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A CROSS-DIFFUSION SYSTEM DERIVED FROM A FOKKER-PLANCK EQUATION WITH PARTIAL AVERAGING

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Abstract. A cross-diffusion system for two compoments with a Laplacian structure is analyzed on the multi-dimensional torus. This system, which was recently suggested by P.-L. Lions, is formally derived from a Fokker-Planck equation for the probability density associated to a multi-dimensional Itō process, assuming that the diffusion coefficients depend on partial averages of the probability density with exponential weights. A main feature is that the diffusion matrix of the limiting cross-diffusion system is generally neither symmetric nor positive definite, but its structure allows for the use of entropy methods. The global-in-time existence of positive weak solutions is proved and, under a simplifying assumption, the large-time asymptotics is investigated.

1. Introduction

The aim of this paper is the analysis of the following cross-diffusion system

$$\partial_t u_i = \Delta\big(a(u_1/u_2)u_i\big) + \mu_i u_i, \quad t > 0, \quad u_i(0) = u_i^0 \geq 0 \quad \text{in } \mathbb{T}^d, \ i = 1, 2, \tag{1}$$

where $\mathbb{T}^d$ is the d-dimensional torus with $d \geq 1$, $a : [0, \infty) \to (0, \infty)$ is a continuously differentiable function, and $\mu_i \in \mathbb{R}$. This system can be formally derived [7] from a $(d+1)$-dimensional Fokker-Planck equation for the probability density $f(x, y, t)$, where $x \in \mathbb{R}^d$, $y \in \mathbb{R}$. The function u_i is obtained from f by partial averaging,

$$u_i(x, t) = \int_{\mathbb{R}} f(x, y, t)e^{\lambda_i y}dy, \quad i = 1, 2,$$

μ_i is a function of λ_i, and $a(u_1/u_2)$ is related to the diffusion coefficients in the Fokker-Planck equation. Strictly speaking, equation (1) holds in $\mathbb{R}^d$ (or on some subset of $\mathbb{R}^d$) but we consider this equation on the torus for the sake of simplicity (and to avoid possible issues with boundary conditions). For details on the derivation, we refer to Section 2.

System (1) has been suggested by P.-L. Lions in [7], and the global-in-time existence of (weak) solutions has been identified as an open problem. In this paper, we solve this problem by applying the entropy method for diffusive equations.

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The underlying Fokker-Planck equation for $f(x,y,t)$ models the time evolution of the value of a financial product in an idealized financial market, depending on various underlying assets or economic values. The function u_i is an average with respect to the variable y, which may be interpreted as the value of an economic parameter, and the exponential weight emphasizes large positive or large negative values of y, depending on the sign of λ_i. We note that partial averaging is also employed to simplify chemical master equations [9]. Here, we are not interested in potential applications, but more in the refinement of mathematical tools to analyze (1).

We assume that there exist $a_0 > 0$ and $p \geq 0$ such that for all $r > 0$,

$$a(r) \geq r|a'(r)|, \quad a(r) \geq \frac{a_0}{r^p + r^{-p}}. \tag{2}$$

The first condition means that a grows at most linearly (see Lemma 6). The second condition is a technical assumption needed for the entropy method (see the proof of Lemma 5). Examples are $a(r) = 1$, which leads to uncoupled heat equations for u_1 and u_2, $a(r) = r^\alpha$ with $0 < \alpha \leq 1$, $a(r) = r^\beta/(1 + r^{\beta-1})$ with $\beta > 0$, and $a(r) = 1/r$. The last example gives the equations

$$\partial_t u_1 = \Delta u_2, \quad \partial_t u_2 = \Delta\left(\frac{u_2^2}{u_1}\right). \tag{3}$$

Surprisingly, this system corresponds (up to a factor) to an energy-transport model for semiconductors. Indeed, introducing the electron density $n := u_1$ and the electron temperature $\theta := u_2/u_1$, equations (3) can be written as

$$\partial_t n = \Delta(n\theta), \quad \partial_t(n\theta) = \Delta(n\theta^2).$$

A class of energy-transport models that includes the above example was analyzed in [13].

Another class of models which resembles (1) are the equations

$$\partial_t u_i = \Delta(p_i(u)u_i), \quad i = 1, \ldots, m, \tag{4}$$

modeling the time evolution of population densities u_i. These systems are analyzed in, e.g., [5, 8], essentially for $m = 2$. In this application, p_i is often given by the sum $p_{i1}(u_1)+p_{i2}(u_2)$, and consequently, the results of [5, 8] do not apply and we need to develop new ideas.

Our first main result is the global-in-time existence of weak solutions to (1).

Theorem 1 (Existence of weak solutions). *Let (2) hold and let $T > 0$, $\alpha \geq p + 4$, $0 \leq a \in C^1([0,\infty))$, $u^0 = (u_1^0, u_2^0) \in L^2(\mathbb{T}^d)^2$ with u_1^0, $u_2^0 \geq 0$ in $\mathbb{T}^d$. Then there exists a solution $u = (u_1, u_2)$ to (1) satisfying $u_i > 0$ in $\mathbb{T}^d$, $t > 0$, and*

$$u_i,\ a(u_1/u_2)u_i \in L^\infty(0,T; L^2(\mathbb{T}^d)),$$
$$\nabla u_i,\ \nabla\big(a(u_1/u_2)u_i\big) \in L^2(0,T; L^2(\mathbb{T}^d)), \quad \partial_t u_i \in L^2(0,T; H^1(\mathbb{T}^d)'), \quad i = 1, 2.$$

If additionally $\mu_i \leq 0$ for $i = 1, 2$, we have the uniform bounds

$$u_i,\ a(u_1/u_2)u_i \in L^\infty(0,\infty; L^2(\mathbb{T}^d)), \quad \nabla u_i,\ \nabla\big(a(u_1/u_2)u_i\big) \in L^2(0,\infty; L^2(\mathbb{T}^d)). \tag{5}$$

As mentioned above, the proof of this theorem is based on entropy methods. These methods have been originally developed to understand the large-time behavior of solutions; see, e.g., [2, 12]. The "entropy" of system (1) is often understood as a convex Lyapunov functional which provides suitable nonlinear gradient estimates. In many situations, and also in the financial context presented here, the "entropy" has no physical counterpart. However, we claim that this notion is appropriate since it naturally generalizes physical situations. For details, we refer to [8].

Our key idea is to employ the functional

$$H[u] = \int_{\mathbb{T}^d} h(u)dx, \quad h(u) = \left(\frac{u_1}{u_2}\right)^{\alpha} u_1^2 + \left(\frac{u_1}{u_2}\right)^{-\alpha} u_2^2 + u_1 - \log u_1 + u_2 - \log u_2, \tag{6}$$

where $\alpha \geq p+4$ and $u = (u_1, u_2) \in (0,\infty)^2$. We will show that

$$\frac{d}{dt}H[u] + \int_{\mathbb{T}^d} \left(\left(\frac{u_1}{u_2}\right)^{\alpha-p} + \left(\frac{u_1}{u_2}\right)^{p-\alpha} \right) \big(|\nabla u_1|^2 + |\nabla u_2|^2\big) dx \leq CH[u] \tag{7}$$

for some constant $C > 0$ which vanishes if $\mu_1 = \mu_2 = 0$. In this situation, the mapping $t \mapsto H[u(t)]$ is nonincreasing; otherwise, for $\mu_i \neq 0$, $t \mapsto H[u(t)]$ is bounded on finite time intervals. We infer from the inequality $x + x^{-1} \geq 2$ for all $x > 0$ uniform bounds for $u_i(t)$ in $H^1(\mathbb{T}^d)$, which are needed for the compactness argument.

The entropy method gives more than just the a priori estimate (7). Indeed, let us write (1) in divergence form:

$$\partial_t u - \operatorname{div}(A(u)\nabla u) = f(u), \quad t > 0, \quad u(0) = u^0 \quad \text{in } \mathbb{T}^d,$$

where the ith component of $\operatorname{div}(A(u)\nabla u)$ equals $\sum_{j=1}^d \sum_{k=1}^2 \partial_j(A_{ik}(u)\partial_j u_k)$, $\partial_j = \partial/\partial x_j$, and $f(u) = (\mu_1 u_1, \mu_2 u_2)^\top$. The diffusion matrix

$$A(u) = \begin{pmatrix} a(u_1/u_2) + (u_1/u_2)a'(u_1/u_2) & -(u_1/u_2)^2 a'(u_1/u_2) \\ a'(u_1/u_2) & a(u_1/u_2) - (u_1/u_2)a'(u_1/u_2) \end{pmatrix} \tag{8}$$

is generally neither symmetric nor positive definite. Since the only eigenvalue of $A(u)$ is given by $\lambda = a(u_1/u_2) > 0$, the system is normally elliptic [1] and local-in-time existence of classical solutions can be expected. The difficulty is to prove the global-in-time existence. The entropy density $h(u)$ allows us to formulate (1) in new variables with a positive semidefinite diffusion matrix. Then, together with the a priori estimates from (7), global existence will be deduced. Indeed, defining the so-called entropy variable $w = (w_1, w_2)$ by $w_i = \partial h/\partial u_i$ ($i = 1, 2$), equation (1) is equivalent to

$$\partial_t u - \operatorname{div}(B(w)\nabla w) = f(u), \quad t > 0, \quad u(0) = u^0 \quad \text{in } \mathbb{T}^d, \tag{9}$$

where $B(w) = A(u)h''(u)^{-1}$ is positive semidefinite (see Lemma 5) and $h''(u)$ is the Hessian matrix of $h(u)$. With this formulation, we obtain

$$\frac{d}{dt}H[u] + \int_{\mathbb{T}^d} \nabla u : h''(u)A(u)\nabla u dx = \int_{\mathbb{T}^d} f(u) \cdot w dx,$$

where $A : B = \sum_{j=1}^d \sum_{k=1}^2 A_{kj}B_{kj}$ for two matrices $A = (A_{kj})$, $B = (B_{kj}) \in \mathbb{R}^{2\times d}$. The right-hand side can be bounded in terms of $H[u]$ (see (14)), and the integral on the left-hand side is related to the corresponding integral in (7).

The proof of Theorem 1 is based on a regularization of (9), the fixed-point theorem of Leray-Schauder, and the de-regularization limit. The compactness is obtained from the entropy estimate (7). This technique is similar to those employed in our works [8, 13]. The novelty here is the (nontrivial) observation that the cross-diffusion system (1) possesses a convex Lyapunov functional, defined by (6). Moreover, compared to [8, 13], we are facing additional technical difficulties due to the quotient u_1/u_2.

The second result concerns the large-time asymptotics in the case $\mu_i = 0$ for $i = 1, 2$. Generally, the idea of the entropy method is to relate the entropy production $-dH/dt$ with the entropy itself and to apply Gronwall's lemma in order to conclude exponential decay to equilibrium. Since we have not found an inequality relating the integral in (7) with the entropy, we obtain a weaker result. We show, still using the entropy inequality (7), convergence to equilibrium without a rate. If $\mu_i < 0$ for $i = 1, 2$, we prove the exponential convergence of $u(t)$ to zero in $H^1(\mathbb{T}^d)'$, see Remark 10. For a discussion of the case $\mu_i > 0$, we refer to Remark 11.

Theorem 2 (Large-time asymptotics)**.** *Let the assumptions of Theorem 1 hold and let $\mu_1 = \mu_2 = 0$, $\alpha > p + 4 + 2(\sqrt{2} - 1)$. Then the solution $u(t) = (u_1, u_2)(t)$ to (1) converges in $L^2(\mathbb{T}^d)$ to $\overline{u} = (\overline{u}_1, \overline{u}_2)$ as $t \to \infty$, where*

$$\overline{u}_i = \frac{1}{\mathrm{meas}(\mathbb{T}^d)} \int_{\mathbb{T}^d} u_i^0 dx, \quad i = 1, 2.$$

The paper is organized as follows. In Section 2, we make precise the derivation of (1) from a Fokker-Planck equation. Some technical results are proved in Section 3. Section 4 is devoted to the proof of Theorem 1, and Theorem 2 is shown in Section 5.

2. Derivation of the cross-diffusion system (1)

We summarize the formal derivation of (1) from a Fokker-Planck equation as presented by P.-L. Lions in [7]. Consider the n-dimensional Itō process $X_t = (X_t^1, \ldots, X_t^n)$ on some probability space, driven by the n-dimensional Wiener process $W_t = (W_t^1, \ldots, W_t^n)$ with respect to some given filtration. We assume that X_t solves the stochastic differential equation

$$dX_t = \widetilde{\mu}_t(X_t)dt + \sigma_t(X_t)dW_t, \quad t > 0,$$

where $\widetilde{\mu}_t = (\widetilde{\mu}_t^1, \ldots, \widetilde{\mu}_t^n)$, and $\sigma_t = (\sigma_t^{ij})_{i,j=1,\ldots,n}$ is an $n \times n$ matrix. It is well known [10, Theorems 7.3.3, 8.2.1] that the probability density $f(x_1, \ldots, x_n, t)$ for X_t satisfies the Fokker-Planck (or forward Kolmogorov) equation

$$\partial_t f = \frac{1}{2} \sum_{i,j=1}^n \frac{\partial^2}{\partial x_i \partial x_j}(D_{ij}(\widehat{x})f) - \sum_{i=1}^n \frac{\partial}{\partial x_i}(\widetilde{\mu}^i(\widehat{x})f), \quad \widehat{x} \in \mathbb{R}^n, \ t > 0,$$

where $D(\widehat{x}) = (D_{ij}(\widehat{x})) = \sigma(\widehat{x})\sigma(\widehat{x})^\top$ is the diffusion tensor and $\widehat{x} = (x_1, \ldots, x_n)$.

In the following, we set $\widetilde{\mu}_t = 0$ and $\sigma_t = \operatorname{diag}(\sigma_1, \ldots, \sigma_n)$. This means that we neglect correlations between the processes. Taking them into account will lead to first-order terms in the final equations; see Remark 3. Under the above simplifications, the Fokker-Planck equation becomes

$$\partial_t f = \frac{1}{2}\sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2}(\sigma_j^2 f), \quad \widehat{x} \in \mathbb{R}^n,\ t > 0. \tag{10}$$

We assume that σ_j is a function of the partial averages

$$u_i(x,t) = \int_{\mathbb{R}} f(x, x_n, t) e^{\lambda_i x_n} dx_n, \quad x = (x_1, \ldots, x_{n-1}),\ i = 1, \ldots, m,$$

where λ_i are some given (pairwise different) parameters. Temporal averages appear, for instance, in the modeling of Asian options. Here, u_i may be interpreted as an average with respect to the ecocnomic parameter x_n. We may employ other weights than the exponential one but this one is mathematically extremely convenient because of the property $\partial u_i/\partial x_n = \lambda_i u_i$ (see Remark 3). Multiplying (10) by $e^{\lambda_i x_n}$ and integrating with respect to $x_n \in \mathbb{R}$, a straightforward calculation shows that u_i solves

$$\partial_t u_i = \frac{1}{2}\sum_{j=1}^{n-1} \frac{\partial^2}{\partial x_j^2}(\sigma_j^2 u_i) + \frac{\lambda_i^2}{2}\sigma_n^2 u_i, \quad i = 1, \ldots, m. \tag{11}$$

We allow σ_j to depend on the partial averages, $\sigma_j = \sigma_j(u_1, \ldots, u_m)$.

We consider only the special case $m = 2$, $\sigma := \sigma_j$ for $j = 1, \ldots, n-1$, and σ_n is constant and positive. Setting $u = (u_1, u_2)$, $\mu_i := \lambda_i^2 \sigma_n/2$, we find that

$$\partial_t u_i = \frac{1}{2}\Delta(\sigma(u)^2 u_i) + \mu_i u_i, \quad x \in \mathbb{R}^{n-1},\ t > 0,\ i = 1, 2. \tag{12}$$

In divergence form, this system is equivalent to

$$\partial_t u = \operatorname{div}(A(u)\nabla u), \quad \text{where } A(u) = \sigma \begin{pmatrix} \sigma + 2\partial_1\sigma u_1 & 2\partial_2\sigma u_1 \\ 2\partial_1\sigma u_2 & \sigma + 2\partial_2\sigma u_2 \end{pmatrix},$$

where $\partial_i \sigma = \partial\sigma/\partial u_i$, $i = 1, 2$. This system is of parabolic type in the sense of Petrovski if the real parts of the eigenvalues of A are nonnegative [1], i.e. if $\sigma + \partial_1\sigma u_1 + \partial_2\sigma u_2 \geq 0$ for all $u \in \mathbb{R}^2$. This requirement is fulfilled if, for instance, σ depends on the quotient u_1/u_2 only. Therefore, we set $\sigma(u)^2 = 2a(u_1/u_2)$. Then

$$\partial_t u_i = \Delta(a(u_1/u_2)u_i) + \mu_i u_i, \quad x \in \mathbb{R}^{n-1},\ t > 0,\ i = 1, 2,$$

is of parabolic type in the sense of Petrovski, and these equations correspond to (1).

Remark 3 (Generalizations)**.** The general model for nonvanishing $\widetilde{\mu}_t^i$ and nondiagonal σ_t is derived as above, and the result reads as

$$\partial_t u_i = \frac{1}{2}\sum_{j,k=1}^{n-1} \frac{\partial^2}{\partial x_j \partial x_k}(D_{jk} u_i) - \frac{1}{2}\sum_{j=1}^{n-1} \frac{\partial}{\partial x_j}\big((2\widetilde{\mu}^j + \lambda_i D_{jn})u_i\big) + \frac{\lambda_i}{2}(2\widetilde{\mu}^n + \lambda_i D_{nn})u_i. \tag{13}$$

Compared to (11), this equation also contains first-order terms. If $\widetilde{\mu}_t^i = 0$ and σ_t is diagonal, we obtain m equations of the type (12). The analysis of systems with more than two components is expected to be much more involved than for those with two components. For instance, the analysis of the cross-diffusion model (4) is rather well understood only in case $m = 2$, while the case $m \geq 3$ requires additional properties [4].

Another generalization concerns nonexponential weights. For instance, we may define

$$u_i = \int_{\mathbb{R}} f(x,t)\sin(\lambda_i x_n)dx_n, \quad i = 1, \ldots, m.$$

Choosing again $\widetilde{\mu}_t^i = 0$ and $\sigma_t = \mathrm{diag}(\sigma_1, \ldots, \sigma_n)$, we find that

$$\partial_t u_i = \frac{1}{2}\sum_{j=1}^{n-1}\frac{\partial^2}{\partial x_j^2}(\sigma_j(u)^2 u_i) - \frac{\lambda_i^2}{2}\sigma_n^2 u_i, \quad i = 1, \ldots, m.$$

This justifies the assumption $\mu_i \in \mathbb{R}$ in (1) but there seems to be no financial interpretation of the trigonometric weight functions. □

3. Some auxiliary lemmas

In this section, we prove some algebraic properties of the matrices $h''(u)$ and $A(u)$ and some estimates related to the entropy density $h(u)$ and the components of $A(u)$. Recall that $h(u)$ is defined in (6) and $A(u)$ in (8).

Lemma 4 (Properties of h)**.** *Let $\alpha > 0$. The function $h : (0,\infty)^2 \to \mathbb{R}^2$, defined in* (6)*, is convex, its derivative h' is invertible, and there exists $C_h > 0$ such that for all $u = (u_1, u_2) \in (0,\infty)^2$,*

$$h(u) \geq u_1^2 + u_2^2, \quad \sum_{i=1}^{2}\mu_i u_i \partial_i h(u) \leq C_h h(u), \tag{14}$$

where we recall that $\partial_i h = \partial h/\partial u_i$.

Proof. We proceed in several steps.

Step 1: h is convex. We compute the first partial derivatives of h,

$$\partial_1 h(u) = (\alpha+2)u_1^{\alpha+1}u_2^{-\alpha} - \alpha u_1^{-\alpha-1}u_2^{\alpha+2} - u_1^{-1} + 1, \tag{15}$$

$$\partial_2 h(u) = (\alpha+2)u_1^{-\alpha}u_2^{\alpha+1} - \alpha u_1^{\alpha+2}u_2^{-\alpha-1} - u_2^{-1} + 1, \tag{16}$$

and the Hessian $h''(u) = H^{(1)} + H^{(2)} + H^{(3)}$, where

$$\begin{aligned}
H^{(1)} &= \begin{pmatrix} (\alpha+2)(\alpha+1)(u_1/u_2)^{\alpha} & -\alpha(\alpha+2)(u_1/u_2)^{\alpha+1} \\ -\alpha(\alpha+2)(u_1/u_2)^{\alpha+1} & \alpha(\alpha+1)(u_1/u_2)^{\alpha+2} \end{pmatrix}, \\
H^{(2)} &= \begin{pmatrix} \alpha(\alpha+1)(u_2/u_1)^{\alpha+2} & -\alpha(\alpha+2)(u_2/u_1)^{\alpha+1} \\ -\alpha(\alpha+2)(u_2/u_1)^{\alpha+1} & (\alpha+2)(\alpha+1)(u_2/u_1)^{\alpha} \end{pmatrix}, \\
H^{(3)} &= \begin{pmatrix} u_1^{-2} & 0 \\ 0 & u_2^{-2} \end{pmatrix}.
\end{aligned} \tag{17}$$

Since $\det H^{(1)} = \alpha(\alpha+2)(u_1/u_2)^{2(\alpha+1)} > 0$, $\det H^{(2)} = \alpha(\alpha+2)(u_2/u_1)^{2(\alpha+1)} > 0$, and the diagonal elements of $H^{(1)}$, $H^{(2)}$ are positive, the matrices $H^{(i)}$, $i=1,2,3$, are positive definite and so does $h''(u)$. Thus, h is convex.

Step 2: h' is invertible. Since the Hessian h'' is nonsingular on $(0,\infty)^2$, h' is one-to-one and the image $R(h')$ is open. If $R(h')$ is also closed, it follows that $R(h') = \mathbb{R}^2$ which means that h' is surjective. For this, let $(w_n) \in R(h')$ for $n \in \mathbb{N}$ such that $w_n \to w$ as $n \to \infty$. We show that $w \in R(h')$. By definition, there exists $u_n > 0$ such that $w_n = h'(u_n)$ for $n \in \mathbb{N}$. The idea is to prove that $(u_n) = (u_{1,n}, u_{2,n})$ is a bounded and strictly positive sequence. This implies that, up to a subsequence, $u_n \to u \in (0,\infty)^2$ as $n \to \infty$. By continuity of h', we infer that $h'(u_n) \to h'(u)$ as $n \to \infty$. We already know that $h'(u_n) = w_n \to w$ which shows that $w = h'(u) \in R(h')$, and $R(h')$ is closed.

It remains to verify that there exist positive constants m, $M > 0$ such that $m \le u_{i,n} \le M$ for all $n \in \mathbb{N}$, $i = 1,2$. We argue by contradiction. Let us assume that (up to a subsequence) $u_{1,n} \to 0$ as $n \to \infty$. Since $(w_{1,n}) = (\partial_1 h(u_n))$ is convergent, we deduce from (15) that $u_{2,n} \to 0$ as well. As a consequence,

$$\alpha u_{1,n} w_{1,n} + (\alpha+2) u_{2,n} w_{2,n} \to 0, \quad (\alpha+2) u_{1,n} w_{1,n} + \alpha u_{2,n} w_{2,n} \to 0.$$

Expanding these expressions yields

$$u_{1,n}^{-\alpha} u_{2,n}^{\alpha+2} \to \frac{1}{2}, \quad u_{1,n}^{\alpha+2} u_{2,n}^{-\alpha} \to \frac{1}{2},$$

and the product also converges, $u_{1,n}^2 u_{2,n}^2 \to 1/4$. This is absurd since (u_n) converges to zero. Therefore, $u_{1,n}$ is strictly positive. With an analogous argument, we conclude that $u_{2,n}$ is strictly positive too.

Let us assume that (up to a subsequence) $u_{1,n} \to \infty$ as $n \to \infty$. Again, the convergence of $(w_{1,n})$ and (15) imply that $u_{2,n} \to \infty$. Consequently,

$$\frac{\alpha}{u_{2,n}} w_{1,n} + \frac{\alpha+2}{u_{1,n}} w_{2,n} \to 0, \quad \frac{\alpha+2}{u_{2,n}} w_{1,n} + \frac{\alpha}{u_{1,n}} w_{2,n} \to 0,$$

from which we infer after expanding these expressions that $u_{2,n}/u_{1,n} \to 0$ and $u_{1,n}/u_{2,n} \to 0$, which is a contradiction. So, $(u_{1,n})$ is bounded, and the same conclusion holds for $(u_{2,n})$.

Step 3: proof of (14). Observing that $x - \log x \ge 1$ for all $x > 0$, it follows that

$$h(u) \ge u_1^2 \bigg(\Big(\frac{u_1}{u_2}\Big)^{\alpha} + \Big(\frac{u_2}{u_1}\Big)^{\alpha+2} \bigg) = u_2^2 \bigg(\Big(\frac{u_1}{u_2}\Big)^{\alpha+2} + \Big(\frac{u_2}{u_1}\Big)^{\alpha} \bigg).$$

The elementary inequality $x^\alpha + (1/x)^{\alpha+2} \ge 2$ for $x > 0$ shows the first inequality in (14):

$$h(u) \ge \frac{u_1^2}{2} \bigg(\Big(\frac{u_1}{u_2}\Big)^{\alpha} + \Big(\frac{u_2}{u_1}\Big)^{\alpha+2} \bigg) + \frac{u_2^2}{2} \bigg(\Big(\frac{u_1}{u_2}\Big)^{\alpha+2} + \Big(\frac{u_2}{u_1}\Big)^{\alpha} \bigg) \ge u_1^2 + u_2^2.$$

For the second inequality in (14), we employ definition (6) of h and the elementary inequality $x - 1 \le 2(x - \log x)$ for $x > 0$ to find that, for some $C_h > 0$,

$$\sum_{i=1}^{2} \mu_i u_i \partial_i h(u) = \big(\mu_1(\alpha+2) - \mu_2\alpha\big) u_1^{\alpha+2} u_2^{-\alpha} + \big(\mu_2(\alpha+2) - \mu_1\alpha\big) u_1^{-\alpha} u_2^{\alpha+2}$$

$$+ \mu_1(u_1-1) + \mu_2(u_2-1)$$
$$\le C_h(u_1^{\alpha+2}u_2^{-\alpha} + u_1^{-\alpha}u_2^{\alpha+2}) + C_h(u_1 - \log u_1 + u_2 - \log u_2).$$

This finishes the proof. □

Next, we prove that $h''(u)A)(u)$ is positive semidefinite. Then $B = A(u)h''(u)^{-1}$ in (9) is positive semidefinite too, since $z^\top A(u)h''(u)^{-1}z = (h''(u)^{-1}z)^\top h''(u)A(u)(h''(u)^{-1}z) \ge 0$ for $z \in \mathbb{R}^2$.

Lemma 5 (Positive semidefiniteness of $h''A$)**.** *Let condition* (2) *hold. If* $\alpha(\alpha+2) > 1$*, the matrix* $h''(u)A(u)$ *is positive semidefinite in* $(0,\infty)^2$*. Furthermore, if additionally* $\alpha \ge p$*, there exists a constant* $\kappa = \kappa(\alpha) > 0$ *such that for all* $u = (u_1, u_2) \in (0,\infty)^2$ *and* $z \in \mathbb{R}^2$*,*

$$z^\top h''(u)A(u)z \ge \kappa\bigg(\Big(\frac{u_1}{u_2}\Big)^{\alpha-p} + \Big(\frac{u_1}{u_2}\Big)^{p-\alpha}\bigg)|z|^2.$$

Proof. Let $\alpha(\alpha+2) > 1$ and let $M^{(i)} = (M^{(i)}_{jk}) := \frac12((H^{(i)}A)^\top + H^{(i)}A)$ be the symmetric part of $H^{(i)}A$, where $H^{(i)}$ with $i = 1,2,3$ is defined in (17). A computation shows that

$$\begin{aligned}
M^{(1)}_{11} &= (\alpha+2)\big((\alpha+1)a(u_1/u_2) + (u_1/u_2)a'(u_1/u_2)\big)(u_1/u_2)^\alpha, \\
\det M^{(1)} &= \big(\alpha(\alpha+2)a(u_1/u_2)^2 - (u_1/u_2)^2a'(u_1/u_2)^2\big)(u_1/u_2)^{2\alpha+2}, \\
M^{(2)}_{11} &= \alpha\big((\alpha+1)a(u_1/u_2) - (u_1/u_2)a'(u_1/u_2)\big)(u_2/u_1)^{\alpha+2}, \\
\det M^{(2)} &= \big(\alpha(\alpha+2)a(u_1/u_2)^2 - (u_1/u_2)^2a'(u_1/u_2)^2\big)(u_2/u_1)^{2\alpha+2}, \\
M^{(3)} &= \begin{pmatrix} (a(u_1/u_2) + (u/u_2)a'(u_1/u_2))u_1^{-2} & 0 \\ 0 & (a(u_1/u_2) - (u_1/u_2)a'(u_1/u_2))u_2^{-2} \end{pmatrix}.
\end{aligned}$$

By the first condition in (2) and the positivity of α, we infer that $M^{(3)}$ is positive semidefinite and $M^{(1)}_{11}$, $M^{(2)}_{11}$ are positive for u, $v > 0$. Moreover, since $\alpha(\alpha+2) > 1$ by assumption, $\det(M^{(1)}) > 0$ and $\det(M^{(2)}) > 0$. Thus, $(h''A)(u)$ is positive semidefinite for all $u \in (0,\infty)^2$.

Now let additionally $\alpha \ge p$. Then the first condition in (2) shows that

$$\begin{aligned}
\frac{\det M^{(1)}}{\operatorname{tr} M^{(1)}} &= \frac{\alpha(\alpha+2)a(u_1/u_2)^2 - (u_1/u_2)^2a'(u_1/u_2)^2}{(\alpha(u_1/u_2)^2+\alpha+2)((\alpha+1)a(u_1/u_2) + (u_1/u_2)a'(u_1/u_2))}(u_1/u_2)^{\alpha+2} \\
&\ge \frac{(\alpha(\alpha+2)-1)a(u_1/u_2)^2}{(\alpha+2)((u_1/u_2)^2+1)(\alpha+2)a(u_1/u_2)}(u_1/u_2)^{\alpha+2} \\
&= k_1(\alpha)\frac{a(u_1/u_2)}{(u_1/u_2)^2+1}(u_1/u_2)^{\alpha+2},
\end{aligned}$$

where $k(\alpha) = (\alpha(\alpha+2)-1)/(\alpha+2)^2$. In a similar way, we find that

$$\frac{\det M^{(2)}}{\operatorname{tr} M^{(2)}} = \frac{\alpha(\alpha+2)a(u_1/u_2)^2 - (u_1/u_2)^2a'(u_1/u_2)^2}{((\alpha+2)(u_1/u_2)^2+\alpha)((\alpha+1)a(u_1/u_2) - (u_1/u_2)a'(u_1/u_2))}(u_2/u_1)^\alpha$$

$$\geq \frac{(\alpha(\alpha+2)-1)a(u_1/u_2)^2}{(\alpha+2)((u_1/u_2)^2+1)(\alpha+2)a(u_1/u_2)}(u_1/u_2)^{-\alpha}$$
$$= k(\alpha)\frac{a(u_1/u_2)}{(u_1/u_2)^2+1}(u_1/u_2)^{-\alpha}.$$

Since $\det M/\operatorname{tr} M$ is a lower bound for the eigenvalues of any symmetric positive definite matrix $M \in \mathbb{R}^{2\times 2}$ (and taking into account that $M^{(3)}$ is positive definite), we deduce that for $z \in \mathbb{R}^2$,

$$z^\top(h''A)(u)z \geq k(\alpha)a(u_1/u_2)\frac{(u_1/u_2)^{\alpha+2}+(u_1/u_2)^{-\alpha}}{(u_1/u_2)^2+1}|z|^2$$
$$\geq \frac{1}{2}k(\alpha)a(u_1/u_2)((u_1/u_2)^\alpha+(u_1/u_2)^{-\alpha})|z|^2.$$

In the last inequality, we have employed the elementary inequality $(x^{\alpha+2}+x^{-\alpha})/(x^2+1) \geq \frac{1}{2}(x^\alpha+x^{-\alpha})$ which is equivalent to $(x^2-1)(x^\alpha-x^{-\alpha}) \geq 0$, and this holds true for all $x > 0$. By the second condition in (2),

$$z^\top(h''A)(u)z \geq \frac{a_0}{2}k(\alpha)\frac{(u_1/u_2)^\alpha+(u_1/u_2)^{-\alpha}}{(u_1/u_2)^p+(u_1/u_2)^{-p}}|z|^2.$$

The inequality $(x^\alpha+x^{-\alpha})/(x^p+x^{-p}) \geq \frac{1}{2}(x^{\alpha-p}+x^{p-\alpha})$ is equivalent to $(x^{\alpha-p}-x^{p-\alpha})(x^p-x^{-p}) \geq 0$, which holds true for $x > 0$ since $\alpha - p \geq 0$ and $p \geq 0$. Therefore,

$$z^\top(h''A)(u)z \geq \frac{a_0}{4}k(\alpha)\big((u_1/u_2)^{\alpha-p}+(u_1/u_2)^{p-\alpha}\big)|z|^2,$$

which concludes the proof with $\kappa = a_0k(\alpha)/4$. □

The following two lemmas concern elementary estimates for $a(r)$.

Lemma 6. *The function a grows at most linearly, i.e., it holds that $a(r)/a(r_0) \leq r/r_0$ for all $r > r_0 > 0$.*

Proof. The first inequality in (2) is equivalent to $-1/\rho \leq a'(\rho)/a(\rho) \leq 1/\rho$ for $\rho > 0$. Integration over $\rho \in (r, r_0)$ yields $-\log(r/r_0) \leq \log(a(r)/a(r_0)) \leq \log(r/r_0)$, and applying the exponential function gives the result. □

Lemma 7. *Let $\alpha \geq 2$. There exists $C_a > 0$ such that for all u_1, $u_2 > 0$,*

$$a\left(\frac{u_1}{u_2}\right)^2(u_1^2+u_2^2) \leq C_a\left(u_1^2+u_2^2+\frac{u_1^4}{u_2^2}\right) \leq 3C_a\left(\left(\frac{u_1}{u_2}\right)^\alpha u_1^2+\left(\frac{u_1}{u_2}\right)^{-\alpha}u_2^2\right),$$

where $C_a = a(1)^2 + \max_{0\leq r\leq 1} a(r)^2$.

Proof. We show the first inequality. Let $u_1/u_2 > 1$. Then, applying Lemma 6 with $r_0 = 1$,

$$a\left(\frac{u_1}{u_2}\right)^2 u_1^2 \leq a(1)^2\frac{u_1^2}{u_2^2}u_1^2, \quad a\left(\frac{u_1}{u_2}\right)^2 u_2^2 \leq a(1)^2u_1^2.$$

If $u_1/u_2 \le 1$, we obtain

$$a\left(\frac{u_1}{u_2}\right)^2(u_1^2+u_2^2) \le \max_{0\le r\le 1} a(r)^2(u_1^2+u_2^2).$$

Adding these inequalities shows the claim with $C_a = a(1)^2 + \max_{0\le r\le 1} a(r)^2$.

Next, we prove the second inequality. For this, let again $u_1/u_2 \le 1$. Then

$$\left(\frac{u_1}{u_2}\right)^2 u_1^2 \le u_1^2 \le u_2^2 \le \left(\frac{u_1}{u_2}\right)^{-\alpha} u_2^2,$$

and thus, $u_1^2 + u_2^2 + u_1^4/u_2^2 \le 3(u_1/u_2)^{-\alpha}u_2^2$. Finally, if $u_1/u_2 > 1$, we take into account the condition $\alpha \ge 2$:

$$u_2^2 < u_1^2 \le \left(\frac{u_1}{u_2}\right)^\alpha u_1^2, \quad \left(\frac{u_1}{u_2}\right)^2 u_1^2 \le \left(\frac{u_1}{u_2}\right)^\alpha u_1^2$$

and consequently $u_1^2 + u_2^2 + u_1^4/u_2^2 \le 3(u_1/u_2)^\alpha u_1^2$. Putting together these inequalities, the lemma follows. □

Lemma 8. *Recall that $A(u) = (A_{ij}(u))$ is given by (8). Then there exists $C_A > 0$, only depending on $a(1)$ and $\max_{0\le r\le 1} a(r)$, such that for all u_1, $u_2 > 0$,*

$$|A(u)| \le C_A\left(1 + \left(\frac{u_1}{u_2}\right)^2 + \left(\frac{u_1}{u_2}\right)^{-2}\right).$$

Proof. Let $u_1/u_2 \ge 1$. We apply the first condition in (2) and Lemma 6 with $r_0 = 1$:

$$|A_{11}(u)| \le 2a(u_1/u_2) \le 2a(1)(u_1/u_2) \le a(1) + a(1)(u_1/u_2)^2.$$

In a similar way, $|A_{22}(u)| \le a(1) + a(1)(u_1/u_2)^2$. Furthermore, for $u_1/u_2 \ge 1$,

$$|A_{12}(u)| \le a(u_1/u_2)(u_1/u_2) \le a(1)(u_1/u_2)^2, \quad |A_{21}(u)| \le a(u_1/u_2)(u_2/u_1) \le a(1).$$

Hence, $|A(u)| \le C(1 + (u_1/u_2)^2)$ for some $C > 0$ depending on $a(1)$ only. If $u_1/u_2 < 1$, we employ again condition (2):

$$\begin{aligned}
&|A_{11}(u)| \le 2a(u_1/u_2) \le 2\max_{0\le r\le 1} a(r), \quad |A_{22}(u)| \le 2a(u_1/u_2) \le 2\max_{0\le r\le 1} a(r),\\
&|A_{12}(u)| \le a(u_1/u_2)(u_1/u_2) \le \max_{0\le r\le 1} a(r),\\
&|A_{21}(u)| \le a(u_1/u_2)(u_2/u_1) \le \max_{0\le r\le 1} a(r)(u_2/u_1) \le \tfrac{1}{2}\max_{0\le r\le 1} a(r)(1 + (u_2/u_1)^2).
\end{aligned}$$

We deduce that $|A(u)| \le C(1 + (u_2/u_1)^2)$ if $u_1/u_2 < 1$, and the proof follows. □

4. Proof of Theorem 1

Let $T > 0$, $N \in \mathbb{N}$, $\tau = T/N$, and $m \in \mathbb{N}$ with $m > d/2$. Then the embedding $H^m(\mathbb{T}^d) \hookrightarrow L^\infty(\mathbb{T}^d)$ is compact. Furthermore, let $w^{k-1} = (w_1^{k-1}, w_2^{k-1}) \in L^\infty(\mathbb{T}^d)^2$ be given and let $u^{k-1} = (h')^{-1}(w^{k-1})$. (If $k = 1$, we define $w^0 = h'(u^0)$.) By Lemma 4, the pair $u^{k-1} = (u_1^{k-1}, u_2^{k-1})$ is well defined and we have $u^{k-1} \in L^\infty(\mathbb{T}^d)^2$. We wish to find $w_k = (w_1^k, w_2^k) \in H^m(\mathbb{T}^d)^2$ such that for all $\phi = (\phi_1, \phi_2) \in H^m(\mathbb{T}^d)^2$,

$$\begin{aligned} &\frac{1}{\tau}\int_{\mathbb{T}^d}(u^k - u^{k-1})\cdot\phi dx + \int_{\mathbb{T}^d}\nabla\phi : B(w^k)\nabla w^k dx \\ &\qquad + \tau\int_{\mathbb{T}^d}(D^m w^k\cdot D^m\phi + w^k\cdot\phi)dx = \sum_{i=1}^2 \mu_i\int_{\mathbb{T}^d}u_i^k\phi_i dx, \end{aligned} \tag{18}$$

where $B(w^k) = A(u^k)h''(u^k)^{-1}$,

$$D^m w^k \cdot D^m\phi := \sum_{|\alpha|=m}\sum_{i=1}^2 D^\alpha u_i^k D^\alpha \phi_i,$$

$\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}_0^d$ is a multiindex and $D^\alpha = \partial^{|\alpha|}/(\partial x_1^{\alpha_1}\cdots\partial x_d^{\alpha_d})$ a partial derivative of order $|\alpha|$.

Step 1: solution of (18). Let $\widehat{w} = (\widehat{w}_1, \widehat{w}_2) \in L^\infty(\mathbb{T}^d)^2$ and $\eta \in [0,1]$ be given. Set $\widehat{u} = (\widehat{u}_1, \widehat{u}_2) := (h')^{-1}(\widehat{w})$. We solve first the linear problem

$$a(w,\phi) = \eta F(\phi) \quad \text{for all } \phi \in H^m(\mathbb{T}^d)^2, \tag{19}$$

where

$$\begin{aligned} a(w,\phi) &= \int_{\mathbb{T}^d}(D^m w^k\cdot D^m\phi + w^k\cdot\phi)dx + \int_{\mathbb{T}^d}\nabla\phi : B(\widehat{w})\nabla w^k dx, \\ F(\phi,\psi) &= -\frac{1}{\tau}\int_{\mathbb{T}^d}(\widehat{u} - u^{k-1})\cdot\phi dx + \sum_{i=1}^2\mu_i\int_{\mathbb{T}^d}\widehat{u}_i^k\phi_i dx. \end{aligned}$$

Since $\widehat{w} \in L^\infty(\mathbb{T}^d)^2$ and h' is continuous in $(0,\infty)^2$, we have $\widehat{u} \in L^\infty(\mathbb{T}^d)^2$. This shows that F is continuous on $H^m(\mathbb{T}^d)$. The bilinear form a is continuous and coercive, by the generalized Poincaré inequality for H^m spaces [11, Chap. 2.1.4, Formula (1.39)] and the positive semidefiniteness of $B(\widehat{w})$ (see Lemma 5). Hence, the Lax-Milgram lemma provides a unique solution $w = (w_1, w_2) \in H^m(\mathbb{T}^d)^2 \hookrightarrow L^\infty(\mathbb{T}^d)^2$ to (19). This defines the fixed-point operator $S : L^\infty(\mathbb{T}^d)^2 \times [0,1] \to L^\infty(\mathbb{T}^d)^2$, $S(\widehat{w},\eta) = w$, where w solves (19).

It holds clearly $S(w,0) = 0$. Standard arguments show that S is continuous (see, e.g., the proof of Lemma 5 in [8]). Because of the compact embedding $H^m(\mathbb{T}^d) \hookrightarrow L^\infty(\mathbb{T}^d)$, the mapping S is even compact. In order to apply the Leray-Schauder fixed-point theorem, it remains to prove a uniform bound for all fixed points of $S(\cdot,\eta)$ in $L^\infty(\mathbb{T}^d)^2$.

Let $w \in L^\infty(\mathbb{T}^d)^2$ be such a fixed point, i.e. a solution to (19) with $\widehat{u}$ replaced by $u := (h')^{-1}(w)$. The uniform bound will be a consequence of the entropy inequality. For this, we employ the test function w in (19):

$$\begin{aligned} &\frac{\eta}{\tau}\int_{\mathbb{T}^d}(u - u^{k-1})\cdot w dx + \int_{\mathbb{T}^d}\nabla w : B(w)\nabla w dx + \tau\int_{\mathbb{T}^d}\big(|D^m w|^2 + |w|^2\big)dx \\ &\qquad = \eta\sum_{i=1}^2\mu_i\int_{\mathbb{T}^d}u_i w_i dx. \end{aligned} \tag{20}$$

By the convexity of h, it follows that

$$h(u) - h(u^{k-1}) \le h'(u) \cdot (u - u^{k-1}) = (u - u^{k-1}) \cdot w.$$

Moreover, by (9) and Lemma 5, we have

$$\int_{\mathbb{T}^d} \nabla w : B(w)\nabla w dx = \int_{\mathbb{T}^d} \nabla u : (h''A)(u)\nabla u dx \ge \kappa \int_{\mathbb{T}^d} \bigg(\Big(\frac{u_1}{u_2}\Big)^{\alpha-p} + \Big(\frac{u_1}{u_2}\Big)^{p-\alpha}\bigg)|\nabla u|^2 dx.$$

Taking into account the second estimate in (14), we infer that

$$\eta \sum_{i=1}^2 \mu_i \int_{\mathbb{T}^d} u_i w_i dx = \eta \sum_{i=1}^2 \mu_i \int_{\mathbb{T}^d} u_i \partial_i h(u) dx \le C_h \int_{\mathbb{T}^d} h(u) dx.$$

Therefore, (20) becomes

$$\begin{aligned} &\frac{\eta}{\tau}\int_{\mathbb{T}^d} h(u)dx + \kappa \int_{\mathbb{T}^d} \bigg(\Big(\frac{u_1}{u_2}\Big)^{\alpha-p} + \Big(\frac{u_1}{u_2}\Big)^{p-\alpha}\bigg)|\nabla u|^2 dx \\ &\quad + \tau \int_{\mathbb{T}^d} \big(|D^m w|^2 + |w|^2\big)dx \le \frac{\eta}{\tau}\int_{\mathbb{T}^d} h(u^{k-1})dx + C_h \int_{\mathbb{T}^d} h(u)dx. \end{aligned} \tag{21}$$

Choosing $\tau < 1/C_h$, this shows that w is uniformly bounded in $H^m(\mathbb{T}^d)$. Thus, we can apply the fixed-point theorem of Leray-Schauder to conclude the existence of a weak solution $w^k := w$ with $u^k = h'(w^k)$ to (18) with $\eta = 1$.

Step 2: a priori estimates. Inequality (21) shows, for $w = w^k$, $u = u^k$, and $\eta = 1$, that

$$\begin{aligned} (1 - C_h\tau)\int_{\mathbb{T}^d} h(u^k)dx + \kappa\tau \int_{\mathbb{T}^d} \bigg(\Big(\frac{u_1^k}{u_2^k}\Big)^{\alpha-p} + \Big(\frac{u_1^k}{u_2^k}\Big)^{p-\alpha}\bigg)|\nabla u^k|^2 dx \\ + \tau^2 \int_{\mathbb{T}^d} \big(|D^m w^k|^2 + |w^k|^2\big)dx \le \int_{\mathbb{T}^d} h(u^{k-1})dx. \end{aligned}$$

We sum (21) for $k = 1, \ldots, j$ and divide the resulting inequality by $1 - C_h\tau$ (recall that we have chosen $\tau < 1/C_h$):

$$\begin{aligned} \int_{\mathbb{T}^d} h(u^j)dx &+ \frac{\kappa\tau}{1 - C_h\tau}\sum_{k=1}^j \int_{\mathbb{T}^d} \bigg(\Big(\frac{u_1^k}{u_2^k}\Big)^{\alpha-p} + \Big(\frac{u_1^k}{u_2^k}\Big)^{p-\alpha}\bigg)|\nabla u^k|^2 dx \\ &+ \frac{\tau^2}{1 - C_h\tau}\sum_{k=1}^j \int_{\mathbb{T}^d} \big(|D^m w^k|^2 + |w^k|^2\big)dx \\ &\le \frac{1}{1 - C_h\tau}\int_{\mathbb{T}^d} h(u^0)dx + \frac{C_h\tau}{1 - C_h\tau}\sum_{k=1}^{j-1}\int_{\mathbb{T}^d} h(u^k)dx. \end{aligned}$$

We apply the discrete Gronwall inequality [3] to obtain for $j\tau \le T$,

$$\int_{\mathbb{T}^d} h(u^j)dx + \tau\sum_{k=1}^j \int_{\mathbb{T}^d}\bigg(\Big(\frac{u_1^k}{u_2^k}\Big)^{\alpha-p} + \Big(\frac{u_1^k}{u_2^k}\Big)^{p-\alpha}\bigg)|\nabla u^k|^2 dx$$
$$+ \tau^2 \sum_{k=1}^j \int_{\mathbb{T}^d}\big(|D^m w^k|^2 + |w^k|^2\big)dx \le C, \tag{22}$$

where $C>0$ denotes a generic constant which is independent of τ (and independent of T if $\mu_i \le 0$).

We define the piecewise constant functions in time $w^{(\tau)}(x,t) = w^k(x)$ and $u^{(\tau)}(x,t) = u^k(x)$ for $x\in\mathbb{T}^d$ and $t\in((k-1)\tau, k\tau]$, $k=1,\ldots,j$. Furthermore, we introduce the shift operator $\sigma_\tau u^{(\tau)}(x,t) = u^{k-1}(x)$ for $x\in\mathbb{T}^d$, $t\in((k-1)\tau,k\tau]$. With this notation, we can rewrite (20) (with $\eta=1$) as

$$\frac{1}{\tau}\int_0^T\int_{\mathbb{T}^d}(u^{(\tau)} - \sigma_\tau u^{(\tau)})\cdot\phi dxdt + \int_0^T\int_{\mathbb{T}^d}\nabla\phi : (h''A)(u^{(\tau)})\nabla u^{(\tau)}dxdt$$
$$+ \tau\int_0^T\int_{\mathbb{T}^d}\big(D^m w^{(\tau)}\cdot D^m\phi + w^{(\tau)}\cdot\phi^{(\tau)}\big)dxdt + \sum_{i=1}^2 \mu_i\int_0^T\int_{\mathbb{T}^d} u_i^{(\tau)}\phi_i dxdt \tag{23}$$

and (22) as

$$\int_{\mathbb{T}^d} h(u^{(\tau)}(t))dx + \int_0^t\int_{\mathbb{T}^d}\bigg(\Big(\frac{u_1^{(\tau)}}{u_2^{(\tau)}}\Big)^{\alpha-p} + \Big(\frac{u_1^{(\tau)}}{u_2^{(\tau)}}\Big)^{p-\alpha}\bigg)|\nabla u^{(\tau)}|^2 dxds$$
$$+ \tau\int_0^t\int_{\mathbb{T}^d}\big(|D^m w^{(\tau)}|^2 + |w^{(\tau)}|^2\big)dxds \le C, \tag{24}$$

where $t\in((j-1)\tau, j\tau]$. It follows that

$$\|w^{(\tau)}\|_{L^2(0,T;H^m(\mathbb{T}^d))} \le C\tau^{-1/2}. \tag{25}$$

By Lemma 7, Lemma 4, and estimate (24), we find that

$$\int_{\mathbb{T}^d}\bigg(\bigg|a\Big(\frac{u_1^{(\tau)}}{u_2^{(\tau)}}\Big)u_1^{(\tau)}\bigg|^2 + \bigg|a\Big(\frac{u_1^{(\tau)}}{u_2^{(\tau)}}\Big)u_2^{(\tau)}\bigg|^2\bigg)dx \le 3C_a\int_{\mathbb{T}^d} h(u^{(\tau)})dx \le C, \tag{26}$$

$$\int_{\mathbb{T}^d}\big((u_1^{(\tau)})^2 + (u_2^{(\tau)})^2\big)dx \le \int_{\mathbb{T}^d} h(u^{(\tau)})dx \le C. \tag{27}$$

Moreover, using Lemma 8 and (24),

$$\int_0^T\int_{\mathbb{T}^d}\Big(\big|\nabla\big(a(u_1^{(\tau)}/u_2^{(\tau)})u_1^{(\tau)}\big)\big|^2 + \big|\nabla\big(a(u_1^{(\tau)}/u_2^{(\tau)})u_2^{(\tau)}\big)\big|^2\Big)dxdt$$
$$= \int_{\mathbb{T}^d}|A(u^{(\tau)})\nabla u^{(\tau)}|^2 dx \le \int_0^T\int_{\mathbb{T}^d}|A(u^{(\tau)})|^2|\nabla u^{(\tau)}|^2 dxdt$$

$$\le C_A \int_0^T \int_{\mathbb{T}^d} \left(1 + \left(\frac{u_1^{(\tau)}}{u_2^{(\tau)}}\right)^4 + \left(\frac{u_2^{(\tau)}}{u_1^{(\tau)}}\right)^4\right) |\nabla u^{(\tau)}|^2 dxdt$$

$$\le C \int_0^T \int_{\mathbb{T}^d} \left(\left(\frac{u_1^{(\tau)}}{u_2^{(\tau)}}\right)^{\alpha-4} + \left(\frac{u_1^{(\tau)}}{u_2^{(\tau)}}\right)^{p-\alpha}\right) |\nabla u^{(\tau)}|^2 dxdt \le C. \tag{28}$$

The last but one inequality follows from the elementary estimate $1 + y^4 \le y^{\alpha-p} + y^{p-\alpha}$ for $y > 0$ which holds because of the assumption $\alpha - p \ge 4$. Estimates (26)-(28) yield for $i = 1, 2$,

$$\|u_i^{(\tau)}\|_{L^\infty(0,T;L^2(\mathbb{T}^d))} \le C, \qquad \|\nabla u_i^{(\tau)}\|_{L^2(0,T;L^2(\mathbb{T}^d))} \le C, \tag{29}$$

$$\|a(u_1^{(\tau)}/u_2^{(\tau)})u_i\|_{L^\infty(0,T;L^2(\mathbb{T}^d))} \le C, \qquad \|\nabla(a(u_1^{(\tau)}/u_2^{(\tau)})u_i^{(\tau)})\|_{L^2(0,T;L^2(\mathbb{T}^d))} \le C. \tag{30}$$

These estimates are uniform in $T > 0$ if $\mu_i \le 0$.

Next, we derive a uniform estimate for the discrete time derivative $(u^{(\tau)} - \sigma_\tau u^{(\tau)})/\tau$. For $\phi \in L^2(0,T; H^m(\mathbb{T}^d))$, we estimate

$$\begin{aligned} \frac{1}{\tau}\left| \int_0^T (u^{(\tau)} - \sigma_\tau u^{(\tau)}) \cdot \phi dxdt \right| &\le \|A(u^{(\tau)})\nabla u^{(\tau)}\|_{L^2(0,T;L^2(\mathbb{T}^d))} \|\nabla \phi\|_{L^2(0,T;L^2(\mathbb{T}^d))} \\ &\quad + \tau \|w^{(\tau)}\|_{L^2(0,T;H^m(\mathbb{T}^d))} \|\phi\|_{L^2(0,T;H^m(\mathbb{T}^d))} \\ &\quad + \max\{\mu_1, \mu_2\} \|u^{(\tau)}\|_{L^2(0,T;L^2(\mathbb{T}^d))} \|\phi\|_{L^2(0,T;L^2(\mathbb{T}^d))} \\ &\le C\|\phi\|_{L^2(0,T;H^m(\mathbb{T}^d))}, \end{aligned}$$

taking into account the bounds (25), (28), and (29). Therefore,

$$\tau^{-1}\|u^{(\tau)} - \sigma_\tau u^{(\tau)}\|_{L^2(0,T;H^m(\mathbb{T}^d)')} \le C. \tag{31}$$

Step 3: limit $\tau \to 0$. Estimates (29) and (31) allow us to apply the Aubin-Lions lemma in the discrete version of [6] to obtain the existence of a subsequence, which is not relabeled, such that, as $\tau \to 0$,

$$u_i^{(\tau)} \to u_i \quad \text{strongly in } L^2(0,T; L^2(\mathbb{T}^d)) \text{ and a.e., } i = 1, 2.$$

Moreover, by (25), (29), and (31), for the same subsequence and $i = 1, 2$,

$$\begin{aligned} w_i^{(\tau)} &\to w_i \quad \text{strongly in } L^2(0,T; H^m(\mathbb{T}^d)), \\ \nabla u_i^{(\tau)} &\rightharpoonup \nabla u_i \quad \text{weakly in } L^2(0,T; L^2(\mathbb{T}^d)), \\ \tau^{-1}(u_i^{(\tau)} - \sigma_\tau u_i^{(\tau)}) &\rightharpoonup \partial_t u_i \quad \text{weakly in } L^2(0,T; H^m(\mathbb{T}^d)'). \end{aligned}$$

The pointwise convergence of $(u_i^{(\tau)})$, Fatou's lemma, and estimate (24) imply that, for a.e. $t \in (0,T)$,

$$\begin{aligned} \sum_{i=1}^2 \int_{\mathbb{T}^d} (u_i(t) - \log u_i(t)) dx &\le \liminf_{\tau \to 0} \sum_{i=1}^2 \int_{\mathbb{T}^d} \big(u_i^{(\tau)}(t) - \log u_i^{(\tau)}(t)\big) dx \\ &\le \liminf_{\tau \to 0} \int_{\mathbb{T}^d} h(u^{(\tau)}(t)) dx \le C. \end{aligned}$$

This means that $u_i > 0$ a.e. in $\mathbb{T}^d \times (0,T)$.

Estimate (26) and (28) show that, up to a subsequence,

$$a(u_1^{(\tau)}/u_2^{(\tau)})u_i^{(\tau)} \rightharpoonup q_i \quad \text{weakly in } L^2(0,T;H^1(\mathbb{T}^d)),\ i=1,2,$$

where $q_i \in L^2(0,T;H^1(\mathbb{T}^d))$. We wish to identify q_i. To this end, let us define $\chi_\varepsilon^{(\tau)} = \mathbf{1}_{\{u_1^{(\tau)}\geq\varepsilon,\, u_2^{(\tau)}\geq\varepsilon\}}$ and $\chi_\varepsilon = \mathbf{1}_{\{u_1\geq\varepsilon,\, u_2\geq\varepsilon\}}$, where $\mathbf{1}_A$ denotes the characteristic function on the set A. Clearly, $\chi_\varepsilon^{(\tau)} \to \chi_\varepsilon$ strongly in $L^s(0,T;L^s(\mathbb{T}^d))$ for all $1 \leq s < \infty$. We infer that

$$\chi_\varepsilon^{(\tau)} a(u_1^{(\tau)}/u_2^{(\tau)})u_i^{(\tau)} \rightharpoonup \chi_\varepsilon a(u_1/u_2)u_i \quad \text{weakly in } L^s(0,T;L^s(\mathbb{T}^d)),\ 1 \leq s < 2.$$

We deduce that $q_i = a(u_1/u_2)u_i$ on the set $\{u_1 \geq \varepsilon,\, u_2 \geq \varepsilon\}$. Since $\varepsilon > 0$ is arbitrary and $u_i > 0$ a.e. in $\mathbb{T}^d \times (0,T)$, this identification holds, in fact, in $\mathbb{T}^d \times (0,T)$.

Consequently, we may perform the limit $\tau \to 0$ in (23) to deduce that u is a weak solution to (1) in $L^2(0,T;H^m(\mathbb{T}^d)')$. However, since $a(u_1/u_2)u_i \in L^2(0,T;H^1(\mathbb{T}^d))$, we employ a density argument to infer that (1) also holds for $L^2(0,T;H^1(\mathbb{T}^d)')$. Since $u_i \in L^2(0,T;H^1(\mathbb{T}^d))$ and $\partial_t u_i \in L^2(0,T;H^1(\mathbb{T}^d)')$, it follows that $u_i \in C^0([0,T];L^2(\mathbb{T}^d))$, so the initial datum is satisfied in $L^2(\mathbb{T}^d)$. Finally, since the bounds are uniform in T if $\mu_i \leq 0$, the statement (5) follows.

5. Proof of Theorem 2

First, we prove the following result.

Lemma 9. *Let u be the weak solution constructed in Theorem 1 and let $\beta \in [0,(\alpha-p-4)/2]$. Then*

$$u_1\left(\frac{u_1}{u_2}\right)^\beta,\ u_2\left(\frac{u_2}{u_1}\right)^\beta,\ u_1\left(\frac{u_1}{u_2}\right)^{\beta+1},\ u_2\left(\frac{u_1}{u_2}\right)^{\beta+1} \in L^2(0,T;H^1(\mathbb{T}^d)) \cap L^\infty(0,T;L^2(\mathbb{T}^d)).$$

These functions are also elements of $L^\infty(0,\infty;L^2(\mathbb{T}^d))$ if $\mu_i \leq 0$, $i=1,2$.

Proof. Let $u^{(\tau)} = (u_1^{(\tau)}, u_2^{(\tau)})$ be a solution to (23). Since

$$1 \leq \left(\frac{u_1^{(\tau)}}{u_2^{(\tau)}}\right)^2 + \left(\frac{u_1^{(\tau)}}{u_2^{(\tau)}}\right)^{-4\beta-2},$$

it follows by equivalence transformations that

$$\left|u_1^{(\tau)}\left(\frac{u_1^{(\tau)}}{u_2^{(\tau)}}\right)^\beta\right|^2 \leq (u_1^{(\tau)})^2\left(\left(\frac{u_1^{(\tau)}}{u_2^{(\tau)}}\right)^{2\beta+2} + \left(\frac{u_1^{(\tau)}}{u_2^{(\tau)}}\right)^{-2\beta-2}\right).$$

Moreover, by distinguishing the cases $u_1^{(\tau)}/u_2^{(\tau)} \geq 1$ and $u_1^{(\tau)}/u_2^{(\tau)} < 1$, one can show that

$$(u_1^{(\tau)})^2\left(\left(\frac{u_1^{(\tau)}}{u_2^{(\tau)}}\right)^{2\beta+2} + \left(\frac{u_1^{(\tau)}}{u_2^{(\tau)}}\right)^{-2\beta-2}\right) \leq (u_1^{(\tau)})^2\left(\frac{u_1^{(\tau)}}{u_2^{(\tau)}}\right)^\alpha + (u_2^{(\tau)})^2\left(\frac{u_1^{(\tau)}}{u_2^{(\tau)}}\right)^{-\alpha}.$$

For this result, we need the assumption $\beta \le \alpha/2$. Consequently, using the entropy estimate (24),

$$\int_{\mathbb{T}^d} \left| u_1^{(\tau)} \left(\frac{u_1^{(\tau)}}{u_2^{(\tau)}} \right)^\beta \right|^2 dx \le \int_{\mathbb{T}^d} h(u^{(\tau)}) dx \le C,$$

where $C > 0$ does not depend on τ. Furthermore, employing the property $\alpha - p \ge 2\beta + 2$, we have

$$\begin{aligned} &\int_0^T \int_{\mathbb{T}^d} \left| \nabla \left(u_1^{(\tau)} \left(\frac{u_1^{(\tau)}}{u_2^{(\tau)}} \right)^\beta \right) \right|^2 dxdt \\ &\quad \le C \int_0^T \int_{\mathbb{T}^d} \left(1 + \left(\frac{u_1^{(\tau)}}{u_2^{(\tau)}} \right)^{2\beta+2} + \left(\frac{u_1^{(\tau)}}{u_2^{(\tau)}} \right)^{-2\beta-2} \right) \big(|\nabla u_1^{(\tau)}|^2 + |\nabla u_2^{(\tau)}|^2 \big) dxdt \\ &\quad \le C \int_0^T \int_{\mathbb{T}^d} \left(\left(\frac{u_1^{(\tau)}}{u_2^{(\tau)}} \right)^{\alpha-p} + \left(\frac{u_1^{(\tau)}}{u_2^{(\tau)}} \right)^{p-\alpha} \right) |\nabla u^{(\tau)}|^2 dxdt \le C. \end{aligned}$$

We conclude that

$$\left\| u_1^{(\tau)} \left(\frac{u_1^{(\tau)}}{u_2^{(\tau)}} \right)^\beta \right\|_{L^\infty(0,T;L^2(\mathbb{T}^d))} + \left\| u_1^{(\tau)} \left(\frac{u_1^{(\tau)}}{u_2^{(\tau)}} \right)^\beta \right\|_{L^2(0,T;H^1(\mathbb{T}^d))} \le C. \tag{32}$$

This implies the weak convergence of (a subsequence of) $((u_1^{(\tau)})^{\beta+1}(u_2^{(\tau)})^{-\beta})$, and arguing as in Step 3 of the proof of Theorem 1, we can identify the weak limit with $u_1^{\beta+1}u_2^{-\beta}$, where u_i is the L^2-limit of $(u_i^{(\tau)})$, $i = 1, 2$. Then (32) shows that $u_1^{\beta+1}u_2^{-\beta} \in L^2(0, T; H^1(\mathbb{T}^d)) \cap L^\infty(0, T; L^2(\mathbb{T}^d))$. If $\mu_i \le 0$, the constant C in (24) does not depend on T and hence, $u_1^{\beta+1}u_2^{-\beta} \in L^\infty(0, \infty; L^2(\mathbb{T}^d))$. In a similar way, the remaining assertions are proved. □

We proceed with the proof of Theorem 2. The proof of Lemma 4 shows that $h_\beta(u) := (u_1/u_2)^\beta u_1^2 + (u_1/u_2)^{-\beta} u_2^2$ for $u = (u_1, u_2)$ is strictly convex in $(0,\infty)^2$. Let $\beta = (\alpha-p-4)/2$. Then $\beta(\beta + 2) > 1$ is equivalent to $\alpha^2 - 2(p + 2)\alpha + (p + 2)^2 - 8 > 0$, and this holds true if $\alpha > p + 2 + \sqrt{8} = p + 4 + 2(\sqrt{2} - 1)$, which is our assumption. By Lemma 5, the matrix $(h_\beta'' A)(u)$ is positive semidefinite. Furthermore, by Lemma 9, $h_\beta'(u) \in L^2(0, T; H^1(\mathbb{T}^d))$ is an admissible test function for (1). We conclude that the relative entropy (also called the Bregman distance)

$$H_\beta[u(t)|\overline{u}] := \int_{\mathbb{T}^d} \big(h_\beta(u(t)) - h_\beta(\overline{u}) - h_\beta'(\overline{u}) \cdot (u(t) - \overline{u}) \big) dx$$

is a Lyapunov functional along solutions to (1). Moreover, by the entropy inequality (24) and assumption $\mu_i = 0$, $\nabla u_i \in L^2(0, \infty; L^2(\mathbb{T}^d))$. Then there exists a sequence (t_n) of positive numbers satisfying $t_n \to \infty$ as $n \to \infty$ such that $\|\nabla u_i(t_n)\|_{L^2(\mathbb{T}^d)} \to 0$ as $n \to \infty$, $i = 1, 2$. Consequently, $u_i(t_n) \to \overline{u}$ in $L^2(\mathbb{T}^d)$.

We wish to show that $H_\beta[u(t_n)|\overline{u}] \to 0$ as $n \to \infty$. Because of the L^2-convergence of $(u_i(t_n))$ to $\overline{u}_i$, it is sufficient to prove that $\int_{\mathbb{T}^d}(h_\beta(u(t_n)) - h_\beta(\overline{u}))dx \to 0$. Using $2h_\beta =$

$h'_\beta(u)\cdot u$, we find that

$$2\int_{\mathbb{T}^d}\big(h_\beta(u(t_n))-h_\beta(\overline{u})\big)dx = \int_{\mathbb{T}^d} h'_\beta(u(t_n))\cdot(u(t_n)-\overline{u})dx + \int_{\mathbb{T}^d}\big(h'_\beta(u(t_n))-h'_\beta(\overline{u})\big)\cdot\overline{u}dx. \tag{33}$$

As $\mu_i=0$, Lemma 9 shows that $h'_\beta(u)\in L^\infty(0,\infty;L^2(\mathbb{T}^d))$ and in particular, the sequences $(\partial_i h_\beta(u(t_n)))$ are bounded in $L^2(\mathbb{T}^d)$, $i=1,2$. Then the convergence $u(t_n)\to\overline{u}$ strongly in $L^2(\mathbb{T}^d)$ implies that the first integral on the right-hand side of (33) converges to zero as $n\to\infty$. We deduce from the bound $h'_\beta(u)\in L^\infty(0,\infty;L^2(\mathbb{T}^d))$ that the sequence $(\partial_i h_\beta(u(t_n)))$ is uniformly integrable in $\mathbb{T}^d$, $i=1,2$. Then, by Vitali's convergence theorem, $\partial_i h_\beta(u(t_n))\to\partial_i h_\beta(\overline{u})$ in $L^1(\mathbb{T}^d)$, $i=1,2$. This shows that the last integral in (33) converges to zero as $n\to\infty$, and this proves the claim.

Since the mapping $t\mapsto H_\beta[u(t)|\overline{u}]$ is monotone, $H_\beta[u(t)|\overline{u}]\to 0$ for all sequences $t\to\infty$. In view of the strict convexity of h_β, we conclude that $u_i(t)\to\overline{u}_i$ strongly in $L^2(\mathbb{T}^d)$ as $t\to\infty$, $i=1,2$. This completes the proof.

Remark 10. If $\mu_i<0$ for $i=1,2$, we can prove the exponential convergence of the solution $u(t)$ to (1) in $H^1(\mathbb{T}^d)'$ by using the dual method. Indeed, let $\phi_i\in L^2(0,T;H^1(\mathbb{T}^d))$ be the unique solution to $-\Delta\phi_i=u_i(t)$ in $\mathbb{T}^d$ and $\int_{\mathbb{T}^d}\phi_i dx=0$, $i=1,2$. Employing $\phi=(\phi_1,\phi_2)$ as a test function in (1), we find after a straightforward computation that

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{T}^d}\big(|\nabla\phi_1|^2+|\nabla\phi_2|^2\big)dx + \int_{\mathbb{T}^d} a(u_1/u_2)(u_1^2+u_2^2)dx = \int_{\mathbb{T}^d}\big(\mu_1|\nabla\phi_1|^2+\mu_2|\nabla\phi_2|^2\big)dx.$$

Then Gronwall's lemma implies that

$$\int_{\mathbb{T}^d}|\nabla\phi(t)|^2dx \le e^{\max\{\mu_1,\mu_2\}t}\int_{\mathbb{T}^d}|\nabla\phi(0)|^2dx, \quad t>0.$$

Since $\|u_i\|_{H^1(\mathbb{T}^d)'}=\|\phi_i\|_{H^1(\mathbb{T}^d)}$, we conclude that $\|u_i(t)\|_{H^1(\mathbb{T}^d)'}\le C\exp(-\kappa t)$ for $t>0$, where $\kappa=-\max\{\mu_1,\mu_2\}>0$ and $C>0$ depends on u^0. □

Remark 11. In the case $\mu_i>0$ for $i=1,2$, we cannot expect equilibration rates, since the solution grows in the L^2 norm as $t\to\infty$. This growth can be made precise if $\mu:=\mu_1=\mu_1>0$. Indeed, $u_i^*=e^{-\mu t}u_i$ solves

$$\partial_t u_i^* = \Delta\big(a(u_1^*/u_2^*)u_i^*\big), \quad t>0, \quad u_i^*(0)=u_i^0 \quad \text{in } \mathbb{T}^d,\ i=1,2,$$

and Theorem 2 shows that $u_i^*(t)\to\overline{u}_i$ in $L^2(\mathbb{T}^d)$ as $t\to\infty$, which translates to $\|e^{-\mu t}u_i(t)-\overline{u}_i\|_{L^2(\mathbb{T}^d)}\to 0$. □

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