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B. Burkotova. I. Rachunkova, E.B. Weinmüller

Institute for Analysis and Scientific Computing
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Institute for Analysis and Scientific Computing
Vienna University of Technology
Wiedner Hauptstraße 8–10
1040 Wien, Austria

E-Mail: admin@asc.tuwien.ac.at
WWW: http://www.asc.tuwien.ac.at
FAX: +43-1-58801-10196

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On singular BVPs with unsmooth data. Part 2: Convergence of the collocation schemes

Jana Burkotová · **Irena Rachůnková** ·
Ewa B. Weinmüller

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Abstract This paper deals with the collocation method applied to solve systems of singular linear ordinary differential equations with variable coefficient matrices and unsmooth inhomogeneities. The classical stage convergence order is shown to hold for the piecewise polynomial collocation applied to boundary value problems with time singularities of the first kind provided that their solutions are appropriately smooth. The question of the existence and uniqueness of solutions to the analytical problems have been investigated in the first part of the paper - On singular BVPs with unsmooth data. Part1: Analysis of the linear case with variable coefficient matrix. The convergence theory is illustrated by numerical examples.

Keywords linear systems of ordinary differential equations · singular boundary value problems · time singularity of the first kind · unsmooth inhomogeneity · collocation method · convergence

Mathematics Subject Classification (2000) 65L05 · 65L10 · 65L20 · 65L60

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Jana Burkotová (✉) · Irena Rachůnková
Department of Mathematics, Faculty of Science, Palacký University Olomouc, 17. listopadu 12, 77146 Olomouc, Czech Republic
E-mail: jana.burkotova@upol.cz
E-mail: irena.rachunkova@upol.cz

Ewa B. Weinmüller
Department for Analysis and Scientific Computing, Vienna University of Technology, Wiedner Hauptstraße 8–10, A-1040 Wien, Austria
E-mail: ewa.weinmueller@tuwien.ac.at

1 Introduction

We are interested in analysing the convergence properties of the polynomial collocation as a numerical approach to solve singular problems of the form

$$y'(t) = \frac{M(t)}{t}y(t) + \frac{f(t)}{t}, \quad B_0y(0) + B_1y(1) = \beta, \tag{1.1}$$

where y is a n-dimensional real function, M is a $n \times n$ matrix function and f is a n-dimensional function which are at least continuous, $M \in C[0,1]$, $f \in C[0,1]$. Moreover, $B_0, B_1 \in \mathbb{R}^{m\times n}$ are constant matrices which are subject to certain restrictions for a well-posed problem, and $\beta \in \mathbb{R}^m$. Note that in general $m \leq n$. In Part 1 [7], we focused our attention on the existence and uniqueness of a solution $y \in C[0,1]$. This smoothness requirement results in general, in $n-m$ additional initial conditions the solution y has to satisfy. We have also specified conditions for f and M which are sufficiently for $y \in C^r[0,1]$, $r \in \mathbb{N}$.

To compute the numerical solution of (1.1) the polynomial collocation [4] was proposed in [8]. See also [21] for second order systems. This was motivated by its advantageous convergence properties for (1.1) while in the presence of a singularity other high order methods show order reductions and become inefficient [9]. Consequently, for singular boundary value problems (BVPs) two open domain MATLAB codes based on collocation have been implemented [3, 12]. The code `sbvp` can be applied to explicit first order ordinary differential equations (ODEs) [3], while `bvpsuite` can be used to solve arbitrary mixed order problems in implicit formulation. Its scope also includes the differential algebraic equations [12]. Both codes were used to numerically simulate singular BVPs important for applications and proved to work dependably and efficiently [5], [11]. This was our motivation to propose the polynomial collocation for the approximation of (1.1).

Due to the very advantageous properties of the collocation method, this approach has been used in a variety of other openly available programs. We enclose some examples for the existing software packages designed to deal with regular and singular ODEs: the standard MATLAB code `bvp4c` [17] and the related solver `bvp5c` [10], two FORTRAN codes, `BVP_SOLVER` specified in [18], and COLNEW described in [1] and based on one of the best established BVP solvers COLSYS [2]. For most of the basic solvers, error estimation routines and grid adaptation strategies implemented in these codes, analytical justification in context of singular systems is given. Typically, to enhance the efficiency of the code, the order of the basic solver varies depending on the tolerances specified by the user.

In [20] local existence and uniqueness analysis was provided for a certain class of nonlinear differential equation of the type $tu(t) = g(t, u(t))$, see also [19]. Also in the nonlinear case the boundary conditions are disregarded and the problem is solved numerically by collocation applied to the integral equation resulting after the integration of the ODE system. It turns out that the global error of the collocation scheme is $O(h^k|\ln h|)$ provided that the problem data is appropriately smooth and h is sufficiently small. Here, k is the number of collocation points.

Collocation proved to be also a useful tool to treat other problem classes, dynamical system in ODEs [14] and algebraic-differential equations, see [15], [16].

For linear problems of type (1.1) with constant coefficient matrices, posed in form of an initial value problem, it was shown in [6] that the convergence order of the collocation is at least equal to the stage order of the method. In the present paper, we intend to generalize the above convergence analysis to the case of a linear BVP with a variable coefficient matrix M and unsmooth inhomogeneity.

The paper is organized as follows: In Section 2 the necessary notation is introduced. The analytical properties of (1.1), discussed in [7], are briefly recapitulated in Section 3. Sections 5 and 6 are devoted to the convergence analysis of the collocation applied to solve initial value problems (IVPs) and terminal value problems (TVPs), respectively. These results are used to show the convergence of the collocation schemes in the context of the general BVPs in Section 7. Finally, in Section 8, we provide numerical examples to illustrate the theory, and in Section 9, we summarize the most important results.

2 Notation

Throughout the paper, the following notation is used. We denote by $\mathbb{R}^n$ and $\mathbb{C}^n$ the n-dimensional vector space of real-valued and complex-valued vectors, respectively, and denote the maximum vector norm by

$$|x| := |(x_1,\dots,x_n)^\top| = \max_{1\le i\le n} |x_i|.$$

We denote by $C_n[0,1]$ the space of continuous real vector-valued functions on $[0,1]$. In this space, we use the maximum norm,

$$\|y\| := \max_{t\in[0,1]} |y(t)|,$$

and the norm restricted to the interval $[0,\delta]$, $\delta > 0$, is denoted by

$$\|y\|_\delta := \max_{t\in[0,\delta]} |y(t)|.$$

$C_n^p[0,\delta]$, $\delta > 0$, is the space of p times continuously differentiable real vector-valued functions on $[0,\delta]$ with the norm

$$\|y\|_{C_n^p[0,\delta]} := \sum_{k=0}^{p} \|y^{(k)}\|_\delta.$$

Furthermore, we denote by $\mathbb{R}^{m\times n}$, $\mathbb{C}^{m\times n}$ the $m\times n$-dimensional space of real-valued, complex-valued matrices, respectively, and denote the corresponding matrix norm by

$$|A| = \max_{1\le i\le m} \sum_{j=1}^{n} |a_{ij}|.$$

Additionally, the space of p-times continuously differentiable real-valued matrix functions on $[0,\delta]$ is denoted by $C^p_{m\times n}[0,\delta]$, $\delta>0$, $p\in\mathbb{N}$. This space is equipped with the norm

$$\|M\|_{C^p_{m\times n}[0,\delta]} := \sum_{k=1}^{p} \|M^{(k)}\|_\delta,$$

where

$$\|M\|_\delta := \max_{t\in[0,\delta]} |M(t)|.$$

If it cannot be confusing, we omit the subscripts m and n for simplicity of notation, and write $C[0,1]=C_n[0,1]$, $C^p[0,1]=C^p_n[0,1]$, $C^p[0,1]=C^p_{m\times n}[0,1]$, etc.

3 Analytical results

Here, we recapitulate the most important analytical properties of the BVP (1.1) [7]. We were mainly interested in deriving general two-point boundary conditions which guarantee certain smoothness requirements for the analytical solution y of (1.1) in the closed interval $[0,1]$. It turned out that the form of such conditions depends on the spectral properties of the coefficient matrix $M(0)$. Therefore we distinguish between three cases where all eigenvalues of $M(0)$, denoted by $\lambda_k=\sigma_k+i\rho_k$, $k=1,\dots n$, have negative real parts, positive real parts or they are equal zero. The case of purely imaginary eigenvalues of $M(0)$ is excluded.

Throughout this section we assume that $f\in C[0,1]$ and the matrix $M\in C[0,1]$ can be written in the form

$$M(t) = M(0) + t^\gamma D(t), \quad \gamma>0, \quad t\in[0,1], \quad D\in C[0,1]. \tag{3.1}$$

Case 1. If all eigenvalues of $M(0)$ have negative real parts then there exists a unique continuous solution y of the BVP (1.1). However, the boundary conditions in (1.1) are here reduced to the initial conditions $M(0)y(0)=-f(0)$, which is necessary and sufficient for $y\in C[0,1]$. Moreover, if $f\in C^r[0,1]$, $D\in C^r[0,1]$ and $\gamma>r$, then $y\in C^r[0,1]$. Let us mention, that the assumption (3.1) can be weakened, see Part 1 [7].

Case 2. In this case all eigenvalues of $M(0)$ are assumed to have positive real parts. Then, provided that $f\in C^1[0,1]$ and $B_1\in\mathbb{R}^{n\times n}$ is nonsingular, there exists a unique continuous solution y of the BVP (1.1). Here, again the general boundary conditions are reduced to a particular form, namely to the terminal conditions $B_1y(1)=\beta$. The smoothness of y depends not only on the smoothness of the inhomogeneity f but also on the size of the smallest positive real part σ_+ of the eigenvalues of $M(0)$. In particular, if $f\in C^{r+1}[0,1]$, $D\in C^r[0,1]$, $\gamma>r$, and $\sigma_+>r$, then $y\in C^r[0,1]$.

Case 3. For the case that all eigenvalues of $M(0)$ are equal zero, we have to assume some special structure in f close to the singularity, namely, $f(t) = O(t^\alpha h(t))$ for $t \to 0$, where $h \in C[0,\delta]$, $\delta > 0$ and $\alpha > 0$. Then, there exists a unique continuous solution y of the BVP (1.1) reduced to an IVP, where $B_1 \equiv 0$ and $B_0\tilde{R} \in \mathbb{R}^{m\times m}$ is nonsingular. The matrix $\tilde{R}$ consists of the linearly independent columns of the projection R onto the m-dimensional space spanned by eigenvectors associated with zero eigenvalues. The necessary and sufficient condition for y to be continuous is here $M(0)y(0) = 0$. Moreover, if $f \in C^r[0,1]$, $D \in C^r[0,1]$, $\alpha \geq r+1$, and $\gamma \geq r+1$, then $y \in C^{r+1}[0,1]$.

It is clear from the previous considerations that the form of the general boundary conditions in (1.1) which are necessary and sufficient for the unique solution $y \in C[0,1]$ of (1.1) depends on the spectrum of $M(0)$. Let S, R, H and N denote the projection onto the subspace spanned by the eigenvectors associated with eigenvalues with positive real parts, the subspace spanned by eigenvectors associated with zero eigenvalues, the subspace spanned by principal eigenvectors associated with zero eigenvalues, and the subspace spanned by eigenvectors associated with eigenvalues with negative real parts, respectively. Moreover, we define $Z := R+H$, $P := R+S$. We also use $\tilde{P}$, $\tilde{R}$ to denote the matrices consisting of the maximal set of linearly independent columns of the respective projections.

In order to formulate the main result for a general BVP (1.1), we have to assume that the inhomogeneity f satisfies $Sf \in C^1[0,1]$ and $Zf(t) = O(t^\alpha h(t))$ for $t \to 0$, where $\alpha > 0$ and h is continuous at zero. Furthermore, we have to assume that the $m\times m$ matrix $B_0\tilde{R} + B_1\tilde{P}$ is nonsingular, where $m = \operatorname{rank} P$. Then, the BVP (1.1) has a unique solution $y \in C[0,1]$. This solution satisfies two initial conditions, $Hy(0) = 0, M(0)Ny(0) = -Nf(0)$ which are necessary and sufficient for $y \in C[0,1]$. For details, we refer the reader to Part 1 [7]. The smoothness result $y \in C^r[0,1]$ can be shown by applying results of the Cases 1–3 to the corresponding projections of the function f and the matrix M.

4 Collocation method

In this section, we introduce a class of collocation methods applied to approximate the solution y of the problem

$$y'(t) = \frac{M(t)}{t}y(t) + \frac{f(t)}{t}, \quad B_0y(0) + B_1y(1) = \beta. \tag{4.1}$$

We assume that the BVP (4.1) has a unique solution in $C[0,1]$. We first choose I, $k \in \mathbb{N}$ and discretize problem (4.1). To this aim, the interval of integration $[0,1]$ is partitioned,

$$\Delta := \{0 = t_0 < t_1 < \ldots < t_{I-1} < t_I = 1,\ t_j = jh,\ j = 0,\ldots,I = 1/h\},$$

and in each subinterval $[t_j, t_{j+1}]$ we introduce k equidistantly spaced collocation nodes $t_{jl} := t_j + u_l h,\ j = 0,\ldots,I-1,\ l = 1,\ldots,k$, where $0 < u_1 < \ldots < u_k \leq 1$. The computational grid including the mesh points and the collocation points is shown in Figure 4.1.

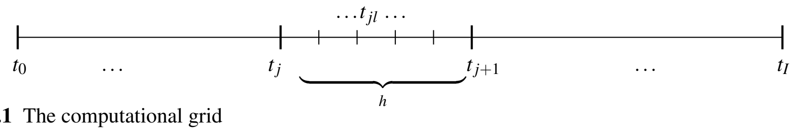

Fig. 4.1 The computational grid

By $\mathscr{P}_{k,h}$, we denote the class of piecewise polynomial functions which are globally continuous on $[0,1]$ and reduce in each subinterval $[t_j,t_{j+1}]$ to a polynomial of degree less or equal to k. We now approximate the analytical solution y by a polynomial function $p \in \mathscr{P}_{k,h}$, such that p satisfies system (1.1) at the collocation points,

$$p'(t_{jl}) - \frac{M(t_{jl})}{t_{jl}} p(t_{jl}) = \frac{f(t_{jl})}{t_{jl}}, \;\; l = 1,\dots,k, \;\; j = 0,\dots,I-1,$$

the boundary conditions

$$B_0 p(0) + B_1 p(1) = \beta,$$

and the continuity relations,

$$p_{j-1}(t_j) = p_j(t_j), \;\; j = 1,\dots,I-1,$$

where $p(t) := p_j(t),\ t \in [t_j,t_{j+1}]$.

For the subsequent analysis, we assume $M \in C^1[0,1]$ which yields

$$M(t) = M(0) + tD(t),\ t \in [0,1],\ D \in C[0,1]. \tag{4.2}$$

Moreover, if $M(0)$ has eigenvalues with positive real parts, we assume that the smallest positive real part $\sigma_+ > 1$. This does not mean a restriction of generality since using the transformation $t = \tau^\mu$, $\mu > 1$, we can enlarge the smallest positive real part according to $\tilde{\sigma}_+ = \mu\sigma_+$, where $\tilde{\sigma}_+$ is the smallest positive real part of the eigenvalues of the transformed system.

In the following sections, we first discuss IVPs and TVPs. Then, we generalize these results to general linear BVPs.

5 Convergence of the collocation scheme for IVPs

Here, we restrict our attention to the class of singular BVPs which can be equivalently expressed as a well-posed IVP, where all boundary conditions are posed at $t = 0$. In this case, we have to assume that the matrix $M(0)$ has only eigenvalues $\lambda := \sigma + i\rho$, with nonpositive real parts, and if $\sigma = 0$ then $\lambda = 0$. These restrictions are necessary to ensure the existence of a well-posed initial value problem, see assumption A.1 in Part 1 [7]. Let H and N denote projections onto the subspace spanned by the principal eigenvectors associated with zero eigenvalues and the subspace spanned by the eigenvectors associated with eigenvalues with negative real parts, respectively.

The underlying IVP has the form

$$y'(t) = \frac{M(t)}{t}y(t) + \frac{f(t)}{t},\ \ B_0y(0) = \beta,\ \ Hy(0) = 0,\ \ M(0)Ny(0) = -Nf(0), \quad (5.1)$$

where $B_0 \in \mathbb{R}^{m\times n}$, $\beta \in \mathbb{R}^m$, and $\operatorname{rank} R = m \le n$.

Collocation methods for linear problems with a smooth inhomogeneity were studied in [8]. In the next lemma, the relevant auxiliary results from Theorem 4.1 [8] are recapitulated.

Lemma 5.1 (Theorem 4.1 in [8]) *Let us consider the collocation scheme,*

$$p'(t_{jl}) - \frac{M(t_{jl})}{t_{jl}}p(t_{jl}) = M^{\mu}(0)\frac{c_{jl}}{t_{jl}^{\nu}},\ \ l = 1,\dots,k,\ \ j = 0,\dots I-1, \quad p(0) = \delta, \quad (5.2)$$

where $\mu,\nu \in \{0,1\}$, $M \in C^1[0,1]$, $\delta \in R^n$ and c_{jl} are arbitrary constants. Then problem (5.2) *has a unique solution $p \in \mathscr{P}_{k,h}$ provided that h is sufficiently small. This solution satisfies*

$$|p(t)| \le const.\left(|\delta| + |\ln(h)|^d|M(0)\delta| + |\ln(h)|^{(\nu(d-\mu))_+}C_I\right),\ t \in [0,1],$$

where d is the dimension of the largest Jordan box of $M(0)$ associated with the eigenvalue $\lambda = 0$,

$$(x)_+ = \begin{cases} x & x \ge 0,\\ 0 & x < 0,\end{cases}$$

and

$$C_I = \max_{0\le j\le I-1}\max_{1\le l\le k}|c_{jl}|.$$

Using Lemma 5.1, we can formulate the convergence result for the collocation method applied to IVP (5.1). For the convergence analysis, we rewrite (5.1) to obtain a more convenient form,

$$y'(t) - \frac{M(t)}{t}y(t) = \frac{f(t)}{t}, \quad y(0) = \delta, \quad (5.3)$$

where

$$B_0\delta = \beta, \quad H\delta = 0, \quad M(0)N\delta = -Nf(0). \quad (5.4)$$

Conditions for the existence of a solution $y \in C^{k+1}$ of problem (5.3), (5.4) are given in Section 3.

Theorem 5.1 *Let us assume that $y \in C^{k+1}[0,1]$ is the unique solution of problem* (5.3)*,* (5.4) *and $M \in C^1[0,1], f \in C[0,1]$. Let the function $p \in \mathscr{P}_{k,h}$ be the unique solution of the collocation scheme,*

$$p'(t_{jl}) - \frac{M(t_{jl})}{t_{jl}}p(t_{jl}) = \frac{f(t_{jl})}{t_{jl}}, \quad l = 1,\dots k,\ j = 0,\dots,I-1, \quad p(0) = \delta.$$

Then

$$\|p - y\| \le const.h^k.$$

Proof: To prove the convergence of the collocation scheme applied to solve IVP (5.3), (5.4), we first define an *error function* $e \in \mathscr{P}_{k,h}$,

$$e'(t_{jl}) := y'(t_{jl}) - p'(t_{jl}), \quad l = 1,\dots,k,\ j = 0,\dots,I-1, \quad e(0) := 0, \tag{5.5}$$

and show that the error function e differs from the global error $p-y$ by $O(h^k)$ terms. Clearly, since the function $e'(t)$ belongs to $\mathscr{P}_{k-1,h}$, it is uniquely determined by its values at k distinct points in each subinterval $[t_j, t_{j+1}]$, $j = 0,\dots I-1$,

$$e'(t) = \sum_{i=1}^{k} l_i\left(\frac{t-t_j}{h}\right) y'(t_{ji}) - p'(t),\ t \in (t_j, t_{j+1}],$$

where

$$\begin{aligned} l_i(t) &= w(t)/\left((t-u_i)w'(u_i)\right),\ i = 1,\dots,k, \\ w(t) &= (t-u_1)(t-u_2)\cdots(t-u_k). \end{aligned} \tag{5.6}$$

For $y \in C^{k+1}[0,1]$, the interpolation error is $O(h^k)$ and hence,

$$e'(t) = y'(t) - p'(t) + O(h^k)$$

which by integration on $[0,t]$ yields

$$e(t) = y(t) - p(t) + O(h^k t),\ t \in [0,1].$$

Therefore, e satisfies the following collocation scheme:

$$\begin{aligned} &e'(t_{jl}) - \frac{M(t_{jl})}{t_{jl}} e(t_{jl}) = \\ &= y'(t_{jl}) - \frac{M(t_{jl})}{t_{jl}} y(t_{jl}) - \left(p'(t_{jl}) - \frac{M(t_{jl})}{t_{jl}} p(t_{jl})\right) - \frac{M(t_{jl})}{t_{jl}} O(t_{jl}h^k) \\ &= \frac{f(t_{jl})}{t_{jl}} - \frac{f(t_{jl})}{t_{jl}} - \frac{M(t_{jl})}{t_{jl}} O(t_{jl}h^k) = O(M(0)h^k + t_{jl}D(t_{jl})h^k) \\ &= O(M(0)h^k), \quad e(0) = 0. \end{aligned}$$

According to Lemma 5.1, with $\mu = 0$, $\nu = 0$, and $c_{jl} = O(h^k)$, the error function $e = O(h^k)$ and since $e(t) = y(t) - p(t) + O(h^k)$, the estimate for the global error $\|p-y\|$ follows. □

For regular ODEs and suitably chosen collocation points (Gaussian, Lobatto, Radau), the superconvergence order in the *mesh points* can be observed. For singular problems considered here, the superconvergence order cannot be expected to hold, in general. Counterexamples in [8] show that the superconvergence order does not hold even for singular problems with a smooth inhomogeneity. The so-called small superconvergence uniform in t will be shown in the next theorem.

Theorem 5.2 *Let us assume that the solution y of* (5.3), (5.4) *satisfies* $y \in C^{k+2}[0,1]$. *Moreover, let the collocation points be chosen in such a way that*

$$\int_0^1 w(s)\,\mathrm{d}s = 0 \tag{5.7}$$

holds. Then, the estimate for the global error given in Theorem 5.1 can be replaced by

$$\|p-y\| \le const.\, h^{k+1}|\ln(h)|^{(d-1)_+}.$$

Proof: Let us consider the error function e defined in (5.5). Since $y \in C^{k+2}[0,1]$, we have for $j = 0,\dots I-1$ and l_i from (5.6),

$$\begin{aligned} e'(t) &= \sum_{i=1}^{k} l_i\left(\frac{t-t_j}{h}\right) y'(t_{ji}) - p'(t) \\ &= y'(t) - p'(t) + \frac{h^k}{k!} w\left(\frac{t-t_j}{h}\right) y^{(k+1)}(t_j) + O(h^{k+1}), \;\; t \in (t_j, t_{j+1}]. \end{aligned}$$

We now integrate e' on $[0,t]$, $t \in (t_j, t_{j+1}]$, and use (5.7) to obtain

$$\begin{aligned} e(t) &= y(t) - p(t) + \sum_{i=0}^{j-1} \frac{h^k}{k!} y^{(k+1)}(t_i) \int_{t_i}^{t_{i+1}} w\left(\frac{s-t_i}{h}\right)\mathrm{d}s \\ &\quad + \frac{h^k}{k!} y^{(k+1)}(t_j) \int_{t_j}^{t} w\left(\frac{s-t_j}{h}\right)\mathrm{d}s + O(th^{k+1}) = y(t) - p(t) + O(h^{k+1}). \end{aligned}$$

This implies

$$\begin{aligned} & e'(t_{jl}) - \frac{M(t_{jl})}{t_{jl}} e(t_{jl}) \\ &= y'(t_{jl}) - \frac{M(t_{jl})}{t_{jl}} y(t_{jl}) - \left(p'(t_{jl}) - \frac{M(t_{jl})}{t_{jl}} p(t_{jl})\right) - \frac{M(t_{jl})}{t_{jl}} O(h^{k+1}) \\ &= -\frac{M(t_{jl})}{t_{jl}} O(h^{k+1}) = O\left(\frac{M(0)}{t_{jl}} h^{k+1}\right) + O(D(t_{jl}) h^{k+1}) \\ &= O\left(\frac{M(0)}{t_{jl}} h^{k+1}\right), \quad e(0) = 0, \end{aligned}$$

and from Lemma 5.1, with $\mu = 1$, $\nu = 1$, and $c_{jl} = O(h^{k+1})$, we conclude

$$|e(t)| \le const.\left(|\ln(h)|^{(d-1)_+} h^{k+1}\right),$$

and thus,

$$\|p-y\| \le const.\left(|\ln(h)|^{(d-1)_+} h^{k+1}\right).$$

□

6 Convergence of the collocation scheme for TVPs

Now we assume that all eigenvalues of matrix $M(0)$ have nonnegative real parts and if zero is an eigenvalue of $M(0)$, then the associated invariant subspace is assumed to be the eigenspace of $M(0)$, cf. assumption A.2 in Part 1 [7].

Under these assumptions, we study the terminal value problem,

$$y'(t) = \frac{M(t)}{t} y(t) + \frac{f(t)}{t}, \quad B_1 y(1) = \beta, \tag{6.1}$$

with $B_1 \in \mathbb{R}^{n\times n}$, $\beta \in \mathbb{R}^n$.

The existence and uniqueness of the respective collocation solution was already studied in [13]. The following lemma covers the case of TVPs with constant matrix $M(0)$ which has only eigenvalues with positive real parts.

Lemma 6.1 (Lemma 3.2 [13]) *Assume that all eigenvalues of $M(0)$ have positive real parts and $M \in C^1[0,1]$. For $\alpha \in \{0,1\}$ and arbitrary constants c_{jl}, there exists a unique polynomial function $p \in \mathscr{P}_{k,h}$ which, for any $0 < b \le 1$, satisfies*

$$p'(t_{jl}) = \frac{M(0)}{t_{jl}} p(t_{jl}) + \frac{c_{jl}}{t_{jl}^{\alpha}}, \quad p(b) = \gamma,\ j = 0,\dots I-1,\ l = 1,\dots k. \tag{6.2}$$

Furthermore,

$$\|p\|_{t_{j+1}} := \max_{0\le s\le t_{j+1}} |p(s)| \le const.(|\gamma| + t_{j+1}^{1-\alpha} C_I),\ j = 0,\dots I-1.$$

Remark 6.1 In the case when the matrix $M(0)$ has zero eigenvalues and the associated invariant subspace coincides with the eigenspace of $M(0)$, for $\alpha \in \{0,1\}$, $0 < b \le 1$, and arbitrary constants c_{jl}, there exists a unique collocation polynomial $p \in \mathscr{P}_{k,h}$ such that

$$p'(t_{jl}) = \frac{c_{jl}}{t_{jl}^{\alpha}}, \quad p(b) = \gamma,\ j = 0,\dots I-1,\ l = 1,\dots k,$$

and

$$\|p\|_{t_{j+1}} \le const.(|\gamma| + t_{j+1}^{1-\alpha} C_I),\ j = 0,\dots I-1.$$

Consequently, for the matrix $M(0)$ whose spectrum consists of eigenvalues with positive real parts and zero eigenvalues with the same algebraic and geometric multiplicity, there exists a unique polynomial function $p \in \mathscr{P}_{k,h}$ satisfying (6.2). Furthermore,

$$\|p\|_{t_{j+1}} \le const.(|\gamma| + t_{j+1}^{1-\alpha} C_I),\ j = 0,\dots I-1.$$

Lemma 6.2 *Assume that $M \in C^1[0,1]$. Then for a sufficiently small h, for $\alpha \in \{0,1\}$, and arbitrary constants c_{jl}, there exists a unique collocation polynomial $p \in \mathscr{P}_{k,h}$ which satisfies*

$$p'(t_{jl}) = \frac{M(t_{jl})}{t_{jl}} p(t_{jl}) + \frac{c_{jl}}{t_{jl}^{\alpha}}, \quad p(1) = \gamma,\ j = 0,\dots I-1,\ l = 1,\dots k.$$

Moreover,

$$\|p\| \le const.(|\gamma| + C_I).$$

Proof: First, let us note that the classical theory yields the existence and uniqueness of a collocation solution r on the interval $[t_1,1]$, $t_1 = h$. In order to show the existence of the solution on $[0,t_1]$ for h small enough, we rewrite the collocation problem as an operator equation $p = \mathscr{K} q$, $\mathscr{K} : \mathscr{P}_{k,h}[0,t_1] \to \mathscr{P}_{k,h}[0,t_1]$, where p is defined for $q \in \mathscr{P}_{k,h}[0,t_1]$ as the solution of the related collocation scheme with the constant coefficient matrix $M(0)$,

$$p'(t_{0l}) = \frac{M(0)}{t_{0l}} p(t_{0l}) + D(t_{0l})q(t_{0l}) + \frac{c_{0l}}{t_{0l}^{\alpha}}, \quad p(t_1) = r(t_1),\ l = 1,\dots k,$$

where D was specified in (4.2). We now show that for a sufficiently small h, the operator $\mathscr{K}$ is a contraction on $\mathscr{P}_{k,h}[0,t_1]$ and therefore, the Banach fixed point theorem can be used. Let $q_1, q_2 \in \mathscr{P}_{k,h}[0,t_1]$. Then $\mathscr{K} q_1$ and $\mathscr{K} q_2$ are solutions of the collocation schemes with $q = q_1$ and $q = q_2$, respectively. Therefore, $v := \mathscr{K} q_1 - \mathscr{K} q_2$ is implicitly defined as the solution of the collocation scheme,

$$v'(t_{0l}) = \frac{M(0)}{t_{0l}} v(t_{0l}) + D(t_{0l})\left(q_1(t_{0l}) - q_2(t_{0l})\right), \quad v(t_1) = 0,\ l = 1,\dots k,$$

According to Lemma 6.1 and Remark 6.1,

$$\|\mathscr{K} q_1 - \mathscr{K} q_2\|_{t_1} \le const.\, h\, \|D\|_{t_1} \|q_1 - q_2\|_{t_1}.$$

For a sufficiently small $h = t_1$, the estimate

$$const.\, t_1 \|D\|_{t_1} =: L < 1$$

holds and thus, $\mathscr{K}$ is a contraction on $\mathscr{P}_{k,h}[0,t_1]$. Consequently, the Banach fixed point theorem ensures the existence of a unique fixed point

$$p = \mathscr{K} p \text{ in } \mathscr{P}_{k,h}[0,t_1].$$

Moreover, the following estimate holds:

$$\|p\|_{t_1} \le const.\, (|r(t_1)| + t_1 \|p\|_{t_1} \|D\|_{t_1} + t_1^{1-\alpha} C_1),$$

and thus,

$$\|p\|_{t_1} \le \frac{1}{1-L} const.\, (|r(t_1)| + t_1^{1-\alpha} C_1),$$

where $C_1 := \max_{1 \le l \le k} |c_{0l}|$. Using the classical theory, we extend the estimate to the whole interval,

$$\|p\| \le const.\, (|\gamma| + C_I).$$

□

We now recapitulate the results of this section: Providing that h is sufficiently small, there exists a unique collocation polynomial $p \in \mathscr{P}_{k,h}$ satisfying

$$\begin{aligned} p'(t_{jl}) &= \frac{M(t_{jl})}{t_{jl}} p(t_{jl}) + \frac{(1-t_{jl})c_{jl}}{t_{jl}} \\ &= \frac{M(t_{jl})}{t_{jl}} p(t_{jl}) + \frac{c_{jl}}{t_{jl}} - c_{jl}, \quad p(1) = \gamma,\ j = 0,\dots I-1,\ l = 1,\dots k, \end{aligned}$$

and

$$\|p\| \le const.(|\gamma| + C_I).$$

We are now able to formulate the convergence result for the TVPs. We consider the TVP (6.1) in the form

$$y'(t) = \frac{M(t)}{t} y(t) + \frac{f(t)}{t},\ y(1) = \delta, \tag{6.3}$$

where $B_1\delta = \beta$.

Theorem 6.1 *Let us assume that $M \in C^1[0,1], f \in C[0,1]$ and $y \in C^{k+1}[0,1]$ is the unique solution of (6.3). Let the function $p \in \mathscr{P}_{k,h}$ satisfy the collocation scheme*

$$p'(t_{jl}) = \frac{M(t_{jl})}{t_{jl}} p(t_{jl}) + \frac{f(t_{jl})}{t_{jl}}, \quad p(1) = \delta,\ j = 0,\dots I-1,\ l = 1,\dots k.$$

Then, provided that h is sufficiently small,

$$\|p - y\| \le const.h^k.$$

Proof: Let us define an error function $e \in \mathscr{P}_{k,h}$ as follows:

$$e'(t_{jl}) := y'(t_{jl}) - p'(t_{jl}),\ j = 0,\dots,I-1,\ l = 1,\dots,k, \quad e(1) = 0.$$

Since the function e' belongs to $\mathscr{P}_{k-1,h}$, it is uniquely determined by

$$e'(t) = \sum_{i=1}^{k} l_i \left(\frac{t-t_j}{h}\right) y'(t_{ji}) - p'(t),\ t \in (t_j, t_{j+1}],$$

where l_i are specified in (5.6). For $y \in C^{k+1}[0,1]$ the interpolation error is $O(h^k)$ and hence, $e'(t) = y'(t) - p'(t) + O(h^k)$. By integration over $[t,1]$ we obtain

$$e(t) = y(t) - p(t) + (1-t)O(h^k).$$

Moreover, we see that e satisfies the following collocation scheme:

$$\begin{aligned} &e'(t_{jl}) - \frac{M(t_{jl})}{t_{jl}} e(t_{jl}) \\ &= y'(t_{jl}) - \frac{M(t_{jl})}{t_{jl}} y(t_{jl}) - \left(p'(t_{jl}) - \frac{M(t_{jl})}{t_{jl}} p(t_{jl})\right) - \frac{M(t_{jl})}{t_{jl}}(1-t_{jl})O(h^k) \\ &= -\frac{M(t_{jl})}{t_{jl}}(1-t_{jl})O(h^k), \qquad e(1) = 0, \end{aligned}$$

and Lemma 6.2 finally yields,

$$\|e\| \le const.\, \|M\| O(h^k).$$

Consequently, $\|y-p\| \le const.\, h^k$. □

7 Convergence of the collocation scheme for BVPs

In this section, we generalize the convergence results derived for IVPs and TVPs to the general BVPs of the form

$$y'(t) = \frac{M(t)}{t} y(t) + \frac{f(t)}{t}, \tag{7.1}$$

$$B_0 y(0) + B_1 y(1) = \beta. \tag{7.2}$$

We allow the spectrum of the matrix $M(0)$ to contains both, eigenvalues with nonpositive and nonnegative real parts. Recall that the above BVP is well-posed if and only if the boundary conditions (7.2) can be equivalently written in a separated fashion,

$$Hy(0) = 0,\ M(0)Ny(0) = -Nf(0),\ Ry(0) = R\gamma,\ Sy(1) = S\gamma \tag{7.3}$$

and therefore, we can restrict our attention to the problem (7.1), (7.3). First, we show the existence and uniqueness of solution to the associated collocation scheme,

$$\begin{aligned} p'(t_{jl}) &= \frac{M(t_{jl})}{t_{jl}} p(t_{jl}) + \frac{f(t_{jl})}{t_{jl}},\ j = 0, \dots I-1,\ l = 1, \dots k, \\ Hp(0) &= 0,\ M(0)Np(0) = -Nf(0),\ Rp(0) = R\gamma,\ Sp(1) = S\gamma \end{aligned} \tag{7.4}$$

and then show that this scheme converges with the classical stage order.

Theorem 7.1 *There exists a unique solution $p \in \mathscr{P}_{k,h}$ of the collocation scheme* (7.4) *provided that h is sufficiently small and $M \in C^1[0,1], f \in [0,1]$. This solution satisfies*

$$\|p\| \le const.\, \Big(|\gamma| + |\ln(h)|^d |M(0)||\gamma| + (|\ln(h)|^d + 1)\|f\| \Big),$$

where d is the dimension of the largest Jordan box of $M(0)$ associated with the eigenvalue $\lambda = 0$.

Proof: In order to show the existence and uniqueness result for p, we study the fixed point equation $p = \mathscr{K}(q)$, $\mathscr{K} : \mathscr{P}_{k,h} \to \mathscr{P}_{k,h}$, where p is defined as a solution of the related collocation scheme with the constant matrix M(0):

$$p'(t_{ij}) = \frac{M(0)}{t_{ij}} p(t_{ij}) + D(t_{ij}) q(t_{ij}) + \frac{f(t_{ij})}{t_{ij}},\ j = 0, \dots I-1,\ l = 1, \dots k,$$

subject to boundary conditions (7.3). In order to decouple the above scheme, we introduce new variables,

$$v(t_{jl}) = E^{-1} p(t_{jl}),\ Q(t_{jl}) = E^{-1} D(t_{jl}),\ g(t_{jl}) = E^{-1} f(t_{jl}),$$

where J is the Jordan canonical form of $M(0)$ and E is the associated matrix of the generalized eigenvectors of $M(0)$. Then, the decoupled system reads:

$$v'(t_{jl}) = \frac{J}{t_{jl}} v(t_{jl}) + Q(t_{jl})q(t_{jl}) + \frac{g(t_{jl})}{t_{jl}},\ j = 0,\dots I-1,\ l = 1,\dots k,$$
$$JV^N v(0) = -V^N g(0),\ V^Z v(0) = E^{-1}R\gamma,\ V^S v(1) = E^{-1}S\gamma,$$

where

$$J = \begin{pmatrix} J^N & 0 & 0 \\ 0 & J^Z & 0 \\ 0 & 0 & J^S \end{pmatrix}$$

and

$$V^N = \begin{pmatrix} I^N & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},\ V^Z = \begin{pmatrix} 0 & 0 & 0 \\ 0 & I^Z & 0 \\ 0 & 0 & 0 \end{pmatrix},\ V^S = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & I^S \end{pmatrix}.$$

Here, J^N is the Jordan block of dimension $\operatorname{rank} N$ associated with the eigenvalues with negative real parts, J^Z is the Jordan block of dimension $\operatorname{rank} H + \operatorname{rank} R$ associated with zero eigenvalues and J^S is the Jordan block of dimension $\operatorname{rank} S$ associated with the eigenvalues with positive real parts. The matrices I^N, I^Z, and I^S are identity matrices of corresponding dimensions.

We can now split the discrete system into an IVP, governed by the blocs J^N and J^Z, associated with the negative real parts and zero eigenvalues of $M(0)$, respectively, and into a TVP, governed by the block J^S, associated with the positive real parts of eigenvalues of $M(0)$. Applying Lemma 5.1 to the IVP and Lemma 6.2 to the TVP, we conclude the existence of a unique collocation solution p of (7.4) which satisfies

$$\|p\| \le const. \left(|\gamma| + |\ln(h)|^d |M(0)||\gamma| + (|\ln(h)|^d + 1)\|f\|\right).$$

□

In the following theorem, we formulate the convergence properties of the collocation solution p to the general BVP (7.4). The proof relays on the techniques developed in Theorem 3.1 [13] for nonlinear problems with smooth inhomogeneities. Therefore, we only discuss the main ideas of this technique and refer to [13] for technical details. Note that here, the situation is easier than in [13] since we deal with a linear problem.

Theorem 7.2 *Let us assume that $y \in C^{k+2}[0,1]$ is the unique solution of the BVP* (7.1)*,* (7.3)*, $f \in C^{k+1}[0,1]$, $M \in C^{k+2}[0,1]$, and $\sigma_+ > k+2$. Let $p \in \mathscr{P}_{k,h}$ be the unique solution of the collocation scheme* (7.4)*. Then,*

$$\|p - y\| \le const.\, h^k.$$

Proof: The main idea of the proof is to derive a representation for the global error $p-y$ of the collocation solution p at all points[1] t_{jl}, $j=0,\dots I-1$, $l=1,\dots k+1$,

$$p(t_{jl}) = y(t_{jl}) + e(t_{jl})h^k + r(t_{jl}), \tag{7.5}$$

where y is the exact solution of (7.1), (7.3), $e \in C[0,1]$ and $r \in \mathscr{P}_{k,h}$. After some tedious calculation cf. Section 3.2 [13], we arrive at the following relation for t_{jl}, $j = 0,\dots I-1$, $l=1,\dots k+1$,

$$\begin{aligned} p'(t_{jl}) &= y'(t_{jl}) + e'(t_{jl})h^k + r'(t_{jl}) \\ &\quad - \frac{1}{(k+1)!}\Omega'(\rho_l)y^{(k+1)}(t_{jl})h^k + (1+\|e''\|)O(h^{k+1}), \end{aligned}$$

where $\Omega(t) := \Pi_{i=1}^{k+1}(t-\rho_i)$. We substitute (7.5) into the collocation scheme (7.4),

$$p'(t_{jl}) = \frac{M(t_{jl})}{t_{jl}}p(t_{jl}) + \frac{f(t_{jl})}{t_{jl}},$$

and obtain

$$\begin{aligned} &y'(t_{jl}) + e'(t_{jl})h^k + r'(t_{jl}) - \frac{1}{(k+1)!}\Omega'(\rho_l)y^{(k+1)}(t_{jl})h^k + (1+\|e''\|)O(h^{k+1}) \\ &= \frac{M(t_{jl})}{t_{jl}}\Big(y(t_{jl}) + e(t_{jl})h^k + r(t_{jl})\Big) + \frac{f(t_{jl})}{t_{jl}}, \end{aligned}$$

or equivalently,

$$\begin{aligned} e'(t_{jl})h^k + r'(t_{jl}) - \frac{1}{(k+1)!}\Omega'(\rho_l)y^{(k+1)}(t_{jl})h^k + (1+\|e''\|)O(h^{k+1}) \\ = \frac{M(t_{jl})}{t_{jl}}\Big(e(t_{jl})h^k + r(t_{jl})\Big), \end{aligned}$$

since y is the exact solution. To determinate a relation defining e, we collect all terms multiplying h^k and obtain,

$$\begin{aligned} &e'(t_{jl}) = \frac{M(t_{jl})}{t_{jl}}e(t_{jl}) + \frac{1}{(k+1)!}\Omega'(\rho_l)y^{(k+1)}(t_j),\ j=0,\dots I-1,\ l=1,\dots k, \\ &He(0)=0,\ M(0)Ne(0)=0,\ Re(0)=0,\ Se(1)=0, \end{aligned} \tag{7.6}$$

on noting that $y^{(k+1)}(t_{jl})h^k = y^{(k+1)}(t_j)h^k + O(h^{k+1})$ holds.

The relation for r follows by collecting all remaining terms,

$$\begin{aligned} &r'(t_{jl}) = \frac{M(t_{jl})}{t_{jl}}r(t_{jl}) + (1+\|e''\|)O(h^{k+1}),\ j=0,\dots I-1,\ l=1,\dots k, \\ &Hr(0)=0,\ M(0)Nr(0)=0,\ Rr(0)=0,\ Sr(1)=0. \end{aligned} \tag{7.7}$$

[1] For technical reasons the mesh is restricted to $u_k < 1$ and $u_{k+1} := 1$.

We now construct an analytical BVP related to (7.6) whose solution is $e \in C[0,1]$,

$$
\begin{aligned}
&e'(t) = \frac{M(t)}{t} e(t) + \frac{1}{(k+1)!} g(t),\\
&He(0) = 0,\ M(0)Ne(0) = -Nf(0),\ Re(0) = 0,\ Se(1) = 0,
\end{aligned}
$$

where $g = g_i(t)$, $t \in [t_j, t_{j+1}]$, $j = 0, \dots I-1$, is an appropriate, piecewise polynomial function satisfying $g_i(t_{jl}) = \Omega'(\rho_l) y^{(k+1)}(t_j)$, $j = 0, \dots I-1$, $l = 1, \dots k$. Therefore, the function e is also a piecewisely defined. The fact that the above BVP has a unique solution $e \in C[0,1] \cap C^{k+2}[t_j, t_{j+1}]$, $j = 0, \dots I-1$, follows from Theorem 2.2 [13]. Moreover, note that $\|e''\| h^{k+1} = O(h^k)$.

It follows from Section 3.1 [13], that there exists a unique solution $r \in \mathscr{P}_{m,h}$ of problem (7.7) such that

$$
\|r\|_{t_{j+1}} \le t_{j+1} O(h^k),\ j = 0, \dots I-1.
$$

We now combine the results for e and r to show the result. Let E be a piecewise polynomial function of degree less or equal k such that

$$
E(t_{jl}) := p(t_{jl}) - y(t_{jl}),\ j = 0, \dots I-1,\ l = 1, \dots k+1.
$$

Then,

$$
E(t) = p(t) - \sum_{i=1}^{k+1} l_i \left(\frac{t - t_j}{h} \right) y(t_{ji}),\ t \in [t_j, t_{j+1}],\ j = 0, \dots I-1.
$$

For $y \in C^{k+2}[0,1]$ the interpolation error is $O(h^{k+1})$ and hence,

$$
E(t) = p(t) - y(t) + O(h^{k+1}),\ t \in [t_j, t_{j+1}],\ j = 0, \dots I-1.
$$

On the other hand, from the error representation (7.5), $E(t_{jl}) = e(t_{jl}) h^k + r(t_{jl})$, $t \in (t_j, t_{j+1}]$, $j = 0, \dots I-1$, and therefore,

$$
\begin{aligned}
E(t) &= \sum_{i=1}^{k+1} l_i \left(\frac{t - t_j}{h} \right) E(t_{ji}) = \sum_{i=1}^{k+1} l_i \left(\frac{t - t_j}{h} \right) \Big(e(t_{ji}) h^k + r(t_{ji}) \Big)\\
&= h^k \sum_{i=1}^{k+1} l_i \left(\frac{t - t_j}{h} \right) e(t_{ji}) + r(t). \qquad (7.8)
\end{aligned}
$$

Since $e \in C^{k+2}(t_j, t_{j+1})$, the interpolation error is $O(h^{k+1})$ and thus,

$$
E(t) = h^k \Big(e(t) + O(h^{k+1}) \Big) + r(t) = h^k \Big(O(1) + O(h^{k+1}) \Big) + O(h^k) = O(h^k),
$$

for $t \in (t_j, t_{j+1})$, $j = 0, \dots, I-1$, For the subinterval endpoints, $t = t_{j+1}$, we have from (7.8)

$$
E(t_{j+1}) = h^k \sum_{i=1}^{k+1} l_i(1) e(t_{ji}) + r(t_{ji}) = O(h^k),\ j = 0, \dots I-1,
$$

and finally, for $t=0$, $j=0$,

$$E(0) := \lim_{t\to 0} E(t) = h^k \sum_{i=1}^{k+1} l_i(0) e(t_{0i}) + r(t_{0i}) = O(h^k).$$

Altogether, $E = O(h^k)$ in $[0,1]$ and the result, $\|p-y\| = O(h^k)$, follows.

□

8 Numerical experiments

In this section, we illustrate the theory by numerical experiments. We have constructed model problems in the IVP, TVP and BVP setting. To calculate the numerical results, we have used the MATLAB code `bvpsuite` and run the code on coherently refined meshes in order to compare the empirically estimated convergence orders with those predicted by the theory.

8.1 Initial value problem

We consider the following initial value problem:

$$y'(t) = \frac{M(t)}{t} y(t) + \frac{f(t)}{t}, \quad \begin{pmatrix} 3 & -2 & 1 \\ -2 & 2 & -1 \\ -2 & 1 & 0 \end{pmatrix} y(0) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

Here

$$M(t) = \begin{pmatrix} 3t - 2\sin t - 4 & -2t + \sin t + 2 & t-1 \\ 3t^2 + 6t - 4\sin t - 8 & -2t^2 - 4t + 2\sin t + 4 & t^2 + 2t - 2 \\ 6t^2 + 6t - 4\sin t - 12 & -4t^2 - 4t + 2\sin t + 8 & 2t^2 + 2t - 4 \end{pmatrix}$$

and

$$f(t) = \begin{pmatrix} -t^2\sin(t) + 2\exp(t) + \sin(t)\cos(t) + 2t\cos^2(t) - t \\ -2t^2\sin(t) - t^2\exp(t) + 4\exp(t) + 2\sin(t)\cos(t) + 4t\cos^2(t) + 2t^2 - 2t \\ -2t^2\sin(t) - 2t^2\exp(t) + 4\exp(t) + 2t\cos^2(t) + 4t^2 - t. \end{pmatrix}.$$

The matrix $M(0)$,

$$M(0) = \begin{pmatrix} -4 & 2 & -1 \\ -8 & 4 & -2 \\ -12 & 8 & -4 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 2 & 2 & 1 \\ 2 & 1 & 2 \end{pmatrix} \begin{pmatrix} -2 & 1 & \\ & -2 & \\ & & 0 \end{pmatrix} \begin{pmatrix} 3 & -2 & 1 \\ -2 & 2 & -1 \\ -2 & 1 & 0 \end{pmatrix}$$

has a double eigenvalue $\lambda_1 = \lambda_2 = -2$, and a simple eigenvalue $\lambda_3 = 0$. The exact solution $y \in C^\infty[0,1]$ of the problem is given and has the form

$$y(t) = \begin{pmatrix} \exp(t) + \sin(t)\cos(t) \\ 2\exp(t) + 2\sin(t)\cos(t) + t^2 \\ 2\exp(t) + \sin(t)\cos(t) + 2t^2. \end{pmatrix},$$

In Tables 8.1 – 8.3, we illustrate the convergence behaviour for the collocation executed with equidistant and Gaussian collocation points. The number of the collocation points k was chosen to vary from 1 to 8. However, in the simulations shown here, we report only on the values 1 to 4 since the results for 5 to 8 are very similar. The maximal global error is computed either in the mesh points,

$$\|Y_h - Y\|_\Delta := \max_{0\le j\le I} |p(t_j) - y(t_j)|,$$

or 'uniformly' in t, $\|Y_h - Y\|_u := \max_{0\le i\le 1.000} |p(\tau_i) - y(\tau_i)|$, $\tau_i = ih,\ h = 10^{-3}$. The order of convergence and the error constant c are estimated using two consecutive meshes with the step sizes h and $h/2$.

From the ansatz, $\|Y_h - Y\| \approx ch^p$ for $h \to 0$, we have

$$\|Y_h - Y\|_\Delta = ch^p, \quad \|Y_{h/2} - Y\|_\Delta = c\left(\frac{h}{2}\right)^p \Rightarrow p = \ln\left(\frac{\|Y_h - Y\|_\Delta}{\|Y_{h/2} - Y\|_\Delta}\right)\frac{1}{\ln(2)}.$$

Having p, we calculate the error constant from $c = \|Y_{h/2} - Y\|_\Delta / \left(\frac{h}{2}\right)^p$.

According to the experiments, the empirical convergence orders very well reflect the theoretical findings. For Gaussian points, we observe the small superconvergence order $k+1$ in the mesh points. The superconvergence order $2k$ in the mesh points does not hold in general. For uniformly spaced equidistant collocation points we observe the order k uniformly in t as we have proven theoretically.

Table 8.1 IVP: Convergence of the collocation scheme, $k = 2$

h	Gaussian, mesh points			equidistant, mesh points			equidistant, uniform		
	$\|Y_h - Y\|_\Delta$	c	p	$\|Y_h - Y\|_\Delta$	c	p	$\|Y_h - Y\|_u$	c	p
$1/2$	1.4e-02	6.8e-02	2.23	4.3e-02	1.6e-01	1.94	4.3e-02	1.6e-01	1.94
$1/4$	3.1e-03	1.9e-01	2.96	1.1e-02	2.0e-01	2.08	1.1e-02	2.0e-01	2.08
$1/8$	4.0e-04	2.1e-01	3.02	2.6e-03	1.7e-01	2.01	2.6e-03	1.7e-01	2.01
$1/16$	4.9e-05	2.1e-01	3.01	6.5e-04	1.6e-01	2.00	6.5e-04	1.6e-01	2.00
$1/32$	6.1e-06	2.0e-01	3.01	1.6e-04	1.6e-01	2.00	1.6e-04	1.6e-01	2.00
$1/64$	7.6e-07	2.0e-01	3.00	4.1e-05	1.7e-01	2.00	4.1e-05	1.7e-01	2.00
$1/128$	9.4e-08	2.0e-01	3.00	1.0e-05	1.7e-01	2.00	1.0e-05	1.7e-01	2.00
$1/256$	1.2e-08	2.0e-01	3.00	2.5e-06	1.7e-01	2.00	2.5e-06	1.7e-01	2.00
$1/512$	1.5e-09	–	–	6.4e-07	–	–	6.4e-07	–	–

8.2 Terminal value problem

As a TVP, we consider the following model problem:

$$y'(t) = \frac{M(t)}{t}y(t) + \frac{f(t)}{t}, \quad \begin{pmatrix} 2 & 1 & -1 \\ 0 & 1 & 0 \\ -1 & -1 & 1 \end{pmatrix} y(1) = \begin{pmatrix} e \\ e \\ 1/15 \end{pmatrix},$$

Table 8.2 IVP: Convergence of the collocation scheme, $k = 3$

h	Gaussian, mesh points			equidistant, mesh points			equidistant, uniform		
	$\lVert Y_h - Y\rVert_\Delta$	c	p	$\lVert Y_h - Y\rVert_\Delta$	c	p	$\lVert Y_h - Y\rVert_u$	c	p
$1/2$	2.7e-04	2.1e-03	2.99	2.8e-02	5.2e-01	4.19	2.8e-02	5.2e-01	4.19
$1/4$	3.3e-05	2.3e-02	4.70	1.6e-03	4.3e-01	4.05	1.6e-03	4.3e-01	4.05
$1/8$	1.3e-06	3.6e-02	4.93	9.4e-05	4.0e-01	4.01	9.4e-05	4.0e-01	4.01
$1/16$	4.2e-08	4.1e-02	4.98	5.8e-06	3.8e-01	4.00	5.8e-06	3.8e-01	4.00
$1/32$	1.3e-09	4.4e-02	4.99	3.6e-07	3.8e-01	4.00	3.6e-07	3.8e-01	4.00
$1/64$	4.2e-11	4.4e-02	4.99	2.3e-08	3.8e-01	4.00	2.3e-08	3.8e-01	4.00
$1/128$	1.3e-12	1.0e-01	5.17	1.4e-09	3.8e-01	4.00	1.4e-09	3.8e-01	4.00
$1/256$	3.7e-14	2.3e-14	-0.08	8.8e-11	3.8e-01	4.00	8.8e-11	3.8e-01	4.00
$1/512$	3.9e-14	–	–	5.5e-12	–	–	5.5e-12	–	–

Table 8.3 IVP: Convergence of the collocation scheme, $k = 4$

h	Gaussian, mesh points			equidistant, mesh points			equidistant,uniform		
	$\lVert Y_h - Y\rVert_\Delta$	c	p	$\lVert Y_h - Y\rVert_\Delta$	c	p	$\lVert Y_h - Y\rVert_u$	c	p
$1/2$	2.2e-05	5.1e-04	4.51	2.1e-03	3.9e-02	4.21	2.1e-03	3.5e-02	4.05
$1/4$	9.7e-07	1.3e-03	5.22	1.1e-04	4.0e-02	4.22	1.3e-04	4.5e-02	4.23
$1/8$	2.6e-08	1.2e-03	5.16	6.1e-06	3.0e-02	4.08	6.8e-06	3.7e-02	4.14
$1/16$	7.3e-10	9.8e-04	5.09	3.6e-07	2.5e-02	4.02	3.8e-07	3.0e-02	4.06
$1/32$	2.1e-11	8.5e-04	5.05	2.2e-08	2.4e-02	4.01	2.3e-08	2.6e-02	4.02
$1/64$	6.5e-13	3.2e-04	4.81	1.4e-09	2.3e-02	4.00	1.4e-09	2.5e-02	4.01
$1/128$	2.3e-14	1.3e-11	1.31	8.7e-11	2.3e-02	4.00	8.8e-11	2.4e-02	4.00
$1/256$	9.3e-15	9.8e-19	-1.65	5.4e-12	2.3e-02	4.00	5.5e-12	2.4e-02	4.00
$1/512$	2.9e-14	–	–	3.4e-13	–	–	3.4e-13	–	–

where

$$M(t) = \begin{pmatrix} 24+2t & 12+t & -12-t \\ -26t & 20-12t & 13t \\ 24-24t & 32-11t & -12+12t \end{pmatrix},$$

$$M(0) = \begin{pmatrix} 24 & 12 & -12 \\ 0 & 20 & 0 \\ 24 & 32 & -12 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{pmatrix} \begin{pmatrix} 12 & & \\ & 20 & \\ & & 0 \end{pmatrix} \begin{pmatrix} 2 & 1 & -1 \\ 0 & 1 & 0 \\ -1 & -1 & 1 \end{pmatrix}$$

and

$$f(t) = \begin{pmatrix} -12\exp(t) + t^{15} \\ -6t\exp(t) \\ -6t\exp(t) - 12\exp(t) + 2t^{15} \end{pmatrix}.$$

The eigenvalues of $M(0)$ are simple, $\lambda_1 = 12$, $\lambda_2 = 20$, and $\lambda_3 = 0$ and the exact solution $y \in C^\infty[0,1]$ reads:

$$y(t) = \begin{pmatrix} \exp(t) + \frac{1}{15}t^{15} \\ t\exp(t) \\ \exp(t) + t\exp(t) + \frac{2}{15}t^{15} \end{pmatrix}$$

Tables 8.4 – 8.6 show the same convergence behaviour as observed for IVPs. For equidistant collocation points the convergence order k holds not only in the mesh points but also uniformly in t. Also, the small superconvergence $k+1$ can be observed in the mesh points, when Gaussian points are used as collocation points.

Table 8.4 TVP: Convergence of the collocation scheme, $k=2$

h	Gaussian, mesh points			equidistant, mesh points			equidistant, uniform		
	$\|Y_h-Y\|_\Delta$	c	p	$\|Y_h-Y\|_\Delta$	c	p	$\|Y_h-Y\|_u$	c	p
1/2	2.8e-01	1.3e+00	2.22	3.7e-01	1.0e+00	1.42	3.7e-01	1.0e+00	1.42
1/4	6.1e-02	4.8e+00	3.15	1.4e-01	1.4e+00	1.64	1.4e-01	1.4e+00	1.64
1/8	6.8e-03	9.1e+00	3.46	4.5e-02	2.3e+00	1.89	4.5e-02	2.3e+00	1.89
1/16	6.2e-04	7.1e+00	3.37	1.2e-02	2.9e+00	1.98	1.2e-02	2.9e+00	1.98
1/32	6.0e-05	4.5e+00	3.24	3.1e-03	3.1e+00	2.00	3.1e-03	3.1e+00	2.00
1/64	6.3e-06	3.0e+00	3.14	7.7e-04	3.1e+00	2.00	7.7e-04	3.1e+00	2.00
1/128	7.2e-07	2.2e+00	3.07	1.9e-04	3.1e+00	2.00	1.9e-04	3.2e+00	2.00
1/256	8.5e-08	1.8e+00	3.04	4.8e-05	3.1e+00	2.00	4.8e-05	3.2e+00	2.00
1/512	1.0e-08	–	–	1.2e-05	–	–	1.2e-05	–	–

Table 8.5 TVP: Convergence of the collocation scheme, $k=3$

h	Gaussian, mesh points			equidistant, mesh points			equidistant, uniform		
	$\|Y_h-Y\|_\Delta$	c	p	$\|Y_h-Y\|_\Delta$	c	p	$\|Y_h-Y\|_u$	c	p
1/2	1.3e-02	1.1e-01	3.03	8.0e-02	1.8e-01	1.18	1.2e-01	2.8e-01	1.29
1/4	1.6e-03	5.8e+00	5.90	3.5e-02	2.6e+00	3.11	4.7e-02	1.5e+00	2.48
1/8	2.7e-05	2.7e-01	4.41	4.1e-03	1.0e+01	3.76	8.5e-03	6.9e+00	3.22
1/16	1.3e-06	6.1e-02	3.88	3.0e-04	1.7e+01	3.94	9.1e-04	2.0e+01	3.61
1/32	8.8e-08	5.4e-02	3.85	2.0e-05	2.0e+01	3.98	7.5e-05	4.0e+01	3.80
1/64	6.1e-09	8.9e-02	3.97	1.3e-06	2.1e+01	4.00	5.3e-06	6.0e+01	3.90
1/128	3.9e-10	1.0e-01	3.99	7.8e-08	2.1e+01	4.00	3.6e-07	7.6e+01	3.95
1/256	2.4e-11	9.3e-02	3.98	4.9e-09	2.1e+01	4.00	2.3e-08	8.7e+01	3.98
1/512	1.6e-12	–	–	3.1e-10	–	–	1.5e-09	–	–

Table 8.6 TVP: Convergence of the collocation scheme, $k=4$

h	Gaussian, mesh points			equidistant, mesh points			equidistant, uniform		
	$\|Y_h-Y\|_\Delta$	c	p	$\|Y_h-Y\|_\Delta$	c	p	$\|Y_h-Y\|_u$	c	p
1/2	5.0e-03	3.0e-01	5.93	9.6e-02	3.5e-01	1.87	9.6e-02	3.5e-01	1.87
1/4	8.1e-05	4.9e-01	6.28	2.6e-02	2.1e+00	3.16	2.6e-02	2.1e+00	3.16
1/8	1.0e-06	9.5e-02	5.49	2.9e-03	7.4e+00	3.77	2.9e-03	7.4e+00	3.77
1/16	2.3e-08	3.5e-02	5.13	2.1e-04	1.2e+01	3.94	2.1e-04	1.2e+01	3.94
1/32	6.7e-10	2.6e-02	5.04	1.4e-05	1.4e+01	3.99	1.4e-05	1.4e+01	3.99
1/64	2.0e-11	2.2e-02	5.00	8.8e-07	1.4e+01	4.00	8.8e-07	1.4e+01	4.00
1/128	6.4e-13	6.7e-04	4.28	5.5e-08	1.5e+01	4.00	5.5e-08	1.5e+01	4.00
1/256	3.3e-14	2.9e-13	0.39	3.4e-09	1.5e+01	4.00	3.4e-09	1.5e+01	4.00
1/512	2.5e-14	–	–	2.1e-10	–	–	2.1e-10	–	–

8.3 Boundary value problem

Finally, we discuss the following BVP:

$$y'(t) = \frac{M(t)}{t}y(t) + \frac{f(t)}{t}, \quad \begin{pmatrix} -2 & 3 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} y(0) + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 1 \end{pmatrix} y(1) = \begin{pmatrix} 1/2 \\ 0 \\ 4/5 \end{pmatrix},$$

where

$$M(t) = \begin{pmatrix} 1-2t & 3t-3\exp(t) & 3\exp(t)-3 \\ 2-2t & 3t-2\exp(t)-2 & 2\exp(t)-2 \\ 2-2t & 3t-2\exp(t)-12 & 2\exp(t)+8 \end{pmatrix},$$

$$M(0) = \begin{pmatrix} 1 & -3 & 0 \\ 2 & -4 & 0 \\ 2 & -14 & 10 \end{pmatrix} = \begin{pmatrix} 1 & 3 & 0 \\ 1 & 2 & 0 \\ 1 & 2 & 1 \end{pmatrix} \begin{pmatrix} -2 & & \\ & -1 & \\ & & 10 \end{pmatrix} \begin{pmatrix} -2 & 3 & 0 \\ 1 & -1 & 0 \\ 0 & -1 & 1 \end{pmatrix},$$

and

$$f(t) = \begin{pmatrix} 3t^{21/2} - t^{-1} + t^{-1}\exp(t) - 3t^{10}\exp(t) + 3t^{10} + \frac{3}{5}\exp(t) - \frac{3}{5} \\ 2t^{21/2} - t^{-1} + t^{-1}\exp(t) - 2t^{10}\exp(t) + 2t^{10} + \frac{2}{5}\exp(t) - \frac{2}{5} \\ 2t^{21/2} - t^{-1} + t^{-1}\exp(t) - 2t^{10}\exp(t) + 2t^{10} + \frac{2}{5}\exp(t) + \frac{8}{5} \end{pmatrix}$$

The matrix $M(0)$ has both positive and negative eigenvalues $\lambda_1 = 10$, $\lambda_2 = -2$, $\lambda_3 = -1$, and the exact solution has the form

$$y(t) = \begin{pmatrix} t^{-2} + t^{-1}\exp(t) - t^{-2}\exp(t) + \frac{6}{23}t^{21/2} \\ t^{-2} + t^{-1}\exp(t) - t^{-2}\exp(t) + \frac{4}{23}t^{21/2} \\ t^{-2} + t^{-1}\exp(t) - t^{-2}\exp(t) + \frac{4}{23}t^{21/2} + t^{10} - \frac{1}{5} \end{pmatrix}.$$

Here, $y \in C^{10}[0,1]$. In Tables 8.7 – 8.9 the convergence order k can be observed uniformly in t. For this model and for the Gaussian points we observe the superconvergence order $O(h^{2k})$. Such high order of convergence can be sometimes observed, but does not hold in general.

Table 8.7 BVP: Convergence of the collocation scheme, $k = 2$

h	Gaussian, mesh points			equidistant, mesh points			equidistant, uniform		
	$\Vert Y_h - Y\Vert_\Delta$	c	p	$\Vert Y_h - Y\Vert_\Delta$	c	p	$\Vert Y_h - Y\Vert_u$	c	p
$1/2$	8.1e-02	3.6e-02	-1.18	5.4e-02	1.1e-02	-2.30	6.1e-01	1.4e+00	1.22
$1/4$	1.8e-01	8.5e+00	2.77	2.6e-01	2.9e+00	1.73	2.6e-01	2.9e+00	1.73
$1/8$	2.7e-02	6.2e+01	3.72	7.9e-02	5.5e+00	2.04	7.9e-02	5.0e+00	1.99
$1/16$	2.0e-03	1.7e+02	4.09	1.9e-02	9.3e+00	2.23	2.0e-02	6.8e+00	2.10
$1/32$	1.2e-04	1.4e+02	4.02	4.1e-03	4.6e+00	2.02	4.6e-03	6.4e+00	2.09
$1/64$	7.4e-06	1.2e+02	4.00	1.0e-03	4.4e+00	2.01	1.1e-03	5.7e+00	2.06
$1/128$	4.6e-07	1.2e+02	4.00	2.5e-04	4.1e+00	2.00	2.6e-04	5.1e+00	2.04
$1/256$	2.9e-08	1.2e+02	4.00	6.2e-05	4.1e+00	2.00	6.4e-05	4.7e+00	2.02
$1/512$	1.8e-09	–	–	1.6e-05	–	–	1.6e-05	–	–

Table 8.8 BVP: Convergence of the collocation scheme, $k = 3$

h	Gaussian, mesh points			equidistant, mesh points			equidistant, uniform		
	$\|Y_h - Y\|_\Delta$	c	p	$\|Y_h - Y\|_\Delta$	c	p	$\|Y_h - Y\|_u$	c	p
1/2	2.6e-02	2.1e-02	-0.35	6.7e-03	4.5e-04	-3.91	4.2e-01	1.7e+00	2.04
1/4	3.4e-02	2.4e+01	4.75	1.0e-01	5.6e+00	2.90	1.0e-01	5.6e+00	2.90
1/8	1.2e-03	2.2e+02	5.81	1.4e-02	3.5e+01	3.78	1.4e-02	1.8e+01	3.45
1/16	2.2e-05	4.1e+02	6.04	9.9e-04	8.3e+01	4.09	1.2e-03	2.5e+01	3.57
1/32	3.4e-07	3.8e+02	6.01	5.8e-05	6.6e+01	4.02	1.0e-04	5.0e+01	3.78
1/64	5.2e-09	3.6e+02	6.00	3.6e-06	6.1e+01	4.01	7.6e-06	8.0e+01	3.89
1/128	8.2e-11	3.6e+02	6.00	2.2e-07	6.0e+01	4.00	5.2e-07	1.0e+02	3.94
1/256	1.3e-12	4.2e-10	1.04	1.4e-08	5.9e+01	4.00	3.4e-08	1.2e+02	3.97
1/512	6.2e-13	–	–	8.7e-10	–	–	2.1e-09	–	–

Table 8.9 BVP: Convergence of the collocation scheme, $k = 4$

h	Gaussian, mesh points			equidistant, mesh points			equidistant, uniform		
	$\|Y_h - Y\|_\Delta$	c	p	$\|Y_h - Y\|_\Delta$	c	p	$\|Y_h - Y\|_u$	c	p
1/2	2.8e-03	2.0e-03	-0.45	4.0e-03	5.5e-04	-2.88	2.5e-01	2.1e+00	3.06
1/4	3.8e-03	5.4e+01	6.89	3.0e-02	6.3e+00	3.87	3.0e-02	6.3e+00	3.87
1/8	3.2e-05	4.7e+02	7.94	2.0e-03	1.3e+01	4.23	2.0e-03	8.6e+00	4.02
1/16	1.3e-07	4.7e+02	7.93	1.1e-04	1.3e+01	4.23	1.2e-04	1.6e+01	4.23
1/32	5.3e-10	5.8e+02	8.00	5.7e-06	7.5e+00	4.06	6.6e-06	1.0e+01	4.12
1/64	2.1e-12	4.1e-06	3.49	3.4e-07	6.1e+00	4.01	3.8e-07	9.1e+00	4.08
1/128	1.9e-13	1.1e-16	-1.53	2.1e-08	5.8e+00	4.00	2.3e-08	7.6e+00	4.05
1/256	5.4e-13	1.5e-13	-0.23	1.3e-09	5.7e+00	4.00	1.4e-09	6.6e+00	4.02
1/512	6.3e-13	–	–	8.3e-11	–	–	8.4e-11	–	–

9 Conclusions

In Part 1 [7], the analytical properties, the existence and uniqueness of smooth solutions, of the following singular BVP with a variable coefficient matrix and unsmooth inhomogeneity were discussed,

$$y'(t) = \frac{M(t)}{t}y(t) + \frac{f(t)}{t}, \quad B_0 y(0) + B_1 y(1) = \beta.$$

In this paper, we have analysed the convergence of the collocation method applied to approximate the solution of the above analytical problem. The convergence behaviour have been investigated separately for general IVPs, TVPs and BVPs. It turned out that the collocation retains its classical stage order k uniformly in t for a scheme with k collocation points, provided that the analytical solutions are appropriately smooth. Moreover, for Gaussian points the so-called small superconvergence order $k+1$ was shown to hold in context of an IVP, whereas, the superconvergence order in the mesh points, $2k$ for Gaussian points, cannot be expected to hold, in general. The theoretical results are supported by the numerical experiments.

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