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M. Gualdani and N. Zamponi

Institute for Analysis and Scientific Computing —  
Vienna University of Technology — TU Wien  
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Institute for Analysis and Scientific Computing  
Vienna University of Technology  
Wiedner Hauptstraße 8–10  
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**E-Mail:** [admin@asc.tuwien.ac.at](mailto:admin@asc.tuwien.ac.at)  
**WWW:** <http://www.asc.tuwien.ac.at>  
**FAX:** +43-1-58801-10196

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# SPECTRAL GAP AND EXPONENTIAL CONVERGENCE TO EQUILIBRIUM FOR A MULTI-SPECIES LANDAU SYSTEM

MARIA GUALDANI AND NICOLA ZAMPONI

ABSTRACT. In this paper we prove new constructive coercivity estimates and convergence to equilibrium for a spatially non-homogeneous system of Landau equations with soft potentials. We show that the nonlinear collision operator conserves each species' mass, total momentum, total energy and that the Boltzmann entropy is nonincreasing along solutions of the system. The entropy decay vanishes if and only if the Boltzmann distributions of the single species are Maxwellians with the same momentum and energy. A linearization of the collision operator is computed, which has the same conservation properties as its nonlinear counterpart. We show that the linearized system dissipates a quadratic entropy, and prove existence of spectral gap and exponential decay of the solution towards the global equilibrium. As a consequence, convergence of smooth solutions of the nonlinear problem toward the unique global equilibrium is shown, provided the initial data are sufficiently close to the equilibrium. Our strategy is based on new spectral gap estimates and uses a strategy similar to [12] based on an hypocoercivity method developed by Mouhot and Neumann in [28].

## 1. INTRODUCTION

This manuscript is concerned with the Cauchy problem for a system of spatially non-homogeneous Landau equations describing collisions in an ideal plasma mixture. The mixture is constituted by  $N \geq 2$  species and each species  $i = 1, \dots, N$  has mass  $m_i$  and is described by a density function  $F_i(x, p, t)$  defined in the phase-space of position and momentum. The vector  $F := (F_1, \dots, F_N)$  is said to be a solution to the multi-species Landau system if each  $F_i$  satisfies

$$\begin{cases} \partial_t F_i + \frac{p}{m_i} \cdot \nabla_x F_i = \sum_{j=1}^N Q_{ij}(F_i, F_j), \\ F(x, p, 0) = F_{\text{in}}(x, p), \end{cases} \quad (1.1)$$

with  $(x, p, t) \in \mathbb{T}^3 \times \mathbb{R}^3 \times \mathbb{R}_+$ . The operator  $Q_{ij}$  is the quadratic Landau collision operator defined as

$$Q_{ij}(F_i, F_j) := \operatorname{div}_p \int_{\mathbb{R}^3} A^{(ij)} \left[ \frac{p}{m_i} - \frac{p'}{m_j} \right] (F_j' \nabla F_i - F_i \nabla F_j') dp'. \quad (1.2)$$

Here we adopt the shortened notation  $F \equiv F(x, p, t)$ ,  $F' \equiv F(x, p', t)$ . The term  $A^{(ij)}[z] = \{a_{ks}^{(ij)}(z)\}$  denotes a positive and symmetric matrix with real-valued entries defined as:

$$A^{(ij)}[z] := C^{(i,j)} \left( \operatorname{Id} - \frac{z \otimes z}{|z|^2} \right) \varphi(|z|), \quad z \neq 0, \quad C^{(i,j)} > 0,$$

which acts as the projection operator onto the space orthogonal to the vector  $z$ . The function  $\varphi(|z|)$  is a scalar valued function determined from the original Boltzmann kernel describing how

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particles interact. If the interaction strength between particles at a distance  $r$  is proportional to  $r^{1-s}$ , then

$$\varphi(|z|) := |z|^{\gamma+2}, \quad \gamma = \frac{(s-5)}{(s-1)}. \quad (1.3)$$

The constant  $C^{(i,j)} > 0$  is positive and symmetric in  $i, j$ , and is proportional to the reduced mass of the system  $m_i m_j / (m_i + m_j)$ . We refer to [25, Chapter 4] for a more accurate derivation and discussion of (1.1). The original Landau system with Coulomb interactions correspond to  $\gamma = -3$ .

The purpose of this paper is to study the spectral gap properties of the linearized operator and to show exponential convergence towards the equilibrium as time grows. We assume throughout this manuscript that  $\gamma \in [-2, 1]$ .

Let us summarize briefly the state of the art concerning the Cauchy problem (1.1) for the mono-species case.

In the homogeneous setting, the cases of Maxwell molecules  $\gamma = 0$  and hard potentials  $\gamma \in (0, 1]$  have been well understood: existence and uniqueness of smooth regular solution and convergence towards the unique equilibrium state have been analyzed in several papers, see [5, 10, 11, 15, 16, 26, 35]. For the spatially non-homogeneous case we refer to Alexandre and Villani [2] for existence of renormalized solutions, to Desvilletle-Villani [17] for *conditional almost exponential* convergence towards equilibrium and to a recent work by Carrapatoso, Tristani and Wu [8] for exponential decay towards equilibrium when initial data are close enough to equilibrium.

The case of soft potentials has been proven to be harder. For moderately soft-potentials  $\gamma \in (-2, 0)$  existence and uniqueness of spatially homogeneous solutions have been proven by Fournier and Guerin [19] and by Guerin [23] using a probabilistic approach, as well as by Wu [36] and by Alexandre, Liao and Lin [1]. Carrapatoso, Tristani and Wu [8] recently showed exponential decay estimates for the linearized semigroup and constructed solutions in a close-to-equilibrium regime to the non-linear inhomogeneous equation. The proof in [8] is based on an abstract method developed by the first author and collaborators in [20].

Global well-posedness theory is still missing for the Coulomb case  $\gamma = -3$ . For the homogeneous setting, Arsenev-Peskov [3] showed existence of weak solutions, uniqueness was later shown by Fournier [18]. Villani [33] proved existence of a new class of solutions, the so called H-solutions, which are defined via the  $L^1$ - bound in time of the entropy production. Recently Alexander, Liao and Lin [1] have shown existence of weak solutions in some weighted  $L^2$ -space under smallness assumption on initial data. Desvillettes [14] has shown that the H-solutions are indeed weak-solutions since they belong to some weighted  $L_t^1 L^p(\mathbb{R}^3)$  space and Carrapatoso, Desvillettes and He [7] have proved time convergence to the associate equilibrium at some explicitly computable rate. For the inhomogeneous setting, Guo [24] and Strain, Guo [30, 31] developed an existence and convergence towards equilibrium theory based on energy methods for initial data close in some Sobolev norm to the equilibrium state. Recently the set of initial data for which this theory is valid has been improved by Carrapatoso and Mischler [9] via a linearization method.

Recently the first author and Guillen have shown, for the Coulomb case, global in time existence of classical solution for a modified *isotropic* homogeneous Landau equation

$$F_t = \operatorname{div}(a[F]\nabla F - F\nabla a[F]),$$

in the case of radially symmetric (but no smallness assumptions!) initial data [22]. Moreover, using the theory of  $A_p$  weights, they showed that solutions to the original Landau equations with general initial data for  $\gamma > -2$  have an instantaneous regularization which does not deteriorate as time increases, with bounds that only depend on the physical quantities, mass, momentum and energy [21].

We believe that this is the first work that concerns system (1.1) and its linearized version. The aim of this work is to extend the spectral analysis valid for the mono-species operator to the multi-species operator with different particles' mass. From a different prospective, the second author and collaborators have recently studied a system of Boltzmann equations for mixtures of mono-atomic particles with same mass in the case of hard and Maxwellian potentials [12]: the authors show an explicit spectral-gap estimate for the linearized collision operator and prove the exponential decay of the solutions towards the global equilibrium by generalizing the hypocoercivity method developed by Mouhot and Neumann in [28] for the mono-species case to the multi-species case.

**1.1. Main results.** The main goal of this paper is to give a constructive proof of exponential decay rate for solutions to the linear system

$$\begin{cases} \partial_t f_i + \frac{p}{m_i} \cdot \nabla_x f_i &= \sum_{j=1}^N L_{i,j}(f), \quad i = 1, \dots, N, \\ f(x, p, 0) &= f_{\text{in}}(x, p), \end{cases} \quad (1.4)$$

with

$$\begin{aligned} L_{ij}(f_i, f_j) &:= \frac{1}{\sqrt{M_i}} \left( Q_{ij}(M_i + \sqrt{M_i} f_i, M_j) + Q_{ij}(M_i, M_j + \sqrt{M_j} f_j) \right) \\ &= \frac{1}{\sqrt{M_i}} \operatorname{div}_p \int \sqrt{M_i M_j'} A^{(ij)} \left[ \frac{p}{m_i} - \frac{p'}{m_j} \right] \\ &\quad \cdot \left( \sqrt{M_j'} \nabla f_i - \sqrt{M_i} \nabla f_j' - f_i \nabla \sqrt{M_j'} + f_j' \nabla \sqrt{M_i} \right) dp', \end{aligned} \quad (1.5)$$

obtained from (1.1) via the perturbative expansion  $F_i = M_i + \sqrt{M_i} f_i$ , with  $M_i$  the Maxwellian equilibrium of the  $i^{\text{th}}$  species

$$M_i(p) := \frac{\rho_i}{(2\pi m_i k_B T)^{3/2}} e^{-\frac{1}{2} \frac{|p|^2}{m_i k_B T}}.$$

We will show that any solution to (1.5) converges exponentially fast to the global equilibrium. The rate of decay is computed explicitly, following an approach already used by the second author and collaborators in [12], which is based upon an abstract method by Mouhot and Neumann [28].

The starting point is the existence of spectral gap for the mono-species linearized collision operator. By exploiting the symmetry properties of the operator we are able to bound the cross terms by relating them with the differences of momentum and energy. Hence a spectral gap for the multi-species linearized operator follows. The hypocoercivity method by Mouhot and Neumann [28] yields convergence to global equilibrium for the solution to the in-homogeneous linearized system.

Define with  $L := (L_1, L_2, \dots, L_N)$  the vector with components  $L_i = \sum_{j=1}^N L_{i,j}$  with  $L_{i,j}$  as in (1.5), and by  $T := (T_1, T_2, \dots, T_N)$  the transport operator,  $T_i f = \frac{p}{m_i} \cdot \nabla_x f_i$ . We also denote by

$\Gamma_i(f_i, f_j)$  the quadratic nonlinear term

$$\Gamma_i(f_i, f_j) := \frac{1}{\sqrt{M_i}} \sum_{j=1}^N Q_{ij}(\sqrt{M_i}f_i, \sqrt{M_j}f_j). \quad (1.6)$$

Let  $\mathcal{H}$  be the space of all functions  $f = (f_1, f_2, \dots, f_N)$  such that the following norm is finite:

$$\begin{aligned} \|f\|_{\mathcal{H}}^2 &:= \sum_{i=1}^N \|\langle p \rangle^{\gamma/2} P \nabla f_i\|_{L^2(\mathbb{R}^3, dp)}^2 + \|\langle p \rangle^{(\gamma+2)/2} (\mathbb{I} - P) \nabla f_i\|_{L^2(\mathbb{R}^3, dp)}^2 \\ &\quad + \|\langle p \rangle^{(\gamma+2)/2} f_i\|_{L^2(\mathbb{R}^3, dp)}^2, \end{aligned}$$

where  $\langle p \rangle := \sqrt{1 + |p|^2}$  and  $P := \frac{p \otimes p}{|p|^2}$ . Note that  $\mathcal{H}$  is a Hilbert space which embeds continuously into  $L^2(\mathbb{R}^3, dp)$ .

Our main results are summarized below.

**Theorem 1.** *There exists an explicitly computable constant  $\lambda > 0$  such that:*

$$-(f, Lf)_{L^2(\mathbb{R}^3, dp)} \geq \lambda \|f - \Pi^L f\|_{\mathcal{H}}^2, \quad f \in D(L),$$

where  $\Pi^L$  is the projection operator on the kernel  $N(L)$  of  $L$ .

The starting point of the proof of Theorem 1 is a coercivity estimate for the part of the operator  $L$  that describes collisions among particles of the same species. Let us denote with  $L^m \equiv (L_{11}, \dots, L_{NN})$  and with  $\Pi^m$  the projection operator onto the null space of  $L^m$ ,  $N(L^m)$ . Estimates of the form

$$C_\gamma \|f - \Pi^m f\|_{\mathcal{H}}^2 \geq -(f, L^m f)_{L^2(\mathbb{R}^3, dp)} \geq \lambda_m \|f - \Pi^m f\|_{\mathcal{H}}^2, \quad f \in D(L^m),$$

have been proven in [4, 13, 24, 27, 29]. Hence the resolvent of  $L^m$  is compact for  $\gamma + 2 > 0$  and there exists a spectral gap in  $L^2$  for  $\gamma \geq -2$ .

The second step in the proof consists in bounding the contribution of  $f^\perp \equiv f - \Pi^m f$  inside the quadratic form  $-(f, L^b f)_{L^2(\mathbb{R}^3, dp)}$ , where  $L^b \equiv L - L^m$  describes collisions between particles of different species:

$$-(f^\perp, L^b f^\perp)_{L^2(\mathbb{R}^3, dp)} \leq C_1 \|f^\perp\|_{\mathcal{H}}^2.$$

In the third step, the contribution of  $f^\parallel \equiv \Pi^m f$  inside the quadratic form  $-(f, L^b f)_{L^2(\mathbb{R}^3, dp)}$  is bounded from below by the differences of momentum  $u_i - u_j$  and differences of energies  $e_i - e_j$ :

$$-(f^\parallel, L^b f^\parallel)_{L^2(\mathbb{R}^3, dp)} \geq C_2 \sum_{i,j=1}^N (|u_i - u_j|^2 + (e_i - e_j)^2), \quad f \in D(L).$$

This result is obtained by exploiting the structure of  $N(L^m)$ .

Finally, for the fourth and last step we recall an estimate from [12], which relates  $u_i - u_j$  and  $e_i - e_j$  to the  $\mathcal{H}$  norms of  $f - \Pi^L f$  and  $f - \Pi^m f$ :

$$\sum_{i,j=1}^N (|u_i - u_j|^2 + (e_i - e_j)^2) \geq C_3 (\|f - \Pi^L f\|_{\mathcal{H}}^2 - 2\|f - \Pi^m f\|_{\mathcal{H}}^2), \quad f \in D(L). \quad (1.7)$$

The above estimate was previously obtained by the authors in [12] for a Boltzmann system. The proof is based on a careful analysis of the different structures of  $N(L^m)$  and  $N(L)$  for the Boltzmann equation, which is intimately connected to its conservation laws. Since the kernel

of the Landau operator has the same structure as its Boltzmann counterpart, we refer to [12, Lemma 15] for the proof of (1.7).

Finally, the non-positivity of  $L^b$  allows us to write

$$\begin{aligned} -(f, Lf)_{L^2(\mathbb{R}^3, dp)} &= -(f, L^m f)_{L^2(\mathbb{R}^3, dp)} - (f, L^b f)_{L^2(\mathbb{R}^3, dp)} \\ &\geq -(f, L^m f)_{L^2(\mathbb{R}^3, dp)} - \eta (f, L^b f)_{L^2(\mathbb{R}^3, dp)} \end{aligned}$$

for an arbitrary  $\eta \in (0, 1]$ . Putting together the results obtained in the previous four steps and choosing  $\eta$  small enough yield the desired spectral gap, concluding the proof of Theorem 1.

**Theorem 2.** *Let  $f^\infty$  be the global equilibrium of the system (1.4), that is,  $f^\infty = \Pi^{L-T} f = \Pi^{L-T} f_{\text{in}}$  where  $\Pi^{L-T}$  is the projection operator on the kernel  $N(L-T)$  of  $L-T$ . There exist explicitly computable constants  $\tau > 0$ ,  $k > 0$  such that:*

$$\|f(t) - f^\infty\|_{H^1(\mathbb{T}^3 \times \mathbb{R}^3)} \leq k e^{-t/\tau}, \quad t > 0. \quad (1.8)$$

Let  $\mathcal{M}(p)$  be the unique equilibrium state to (1.1). Assume there exists an  $\varepsilon > 0$  such that

$$\left\| \frac{1}{\sqrt{\mathcal{M}}} (F_{\text{in}} - \mathcal{M}) \right\|_{L^2(\mathbb{T}^3 \times \mathbb{R}^3)} \leq \varepsilon,$$

then the nonlinear problem (1.1) has an unique solution  $F(x, p, t)$  which decays exponentially fast towards the global equilibrium with a constant rate  $\lambda > 0$  that only depends on the linearized part of the operator:

$$\left\| \frac{1}{\sqrt{\mathcal{M}}} (F - \mathcal{M}) \right\|_{L^2(\mathbb{T}^3 \times \mathbb{R}^3)} \leq C_{\text{in}} \varepsilon e^{-\lambda t}, \quad t > 0.$$

**Remark 1.** *The global equilibrium states  $\mathcal{M}(p)$  and  $f^\infty(p)$  are computed in Theorem 4 and Theorem 6 respectively.*

In order to prove Theorem 2 we use the method developed in [28] which (i) relates coercivity estimates on  $L$  to the evolution of the corresponding semigroup in the Sobolev space  $H^1(\mathbb{T}^3 \times \mathbb{R}^3)$ , and (ii) combines spectral gap estimates for the linearized operator with bounds of the nonlinear terms to obtain asymptotic-in-time estimates for the non-linear problem when initial data are sufficiently close to the equilibrium. We summarize the method in the theorem below:

**Theorem 3.** [28, Thr. 1.1, Thr. 4.1]

- Let  $L$  be a linear operator. Assume there exists a suitable decomposition  $L = K - \Lambda$  such that

- (i)  $\nu_1 \|f\|_{\mathcal{H}}^2 \leq (f, \Lambda f)_{L^2(\mathbb{R}^3, dp)} \leq \nu_2 \|f\|_{\mathcal{H}}^2$ ,
- (ii)  $(\nabla_p \Lambda f, \nabla_p f)_{L^2(\mathbb{R}^3, dp)} \geq \nu_3 \|\nabla_p f\|_{\mathcal{H}}^2 - \nu_4 \|f\|_{L^2_p}^2$ ,
- (iii)  $(\nabla_p K f, \nabla_p f)_{L^2(\mathbb{R}^3, dp)} \leq C(\delta) \|f\|_{L^2(\mathbb{R}^3, dp)}^2 + \delta \|\nabla_p f\|_{L^2(\mathbb{R}^3, dp)}^2$ ,
- (iv)  $|(f, Lg)_{L^2_p}| \leq C \|f\|_{\mathcal{H}} \|g\|_{\mathcal{H}}$ ,
- (v)  $-(f, Lf)_{L^2(\mathbb{R}^3, dp)} \geq \lambda \|f - \Pi^L f\|_{\mathcal{H}}^2$ .

Then  $\mathcal{L} := L - v \cdot \nabla_x$  generates a strongly continuous evolution semi-group which satisfies

$$\|e^{\mathcal{L}t} (\Pi - \Pi^{\mathcal{L}})\|_{H^1(\mathbb{T}^3 \times \mathbb{R}^3)} \leq k e^{-t/\tau},$$

for some explicit constants  $k$  and  $\tau$  that only depend on the constants appearing in (i) – (v).

- Consider the nonlinear problem

$$F_t + v \cdot \nabla_x F = Q(F, F), \quad (1.9)$$

and denote by  $F^\infty$  the unique global equilibrium to (1.9). Let

$$\Gamma(f, f) + Lf := \frac{1}{\sqrt{F^\infty}} Q(F^\infty + f\sqrt{F^\infty}, F^\infty + f\sqrt{F^\infty}),$$

with  $Lf$  a linear operator satisfying (v) above, and  $\Gamma(f, f)$  such that

$$(vi) \quad \|\Gamma(f, f)\|_{W^{k,2}(\mathbb{T}^3 \times \mathbb{R}^3)} \leq C \|f\|_{W^{k,2}(\mathbb{T}^3 \times \mathbb{R}^3)} \left( \sum_{l,j \leq k} \|\partial_x^l \partial_v^j f\|_{L^2(\mathbb{T}^3, \mathcal{H})} \right)^{1/2},$$

for some  $k \geq 0$ . Then (1.9) has an unique smooth solution that decays exponentially fast towards  $F^\infty$ :

$$\left\| \frac{1}{\sqrt{F^\infty}} (F - F^\infty) \right\|_{L^2(\mathbb{T}^3 \times \mathbb{R}^3)} \leq C_{in} \varepsilon e^{-t/\tau}, \quad t > 0,$$

provided the initial data  $F_{in}$  satisfies

$$\left\| \frac{1}{\sqrt{F^\infty}} (F_{in} - F^\infty) \right\|_{L^2(\mathbb{T}^3 \times \mathbb{R}^3)} \leq \varepsilon.$$

Conditions (i) – (iii) state that  $\Lambda$  is coercive (in some sense) on the space  $\mathcal{H}$ , while  $K$  has a regularizing property. Assumption (v) is exactly the spectral gap proved in Theorem 1. For the propose of our theorem, condition (vi) will be relaxed, and we will use an estimate proved for the mono-species case by Guo in [24, Thr. 3].

An alternative (and perhaps easier) way of proving Theorem 1 and Theorem 2 is to show that  $K$  is compact and  $\Lambda$  is coercive, see [12, Lemma 10]. However this method is non-constructive, in the sense that the size of both the spectral gap and rate of convergence will be only given implicitly. For completeness we add the proof of compactness of  $K$  in the Appendix. In the following sections we will adopt the procedure outlined earlier that will allow for constructive estimates.

**1.2. Outline.** The rest of the paper is organized as follows: after brief summary of the conservation properties for the non-linear system, Section 2 concerns the formulation of the linearized system and its properties. Section 3 contains the proof to Theorem 1. In Section 4 we present the proof of Theorem 2. Exponential decay is proven with an explicit rate. Finally, in the Appendix we prove the compactness of the operator  $K$ .

We conclude by mentioning that among the several open problems, the one about estimates in the case of very soft potentials  $\gamma < -2$  is a particularly interesting question.

**1.3. Notation.** Vectors in  $\mathbb{R}^3$  will be denoted by  $v, v', p, p'$  and so on, the inner product between  $v$  and  $w$  will be written  $(v, w)$ . The identity matrix will be noted by  $\mathbb{I}$ , the trace of a matrix  $X$  will be denoted  $\text{Tr}(X)$ . The initial condition for the Cauchy problem will always be denoted by  $f_{in}$ . Unless otherwise specified,  $\int dp \equiv \int_{\mathbb{R}^3} dp$ ,  $\int dx \equiv \int_{\mathbb{T}^3} dx$ . The space  $L_p^2$  denotes the classical Lebesgue spaces  $L^2(\mathbb{R}^3)$  with respect to the variable  $p$ . We denote by  $H_{x,p}^1$  the Sobolev



space  $H^1(\mathbb{T}^3 \times \mathbb{R}^3)$  with respect to the variable  $x$  and  $p$  and by  $L_x^2 \mathcal{H}$  the space of all functions with finite norm  $\|\cdot\|_{\mathcal{H}} \| \cdot \|_{L^2(\mathbb{T}^3)}$ .

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## 2. CONSERVED QUANTITIES AND LINEARIZATION

In this section we first outline the conservation laws and entropy decay property which hold for (1.1). Then we present a linearization of (1.1) around an equilibrium state and show that the new linear system also satisfies conservation of mass, total momentum and total energy.

**Theorem 4.** *Let  $f_i$ ,  $i = 1, \dots, N$  be a solution to (1.1)-(1.2). The mass, the total momentum and energy of the system are conserved over time, i.e.*

$$\frac{d}{dt} \int f_i dp = \frac{d}{dt} \sum_{i=1}^N \int p f_i dp = \frac{d}{dt} \sum_{i=1}^N \int \frac{|p|^2}{2m_i} f_i dp = 0.$$

In addition the Boltzmann entropy functional  $H(f_1, f_2, \dots, f_N)$  defined as

$$H(f_1, f_2, \dots, f_N) := \int \sum_{i=1}^N f_i \log \frac{f_i}{m_i^3} dp$$

decreases along solutions to (1.1), and it is constant (that is, the entropy production vanishes) if and only if the distribution functions  $(f_1, \dots, f_N)$  are Maxwellians  $(\mathcal{M}_1, \dots, \mathcal{M}_N)$  of the form:

$$\mathcal{M}_i(x, p) = \frac{\rho_i(x)}{(2\pi m_i k_B T(x))^{3/2}} e^{-\frac{|p - m_i u(x)|^2}{2m_i k_B T(x)}},$$

where  $k_B$  is the Boltzmann constant. The density  $\rho_i(x)$ , velocity  $u(x)$  and temperature  $T(x)$  are uniquely determined by the conservation properties:

$$T(x) = \frac{1}{\sum_i \rho_i} \sum_{i=1}^N \int \frac{|p - m_i u|^2}{3m_i k_B} f_i dp, \quad u(x) = \frac{1}{\sum_i \rho_i m_i} \sum_{i=1}^N \int p f_i dp, \quad \rho_i(x) = \int f_i dp.$$

The only local equilibrium that satisfies (1.1)-(1.2) is the global equilibrium

$$\mathcal{M}_i(p) = \frac{\bar{\rho}_i}{(2\pi m_i k_B T_\infty)^{3/2}} e^{-\frac{(p - m_i u_\infty)^2}{2m_i k_B T_\infty}},$$

with  $\bar{\rho}_i$ ,  $T_\infty$  and  $u_\infty$  constants uniquely determined by the conservation properties:

$$T_\infty = \frac{1}{\sum_i \bar{\rho}_i} \sum_{i=1}^N \int \int \frac{|p - m_i u|^2}{3m_i k_B} f_i dp dx, \quad u_\infty = \frac{1}{\sum_i \bar{\rho}_i m_i} \sum_{i=1}^N \int \int p f_i dp dx, \quad \bar{\rho}_i = \int \int f_i dx dp.$$

*Proof.* The mass conservation follows immediately from the divergence structure of the collision operators. We first show total momentum conservation. Integration by parts yields:

$$\begin{aligned} \int p Q_{ij}(f_i, f_j) dp &= - \iint A^{(ij)} \left[ \frac{p}{m_i} - \frac{p'}{m_j} \right] (f'_j \nabla f_i - f_i \nabla f'_j) dp dp' \\ &= \iint f_i f'_j (\operatorname{div}_p A^{(ij)} \left[ \frac{p}{m_i} - \frac{p'}{m_j} \right] - \operatorname{div}_{p'} A^{(ij)} \left[ \frac{p}{m_i} - \frac{p'}{m_j} \right]) dp dp' \\ &= \left( \frac{1}{m_i} + \frac{1}{m_j} \right) \iint f_i f'_j (\operatorname{div}_w A^{(ij)}[w])|_{w=\frac{p}{m_i} - \frac{p'}{m_j}} dp dp' =: I_{ij}. \end{aligned}$$

Applying the transformation  $p \leftrightarrow p'$  inside  $I_{ij}$  and noticing that  $w \in \mathbb{R}^3 \mapsto \operatorname{div}_w A^{(ij)}[w]$  is an odd function, we find that  $I_{ij}$  is skew-symmetric:  $I_{ij} = -I_{ji}$ . Hence, summing up the above equality w.r.t.  $i, j = 1, \dots, N$  we get

$$\sum_{i,j=1}^N \int p Q_{ij}(f_i, f_j) dp = \sum_{i,j=1}^N I_{ij} = 0,$$

due to the skew-symmetry of  $I_{ij}$ .

Similarly, for the conservation of the total energy, integration by parts yields:

$$\begin{aligned} \int \frac{|p|^2}{2} Q_{ij}(f_i, f_j) dp &= - \iint p \cdot A^{(ij)} \left[ \frac{p}{m_i} - \frac{p'}{m_j} \right] (f'_j \nabla f_i - f_i \nabla f'_j) dp dp' \\ &= \iint f_i f'_j (\operatorname{div}_p (A^{(ij)} \left[ \frac{p}{m_i} - \frac{p'}{m_j} \right] p) - \operatorname{div}_{p'} (A^{(ij)} \left[ \frac{p}{m_i} - \frac{p'}{m_j} \right] p)) dp dp' \\ &= \iint f_i f'_j \operatorname{tr}(A^{(ij)} \left[ \frac{p}{m_i} - \frac{p'}{m_j} \right]) dp dp' + \\ &\quad + \iint f_i f'_j p \cdot (\operatorname{div}_p A^{(ij)} \left[ \frac{p}{m_i} - \frac{p'}{m_j} \right] - \operatorname{div}_{p'} A^{(ij)} \left[ \frac{p}{m_i} - \frac{p'}{m_j} \right]) dp dp' \\ &= \iint f_i f'_j \operatorname{tr}(A^{(ij)} \left[ \frac{p}{m_i} - \frac{p'}{m_j} \right]) dp dp' + \\ &\quad + \left( \frac{1}{m_i} + \frac{1}{m_j} \right) \iint f_i f'_j p \cdot (\operatorname{div}_w A^{(ij)}[w])|_{w=\frac{p}{m_i} - \frac{p'}{m_j}} dp dp'. \end{aligned}$$

It follows:

$$\begin{aligned} \sum_{i,j=1}^N \frac{1}{m_i} \int \frac{|p|^2}{2} Q_{ij}(f_i, f_j) dp &= \sum_{i,j=1}^N \frac{1}{m_i} \iint f_i f'_j \operatorname{tr}(A^{(ij)} \left[ \frac{p}{m_i} - \frac{p'}{m_j} \right]) dp dp' + \\ &\quad + \sum_{i,j=1}^N \left( \frac{1}{m_i} + \frac{1}{m_j} \right) \iint f_i f'_j \frac{p}{m_i} \cdot (\operatorname{div}_w A^{(ij)}[w])|_{w=\frac{p}{m_i} - \frac{p'}{m_j}} dp dp'. \end{aligned} \tag{2.1}$$

By applying the transformation  $(p, i) \leftrightarrow (p', j)$  in the terms on the right-hand side of (2.1) we deduce:

$$\begin{aligned} \sum_{i,j=1}^N \frac{1}{m_i} \int \frac{|p|^2}{2} Q_{ij}(f_i, f_j) dp &= \frac{1}{2} \sum_{i,j=1}^N \left( \frac{1}{m_i} + \frac{1}{m_j} \right) \iint f_i f_j' \operatorname{tr}(A^{(ij)} \left[ \frac{p}{m_i} - \frac{p'}{m_j} \right]) dp dp' + \\ &+ \frac{1}{2} \sum_{i,j=1}^N \left( \frac{1}{m_i} + \frac{1}{m_j} \right) \iint f_i f_j' (w \cdot \operatorname{div}_w A^{(ij)}[w])|_{w=\frac{p}{m_i}-\frac{p'}{m_j}} dp dp' = 0, \end{aligned}$$

since  $w \cdot \operatorname{div}_w A^{(ij)}[w] = -\operatorname{tr} A^{(ij)}[w]$  for  $w \in \mathbb{R}^3$ . The total energy conservation follows.

Finally, we show that the entropy functional  $H$  is decreasing as time increases:

$$\begin{aligned} -\frac{d}{dt} H(f_1, f_2, \dots, f_N) &= -\sum_{i,j=1}^N \int (\log f_i + 1) Q_{ij}(f_i, f_j) dp \\ &= \sum_{i,j=1}^N \iint \frac{\nabla f_i}{f_i} \cdot A^{(ij)} \left[ \frac{p}{m_i} - \frac{p'}{m_j} \right] (f_j' \nabla f_i - f_i \nabla f_j') dp dp' \\ &= \sum_{i,j=1}^N \iint f_i f_j' \frac{\nabla f_i}{f_i} \cdot A^{(ij)} \left[ \frac{p}{m_i} - \frac{p'}{m_j} \right] \left( \frac{\nabla f_i}{f_i} - \frac{\nabla f_j'}{f_j'} \right) dp dp'. \end{aligned}$$

By exchanging  $i \leftrightarrow j$  and  $p \leftrightarrow p'$  we obtain:

$$\begin{aligned} -\frac{d}{dt} H &= \sum_{i,j=1}^N \iint f_i f_j' \frac{\nabla f_i}{f_i} \cdot A^{(ij)} \left[ \frac{p}{m_i} - \frac{p'}{m_j} \right] \left( \frac{\nabla f_i}{f_i} - \frac{\nabla f_j'}{f_j'} \right) dp dp' \\ &= -\sum_{i,j=1}^N \iint f_i f_j' \frac{\nabla f_j'}{f_j'} \cdot A^{(ij)} \left[ \frac{p}{m_i} - \frac{p'}{m_j} \right] \left( \frac{\nabla f_i}{f_i} - \frac{\nabla f_j'}{f_j'} \right) dp dp' \\ &= \frac{1}{2} \sum_{i,j=1}^N \iint f_i f_j' \left( \frac{\nabla f_i}{f_i} - \frac{\nabla f_j'}{f_j'} \right) \cdot A^{(ij)} \left[ \frac{p}{m_i} - \frac{p'}{m_j} \right] \left( \frac{\nabla f_i}{f_i} - \frac{\nabla f_j'}{f_j'} \right) dp dp' \geq 0, \end{aligned}$$

since  $A^{(ij)}$  is a positive definite matrix.

Hence,  $\frac{d}{dt} H = 0$  if and only if  $\frac{\nabla f_i}{f_i} - \frac{\nabla f_j'}{f_j'}$  lies in the kernel of  $A^{(ij)} \left[ \frac{p}{m_i} - \frac{p'}{m_j} \right]$ , that is, if and only if there exists a scalar function  $\lambda_{ij}[v, v'] : \mathbb{R}^3 \times \mathbb{R}^3 \rightarrow \mathbb{R}$  such that

$$\frac{\nabla f_i}{f_i} - \frac{\nabla f_j'}{f_j'} = \lambda_{ij} \left[ \frac{p}{m_i}, \frac{p'}{m_j} \right] \left( \frac{p}{m_i} - \frac{p'}{m_j} \right). \quad (2.2)$$

We next show that the matrix  $\{\lambda_{ij}[\frac{p}{m_i}, \frac{p}{m_i}]\}_{i,j}$  is constant for all  $i$  and  $j$ . Applying the transformation  $(p, i) \leftrightarrow (p', j)$  in (2.2) we get

$$\lambda_{ij} \left[ \frac{p}{m_i}, \frac{p'}{m_j} \right] = \lambda_{ji} \left[ \frac{p'}{m_j}, \frac{p}{m_i} \right],$$

which implies

$$\lambda_{ij} \left[ \frac{p}{m_i}, \frac{p}{m_i} \right] = \lambda_{ji} \left[ \frac{p}{m_i}, \frac{p}{m_i} \right].$$

We differentiate (2.2) w.r.t.  $p$  and obtain:

$$D^2 \log f_i(p) = \nabla_p \lambda_{ij} \left[ \frac{p}{m_i}, \frac{p'}{m_j} \right] \otimes \left( \frac{p}{m_i} - \frac{p'}{m_j} \right) + \frac{1}{m_i} \lambda_{ij} \left[ \frac{p}{m_i}, \frac{p'}{m_j} \right] \mathbb{I}.$$

Consequently for  $p'/m_j = p/m_i$ ,

$$\partial_{p_k p_s}^2 \log f_i(p) = \frac{1}{m_i} \lambda_{ij} \left[ \frac{p}{m_i}, \frac{p}{m_i} \right] \delta_{ks}, \quad k, s = 1, 2, 3. \quad (2.3)$$

Differentiation of (2.3) leads to:

$$\partial_{p_\ell} \partial_{p_k p_s}^2 \log f_i(p) = \frac{1}{m_i} \partial_{p_\ell} \lambda_{ij} \left[ \frac{p}{m_i}, \frac{p}{m_i} \right] \delta_{ks}, \quad k, s, \ell = 1, 2, 3.$$

Since the order of the derivatives on the left hand side is interchangeable (assuming enough smoothness for  $f_i$ ), one deduces that

$$\partial_{p_\ell} \lambda_{ij} \left[ \frac{p}{m_i}, \frac{p}{m_i} \right] \delta_{ks} = \partial_{p_k} \lambda_{ij} \left[ \frac{p}{m_i}, \frac{p}{m_i} \right] \delta_{\ell s}, \quad k, s, \ell = 1, 2, 3,$$

which is consistent if and only if  $v \in \mathbb{R}^3 \mapsto \lambda_{ij}[v, v]$  is constant.

Moreover, (2.3) implies that, for  $i = 1, \dots, N$ ,  $\lambda_{ij}$  does not depend on  $j$ . Summarizing, we have found that  $\lambda_{i,j}[v, v]$  is constant, symmetric in  $i, j$  and does not depend on  $j$ . Hence  $\lambda_{i,j}[v, v] \equiv -\alpha^{(2)}$ ,  $\alpha^{(2)} \in \mathbb{R}$ , for  $i, j = 1, \dots, N$ ,  $v \in \mathbb{R}^3$ . This fact and (2.3) imply that  $\log f_i(p)$  is a second order polynomial in  $p$ :

$$\log f_i(p) = \alpha_i^{(0)} + \alpha_i^{(1)} \cdot p - \alpha^{(2)} \frac{|p|^2}{2m_i}, \quad i = 1, \dots, N. \quad (2.4)$$

From (2.2) and (2.4) it follows:

$$\alpha_i^{(1)} - \alpha_j^{(1)} - \alpha^{(2)} \left( \frac{p}{m_i} - \frac{p'}{m_j} \right) = -\alpha^{(2)} \left( \frac{p}{m_i} - \frac{p'}{m_j} \right),$$

which leads to  $\alpha_i^{(1)} = \alpha_j^{(1)}$ ,  $i, j = 1, \dots, N$  after evaluation for  $p'/m_j = p/m_i$ . We conclude that

$$\log f_i(p) = \alpha_i^{(0)} + \alpha^{(1)} \cdot p - \alpha^{(2)} \frac{|p|^2}{2m_i}, \quad i = 1, \dots, N.$$

Conservation of mass, momentum and energy uniquely determine the constants  $\alpha_i^{(0)}$ ,  $\alpha^{(1)}$  and  $\alpha^{(2)}$ . □

**Linearization around the equilibrium.** We now linearize the collision operator  $Q$  around the Maxwellians  $(M_1, \dots, M_N)$  defined as

$$M_i(p) = \frac{\rho_i}{(2\pi m_i k_B T)^{3/2}} e^{-\frac{1}{2} \frac{|p|^2}{m_i k_B T}}.$$

It holds:

$$\sum_{j=1}^N Q_{ij}(M_i + \sqrt{M_i} f_i, M_j + \sqrt{M_j} f_j) = \sum_{j=1}^N Q_{ij}(M_i, \sqrt{M_j} f_j) + Q_{ij}(\sqrt{M_i} f_i, M_j) + \text{lower order terms.}$$

Let us first compute:

$$\begin{aligned} Q_{ij}(\sqrt{M_i}f_i, M_j) &= \operatorname{div}_p \int A^{(ij)} \left[ \frac{p}{m_i} - \frac{p'}{m_j} \right] (M'_j \nabla(\sqrt{M_i}f_i) - \sqrt{M_i}f_i \nabla M'_j) dp' \\ &= \operatorname{div}_p \int A^{(ij)} \left[ \frac{p}{m_i} - \frac{p'}{m_j} \right] \left( (M'_j \nabla \sqrt{M_i} - \sqrt{M_i} \nabla M'_j) f_i + M'_j \sqrt{M_i} \nabla f_i \right) dp'. \end{aligned}$$

Rewriting

$$\begin{aligned} M'_j \nabla \sqrt{M_i} - \sqrt{M_i} \nabla M'_j &= \sqrt{M_i} M'_j \left( -\frac{1}{2} \nabla \log M'_j - \frac{1}{2k_B T} \left( \frac{p}{m_i} - \frac{p'}{m_j} \right) \right) \\ &= -\sqrt{M_i M'_j} \nabla \sqrt{M'_j} - \frac{\sqrt{M_i M'_j}}{2k_B T} \left( \frac{p}{m_i} - \frac{p'}{m_j} \right), \end{aligned} \quad (2.5)$$

it follows:

$$Q_{ij}(\sqrt{M_i}f_i, M_j) = \operatorname{div}_p \int \sqrt{M_i M'_j} A^{(ij)} \left[ \frac{p}{m_i} - \frac{p'}{m_j} \right] \left( \sqrt{M'_j} \nabla f_i - f_i \nabla \sqrt{M'_j} \right) dp', \quad (2.6)$$

since  $A^{(ij)} \left[ \frac{p}{m_i} - \frac{p'}{m_j} \right] \left( \frac{p}{m_i} - \frac{p'}{m_j} \right) \equiv 0$ . We now consider

$$\begin{aligned} Q_{ij}(M_i, \sqrt{M_j}f_j) &= \operatorname{div}_p \int A^{(ij)} \left[ \frac{p}{m_i} - \frac{p'}{m_j} \right] \left( \sqrt{M'_j} f'_j \nabla M_i - M_i \nabla \left( \sqrt{M'_j} f'_j \right) \right) dp' \\ &= \operatorname{div}_p \int A^{(ij)} \left[ \frac{p}{m_i} - \frac{p'}{m_j} \right] \left( f'_j \left( \sqrt{M'_j} \nabla M_i - M_i \nabla \sqrt{M'_j} \right) - M_i \sqrt{M'_j} \nabla f'_j \right) dp'. \end{aligned}$$

Using similar calculations as in (2.5) one gets

$$\sqrt{M'_j} \nabla M_i - M_i \nabla \sqrt{M'_j} = \sqrt{M_i M'_j} \nabla \sqrt{M_i} + \frac{M_i \sqrt{M'_j}}{2k_B T} \left( \frac{p}{m_i} - \frac{p'}{m_j} \right),$$

which implies

$$Q_{ij}(M_i, \sqrt{M_j}f_j) = \operatorname{div}_p \int \sqrt{M_i M'_j} A^{(ij)} \left[ \frac{p}{m_i} - \frac{p'}{m_j} \right] \left( f'_j \nabla \sqrt{M_i} - \sqrt{M_i} \nabla f'_j \right) dp'. \quad (2.7)$$

Adding (2.6) with (2.7) (and dividing by  $\sqrt{M_i}$ ) we obtain the linearized collision operator:

$$L_i(f_1, \dots, f_n) = \sum_{j=1}^N L_{ij}(f_i, f_j),$$

with

$$\begin{aligned} L_{ij}(f_i, f_j) &:= \frac{1}{\sqrt{M_i}} \operatorname{div}_p \int \sqrt{M_i M'_j} A^{(ij)} \left[ \frac{p}{m_i} - \frac{p'}{m_j} \right] \\ &\quad \cdot \left( \sqrt{M'_j} \nabla f_i - \sqrt{M_i} \nabla f'_j - f_i \nabla \sqrt{M'_j} + f'_j \nabla \sqrt{M_i} \right) dp'. \end{aligned} \quad (2.8)$$

We briefly recall the conserved quantities for  $L_i$ :

**Theorem 5.** *Let  $f_i$ ,  $i = 1, \dots, N$  be the solution to the linear system:*

$$\begin{cases} \partial_t f_i + \frac{p}{m_i} \cdot \nabla_x f_i = \sum_{j=1}^N L_{ij}(f_i, f_j), \\ f(x, p, 0) = f_{\text{in}}(x, p), \end{cases}$$

with  $L_{ij}$  defined as in (2.8). The mass  $\int \sqrt{M_i} f_i dp$ , total momentum  $\sum_{i=1}^N \int p \sqrt{M_i} f_i dp$  and total energy  $\sum_{i=1}^N \int (|p|^2/2m_i) \sqrt{M_i} f_i dp$  are constant in time.

*Proof.* The mass of each function  $\sqrt{M_i} f_i$  is conserved because of the divergence form of the operator. Moreover, with an integration by parts we can deduce

$$\begin{aligned} & \int p \sum_{i=1}^N \sqrt{M_i} L_i(f_1, \dots, f_N) dp \\ &= - \sum_{i,j=1}^N \iint \sqrt{M_i M_j'} A^{(ij)} \left[ \frac{p}{m_i} - \frac{p'}{m_j} \right] \\ & \quad \cdot \left( \sqrt{M_j'} \nabla f_i - \sqrt{M_i} \nabla f_j' - f_i \nabla \sqrt{M_j'} + f_j' \nabla \sqrt{M_i} \right) dp dp' = 0, \end{aligned}$$

because the quantity inside the integral is antisymmetric for the transformation  $(i, p) \leftrightarrow (j, p')$ . Finally, the same transformation and another integration by parts allow us to write:

$$\begin{aligned} & \int \sum_{i=1}^N \frac{|p|^2}{2m_i} \sqrt{M_i} L_i(f_1, \dots, f_N) dp \\ &= - \sum_{i,j=1}^N \iint \sqrt{M_i M_j'} \frac{p}{m_i} \cdot A^{(ij)} \left[ \frac{p}{m_i} - \frac{p'}{m_j} \right] \\ & \quad \cdot \left( \sqrt{M_j'} \nabla f_i - \sqrt{M_i} \nabla f_j' - f_i \nabla \sqrt{M_j'} + f_j' \nabla \sqrt{M_i} \right) dp dp' \\ &= - \sum_{i,j=1}^N \iint \frac{1}{2} \sqrt{M_i M_j'} \left( \frac{p}{m_i} - \frac{p'}{m_j} \right) \cdot A^{(ij)} \left[ \frac{p}{m_i} - \frac{p'}{m_j} \right] \\ & \quad \cdot \left( \sqrt{M_j'} \nabla f_i - \sqrt{M_i} \nabla f_j' - f_i \nabla \sqrt{M_j'} + f_j' \nabla \sqrt{M_i} \right) dp dp' = 0, \end{aligned}$$

since  $A^{(ij)} \left[ \frac{p}{m_i} - \frac{p'}{m_j} \right] \left( \frac{p}{m_i} - \frac{p'}{m_j} \right) = 0$ . The proof is complete.  $\square$

**Structure of the linearized collision operator.** We first show that  $L_{ij}$  can be rewritten in the following form:

$$L_{ij}(f_i, f_j) = \frac{1}{\sqrt{M_i}} \operatorname{div}_p \int M_i M_j' A^{(ij)} \left[ \frac{p}{m_i} - \frac{p'}{m_j} \right] \left( \nabla \left( \frac{f_i}{\sqrt{M_i}} \right) - \nabla \left( \frac{f_j'}{\sqrt{M_j'}} \right) \right) dp'. \quad (2.9)$$

To prove (2.9) we first notice that:

$$\nabla \log \sqrt{M_j'} = -\frac{1}{2k_B T} \frac{p'}{m_j} = \frac{1}{2k_B T} \left( \frac{p}{m_i} - \frac{p'}{m_j} \right) + \nabla \log \sqrt{M_i}.$$

It follows that the term  $\sqrt{M'_j} \nabla f_i - f_i \nabla \sqrt{M'_j}$  inside (2.8) can be rewritten as:

$$\begin{aligned} \sqrt{M'_j} \nabla f_i - f_i \nabla \sqrt{M'_j} &= \sqrt{M_i M'_j} \left( \frac{\nabla f_i}{\sqrt{M_i}} - \frac{f_i}{\sqrt{M_i}} \nabla \log \sqrt{M'_j} \right) \\ &= \sqrt{M_i M'_j} \left( \frac{\nabla f_i}{\sqrt{M_i}} - \frac{f_i}{\sqrt{M_i}} \nabla \log \sqrt{M_i} \right) + \frac{f_i \sqrt{M'_j}}{2k_B T} \left( \frac{p}{m_i} - \frac{p'}{m_j} \right) \\ &= \sqrt{M_i M'_j} \nabla \left( \frac{f_i}{\sqrt{M_i}} \right) + \frac{f_i \sqrt{M'_j}}{2k_B T} \left( \frac{p}{m_i} - \frac{p'}{m_j} \right). \end{aligned}$$

The other term  $\sqrt{M_i} \nabla f'_j - f'_j \nabla \sqrt{M_i}$  is treated in a similar way. This shows that (2.9) and (2.8) are equivalent formulations.

We will now decompose the operator  $L = (L_1, L_2, \dots, L_N)$  as  $L = L^m + L^b$ , where  $L^m$  and  $L^b$  respectively describe collisions between particles of the same species and of different species. More precisely,

$$\begin{aligned} L^m(f) &:= (L_{11}(f_1, f_1), \dots, L_{NN}(f_N, f_N)), \\ L^b(f) &:= \left( \sum_{j \neq 1} L_{1j}(f_1, f_j), \dots, \sum_{j \neq N} L_{Nj}(f_N, f_j) \right). \end{aligned}$$

**Theorem 6.** *Both operators  $L^m$  and  $L^b$  are negative semidefinite. Moreover  $f \in N(L)$  if and only if*

$$f_i = M_i^{1/2} \left( \beta_i^{(0)} + \beta^{(1)} \cdot p + \beta^{(2)} \frac{|p|^2}{2m_i} \right), \quad i = 1, \dots, N,$$

for some  $p$ -independent real coefficients  $\beta_i^{(0)}$ ,  $i = 1, \dots, N$ ,  $\beta^{(1)}$  and  $\beta^{(2)}$ , and  $f \in N(L^m)$  if and only if:

$$f_i = M_i^{1/2} \left( \alpha_i^{(0)} + \alpha_i^{(1)} \cdot p + \alpha_i^{(2)} |p|^2 \right), \quad i = 1, \dots, N,$$

for some  $p$ -independent real coefficients  $\alpha_i^{(0)}$ ,  $\alpha_i^{(1)}$ ,  $\alpha_i^{(2)}$ ,  $i = 1, \dots, N$ .

*Proof.* A change of variable  $p \leftrightarrow p'$  allows to write

$$\begin{aligned} (f, L^m f)_{L_p^2} &:= \sum_{i=1}^N (f_i, L_{ii}(f_i, f_i))_{L_p^2} \\ &= -\frac{1}{2} \sum_{i=1}^N \int \int M_i M'_i A^{(ii)} \left[ \frac{p}{m_i} - \frac{p'}{m_i} \right] \left( \nabla \left( \frac{f_i}{\sqrt{M_i}} \right) - \nabla \left( \frac{f'_i}{\sqrt{M'_i}} \right) \right) \\ &\quad \cdot \left( \nabla \left( \frac{f_i}{\sqrt{M_i}} \right) - \nabla \left( \frac{f'_i}{\sqrt{M'_i}} \right) \right) dp dp' \leq 0. \end{aligned}$$

Using the same change of variable, for each  $i \neq j$  one can show that

$$\begin{aligned} & (f_i, L_{ij}(f_i, f_j))_{L_p^2} + (f_j, L_{ji}(f_j, f_i))_{L_p^2} \\ &= - \int \int M_i M_j' A^{(ii)} \left[ \frac{p}{m_i} - \frac{p'}{m_j} \right] \left( \nabla \left( \frac{f_i}{\sqrt{M_i}} \right) - \nabla \left( \frac{f_j'}{\sqrt{M_j'}} \right) \right) \\ & \quad \cdot \left( \nabla \left( \frac{f_i}{\sqrt{M_i}} \right) - \nabla \left( \frac{f_j'}{\sqrt{M_j'}} \right) \right) dp dp' \leq 0, \end{aligned} \quad (2.10)$$

which yields  $(f, L^b f)_{L_p^2} := \sum_{\substack{i,j=1 \\ j \neq i}}^N (f_i, L_{ij}(f_i, f_j))_{L_p^2} \leq 0$  for all  $f \in D(L)$ .

It is clear that  $(f, L^m f)_{L_p^2} = 0$  if and only if

$$\nabla \left( \frac{f_i}{\sqrt{M_i}} \right) - \nabla \left( \frac{f_i'}{\sqrt{M_i'}} \right) = \mu_{ij}[p, p'] \left( \frac{p}{m_i} - \frac{p'}{m_i} \right).$$

By employing the same method that was used to solve (2.2) we find that  $f \in N(L^m)$  if and only if:

$$f_i = M_i^{1/2} \left( \alpha_i^{(0)} + \alpha_i^{(1)} \cdot p + \alpha_i^{(2)} |p|^2 \right), \quad i = 1, \dots, N, \quad (2.11)$$

for some  $p$ -independent real coefficients  $\alpha_i^{(0)}, \alpha_i^{(1)}, \alpha_i^{(2)}, i = 1, \dots, N$ . Eq. (2.11) is a complete characterization of  $N(L^m)$ . A similar strategy yields the description of the kernel of  $L$ :  $f \in N(L)$  if and only if

$$f_i = M_i^{1/2} \left( \beta_i^{(0)} + \beta_i^{(1)} \cdot p + \beta_i^{(2)} \frac{|p|^2}{2m_i} \right), \quad i = 1, \dots, N, \quad (2.12)$$

for some  $p$ -independent real coefficients  $\beta_i^{(0)}, i = 1, \dots, N, \beta^{(1)}$  and  $\beta^{(2)}$ . Eq. (2.12) is a complete characterization of  $N(L)$ . □

### 3. PROOF OF THEOREM 1

This section is devoted to the proof of Theorem 1 which states that the multi-species linearized Landau collision operator  $L = (L_1, L_2, \dots, L_N)$  defined as in (2.8) has a spectral gap in the Hilbert space  $\mathcal{H}$ .

The starting point in the proof is the already known spectral gap for the mono-species operator proven in several works, including [24, 29] and summarized in the next lemma.

**Lemma 1.** *There exists an explicitly computable constant  $\lambda_m > 0$  such that:*

$$-(f, L^m f) \geq \lambda_m \|f - \Pi^m f\|_{\mathcal{H}}^2 \quad f \in D(L^m),$$

where  $\Pi^m$  denotes the projection operator onto the subspace  $N(L^m)$ .

We will now follow an approach similar to the one formulated in [12]. We first write

$$f = f^{\parallel} + f^{\perp},$$

with

$$f^{\parallel} := \Pi^m f, \quad f^{\perp} := (\mathbb{I} - \Pi^m) f.$$



From (2.10) it follows:

$$-(f, L^b f)_{L_p^2} = \frac{1}{2} \sum_{\substack{i,j=1 \\ j \neq i}}^N \iint M_i M'_j (w_p + w_o) \cdot A^{(ij)} \left[ \frac{p}{m_i} - \frac{p'}{m_j} \right] (w_p + w_o) dp dp',$$

with

$$w_p := \nabla \left( \frac{f_i^\parallel}{\sqrt{M_i}} \right) - \nabla \left( \frac{(f_j^\parallel)'}{\sqrt{M'_j}} \right), \quad w_o := \nabla \left( \frac{f_i^\perp}{\sqrt{M_i}} \right) - \nabla \left( \frac{(f_j^\perp)'}{\sqrt{M'_j}} \right).$$

Since  $A^{(ij)}$  is symmetric and positive definite, Young's inequality yields

$$\frac{1}{2} w_p \cdot A^{(ij)} w_o + \frac{1}{2} w_o \cdot A^{(ij)} w_p = w_p \cdot A^{(ij)} w_o \geq -\frac{1}{4} w_p \cdot A^{(ij)} w_p - w_o \cdot A^{(ij)} w_o,$$

and

$$\begin{aligned} -(f, L^b f)_{L_p^2} &\geq \frac{1}{4} \sum_{\substack{i,j=1 \\ j \neq i}}^N \iint M_i M'_j w_p \cdot A^{(ij)} \left[ \frac{p}{m_i} - \frac{p'}{m_j} \right] w_p dp dp' \\ &\quad - \frac{1}{2} \sum_{\substack{i,j=1 \\ j \neq i}}^N \iint M_i M'_j w_o \cdot A^{(ij)} \left[ \frac{p}{m_i} - \frac{p'}{m_j} \right] w_o dp dp' \\ &= -\frac{1}{2} (f^\parallel, L^b f^\parallel)_{L_p^2} - \frac{1}{2} \sum_{\substack{i,j=1 \\ j \neq i}}^N \iint M_i M'_j w_o \cdot A^{(ij)} \left[ \frac{p}{m_i} - \frac{p'}{m_j} \right] w_o dp dp'. \end{aligned} \quad (3.1)$$

Let us estimate the second term on the right-hand side of (3.1). Applying Young's inequality one more time we get

$$\begin{aligned} &\frac{1}{2} \sum_{\substack{i,j=1 \\ j \neq i}}^N \iint M_i M'_j w_o \cdot A^{(ij)} \left[ \frac{p}{m_i} - \frac{p'}{m_j} \right] w_o dp dp' \\ &\leq \sum_{\substack{i,j=1 \\ j \neq i}}^N \iint M_i M'_j \nabla \left( \frac{f_i^\perp}{\sqrt{M_i}} \right) \cdot A^{(ij)} \left[ \frac{p}{m_i} - \frac{p'}{m_j} \right] \nabla \left( \frac{f_i^\perp}{\sqrt{M_i}} \right) dp dp' \\ &\quad + \sum_{\substack{i,j=1 \\ j \neq i}}^N \iint M_i M'_j \nabla \left( \frac{(f_j^\perp)'}{\sqrt{M'_j}} \right) \cdot A^{(ij)} \left[ \frac{p}{m_i} - \frac{p'}{m_j} \right] \nabla \left( \frac{(f_j^\perp)'}{\sqrt{M'_j}} \right) dp dp' \\ &= 2 \sum_{\substack{i,j=1 \\ j \neq i}}^N \iint M_i M'_j \nabla \left( \frac{f_i^\perp}{\sqrt{M_i}} \right) \cdot A^{(ij)} \left[ \frac{p}{m_i} - \frac{p'}{m_j} \right] \nabla \left( \frac{f_i^\perp}{\sqrt{M_i}} \right) dp dp'. \end{aligned}$$

Since

$$\nabla \left( \frac{f_i^\perp}{\sqrt{M_i}} \right) = \frac{\nabla f_i^\perp}{\sqrt{M_i}} - \frac{f_i^\perp}{\sqrt{M_i}} \nabla \log \sqrt{M_i},$$

we have

$$\begin{aligned}
& \frac{1}{2} \sum_{\substack{i,j=1 \\ j \neq i}}^N \iint M_i M'_j w_o \cdot A^{(ij)} \left[ \frac{p}{m_i} - \frac{p'}{m_j} \right] w_o dp dp' \\
& \leq 4 \sum_{\substack{i,j=1 \\ j \neq i}}^N \iint M'_j \nabla f_i^\perp \cdot A^{(ij)} \left[ \frac{p}{m_i} - \frac{p'}{m_j} \right] \nabla f_i^\perp dp dp' \\
& \quad + 4 \sum_{\substack{i,j=1 \\ j \neq i}}^N \iint M'_j (f_i^\perp)^2 \nabla \log \sqrt{M_i} \cdot A^{(ij)} \left[ \frac{p}{m_i} - \frac{p'}{m_j} \right] \nabla \log \sqrt{M_i} dp dp' \\
& \leq \sum_{i=1}^N \int \nabla f_i^\perp \cdot \mathcal{A}^{(i)} \nabla f_i^\perp dp + \sum_{i=1}^N \int (f_i^\perp)^2 \mathcal{B}^{(i)} dp, \tag{3.2}
\end{aligned}$$

with

$$\mathcal{A}^{(i)} := 4 \sum_{j=1}^N \int M'_j A^{(ij)} \left[ \frac{p}{m_i} - \frac{p'}{m_j} \right] dp', \quad \mathcal{B}^{(i)} := \nabla \log \sqrt{M_i} \cdot \mathcal{A}^{(i)} \nabla \log \sqrt{M_i}.$$

From [8, Lemma 2.3] we deduce that:

$$\nabla f_i^\perp \cdot \mathcal{A}^{(i)} \nabla f_i^\perp \leq C \left( \langle p \rangle^\gamma |P \nabla f_i^\perp|^2 + \langle p \rangle^{\gamma+2} |(I-P) \nabla f_i^\perp|^2 \right), \tag{3.3}$$

$$\mathcal{B}^{(i)} \leq C \langle p \rangle^{\gamma+2}. \tag{3.4}$$

Inequalities (3.2), (3.3) and (3.4) imply:

$$\frac{1}{2} \sum_{\substack{i,j=1 \\ j \neq i}}^N \iint M_i M'_j w_o \cdot A^{(ij)} \left[ \frac{p}{m_i} - \frac{p'}{m_j} \right] w_o dp dp' \leq C_1 \|f^\perp\|_{\mathcal{H}}^2, \tag{3.5}$$

for some explicitly computable constant  $C_1 > 0$ . In summary we have shown that

$$-(f, L^b f)_{L_p^2} \geq -\frac{1}{2} (f^\parallel, L^b f^\parallel)_{L_p^2} - C_1 \|f^\perp\|_{\mathcal{H}}^2. \tag{3.6}$$

We are now ready to prove the next lemma:

**Lemma 2.** *For each  $f \in D(L)$  and  $\eta \in (0, 1]$  we have*

$$-(f, Lf)_{L_p^2} \geq (\lambda_m - \eta C_1) \|f^\perp\|_{\mathcal{H}}^2 - \frac{\eta}{2} (f^\parallel, L^b f^\parallel)_{L_p^2}.$$

*Proof.* Using the decomposition  $L = L^m + L^b$  we get,

$$\begin{aligned}
-(f, Lf)_{L_p^2} &= -(f, L^m f)_{L_p^2} - (f, L^b f)_{L_p^2} \\
&\geq -(f, L^m f)_{L_p^2} - \eta (f, L^b f)_{L_p^2}
\end{aligned}$$

for each  $\eta \in (0, 1]$ , since  $L^b$  is a negative semidefinite operator, as shown in Theorem 6. Finally Lemma 1 and (3.6) imply

$$-(f, Lf)_{L_p^2} \geq \lambda_m \|f^\perp\|_{\mathcal{H}}^2 - \eta \left( \frac{1}{2} (f^\parallel, L^b f^\parallel)_{L_p^2} + C_1 \|f^\perp\|_{\mathcal{H}}^2 \right),$$

which finishes the proof. □

We focus now our attention on  $(f^\parallel, L^b f^\parallel)_{L_p^2}$ . From (2.11) it follows

$$f_i^\parallel = (\Pi^m f)_i = M_i^{1/2} \left( \alpha_i + u_i \cdot p + e_i \frac{|p|^2}{2m_i} \right), \quad i = 1, \dots, N, \quad (3.7)$$

for a suitable choice of  $\alpha_i, u_i, e_i$ . We get:

$$\begin{aligned} - (f^\parallel, L^b f^\parallel)_{L_p^2} &= \frac{1}{2} \sum_{\substack{i,j=1 \\ j \neq i}}^N \iint M_i M_j' \left( u_i - u_j + e_i \frac{p}{m_i} - e_j \frac{p'}{m_j} \right) \\ &\quad \cdot A^{(ij)} \left[ \frac{p}{m_i} - \frac{p'}{m_j} \right] \left( u_i - u_j + e_i \frac{p}{m_i} - e_j \frac{p'}{m_j} \right) dp dp'. \end{aligned}$$

We first notice that

$$\begin{aligned} &(u_i - u_j) \cdot \iint M_i M_j' A^{(ij)} \left[ \frac{p}{m_i} - \frac{p'}{m_j} \right] \left( e_i \frac{p}{m_i} - e_j \frac{p'}{m_j} \right) dp dp' \\ &= (u_i - u_j) \cdot \frac{e_i}{m_i} \int M_i p \left( \int M_j' A^{(ij)} \left[ \frac{p}{m_i} - \frac{p'}{m_j} \right] dp' \right) dp \\ &\quad - (u_i - u_j) \cdot \frac{e_j}{m_j} \int M_j' p' \left( \int M_i A^{(ij)} \left[ \frac{p}{m_i} - \frac{p'}{m_j} \right] dp \right) dp'. \end{aligned}$$

Since  $\int M_i A^{(ij)} \left[ \frac{p}{m_i} - \frac{p'}{m_j} \right] dp$  is an even function with respect to  $p'$  and  $M_j' p'$  is odd, the integral of the product of the two vanishes. This implies

$$(u_i - u_j) \cdot \iint M_i M_j' A^{(ij)} \left[ \frac{p}{m_i} - \frac{p'}{m_j} \right] \left( e_i \frac{p}{m_i} - e_j \frac{p'}{m_j} \right) dp dp' = 0.$$

Hence we are left with

$$\begin{aligned} - (f^\parallel, L^b f^\parallel)_{L_p^2} &= \sum_{i,j=1}^N (u_i - u_j) \cdot \iint M_i M_j' A^{(ij)} \left[ \frac{p}{m_i} - \frac{p'}{m_j} \right] dp dp' (u_i - u_j) \\ &\quad + \sum_{i,j=1}^N \frac{(e_i - e_j)^2}{4} \iint M_i M_j' \left( \frac{p}{m_i} + \frac{p'}{m_j} \right) \cdot A^{(ij)} \left[ \frac{p}{m_i} - \frac{p'}{m_j} \right] \left( \frac{p}{m_i} + \frac{p'}{m_j} \right) dp dp', \end{aligned}$$

after rewriting  $\left( e_i \frac{p}{m_i} - e_j \frac{p'}{m_j} \right)$  as

$$\left( e_i \frac{p}{m_i} - e_j \frac{p'}{m_j} \right) = \left( \frac{p}{m_i} + \frac{p'}{m_j} \right) \frac{(e_i - e_j)}{2} + \left( \frac{p}{m_i} - \frac{p'}{m_j} \right) \frac{(e_i + e_j)}{2}.$$

It is easy to see that, for  $i, j = 1, \dots, N$ , the matrix

$$\mathcal{A}^{(ij)} \equiv \iint M_i M_j' A^{(ij)} \left[ \frac{p}{m_i} - \frac{p'}{m_j} \right] dp dp'$$

is positive definite, while

$$\mathcal{B}^{(ij)} \equiv \frac{1}{4} \iint M_i M_j' \left( \frac{p}{m_i} + \frac{p'}{m_j} \right) \cdot A^{(ij)} \left[ \frac{p}{m_i} - \frac{p'}{m_j} \right] \left( \frac{p}{m_i} + \frac{p'}{m_j} \right) dp dp' > 0.$$

We conclude:

**Lemma 3.** *There exists an explicitly computable constant  $C_2 > 0$  such that:*

$$-(f^\parallel, L^b f^\parallel)_{L_p^2} \geq C_2 \sum_{i,j=1}^N (|u_i - u_j|^2 + (e_i - e_j)^2), \quad f \in D(L),$$

where the  $p$ -independent quantities  $u_i, e_i$  are related to  $f$  through (3.7).

The last step in the proof of the spectral gap for  $L$  is the result shown in the next lemma.

**Lemma 4.** *There exists an explicitly computable constant  $C_3 > 0$  such that*

$$\sum_{i,j=1}^N (|u_i - u_j|^2 + (e_i - e_j)^2) \geq C_3 \left( \|f - \Pi^L f\|_{\mathcal{H}}^2 - 2\|f^\perp\|_{\mathcal{H}}^2 \right), \quad f \in D(L), \quad (3.8)$$

where the  $p$ -independent quantities  $u_i, e_i$  are related to  $f$  through (3.7).

For the proof of Lemma 4 we refer directly to the one of Lemma 15 in [12]. In such lemma the authors prove (3.8) for  $f$  solution to a multi-species linearized Boltzmann operator. The proof only relies on the structure of  $N(L)$  and  $N(L^m)$ , which is the same in both multi-species Boltzmann system studied in [12] and the Landau systems considered in this manuscript.

Summarizing, Lemmas 2, 3 and 4 imply

$$-(f, Lf)_{L_p^2} \geq \frac{\eta}{2} C_2 C_3 \|f - \Pi^L f\|_{\mathcal{H}}^2 + (\lambda_m - \eta(C_1 + C_2 C_3)) \|f^\perp\|_{\mathcal{H}}^2 \quad f \in \mathcal{H}.$$

Choosing  $\eta = \min\{1, \lambda_m/(C_1 + C_2 C_3)\}$  we obtain the desired spectral gap with

$$\lambda = \frac{C_2 C_3}{2} \min \left\{ 1, \frac{\lambda_m}{C_1 + C_2 C_3} \right\}.$$

This finishes the proof of Theorem 1.

#### 4. EXPONENTIAL DECAY TO GLOBAL EQUILIBRIUM

This section is devoted to Theorem 2. The proof relies on the spectral gap of Theorem 1 and on the hypocoercivity method by Mouhot and Neumann [28]. We have to show that there exists a suitable decomposition of  $L$  for which conditions (i) – (iv) in Theorem 3 hold.

We preliminarily observe that  $L$  is bounded w.r.t. the  $\mathcal{H}$  norm, that is:

$$|(f, Lg)_{L_p^2}| \leq C \|f\|_{\mathcal{H}} \|g\|_{\mathcal{H}}, \quad f, g \in D(L). \quad (4.1)$$

Relation (4.1) can be showed by arguing as in the proof of (3.5).

Using formulation (2.9), the operator  $L$  can be rewritten as  $L = K - \Lambda$  with:

$$\begin{aligned} K_i(f) &:= -\frac{1}{\sqrt{M_i}} \sum_{j=1}^N \operatorname{div}_p \int M_i M_j' A^{(ij)} \left[ \frac{p}{m_i} - \frac{p'}{m_j} \right] \nabla \left( \frac{f_j'}{\sqrt{M_j'}} \right) dp', \\ \Lambda_i(f) &:= -\frac{1}{\sqrt{M_i}} \sum_{j=1}^N \operatorname{div}_p \int M_i M_j' A^{(ij)} \left[ \frac{p}{m_i} - \frac{p'}{m_j} \right] \nabla \left( \frac{f_i}{\sqrt{M_i}} \right) dp'. \end{aligned} \quad (4.2)$$

For the operator  $\Lambda$  we will use the following estimates proven by Guo in [24]: for each  $f \in D(L)$  we have

$$c_1 \|f\|_{\mathcal{H}}^2 \leq (f, \Lambda f)_{L_p^2} \leq c_2 \|f\|_{\mathcal{H}}^2, \quad (4.3)$$

$$(\nabla f, \nabla \Lambda f)_{L_p^2} \geq c_3 \|\nabla f\|_{\mathcal{H}}^2 - c_4 \|f\|_{L_p^2}^2. \quad (4.4)$$

Concerning  $K$ , we need the following lemma:

**Lemma 5.** *For every  $\delta > 0$  there exists a constant  $C(\delta) > 0$  such that:*

$$(\nabla f, \nabla K f)_{L_p^2} \leq \delta \|\nabla f\|_{L_p^2}^2 + C(\delta) \|f\|_{L_p^2}^2. \quad (4.5)$$

*Proof.* We first observe that

$$\operatorname{div}_z A^{(ij)}[z] = -2C^{(ij)}|z|^\gamma z, \quad \text{and} \quad \nabla M_i = -\frac{p}{m_i k_B T} M_i.$$

Then  $K$  can be rewritten as:

$$K_{i,j}(f) = -\frac{1}{\sqrt{M_i}} \int \operatorname{div}_p \left( M_i A^{(ij)} \left[ \frac{p}{m_i} - \frac{p'}{m_j} \right] \right) M_j' \nabla \left( \frac{f_j'}{\sqrt{M_j'}} \right) dp' \quad (4.6)$$

$$\begin{aligned} &= \int \omega^{(ij)} \cdot \sqrt{M_j'} \nabla \left( \frac{f_j'}{\sqrt{M_j'}} \right) dp' \\ &= \int \omega^{(ij)} \cdot \left( \nabla f_j' + f_j' \frac{p'}{m_j k_B T} \right) dp', \end{aligned} \quad (4.7)$$

with the kernel  $\omega^{(ij)}$  defined as:

$$\omega^{(ij)} := \sqrt{M_i M_j'} \left( A^{(ij)} \left[ \frac{p}{m_i} - \frac{p'}{m_j} \right] \frac{p}{m_i k_B T} + \frac{2C^{(ij)}}{m_i} \left| \frac{p}{m_i} - \frac{p'}{m_j} \right|^\gamma \left( \frac{p}{m_i} - \frac{p'}{m_j} \right) \right).$$

It is useful to estimate  $\omega^{(ij)}$  and its Jacobian. Since

$$|A^{(ij)}[z]v| \leq C^{(ij)}|z|^{\gamma+2}|v|,$$

we have

$$|\omega^{(ij)}| \leq \sqrt{M_i M_j'} \left( \frac{|p|}{m_i k_B T} + \frac{2C^{(ij)}}{m_i} \right) \left( \left| \frac{p}{m_i} - \frac{p'}{m_j} \right|^{\gamma+2} + \left| \frac{p}{m_i} - \frac{p'}{m_j} \right|^{\gamma+1} \right). \quad (4.8)$$

Taking into account that

$$\nabla \sqrt{M_i} = -\sqrt{M_i} \frac{p}{2m_i k_B T}, \quad \left| \nabla \otimes A^{(ij)} \left[ \frac{p}{m_i} - \frac{p'}{m_j} \right] \right| \leq C \left| \frac{p}{m_i} - \frac{p'}{m_j} \right|^{\gamma+1},$$

for some suitable polynomial  $q(|p|)$  we have

$$|\nabla_p \otimes \omega^{(ij)}| \leq \sqrt{M_i M_j'} q(|p|) \left( \left| \frac{p}{m_i} - \frac{p'}{m_j} \right|^\gamma + \left| \frac{p}{m_i} - \frac{p'}{m_j} \right|^{\gamma+2} \right). \quad (4.9)$$

Let us now introduce an arbitrary parameter  $\varepsilon > 0$  and a cutoff function  $\psi_\varepsilon : [0, \infty) \rightarrow [0, 1]$  such that  $\psi_\varepsilon \in C^1([0, \infty))$ ,  $\psi_\varepsilon(x) = 1$  for  $0 \leq x \leq \varepsilon$ ,  $\psi_\varepsilon(x) = 0$  for  $x \geq 2\varepsilon$ ,  $|\psi_\varepsilon'| \leq C\varepsilon^{-1}\chi_{(0, 2\varepsilon)}$ .

Moreover let us define  $\Psi_\varepsilon^{(ij)}(p, p') = \psi_\varepsilon \left( \left| \frac{p}{m_i} - \frac{p'}{m_j} \right| \right)$ .

We write  $K = K^{(I)} + K^{(II)}$ , where:

$$K_i^{(I)}(f) = \sum_{j=1}^N \int \left(1 - \Psi_\varepsilon^{(ij)}\right) \omega^{(ij)} \cdot \left(\nabla f'_j + f'_j \frac{p'}{m_j k_{BT}}\right) dp',$$

$$K_i^{(II)}(f) = \sum_{j=1}^N \int \Psi_\varepsilon^{(ij)} \omega^{(ij)} \cdot \left(\nabla f'_j + f'_j \frac{p'}{m_j k_{BT}}\right) dp'.$$

The function  $\omega^{(ij)}$  is smooth in the region  $\{|p/m_i - p'/m_j| > 2\varepsilon\}$ , thus

$$(1 + |p'|)\Delta_p \otimes \left(\left(1 - \Psi_\varepsilon^{(ij)}\right) \omega^{(ij)}\right) \in L_{p,p'}^\infty.$$

It follows:

$$\begin{aligned} (\nabla f, \nabla K^{(I)} f)_{L_p^2} &= \sum_{i,j=1}^N \iint \nabla_p f_i \cdot \left(\nabla_p \otimes \left(\left(1 - \Psi_\varepsilon^{(ij)}\right) \omega^{(ij)}\right)\right) \left(\nabla f'_j + f'_j \frac{p'}{m_j k_{BT}}\right) dp dp' \\ &= - \sum_{i,j=1}^N \iint f_i \left(\Delta_p \left(\left(1 - \Psi_\varepsilon^{(ij)}\right) \omega^{(ij)}\right)\right) \left(\nabla f'_j + f'_j \frac{p'}{m_j k_{BT}}\right) dp dp' \\ &\leq C(\varepsilon) \|f\|_{L_p^2} \left(\|\nabla f\|_{L_p^2} + \|f\|_{L_p^2}\right) \leq \delta \|\nabla f\|_{L_p^2}^2 + \delta^{-1} C(\varepsilon) \|f\|_{L_p^2}^2. \end{aligned} \tag{4.10}$$

Let us now estimate the term:

$$\begin{aligned} (\nabla f, \nabla K^{(II)} f)_{L_p^2} &= \sum_{i,j=1}^N \iint \nabla_p f_i \cdot \left(\nabla_p \otimes \left(\Psi_\varepsilon^{(ij)} \omega^{(ij)}\right)\right) \left(\nabla f'_j + f'_j \frac{p'}{m_j k_{BT}}\right) dp dp' \\ &= \sum_{i,j=1}^N \iint \Psi_\varepsilon^{(ij)} \nabla_p f_i \cdot \left(\nabla_p \otimes \omega^{(ij)}\right) \left(\nabla f'_j + f'_j \frac{p'}{m_j k_{BT}}\right) dp dp' \\ &\quad + \sum_{i,j=1}^N \iint (\nabla_p f_i \cdot \nabla_p \Psi_\varepsilon^{(ij)}) \left(\omega^{(ij)} \cdot \left(\nabla f'_j + f'_j \frac{p'}{m_j k_{BT}}\right)\right) dp dp'. \end{aligned}$$

Our aim is now to bound  $(\nabla f, \nabla K^{(II)} f)_{L_p^2}$  with respect to the  $L^2$ -norms of  $f$  and  $\nabla f$ . By using (4.9) we get

$$\begin{aligned} \Psi_\varepsilon^{(ij)} (1 + |p'|) |\nabla_p \otimes \omega^{(ij)}| &\leq \Psi_\varepsilon^{(ij)} (1 + |p'|) \sqrt{M_i M'_j} q(|p|) \left( \left| \frac{p}{m_i} - \frac{p'}{m_j} \right|^\gamma + \left| \frac{p}{m_i} - \frac{p'}{m_j} \right|^{\gamma+2} \right) \\ &\leq C \left( \left| \frac{p}{m_i} - \frac{p'}{m_j} \right|^\gamma + \left| \frac{p}{m_i} - \frac{p'}{m_j} \right|^{\gamma+2} \right) \chi_{\left| \frac{p}{m_i} - \frac{p'}{m_j} \right| \leq 2\varepsilon} =: \Phi_{1,\varepsilon} \left( \frac{p}{m_i} - \frac{p'}{m_j} \right). \end{aligned}$$

Assumption  $\gamma + 2 \geq 0$  implies

$$\|\Phi_{1,\varepsilon}\|_{L^1(\mathbb{R}^3)} \leq C(\varepsilon^{\gamma+3} + \varepsilon^{\gamma+5}) \leq C\varepsilon,$$

which yields

$$\begin{aligned}
& \sum_{i,j=1}^N \iint \Psi_\varepsilon^{(ij)} \nabla_p f_i \cdot (\nabla_p \otimes \omega^{(ij)}) \left( \nabla f'_j + f'_j \frac{p'}{m_j k_B T} \right) dp dp' \\
& \leq C \sum_{i,j=1}^N \iint \Psi_\varepsilon^{(ij)} (1 + |p'|) |\nabla_p \otimes \omega^{(ij)}| |\nabla_p f_i| (|\nabla f'_j| + f'_j) dp dp' \\
& \leq C \sum_{i,j=1}^N \iint \Phi_{1,\varepsilon} \left( \frac{p}{m_i} - \frac{p'}{m_j} \right) |\nabla_p f_i| (|\nabla f'_j| + f'_j) dp dp' \\
& = C \sum_{i,j=1}^N \int |\nabla_p f_i| (\Phi_{1,\varepsilon} * (|\nabla f_j| + f_j)) dp \\
& \leq C\varepsilon (\|\nabla f\|_{L^2}^2 + \|f\|_{L^2}^2).
\end{aligned}$$

From the estimate on  $\omega^{(ij)}$  it follows

$$\begin{aligned}
& |\nabla_p \Psi_\varepsilon^{(ij)}| (1 + |p'|) |\omega^{(ij)}| \leq \\
& \leq \frac{C}{\varepsilon} (1 + |p'| + |p|) \sqrt{M_i M'_j} \left( \left| \frac{p}{m_i} - \frac{p'}{m_j} \right|^{\gamma+2} + \left| \frac{p}{m_i} - \frac{p'}{m_j} \right|^{\gamma+1} \right) \chi_{\left| \frac{p}{m_i} - \frac{p'}{m_j} \right| \leq 2\varepsilon} \\
& \leq \frac{C}{\varepsilon} \left( \left| \frac{p}{m_i} - \frac{p'}{m_j} \right|^{\gamma+2} + \left| \frac{p}{m_i} - \frac{p'}{m_j} \right|^{\gamma+1} \right) \chi_{\left| \frac{p}{m_i} - \frac{p'}{m_j} \right| \leq 2\varepsilon} \\
& =: \Phi_{2,\varepsilon} \left( \frac{p}{m_i} - \frac{p'}{m_j} \right).
\end{aligned}$$

As before, assumption  $\gamma + 2 \geq 0$  implies

$$\|\Phi_{2,\varepsilon}\|_{L^1(\mathbb{R}^3)} \leq C(\varepsilon^{\gamma+3} + \varepsilon^{\gamma+4}) \leq C\varepsilon,$$

and therefore

$$\begin{aligned}
& \sum_{i,j=1}^N \iint (\nabla_p f_i \cdot \nabla_p \Psi_\varepsilon^{(ij)}) \left( \omega^{(ij)} \cdot \left( \nabla f'_j + f'_j \frac{p'}{m_j k_B T} \right) \right) dp dp' \\
& \leq \sum_{i,j=1}^N \int |\nabla_p f_i| (\Phi_{2,\varepsilon} * (|\nabla f_j| + f_j)) dp \\
& \leq C\varepsilon (\|\nabla f\|_{L^2}^2 + \|f\|_{L^2}^2).
\end{aligned}$$

Summarizing

$$(\nabla f, \nabla K^{(II)} f)_{L^2} \leq C\varepsilon (\|\nabla f\|_{L^2}^2 + \|f\|_{L^2}^2), \quad (4.11)$$

which, together with (4.10), yields (4.5).  $\square$

Relations (4.1)–(4.5) and the spectral gap allow us to apply Theorem 3, which yields (1.8).

We now show the second part of Theorem 2. The non-linear terms  $\Gamma_i(f_i, f_j)$ , defined as

$$\Gamma_i(f_i, f_j) = \frac{1}{\sqrt{M_i}} \sum_{j=1}^N Q_{ij}(M_i + \sqrt{M_i}f_i, M_j + \sqrt{M_j}f_j),$$

can be rewritten as

$$\Gamma_i(f_i, f_j) = \sum_{j=1}^N \Theta_i(f_i, f_j),$$

with

$$\begin{aligned} \Theta_i(f_i, f_j) = & \operatorname{div}_p \left( \int A^{(ij)} \left[ \frac{p}{m_i} - \frac{p'}{m_j} \right] \sqrt{M'_j} f'_j dp' \cdot \nabla f_i \right) \\ & - \operatorname{div}_p \left( f_i \int A^{(ij)} \left[ \frac{p}{m_i} - \frac{p'}{m_j} \right] \sqrt{M'_j} \nabla f'_j dp' \right) \\ & - \int A^{(ij)} \left[ \frac{p}{m_i} - \frac{p'}{m_j} \right] \frac{p'}{m_j} \sqrt{M'_j} f'_j dp' \cdot \nabla f_i \\ & + f_i \int A^{(ij)} \left[ \frac{p}{m_i} - \frac{p'}{m_j} \right] \frac{p'}{m_j} \sqrt{M'_j} \cdot \nabla f'_j dp'. \end{aligned}$$

We now recall an estimate by Guo in [24, Thr. 3] which states that the inner product  $(\Theta_i(f_i, f_j), f_i)_{L^2_{x,p}}$  can be bounded by the  $L^2_{x,p}$  and  $L^2_x \mathcal{H}$  norms of  $f_i$  and  $f_j$ ; more precisely

$$(\Theta_i(f_i, f_j), f_i)_{L^2_{x,p}} \leq C \left( \|f_i\|_{L^2_{x,p}} \|f_j\|_{L^2_x \mathcal{H}} + \|f_j\|_{L^2_{x,p}} \|f_i\|_{L^2_x \mathcal{H}} \right) \|f_i\|_{L^2_x \mathcal{H}}.$$

Therefore

$$(\Gamma_i(f_i, f_j), f_i)_{L^2_{x,p}} \leq C \|f_i\|_{L^2_{x,p}} \|f_i\|_{L^2_x \mathcal{H}} \left( \sum_{i=1}^N \|f_j\|_{L^2_x \mathcal{H}} \right) + \|f_i\|_{L^2_x \mathcal{H}}^2 \left( \sum_{i=1}^N \|f_j\|_{L^2_{x,p}} \right),$$

which implies

$$(\Gamma(f, f), f)_{L^2_{x,p}} := \sum_{i=1}^N (\Gamma_i(f_i, f_j), f_i)_{L^2_{x,p}} \leq C \|f\|_{L^2_{x,p}} \|f\|_{L^2_x \mathcal{H}}^2. \quad (4.12)$$

Define now the function  $f := \frac{F-\mathcal{M}}{\sqrt{\mathcal{M}}}$  with  $\mathcal{M}(p)$  and  $F$  respectively the unique equilibrium state and the unique smooth solution to (1.1). The function  $f = (f_1, f_2, \dots, f_N)$  solves

$$\partial_t f_i + \frac{p}{m_i} \cdot \nabla_x f_i = \sum_{j=1}^N L_{ij}(f_i, f_j) + \Gamma_i(f_i, f_j).$$

Thanks to Theorem 1 and (4.12) one can deduce

$$\frac{1}{2} \partial_t \|f\|_{L^2_{x,p}}^2 \leq -\lambda \|f\|_{L^2_x \mathcal{H}}^2 + C \|f\|_{L^2_{x,p}} \|f\|_{L^2_x \mathcal{H}}^2.$$



The above differential inequality can be solved by simple iteration method: since  $\|f_{\text{in}}\|_{L_x^2 \mathcal{H}} \leq \varepsilon$ , there exists a positive time  $T_0$  such that  $\|f\|_{L_{x,p}^2} \leq 2\varepsilon$  for all  $t \in [0, T_0]$ . Hence any solution to

$$\frac{1}{2} \partial_t \|h\|_{L_{x,p}^2}^2 = -\frac{\lambda}{2} \|h\|_{L_x^2 \mathcal{H}}^2, \quad \|h_{\text{in}}\|_{L_x^2 \mathcal{H}} = \varepsilon,$$

satisfies  $\|f\|_{L_{x,p}^2}^2 \leq \|h\|_{L_{x,p}^2}^2 \leq \varepsilon e^{-\lambda t}$  for  $t \in [0, T_0]$ , taking into account that the  $L_x^2 \mathcal{H}$ -norm controls the  $L_{x,p}^2$ -norm. At time  $T_0$  we can restart the same process since  $\|f(\cdot, T_0)\|_{L_x^2 \mathcal{H}} \leq \varepsilon$ . This finishes the proof of Theorem 2.

## 5. APPENDIX

**Lemma 6.** *The operator  $K : L_p^2 \rightarrow L_p^2$  defined in (4.2) is compact.*

*Proof.* We will show that  $K$  is the limit, in the operator norm, of a sequence of Hilbert-Schmidt operators. From (4.6) it follows:

$$K_i(f) = \sum_{j=1}^N \int k^{(ij)}(p, p') f_j(p') dp', \quad k^{(ij)}(p, p') = \frac{p'}{m_j k_B T} \cdot \omega^{(ij)} - \text{div}_{p'} \omega^{(ij)}.$$

The following estimate is a consequence of (4.8) and (4.9):

$$\begin{aligned} |k^{(ij)}(p, p')| &\leq C (M_i(p) M_j(p'))^{1/4} \left( \left| \frac{p}{m_i} - \frac{p'}{m_j} \right|^\gamma + \left| \frac{p}{m_i} - \frac{p'}{m_j} \right|^{\gamma+2} \right) \\ &\leq CW \left( \frac{p}{m_i} - \frac{p'}{m_j} \right), \\ W(z) &\equiv e^{-\delta|z|^2} (|z|^\gamma + |z|^{\gamma+2}), \end{aligned}$$

for some suitable constant  $\delta > 0$ .

Let  $\xi_n$  be the characteristic function of the ball  $B(0, \frac{1}{n})$ , and let us define the sequence of operators  $K^{(n)} = (K_1^{(n)}, \dots, K_N^{(n)}) : L_p^2 \rightarrow L_p^2$ ,

$$\begin{aligned} K_i^{(n)}(f) &= \sum_{j=1}^N \int k_n^{(ij)}(p, p') f_j(p') dp', \\ k_n^{(ij)}(p, p') &= k^{(ij)}(p, p') \left( 1 - \xi_n \left( \frac{p}{m_i} - \frac{p'}{m_j} \right) \right). \end{aligned}$$

It is clear that  $k_n^{(ij)} \in L_{p,p'}^2$ , so  $K^{(n)}$  is a Hilbert-Schmidt operator for all  $n \in \mathbb{N}$ . In particular  $K^{(n)}$  is compact. Let us now estimate:

$$\begin{aligned} \left| K_i(f) - K_i^{(n)}(f) \right| &\leq \int |k^{(ij)}(p, p')| \xi_n \left( \frac{p}{m_i} - \frac{p'}{m_j} \right) |f_j(p')| dp' \\ &\leq \sum_{j=1}^N (W \xi_n) * f_j. \end{aligned}$$

It follows:

$$\frac{\|K(f) - K^{(n)}(f)\|_{L^2}}{\|f\|_{L^2}} \leq C \|W \xi_n\|_{L^1} = C \int_{\{|z| < 1/n\}} e^{-\delta|z|^2} (|z|^\gamma + |z|^{\gamma+2}) dz \leq \frac{C}{n},$$

because  $\gamma + 2 \geq 0$ . This means that  $K^{(n)} \rightarrow K$  strongly in  $\mathcal{L}(L_p^2)$ , which implies that  $K$  is compact. This finishes the proof.  $\square$

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DEPARTMENT OF MATHEMATICS, GEORGE WASHINGTON UNIVERSITY, 801 22ND STREET, NW WASHINGTON DC, 20052, USA

*E-mail address:* `gualdani@gwu.edu`

INSTITUTE FOR ANALYSIS AND SCIENTIFIC COMPUTING, VIENNA UNIVERSITY OF TECHNOLOGY, WIEDNER HAUPTSTRASSE 8-10, 1040 WIEN, AUSTRIA

*E-mail address:* `nicola.zamponi@tuwien.ac.at`