Throughout, \(| \cdot |\) denotes the absolute value of scalars, the Euclidean norm of vectors in \( \mathbb{R}^d \), as well as the \( d \)-dimensional measure of a set in \( \mathbb{R}^d \). Moreover, \# denotes the cardinality of a set as well as the multiplicity of a knot within a given knot vector. We write \( A \lesssim B \) to abbreviate \( A \leq cB \) with some generic constant \( c > 0 \) which is clear from the context. Moreover, \( A \simeq B \) abbreviates \( A \lesssim B \lesssim A \). Throughout, mesh-related quantities have the same index, e.g., \( X_{\bullet} \) is the ansatz space corresponding to the mesh \( T_{\bullet} \). The analogous notation is used for meshes \( T_{\sigma}, T_{\tau} \) or \( T_{\ell} \) etc. Moreover, we use \( \hat{\cdot} \) to transfer quantities in the physical domain \( \Omega \) to the parameter domain \( \hat{\Omega} \), e.g., we write \( \hat{T} \) for the set of all admissible meshes in the parameter domain instead of \( T \) which denotes the set of all admissible meshes in the physical domain.

2. Axioms of adaptivity (revisited)

The aim of this section is to formulate an adaptive algorithm (Algorithm 2.3) for conforming FEM discretizations of our model problem (1.1), where adaptivity is driven by the residual \( a \) posteriori error estimator (see (2.9) below). We identify the crucial properties of the underlying meshes, the mesh-refinement, as well as the finite element spaces which ensure that the residual error estimator fits into the general framework of [CFPP14] and which hence guarantee optimal convergence behavior of the adaptive algorithm. The main result of this section is Theorem 2.4 which is proved in Section 4.

2.1. Meshes. Throughout, \( T_{\bullet} \) is a mesh of \( \Omega \) in the following sense:

- \( T_{\bullet} \) is a finite set of compact Lipschitz domains;
- for all \( T, T' \in T_{\bullet} \) with \( T \neq T' \), the intersection \( T \cap T' \) has measure zero;
- \( \Omega = \bigcup_{T \in T_{\bullet}} T \), i.e., \( T_{\bullet} \) is a partition of \( \Omega \).

We suppose that there is a countably infinite set \( \mathcal{T} \) of admissible meshes. For \( T_{\bullet} \in \mathcal{T} \) and \( \omega \subseteq \Omega \), we define the patches of order \( k \in \mathbb{N}_0 \) inductively by

\[
\pi^0_{\bullet}(\omega) := \omega, \quad \pi^k_{\bullet}(\omega) := \bigcup \{ T \in T_{\bullet} : T \cap \pi^{k-1}_{\bullet}(\omega) \neq \emptyset \}.
\]  

(2.1)

The corresponding set of elements is

\[
\Pi^k_{\bullet}(\omega) := \{ T \in T_{\bullet} : T \subseteq \pi^k_{\bullet}(\omega) \}, \quad \text{i.e.,} \quad \pi^k_{\bullet}(\omega) = \bigcup \Pi^k_{\bullet}(\omega).
\]  

(2.2)

To abbreviate notation, we let \( \pi_{\bullet}(\omega) := \pi^1_{\bullet}(\omega) \) and \( \Pi_{\bullet}(\omega) := \Pi^1_{\bullet}(\omega) \). For \( S_{\bullet} \subseteq T_{\bullet} \), we define \( \pi^k_{\bullet}(S_{\bullet}) := \pi^k_{\bullet}(\bigcup S_{\bullet}) \) and \( \Pi^k_{\bullet}(S_{\bullet}) := \Pi^k_{\bullet}(\bigcup S_{\bullet}) \).

We suppose that there exist \( C_{\text{shape}}, C_{\text{patch}}, C_{\text{trace}} > 0 \) such that all meshes \( T_{\bullet} \in \mathcal{T} \) satisfy the following three properties (M1)–(M3):

(M1) Shape regularity. For all \( T \in T_{\bullet} \) and all \( T' \in \Pi_{\bullet}(T) \), it holds that \( C_{\text{shape}}^{-1} |T| \leq |T'| \leq C_{\text{shape}} |T'| \), i.e., neighboring elements have comparable size.

(M2) Bounded element patch. For all \( T \in T_{\bullet} \), it holds that \#\( \Pi_{\bullet}(T) \leq C_{\text{patch}} \), i.e., the number of elements in a patch is uniformly bounded.

(M3) Trace inequality. For all \( T \in T_{\bullet} \) and all \( v \in H^1(\Omega) \), it holds that \( \|v\|_{L^2(\partial T)}^2 \leq C_{\text{trace}} (|T|^{-1/d} \|v\|_{L^2(T)}^2 + |T|^{1/d} \|
abla v\|_{L^2(T)}^2) \).
We associate a corresponding refinement \( T \), i.e., at least the marked elements have been refined. We define \( \text{refine}(T) \) as the set of all \( T \) such that there exist meshes \( T_0, \ldots, T_J \) and marked elements \( M_0, \ldots, M_{J-1} \) with \( T = T_J = \text{refine}(T_{J-1}, M_{J-1}), \ldots, T_1 = \text{refine}(T_0, M_0) \) and \( T_0 = T \). Here, we formally allow \( J = 0 \), i.e., \( T \in \text{refine}(T) \). We assume that there exists a fixed initial mesh \( T_0 \in T \) with \( T = \text{refine}(T_0) \).

We suppose that there exist \( C_\text{son} \geq 2 \) and \( 0 < q_\text{son} < 1 \) such that all meshes \( T_\ell \in T \) satisfy for arbitrary marked elements \( M_\ell \subseteq T_\ell \) with corresponding refinement \( T_\ell := \text{refine}(T_\ell, M_\ell) \), the following elementary properties (R1)–(R3):

- **(R1)** Bounded number of sons. It holds that \( \#T_\ell \leq C_\text{son} \#T_0 \), i.e., one step of refinement leads to a bounded increase of elements.
- **(R2)** Father is union of sons. It holds that \( T = \bigcup \{ T' \in T_\ell : T' \subseteq T \} \) for all \( T \in T_\ell \), i.e., each element \( T \) is the union of its successors.
- **(R3)** Reduction of sons. It holds that \( |T'| \leq q_\text{son} |T| \) for all \( T \in T_\ell \) and all \( T' \in T_\ell \) with \( T' \subseteq T \), i.e., successors are uniformly smaller than their father.

By induction and the definition of \( \text{refine}(T_\ell) \), one easily sees that (R2)–(R3) remain valid if \( T_\ell \) is an arbitrary mesh in \( \text{refine}(T) \). In particular, (R2)–(R3) imply that each refined element \( T \in T_\ell \setminus T_0 \) is split into at least two sons, wherefore

\[
\#(T_\ell \setminus T_0) \leq \#T_0 - \#T_\ell \quad \text{for all } T_\ell \in \text{refine}(T).
\]

Besides (R1)–(R3), we suppose the following less trivial requirements (R4)–(R5) with a generic constant \( C_\text{clo} > 0 \):

- **(R4)** Closure estimate. If \( M_\ell \subseteq T_\ell \) and \( T_{\ell+1} = \text{refine}(T_\ell, M_\ell) \) for all \( \ell \in \mathbb{N}_0 \), then

\[
\#T_\ell - \#T_0 \leq C_\text{clo} \sum_{\ell=0}^{L-1} \#M_\ell \quad \text{for all } L \in \mathbb{N}.
\]

- **(R5)** Overlay estimate. For all \( T_\ell, T_\ell' \in T \), there exists a common refinement \( T_\ell' \in \text{refine}(T_\ell) \cap \text{refine}(T_\ell') \) such that

\[
\#T_\ell' \leq \#T_\ell + \#T_\ell' - \#T_0.
\]

### 2.3. Finite element space

With each \( T_\ell \in T \), we associate a finite dimensional space \( X_\ell \subseteq \{ v \in H^1_0(\Omega) : v|_T \in H^2(T) \} \) for all \( T \in T_\ell \).

Let \( U_\ell \in X_\ell \) be the corresponding Galerkin approximation to the solution \( u \in H^1_0(\Omega) \), i.e.,

\[
\langle U_\ell, V_\ell \rangle_L = \int_\Omega fV_\ell \, dx \quad \text{for all } V_\ell \in X_\ell.
\]

We note the Galerkin orthogonality

\[
\langle u - U_\ell, V_\ell \rangle_L = 0 \quad \text{for all } V_\ell \in X_\ell
\]

as well as the resulting Céa-type quasi-optimality

\[
\|u - U_\ell\|_{H^1(\Omega)} \leq C_\text{Céa} \min_{V_\ell \in X_\ell} \|u - V_\ell\|_{H^1(\Omega)}, \quad \text{with } C_\text{Céa} := \frac{\|A\|_{L^\infty(\Omega)} + \|b\|_{L^\infty(\Omega)} + \|c\|_{L^\infty(\Omega)}}{C_\text{st}}.
\]
We suppose that there exist constants $C_{\text{inv}} > 0$ and $k_{\text{loc}}, k_{\text{proj}} \in \mathbb{N}_0$ such that the following properties (S1)–(S3) hold for all $\mathcal{T} \in \mathcal{T}$:

**S1** Inverse estimate. For all $i, j \in \{0, 1, 2\}$ with $j \leq i$, all $V_\bullet \in \mathcal{X}_\bullet$ and all $T \in \mathcal{T}_\bullet$, it holds that $|T|^{(i-j)/d} \|V_\bullet\|_{H^i(T)} \leq C_{\text{inv}} \|V_\bullet\|_{H^j(T)}$.

**S2** Refinement guarantees nestedness. For all $\mathcal{T}_\circ \in \text{refine}(\mathcal{T}_\bullet)$, it holds that $\mathcal{X}_\bullet \subseteq \mathcal{X}_\circ$.

**S3** Local domain of definition. With $\Pi_{\text{loc}}^\bullet := \Pi_{\text{loc}}^{k_{\text{loc}}}$, $\pi_{\text{loc}}^\bullet := \pi_{\text{loc}}^{k_{\text{loc}}}$ and $\pi_{\text{proj}}^\bullet := \pi_{\text{proj}}^{k_{\text{proj}}}$, it holds for all $\mathcal{T}_\circ \in \text{refine}(\mathcal{T}_\bullet)$ and all $T \in \mathcal{T}_\bullet \setminus \Pi_{\text{loc}}^\bullet(\mathcal{T}_\bullet \setminus \mathcal{T}_\circ) \subseteq \mathcal{T}_\bullet \cap \mathcal{T}_\circ$, that $V_\bullet|_{\mathcal{T}_{\text{proj}}(T)}(T) \in \{V_\bullet|_{\mathcal{T}_{\text{proj}}(T)} : V_\bullet \in \mathcal{X}_\bullet\}$.

Besides (S1)–(S3), we suppose that there exist constants $C_{\text{sz}} > 0$ as well as $k_{\text{app}} \in \mathbb{N}_0$ such that for all $\mathcal{T} \in \mathcal{T}$, there exists a Scott-Zhang-type projector $J_\bullet : H^1_0(\Omega) \rightarrow \mathcal{X}_\bullet$ with the following properties (S4)–(S6):

**S4** Local projection property. With $k_{\text{proj}} \in \mathbb{N}_0$ from (S3), let $\pi_{\text{proj}}^\bullet := \pi_{\text{proj}}^{k_{\text{proj}}}$. For all $v \in H^1_0(\Omega)$ and $T \in \mathcal{T}_\bullet$, it holds that $(J_\bullet v)|_T = v|_T$, if $v|_{\mathcal{T}_{\text{proj}}(T)} \in \{V_\bullet|_{\mathcal{T}_{\text{proj}}(T)} : V_\bullet \in \mathcal{X}_\bullet\}$.

**S5** Local $L^2$-approximation property. Let $\pi_{\text{app}}^\bullet := \pi_{\text{app}}^{k_{\text{app}}}$. For all $T \in \mathcal{T}$ and all $v \in H^1_0(\Omega)$, it holds that $\|(1 - J_\bullet)v\|_{L^2(T)} \leq C_{\text{sz}} \|v\|_{H^1(\pi_{\text{app}}^\bullet(T))}$.

**S6** Local $H^1$-stability. Let $\pi_{\text{grad}}^\bullet := \pi_{\text{grad}}^{k_{\text{grad}}}$. For all $T \in \mathcal{T}_\bullet$ and $v \in H^1_0(\Omega)$, it holds that $\|\nabla J_\bullet v\|_{L^2(T)} \leq C_{\text{sz}} \|v\|_{H^1(\pi_{\text{grad}}^\bullet(T))}$.

### 2.4. Error estimator

Let $\mathcal{T} \in \mathcal{T}$ and $T_1 \in \mathcal{T}_1$. For almost every $x \in \partial T_1 \cap \Omega$, there exists a unique element $T_2 \in \mathcal{T}_1$ with $x \in T_1 \cap T_2$. We denote the corresponding outer normal vectors by $\nu_1$ resp. $\nu_2$ and define the normal jump as

$$[A \nabla U_\bullet \cdot \nu](x) = A \nabla U_\bullet|_{T_1}(x) \cdot \nu_1(x) + A \nabla U_\bullet|_{T_2}(x) \cdot \nu_2(x). \tag{2.8}$$

With this definition, we employ the residual a posteriori error estimator

$$\eta_\bullet := \eta_\bullet(\mathcal{T}_\circ) \quad \text{with} \quad \eta_\bullet(S) := \sum_{T \in S} \eta_\bullet(T)^2 \text{ for all } S \subseteq \mathcal{T}_\circ, \tag{2.9a}$$

where, for all $T \in \mathcal{T}_\bullet$, the local refinement indicators read

$$\eta_\bullet(T)^2 := |T|^{2/d} f + \text{div } A \nabla U_\bullet - b \cdot \nabla U_\bullet - c U_\bullet |_{L^2(T)} + |T|^{1/d} \|A \nabla U_\bullet \cdot \nu\|_{L^2(\partial T \cap \Omega)}. \tag{2.9b}$$

We refer, e.g., to the monographs [AO00, Ver13] for the analysis of the residual a posteriori error estimator (2.9) in the frame of standard FEM with piecewise polynomials of fixed order.

**Remark 2.1.** If $\mathcal{X}_\bullet \subset C^1(\overline{\Omega})$, then the jump contributions in (2.9) vanish and $\eta_\bullet(T)$ consists only of the volume residual; see [BG16b] in the frame of IGAFEM. \[\square\]

### 2.5. Data oscillations

The definition of the data oscillations corresponding to the residual error estimator (2.9) requires some further notation. Let $\mathcal{P}(\Omega) \subset H^1(\Omega)$ be a fixed discrete subspace. We suppose that there exists $C_{\text{inv}}'$ such that the following property (O1) holds for all $\mathcal{T}_\bullet \in \mathcal{T}$:

**O1** Inverse estimate in dual norm. For all $W \in \mathcal{P}(\Omega)$, it holds that $|T|^{-1/d} \|W\|_{H^{-1}(T)} \leq C_{\text{inv}}' \|W\|_{L^2(T)}$, where $\|W\|_{H^{-1}(T)} := \sup \{ \int_T W v \, dx : w \in H^1_0(T) \wedge \|\nabla v\|_{L^2(T)} = 1 \}$. 


Besides (O1), we suppose that there exists $C_{\text{lift}} > 0$ such that for all $T \in \mathcal{T}$ and all $T, T' \in \mathcal{T}$ with $(d - 1)$-dimensional intersection $E := T \cap T'$, there exists a lifting operator $L_{*,E} : \{W|_E : W \in \mathcal{P}(\Omega)\} \to H^1_0(T \cup T')$ with the following properties (O2)–(O4):

(02) Dual inequality. For all $W \in \mathcal{P}(\Omega)$, it holds that $\int_{E} W^2 \, dx \leq C_{\text{lift}} \int_{E} L_{*,E}(W|_E)W \, dx$.

(03) $L^2$-control. For all $W \in \mathcal{P}(\Omega)$, it holds that $\|L_{*,E}(W|_E)\|_{L^2(T \cup T')}^2 \leq C_{\text{lift}} |T \cup T'|^{1/d}\|W\|_{L^2(E)}^2$.

(04) $H^1$-control. For all $W \in \mathcal{P}(\Omega)$, it holds that $\|\nabla L_{*,E}(W|_E)\|_{L^2(T \cup T')} \leq C_{\text{lift}} |T \cup T'|^{-1/d}\|W\|_{L^2(E)}$.

Let $T_*, \mathcal{T} \in \mathcal{T}$. For $T \in \mathcal{T}$, we define the $L^2$-orthogonal projection $P_{*,T} : L^2(T) \to \{W|_T : W \in \mathcal{P}(\Omega)\}$. For an interior edge $E \in \mathcal{E}_{*,T} := \{T \cap T' : T' \in \mathcal{T} \land \dim(T \cap T') = d - 1\}$, we define the $L^2$-orthogonal projection $P_{*,E} : L^2(E) \to \{W|_E : W \in \mathcal{P}(\Omega)\}$. Note that $\bigcup \mathcal{E}_{*,T} = \partial T \cap \Omega$. For $V_* \in \mathcal{X}_*$, we define the corresponding oscillations

$$\text{osc}_*(V_*) := \text{osc}_*(V_*, \mathcal{T}_*) \quad \text{with} \quad \text{osc}_*(V_*, \mathcal{S}_*)^2 := \sum_{T \in \mathcal{S}_*} \text{osc}_*(V_*, T)^2 \quad \text{for all } \mathcal{S}_* \subseteq \mathcal{T}_*.$$  

(2.10a)

where, for all $T \in \mathcal{T}$, the local oscillations read

$$\text{osc}_*(V_*, T)^2 := |T|^{2/d}\| (1 - P_{*,T})(f + \text{div} A\nabla V_* - b \cdot \nabla V_* - cV_*)\|_{L^2(T)}^2 + \sum_{E \in \mathcal{E}_{*,T}} |T|^{1/d}\| (1 - P_{*,E})[A\nabla V_* \cdot \nu]\|_{L^2(E)}^2.\quad (2.10b)$$

We refer, e.g., to [NV12] for the analysis of oscillations in the frame of standard FEM with piecewise polynomials of fixed order.

**Remark 2.2.** If $\mathcal{X}_* \subset C^1(\Omega)$, then the jump contributions in (2.10) vanish and $\text{osc}_*(V_*, T)$ consists only of the volume oscillations; see [BG16b] in the frame of IGAFEM.

\[\square\]

### 2.6. Adaptive algorithm

We consider the common formulation of an adaptive mesh-refining algorithm; see, e.g., [CFPP14, Algorithm 2.2].

**Algorithm 2.3. Input:** Adaptivity parameter $0 < \theta \leq 1$ and marking constant $C_{\text{min}} \geq 1$.

**Loop:** For each $\ell = 0, 1, 2, \ldots$, iterate the following steps (i)–(iv):

(i) Compute Galerkin approximation $U_\ell \in \mathcal{X}_\ell$.

(ii) Compute refinement indicators $\eta_\ell(T)$ for all elements $T \in \mathcal{T}_\ell$.

(iii) Determine a set of marked elements $\mathcal{M}_\ell \subseteq \mathcal{T}_\ell$ which has up to the multiplicative constant $C_{\text{min}}$ minimal cardinality such that $\theta \eta_\ell^2 \leq \eta_\ell(\mathcal{M}_\ell)^2$.

(iv) Generate refined mesh $\mathcal{T}_{\ell+1} := \text{refine}(\mathcal{T}_\ell, \mathcal{M}_\ell)$.

**Output:** Refined meshes $\mathcal{T}_\ell$ and corresponding Galerkin approximations $U_\ell$ with error estimators $\eta_\ell$ for all $\ell \in \mathbb{N}_0$.

### 2.7. Main theorem on rate optimal convergence

We define

$$\mathcal{T}(N) := \{\mathcal{T}_* \in \mathcal{T} : \#\mathcal{T}_* - \#\mathcal{T}_0 \leq N\} \quad \text{for all } N \in \mathbb{N}_0 \quad (2.11)$$

and for all $s > 0$

$$\|u\|_{A_*} := \sup_{N \in \mathbb{N}_0} \min_{\mathcal{T}_* \in \mathcal{T}(N)} (N + 1)^s \eta_\ell \in [0, \infty], \quad (2.12)$$
and
\[ \|u\|_{B_s} := \sup_{N \in \mathbb{N}_0} \left( \min_{\mathcal{T}_\nu \in \mathcal{T}(N)} (N+1)^s \inf_{V \in \mathcal{A}_s} \left( \|u - V\|_{H^1(\Omega)} + \text{osc}_{\bullet}(V) \right) \right) \in [0, \infty]. \]  

By definition, \( \|u\|_{A_s} < \infty \) (resp. \( \|u\|_{B_s} < \infty \)) implies that the error estimator \( \eta_{\bullet} \) (resp. the total error) on the optimal meshes \( \mathcal{T} \) decays at least with rate \( O\left((\#\mathcal{T})^{-s}\right) \). The following main theorem states that each possible rate \( s > 0 \) is in fact realized by Algorithm 2.3. The (sketch of the) proof is given in Section 4. It is split into eight steps and builds upon the analysis of [CFPP14].

**Theorem 2.4.**  
(i) Suppose (M2)–(M3) and (S5)–(S6). Then, the residual error estimator (2.9) satisfies reliability, i.e., there exists a constant \( C_{\text{rel}} > 0 \) such that
\[ \|u - U_{\bullet}\|_{H^1(\Omega)} + \text{osc}_{\bullet}(U_{\bullet}) \leq C_{\text{rel}} \eta_{\bullet} \quad \text{for all } \mathcal{T} \in \mathcal{T}. \]  
(ii) Suppose (M1)–(M3), (S1), and (O1)–(O4). Then, the residual error estimator satisfies efficiency, i.e., there exists a constant \( C_{\text{eff}} > 0 \) such that
\[ C_{\text{eff}}^{-1} \eta_{\bullet} \leq \inf_{V \in \mathcal{A}_s} \left( \|u - V\|_{H^1(\Omega)} + \text{osc}_{\bullet}(V) \right). \]  
(iii) Suppose (M1)–(M3), (R2)–(R3), and (S1)–(S6). Then, for arbitrary \( 0 < \theta \leq 1 \) and \( C_{\text{min}} \geq 1 \), there exist constants \( C_{\text{lin}} > 0 \) and \( 0 < q_{\text{lin}} < 1 \) such that the estimator sequence of Algorithm 2.3 guarantees linear convergence in the sense of
\[ \eta_{\ell+1}^2 \leq C_{\text{lin}} q_{\text{lin}}^\ell \eta_0^2 \quad \text{for all } j, \ell \in \mathbb{N}_0. \]  
(iv) Suppose (M1)–(M3), (R1)–(R5), and (S1)–(S6). Then, there exists a constant \( 0 < \theta_{\text{opt}} \leq 1 \) such that for all \( 0 < \theta < \theta_{\text{opt}} \), all \( C_{\text{min}} \geq 1 \), and all \( s > 0 \), there exist constants \( c_{\text{opt}}, C_{\text{opt}} > 0 \) such that
\[ c_{\text{opt}} \|u\|_{A_s} \leq \sup_{\ell \in \mathbb{N}_0} (\#T_\ell - \#T_0 + 1)^s \eta_\ell \leq C_{\text{opt}} \|u\|_{A_s}, \]  
i.e., the estimator sequence will decay with each possible rate \( s > 0 \).

All involved constants \( C_{\text{rel}}, C_{\text{eff}}, C_{\text{lin}}, q_{\text{lin}}, \theta_{\text{opt}}, c_{\text{opt}}, \) and \( C_{\text{opt}} \) depend only on the assumptions made as well as the coefficients of the differential operator \( \mathcal{L} \) and \( \text{diam}(\Omega) \), where \( C_{\text{lin}}, q_{\text{lin}} \) depend additionally on \( \theta \) and the sequence \((U_\ell)_{\ell \in \mathbb{N}_0} \), and \( c_{\text{opt}}, C_{\text{opt}} \) depend furthermore on \( C_{\text{min}} \), and \( s > 0 \).

**Remark 2.5.** If the assumptions of Theorem 2.4 (i)–(ii) are satisfied, there holds in particular
\[ C_{\text{eff}}^{-1} \|u\|_{A_s} \leq \|u\|_{B_s} \leq C_{\text{rel}} \|u\|_{A_s} \quad \text{for all } s > 0. \]  

**Remark 2.6.** Note that almost minimal cardinality of \( \mathcal{M}_\ell \) in Step (iii) of Algorithm 2.3 is only required to prove optimal convergence behavior (2.17), while linear convergence (2.16) formally allows \( C_{\text{min}} = \infty \), i.e., it suffices that \( \mathcal{M}_\ell \) satisfies the Dörfler marking criterion in Step (iii). We refer to [CFPP14, Section 4.3–4.4] for details.

**Remark 2.7.** (a) If the bilinear form \( \langle \cdot, \cdot \rangle_{\mathcal{L}} \) is symmetric, \( C_{\text{lin}}, q_{\text{lin}} \) as well as \( c_{\text{opt}}, C_{\text{opt}} \) are then independent of \((U_\ell)_{\ell \in \mathbb{N}_0} \); see Remark 4.1 below.

(b) If the bilinear form \( \langle \cdot, \cdot \rangle_{\mathcal{L}} \) is non-symmetric, there exists an index \( \ell_0 \in \mathbb{N}_0 \) such that the constants \( C_{\text{lin}}, q_{\text{lin}} \) as well as \( c_{\text{opt}}, C_{\text{opt}} \) are independent of \((U_\ell)_{\ell \in \mathbb{N}_0} \), if (2.16)–(2.17) are formulated only for \( \ell \geq \ell_0 \). We refer to the recent work [BHP17, Theorem 19].
Remark 2.8. If $\mathcal{X} \subset C^1(\Omega)$, all jump contributions vanish; see Remark 2.1 and Remark 2.2. In this case, the assumptions (O2)–(O4) are not necessary for the proof of (2.15).

Remark 2.9. (a) Let $h_\ell := \max_{T \in T_\ell} |T|^{1/d}$ be the maximal mesh-width. Then, $h_\ell \to 0$ as $\ell \to \infty$, ensures that $X_\infty := \bigcup_{\ell \in \mathbb{N}_0} X_\ell = H_0^1(\Omega)$. To see this, recall that (S2) ensures that $\bigcup_{\ell \in \mathbb{N}_0} X_\ell$ is a vector space and, in particular, convex. By Mazur’s lemma (see, e.g., [Rud91, Theorem 3.12]), it is thus sufficient to show that $\bigcup_{\ell \in \mathbb{N}_0} X_\ell$ is weakly dense in $H_0^1(\Omega)$. Let $v \in H_0^1(\Omega)$. The Banach-Alaoglu theorem (see, e.g., [Rud91, Theorem 3.15]) together with (M2) and (S5)–(S6) proves that each subsequence $(J_{\ell_m} v)_{m \in \mathbb{N}_0}$ admits a further subsequence $(J_{\ell_m} v)_{m \in \mathbb{N}_0}$ which is weakly convergent in $H_0^1(\Omega)$ towards some limit $w \in H_0^1(\Omega)$. The Rellich compactness theorem hence implies $\|w - J_{\ell_m} v\|_{L^2(\Omega)} \to 0$ as $n \to \infty$. On the other hand, (S5) together with (M2), (R1)–(R3), and $h_\ell \to 0$ shows that $\|v - J_{\ell_m} v\|_{H^1(\Omega)} \leq h_\ell \|v\|_{H^1(\Omega)} \to 0$ as $\ell \to \infty$. Together with the uniqueness of limits, these two observations conclude $v = w$. Overall, each subsequence $(J_{\ell_m} v)_{m \in \mathbb{N}_0}$ of $(J_{\ell} v)_{\ell \in \mathbb{N}}$ admits a further subsequence $(J_{\ell_m} v)_{m \in \mathbb{N}_0}$ which converges weakly in $H_0^1(\Omega)$ to $v$. Basic calculus thus yields that $J_{\ell} v \rightharpoonup v$ weakly in $H_0^1(\Omega)$ as $\ell \to \infty$. This concludes the proof.

(b) We note that the latter observation allows to follow the ideas of [BHP17] and to show that the adaptive algorithm yields convergence even if the bilinear form $\langle \cdot, \cdot \rangle_{\mathcal{L}}$ is only elliptic up to some compact perturbation, provided that the continuous problem is well-posed. This includes, e.g., adaptive FEM for the Helmholtz equation. For details, the reader is referred to [BHP17].

3. Hierarchical setting

In this section, we recall the definition of hierarchical (B-)splines from [VGJS11] and propose a local mesh-refinement strategy. The main result of this section is Theorem 3.8 which states that hierarchical splines together with the proposed mesh-refinement strategy fit into the abstract setting of Section 2 and are hence covered by Theorem 2.4. The proof of Theorem 3.8 is given in Section 5.

3.1. Nested tensor meshes and splines. We define the parameter domain $\tilde{\Omega} := (0,1)^d$. Let $p_1, \ldots, p_d \geq 1$ be fixed polynomial degrees with $p := \max_{i=1,\ldots,d} p_i$. Let $\vec{\mathcal{K}}^0$ be an arbitrary fixed $d$-dimensional vector of $p_i$-open knot vectors with multiplicity smaller or equal to $p_i$ for the interior knots, i.e.,

$$\vec{\mathcal{K}}^0 = (\vec{\mathcal{K}}^0_1, \ldots, \vec{\mathcal{K}}^0_d),$$

where $\vec{\mathcal{K}}^0_i = (t_{i,0}^0, \ldots, t_{i,N_i^0+p_i}^0)$ is a non-decreasing vector in $[0,1]$ such that $t_{i,0}^0 = \cdots = t_{i,p_i}^0 = 0$, $t_{i,-N_i^0}^0 = \cdots = t_{i,-N_i^0+p_i}^0$, and $\#(t_{i,j}) := \# \{ k \in \{0, \ldots, N_i^0 + p_i \} : t_{i,k}^0 = t_{i,j}^0 \} \leq p_i$ for $j = p_i + 1, \ldots, N_i^0 - 1$. For $k \in \mathbb{N}_0$, we recursively define $\hat{\mathcal{K}}^{k+1}$ as the uniform $h$-refinement of $\hat{\mathcal{K}}^k$, i.e., it is obtained by inserting the knot $\frac{t_{i,j}^k + t_{i,j+1}^k}{2}$ of multiplicity one in each knot span $[t_{i,j}^k, t_{i,j+1}^k]$ with $t_{i,j}^k \neq t_{i,j+1}^k$. Let $\vec{\mathcal{B}}^k$ be the corresponding tensor-product B-spline basis, i.e.,

$$\vec{\mathcal{B}}^k = \{ \vec{\mathcal{B}}^k_{j_1,\ldots,j_d} : j_i \in \{1, \ldots, N_i^k \} \}. $$

(3.2)
where for \((s_1, \ldots, s_d) \in \mathbb{R}^d\)
\[
\hat{B}^k_{j_1, \ldots, j_d}(s_1, \ldots, s_d) := \prod_{i=1}^d \hat{B}(s_i | t^k_{i,j_i-1}, \ldots, t^k_{i,j_i+p_i}),
\]  
where \(\hat{B}(.|t^k_{i,j_i-1}, \ldots, t^k_{i,j_i+p_i})\) denotes the one-dimensional B-spline corresponding to the local knot vector \((t^k_{i,j_i-1}, \ldots, t^k_{i,j_i+p_i})\). It is well known that the function in \(\hat{B}^k\) have support \(\text{supp}(\hat{B}^k_{j_1, \ldots, j_d}) = [t^k_{1,j_1-1}, t^k_{1,j_1}] \times \cdots \times [t^k_{d,j_d-1}, t^k_{d,j_d}] \subseteq [0, 1]^d\), form a partition of unity, and are even locally linearly independent, i.e., for any open set \(O \subseteq [0, 1]^d\), the restricted B-splines \(\{\hat{\beta}|_O : \hat{\beta} \in \hat{B}^k \wedge \text{supp}(\hat{\beta}) \cap O \neq \emptyset\}\) are linearly independent. Let \(\hat{\mathcal{F}}^k := \text{span}(\hat{B}^k)\). This yields a nested sequence of tensor-product \(d\)-variate spline function spaces \((\hat{\mathcal{F}}^k)_{k \in \mathbb{N}_0}\) that are at least Lipschitz continuous
\[
\hat{\mathcal{F}}^k \subset \hat{\mathcal{F}}^{k+1} \subset W^{1, \infty}(\hat{\Omega}).
\]  
In particular, each \(\hat{\mathcal{F}}^k \subset \hat{\mathcal{F}}^{k+1}\) can be written as linear combination of functions in \(\hat{B}^{k+1}\), i.e., it has a unique representation of the form
\[
\hat{\beta}^k = \sum_{\hat{\beta}^{k+1} \in \hat{\mathcal{F}}^{k+1}} c_{\hat{\beta}^{k+1}} \hat{\beta}^{k+1}.
\]  
By the knot insertion procedure, one can show that these coefficients are nonnegative, i.e., \(c_{\hat{\beta}^{k+1}} \geq 0\), see, e.g., [BdVBSV14, Section 2.1.3]. With
\[
\hat{T}^k_{j_1, \ldots, j_d} := [t^k_{1,j_1-1}, t^k_{1,j_1}] \times \cdots \times [t^k_{d,j_d-1}, t^k_{d,j_d}], \quad \text{for all } j_i = 1, \ldots, N^k_i + p_i, \quad i = 1, \ldots, d.
\]
we define the corresponding set of all non-trivial closed cells of level \(k\) as
\[
\hat{T}^k := \{\hat{T}^k_{j_1, \ldots, j_d} : |\hat{T}^k_{j_1, \ldots, j_d}| > 0 \wedge j_i = 1, \ldots, N^k_i + p_i \wedge i = 1, \ldots, d\}.\]
Each function in \(\hat{\mathcal{F}}^k\) is a \(\hat{T}^k\) piecewise polynomial, where the smoothness across the boundary of an element \(\hat{T}\) depends only on the corresponding knot multiplicities. For a more detailed presentation of tensor-product splines, we refer to, e.g., [DB01, Sch07, BdVBSV14].

### 3.2. Hierarchical meshes and splines in the parameter domain \(\hat{\Omega}\)

Meshes \(\hat{\mathcal{T}}\) and corresponding spaces \(\hat{\mathcal{X}}\) are defined through their counterparts on the parameter domain \(\hat{\Omega} := (0, 1)^d\). Let \((\hat{\Omega}^k)_{k \in \mathbb{N}_0}\) be a nested sequence of closed subsets of \(\hat{\mathcal{X}} = [0, 1]^d\) such that
\[
\hat{\Omega}^k = \hat{\Omega}^0 \quad \text{and} \quad \hat{\Omega}^k \supseteq \hat{\Omega}^{k+1}.
\]
We suppose that for \(k > 0\) each \(\hat{\Omega}^k\) is the union of a selection of cells of level \(k-1\), i.e.,
\[
\hat{\Omega}^k = \bigcup \{\hat{T} \in \hat{T}^{k-1} : \hat{T} \subseteq \hat{\Omega}^k\}.
\]
Moreover, we assume the existence of some minimal \(M > 0\) such that \(\hat{\Omega}^{M} = \emptyset\). Then, we define the mesh in the parameter domain
\[
\hat{\mathcal{T}} := \bigcup \{\hat{T} \in \hat{T}^k : \hat{T} \subseteq \hat{\Omega}^k \wedge \hat{T} \not\subseteq \hat{\Omega}^{k+1}\}.
\]
Note that \(\hat{T}^k \cap \hat{T}^{k'} = \emptyset\) for \(k \neq k' \in \mathbb{N}_0\). For \(\hat{T} \in \hat{\mathcal{T}}\), there exists a unique level \(\hat{T} := k \in \mathbb{N}_0\) with \(\hat{T} \subseteq \hat{\Omega}^k\) and \(\hat{T} \not\subseteq \hat{\Omega}^{k+1}\). Note that \(\hat{\mathcal{T}}\) is a mesh of \(\hat{\Omega}\) in the sense of Section 2.1.
Figure 3.1. A two-dimensional hierarchical mesh $\hat{T}$ on the parameter domain is depicted, where $\hat{\Omega}_*^4 = \emptyset$. The corresponding domains $\hat{\Omega}_0^* \supseteq \hat{\Omega}_1^* \supseteq \hat{\Omega}_2^* \supseteq \hat{\Omega}_3^*$ are highlighted in black, red, blue and green.

With these preparations, one inductively defines the set of all hierarchical B-splines in the parameter domain $\hat{H}_* := \hat{H}_*^{M_*-1}$ as follows:

(i) Define $\hat{H}_0^* := \hat{B}_0^*$. 
(ii) For $k = 0, \ldots, M_* - 2$, define $\hat{H}_k^* + 1 := \text{old}(\hat{H}_k^*) \cup \text{new}(\hat{H}_k^*)$, where
\[
\text{old}(\hat{H}_k^*) := \{ \hat{\beta} \in \hat{H}_k^* : \text{supp}(\hat{\beta}) \not\subseteq \hat{\Omega}_k^* \},
\]
\[
\text{new}(\hat{H}_k^*) := \{ \hat{\beta} \in \hat{B}_k^* : \text{supp}(\hat{\beta}) \subseteq \hat{\Omega}_k^* \}.
\]

One can prove that the so-called hierarchical basis $\hat{H}_*$ is linearly independent; see [VGJS11, Lemma 2]. By definition, it holds that
\[
\hat{H}_* = \bigcup_{k \in \mathbb{N}_0} \{ \hat{\beta} \in \hat{B}_k^* : \text{supp}(\hat{\beta}) \subseteq \hat{\Omega}_k^* \wedge \text{supp}(\hat{\beta}) \not\subseteq \hat{\Omega}_k^* \}.
\]

Note that $\hat{B}_k^* \cap \hat{B}_{k'}^* = \emptyset$ for $k \neq k' \in \mathbb{N}_0$. For $\hat{\beta} \in \hat{H}_*$, there exists a unique level($\hat{\beta}$) := $k \in \mathbb{N}_0$ with supp($\hat{\beta}$) $\subseteq \hat{\Omega}_k^*$ and supp($\hat{\beta}$) $\not\subseteq \hat{\Omega}_{k+1}^*$.

The hierarchical basis $\hat{H}_*$ and the mesh $\hat{T}$ are compatible in the following sense: For all $\hat{\beta} \in \hat{H}_*$, the corresponding support can be written as union of elements in $\hat{T}_{\text{level}(\hat{\beta})}$, i.e.,
\[
\text{supp}(\hat{\beta}) = \bigcup_{\hat{T} \in \hat{T}_{\text{level}(\hat{\beta})}} \{ \hat{T} \subseteq \text{supp}(\hat{\beta}) \}.
\]

Each such element $\hat{T} \in \hat{T}_{\text{level}(\hat{\beta})}$ with $\hat{T} \subseteq \text{supp}(\hat{\beta}) \subseteq \hat{\Omega}_{\text{level}(\hat{\beta})}^*$ satisfies $\hat{T} \in \hat{T}_*$ or $\hat{T} \subseteq \hat{\Omega}_{\text{level}(\hat{\beta})+1}^*$. In either case, we see that $\hat{T}$ can be written as union of elements in $\hat{T}_*$ with level greater or equal to level($\hat{\beta}$). Altogether, we have
\[
\text{supp}(\hat{\beta}) = \bigcup_{k \geq \text{level}(\hat{\beta})} \{ \hat{T} \in \hat{T}_* \cap \hat{T}_k^* : \hat{T} \subseteq \text{supp}(\hat{\beta}) \}.
\]
Moreover, supp(\(\hat{\beta}\)) must contain at least one element of level level(\(\hat{\beta}\)), otherwise one would get the contradiction supp(\(\hat{\beta}\)) \(\subseteq \hat{\Omega}_{k-1}^*\). In particular, this shows that

\[
\text{level}(\hat{\beta}) = \min_{\hat{T} \in \hat{T}_*} \text{level}(\hat{T}) \quad \text{for all } \hat{\beta} \in \hat{\mathcal{H}}_*.
\]

(3.15)

Define the space of hierarchical splines in the parameter domain by \(\hat{\mathcal{Y}}_* := \text{span}(\hat{\mathcal{H}}_*)\). According to (3.14), each \(\hat{V}_* \in \hat{\mathcal{Y}}_*\) is a \(\hat{T}_*\)-piecewise tensor polynomial of degree \((p_1, \ldots, p_d)\). We define our ansatz space in the parameter domain as

\[\hat{\mathcal{X}}_* := \{\hat{V}_* \in \hat{\mathcal{Y}}_* : \hat{V}_*|_{\partial \hat{\Omega}} = 0\} \subset \hat{\mathcal{Y}}_* \subset \{\hat{v} \in W_0^{1,\infty}(\hat{\Omega}) : \hat{v}|_{\hat{T}} \in C^2(\hat{T}) \text{ for all } \hat{T} \in \hat{T}_*\}. \quad (3.16)\]

Note that this specifies the abstract setting of Section 2.3. For a more detailed introduction to hierarchical meshes and splines, we refer to, e.g., [VGJS11, BG15, SM16].

### 3.3. Basis of \(\hat{\mathcal{X}}_*\).

In this section, we characterize a basis of the hierarchical splines \(\hat{\mathcal{X}}_*\) that vanish on the boundary. To this end, we first determine the restriction of the hierarchical basis \(\hat{\mathcal{H}}_*\) to a facet of the boundary. It turns out that this restriction coincides with the set of \((d-1)\)-dimensional hierarchical B-splines.

**Proposition 3.1.** Let \(\hat{T}_*\) be an arbitrary hierarchical mesh on the parameter domain \(\hat{\Omega}\). For \(E = [0,1]^{I-1} \times \{e\} \times [0,1]^{d-I}\) with some \(I \in \{1, \ldots, d\}\) and some \(e \in \{0,1\}\), set \(\hat{K}_0^*|_E := (\hat{K}_1^0, \ldots, \hat{K}_{I-1}^0, \hat{K}_{I+1}^0, \ldots, \hat{K}_d^0)\), and \(\hat{\Omega}^k|_E := \{(s_1, \ldots, s_{I-1}, s_{I+1}, \ldots, s_d) : (s_1, \ldots, s_d) \in \hat{\Omega}^k \cap E\}\) for \(k \in \mathbb{N}_0\). Moreover, let \(\hat{T}_*|_E\) be the corresponding hierarchical mesh and \(\hat{\mathcal{H}}_*|_E\) the corresponding hierarchical basis. Then, there holds \(\hat{\mathcal{H}}_*|_E = \{\hat{\beta}|_E : \hat{\beta} \in \hat{\mathcal{H}}_* \wedge \hat{\beta}|_E \neq 0\}\). Moreover, the restriction \(\hat{\mathcal{H}}_* \rightarrow \hat{\mathcal{H}}_*|_E\) is essentially injective, i.e., for \(\hat{\beta}_1, \hat{\beta}_2 \in \hat{\mathcal{H}}_*\) with \(\hat{\beta}_1 \neq \hat{\beta}_2\) and \(\hat{\beta}_1|_E \neq 0\), it follows that \(\hat{\beta}_1|_E \neq \hat{\beta}_2|_E\).

**Proof.** We prove the assertion in two steps.

**Step 1:** Let \(k \in \mathbb{N}_0\). We recall that the knot vectors \(\hat{K}_i^k\) are \(p_i\)-open. In particular, this implies that the corresponding one-dimensional B-splines \(\hat{B}_i^k\) are interpolatory at the end points \(e \in \{0,1\}\). This means that the first resp. last \(B\)-spline in \(\hat{B}_i^k\) (i.e., \(B(\cdot|_{t_i^k, t_{i,N_i+1}^k})\)) resp. \(B(\cdot|_{t_i^k, t_{i,N_i-1}^k})\)) is equal to one at \(0\) resp. \(1\) and that all other B-splines of \(\hat{B}_i^k\) vanish at these points; see, e.g., [Sch16, Lemma 2.1].

**Step 2:** We consider arbitrary \(d > 1\). For \(k \in \mathbb{N}_0\), let \(\hat{B}_k^*|_E\) be the set of tensor product B-splines induced by the reduced knots \(\hat{K}_j^*|_E\) which are defined analogously to \(\hat{K}_0^*|_E\). Since \(\hat{K}_j^*\) is \(p_j\)-open, it holds that \(\hat{B}_k^*|_E = \{\hat{\beta}|_E : \hat{\beta} \in \hat{B}_k^* \wedge \hat{\beta}|_E \neq 0\}\); see also Step 1. Then, the identity (3.12) shows

\[
\hat{\mathcal{H}}_*|_E = \bigcup_{k \in \mathbb{N}_0} \left\{ \hat{\beta}|_E : \hat{\beta} \in \hat{B}_k^* \wedge \hat{\beta}|_E \neq 0 \wedge \text{supp}(\hat{\beta}|_E) \subseteq \hat{\Omega}_{k-1}^* \wedge \text{supp}(\hat{\beta}|_E) \not\subseteq \hat{\Omega}_{k+1}^* \right\}. \quad (3.17)
\]

Let \(\hat{\beta} \in \hat{B}_k^*\) for some \(k \in \mathbb{N}_0\) with \(\hat{\beta}|_E \neq 0\). We set \(J := 0\) for \(e = 0\) resp. \(J := N_I - 1\) for \(e = 1\). Since \(\hat{B}(e|_{t_j^k, t_{j+p_j}^k})\) does not vanish only if \(j = J\) (see Step 1), \(\hat{\beta}\) must be

\footnote{Actually, the set on left-hand side consists of functions defined on \([0,1]^{d-1}\), whereas the right-hand side functions are defined on \(E\). However, clearly these functions can be identified.}
Proof. Linear independence as well as 

\[
\text{equivalent to } \text{supp}(\hat{B}(-|t^k_{i,j}, \ldots, t^k_{i,j+p_{i+1}})) = [t^k_{i,j}, t^k_{i,j+p_{i+1}}].
\]

This shows that \( \text{supp}(\hat{\beta}) \) is the union of elements \( \hat{T} \in \hat{T}^k \) with non-empty intersection with \( E \). Hence \( \text{supp}(\hat{\beta}|_E) \subseteq \hat{\Omega}_k^k|_E \) is equivalent to \( \text{supp}(\hat{\beta}) \subseteq \hat{\Omega}_k^k \), and \( \text{supp}(\hat{\beta}|_E) \not\subseteq \hat{\Omega}_k^{k+1}|_E \) is equivalent to \( \text{supp}(\hat{\beta}) \not\subseteq \hat{\Omega}_k^{k+1} \). Therefore, (3.17) becomes

\[
\hat{\mathcal{H}}|_E = \bigcup_{k \in \mathbb{N}_0} \{ \beta|_E : \hat{\beta} \in \hat{\mathcal{B}}^k \land \beta|_E \neq 0 \land \text{supp}(\beta) \subseteq \Omega_k^k \land \text{supp}(\hat{\beta}) \not\subseteq \Omega_{k+1}^{k+1} \}.
\]

Together with (3.12), this shows \( \hat{\mathcal{H}}|_E = \{ \beta|_E : \beta \in \hat{\mathcal{H}} \land \beta|_E \neq 0 \} \). Finally, let \( \hat{\beta}_1, \hat{\beta}_2 \in \hat{\mathcal{H}} \) with \( \beta|_E \neq 0 \). If \( \hat{\beta}_1|_E = \hat{\beta}_2|_E \), then (3.18) already implies \( \hat{\beta}_1 = \hat{\beta}_2 \). This concludes the proof.

Corollary 3.2. Let \( \hat{T}_\bullet \) be an arbitrary hierarchical mesh on the parameter domain \( \hat{\Omega} \). Then, \( \{ \hat{\beta} \in \hat{\mathcal{H}}_\bullet : \beta|_{\partial \hat{\Omega}} = 0 \} \) is a basis of \( \hat{\mathcal{H}}_\bullet \).

Proof. Linear independence as well as \( \{ \hat{\beta} \in \hat{\mathcal{H}}_\bullet : \beta|_{\partial \hat{\Omega}} = 0 \} \subseteq \mathcal{X}_\bullet \) are obvious. To see \( \mathcal{X}_\bullet \subseteq \text{span}\{ \hat{\beta} \in \hat{\mathcal{H}}_\bullet : \beta|_{\partial \hat{\Omega}} = 0 \} \), let \( \hat{V}_\bullet \in \hat{\mathcal{X}} \). Consider the unique representation \( \hat{V}_\bullet = \sum_{\beta \in \hat{\mathcal{H}}_\bullet} c_\beta \hat{\beta} \) with \( c_\beta \in \mathbb{R} \). For arbitrary \( \hat{\beta} \in \hat{\mathcal{H}}_\bullet \) with \( \beta|_{\partial \hat{\Omega}} \neq 0 \), we have to prove \( c_\beta = 0 \), i.e., we have to show the implication

\[
\sum_{\beta \in \hat{\mathcal{H}}_\bullet} c_\beta \beta|_{\partial \hat{\Omega}} = 0 \quad \Rightarrow \quad (\forall \hat{\beta} \in \hat{\mathcal{H}}_\bullet \text{ with } \beta|_{\partial \hat{\Omega}} \neq 0 \quad c_\beta = 0).
\]

Let \( E = [0, 1]^I \times \{ e \} \times [0, 1]^{d-I} \) with \( I \subseteq \{ 1, \ldots, d \} \) and \( \sum_{\beta \in \hat{\mathcal{H}}_\bullet} c_\beta \beta|_E = 0 \). According to Proposition 3.1, the family \( \{ \beta|_E : \beta \in \hat{\mathcal{H}} \land \beta|_E \neq 0 \} \) is linearly independent. Hence, \( \hat{c}_\beta = 0 \) for \( \hat{\beta} \in \hat{\mathcal{H}}_\bullet \) with \( \beta|_E \neq 0 \). Since \( \partial \hat{\Omega} \) is the union of such facets \( E \), this concludes the proof.

3.4. Admissible meshes in the parameter domain \( \hat{\Omega} \). Let \( \hat{T}_\bullet \) be an arbitrary hierarchical mesh. We define the set of all neighbors of an element \( \hat{T} \in \hat{T}_\bullet \) as

\[
\mathcal{N}(\hat{T}) := \{ \hat{T}' \in \hat{T}_\bullet : \exists \hat{\beta} \in \hat{\mathcal{H}}_\bullet \land \hat{T}, \hat{T}' \subseteq \text{supp}(\hat{\beta}) \},
\]

According to (3.14), the condition \( \hat{T}, \hat{T}' \subseteq \text{supp}(\hat{\beta}) \) is equivalent to \( \hat{T} \cap \text{supp}(\hat{\beta}) \neq 0 \neq \hat{T}' \cap \text{supp}(\hat{\beta}) \). We call \( \hat{T}_\bullet \) admissible if

\[
|\text{level}(\hat{T}) - \text{level}(\hat{T}')| \leq 1 \quad \text{for all } \hat{T}, \hat{T}' \in \hat{T}_\bullet \text{ with } \hat{T}' \in \mathcal{N}(\hat{T}).
\]

Let \( \hat{\mathcal{T}}^k \) be the set of all admissible hierarchical meshes in the parameter domain. Clearly, \( \hat{\mathcal{T}}^k \in \hat{T}_\bullet \) for all \( k \in \mathbb{N}_0 \). Moreover, admissible meshes satisfy the following interesting properties which are also important for an efficient implementation of IGAFEM with hierarchical splines.
Proposition 3.3. Let \( \hat{T} \subset \hat{\Omega} \). Then, the support of any basis function \( \hat{\beta} \in \hat{\mathcal{H}}_\ast \) is the union of at most \( 2^d(p+1)^d \) elements \( \hat{T}' \subset \hat{T} \). Moreover, for any \( \hat{T} \subset \hat{T} \), there are at most \( 2(p+1)^d \) basis functions \( \hat{\beta}' \in \hat{\mathcal{H}}_\ast \) that have support on \( \hat{T} \), i.e., \( |\text{supp}(\hat{\beta}') \cap \hat{T}| > 0 \).

**Proof.** We abbreviate \( k := \text{level}(\hat{\beta}) \). By (3.15), there exists \( \hat{T}' \subset \text{supp}(\hat{\beta}) \) with level(\( \hat{T}' \)) = \( k \). Admissibility of \( \hat{T} \) together with (3.14) shows that level(\( \hat{T}' \)) \( \in \{k, k+1\} \) for all \( \hat{T}' \subset \hat{T} \) with \( \hat{T}' \subset \text{supp}(\hat{\beta}) \). Since \( \hat{\beta} \) is an element of \( \hat{\mathcal{B}}^k \), its support is the union of at most \( 2^d(p+1)^d \) elements in \( \hat{T}^{k+1} \). This proves the first assertion. For \( \hat{\beta}' \in \hat{\mathcal{H}}_\ast \) and \( \hat{T} \subset \hat{T} \), with \( |\text{supp}(\hat{\beta}') \cap \hat{T}| > 0 \), the characterization (3.14) proves \( \hat{T} \subset \text{supp}(\hat{\beta}') \). Hence, (3.15) together with admissibility of \( \hat{T} \) proves that level(\( \hat{\beta}' \)) = \( k \) := level(\( \hat{T} \)) or level(\( \hat{\beta}' \)) = \( k - 1 \). With \( \hat{\mathcal{B}}^{-1} := \hat{\mathcal{B}}^0 \), there are at most \( (p+1)^d \) basis functions in \( \hat{\mathcal{B}}^{k-1} \) and \( (p+1)^d \) basis functions in \( \hat{\mathcal{B}}^k \) that have support on the element \( \hat{T} \). This concludes the proof. \( \square \)

Remark 3.4. Since the support of any \( \hat{\beta} \in \hat{\mathcal{H}}_\ast \) is connected, Proposition 3.3 particularly shows that \( \hat{T}' \subset \text{supp}(\hat{\beta}) \) for an element \( \hat{T}' \subset \hat{T} \) implies that \( \text{supp}(\hat{\beta}) \subset \hat{\mathcal{P}}^2(p+1)(\hat{T}') \). Moreover, we recall that \( \hat{T}' \subset \text{supp}(\hat{\beta}) \) is equivalent to \( |\hat{T}' \cap \text{supp}(\hat{\beta})| > 0 \); see (3.14). \( \square \)

3.5. Refinement in the parameter domain \( \hat{\Omega} \). We define the initial mesh \( \hat{T}_0 := \hat{T}^0 \). Note that \( \hat{T}_0 \) is a hierarchical mesh with \( \hat{\Omega}_0^k = \emptyset \) for all \( k > 0 \). We say that a hierarchical mesh \( \hat{T}_0 \) is finer than another hierarchical mesh \( \hat{T}_0 \) if \( \hat{\Omega}_k^k \subset \hat{\Omega}_k^k \) for all \( k \in \mathbb{N} \). This just means that \( \hat{T}_0 \) is obtained from \( \hat{T}_0 \) by iterative dyadic bisections of the elements in \( \hat{T}_0 \). To bisect an element \( \hat{T} \subset \hat{T}_0 \), one just has to add it to the set \( \hat{\Omega}_\ast^{\text{level}(\hat{T})+1} \), see (3.24) below. In this case, the corresponding spaces are nested, i.e.,

\[
\hat{\mathcal{Y}}_0 \subset \hat{\mathcal{Y}}_0 \quad \text{and} \quad \hat{\mathcal{X}}_0 \subset \hat{\mathcal{X}}_0.
\]  
(3.21)

For a proof, see, e.g., [SM16, Corollary 2]. In particular, this implies

\[
\hat{\mathcal{Y}}_0 \subset \hat{\mathcal{Y}}_0 \subset \hat{\mathcal{Y}}^{M-1}.
\]
(3.22)

Next, we present a concrete refinement algorithm to specify the setting of Section 2.2. To this end, we first define for \( \hat{T} \subset \hat{T}_0 \subset \hat{T} \) the set of its bad neighbors

\[
\hat{\mathcal{N}}_\ast^{\text{bad}}(\hat{T}) := \{ \hat{T}' \subset \hat{T}_0(\hat{T}) : \text{level}(\hat{T}') = \text{level}(\hat{T}) - 1 \}.
\]
(3.23)

**Algorithm 3.5.** **Input:** Hierarchical mesh \( \hat{T}_0 \), marked elements \( \hat{\mathcal{M}}_\ast := \hat{\mathcal{M}}_0 \subset \hat{T}_0 \).

(i) Iterate the following steps (a)–(b) for \( i = 0, 1, 2, \ldots \) until \( \hat{\mathcal{U}}_i(\hat{T}) = \emptyset \):

(a) Define \( \hat{\mathcal{U}}_i(\hat{T}) := \bigcup_{\hat{T} \in \hat{\mathcal{M}}_i(\hat{T})} \{ \hat{T}' \subset \hat{T} \setminus \hat{\mathcal{M}}_i(\hat{T}) : \text{level}(\hat{T}') = \text{level}(\hat{T}) - 1 \} \).

(b) Define \( \hat{\mathcal{M}}_{i+1} := \hat{\mathcal{M}}_i \cup \hat{\mathcal{U}}_i \).

(ii) Dyadically bisect all \( \hat{T} \subset \hat{\mathcal{M}}_i(\hat{T}) \) by adding \( \hat{T} \) to the set \( \hat{\Omega}_\ast^{\text{level}(\hat{T})+1} \) and obtain a finer hierarchical mesh \( \hat{T}_0 = \text{refine}(\hat{T}_0, \hat{\mathcal{M}}_\ast) \), where

\[
\hat{\Omega}_k^k = \hat{\Omega}_k^k \cup \{ \hat{T} \subset \hat{\mathcal{M}}_i(\hat{T}) : \text{level}(\hat{T}) = k - 1 \} \quad \text{for all } k \in \mathbb{N}.
\]
(3.24)

**Output:** Refined mesh \( \hat{T}_0 := \text{refine}(\hat{T}_0, \hat{\mathcal{M}}_\ast) \).
Algorithm [BG16b]

Algorithm 3.5

Figure 3.2. An initial mesh $\hat{T}_0$ with only one element $[0, 1]^2$ is locally refined in the lower left corner using the refinement of [BG16b] (left) resp. the refinement of Algorithm 3.5 (right); see Remark 3.6. Consider the lowest-order case $(p_1, p_2) = (1, 1)$. By repetitive refinement via [BG16b], the number of elements in the support of the hierarchical B-spline $\hat{B}(s_1|0, 1/2, 1)\hat{B}(s_2|0, 1/2, 1)$ grows to infinity. Moreover, the number of hierarchical B-splines with support on the element in the lower left corner grows to infinity. This is not the case if one uses Algorithm 3.5, see also Proposition 3.3.

Clearly, $\text{refine}(\hat{T}_0, \hat{M}_0)$ is finer than $\hat{T}_0$. For any hierarchical mesh $\hat{T}_0$, we define $\text{refine}(\hat{T}_0)$ as the set of all hierarchical meshes $\hat{T}_0$ such that there exist hierarchical meshes $\hat{T}_{(0)}, \ldots, \hat{T}_{(J)}$ and marked elements $\hat{M}_{(0)}, \ldots, \hat{M}_{(J-1)}$ with $\hat{T}_0 = \hat{T}_{(J)} = \text{refine}(\hat{T}_{(J-1)}, \hat{M}_{(J-1)}), \ldots, \hat{T}_{(1)} = \text{refine}(\hat{T}_{(0)}, \hat{M}_{(0)}))$, and $\hat{T}_{(0)} = \hat{T}_0$. Here, we formally allow $J = 0$, i.e., $\hat{T}_0 \in \text{refine}(\hat{T}_0)$. Proposition 5.2 below will show that $\hat{T}_0 = \text{refine}(\hat{T}_0)$, i.e., starting from $\hat{T}_0 = \hat{T}_0$, all admissible meshes $\hat{T}_0$ can be generated by iterative refinement via Algorithm 3.5.

Remark 3.6. [BG16b, BGMP16] studied a related refinement strategy, where $N_\bullet(\hat{T})$ of (3.19) and $N_\bullet^{\text{had}}(\hat{T})$ from (3.23) are replaced by

$$\tilde{N}_\bullet(\hat{T}) := \{ \hat{T}' \in \hat{T}_\bullet : \exists \hat{\beta} \in \hat{B}^{\text{level}(\hat{T})} \text{ with } |\hat{T} \cap \text{supp}(\hat{\beta})| \neq 0 \neq |\hat{T}' \cap \text{supp}(\hat{\beta})| \},$$

$$\tilde{N}_\bullet^{\text{had}}(\hat{T}) := \{ \hat{T}' \in \tilde{N}_\bullet : \text{level}(\hat{T}') = \text{level}(\hat{T}) - 1 \}. \quad (3.25)$$

There, the refinement strategy was designed for truncated hierarchical B-splines; see Section 5.9. Compared to the hierarchical B-splines $\hat{H}_\bullet$, those have generically a smaller, but also more complicated and not necessarily connected support. [BG16b, Corollary 17] shows that the generated meshes are strictly admissible in the sense of [BG16b, BGMP16], i.e., for all $k \in \mathbb{N}$, it holds that

$$\hat{\Omega}_k^\bullet \subseteq \bigcup \{ \hat{T} \in \hat{T}_k^k : \forall \hat{\beta} \in \hat{B}^{k-1} \quad (\hat{T} \subseteq \text{supp}(\hat{\beta}) \Rightarrow \text{supp}(\hat{\beta}) \subseteq \hat{\Omega}_k^k) \}. \quad (3.26)$$

This definition actually goes back to [GJS14, Appendix A]. According to [BG16b, Section 2.4], strictly admissible meshes satisfy a similar version of Proposition 3.3 for truncated hierarchical B-splines. However, the example from Figure 3.2 shows that the proposition fails for hierarchical B-splines and the refinement strategy from [BG16b]. In particular, strictly admissible meshes are not necessarily admissible in the sense of Section 3.4. □
3.6. Hierarchical meshes and splines in the physical domain $\Omega$. To transform the definitions in the parameter domain to the physical domain, we assume that we are given 
\[ \gamma : \tilde{\Omega} \rightarrow \Omega \text{ with } \gamma \in W^{1,\infty}(\tilde{\Omega}) \cap C^2(\tilde{T}_0) \text{ and } \gamma^{-1} \in W^{1,\infty}(\Omega) \cap C^2(T_0), \] (3.27)
where $C^2(T_0) := \{v : \tilde{T} \rightarrow \mathbb{R} : v|_T \in C^2(T) \text{ for all } T \in T_0\}$. Consequently, there exists $C_\gamma > 0$ such that for all $i, j, k \in \{1, \ldots, d\}$
\[ \left\| \frac{\partial}{\partial t} \gamma_i \right\|_{L^\infty(\tilde{\Omega})} \leq C_\gamma, \quad \left\| \frac{\partial}{\partial x_j} \gamma_i \right\|_{L^\infty(\tilde{\Omega})} \leq C_\gamma, \quad \left\| \frac{\partial^2}{\partial x_j \partial x_k} \gamma_i \right\|_{L^\infty(\tilde{\Omega})} \leq C_\gamma, \] (3.28)
where $\gamma_i$ resp. $(\gamma^{-1})_i$ denotes the $i$-th component of $\gamma$ resp. $\gamma^{-1}$. All previous definitions can now also be made in the physical domain, just by pulling them from the parameter domain via the diffeomorphism $\gamma$. For these definitions, we drop the symbol $\tilde{}$. If $\tilde{T}_* \in \hat{T}$, we define the corresponding mesh in the physical domain as $T_* := \{\gamma(\hat{T}) : \hat{T} \in \hat{T}_*\}$. In particular, we have $T_0 = \{\gamma(\hat{T}) : \hat{T} \in \hat{T}_0\}$. Moreover, let $T_* := \{T_* : \hat{T}_* \in \hat{T}_*\}$ be the set of admissible meshes in the physical domain. If now $M_* \subseteq T_*$ with $T_* \in T$, we abbreviate $\hat{M}_* := \{\gamma^{-1}(T) : T \in M_*\}$ and define $\text{refine}(T_*, M_*):=\{\gamma(\hat{T}) : \hat{T} \in \text{refine}(\hat{T}_*, \hat{M}_*)\}$.

For $T_* \in T$, let $X_* := \{\hat{V}_* \circ \gamma^{-1} : \hat{V}_* \in \hat{X}_*\}$ be the corresponding hierarchical spline space. By regularity of $\gamma$, we especially obtain
\[ X_* \subset \{v \in H^1_0(\Omega) : v|_T \in H^2(T) \text{ for all } T \in T_*\}. \] (3.29)

3.7. Main result. Before we come to the main result of this work, we fix polynomial orders $(q_1, \ldots, q_d)$ and define for $T_* \in T$ the space of transformed polynomials
\[ \mathcal{P}(\Omega) := \{\hat{V} \circ \gamma : \hat{V} \text{ is a tensor polynomial of order } (q_1, \ldots, q_d)\} \] (3.30)

Remark 3.7. In order to obtain higher-order oscillations, the natural choice of the polynomial orders is $q_i \geq 2p_i - 1$; see, e.g., [NV12, Section 3.1]. If $X_* \subset C^1(\Omega)$, it is sufficient to choose $q_i \geq 2p_i - 2$; see Remark 2.2. \hfill \square

Altogether, we have specified the abstract framework of Section 2 to hierarchical meshes and splines. The following theorem is the main result of the present work. It shows that all assumptions of Theorem 2.4 are satisfied for the present IGAFEM approach. The proof is given in Section 5.

Theorem 3.8. Hierarchical splines on admissible meshes satisfy the abstract assumptions (M1)–(M3), (R1)–(R5), and (S1)–(S6) from Section 2, where the constants depend only on $d$, $C_\gamma$, $\tilde{T}_0$, and $(p_1, \ldots, p_d)$. Moreover, the piecewise polynomials $\mathcal{P}(\Omega)$ from (3.30) on admissible meshes satisfy the abstract assumptions (O1)–(O4), where the constants depend only on $d$, $C_\gamma$, $\tilde{T}_0$, and $(q_1, \ldots, q_d)$. By Theorem 2.4, this implies reliability (2.14) as well as efficiency (2.15) of the error estimator, linear convergence (2.16), and quasi-optimal convergence rates (2.17) for the adaptive strategy from Algorithm 2.3.

3.8. Generalization to rational hierarchical splines. One can easily verify that all theoretical results of this work are still valid if one replaces the ansatz space $X_*$ by rational
hierarchical splines, i.e., by the set

\[ X^W_0 := \left\{ V^\bullet_0 : V^\bullet \in X_0 \right\}, \quad (3.31) \]

where \( W_0 \) is a fixed positive weight function in the initial ansatz space \( X_0 \). In this case, the corresponding basis consists of NURBS instead of B-splines. Indeed, the mesh properties \((M1)-(M3)\) as well as the refinement properties \((R1)-(R5)\) from Section 2 are independent of the discrete spaces. To verify the validity of Theorem 3.8 in the NURBS setting, it thus only remains to verify the properties \((S1)-(S6)\) for the NURBS finite element spaces. The inverse estimate \((S1)\) follows similarly as in Section 5.6 since we only consider a fixed and thus uniformly bounded weight function \( W_0 \in X_0 \). The properties \((S2)-(S3)\) depend only on the numerator of the NURBS functions and thus transfer. To see \((S4)-(S6)\), one can proceed as in Section 5.10, where the corresponding Scott-Zhang type operator \( J_{V^\bullet_0} : L^2(\Omega) \rightarrow X^W_0 \) now reads

\[ J_{V^\bullet_0} v := \frac{J_{V^\bullet}(vW^\bullet_0)}{W^\bullet_0} \quad \text{for all } v \in L^2(\Omega). \quad (3.32) \]

With this definition, Lemma 5.9 holds accordingly, and \((S4)-(S6)\) are proved as in Section 5.10.

4. Sketch of proof of Theorem 2.4

In the following seven subsections we sketch the proof of Theorem 2.4, where we build upon the analysis of [CFPP14]. Recall the residual a posteriori error estimator \( \eta_\bullet \) from Section 2.4.

4.1. Discrete reliability. Under the assumptions \((M2)-(M3)\), and \((S2)-(S6)\), we show that there exists \( C_{drel}, C_{ref} \geq 1 \) such that for all \( T^\bullet \in T \) and all \( T^\circ \in \text{refine}(T^\bullet) \), the subset \( R^\bullet:={\Pi^\bullet}_{\text{loc}}(T^\bullet \setminus T^\circ) \subseteq T_\bullet \) satisfies

\[ \left\| U^\circ - U^\bullet \right\|_{H^1(\Omega)} \leq C_{drel} \eta_\bullet (R^\bullet), \quad T^\bullet \setminus T^\circ \subseteq R^\bullet, \quad \text{and} \quad \#R^\bullet \leq C_{ref} \#(T^\bullet \setminus T^\circ). \]

The last two properties are obvious with \( C_{ref} = C_{\text{patch}} \) by validity of \((M2)\) and \((S3)\). For the first property, we argue as in [Ste07, Theorem 4.1]: Ellipticity \((1.4)\), \( e^\circ := U^\circ - U^\bullet \in X^\circ \) (which follows from \((S2)\)), and Galerkin orthogonality \((2.6)\) with \( V^\bullet := J_{V^\bullet} e^\circ \in X_\bullet \) prove

\[ \left\| U^\circ - U^\bullet \right\|_{H^1(\Omega)} \lesssim \langle e^\circ , e^\circ \rangle_L = \langle e^\circ , (1 - J^\bullet) e^\circ \rangle_L = \int_\Omega f(1 - J^\bullet) e^\circ \, dx - \langle U^\bullet , (1 - J^\bullet) e^\circ \rangle_L. \]

We split \( \Omega \) into elements \( T \in T_\bullet \) and apply elementwise integration by parts, where we denote the outer normal vector by \( \nu \). With \( U^\bullet |_{T} \in H^2(T) \), this leads to

\[ = \sum_{T \in T_\bullet} \left( \int_T f(1 - J^\bullet) e^\circ \, dx - \int_T (- \text{div}(A \nabla U^\bullet) + b \nabla U^\bullet + c U^\bullet)(1 - J^\bullet) e^\circ \, dx \right. \]

\[ \left. + \int_{\partial T} (A \nabla U^\bullet \cdot \nu)(1 - J^\bullet) e^\circ \, ds \right). \quad (4.1) \]

The properties \((S3)-(S4)\) immediately prove for any \( V^\circ \in X^\circ \)

\[ J^\bullet V^\circ = V^\circ \quad \text{on } \Omega \setminus \pi_{\text{loc}}^\bullet(\mathcal{T}_\bullet \setminus T^\circ) = \Omega \setminus \mathcal{R}^\bullet = \bigcup(\mathcal{T}_\bullet \setminus \mathcal{R}^\bullet). \]
Hence, the sum in (4.1) reduces from $T \in \mathcal{T}_*$ to $T \in \mathcal{R}_*$. Recall that $(1 - J_*)e_0 \in \mathcal{X}_0 \subset H^1_0(\Omega)$ with $(1 - J_*)e_0 = 0$ on $\partial(\bigcup \mathcal{R}_*)$. We define the set of facets $\mathcal{E}_* := \{ T_1 \cap T_2 : T_1, T_2 \in \mathcal{R}_*, T_1 \neq T_2 \wedge | T_1 \cap T_2 | > 0 \}$, where $| \cdot |$ denotes the $(d - 1)$-dimensional measure. Almost all $x \in \bigcup \mathcal{E}_*$ belong to precisely two elements with opposite normal vectors. Hence,

$$\sum_{T \in \mathcal{R}_*} \int_{\partial T} (A \nabla U_* \cdot \nu) (1 - J_*)e_0 \, ds = \sum_{T \in \mathcal{R}_*} \int_{\partial T \cap \Omega} (A \nabla U_* \cdot \nu) (1 - J_*)e_0 \, ds \leq \sum_{E \in \mathcal{E}_*} \int_E \left[ (A \nabla U_* \cdot \nu) (1 - J_*)e_0 \right] \, ds = \frac{1}{2} \sum_{T \in \mathcal{R}_*} \int_{\partial T \cap \Omega} \left[ (A \nabla U_* \cdot \nu) (1 - J_*)e_0 \right] \, ds.$$

Altogether, we have derived

$$\| U_o - U_* \|^2_{H^1(\Omega)} \lesssim \sum_{T \in \mathcal{R}_*} \left( \int_T (f - L U_*) (1 - J_*)e_0 \, dx + \int_{\partial T \cap \Omega} \left[ (A \nabla U_* \cdot \nu) (1 - J_*)e_0 \right] \, ds \right) \leq \sum_{T \in \mathcal{R}_*} \left( |T|^{1/d} \| f - L U_* \|_{L^2(T)} |T|^{1/d} \| (1 - J_*)e_0 \|_{L^2(T)} \right) + |T|^{1/(2d)} \| (A \nabla U_* \cdot \nu) \|_{L^2(\partial T \cap \Omega)} |T|^{1/(2d)} \| (1 - J_*)e_0 \|_{L^2(\partial T \cap \Omega)} \right).$$

(4.2)

We abbreviate $\pi_{\text{max}}^\text{rel} := \pi_{\text{max}}(\pi_{\text{app}}, \pi_{\text{grad}})$. By (M3), (S5), and (S6), we have

$$|T|^{1/d} \| (1 - J_*)e_0 \|_{L^2(T)} + |T|^{1/(2d)} \| (1 - J_*)e_0 \|_{L^2(\partial T \cap \Omega)} \lesssim \| U_o - U_* \|_{H^1(\pi_{\text{max}}^\text{rel}(T))}.$$

Plugging this into (4.2) and using the Cauchy-Schwarz inequality, we obtain

$$\| U_o - U_* \|^2_{H^1(\Omega)} \lesssim \left( \sum_{T \in \mathcal{R}_*} \eta(T)^2 \right) \left( \sum_{T \in \mathcal{R}_*} \| U_o - U_* \|^2_{H^1(\pi_{\text{max}}^\text{rel}(T))} \right)^{1/2}.$$

With (M2), the second factor is controlled by $\| U_o - U_* \|_{H^1(\Omega)}$. This concludes the current section, and $C_{\text{drel}}$ depends only on $C_{\text{cell}}$, (M2)–(M3), and (S2)–(S6).

4.2. Reliability (2.14). Note that $\text{osc}_* \lesssim \eta_*$ follows immediately from their definitions (2.9) – (2.10). If one replaces $U_o \in \mathcal{X}_0$ by the exact solution $u \in H^1_0(\Omega)$ and $\mathcal{R}_*$ by $\mathcal{T}_*$, reliability (2.14) follows along the lines of Section 4.1, but now, (S2)–(S4) are not needed for the proof.

4.3. Efficiency (2.15). As in [NV12, Theorem 7], the assumptions (M1)–(M2) and (O1)–(O4) imply that

$$\eta_* \lesssim \| u - U_* \|_{H^1(\Omega)} + \text{osc}_*(U_*). \quad (4.3)$$

As in [CKNS08, Proposition 3.3], the assumptions (M1)–(M3) and (S1) imply that

$$\text{osc}(U_*) \lesssim \text{osc}(V_*) + \| U_* - V_* \|_{H^1(\Omega)} \quad \text{for all } V_* \in \mathcal{X}_*. \quad (4.4)$$

The Céa lemma (2.7) and (4.4) show that

$$\| u - U_* \|_{H^1(\Omega)} + \text{osc}_*(U_*) \overset{(4.4)}{\lesssim} \| u - U_* \|_{H^1(\Omega)} + \text{osc}_*(V_*) + \| u - V_* \|_{H^1(\Omega)} \overset{(2.7)}{\lesssim} \| u - V_* \|_{H^1(\Omega)} + \text{osc}_*(V_*) \quad \text{for all } V_* \in \mathcal{X}_*.$$
This proves \( \|u - U_\bullet\|_{H^1(\Omega)} + \text{osc}_\bullet(U_\bullet) \simeq \inf_{V_\bullet \in X_\bullet}(\|u - V_\bullet\|_{H^1(\Omega)} + \text{osc}_\bullet(V_\bullet)) \). Combining these observations, we conclude (2.15), where \( C_{\text{eff}} \) depends only on \( C_{\text{eff}}, (M1)-(M3), (S1) \) and (O1)-(O4).

4.4. Stability on non-refined elements. As in [CKNS08, Corollary 3.4], the assumptions (M1)–(M3) and (S1) imply the existence of \( C_{\text{stab}} \geq 1 \) such that for all \( T_\bullet \in \mathbb{T} \), all subsets \( S \subseteq T_\bullet \cap T_\circ \) of non-refined elements, it holds that

\[
\|\eta_\bullet(S) - \eta_\circ(S)\| \leq C_{\text{stab}}\|U_\bullet - U_\circ\|_{H^1(\Omega)}.
\]

The constant \( C_{\text{stab}} \) depends only on (M1)–(M3), (S1), as well as on \( \|A\|_{W^{1,\infty}(\Omega)}, \|b\|_{L^\infty(\Omega)}, \|c\|_{L^\infty(\Omega)}, \|\eta\|_{L^\infty(\Omega)}, \text{diam}(\Omega) \).

4.5. Reduction on refined elements. As in [CKNS08, Corollary 3.4], the assumptions (M1)–(M3), (R2)–(R3), and (S1) imply the existence of \( C_{\text{red}} \geq 1 \) and \( 0 < q_{\text{red}} < 1 \) such that for all \( T_\bullet \in \mathbb{T} \) and all subsets \( T_\circ \subseteq \text{refine}(T_\bullet) \), it holds that

\[
\eta_\circ(T_\circ \setminus T_\bullet)^2 \leq q_{\text{red}}\eta_\bullet(T_\circ \setminus T_\bullet)^2 + C_{\text{red}}\|U_\circ - U_\bullet\|_{H^1(\Omega)}^2.
\]

The constants \( C_{\text{red}} \) and \( q_{\text{red}} = q_{\text{est}}^{1/d} \) depend only on (M1)–(M3), (R2)–(R3), (S1) as well as on \( \|A\|_{W^{1,\infty}(\Omega)}, \|b\|_{L^\infty(\Omega)}, \|c\|_{L^\infty(\Omega)}, \text{diam}(\Omega) \). Note that \( [A\nabla U_\bullet, \nu] = 0 \) on \( (\partial T' \setminus \partial T) \cap \Omega \) for all \( s \in T' \subseteq T \) of an element \( T \in \mathbb{T} \), since \( U_\bullet|_{T} \in H^2(T) \).

4.6. Estimator reduction principle. Choose sufficiently small \( \delta > 0 \) such that \( 0 < q_{\text{est}} := (1+\delta)(1-(1-q_{\text{red}})\theta) \) and define \( C_{\text{est}} := C_{\text{red}}(1+\delta^{-1})C_{\text{stab}} \). With Section 4.4–4.5, elementary calculation shows that

\[
\eta_{\ell+1}^2 \leq q_{\text{est}}\eta_{\ell}^2 + C_{\text{est}}\|U_{\ell+1} - U_{\ell}\|_{H^1(\Omega)}^2 \quad \text{for all } \ell \in \mathbb{N}_0;
\]

(4.5) see [CFPP14, Lemma 4.7]. Nestedness (S2) ensures that \( X_\infty := \bigcup_{\ell \in \mathbb{N}_0} X_\ell \) is a closed subspace of \( H^1_0(\Omega) \) and hence admits a unique Galerkin solution \( U_\infty \in X_\infty \). Note that \( U_\ell \) is also a Galerkin approximation of \( U_\infty \). Hence, the Céa lemma (2.7) with \( u \) replaced by \( U_\infty \) and a density argument prove \( \|U_\infty - U_{\ell}\|_{H^1(\Omega)} \to 0 \) as \( \ell \to \infty \). Elementary calculus, estimator reduction (4.5), and Section 4.2 thus prove \( \|u - U_{\ell}\|_{H^1(\Omega)} \lesssim \eta_{\ell} \to 0 \) as \( \ell \to \infty \). This proves \( U_\ell \to U_\infty = u \); see [AFLP12, Section 2] for the detailed argument.

4.7. General quasi-orthogonality. By use of reliability from Section 4.2, the plain convergence result from Section 4.6 and a perturbation argument (since the non-symmetric part of \( \mathcal{L} \) is compact), it is shown in [FFP14, Proof of Theorem 8] that

\[
\sum_{k=\ell}^{N} (\|U_{k+1} - U_k\|_{H^1(\Omega)}^2 - \varepsilon \|u - U_k\|_{H^1(\Omega)}^2) \leq C(\varepsilon) \eta_{\ell}^2 \quad \text{for all } 0 \leq \ell \leq N \text{ and all } \varepsilon > 0.
\]

(4.6)

The constant \( C(\varepsilon) \) depends on \( \varepsilon > 0 \), the operator \( \mathcal{L} \), the sequence \( (U_\ell)_{\ell \in \mathbb{N}_0} \), and \( C_{\text{red}} \). See also [CFPP14, Proposition 6.1] for a short paraphrase of the proof.

Remark 4.1. If the bilinear form \( \langle \cdot, \cdot \rangle_{\mathcal{L}} \) is symmetric, (4.6) follows from the Pythagoras theorem in the \( \mathcal{L} \)-induced energy norm \( \|v\|_{\mathcal{E}}^2 := \langle v, v \rangle_{\mathcal{L}} \) and norm equivalence

\[
\sum_{k=\ell}^{N} \|U_{k+1} - U_k\|_{H^1(\Omega)}^2 \simeq \sum_{k=\ell}^{N} \|U_{k+1} - U_k\|_{\mathcal{E}}^2 = \|u - U_{\ell}\|_{\mathcal{E}}^2 - \|u - U_N\|_{\mathcal{E}}^2 \lesssim \|u - U_{\ell}\|_{H^1(\Omega)}^2.
\]
Together with reliability (2.14), this proves (4.6) even for $\varepsilon = 0$, and $C(\varepsilon) \simeq C_{\text{ref}}^2$ is independent of the sequence $(U_\ell)_{\ell \in \mathbb{N}_0}$. 

4.8. Linear convergence with optimal rates. The remaining claims (2.16)–(2.17) follow from [CFPP14, Theorem 4.1] which only relies on (R1), (2.3), (R4)–(R5), stability (Section 4.4), reduction (Section 4.5), quasi-orthogonality (Section 4.7), discrete reliability (Section 4.1), and reliability (Section 4.2).

5. Proof of Theorem 3.8

5.1. Admissibility and refine. In this section, we show that, given a mesh $\hat{T}_0 \in \hat{\mathcal{T}}$, iterative application of the refinement Algorithm 3.5 generates exactly the set of all admissible meshes $\hat{T}_0$ that are finer than $\mathcal{T}_0$. In particular, this implies that $\hat{\mathcal{T}}$ coincides with the set of all admissible hierarchical meshes that are finer than $\mathcal{T}_0$, which we have already been mentioned in Section 3.5. We start with the following lemma.

Lemma 5.1. Let $\hat{T}_0$ and $\hat{T}_\beta$ be hierarchical meshes such that $\hat{T}_\beta$ is finer than $\hat{T}_0$, i.e., $\hat{\Omega}_\beta \subset \hat{\Omega}_\beta$ for all $k \in \mathbb{N}_0$. Then, for all $\hat{\beta}_0 \in \hat{\mathcal{H}}_0$ there exists $\hat{\beta}_0 \in \hat{\mathcal{H}}_0$ with supp($\hat{\beta}_0$) $\subset$ supp($\hat{\beta}_0$).

Proof. Clearly, we may assume $\hat{\beta}_0 = \hat{\mathcal{H}}_0 \setminus \hat{\mathcal{H}}_0$. Let $k := \text{level}(\hat{\beta}_0)$ and define $\hat{\beta}_0 := \hat{\beta}_0$. Since $\hat{\beta}_0 \in \hat{\mathcal{H}}_0$, (3.12) implies that supp($\hat{\beta}_0$) $\setminus \hat{\Omega}_0^{k+1} \neq \emptyset$ and supp($\hat{\beta}_0$) $\subset \hat{\Omega}_0^k$. Since $\hat{\beta}_0 \notin \hat{\mathcal{H}}_0$, (3.12) implies that supp($\hat{\beta}_0$) $\setminus \hat{\Omega}_0^{k+1} = \emptyset$ or supp($\hat{\beta}_0$) $\notin \hat{\Omega}_0^k$. However, $\hat{\Omega}_0^{k+1} \subset \hat{\Omega}_0^{k+1}$ and supp($\hat{\beta}_0$) $\setminus \hat{\Omega}_0^{k+1} \neq \emptyset$ imply that supp($\hat{\beta}_0$) $\setminus \hat{\Omega}_0^{k+1} = \emptyset$. Hence, we have supp($\hat{\beta}_0$) $\subset \hat{\Omega}_0^k$, which especially implies $k > 0$. This is equivalent to supp($\hat{\beta}_0$) $\setminus \hat{\Omega}_0^k \neq \emptyset$. Clearly, there exists $\hat{\beta}_0 \in B^k$ with supp($\hat{\beta}_0$) $\subset$ supp($\hat{\beta}_0$). If $\hat{\beta}_0 \in \hat{\mathcal{H}}_0$, we are done. Otherwise, supp($\hat{\beta}_0$) $\neq \emptyset$, which especially implies $k > 0$. This is equivalent to supp($\hat{\beta}_0$) $\setminus \hat{\Omega}_0^k \neq \emptyset$. Inductively, we obtain a sequence $\hat{\beta}_0, \ldots, \hat{\beta}_0$ with $\hat{\beta}_0 \in B^k$ and supp($\hat{\beta}_0$) $\supset \cdots \supset$ supp($\hat{\beta}_0$), where $\hat{\beta}_0 \in \hat{\mathcal{H}}_0$ for some $K \geq 0$.

Proposition 5.2. If $\hat{T}_0 \in \hat{\mathcal{T}}$, then refine($\hat{T}_0$) coincides with the set of all admissible hierarchical meshes $\hat{T}_0 \in \hat{\mathcal{T}}$ that are finer than $\hat{T}_0$.

Proof. We prove the assertion in four steps.

Step 1: We show that $\hat{T}_0 := \text{refine}(\hat{T}_0, \hat{\mathcal{M}}_\ast) \in \hat{\mathcal{T}}$ for any $\hat{\mathcal{M}}_\ast \subset \hat{\mathcal{T}}$. Let $\hat{T}, \hat{T}' \in \hat{\mathcal{T}}$ with $\hat{T}' \in \mathcal{N}_0(\hat{T})$, i.e., there exists $\hat{\beta}_0 \in \mathcal{H}_0$ with $|\hat{T} \cap \text{supp}(\hat{\beta}_0)| \neq 0 \neq |\hat{T}' \cap \text{supp}(\hat{\beta}_0)|$; see (3.19).

By Lemma 5.1, there exists some (not necessarily unique) $\hat{\beta}_0 \in \mathcal{H}_0$ with supp($\hat{\beta}_0$) $\subset$ supp($\hat{\beta}_0$). We consider four different cases.

(i) Let $\hat{T}, \hat{T}' \in \hat{T}_0$. Then, $|\hat{T} \cap \text{supp}(\hat{\beta}_0)| \neq 0 \neq |\hat{T}' \cap \text{supp}(\hat{\beta}_0)|$, i.e., $\hat{T}' \in \mathcal{N}_0(\hat{T})$ and hence $|\text{level}(\hat{T}) - \text{level}(\hat{T}')| \leq 1$ by $\hat{T}_0 \in \hat{\mathcal{T}}$.

(ii) Let $\hat{T}, \hat{T}' \in \hat{T}_0 \setminus \hat{T}_0$. Let $\hat{T}, \hat{T}' \in \hat{T}_0$ with $\hat{T} \notin \hat{T}_0, \hat{T}' \notin \hat{T}_0$. Then, it holds that $\text{level}(\hat{T}) = \text{level}(\hat{T}_0) + 1, \text{level}(\hat{T}') = \text{level}(\hat{T}_0) + 1$ as well as $|\hat{T} \cap \text{supp}(\hat{\beta}_0)| \neq 0 \neq |\hat{T}' \cap \text{supp}(\hat{\beta}_0)|$.
shows that supp(\(\hat{\beta}_\bullet\)) from admissibility in the sense of Section 3.4 in combination with the following lemma.

The mesh properties (M1)–(M3) essentially follow Step 3:

By definition of \(\hat{\beta}_\bullet\), we have \(\hat{\beta}_\bullet \subseteq \text{supp}(\hat{\beta})\) for some \(\hat{\beta} \in \mathcal{H}_\bullet\) as well as level(\(\hat{T}(J-1)\)) = level(\(\hat{T}(J)\)) - 1. Hence, (3.15) and \(\hat{T}_\bullet \in \hat{T}\) show \(k := \text{level}(\hat{\beta}) = \text{level}(\hat{T}(J-1))\). Since \(\hat{T}(J-1) \in \mathcal{T}_\bullet\), (3.10) implies \(\hat{T}(J-1) \notin \hat{\Omega}_0^{k+1}\) and hence \(\text{supp}(\hat{\beta}) \notin \hat{\Omega}_0^{k+1}\). Moreover, (3.12) shows \(\text{supp}(\hat{\beta}) \subseteq \hat{\Omega}_0^k \subseteq \hat{\Omega}_0^{k+1}\). Using (3.12) again, we see \(\hat{\beta} \in \hat{\mathcal{H}}_0\). Together with \(\hat{T}_0(J) \subseteq \hat{\Omega}_0^{k+1}\) and \(\text{level}(\hat{T}(J)) \geq \text{level}(\hat{T}(J)) + 1 = \text{level}(\hat{T}(J-1)) + 2\), this contradicts admissibility of \(\hat{T}_0 \in \hat{T}\) and concludes the proof.

Step 4: Let again \(\hat{T}_\bullet \in \hat{T}\) be an arbitrary admissible mesh that is finer than \(\hat{T}_\bullet\). Step 3 together with Step 2 shows that we can iteratively refine \(\hat{T}_\bullet\) and obtain a sequence \(\hat{T}_0, \ldots, \hat{T}_J\) with \(\hat{T}_0 = \hat{T}_0, \hat{T}_J+1 = \text{refine}(\hat{T}_J, \{\hat{T}_J\})\) with some \(\hat{T}_{j} \in \hat{T}_{j} \setminus \hat{T}_{j+1}\) for \(j = 1, \ldots, J - 1\) and \(\hat{T}_{J} = \hat{T}_\bullet\). By definition, this proves \(\hat{T}_\bullet \in \text{refine}(\hat{T}_\bullet)\).

5.2. Verification of (M1)–(M3). The mesh properties (M1)–(M3) essentially follow from admissibility in the sense of Section 3.4 in combination with the following lemma.

**Lemma 5.3.** Let \(\hat{T}_\bullet\) be an arbitrary hierarchical mesh in the parameter domain. Then,

\[
\Pi_\bullet(\hat{T}) \subseteq \mathcal{N}_\bullet(\hat{T}) \quad \text{for all } \hat{T} \in \mathcal{T}_\bullet.
\]

**Proof.** Let \(\hat{T}' \in \Pi_\bullet(\hat{T})\), i.e., \(\hat{T}' \in \mathcal{T}_\bullet\) with \(\hat{T} \cap \hat{T}' \neq \emptyset\). We abbreviate \(k := \text{level}(\hat{T})\). Since all knot multiplicities are smaller than \(p + 1\), there exists \(\hat{\beta}_k \in \hat{\mathcal{B}}_k^k\) such that \(|\hat{T} \cap \text{supp}(\hat{\beta}_k)| \neq 0 \neq |\hat{T}' \cap \text{supp}(\hat{\beta}_k)|\). If \(\hat{\beta}_k \in \hat{\mathcal{H}}_\bullet\), then \(\hat{T} \in \mathcal{N}_\bullet(\hat{T})\). If \(\hat{\beta}_k \notin \hat{\mathcal{H}}_\bullet\), the characterization (3.12) shows that \(\text{supp}(\hat{\beta}_k) \subseteq \hat{\Omega}_0^k\) or \(\text{supp}(\hat{\beta}_k) \subseteq \hat{\Omega}_0^{k+1}\). By choice of \(k\), it holds that \(\hat{T} \subseteq \text{supp}(\hat{\beta}_k)\).
In view of (3.10), \( \hat{T} \in \hat{T}_h \) implies \( \hat{T} \not\subset \hat{\Omega}_k^{k+1} \). Hence, \( \text{supp}(\hat{\beta}^k) \not\subset \hat{\Omega}_k \) and, in particular, \( k > 0 \). Next, there exists \( \hat{\beta}^{k-1} \in \hat{\beta}^{k-1} \) such that \( \text{supp}(\hat{\beta}^k) \not\subset \text{supp}(\hat{\beta}^{k-1}) \). If \( \hat{\beta}^{k-1} \in \hat{H}_h \), then \( \hat{T} \in \hat{N}_h(\hat{T}) \). If \( \hat{\beta}^{k-1} \not\subset \hat{H}_h \), there holds again either \( \text{supp}(\hat{\beta}^{k-1}) \not\subset \text{supp}(\hat{\beta}^{k-1}) \) or \( \text{supp}(\hat{\beta}^{k-1}) \not\subset \hat{\Omega}_k \). Due to \( \text{supp}(\hat{\beta}^k) \not\subset \hat{\Omega}_k \), the second case is not possible. Hence, \( \text{supp}(\hat{\beta}^{k-1}) \not\subset \hat{\Omega}_k \) and, in particular, \( k-1 > 0 \). We proceed in the same way to get a sequence \( \hat{\beta}^k, \ldots, \hat{\beta}^K \) with \( \hat{\beta}^j \in \hat{\beta}^j \) and \( \text{supp}(\hat{\beta}^K) \supset \cdots \supset \text{supp}(\hat{\beta}^k) \), where \( \hat{\beta}^K \in \hat{H}_h \) for some \( K \geq 0 \). \( \square \)

We define the patches \( \pi_\ast(\cdot) \) and \( \Pi_\ast(\cdot) \) in the parameter domain analogously to the patches in the physical domain, see Section 2.1.

With Lemma 5.3, one can easily verify that \( T \) satisfies (M1)–(M3): Let \( T_0 \in T \). We start with (M1). Let \( T \in \mathcal{T}_h \) and \( T' \in \Pi_\ast(T) \). Lemma 5.3 and admissibility show for the corresponding elements \( \hat{T}, \hat{T}' \) in the parameter domain that \( |\text{level}(\hat{T}) - \text{level}(\hat{T}')| \leq 1 \), wherefore \( |\hat{T}| \simeq |\hat{T}'| \). Regularity (3.28) of the transformation \( \gamma \) finally yields \( |T| \simeq |T'| \). The constant \( C_{\text{shape}} \) depends only on \( d, C_\gamma \), and \( T_0 \).

To prove (M2), let \( T \in \mathcal{T}_h \) and \( T' \in \Pi_\ast(T) \). As before, we have \( |\text{level}(\hat{T}) - \text{level}(\hat{T}')| \leq 1 \) for the corresponding elements in the parameter domain. With this, one easily sees that \( \#\Pi_\ast(T) \leq C_{\text{patch}} \) with a constant \( C_{\text{patch}} > 0 \) that depends only on the dimension \( d \).

Regularity (3.28) of \( \gamma \) shows that it is sufficient to prove (M3) for hyperrectangles \( \hat{T} \) in the parameter domain. There, the trace inequality (M3) is well-known; see, e.g., [Era05, Satz 3.4.5]. The constant \( C_{\text{trace}} \) depends only on \( d, C_\gamma \), and \( T_0 \).

5.3. Verification of (R1)–(R3). Let \( T_0 \in T \), \( T_0 \in \text{refine}(\mathcal{T}_h) \), and \( T \in \mathcal{T}_h \). (R1) is trivially satisfied with \( C_{\text{son}} = 2^d \), since each refined element is split into exactly \( 2^d \) elements. Moreover, the union of sons property (R2) holds by definition.

To see the reduction property (R3), let \( T' \in \mathcal{T}_h \) with \( T' \subset T \). Since each refined element is split into \( 2^d \) elements, we have for the corresponding elements in the parameter domain \( |\hat{T'}| \leq 2^{-d}|\hat{T}| \). Next, we prove \( |T'| \leq q_{\text{son}}|T| \) with a constant \( 0 < q_{\text{son}} < 1 \) which depends only on \( d \) and \( C_\gamma \). Indeed, we even prove for arbitrary measurable sets \( \hat{S}' \subset \hat{S} \subset \hat{\Omega} \) and \( S := \gamma(\hat{S}), S' := \gamma(\hat{S}') \) that \( 0 < |\hat{S}'| \leq 2^{-d}|\hat{S}| \) implies \( |S'| \leq q_{\text{son}}|S| \). To see this, we argue by contradiction and assume that there are two sequences of such sets \( (\hat{S}_n)_{n \in \mathbb{N}} \) and \( (\hat{S}'_n)_{n \in \mathbb{N}} \) with \( |S'_n|/|S_n| \rightarrow 1 \). This implies \( |S_n \setminus S'_n|/|S_n| \rightarrow 0 \) and yields the contradiction

\[
1 - 2^{-d} \leq \frac{|\hat{S}_n \setminus \hat{S}'_n|}{|\hat{S}_n|} \simeq \int_{\hat{S}_n \setminus \hat{S}'_n} |\text{det} \, D\gamma(t)| \, dt \int_{\hat{S}_n} |\text{det} \, D\gamma(t)| \, dt = \frac{|S_n \setminus S'_n|}{|S_n|} \rightarrow 0.
\]

5.4. Verification of (R4). The proof of the closure estimate (R4) goes back to the seminal works [BDD04, Ste08]. Our analysis builds on [BGMP16, Section 3] which proves (R4) for the refinement strategy of [BG16b]; see also Remark 3.6. The following auxiliary result states that \( \text{refine}(\cdot, \cdot) \) is equivalent to iterative refinement of one single element. For a mesh in the parameter domain \( \hat{T}_h \in \hat{T} \) and an arbitrary set \( \hat{\mathcal{M}}_h \), we define \( \text{refine}(\hat{T}_h, \hat{\mathcal{M}}_h) := \text{refine}(\hat{T}_h, \hat{\mathcal{M}}_h \cap \hat{T}_h) \) and note that \( \text{refine}(\hat{T}_h, \emptyset) = \hat{T}_h \).

Lemma 5.4. Let \( \hat{T}_h \in \hat{T} \) and \( \hat{\mathcal{M}}_h = \{\hat{T}_1, \ldots, \hat{T}_n\} \subseteq \hat{T}_h \). Then, it holds that

\[
\text{refine}(\hat{T}_h, \hat{\mathcal{M}}_h) = \text{refine}(\text{refine}(\ldots \text{refine}(\hat{T}_h, \{\hat{T}_1\}, \ldots, \{\hat{T}_{n-1}\}), \{\hat{T}_n\})).
\]
Proof. We only show that $\text{refine}(\widehat{T}, \widehat{M}) = \text{refine}(\text{refine}(\widehat{T}, \{\widehat{T}\}), \widehat{M} \setminus \{\widehat{T}\})$, and then (5.2) follows by induction. We define
\[
\widehat{T}(1) := \text{refine}(\widehat{T}, \{\widehat{T}\}), \quad \widehat{T}(2) := \text{refine}(\widehat{T}(1), \widehat{M} \setminus \{\widehat{T}\}),
\]
\[
\widehat{M}(0) := \widehat{M}(0) := \{\widehat{T}\}, \quad \widehat{M}(1) := \widehat{M}(1) := \widehat{M}(1) := \widehat{M} \setminus \{\widehat{T}\}.
\]
For $i \in \mathbb{N}_0$, we introduce the following notation which is conform with that of Algorithm 3.5:
\[
\widehat{M}(i+1) := \widehat{M}(i) \cup \bigcup_{\widehat{T} \in \widehat{M}(i)} \mathcal{N}_\text{bad}^\ast(\widehat{T}), \quad \widehat{M}(i+1) := \widehat{M}(i) \cup \bigcup_{T \in \widehat{M}(i)} \mathcal{N}_\text{bad}^\ast(\widehat{T}),
\]
\[
\widehat{M}(i+1) := \widehat{M}(i) \cup \bigcup_{\widehat{T} \in \widehat{M}(i)} \mathcal{N}_\text{bad}^\ast(\widehat{T}).
\]
Finally, we set
\[
\widehat{M} = \bigcup_{i \in \mathbb{N}_0} \widehat{M}(i)
\]
With these notations, we have
\[
\widehat{T} \setminus \widehat{T}(1) = \widehat{M}(0), \quad \widehat{T}(1) \setminus \widehat{T}(2) = \widehat{M}(1),
\]
\[
\widehat{T} \setminus \text{refine}(\widehat{T}, \widehat{M} \setminus \{\widehat{T}\}) = \widehat{M}(0) \cup \widehat{M}(1).
\]
To conclude the proof, we will prove that $\widehat{M}(0) \cup \widehat{M}(1) = \widehat{M}(0) \cup \widehat{M}(1)$ To this end, we split the proof into three steps.

**Step 1:** We first prove $\widehat{M}(1) \subset \widehat{T}$ by induction. Clearly, we have $\widehat{M}(1) \subset \widehat{T}$. Now, let $i \in \mathbb{N}_0$ and suppose $\widehat{M}(i) \subset \widehat{T}$. To see $\widehat{M}(i+1) \subset \widehat{T}$, we argue by contradiction and assume that there exists $\widehat{T} \in \widehat{M}(i)$ and $\widehat{T}' \in \mathcal{N}_\text{bad}^\ast(\widehat{T}) \setminus \widehat{T}$. By Lemma 5.1, the unique father element $\widehat{T} \in \widehat{T}$, with $\widehat{T}' \subset \widehat{T}$ satisfies $\widehat{T} \in \mathcal{N}_\ast(\widehat{T})$. Therefore, admissibility of $\widehat{T}$ proves $\text{level}(\widehat{T}) - \text{level}(\widehat{T}) = 1$, which contradicts
\[
\text{level}(\widehat{T}) = \text{level}(\widehat{T}) - 1 = \text{level}(\widehat{T}) = 2.
\]

**Step 2:** Let $\widehat{T} \in \widehat{M}(1)$. In this step, we will prove that
\[
\widehat{M}(0) \cup \mathcal{M}_\text{bad}(\widehat{T}) = \widehat{M}(0) \cup \mathcal{M}_\text{bad}(\widehat{T}).
\]
By Step 1, we have $\widehat{T} \in \widehat{T}$. Lemma 5.1 proves $\mathcal{M}_\text{bad}(\widehat{T}) \subset \widehat{T} \subset \mathcal{M}_\text{bad}(\widehat{T})$. Using Step 1 again, we see $\mathcal{M}_\text{bad}(\widehat{T}) \subset \widehat{M}(0) \subset \widehat{T}$ and conclude “$\subset$” in (5.3). To see “$\supset$”, let $\widehat{T}' \in \mathcal{M}_\text{bad}(\widehat{T}) \setminus \widehat{M}(0)$. Note that $\widehat{T}' \in \widehat{T} \setminus \{\widehat{T}\}$ since $\widehat{T} \setminus \{\widehat{T}\} = \mathcal{M}(0)$. There exists $\widehat{\beta} \in \mathcal{H}$ with $\widehat{T}, \widehat{T}' \subset \text{supp}(\widehat{\beta})$. By admissibility of $\widehat{T} \in \widehat{T}$, $\text{level}(\widehat{T}') = \text{level}(\widehat{T}) - 1$, and (3.15), we see that $\text{level}(\widehat{\beta}) = \text{level}(\widehat{T}') = k$. Hence, (3.12) yields that $\text{supp}(\widehat{\beta}) \subset \widehat{\Omega}_k^\ast$ as well as $\text{supp}(\widehat{\beta}) \subset \widehat{\Omega}_k^\ast$. The definition of $k'$ and (3.10) show that $\widehat{T}' \subset \widehat{N}_k$. We conclude $\text{supp}(\widehat{\beta}) \subset \widehat{\Omega}_k \subset \widehat{\Omega}_k$ and $\text{supp}(\widehat{\beta}) \subset \widehat{\Omega}_k$, since $\widehat{T}(1) \equiv T \subset \text{supp}(\widehat{\beta})$. Therefore, (3.12) shows $\widehat{\beta} \in \mathcal{H}(1)$. Altogether, we have $\widehat{T}' \in \mathcal{M}_\text{bad}(\widehat{T})$. 


Step 3: Finally, we prove $\widehat{M}^{(\text{end})}_{(0)} \cup \widehat{M}^{(i)}_{(1)} = \widehat{M}^{(\text{end})}_{(0)} \cup \widehat{M}^{(i)}_{(1)}$ by induction on $i \in \mathbb{N}_0$. In particular, this will imply $\widehat{M}^{(\text{end})}_{(0)} \cup \widehat{M}^{(\text{end})}_{(1)} = \widehat{M}^{(\text{end})}_{(0)} \cup \widehat{M}^{(\text{end})}_{(1)}$. For $i = 0$, the claim follows from $\widehat{M}^{(0)}_{(1)} = \widehat{M}^{(0)}_{(1)}$. By Step 2, the induction step works as follows:

\[
\widehat{M}^{(\text{end})}_{(0)} \cup \widehat{M}^{(i+1)}_{(1)} = \widehat{M}^{(\text{end})}_{(0)} \cup \widehat{M}^{(i)}_{(1)} \cup \bigcup_{\hat{T} \in \widehat{M}^{(\text{end})}_{(1)} \cup \widehat{M}^{(i)}_{(1)}} \mathcal{N}^\text{bad}(\hat{T})
\]

\[
= \widehat{M}^{(\text{end})}_{(0)} \cup \widehat{M}^{(i)}_{(1)} \cup \bigcup_{\hat{T} \in \widehat{M}^{(\text{end})}_{(0)} \cup \widehat{M}^{(i)}_{(1)}} \mathcal{N}^\text{bad}(\hat{T})
\]

\[
= \widehat{M}^{(\text{end})}_{(0)} \cup \widehat{M}^{(i+1)}_{(1)}
\]

This concludes the proof. \qed

Let $\mathcal{T} \in \widehat{T}$. For $\hat{T}, \hat{T}' \in \mathcal{T}$, let $\text{dist}(\hat{T}, \hat{T}')$ be the Euclidean distance of their midpoints in the parameter domain. Let $\hat{T} \in \mathcal{T}$ and $\hat{T}' \in \mathcal{N}_s(\hat{T})$. Hence, there is $\hat{\beta} \in \mathcal{H}$ such that $\hat{T}, \hat{T}' \subseteq \text{supp}(\hat{\beta})$. In particular, it holds that $\text{dist}(\hat{T}, \hat{T}') \leq \text{diam}(\text{supp}(\hat{\beta}))$. By admissibility of $\mathcal{T}$ and (3.15), we see $|\text{level}(\hat{\beta}) - \text{level}(\hat{T})| \leq 1$. This proves

\[
\text{dist}(\hat{T}, \hat{T}') \leq C_{\text{diam}} 2^{-\text{level}(\hat{T})},
\]

where $C_{\text{diam}} > 0$ depends only on $d$, $\hat{T}_0$ and $(p_1, \ldots, p_d)$. With this observation, we can prove the following lemma. The proof follows the lines of [BGMP16, Lemma 11], but is also included here for completeness.

**Lemma 5.5.** Let $\mathcal{T} \in \widehat{T}$ and $\hat{T}' \in \mathcal{T}$. With $\mathcal{T} = \text{refine}(\mathcal{T}, \{\hat{T}'\})$, it holds that

\[
\text{dist}(\hat{T}, \hat{T}') \leq 2^{-\text{level}(\hat{T})} C_{\text{dist}} \quad \text{for all } \hat{T} \in \mathcal{T} \setminus \mathcal{T},
\]

where $C_{\text{dist}} > 0$ depends only on $d$, $\hat{T}_0$ and $(p_1, \ldots, p_d)$.

**Proof.** $\hat{T} \in \mathcal{T} \setminus \mathcal{T}$ implies the existence of a sequence $\hat{T}' = \hat{T}, \hat{T}_1, \ldots, \hat{T}_0$ such that $\hat{T}_{j-1} \in \mathcal{N}_s^\text{bad}(\hat{T}_j)$ and $\hat{T}$ is a child of $\hat{T}_0$, i.e., $\hat{T} \subseteq \hat{T}_0$ and $\text{level}(\hat{T}) = \text{level}(\hat{T}_0) + 1$. Since $\text{level}(\hat{T}_{j-1}) = \text{level}(\hat{T}_j) - 1$, it follows

\[
\text{level}(\hat{T}_j) = \text{level}(\hat{T}_0) + j.
\]

The triangle inequality proves

\[
\text{dist}(\hat{T}, \hat{T}') \leq \text{dist}(\hat{T}, \hat{T}_0) + \text{dist}(\hat{T}_0, \hat{T}') \leq \text{dist}(\hat{T}, \hat{T}_0) + \sum_{j=1}^{J} \text{dist}(\hat{T}_j, \hat{T}_{j-1})
\]

Further, there exists a constant $C > 0$ which depends only on $\hat{T}_0$ and $d$, such that

\[
\text{dist}(\hat{T}, \hat{T}_0) \leq C 2^{-\text{level}(\hat{T})}.
\]
With (5.4) and (5.6), we see
\[ \sum_{j=1}^{J} \text{dist}(\tilde{T}_j, \tilde{T}_{j-1}) \leq C_{\text{diam}} \sum_{j=1}^{J} 2^{-\text{level}(\tilde{T}_j)}, \] (5.4)
\[ \sum_{j=1}^{J} 2^{-\text{level}(\tilde{T}_j)-j} \leq C_{\text{diam}} 2^{-\text{level}(\tilde{T})-1}, \] (5.6)
which concludes the proof. \[ \square \]

Finally, let \( \hat{T}_0 \in \hat{T} \) and \( \hat{T} \in \hat{T}_0 \). We abbreviate \( \hat{T}_0 = \text{refine}(\hat{T}, \{\hat{T}\}) \). Then, there holds
\[ \text{level}(\hat{T}''') \leq \text{level}(\hat{T}) + 1 \] for all refined elements \( \hat{T}'' \in \hat{T}_{\hat{T}_0} \). \( \hat{T}_0 \). To see this, note that all elements \( \hat{T}'' \in \hat{T}_{\hat{T}_0} \) which are refined, satisfy \( \hat{T}'' \neq \hat{T} \) or \( \text{level}(\hat{T}''') \leq \text{level}(\hat{T}) + 1 \). Therefore, their children satisfy \( \text{level}(\hat{T}) \leq \text{level}(\hat{T}) + 1 \). With this last observation, we can argue as in the proof of [BGMP16, Theorem 12] to show the closure estimate (R4). The constant \( C_{\text{clo}} > 0 \) depends only on \( d, \hat{T}_0, \) and \( (p_1, \ldots, p_d) \).

5.5. Verification of (R5). We prove (R5) in the parameter domain \( \hat{\Omega} \). Let \( \hat{T}, \hat{T}_0 \in \hat{T} \) be two admissible hierarchical meshes. We define the overlay
\[ \hat{T}_0 := \{ \hat{T} \in \hat{T} : \exists \hat{T}_0 \in \hat{T}_0, \hat{T} \subseteq \hat{T}_0 \} \cup \{ \hat{T} \in \hat{T} : \exists \hat{T}_0 \in \hat{T}_0, \hat{T} \subseteq \hat{T}_0 \}. \] (5.8)
Note that \( \hat{T}_0 \) is a hierarchical mesh with hierarchical domains \( \hat{\Omega}_0^k = \hat{\Omega}_0^k \cup \hat{\Omega}_0^k \) for \( k \in \mathbb{N}_0 \). In particular, \( \hat{T}_0 \) is finer than \( \hat{T} \). Moreover, the overlay estimate easily follows from the definition of \( \hat{T}_0 \). It remains to prove that \( \hat{T}_0 \) is admissible. To see this, let \( \hat{T}, \hat{T}' \in \hat{T}_0 \) with \( \hat{T}' = \{ \hat{T} \in \hat{T} \) admissible. To see this, let \( \hat{T}, \hat{T}' \in \hat{T}_0 \) with \( \hat{T}' \in \hat{T}_0 \), i.e., there exists \( \hat{T} \subseteq \hat{T}_0 \) such that \( \hat{T} \cap \text{supp}(\hat{T}_0) \neq \emptyset \). Without loss of generality, we suppose \( \text{level}(\hat{T}) = \text{level}(\hat{T}') \) and \( \hat{T} \in \hat{T}_0 \). If \( \hat{T} \in \hat{T}_0 \), Lemma 5.1 shows \( \hat{T} \subseteq \hat{T}_0 \), and admissibility of \( \hat{T}_0 \) implies \( |\text{level}(\hat{T}) - \text{level}(\hat{T}')| \leq 1 \). Now, let \( \hat{T} \in \hat{T}_0 \). By definition of the overlay, there exists \( \hat{T} \subseteq \hat{T}_0 \), with \( \hat{T} \subseteq \hat{T}_0 \) and \( \text{level}(\hat{T}_0) \leq \text{level}(\hat{T}') \). Further, Lemma 5.1 provides some (not necessarily unique) \( \hat{T} \subseteq \hat{T}_0 \) such that \( \text{supp}(\hat{T}_0) \subseteq \hat{T}_0 \). Hence, \( |\hat{T}_0 \cap \text{supp}(\hat{T}_0) \neq \emptyset \) \( \neq |\hat{T}_0 \cap \text{supp}(\hat{T}_0)| \), i.e., \( \hat{T}_0 \subseteq \hat{T}_0 \). Since \( \hat{T}_0 \subseteq \hat{T}_0 \), it follows that \( |\text{level}(\hat{T}) - \text{level}(\hat{T}_0)| \leq 1 \). Altogether, we see that
\[ |\text{level}(\hat{T}) - \text{level}(\hat{T}')| = |\text{level}(\hat{T}) - \text{level}(\hat{T}_0)| \leq |\text{level}(\hat{T}) - \text{level}(\hat{T}_0)| \leq 1. \]
This concludes the proof of (R5).

5.6. Verification of (S1). Let \( T \in T_{\hat{T}_0} \in T \). Let \( V : X \to X \). Define \( \hat{V} := V \circ \gamma \in \hat{X} \subseteq \hat{X} \) and \( \hat{T} := \gamma^{-1}(T) \in \hat{T}_0 \). Regularity (3.28) of \( \gamma \) proves for \( i \in \{0, 1, 2\} \)
\[ ||V||_{H^i(T)} \simeq ||\hat{V}||_{H^i(\hat{T})}, \] (5.9)
where the hidden constants depend only on \( d \) and \( C_{\gamma} \). Since \( \hat{V} \) is a \( \hat{T}_0 \)-piecewise tensor polynomial, there holds for \( i, j \in \{0, 1, 2\} \) with \( j \leq i \) that
\[ ||T^{(i-j)/d}||_{H^i(\hat{T})} \lesssim ||\hat{V}||_{H^i(\hat{T})}, \] (5.10)
where the hidden constant depends only on \( d, \hat{T}_0, \) and \( (p_1, \ldots, p_d) \). Together, (5.9)–(5.10) conclude the proof of (S1), where \( C_{\text{inv}} \) depends only on \( d, \hat{T}_0, \) and \( (p_1, \ldots, p_d) \).
5.7. Verification of (S2). In (3.21), we already saw that \( \mathcal{T}_0 \in \text{refine}(\mathcal{T}_\bullet) \) with \( \mathcal{T}_\bullet \in \mathcal{T} \) implies nestedness of the corresponding ansatz spaces \( X_\bullet \subseteq X_\circ \).

5.8. Verification of (S3). We show the assertion in the parameter domain. For arbitrary but fixed \( k_{\text{proj}} \in \mathbb{N}_0 \) (which will be fixed later in Section 5.10 to be \( k_{\text{proj}} := 2(p+1) \)), we set \( k_{\text{loc}} := k_{\text{proj}} + 2(p+1) \). Let \( \hat{\mathcal{T}}_\bullet \in \hat{\mathcal{T}}, \hat{\mathcal{T}}_o \in \text{refine}(\hat{\mathcal{T}}_\bullet) \), and \( \hat{V}_o \in X_\circ \). We define the patch functions \( \pi_k \) and \( \Pi_k \) in the parameter domain analogously to the patch functions in the physical domain, see Section 2.1. Let \( \hat{T} \in \hat{\mathcal{T}}_\bullet \setminus \Pi_{\text{loc}}^{\text{loc}}(\hat{\mathcal{T}}_\bullet \setminus \hat{\mathcal{T}}_o) \), where \( \Pi_{\text{loc}} := \Pi_{\text{loc}}^{\text{loc}} \). First, we show that

\[
\Pi_{\text{loc}}^{\text{loc}}(\hat{T}) \subseteq \hat{T}_o \cap \hat{T}_o.
\] (5.11)

To this end, we argue by contradiction and assume that there exists \( \hat{T}' \in \Pi_{\text{loc}}^{\text{loc}}(\hat{T}) \) with \( \hat{T}' \not\subseteq \hat{T}_o \cap \hat{T}_o \). This is equivalent to \( \hat{T} \in \Pi_{\text{loc}}^{\text{loc}}(\hat{T}') \) and \( \hat{T}' \in \hat{T}_o \setminus \hat{T}_o \). This implies \( \hat{T} \in \Pi_{\text{loc}}^{\text{loc}}(\hat{T}_o \setminus \hat{T}_o) \), contradicts \( \hat{T} \in \hat{T}_o \setminus \Pi_{\text{loc}}^{\text{loc}}(\hat{T}_o \setminus \hat{T}_o) \), and hence proves (5.11). Again, we abbreviate \( \pi^{\text{proj}} := \pi_{k_{\text{proj}}}^{\text{proj}} \).

According to Corollary 3.2, it holds that

\[
\{ V_{\mid \pi^{\text{proj}}}(\hat{T}) : V_\circ \in X_\circ \} = \text{span}\{ \hat{\beta}_{\mid \pi^{\text{proj}}}(\hat{T}) : \hat{\beta} \in \hat{H}_\circ \land \hat{\beta}|_{\partial \hat{\Omega}} = 0 \land |\text{supp}(\hat{\beta}) \cap \pi^{\text{proj}}(\hat{T})| > 0 \},
\]
as well as

\[
\{ V_{\mid \pi^{\text{proj}}}(\hat{T}) : V_\circ \in X_\circ \} = \text{span}\{ \hat{\beta}_{\mid \pi^{\text{proj}}}(\hat{T}) : \hat{\beta} \in \hat{H}_\circ \land \hat{\beta}|_{\partial \hat{\Omega}} = 0 \land |\text{supp}(\hat{\beta}) \cap \pi^{\text{proj}}(\hat{T})| > 0 \}.
\]

We will prove

\[
\{ \hat{\beta} \in \hat{H}_\circ : |\text{supp}(\hat{\beta}) \cap \pi^{\text{proj}}(\hat{T})| > 0 \} = \{ \hat{\beta} \in \hat{H}_\circ : |\text{supp}(\hat{\beta}) \cap \pi^{\text{proj}}(\hat{T})| > 0 \},
\] (5.12)

which will conclude (S3). First let \( \hat{\beta} \) be an element of the left set. By Remark 3.4, this implies \( \text{supp}(\hat{\beta}) \subseteq \pi_{\text{loc}}^{\text{loc}}(\hat{T}) \). Together with (5.11), we see \( \text{supp}(\hat{\beta}) \subseteq \bigcup(\hat{T}_o \cap \hat{T}_o) \). This proves that no element within \( \text{supp}(\hat{\beta}) \) is changed during refinement, i.e., \( \hat{\Omega}_k \cap \text{supp}(\hat{\beta}) = \hat{\Omega}_k \cap \text{supp}(\hat{\beta}) \) for all \( k \in \mathbb{N}_0 \). Thus, (3.12) proves \( \hat{\beta} \in \hat{H}_\circ \). The proof works the same if we start with some \( \beta \) in the right set. This proves (5.12) and therefore (S3).

5.9. Truncated hierarchical B-splines. We will define some Scott-Zhang type operator \( J_\bullet \) similarly as in [SM16] with the help of so-called truncated hierarchical B-splines (THB-splines) introduced in [GJS12]. In this section, we recall their definition and list some basic properties. For a more detailed presentation, we refer to, e.g., [GJS12, SM16].

Let \( \hat{\mathcal{T}}_\circ \) be an arbitrary hierarchical mesh in the parameter domain. For \( k \in \mathbb{N}_0 \), we define the truncation \( \text{trunc}_{\text{loc}}^{k+1}(\hat{\mathcal{T}}_\circ) : \hat{\mathcal{T}}_k \rightarrow \hat{\mathcal{T}}_{k+1} \) as follows:

\[
\text{trunc}_{\text{loc}}^{k+1}(\hat{\mathcal{T}}_\circ) := \sum_{\hat{\beta} \in \hat{\mathcal{B}}^{k+1}} c_\beta \hat{\beta} \text{ for all } \hat{\mathcal{T}}_k = \sum_{\hat{\beta} \in \hat{\mathcal{B}}^{k+1}} c_\beta \hat{\beta} \in \hat{\mathcal{T}}_k \subseteq \hat{\mathcal{T}}_{k+1},
\] (5.13)
i.e., truncation is defined via the (unique) basis representation of \( \hat{\mathcal{T}}_k \in \hat{\mathcal{T}}_k \) with respect \( \hat{\mathcal{B}}^{k+1} \). Recall that \( M_\bullet \in \mathbb{N} \) is the minimal integer such that \( \hat{\Omega}_\bullet^{M_\bullet} = \emptyset \). For all \( \hat{\beta} \in \hat{H}_\bullet \), the corresponding truncated hierarchical B-spline reads

\[
\text{Trunc}_{\text{loc}}(\hat{\beta}) := \text{trunc}_{\text{proj}}^{M_\bullet-1}(\text{trunc}_{\text{proj}}^{M_\bullet-2}(\ldots(\text{trunc}_{\text{proj}}^{\text{level}(\hat{\beta})+1}(\hat{\beta})\ldots)),
\] (5.14)

As the set \( \hat{H}_\bullet \), the set of THB-splines \( \{ \text{Trunc}_{\text{loc}}(\hat{\beta}) : \hat{\beta} \in \hat{H}_\bullet \} \) forms a basis of the space of hierarchical splines \( \hat{H}_\bullet \). In Section 3.1, we mentioned that each basis function in \( \hat{\mathcal{B}}^k \)
is the linear combination of basis functions $\hat{B}^{k+1}$, where the corresponding coefficients are nonnegative; see (3.5). For $\hat{\beta} \in \hat{\mathcal{H}}$, this proves

$$0 \leq \text{Trunc}_{\bullet}(\hat{\beta}) \leq \hat{\beta} \leq 1,$$

and in particular $\text{supp}(\text{Trunc}_{\bullet}(\hat{\beta})) \subseteq \text{supp}(\hat{\beta})$. With this and the fact that the THB-splines are a basis of $\hat{\mathcal{Y}}$. Corollary 3.2 proves

$$\hat{\mathcal{X}} \times \text{span}\{\text{Trunc}_{\bullet}(\hat{\beta}) : \hat{\beta} \in \hat{\mathcal{H}} \land \hat{\beta}|_{\partial \Omega} = 0\},$$

where the set on the right-hand side is even a basis of $\hat{\mathcal{X}}$. The following proposition shows that for an admissible mesh $\hat{T} \in \hat{T}$, the full truncation $\text{Trunc}_{\bullet}$ reduces to $\text{Trunc}_{\bullet}^\text{level}(\hat{\beta}+1)$.

**Proposition 5.6.** Let $\hat{T} \in \hat{T}$ and $\hat{\beta} \in \hat{\mathcal{H}}$. Then, it holds that

$$\text{Trunc}_{\bullet}(\hat{\beta}) = \text{Trunc}_{\bullet}^\text{level}(\hat{\beta}+1).$$

**Proof.** We prove the proposition in three steps.

**Step 1:** Let $k' < k'' \in \mathbb{N}_0$ and $\hat{\beta}' \in \hat{B}^{k'}$ with representation $\hat{\beta}' = \sum_{\beta'' \in B^{k''}} c_{\beta''} \hat{\beta}''$. Let $\hat{\beta}'' \in \hat{B}^{k''}$ such that $c_{\beta''} \neq 0$. Then, local linear independence (with the open set $O = (0, 1)^d \setminus \text{supp}(\hat{\beta}'')$) of $\hat{B}^{k''}$ implies $\text{supp}(\hat{\beta}'') \subseteq \text{supp}(\hat{\beta}')$.

**Step 2:** We prove (5.17). We abbreviate $k = \text{level}(\hat{\beta})$. Let $\hat{\beta} = \sum_{\beta'' \in B^{k+1}} c_{\beta''} \hat{\beta}''$. Let $\hat{\beta}'' \in \hat{B}^{k+1}$ with $\text{supp}(\hat{\beta}'') \supseteq \hat{\mathcal{X}}^{k+1}$ and $c_{\beta''} \neq 0$. By Step 1, this proves $\text{supp}(\hat{\beta}') \subseteq \text{supp}(\hat{\beta})$. For $k'' > k + 1$, we consider the representation

$$\text{trunc}^{k''}(\hat{\beta}') = \sum_{\beta'' \in B^{k''}} c_{\beta''} \hat{\beta}'' \text{, where } \hat{\beta}' = \sum_{\beta'' \in B^{k''}} c_{\beta''} \hat{\beta}''.$$

If $\hat{\beta}'' \in \hat{B}^{k''}$ with $\text{supp}(\hat{\beta}'') \subseteq \hat{\mathcal{X}}^{k''}$, let $\hat{T}'' \in \hat{T}^{k''}$ with $\hat{T}'' \subseteq \text{supp}(\hat{\beta}'')$. (3.10) shows the existence of an element $\hat{T} \in \hat{T}^k$ with $\text{level}(\hat{T}) \geq k''$ such that $\hat{T} \subseteq \hat{T}''$. To see $c_{\beta''} = 0$, we argue by contradiction and assume $c_{\beta''} \neq 0$. By Step 1, this implies $\hat{T} \subseteq \text{supp}(\hat{\beta}'') \subseteq \text{supp}(\hat{\beta}') \subseteq \text{supp}(\hat{\beta})$. Due to (3.15), this contradicts admissibility of $\hat{T}$. This proves $c_{\beta''} = 0$. Overall, we conclude $\text{trunc}^{k''}(\hat{\beta}') = \hat{\beta}'$, and thus $\text{trunc}^{k''}(\text{trunc}^{k+1}(\hat{\beta})) = \text{trunc}^{k+1}(\hat{\beta})$ as well as (5.17).

**Remark 5.7.** Actually, the proposed refinement strategy of Algorithm 3.5 was designed for hierarchical B-splines; see also Proposition 3.3. However, (5.15) implies that Proposition 3.3 holds accordingly for truncated hierarchical B-splines. Moreover, if one applies the refinement strategy of Algorithm 3.5, (5.17) shows that the computation of the truncated hierarchical B-splines greatly simplifies.

**5.10. Verification of (S4)–(S6).** Given $T \in \mathcal{T}$, we are finally able to introduce a suitable Scott-Zhang-type operator $J : H^1_0(\Omega) \to \hat{\mathcal{X}}$ which satisfies (S4)–(S6). To this end, it is sufficient to construct a corresponding operator $\hat{J} : H^1_0(\Omega) \to \hat{\mathcal{X}}$ in the parameter space, and to define

$$J_{\bullet}v := (\hat{J}_{\bullet}(v \circ \gamma)) \circ \gamma^{-1} \text{ for all } v \in H^1_0(\Omega).$$
By regularity (3.28) of \( \gamma \), the properties (S4)–(S6) immediately transfer from the parameter domain \( \hat{\Omega} \) to the physical domain \( \Omega \). Recall that \( \hat{B}^k \cap \hat{B}^{k'} = \emptyset \) for \( k \neq k' \). For \( k \in \mathbb{N}_0 \) and \( \hat{\beta} \in \hat{B}^k \), let \( \hat{T}_{\beta} \in \hat{T}^k \) be an arbitrary but fixed element with \( \hat{T}_{\beta} \subseteq \text{supp}(\hat{\beta}) \). If \( \hat{\beta} \in \hat{\mathcal{H}}_* \), we additionally require\(^2\) \( \hat{T}_{\beta} \in \hat{T}_* \), which is possible due to (3.15). By local linear independence and continuity of \( \hat{B}^k \) (see Section (3.1)), also the restricted basis functions \( \{ \hat{\beta}_{|T_{\beta}} : \hat{\beta} \in \hat{B}^k \cap \hat{\beta}|_{T_{\beta}} \neq 0 \} \) are linearly independent. Hence, the Riesz theorem guarantees the existence and uniqueness of some \( \hat{\beta}^* \in \{ \hat{V}^k |_{T_{\beta}} : \hat{V}^k \in \hat{\mathcal{V}}^k \} \) such that

\[
\int_{T_{\beta}} \hat{\beta}^* \hat{\beta}' \, dt = \delta_{\hat{\beta}^*, \hat{\beta}'} \quad \text{for all } \hat{\beta}' \in \hat{B}^k.
\]  
(5.19)

These dual basis functions \( \hat{\beta}^* \) satisfy the following scaling property.

**Lemma 5.8.** There exists \( C_{\text{dual}} > 0 \) such that for all \( k \in \mathbb{N}_0 \) and all \( \hat{\beta} \in \hat{B}^k \), it holds that

\[
\| \hat{\beta}^* \|_{L^\infty(T_{\beta})} \leq C_{\text{dual}} |T_{\beta}|^{-1}.
\]  
(5.20)

The constant \( C_{\text{dual}} \) depends only on \( d \), \( \hat{T}_0 \) and \( (p_1, \ldots, p_d) \).

**Proof.** Recall that \( \hat{T}_{\beta} \) is a rectangle of the form \( \hat{T}_{\beta}^k = [t_{1,\ell_1}^k, t_{1,\ell_1}^k + 1] \times \cdots \times [t_{d,\ell_d}^k, t_{d,\ell_d}^k + 1] \). We abbreviate \( C := |T_{\beta}|^{1/d}, (a_1, \ldots, a_d) := (t_{1,\ell_1}^k, \ldots, t_{d,\ell_d}^k) \) and define the normalized element \( \hat{T}_{\beta} := (\hat{T}_{\beta} - (a_1, \ldots, a_d))/C \) and the corresponding affine transformation \( \Phi : \hat{T}_{\beta} \to \hat{T}_{\beta} \). We apply the transformation formula to see \( \int_{T_{\beta}} \hat{\beta}^* \hat{\beta}' \, dt = C^d \int_{\hat{T}_{\beta}} (\hat{\beta}^* \circ \Phi)(\hat{\beta} \circ \Phi) \, dt \). Therefore, the Riesz theorem implies that \( \hat{\beta}^* = (\hat{\beta}^* \circ \Phi^{-1})/C^d \), where \( \hat{\beta}^* \) is the unique element in \( \hat{B}^k := \{ \hat{\beta} \circ \Phi : \hat{\beta} \in \hat{B}^k \} \setminus \{0\} \) such that

\[
\int_{T_{\beta}} \hat{\beta}^* \hat{\beta}' \, dt = \delta_{\hat{\beta}^*, \hat{\beta}'} \quad \text{for all } \hat{\beta}' \in \hat{B}^k.
\]

By (3.3), each \( \hat{\beta}' \in \hat{B}^k \) is of the form

\[
\hat{\beta}'(s_1, \ldots, s_d) = \prod_{i=1}^d B(s_i|t_{i,j_i}^k, \ldots, t_{i,j_i}^k + p_i + 1) \text{ with } (s_1, \ldots, s_d) = (s_1, \ldots, s_d) C + (a_1, \ldots, a_d).
\]

We only have to consider \( \hat{\beta}' \) that are supported on \( \hat{T}_{\beta} \). As the support of any \( B(\cdot|t_{i,j_i}^k, \ldots, t_{i,j_i}^k + p_i + 1) \) is just \( [t_{i,j_i}^k, t_{i,j_i}^k + p_i + 1] \), it is sufficient to consider \( j_i = \ell_i - p_i, \ldots, \ell_i \). By the definition of B-splines, one immediately sees that an affine transformation in the parameter domain can just be passed to the knots, i.e.,

\[
B(s_i|t_{i,j_i}^k, \ldots, t_{i,j_i}^k + p_i + 1) = B(s_i|t_{i,j_i}^k - a_i/C, \ldots, (t_{i,j_i}^k + p_i + 1 - a_d)/C).
\]

Altogether, we see that \( \hat{\beta}^* \) depends only on the knots

\[
\left( \frac{t_{i,j_i}^k - a_i}{C}, \ldots, \frac{t_{i,j_i}^k + p_i + 1 - a_d}{C} : i = 1, \ldots, d, j_i = \ell_i - p_i, \ldots, \ell_i \right).
\]

\(^2\)Therefore, the elements \( \hat{T}_{\beta} \) depend additionally on the considered mesh \( \hat{T}_* \).
Since we only use global dyadic bisection between two consecutive levels, we see that these knots depend only on \( d, \hat{T}_0 \) and \( (p_1, \ldots, p_d) \) but not on the level \( k \). This shows \( \| \hat{\beta} \|_{L^\infty(T_{\beta})} \lesssim 1 \), where the hidden constant depends only on \( d, \hat{T}_0 \) and \( (p_1, \ldots, p_d) \).

We adopt the approach of [SM16]. For \( \hat{\nu} \in L^2(\hat{\Omega}) \), we abbreviate \( (\hat{\beta}^\ast, \hat{\nu}) := \int_{\hat{T}_{\beta}} \hat{\beta}^\ast \hat{\nu} \, dt \) and define

\[
\hat{J}^k : L^2(\hat{\Omega}) \to \hat{Y}^k, \quad \hat{J}^k \hat{\nu} := \sum_{\hat{\beta} \in B_k} \langle \hat{\beta}^\ast, \hat{\nu} \rangle \hat{\beta},
\]

\[
\hat{J}_* : L^2(\hat{\Omega}) \to \hat{X}_*, \quad \hat{J}_* \hat{\nu} := \sum_{k=0}^{M_\ast-1} \sum_{\hat{\beta} \in \hat{H}_\ast \cap \beta_{\hat{\Omega}} = 0 \atop \text{level}(\hat{\beta}) = k} \langle \hat{\beta}^\ast, \hat{\nu} \rangle \text{Trunc}_* (\hat{\beta}).
\]

Before we prove the properties (S4)–(S6), we collect some basic properties of \( \hat{J}_* \).

**Lemma 5.9.** Let \( \hat{T}_* \in \hat{T} \). Then, \( \hat{J}_* \) is a projection, i.e.,

\[
\hat{J}_* \hat{\nu} = \hat{\nu} \quad \text{for all } \hat{\nu} \in \hat{X}_*.
\]

Moreover, \( \hat{J}_* \) is locally \( L^2 \)-stable, i.e., there exists \( C_J > 0 \) such that for all \( \hat{T} \in \hat{T}_* \)

\[
\| \hat{J}_* \hat{\nu} \|_{L^2(\hat{T})} \leq C_J \| \hat{\nu} \|_{L^2(\hat{\Omega}^{(p+1)}(\hat{T}))} \quad \text{for all } \hat{\nu} \in L^2(\hat{\Omega}).
\]

The constant \( C_J \) depends only on \( d, \hat{T}_0 \), and \( (p_1, \ldots, p_d) \).

**Proof.** We prove the lemma in three steps.

**Step 1:** The projection property (5.23) can be proved as in [SM16, Theorem 4]. There, a corresponding projection onto \( \hat{\nu}_* \) instead of \( \hat{\nu}_* = \hat{\nu}_* \cap H_0^1(\hat{\Omega}) \) is considered. However, with (5.16), the proof works exactly the same.

**Step 2:** We prove (5.24). The triangle inequality proves that

\[
\| \hat{J}_* \hat{\nu} \|_{L^2(\hat{T})} \leq \sum_{k=0}^{M_\ast-1} \sum_{\hat{\beta} \in \hat{H}_\ast \cap \beta_{\hat{\Omega}} = 0 \atop \text{supp}(\hat{\beta}) \cap \hat{T} > 0 \atop \text{level}(\hat{\beta}) = k} \| \hat{\beta}^\ast \|_{L^2(\hat{T}_{\beta})} \| \hat{\nu} \|_{L^2(\hat{T}_{\beta})} \| \text{Trunc}_* (\hat{\beta}) \|_{L^2(\hat{T})}.
\]

By Remark 3.4, it holds that \( \text{supp}(\hat{\beta}) \subseteq \pi^{2(p+1)}(\hat{T}) \) if \( \hat{\beta} \in \hat{H}_\ast \) with \( |\text{supp}(\hat{\beta}) \cap \hat{T}| > 0 \). Therefore, we obtain that

\[
\| \hat{J}_* \hat{\nu} \|_{L^2(\hat{T})} \leq \| \hat{\nu} \|_{L^2(\pi^{2(p+1)}(\hat{T}))} \sum_{\hat{\beta} \in \hat{H}_\ast \atop \text{supp}(\hat{\beta}) \subseteq \pi^{2(p+1)}(\hat{T})} \| \hat{\beta}^\ast \|_{L^2(\hat{T}_{\beta})} \| \text{Trunc}_* (\hat{\beta}) \|_{L^2(\hat{T})}.
\]

We consider the set \( \{ \hat{\beta} \in \hat{H}_\ast : \text{supp}(\hat{\beta}) \subseteq \pi^{2(p+1)}(\hat{T}) \} \). Since the support of each basis function in \( \hat{H}_\ast \) consists of elements in \( \hat{T}_\ast \), see (3.14), this set is a subset of \( \{ \hat{\beta} \in \hat{H}_\ast : \exists \hat{T}' \in \Pi^{2(p+1)}(\hat{T}) \text{ with } \hat{T}' \subseteq \text{supp}(\hat{\beta}) \} \). By (M2) and Proposition 3.3, the cardinality of the latter
set is bounded by a constant \( C > 0 \) that depends only on \( d \) and \((p_1, \ldots, p_d)\). With (M1), (5.15), and (5.20), we see that for \( \widehat{\beta} \in \widehat{\mathcal{H}}_\bullet \) with \( \text{supp}(\widehat{\beta}) \subseteq \pi_{x^{(p+1)}}(\widehat{T}) \), it holds that

\[
\|\widehat{\beta}^*\|_{L^2(\widehat{T})} \|\text{Trunc}_\bullet(\widehat{\beta})\|_{L^2(\widehat{T})} \leq |\widehat{T}|^{1/2} |\widehat{T}|^{1/2}\|\widehat{\beta}^*\|_{L^\infty(\widehat{T})} \lesssim 1.
\]

The hidden constant depends only on \( d \), \( \widehat{T}_0 \), and \((p_1, \ldots, p_d)\). 

We prove (S4) in the parameter domain. Let \( \widehat{T} \in \widehat{T}_\bullet, \widehat{v} \in H^1_0(\widehat{\Omega}) \), and \( \widehat{V}_\bullet \in \widehat{X}_\bullet \) such that \( \widehat{v}|_{\pi_{x^{(p+1)}}(\widehat{T})} = \widehat{V}_\bullet|_{\pi_{x^{(p+1)}}(\widehat{T})} \) with \( k_{\text{proj}} := 2(p+1) \). Remark 3.4 shows that for \( \widehat{\beta} \in \widehat{\mathcal{H}}_\bullet \) with \( |\text{supp}(\widehat{\beta}) \cap \widehat{T}| > 0 \), it holds that \( \text{supp}(\widehat{\beta}) \subseteq \pi_{x^{(p+1)}}(\widehat{T}) \). With this, (5.15), and the projection property (5.23) of \( \widehat{J}_\bullet \), we conclude that

\[
(\widehat{J}_\bullet \widehat{v})|_{\widehat{T}} = \sum_{k=0}^{M_\bullet - 1} \sum_{\substack{\widehat{\beta} \in \widehat{\mathcal{H}}_\bullet, \text{level}(\widehat{\beta}) = k \\text{level}(\widehat{\beta}) = k}} \langle \widehat{\beta}^*, \widehat{v} \rangle \text{Trunc}_\bullet(\widehat{\beta})|_{\widehat{T}}
\]

\[
= \sum_{k=0}^{M_\bullet - 1} \sum_{\substack{\widehat{\beta} \in \widehat{\mathcal{H}}_\bullet, \text{level}(\widehat{\beta}) = k \\text{supp}(\widehat{\beta}) \subseteq \pi_{x^{(p+1)}}(\widehat{T})}} \langle \widehat{\beta}^*, \widehat{V}_\bullet \rangle \text{Trunc}_\bullet(\widehat{\beta})|_{\widehat{T}} = (\widehat{J}_\bullet \widehat{V}_\bullet)|_{\widehat{T}} = \widehat{V}_\bullet|_{\widehat{T}} = \widehat{v}|_{\widehat{T}}.
\]

Next, we prove (S5). Let \( \widehat{T} \in \widehat{T}_\bullet, \widehat{v} \in H^1_0(\widehat{\Omega}) \), and \( \widehat{V}_\bullet \in \widehat{X}_\bullet \). By (5.23)–(5.24), it holds that

\[
\|(1 - \widehat{J}_\bullet) \widehat{v}\|_{L^2(\widehat{T})} \overset{(5.23)}{=} \|\widehat{v} - \widehat{V}_\bullet\|_{L^2(\widehat{T})} + \|\widehat{J}_\bullet(\widehat{v} - \widehat{V}_\bullet)\|_{L^2(\widehat{T})} \overset{(5.24)}{\lesssim} \|\widehat{v} - \widehat{V}_\bullet\|_{L^2(\pi_{x^{(p+1)}}(\widehat{T}))}.
\]

To proceed, we distinguish between two cases, first, \( \pi_{x^{(p+1)}}(\widehat{T}) \cap \partial \widehat{\Omega} = \emptyset \) and, second, \( \pi_{x^{(p+1)}}(\widehat{T}) \cap \partial \widehat{\Omega} \neq \emptyset \), i.e., if \( \widehat{T} \) is far away from the boundary or not. Note that these cases are equivalent to \( |\pi_{x^{(p+1)}}(\widehat{T}) \cap \partial \widehat{\Omega}| = 0 \) resp. \( |\pi_{x^{(p+1)}}(\widehat{T}) \cap \partial \widehat{\Omega}| > 0 \), since the elements in the parameter domain are rectangular.

In the first case, we proceed as follows: (3.22) especially proves \( 1 \in \widehat{V}_\bullet \) with \( 1 = \sum_{\substack{\widehat{\beta} \in \widehat{\mathcal{H}}_\bullet \\text{level}(\widehat{\beta}) = 0 \\text{level}(\widehat{\beta}) = 0}} c_{\widehat{\beta}} \widehat{\beta} \) on \( \widehat{\Omega} \). With Remark 3.4, we see that \( |\text{supp}(\widehat{\beta}) \cap \pi_{x^{(p+1)}}(\widehat{T})| > 0 \) implies \( \text{supp}(\widehat{\beta}) \subseteq \pi_{x^{(p+1)}}(\widehat{T}) \). Therefore, the restriction satisfies

\[
1 = \sum_{\substack{\widehat{\beta} \in \widehat{\mathcal{H}}_\bullet \\text{level}(\widehat{\beta}) = 0}} c_{\widehat{\beta}} \widehat{\beta} \big|_{\pi_{x^{(p+1)}}(\widehat{T})} = \sum_{\substack{\widehat{\beta} \in \widehat{\mathcal{H}}_\bullet \\text{level}(\widehat{\beta}) = 0 \\text{supp}(\widehat{\beta}) \cap \pi_{x^{(p+1)}}(\widehat{T}) > 0}} c_{\widehat{\beta}} \widehat{\beta} \big|_{\pi_{x^{(p+1)}}(\widehat{T})} = \sum_{\substack{\widehat{\beta} \in \widehat{\mathcal{H}}_\bullet \\text{level}(\widehat{\beta}) = 0 \\text{supp}(\widehat{\beta}) \cap \pi_{x^{(p+1)}}(\widehat{T}) > 0}} c_{\widehat{\beta}} \widehat{\beta} \big|_{\pi_{x^{(p+1)}}(\widehat{T})}.
\]

We define

\[
\widehat{V}_\bullet := \widehat{v}|_{\pi_{x^{(p+1)}}(\widehat{T})} \sum_{\substack{\widehat{\beta} \in \widehat{\mathcal{H}}_\bullet \\text{level}(\widehat{\beta}) > 0 \\text{supp}(\widehat{\beta}) \subset \pi_{x^{(p+1)}}(\widehat{T})}} c_{\widehat{\beta}} \widehat{\beta}, \quad \text{where} \quad \widehat{v}|_{\pi_{x^{(p+1)}}(\widehat{T})} := \big|\pi_{x^{(p+1)}}(\widehat{T})\big|^{-1} \int_{\pi_{x^{(p+1)}}(\widehat{T})} \widehat{v} \ dt.
\]

In the second case, we set \( \widehat{V}_\bullet := 0 \). For the first case, the Poincaré inequality combined with admissibility concludes the proof, whereas we use the Friedrichs inequality combined with
affine transformation with the shape of \( \pi \) for different configurations for the set \( p \) and \( (p, q) \), the number of different patch shapes is bounded itself by a constant which again depends only on \( p \) and \( (p_1, \ldots, p_d) \).

Finally, we prove (S6). Let again \( \hat{T} \in \mathcal{T}_h \), \( \hat{\gamma} \) that are constant on \( \hat{T} \), the projection property (5.23) implies that

\[
\| \nabla \hat{\gamma} \|_{L^2(\hat{T})} \stackrel{(5.23)}{=} \| \nabla \hat{\gamma}(\hat{\gamma} - \hat{\gamma} \hat{\gamma}) \|_{L^2(\hat{T})} \leq |\hat{T}|^{-1/d} \| \hat{\gamma}(v - \hat{\gamma} \hat{\gamma}) \|_{L^2(\hat{T})} \leq |\hat{T}|^{-1/d} \| v - \hat{\gamma} \hat{\gamma} \|_{L^2(\hat{T})}.
\]

Arguing as before and using (5.25), we conclude the proof.

### 5.11. Verification of (O1)

The inverse estimate (O1) follows from a standard scaling argument together with the regularity (3.28) of \( \gamma \) and the Poincaré-Friedrichs inequality on \( \gamma^{-1}(T) \). The constant \( C'_{\text{inv}} \) depends only on \( d, C_\gamma, \hat{T}_0 \), and \( (q_1, \ldots, q_d) \).

### 5.12. Verification of (O2)-(O4)

This section adapts [NV12, Section 3.4]. Let \( W \in \mathcal{P}(\Omega), \mathcal{T}_h \in \mathcal{T}, \) and \( T, T' \in \mathcal{T}_h \) with \((d - 1)\)-dimensional intersection \( E := T \cap T' \). We set \( \hat{W} := W \circ \gamma, \hat{T} := \gamma^{-1}(T), \hat{T}' := \gamma^{-1}(T'), \) and \( \hat{E} := \gamma^{-1}(E) \). Let \( \gamma_{T'} : \mathbb{R}^d \to \mathbb{R}^d \) be the affine transformation with \( \hat{T} := \gamma_{T'}^{-1}(\hat{T}) = [0, 1]^d \). Due to admissibility of \( \mathcal{T}_h \), the number of different configurations for the set \( \hat{T}' := \gamma_{T'}^{-1}(\hat{T}) \) is uniformly bounded by a constant that depends only on \( d \) and \( \hat{T}_0 \). We fix some cut-off function \( \varphi \in H^1_0(\hat{T} \cup \hat{T}') \) with \( \varphi > 0 \) almost everywhere on \( \hat{E} := \gamma_{T'}^{-1}(\hat{E}) \). We define \( \varphi := \hat{\varphi} \circ \gamma_{T'}^{-1} \circ \gamma^{-1} \), and \( \hat{W} := W \circ \gamma \circ \gamma_{T'} \). Due to the finite dimension of the polynomial space \( \hat{\mathcal{P}}(\hat{T} \cup \hat{T}') := \{ \hat{W}' \circ \gamma^{-1} \circ \gamma^{-1} : \hat{W}' \in \mathcal{P}(\hat{E}) \} \), there exists \( \hat{W}' \in \hat{\mathcal{P}}(\hat{T} \cup \hat{T}') \) with \( \hat{W}|_{\hat{E}} = \hat{W}'|_{\hat{E}} \) and \( \| \hat{W}' \varphi \|_{L^2(\hat{T} \cup \hat{T}')} \leq \| \hat{W} \|_{L^2(\hat{E})} \). Finally, we set \( \hat{W}' := \hat{W}' \circ \gamma_{T'}^{-1} \circ \gamma^{-1} \), and \( \hat{J}_{\mathcal{T}_h} := \hat{W}' \varphi \). Standard scaling arguments prove that (O2)-(O4) are satisfied, where the constants depend only on \( d, C_\gamma, \hat{T}_0 \), and \( (q_1, \ldots, q_d) \).

### 6. Numerical examples

In this section, we apply our adaptive algorithm to the 2D Poisson model problem

\[
-\Delta u = f \quad \text{in } \Omega, \\
\quad u = 0 \quad \text{on } \partial \Omega.
\]

on different NURBS surfaces \( \Omega \subset \mathbb{R}^2 \). On the parameter domain \( \hat{\Omega} = (0, 1)^d \), let \( p_1^I, p_2^I \geq 1 \) be fixed polynomial degrees and \( \hat{\mathcal{K}}^{\gamma} \) be an arbitrary fixed 2-dimensional vector of \( p_i^I \)-open knot vectors with multiplicity smaller or equal to \( p_i^I \) for the interior knots, i.e.,

\[
\hat{\mathcal{K}}^{\gamma} = (\hat{\mathcal{K}}_1^{\gamma}, \hat{\mathcal{K}}_2^{\gamma}).
\]
see also Section 3.1. Let $\hat{B}^\gamma$ be the corresponding tensor-product B-spline basis, i.e.,

$$\hat{B}^\gamma = \{\hat{B}_{j_1,j_2} : j_i \in \{0, \ldots, N_i^\gamma - 1\}\}, \tag{6.3}$$

where $\hat{B}_{j_1,j_2}$ is defined as in (3.3). For given points in the plane $c_{j_1,j_2}^\gamma \in \mathbb{R}^2$ and positive weights $w_{j_1,j_2}^\gamma > 0$ for $j_i \in \{0, \ldots, N_i^\gamma - 1\}$, a NURBS-surface $\Omega$ is defined via a parametrization $\gamma : \hat{\Omega} \to \Omega$ of the form

$$\gamma(s_1, s_2) = \sum_{j_i \in \{0, \ldots, N_i^\gamma - 1\}} w_{j_1,j_2}^\gamma c_{j_1,j_2}^\gamma \hat{B}_{j_1,j_2}^\gamma(s_1, s_2). \tag{6.4}$$

### 6.1. Square

In the first experiment, we consider the unit square $\Omega = (0,1)^2$, where we choose $p_1^\gamma, p_2^\gamma = 1$ and $\hat{K}_1^\gamma = \hat{K}_2^\gamma = (0,0,1,1)$. We set the control points

$$c_{1,1}^\gamma = (0,0), \quad c_{2,1}^\gamma = (1,0), \quad c_{1,2}^\gamma = (0,1), \quad c_{2,2}^\gamma = (1,1)$$

and all weights equal to 1. We choose $\mathcal{f}$ as in [BG16a, Example 7.4] such that the exact solution of (6.1) is given by $u(x_1, x_2) = x_1^2 x_2 + x_2^2$, which implies $u \in H^2(\Omega) \setminus H^3(\Omega)$. To create the initial ansatz space with spline degrees $p_1 = p_2 \in \{2,3,4\}$, we choose the initial knot vectors $\hat{K}_1^0 = \hat{K}_2^0 = (0,0,1,1,\ldots,1)$, where the multiplicity of 0 and 1 is $p_1 + 1 = p_2 + 1$. We apply Algorithm 2.3 for the spline degrees 2, 3, and 4 with uniform ($\theta = 1$) and adaptive ($\theta = 0.5$) mesh-refinement. Some adaptively generated hierarchical meshes are shown in Figure 6.1. In Figure 6.2, we plot the energy error $\|\nabla u - \nabla U_{\ell}\|_{L^2(\Omega)}$ as well as the error estimator $\eta_{\ell}$ against the number of elements $\#\mathcal{T}_\ell$. Due to the regularity of $u$, we obtain a suboptimal convergence rate for $p_1 = p_2 > 2$. The adaptive strategy on the other hand recovers the optimal convergence of the energy error and the error estimator, i.e., $\mathcal{O}((\#\mathcal{T}_\ell)^{-p/2})$.

### 6.2. L-shape

To obtain the L-shaped domain $\Omega = (0,1)^2 \setminus ([0,0.5] \times [0,0.5])$, we choose $p_1^\gamma, p_2^\gamma = 1$ and $\hat{K}_1^\gamma = \hat{K}_2^\gamma = (0,0,1,1,1)$. Moreover, we choose the control points

$$c_{1,1}^\gamma = (0,0.5), \quad c_{2,1}^\gamma = (0.5,0.5), \quad c_{3,1}^\gamma = (0.5,0),$$

$$c_{1,2}^\gamma = (0,1), \quad c_{2,2}^\gamma = (1,1), \quad c_{3,2}^\gamma = (1,0),$$

and set all weights equal to 1. We consider the Poisson problem (6.1) with $f = 1$. For the initial ansatz space with spline degrees $p_1 = p_2 \in \{2,3,4\}$, we choose the initial knot vectors $\hat{K}_1^0 = (0,\ldots,0,0.5,\ldots,0.5,1,\ldots,1)$ and $\hat{K}_2^0 = (0,\ldots,0,1,\ldots,1)$, where the multiplicity of 0 and 1 is $p_1 + 1 = p_2 + 1$, whereas the multiplicity of 0.5 is $p_1$. As a consequence, the ansatz functions are only continuous at $\{0.5\} \times [0,1]$, but not continuously differentiable. We compare uniform ($\theta = 1$) and adaptive ($\theta = 0.4$) mesh-refinement. In Figure 6.3, one can see some adaptively generated hierarchical meshes. In Figure 6.4 we plot again the energy error $\|\nabla u - \nabla U_{\ell}\|_{L^2(\Omega)}$ and the error estimator $\eta_{\ell}$ against the number of elements $\#\mathcal{T}_\ell$. The uniform approach leads to a suboptimal convergence rate, since the reentrant corner at $(0.5,0.5)$ causes a generic singularity of the solution $u$. However, the adaptive strategy recovers the optimal convergence rate $\mathcal{O}((\#\mathcal{T}_\ell)^{-p/2})$.  

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Figure 6.1. Hierarchical meshes generated by Algorithm 2.3 with $\theta = 0.5$ for the problem of Section 6.1 for $p_1 = p_2 = 4$.

6.3. Quarter ring. We construct the NURBS-surface given in polar coordinates $\Omega = \{(r, \varphi) \mid 0.5 < r < 1 \land 0 < \varphi < \pi/2\}$ by choosing $p_{1,2}^\gamma = 2$ and $\hat{K}_1^\gamma = (0, 0, 0, 1, 1, 1)$, $\hat{K}_2^\gamma = (0, 0, 1, 1)$. Moreover, we choose the control points $c_{1,1}^\gamma = (0, 0.5)$, $c_{2,1}^\gamma = (0.5, 0.5)$, $c_{3,1}^\gamma = (0.5, 0)$, $c_{1,2}^\gamma = (0, 1)$, $c_{2,2}^\gamma = (1, 1)$, $c_{3,2}^\gamma = (1, 0)$, and set all weights, except $w_{2,1}^\gamma$, $w_{2,2}^\gamma := 1/\sqrt{2}$, equal to 1. As right-hand side in (6.1), we choose the indicator function $f = \chi_S$, where $S = \{(r, \varphi) \mid 0.5 < r < 0.75 \land 0 < \varphi < \pi/4\} = \gamma([0.5,1] \times [0,0.5])$. For the initial ansatz space with spline degrees $p_1 = p_2 \in \{2, 3, 4\}$, we choose the initial knot vectors $\hat{K}_0^\gamma = \hat{K}_0^\gamma = (0, \ldots, 0, 0.5, \ldots, 0.5, 1, \ldots, 1)$, where the multiplicity of 0 and 1 is $p_1 + 1 = p_2 + 1$, whereas the multiplicity of 0.5 is $p_1 = p_2$. As a consequence the ansatz functions are only continuous at $\{0.5\} \times [0, 1]$ and $[0, 1] \times \{0.5\}$. We compare uniform ($\theta = 1$) and adaptive ($\theta = 0.8$) mesh-refinement. Some adaptively generated hierarchical meshes are shown in Figure 6.5. In Figure 6.6, we plot the energy error $\|\nabla u - \nabla U\|_{L^2(\Omega)}$ and the error estimator $\eta_\ell$ against the number of elements $\#T_\ell$. For $p_1 = p_2 > 2$, uniform refinement leads to suboptimal convergence rate. However, the adaptive approach regains the optimal rate $O((\#T_\ell)^{-p/2})$.

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References


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**Figure 6.2.** Energy error $\| \nabla u - \nabla U_\ell \|_{L^2(\Omega)}$ and error estimator $\eta_\ell$ for the problem of Section 6.1 on the unit square for uniform ($\theta = 1$) and adaptive ($\theta = 0.5$) mesh-refinement and different spline orders $p_1 = p_2 = p \in \{2, 3, 4\}$, where adaptivity always regains the respective optimal convergence rate.

$\mathcal{T}_6$ $\mathcal{T}_9$ $\mathcal{T}_{11}$

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**Figure 6.3.** Hierarchical meshes generated by Algorithm 2.3 with $\theta = 0.4$ for the problem of Section 6.2 for $p_1 = p_2 = 2$.

$\mathcal{T}_6$ $\mathcal{T}_9$ $\mathcal{T}_{11}$

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**Figure 6.4.** Energy error $\| \nabla u - \nabla U_\ell \|_{L^2(\Omega)}$ and error estimator $\eta_\ell$ for the problem of Section 6.3 on the L-shape for uniform ($\theta = 1$) and adaptive ($\theta = 0.5$) mesh-refinement and different spline orders $p_1 = p_2 = p \in \{2, 3, 4\}$, where adaptivity always regains the respective optimal convergence rate.

34
$T_4 = 265$

$T_5 = 718$

$T_6 = 1777$

**Figure 6.5.** Hierarchical meshes generated by Algorithm 2.3 with $\theta = 0.5$ for the problem of Section 6.3 for $p_1 = p_2 = 3$.

**Figure 6.6.** Energy error $\|\nabla u - \nabla U_\ell\|_{L^2(\Omega)}$ and error estimator $\eta_\ell$ for the problem of Section 6.3 on the quarter ring for uniform ($\theta = 1$) and adaptive ($\theta = 0.8$) mesh-refinement and different spline orders $p_1 = p_2 = p \in \{2, 3, 4\}$, where adaptivity always regains the respective optimal convergence rate.

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