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Post-Newtonian Corrections to the Newtonian Predictions for the Motion of Designated Targets with respect to Space Based APT Laser Systems

Jose M. Gambi, Maria L. García del Pino, Jürgen Gschwindl and Ewa B. Weinmüller

Abstract Numerical experiments are carried out to validate the short/long term differences between the solutions of the Newtonian equations for the relative motion of middle size targets in space with respect to space based APT systems and the respective solutions of a system of non-linear post-Newtonian equations. This system has been introduced in the ECMI 2016 contribution *Non-linear post-Newtonian equations for the motion of designated targets with respect to space based APT laser systems*. Two auxiliary systems of post-Newtonian equations are used to carry out the validation. The simulations are made under the following assumptions: (i) the structures of the Earth surrounding space are respectively the Euclidean and that of the post-Newtonian approximation to the exterior Schwarzschild field; (ii) the targets are on equatorial circular orbits, and (iii) the APT systems are ECI oriented inertial-guided systems placed onboard HEO, MEO and LEO satellites in equatorial orbits about the Earth. The APT systems have initially been placed at short and successively increasing distances from the targets.

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1 Introduction

Numerical solutions of the Newtonian equations for the Earth Centered Inertial (ECI) relative motions of middle size targets with respect to space based laser Acquisition, Pointing and Tracking (APT) systems are used to estimate the short/long term differences between the predictions provided by these equations and those provided by three systems of post-Newtonian equations.

The first post-Newtonian system describes the relative motions as differences of the ECI post-Newtonian equations for the targets and systems in the Earth Schwarzschild field, thus resembling the procedure followed to derive the Newtonian equations [6]; the second, by means of the linear approximation to Synge's equations for the relative motion in Fermi coordinates derived by Gambi et al. [1], and the third, by means of the equations introduced in the contribution *Non-linear post-Newtonian equations for the motion of designated targets with respect to space based APT laser systems* presented in this ECMI 2016 Minisymposium.

The estimations are made under the assumption that the structures of space-time about the Earth are respectively those of the Newtonian and of the post-Newtonian approximation to the exterior Schwarzschild field. The APT systems are assumed to be ECI oriented inertial-guided systems that, from the ranging and angle measurements made, finally assign Cartesian coordinates $(X_{(\alpha)}, t) = (x^{\alpha 2} - x^{\alpha 1}, t)$, where $(x^{\alpha 2}, t)$ and $(x^{\alpha 1}, t)$ ($\alpha = 1, 2, 3$) are ECI coordinates of the targets and the systems, respectively. To consider simulations resembling actual configurations, the APT systems are assumed to be onboard GEO satellites, and on LEO and MEO satellites in equatorial orbits about the Earth; the targets are assumed to be on equatorial circular orbits, and finally, the systems are assumed to be initially placed at short, and successively increasing distances from the targets.

The computations here carried out for each configuration and the magnitudes analyzed correspond to (i) the ECI Newtonian and post-Newtonian orbits of S and T (S stands for the APT systems, and T , for the targets); (ii) the Newtonian and post-Newtonian relative orbits of T with respect to S on the basis of the three different systems of equations mentioned above; (iii) the ranges from the ECI Newtonian to the three ECI post-Newtonian positions of S ; (iv) the ranges from the ECI Newtonian to the three ECI post-Newtonian positions of T ; (v) the differences of the Newtonian and post-Newtonian distances from the ECI center to S ; (vi) the differences of the Newtonian and post-Newtonian distances from the ECI center to T ; (vii) the distances from the Newtonian to the post-Newtonian linear and non-linear relative positions of T with respect to S , and finally, (viii) the distances from the ECI post-Newtonian position of T to the position of T obtained by adding the ECI post-Newtonian position of S to the relative post-Newtonian position of T with respect to S derived from the non-linear equations.

2 The Systems of Equations

The notations used in the papers cited above were unified in this work by making $(x^1, x^2) = (x, y)$, $(X_{(1)}, X_{(2)}) = (x^{1^2} - x^{1^1}, x^{2^2} - x^{2^1}) = (X, Y)$, $(x_1, y_1) = (x^{1^1}, x^{2^1})$ and $(x_2, y_2) = (x^{1^2}, x^{2^2})$ throughout. The ECI and relative Newtonian equations whose solutions are considered as reference of the subsequent results are the known classical equations (see e.g. [2])

$$\frac{d^2x}{dt^2} = \frac{-mx}{r^3}, \quad \frac{d^2y}{dt^2} = \frac{-my}{r^3}, \quad (1)$$

$$\frac{d^2X}{dt^2} = -m \left(\frac{x_2}{r_2^3} - \frac{x_1}{r_1^3} \right), \quad \frac{d^2Y}{dt^2} = -m \left(\frac{y_2}{r_2^3} - \frac{y_1}{r_1^3} \right), \quad (2)$$

where $r^2 = (x)^2 + (y)^2$, $r_1^2 = (x_1)^2 + (y_1)^2$ and $r_2^2 = (x_2)^2 + (y_2)^2$.

The first system of relative post-Newtonian equations are derived from the difference between the equations of the geodesics for S and T in the weak approximation to the Earth Schwarzschild field [7]. After a straightforward calculation, we have that the equations of the geodesics are [6]

$$\begin{aligned} \frac{d^2x}{dt^2} &= \frac{-m}{r^3} \left[1 - \frac{2m}{r^2} + \left(1 - \frac{3(x)^2}{r^2} \right) \left(\frac{dx}{dt} \right)^2 - \frac{6xy}{r^2} \frac{dx}{dt} \frac{dy}{dt} + \left(1 - \frac{3(y)^2}{r^2} \right) \left(\frac{dy}{dt} \right)^2 \right] x, \\ \frac{d^2y}{dt^2} &= \frac{-m}{r^3} \left[1 - \frac{2m}{r^2} + \left(1 - \frac{3(x)^2}{r^2} \right) \left(\frac{dx}{dt} \right)^2 - \frac{6xy}{r^2} \frac{dx}{dt} \frac{dy}{dt} + \left(1 - \frac{3(y)^2}{r^2} \right) \left(\frac{dy}{dt} \right)^2 \right] y, \end{aligned} \quad (3)$$

so that

$$\frac{d^2X}{dt^2} = \frac{d^2x_2}{dt^2} - \frac{d^2x_1}{dt^2}, \quad \frac{d^2Y}{dt^2} = \frac{d^2y_2}{dt^2} - \frac{d^2y_1}{dt^2}. \quad (4)$$

The linear relative equations are [1]

$$\begin{aligned} \frac{d^2X}{dt^2} &= -mX \int_0^1 \left(1 - \frac{3[x_1(1-u) + x_2u]^2}{r(u)^2} \right) \frac{1-2u+3u^2}{r(u)^3} du \\ &+ mY \int_0^1 \left(\frac{3x_1y_1(1-u)^2 + 3(1-u)u(x_1y_2 + y_1x_2) + 3x_2y_2u^2}{r(u)^2} \right) \frac{1-2u+3u^2}{r(u)^3} du, \\ \frac{d^2Y}{dt^2} &= -mY \int_0^1 \left(1 - \frac{3[y_1(1-u) + y_2u]^2}{r(u)^2} \right) \frac{1-2u+3u^2}{r(u)^3} du \\ &+ mX \int_0^1 \left(\frac{3x_1y_1(1-u)^2 + 3(1-u)u(x_1y_2 + y_1x_2) + 3x_2y_2u^2}{r(u)^2} \right) \frac{1-2u+3u^2}{r(u)^3} du, \end{aligned} \quad (5)$$

where $r(u)^2 = [x_1(1-u) + x_2u]^2 + [y_1(1-u) + y_2u]^2$.

Finally, the relative non-linear equations are

$$\begin{aligned}
\frac{d^2X}{dt^2} &= -mX \int_0^1 \left(1 - \frac{3[x_1(1-u) + x_2u]^2}{r(u)^2}\right) \frac{1-2u+3u^2}{r(u)^3} du \\
+mY \int_0^1 &\left(\frac{3x_1y_1(1-u)^2 + 3(1-u)u(x_1y_2 + y_1x_2) + 3x_2y_2u^2}{r(u)^2}\right) \frac{1-2u+3u^2}{r(u)^3} du \\
-9mX^2 \int_0^1 &\left(1 - \frac{5[x_1(1-u) + x_2u]^2}{3r(u)^2}\right) \frac{x_1(1-u)^2u + x_2(1-u)u^3}{r(u)^5} du \\
-6mXY \int_0^1 &\left(1 - \frac{5[x_1(1-u) + x_2u]^2}{r(u)^2}\right) \frac{y_1(1-u)^2u + y_2(1-u)u^3}{r(u)^5} du \\
-3mY^2 \int_0^1 &\left(1 - \frac{5[y_1(1-u) + y_2u]^2}{r(u)^2}\right) \frac{x_1(1-u)^2u + x_2(1-u)u^3}{r(u)^5} du, \\
\frac{d^2Y}{dt^2} &= -mY \int_0^1 \left(1 - \frac{3[y_1(1-u) + y_2u]^2}{r(u)^2}\right) \frac{1-2u+3u^2}{r(u)^3} du \\
+mX \int_0^1 &\left(\frac{3x_1y_1(1-u)^2 + 3(1-u)u(x_1y_2 + y_1x_2) + 3x_2y_2u^2}{r(u)^2}\right) \frac{1-2u+3u^2}{r(u)^3} du \\
-3mX^2 \int_0^1 &\left(1 - \frac{5[x_1(1-u) + x_2u]^2}{r(u)^2}\right) \frac{y_1(1-u)^2u + y_2(1-u)u^3}{r(u)^5} du \\
-6mXY \int_0^1 &\left(1 - \frac{5[y_1(1-u) + y_2u]^2}{r(u)^2}\right) \frac{x_1(1-u)^2u + x_2(1-u)u^3}{r(u)^5} du \\
-9mY^2 \int_0^1 &\left(1 - \frac{5[y_1(1-u) + y_2u]^2}{3r(u)^2}\right) \frac{y_1(1-u)^2u + y_2(1-u)u^3}{r(u)^5} du, \quad (6)
\end{aligned}$$

which are deduced by means of simple computations from the two first equations introduced in the contribution *Non-linear post-Newtonian equations for the motion of designated targets with respect to space based APT laser systems* mentioned above.

3 Numerical Simulations

In searching for patterns from the data produced by the numerical simulations made, each experiment was included into one of the three groups corresponding to the LEO-MEO-GEO classification. In turn, each group was split into two subgroups according to the ratio of the orbital radii of S and T , r_S and r_T ($r_S/r_T > 1$, $r_S/r_T < 1$).

Finally, the simulations in each of the resultant subgroups were ordered according to the initial distance from S to T . As a result, we found two family of patterns associated to the ratio r_S/r_T .

The following figures illustrate the results; they correspond to a rather standard S - T configuration for which $r_S/r_T > 1$ (see e.g. [3, 4, 5]): T is in LEO , 631 Km above the Earth, and S is initially placed 200 Km above T . (Note that these are approximated data. In fact, for the sake of accuracy, the equations were computed after having made $c = G = 1$, so that both data and results were given, resp. obtained, in seconds, and the accuracy achieved was 10^{-10} . We note finally that the results shown in these figures are in cm).

Fig. 1, left shows the ECI Newtonian orbits of T (red) and S (blue) derived from (1). The green line represents the Earth surface, and the black solid line, the distance from S to T after two hours orbiting. On the right, it is shown the corresponding relative orbit of T with respect to S , derived from (2). The results are in cm, as was said previously.

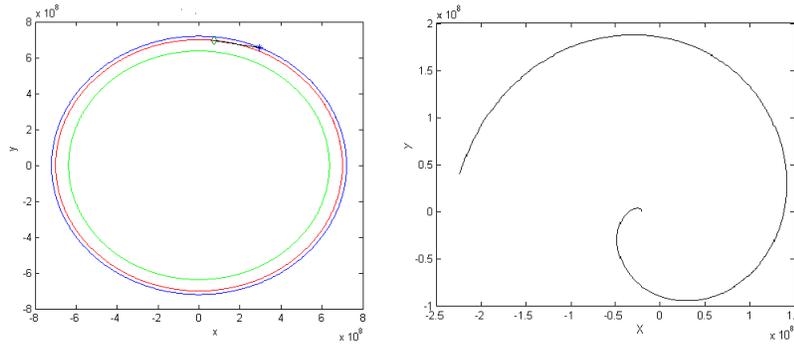


Fig. 1 ECI Newtonian orbits of S and T and relative Newtonian orbit of T with respect to S

Fig. 2 shows the distances, D , from the ECI Newtonian to the ECI post-Newtonian positions of T , and the differences, d , between the respective orbital radius of T . The orbital time is again two hours and the results appear in cm to facilitate the physical interpretation of the graphs.

The Newtonian and the three post-Newtonian relative orbits of T with respect to S are plotted in Fig. 3, left. The orbit showing a clear deviation (the green trajectory) corresponds to the solution of (5), which are the linear equations. Since all the experiments carried out lead to similar (time-increasing) deviations from the early stages of the integrations, system (5) is disregarded as candidate to account for the post-Newtonian corrections. Since the solutions of (2) and of (4), (6), i.e., of the other two post-Newtonian equations, are undistinguishable from each other in this graphic (they are represented by the red line) and in all similar graphics, deep analysis of these solutions were necessary.

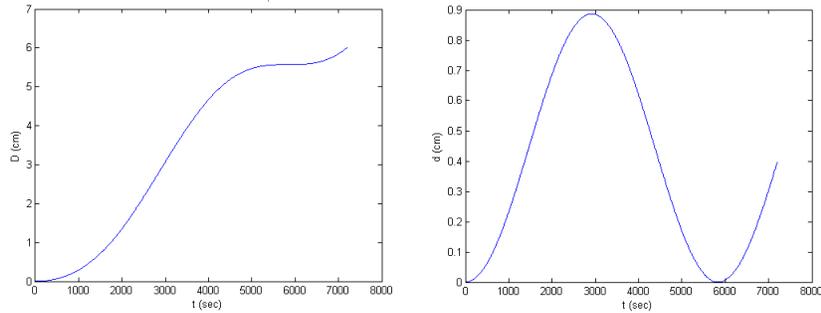


Fig. 2 Distances from the ECI Newtonian to the ECI post-Newtonian positions of T and differences between the orbital radius

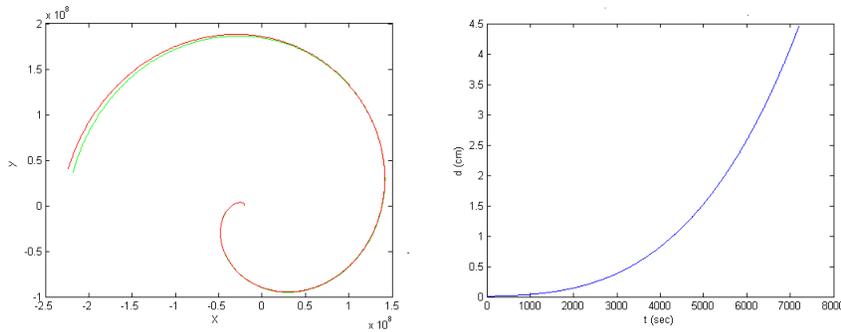


Fig. 3 Relative orbits of T w.r.t. S and distances from the p-N positions of T to the p-N positions of S plus the solutions of (6)

From the order of the differences of the respective solutions we found that the size of the differences between the solutions of (2) and (4) are not large enough for the curved structure of the space-time about the Earth, which is given by the Riemann tensor, manifests throughout them. In fact, as Fig. 3 on the right shows, the distances from the positions of T derived from (3) to the positions derived by adding the positions of S , also obtained with (3), to the solution of (6) are the only corrections that satisfy this requirement. Therefore, we can conclude that (4) must also be disregarded, since, unlike for the role played by equations (2) with respect to (1), equations (4) cannot play the same role with respect to equations (3); or shortly said, (4) cannot yield relative motions within the post-Newtonian framework.

Table 1 shows the size of some differences between the solutions of (2) and (6). They are depicted to illustrate the magnitude of the post-Newtonian corrections corresponding to this example. We note to this respect that orbiting along two hours (the time considered here) is equivalent to 1.23 revolutions of T about the Earth, approximately.

Table 1 Post-Newtonian corrections from equations (6) ^a

Time	Correction	Time	Correction
4555.102	1.03	6024.490	2.43
4848.979	1.26	6318.367	2.78
5142.857	1.52	6612.245	3.16
5436.735	1.80	6906.122	3.58
5730.612	2.10	7200	4.03

^a Time in sec, and corrections, in cm

4 Conclusions

From the simulations carried out in this work, we conclude that for effective tracking of middle size targets, the non-linear system (6) is the appropriate for an APT systems to correct the Newtonian predictions used at practically any instant and any S - T distance, particularly at the late stages of the tracking. The reason is that the corrections due to the non-linear terms included in these equations make the size of the total post-Newtonian corrections to fit very neatly with the size of the gravitational corrections corresponding to the post-Newtonian approximation. In fact, the underlying curved structure of space-time accounted for this approximation becomes apparent through this system of equations. Further, the corrections become measurable shortly after the initial integration instants. Therefore, they can be taken into consideration to increase the pointing accuracy of the APT systems, since the size of these corrections may be larger than the size of the targets in the scenarios considered.

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