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REGULAR GALERKIN APPROXIMATIONS OF HOLOMORPHIC T-GÅRDING OPERATOR EIGENVALUE PROBLEMS

MARTIN HALLA

Abstract. In this paper we deal with holomorphic Fredholm operator eigenvalue problems and their approximations by Galerkin schemes. We consider operator functions, which don't have the structure "coercive+compact". In this case the regularity (in sense of [O. Karma, Numer. Funct. Anal. Optim. 17 (1996)]) of Galerkin approximations is not unconditionally satisfied. We assume that this structure is regained by multiplication with an appropriate operator. We analyze in which sense this operator has to be approximable by operators mapping from the Galerkin spaces into themselves in order to ensure the regularity of Galerkin approximations. We report a sufficient condition which is verifiable in praxis and hence provide a new tool for the numerical analysis of holomorphic Fredholm operator eigenvalue problems.

1. Introduction

For an open subset $\Lambda \subset \mathbb{C}$ and the space of bounded linear operators $\mathcal{L}(X)$ from a Hilbert space X to itself, we consider holomorphic Fredholm operator functions $A(\cdot)\colon \Lambda \to \mathcal{L}(X)$ and the eigenvalue problem to find $(\lambda, x) \in \Lambda \times X$ such that $A(\lambda)x = 0$. The analysis of approximations for holomorphic Fredholm operator eigenvalue problems has a long history [**13**], [**20**], [**14**], [**21**], [**15**], [**16**] and is usually performed in the framework of discrete approximation schemes [**18**] and regular approximations of operator functions [**12**], [**1**]. In this framework a complete convergence analysis and asymptotic error estimates for eigenvalues are given by Karma in [**15**], [**16**]. If the discrete approximation scheme is chosen as a Galerkin scheme, then the assumptions of [**15**], [**16**] reduce to a single assumption: the regular approximation property (see Lem. 2.7 for the meaning of regularity). If for every $\lambda \in \Lambda$ the operator $A(\lambda)$ is a compact perturbation of a coercive operator (by a coercive operator A we mean that $|\langle Ax, x\rangle_X| \geq c\|x\|_X^2$ for a $c > 0$ and all $x \in X$), then the regularity of $A_n(\cdot)$ is unconditionally satisfied.

However, if $A(\cdot)$ is not of this kind the question of spectrally converging approximations is very delicate. This can already be observed for linear eigenvalue problems, see e.g. [**4**]. A framework which in particular incorporates a huge class of important linear eigenvalue problems for such difficult operators is finite element exterior calculus [**2**], [**3**]. This framework yields sufficient conditions on the Galerkin spaces X_n to obtain spectral convergence.

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In general the analysis of an operator A starts by verifying its Fredholmness. A feasible way to do this is to construct a bijective operator T such that T^*A is Gårding (T^* denotes the adjoint operator of T), i.e. T^*A is a compact perturbation of a coercive operator. In this case we call A to be T-Gårding. If A is T-Gårding, it is not only Fredholm but also of index zero. The notion of T-Gårding is closely related to T-coercivity, which was introduced [**7**] and further investigated e.g. in [**17**], [**5**], [**6**], [**9**]. For an operator A to be T-coercive means that T^*A is already coercive. However, in eigenvalue problems the operator $A(\lambda)$ will be in general not bijective (precisely at the eigenvalues of $A(\cdot)$). Thus the nomenclature of T-coercivity is not meaningful for our purposes.

In the finite element community it is commonly agreed that in order to obtain stable Galerkin approximations $A_n(\cdot)$ for non-Gårding operator functions $A(\cdot)$, the approximations $A_n(\cdot)$ should mimic the properties of the approximated operator function $A(\cdot)$ as much as possible. It is also known (see e.g. [**4**] for mixed finite element methods) that generally the requirements on $A_n(\cdot)$ to be a converging approximation on the eigenvalue problem are stronger than to be a converging approximation on the source problem (i.e. for $(\lambda, y) \in \Lambda \times X$ find $x \in X$ s.t. $A(\lambda)x = y$). A way to capture the notion of T-Gårding operators for Galerkin approximations is to demand the existence of bounded linear operators T_n mapping from the Galerkin space X_n to itself with prescribed assumptions.

In this paper we establish a sufficient condition on T_n to obtain regular approximations A_n, i.e. the theory of [**15**], [**16**] can be applied which yields spectral convergence. The condition is

$$\lim_{n\to\infty} \|T - T_n\|_n = 0,$$

where

$$\|T - T_n\|_n := \sup_{\substack{x_n \in X_n,\\ \|x_n\|_X = 1}} \|(T - T_n)x_n\|_X.$$

This condition enables to verify the regularity of approximations in praxis and yields new possibilities to perform numerical analysis for non-linear eigenvalue problems.

An intended application of this work is the convergence analysis of boundary element discretizations of boundary integral formulations of Maxwell eigenvalue problems, see [**22**] for numerical experiments. Although the Maxwell eigenvalue problem is of linear nature, its formulation as boundary integral equation becomes non-linear due to the dependency of the fundamental solution on the frequency.

A further possible application of this work is the finite element error analysis for partial differential equations with sign-changing coefficients in the principal part of the differential operator. Therein the sign-change of the coefficient destroys the coercivity structure. Such problems occur if meta materials are coupled with classical ones [**7**], [**5**], [**6**]. Note that the negative coefficients in these references stem from prefactors of the kind $(1 - 1/\omega^2)^{-1}$ with ω^2 being the eigenvalue parameter. Hence eigenvalue problems for such configurations are indeed non-linear.

Casually, this paper can serve as an intermediate reference for colleagues in the numerical analysis community who want to apply [**15**], [**16**] to Galerkin schemes, but are very unfamiliar with the notion of discrete approximation schemes and its notation. This is also the motivation for some rather pedagogical lemmata and remarks at the end of this paper.

The remain of the paper is structured as follows. In Section 2 we introduce the nomenclature of convergence in discrete norm, $T(\cdot)$-Gårding operator functions and $T(\cdot)$-compatible approximations and study their properties. In Section 3 we establish the result that $T(\cdot)$-compatible approximations of $T(\cdot)$-Gårding operator functions lead to spectral convergence. We also prove a new asymptotic error estimate on the eigenspaces for Galerkin approximations, which was previously only known for operator functions of a special kind [**19**].

2. Preliminary results

First we set our notation. Let X be a Hilbert space over $\mathbb{C}$ with scalar product $\langle .,.\rangle_X$ and induced norm $\|.\|_X$. Let $\mathcal{L}(X)$ be the space of bounded linear operators from X to X with operator norm $\|.\|_{\mathcal{L}(X}$, i.e. $\|A\|_{\mathcal{L}(X)} := \sup_{x\in X\setminus\{0\}} \frac{\|Ax\|_X}{\|x\|_X}$ for $A \in \mathcal{L}(X)$. For an operator A let A^* be the adjoint operator.

Definition 2.1. *Let $A, T \in \mathcal{L}(X)$ and T be bijective. Then A is*

1. *coercive, if $\inf_{x\in X\setminus\{0\}} |\langle Ax, x\rangle_X|/\|x\|^2 > 0$,*
2. *Gårding, if there exists a compact operator $K \in \mathcal{L}(X)$ such that $A + K$ is coercive,*
3. *T-Gårding, if there exists a compact operator $K \in \mathcal{L}(X)$ such that T^*A+K is coercive.*

Definition 2.2 ($T(\cdot)$-Gårding operator function)**.** *Let Λ be an open subset of $\mathbb{C}$ and $A(\cdot), T(\cdot)\colon \Lambda \to \mathcal{L}(X)$ be such that $T(\lambda)$ is bijective for all $\lambda \in \Lambda$. $A(\cdot)$ is $T(\cdot)$-Gårding if there exists an operator function $K(\cdot)\colon \Lambda \to \mathcal{L}(X)$ such that*

$$T(\lambda)^* A(\lambda) + K(\lambda)$$

is coercive and $K(\lambda)$ is compact for all $\lambda \in \Lambda$.

Remark 2.3. *We could expand the previous definitions and all following results in this paper to operators A (and operator functions likewise) which yield two bijective operators T_L and T_R, such that $T_L^* A T_R$ is Gårding. All forthcoming assumptions on T would have to be posed likewise on T_L and T_R to yield the same results. However, we don't know of possible applications where the concept of T-Gårding wouldn't suffice but the concept of T_L-T_R-Gårding would. Thus we stick to the notion of only one operator T, which simplifies the presentation.*

2.1. **Discrete convergence.** Let $(X_n)_{n\in\mathbb{N}} \subset X^{\mathbb{N}}$ be a sequence of closed subspaces with orthogonal projections P_n onto X_n, such that $(P_n)_{n\in\mathbb{N}}$ converges point-wise to the identity, i.e. $\lim_{n\to\infty} \|x - P_n x\|_X \to 0$ for all $x \in X$. Let $\mathcal{L}(X_n)$ be the spaces of bounded linear operators from X_n to X_n. For a bounded linear operator $T\colon X \to X$ or $T\colon X_n \to X_n, n \in \mathbb{N}$ or a sum of such define

$$\|T\|_n := \sup_{x_n\in X_n\setminus\{0\}} \frac{\|Tx_n\|_X}{\|x_n\|_X}. \tag{1}$$

Let $T \in \mathcal{L}(X)$ and $(T_n)_{n\in\mathbb{N}}$ be a sequence of operators with $T_n \in \mathcal{L}(X_n)$. We say that T_n converges to T in discrete norm, if

$$\lim_{n\to\infty} \|T - T_n\|_n = 0. \tag{2}$$

We mention that the discrete norm $\|\cdot\|_n$ has already appeared in [**10**], however in a different context. Let us collect some basic facts, which immediately follow from the definition of convergence in discrete norm.

i) For $i=1,2$ let $T^i \in \mathcal{L}(X)$, $(T^i_n)_{n\in\mathbb{N}}$ be sequences with $T^i_n \in \mathcal{L}(X_n)$ such that T^i_n converges to T^i in discrete norm. Then for any $c\in\mathbb{C}$, $T^1_n + cT^2_n$ converges to T^1+cT^2 in discrete norm.
ii) Let $T\in\mathcal{L}(X)$ be X_n invariant for all $n\in\mathbb{N}$, i.e. $Tx_n \in X_n$ for all $x_n\in X_n$ and all $n\in\mathbb{N}$. Let $T_n := P_nTP_n|_{X_n}$. Then $\|T-T_n\|_n = 0$ for all $n\in\mathbb{N}$.
iii) Let $K\in\mathcal{L}(X)$ be compact and set $K_n := P_nKP_n|_{X_n}$. Then K_n converges to K in discrete norm.

In preparation of our forthcoming analysis we formulate the next lemma.

Lemma 2.4. *Let $T\in\mathcal{L}(X)$ and $(T_n)_{n\in\mathbb{N}}$ be a sequence of operators with $T_n\in\mathcal{L}(X_n)$ and $\lim_{n\to\infty}\|T-T_n\|_n = 0$. Then there exist a constant $c>0$ and an index $n_0\in\mathbb{N}$ such that $\|T_n\|_{\mathcal{L}(X_n)}, \|T_n\|^{-1}_{\mathcal{L}(X_n)} \le c$ for all $n>n_0$. If T is bijective, then there exist a constant $c>0$ and an index $n_0\in\mathbb{N}$ such that T_n is also bijective for all $n>n_0$ and $\|(T_n)^{-1}\|_{\mathcal{L}(X_n)} \le c$.*

Proof. The claims follow from the inequalities
$$\begin{aligned}
\|T_n\|_{\mathcal{L}(X_n)} &\le \|T\|_X + \|T-T_n\|_n,\\
\|T_n\|_{\mathcal{L}(X_n)} &\ge \|T\|_X - \|T-T_n\|_n,\\
\inf_{x_n\in X_n, \|x_n\|=1} \|T_nx_n\|_X &\ge \inf_{x\in X,\|x\|=1} \|Tx\|_X - \|T-T_n\|_n.
\end{aligned}$$
□

2.2. **Compatible approximations.** We define in the following what we mean by $T(\cdot)$-compatible approximations of $T(\cdot)$-Gårding operator functions. The assumption on the Galerkin approximations to be $T(\cdot)$-compatible will be sufficient to ensure their regularity.

Definition 2.5 ($T(\cdot)$-compatible approximation)**.** *Let X be a Hilbert space and $\mathcal{L}(X)$ be the space of bounded linear operators from X to X. Let $(X_n)_{n\in\mathbb{N}} \subset X^{\mathbb{N}}$ be a sequence of closed subspaces with orthogonal projections P_n onto X_n, such that $(P_n)_{n\in\mathbb{N}}$ converges point-wise to the identity, i.e.* $\lim_{n\to\infty}\|x-P_nx\|_X \to 0$ *for all $x\in X$. Let $\Lambda\subset\mathbb{C}$ be open and $A(\cdot)\colon \Lambda\to\mathcal{L}(X)$ be $T(\cdot)$-Gårding (see Def. 2.2). Then we call the sequence of Galerkin approximations $(A_n(\cdot) := P_nA(\cdot)P_n|_{X_n}\colon \Lambda \to \mathcal{L}(X_n))_{n\in\mathbb{N}}$ a $T(\cdot)$–compatible approximation of $A(\cdot)$, if $(A_n(.))_{n\in\mathbb{N}}$ is a sequence of Fredholm operator functions and there exists a sequence of operator functions $(T_n(\cdot))_{n\in\mathbb{N}}$ with $T_n(\cdot)\colon\Lambda\to\mathcal{L}(X_n)$ for each $n\in\mathbb{N}$, such that*
$$\lim_{n\to\infty}\|T(\lambda)-T_n(\lambda)\|_n = 0 \tag{3}$$
for all $\lambda\in\Lambda$.

Lemma 2.6 (Stability of $T(\cdot)$-compatible approximations)**.** *Let $\Lambda\subset\mathbb{C}$ be open and $A(\cdot)\colon\Lambda\to\mathcal{L}(X)$ be a $T(\cdot)$-Gårding operator function with $K(\cdot)$ as in Def. 2.2. Let $(A_n(\cdot))_{n\in\mathbb{N}}$ be a $T(\cdot)$-compatible approximation of $A(\cdot)$ and let $\tilde K(\lambda) := (T(\lambda)^*)^{-1}K(\lambda)$ for $\lambda\in\Lambda$. Then for every $\lambda\in\Lambda$ there exist $n_0\in\mathbb{N}$ and $c>0$, such that $A_n(\lambda)+P_n\tilde K(\lambda)P_n|_{X_n}$ is invertible and*
$$\|\big(A_n(\lambda)+P_n\tilde K(\lambda)P_n|_{X_n}\big)^{-1}\|_{\mathcal{L}(X_n)} \le c$$

for all $n > n_0$.

Proof. Let n be large enough such that $T_n(\lambda)$ is bijective (see Lem. 2.4). We compute

$$\begin{aligned}
&\inf_{x_n\in X_n}\sup_{y_n\in X_n}\frac{|\langle (A(\lambda)+\tilde K(\lambda))x_n, y_n\rangle_X|}{\|x_n\|_X\|y_n\|_X}\\
&\qquad\ge \inf_{x_n\in X_n}\sup_{y_n\in X_n}\frac{|\langle T_n(\lambda)^*(A(\lambda)+\tilde K(\lambda))x_n, y_n\rangle_X|}{\|T_n(\lambda)\|_{\mathcal{L}(X_n)}\|x_n\|_X\|y_n\|_X}\\
&\qquad\ge \inf_{x_n\in X_n}\sup_{y_n\in X_n}\frac{|\langle (T(\lambda)^*A(\lambda)+K(\lambda))x_n, y_n\rangle_X|}{\|T_n(\lambda)\|_{\mathcal{L}(X_n)}\|x_n\|_X\|y_n\|_X}\\
&\qquad\qquad -\frac{\|A(\lambda)+\tilde K(\lambda)\|_{\mathcal{L}(X)}}{\|T_n(\lambda)\|_{\mathcal{L}(X_n)}}\|T(\lambda)-T_n(\lambda)\|_n\\
&\qquad\ge c(\lambda)\|T_n(\lambda)\|^{-1}_{\mathcal{L}(X_n)}-\mathcal{O}(\|T(\lambda)-T_n(\lambda)\|_n)
\end{aligned}$$

with coercivity constant

$$c(\lambda) := \inf_{x\in X\setminus\{0\}} |\langle (T(\lambda)^*A(\lambda)+K(\lambda))x, x\rangle_X|/\|x\|_X^2 > 0.$$

Since $\|T_n(\lambda)\|^{-1}_{\mathcal{L}(X_n)}$ is uniformly bounded from below (see Lem. 2.4) and $\lim_{n\to\infty}\|T(\lambda)-T_n(\lambda)\|_n = 0$ by assumption, it follows the existence of $n_0\in\mathbb{N}$ and $\tilde c > 0$ such that

$$\inf_{x_n\in X_n}\sup_{y_n\in X_n}\frac{|\langle (A(\lambda)+\tilde K(\lambda))x_n, y_n\rangle_X|}{\|x_n\|_X\|y_n\|_X}\ge\tilde c.$$

The inequality

$$\inf_{y_n\in X_n}\sup_{x_n\in X_n}\frac{|\langle (A(\lambda)+\tilde K(\lambda))x_n, y_n\rangle_X|}{\|x_n\|_X\|y_n\|_X}\ge\tilde c$$

for all $n > n_0$ can be proven similar. The bijectivity of $A_n(\lambda)+P_n\tilde K(\lambda)P_n|_{X_n}$ and the estimate $\|\big(A_n(\lambda)+P_n\tilde K(\lambda)P_n|_{X_n}\big)^{-1}\|_{\mathcal{L}(X_n)}\le 1/\tilde c$ for $n > n_0$ follow from the famous equivalences to inf-sup-conditions, see e.g. [**8**]. ☐

Lemma 2.7 (Regularity of $T(\cdot)$-compatible approximations)**.** *Let $\Lambda\subset\mathbb{C}$ be open and let the sequence $(A_n(\cdot))_{n\in\mathbb{N}}$ be a $T(\cdot)$-compatible approximation of a $T(\cdot)$-Gårding operator function $A(\cdot)\colon\Lambda\to\mathcal{L}(X)$. Then for every $\lambda\in\Lambda$, $(A_n(\lambda))_{n\in\mathbb{N}}$ is a regular approximation of $A(\lambda)$ in the sense of [**15**], i.e. if $(x_n)_{n\in\mathbb{N}}$ is a sequence with $x_n\in X_n$ and $\|x_n\|_X\le 1$ for all $n\in\mathbb{N}$ and $\big(A_n(\lambda)x_n\big)_{n\in\mathbb{N}}$ is a compact sequence, then this implies that the sequence $(x_n)_{n\in\mathbb{N}}$ is already compact.*

Proof. Let $\lambda\in\Lambda$ be given. W.l.o.g. let $\big(A_n(\lambda)x_n\big)_{n\in\mathbb{N}}$ and $y\in X$ be such that $\lim_{n\to\infty}\|y-A_n(\lambda)x_n\|_X = 0$. Let $K(\cdot)$ be as in Def. 2.2 and let $\tilde K(\lambda) := (T(\lambda)^*)^{-1}K(\lambda)$ for $\lambda\in\Lambda$. Let P_n be the orthogonal projection onto X_n. Let $\tilde A(\lambda) := A(\lambda)+\tilde K(\lambda)$ and $\tilde A_n(\lambda) := P_n\tilde A(\lambda)P_n|_{X_n}$. Since $\tilde K(\lambda)$ is compact and $\|x_n\|_X\le 1$ for all $n\in\mathbb{N}$, there exist a subsequence $(x_{n(m)})_{m\in\mathbb{N}}$ and $w\in X$ such that $\lim_{m\to\infty}\|w-\tilde K(\lambda)x_{n(m)}\|_X = 0$. It follows

$$\lim_{m\to\infty}\|\tilde A_{n(m)}(\lambda)x_{n(m)}-(y+w)\|_X = 0.$$

Due to Lem. 2.6 there exist $c > 0$ and $m_0 \in \mathbb{N}$, such that for all $m > m_0$ the operator $\tilde{A}_{n(m)}(\lambda)$ is invertible and $\|\tilde{A}_{n(m)}(\lambda)^{-1}\|_{\mathcal{L}(X_{n(m)})} \leq c$. For $m > m_0$ we compute

$$\begin{aligned}\|x_{n(m)} - &\tilde{A}(\lambda)^{-1}(y+w)\|_X \\ &\leq \|x_{n(m)} - P_{n(m)}\tilde{A}(\lambda)^{-1}(y+w)\|_X + \|(\mathrm{Id} - P_{n(m)})\tilde{A}(\lambda)^{-1}(y+w)\|_X \\ &\leq c\|\tilde{A}_{n(m)}(\lambda)x_{n(m)} - \tilde{A}_{n(m)}(\lambda)P_{n(m)}\tilde{A}(\lambda)^{-1}(y+w)\|_X \\ &+ \|(\mathrm{Id} - P_{n(m)})\tilde{A}(\lambda)^{-1}(y+w)\|_X.\end{aligned}$$

Since $P_{n(m)}$ converge point-wise to the identity and due to the definition of $\tilde{A}_n(\lambda)$, the claim follows. $\square$

3. Spectral convergence

We are now able to prove the main Theorem 3.1 of this paper. The notion of $T(\cdot)$-compatible approximations of $T(\cdot)$-Gårding operator functions allows us to apply the theory of Karma [**15**], [**16**], which yields most results of Thm. 3.1.

Let $\Lambda \subset \mathbb{C}$ be open. An operator function $A(\cdot)\colon \Lambda \to \mathcal{L}(X)$ is called holomorphic, if it is complex-differentiable. For a Fredholm operator function $A(\cdot)\colon \Lambda \to \mathcal{L}(X)$ define the resolvent set

$$\rho\big(A(\cdot)\big) := \{\lambda \in \lambda\colon A(\lambda) \text{ is bijective}\}$$

and the spectrum

$$\sigma\big(A(\cdot)\big) := \Lambda \setminus \rho\big(A(\cdot)\big).$$

We recall [**11**], [**15**] that if $A(\cdot)\colon \Lambda \to \mathcal{L}(X)$ is a holomorphic Fredholm operator function, then the index of $A(\lambda)$ is constant on Λ and if the resolvent set $\rho\big(A(\cdot)\big)$ is not empty, then

i) the index of$A(\lambda)$ vanishes for all $\lambda \in \Lambda$,
ii) the spectrum $\sigma\big(A(\cdot)\big)$ has no cluster points in Λ,
iii) every $\lambda \in \sigma\big(A(\cdot)\big)$ is an eigenvalue of $A(\cdot)$, i.e. $\ker A(\lambda) \neq \{0\}$ for $\lambda \in \sigma\big(A(\cdot)\big)$,
iv) the operator function $A^{-1}(\cdot)$, defined on $\rho\big(A(\cdot)\big)$ by $A^{-1}(\lambda) := A(\lambda)^{-1}$ for all $\lambda \in \rho\big(A(\cdot)\big)$, is holomorphic on $\rho\big(A(\cdot)\big)$ and has poles of finite order at every point $\lambda_0 \in \sigma\big(A(\cdot)\big)$.

For a holomorphic Fredholm operator function $A(\cdot)\colon \Lambda \to \mathcal{L}(X)$ with non-empty resolvent set, we consider the eigenvalue problem to

$$\text{find } (\lambda, u) \in \Lambda \times X \text{ such that} \qquad A(\lambda)u = 0. \tag{4}$$

For solutions (λ, u) to (4) we call λ an eigenvalue and u an eigenfunction.

Theorem 3.1 (Spectral convergence)**.** *Let $\Lambda \subset \mathbb{C}$ be open, X be a Hilbert space and $\mathcal{L}(X)$ be the space of bounded linear operators from X to X. Let $A(\cdot)\colon \Lambda \to \mathcal{L}(X)$ be a holomorphic $T(\cdot)$-Gårding operator function (see Def. 2.2) with non-empty resolvent set $\rho\big(A(\cdot)\big) \neq \emptyset$. Let $(X_n)_{n\in\mathbb{N}} \subset X^{\mathbb{N}}$ be a sequence of closed subspaces with orthogonal projections P_n onto X_n, such that $(P_n)_{n\in\mathbb{N}}$ converges point-wise to the identity, i.e. $\lim_{n\to\infty} \|x - P_n x\|_X = 0$ for all $x \in X$. Let $A_n(\cdot)\colon \Lambda \to \mathcal{L}(X_n)$ be the Galerkin approximation of $A(\cdot)$ defined by $A_n(\lambda) := P_n A(\lambda) P_n|_{X_n}$ for every $\lambda \in \Lambda$. Assume that $(A_n(\cdot))_{n\in\mathbb{N}}$ is a $T(\cdot)$-compatible approximation of $A(\cdot)$ (see Def. 2.5). Then the following results hold.*

i) For every eigenvalue λ_0 *of* $A(\cdot)$ *exists a sequence* $(\lambda_n)_{n\in\mathbb{N}}$ *converging to* λ_0 *with* λ_n *being an eigenvalue of* $A_n(\cdot)$ *for almost all* $n \in \mathbb{N}$.

ii) Let $(\lambda_n, x_n)_{n\in\mathbb{N}}$ *be a sequence of normalized eigenpairs of* $A_n(\cdot)$, *i.e.* $A_n(\lambda_n)x_n = 0$ *and* $\|x_n\|_X = 1$, *so that* $\lambda_n \to \lambda_0 \in \Lambda$, *then*
   a) λ_0 *is an eigenvalue of* $A(\cdot)$,
   b) $(x_n)_{n\in\mathbb{N}}$ *is a compact sequence and its cluster points are normalized eigenelements of* $A(\lambda_0)$.

iii) For every compact $\tilde{\Lambda} \subset \rho(A)$ *the sequence* $(A_n(\cdot))_{n\in\mathbb{N}}$ *is stable on* $\tilde{\Lambda}$, *i.e. there exist* $n_0 \in \mathbb{N}$ *and* $c > 0$ *such that* $\|A_n(\lambda)^{-1}\|_{\mathcal{L}(X_n)} \leq c$ *for all* $n > n_0$ *and all* $\lambda \in \tilde{\Lambda}$.

iv) For every compact $\tilde{\Lambda} \subset \Lambda$ *with boundary* $\partial\tilde{\Lambda} \subset \rho\big(A(\cdot)\big)$ *exist an index* $n_0 \in \mathbb{N}$ *such that*

$$\dim G\big(A(\cdot), \lambda_0\big) = \sum_{\lambda_n \sigma(A_n(\cdot))\cap\tilde{\Lambda}} \dim G\big(A_n(\cdot), \lambda_n\big).$$

for all $n > n_0$, *where* $G\big(B(\cdot), \lambda\big)$ *denotes the generalized eigenspace of an operator function* $B(\cdot)$ *at* $\lambda \in \Lambda$.

Let $\tilde{\Lambda} \subset \Lambda$ *be a compact set with boundary* $\partial\tilde{\Lambda} \subset \rho\big(A(\cdot)\big)$, $\tilde{\Lambda} \cap \sigma\big(A(\cdot)\big) = \{\lambda_0\}$ *and*

$$\delta_n := \max_{\substack{x_0 \in \ker G(A(\cdot),\lambda_0) \\ \|x_0\|_X \leq 1}} \inf_{x_n \in X_n} \|x_0 - x_n\|_X,$$

$$\delta_n^* := \max_{\substack{x_0 \in \ker G\left(A^*(\cdot),\overline{\lambda_0}\right) \\ \|x_0\|_X \leq 1}} \inf_{x_n \in X_n} \|x_0 - x_n\|_X,$$

where $\overline{\lambda_0}$ *denotes the complex conjugate of* λ_0 *and* $A^*(\cdot)$ *the adjoint operator function of* $A(\cdot)$ *defined by* $A^*(\lambda) := A(\lambda)^*$ *for all* $\lambda \in \Lambda$. *Then there exist* $n \in \mathbb{N}$ *and* $c > 0$ *such that for all* $n > n_0$

v)

$$|\lambda_0 - \lambda_n| \leq c(\delta_n \delta_n^*)^{1/\varkappa(A(\cdot),\lambda_0)}$$

for all $\lambda_n \in \sigma\big(A_n(\cdot)\big) \cap \tilde{\Lambda}$, *where* $\varkappa(A(\cdot), \lambda_0)$ *denotes the maximal length of a Jordan chain of the eigenvalue* λ_0,

vi)

$$|\lambda_0 - \bar{\lambda}_n| \leq c\delta_n\delta_n^*,$$

where $\bar{\lambda}_n$ *is the weighted mean of all the eigenvalues of* $A_n(\cdot)$ *in* $\tilde{\Lambda}$

$$\bar{\lambda}_n = \sum_{\lambda\in\sigma(A_n(\cdot))\cap\tilde{\Lambda}} \lambda \, \frac{\dim G\big(A_n(\cdot), \lambda\big)}{\dim G\big(A(\cdot), \lambda_0\big)},$$

vii)

$$\inf_{x_0 \in \ker A(\lambda_0)} \|x_n - x_0\|_X \leq c\Big(|\lambda_n - \lambda_0| + \max_{\substack{y_0 \in \ker A(\lambda_0) \\ \|y_0\|_X \leq 1}} \inf_{y_n \in X_n} \|y_0 - y_n\|_X\Big)$$

for all $\lambda_n \in \sigma\big(A_n(\cdot)\big) \cap \tilde{\Lambda}$ *and all* $x_n \in \ker A_n(\lambda_n)$ *with* $\|x_n\|_X = 1$.

Proof. The first three claims follow with [**15**][Thm. 2], if we can proof that the required assumptions are satisfied. First of all a Galerkin scheme is a discrete approximation scheme due to Lem. 3.3. The operator function $A(\cdot)$ is holomorphic by assumption. It follows that $A_n(\cdot) := P_n A(\cdot) P_n|_{X_n}$ is also holomorphic. Since $A(\cdot)$ is $T(\cdot)$-Gårding, it is Fredholm valued. $A_n(\cdot)$ is Fredholm valued by assumption (see Def. 2.5). Assumption b1 $\rho\big(A(\cdot)\big) \neq \emptyset$ is also an assumption of this theorem. Assumption b2 follows from Lem. 2.6 (at least for sufficiently large n). Assumption b3 follows from $\|A_n(\lambda)\|_{\mathcal{L}(X_n)} \leq \|A(\lambda)\|_{\mathcal{L}(X)}$. Assumption b4 follows from the point-wise convergence of the projections P_n. Assumption b5 follows from Lem. 2.7.

The fourth claim follows with [**15**][Thm. 3], if we can proof the required assumption (R). We can chose r_n as injection, i.e. $r_n x_n := x_n$. Hence $\|r_n\| = 1$. Since $p_n = P_n$ ii) follows from the point-wise convergence of the projections P_n.

The fifth and sixth claim follow with [**16**][Thm. 2, Thm. 3], if we can proof their required assumptions. Assumption a1-a4 are canonical satisfied by Galerkin schemes. We already proved that Assumptions b1-b5 are satisfied. We can chose $p'_n = p_n = q'_n = q_n = P_n$. For [**16**][Thm. 3] we can chose the same r_n as before.

For the proof of the seventh claim we actually don't need the notion of this paper and only require the assumptions of [**15**][Thm. 2, Thm. 3], [**16**][Thm. 2, Thm. 3] plus the Galerkin setting. Thus we refer to Lem. 3.4. ☐

Remark 3.2. *If $A(\cdot)\colon \Lambda \to \mathcal{L}(X)$ is a holomorphic operator function and for every $\lambda \in \Lambda$ exist $c(\lambda) > 0$ and a compact operator $K \in \mathcal{L}(X)$ such that $(A(\lambda) + K(\lambda)$ is coercive, then $A(\cdot)$ is a* Id-*Gårding operator.*

If $(X_n)_{n\in\mathbb{N}} \subset X^{\mathbb{N}}$ is a sequence of closed subspaces with orthogonal projections P_n onto X_n, such that $(P_n)_{n\in\mathbb{N}}$ converges point-wise to the identity and $A_n(\cdot)\colon \Lambda \to \mathcal{L}(X_n)$ is defined by $A_n(\lambda) := P_n A(\lambda) P_n|_{X_n}$ for all $\lambda \in \Lambda$, then $\big(A(\cdot)_n\big)_{n\in\mathbb{N}}$ is a Id-*compatible approximation of $A(\cdot)$.*

The next lemmata are completely independent from the concept of $T(\cdot)$-Gårding operators and $T(\cdot)$-compatible approximations.

Lemma 3.3. *Let $\Lambda \subset \mathbb{C}$ be open, X be a Hilbert space and $\mathcal{L}(X)$ be the space of bounded linear operators from X to X. Let $A(\cdot)\colon \Lambda \to \mathcal{L}(X)$ be a holomorphic operator function and let $(X_n)_{n\in\mathbb{N}} \subset X^{\mathbb{N}}$ be a sequence of closed subspaces with orthogonal projections P_n onto X_n, such that $(P_n)_{n\in\mathbb{N}}$ converges point-wise to the identity. Then the Galerkin scheme $\big(P_n A(\cdot) P_n|_{X_n}\big)_{n\in\mathbb{N}}$ is a discrete approximation scheme in the sense of [**15**].*

Proof. For a Galerkin scheme it holds with the notation of [**15**]

$$U = V = X, \qquad X_n = Y_n = X_n, \qquad A_n(\cdot) = P_n A(\cdot) P_n|_{X_n}, \qquad p_n = q_n = P_n.$$

Assumptions a1)-a4) of [**15**] follow all from the point-wise convergence of P_n. ☐

Lemma 3.4. *Let $\Lambda \subset \mathbb{C}$ be open, X be a Hilbert space and $\mathcal{L}(X)$ be the space of bounded linear operators from X to X. Let $A(\cdot)\colon \Lambda \to \mathcal{L}(X)$ be a holomorphic operator function and $(X_n)_{n\in\mathbb{N}} \subset X^{\mathbb{N}}$ be a sequence of closed subspaces with orthogonal projections P_n onto X_n, such that $(P_n)_{n\in\mathbb{N}}$ converges point-wise to the identity, i.e. $\lim_{n\to\infty} \|x - P_n x\|_X = 0$ for all $x \in X$. Let $A_n(\cdot)\colon \Lambda \to \mathcal{L}(X_n)$ be the Galerkin approximation of $A(\cdot)$ defined by $A_n(\lambda) := P_n A(\lambda) P_n|_{X_n}$ for every $\lambda \in \Lambda$. Let the assumptions of [**15**][Thm. 2, Thm. 3], [**16**][Thm. 2, Thm. 3] be satisfied. Let $\tilde{\Lambda} \subset \Lambda$ be a compact set with boundary $\partial\tilde{\Lambda} \subset \rho\big(A(\cdot)\big)$ and $\tilde{\Lambda} \cap \sigma\big(A(\cdot)\big) = \{\lambda_0\}$. Then there exist $n \in \mathbb{N}$ and $c > 0$ such that for all $n > n_0$ such that*

$$\inf_{x_0 \in \ker A(\lambda_0)} \|x_n - x_0\|_X \leq c\Big(|\lambda_n - \lambda_0| + \max_{\substack{y_0 \in \ker A(\lambda_0) \\ \|y_0\|_X \leq 1}} \inf_{y_n \in X_n} \|y_0 - y_n\|_X\Big) \tag{5}$$

for all $\lambda_n \in \sigma\big(A_n(\cdot)\big) \cap \tilde{\Lambda}$ *and all* $x_n \in \ker A_n(\lambda_n)$ *with* $\|x_n\|_X = 1$.

Proof. We proceed as in [**19**]: Thm. 4.3.7 of [**19**] requires a special form of the operator function $A(\cdot)$. However its proof uses this assumption only to apply Lem. 4.2.1 of [**19**]. Hence we prove [**19**][Lem. 4.2.1] without the assumption on the special form of $A(\cdot)$. Again the proof of [**19**][Lem. 4.2.1] doesn't need the special form of $A(\cdot)$ to establish [**19**][Eq. (4.25)]. We continue the proof of [**19**][Lem. 4.2.1] at this point.

Let now $\{y_n\}$ be an orthonormal basis of $\big(\operatorname{range} A(\lambda_0)\big)^\perp$ and $\{z_n\}$ be an orthonormal basis of $\ker A(\lambda_0)$. Let $K \in \mathcal{L}(X)$ be defined as $Kx := \sum_n y_n \langle x, z_n\rangle_X$, $x \in X$ and $\tilde{A}\colon \Lambda \to \mathcal{L}(X)$ be defined as $\tilde{A}(\lambda)x := A(\lambda)x + Kx$. Then K is compact and $\tilde{A}(\cdot)$ is bijective at λ_0. Since $A(\cdot)$, $A_n(\cdot) := P_nA(\cdot)P_n$ fulfill the assumptions of [**15**][Thm. 2] so do $\tilde{A}(\cdot)$, $\tilde{A}_n(\cdot) := P_n\tilde{A}(\cdot)P_n$, see e.g. [**15**][p. 367] for the regularity of $\tilde{A}_n(\cdot)$. Since the resolvent set of a Fredholm operator function is open, [**15**][Thm. 2 (3)] yields $\|\tilde{A}_n(\lambda)^{-1}\| \leq c$ for a $c > 0$ and all λ in a neighborhood of λ_0. Thus for sufficiently large $n \in \mathbb{N}$ it holds

$$\begin{aligned}
\|x^0 - x_{n_k}\|_X &\leq \|(\operatorname{Id} - P_n)x^0\|_X + \|P_n(x^0 - x_{n_k})\|_X \\
&\leq \|(\operatorname{Id} - P_n)x^0\|_X + c\|\tilde{A}_n(\lambda_{n_k})P_n(x^0 - x_{n_k})\|_X \\
&\leq \|(\operatorname{Id} - P_n)x^0\|_X + c\|P_nA(\lambda_{n_k})P_n(x^0 - x_{n_k})\|_X \\
&\qquad + c\|P_nKP_n(x^0 - x_{n_k})\|_X.
\end{aligned}$$

The first term $\|(\operatorname{Id} - P_n)x^0\|_X$ converges to zero due to the point-wise convergence of P_n. For the second term $c\|P_nA(\lambda_{n_k})P_n(x^0 - x_{n_k})\|_X$ it holds

$$\|P_nA(\lambda_{n_k})P_n(x^0 - x_{n_k})\|_X \leq \|P_nA(\lambda_{n_k})P_nx^0\|_X + \|P_nA(\lambda_{n_k})x_{n_k}\|_X.$$

Since $x^0 \in \ker A(\lambda_0)$ and due to the point-wise convergence of P_n it holds $\|P_nA(\lambda_{n_k})P_nx^0\|_X \to 0$. Further $\|P_nA(\lambda_{n_k})x_{n_k}\|_X \to 0$ by assumption of [**19**][Lem. 4.2.1]. The third term $c\|P_nKP_n(x^0 - x_{n_k})\|_X$ can be estimated as follows

$$\begin{aligned}
\|P_nKP_n(x^0 - x_{n_k})\|_X &\leq \|KP_n(x^0 - x_{n_k})\|_X \\
&\leq \|K(\operatorname{Id} - P_n)x^0\|_X + \|K(x^0 - x_{n_k})\|_X.
\end{aligned}$$

Due to the point-wise convergence of P_n it holds $\|K(\operatorname{Id} - P_n)x^0\|_X \to 0$. Since K is compact $\|K(x^0 - x_{n_k})\|_X \to 0$ follows from the weak convergence of $x_{n_k} \rightharpoonup x^0$. Hence the claim is proven. □

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