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Rate optimal adaptive FEM with inexact solver for strongly monotone operators

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(joint work with Gregor Gantner, Alexander Haberl, and Bernhard Stiftner)

Meanwhile, the mathematical understanding of adaptive FEM has reached a mature state; see [9, 14, 3, 15, 6, 11] for some milestones for linear elliptic PDEs, [17, 8, 2, 13] for non-linear problems, and [4] for some general framework. Optimal adaptive FEM with inexact solvers has already been addressed in [15, 1, 4] for linear PDEs and in [5] for eigenvalue problems. However, for problems involving nonlinear operators, optimal adaptive FEM with inexact solvers has not been analyzed yet. Our work [12] aims to close the gap between convergence analysis (e.g. [4]) and empirical evidence (e.g. [10]) by analyzing an algorithm from [7].

Model problem. We follow [4] and present our results from [12] in an abstract framework, while precise examples for our setting are given, e.g., in [2, 13]. Let $H$ be a separable Hilbert space over $K \in \{\mathbb{R}, \mathbb{C}\}$ with norm $\| \cdot \|$ and scalar product $(\cdot, \cdot)_H$. With the duality pairing $\langle \cdot, \cdot \rangle$ between $H$ and its dual $H^*$, let $A : H \to H^*$ be a nonlinear operator which satisfies the following assumptions:

(O1) $A$ is strongly monotone: There exists $\alpha > 0$ such that
$$\alpha \| u - v \|^2 \leq \text{Re} \langle Au - Av, u - v \rangle$$
for all $u, v \in H$.

(O2) $A$ is Lipschitz continuous: There exists $L > 0$ such that
$$\| Au - Av \|_* := \sup_{w \in H \setminus \{0\}} \frac{\langle Au - Av, w \rangle}{\|w\|} \leq L \| u - v \|$$
for all $u, v \in H$.

(O3) $A$ has a potential: There exists a Gateaux differentiable function $P : H \to K$ with Gateaux derivative $dP = A$, i.e.,
$$\langle Au, v \rangle = \lim_{r \to 0} \frac{P(u + rv) - P(u)}{r}$$
for all $u, v \in H$.

Let $F \in H^*$. According to the main theorem on strongly monotone operators [18], (O1)–(O2) imply the existence and uniqueness of $u^* \in H$ such that

$$\langle Au^*, v \rangle = \langle F, v \rangle$$
for all $v \in H$.

To sketch the proof, let $I : H \to H^*$ denote the Riesz mapping defined by $\langle Iv, v \rangle = (u, v)_H$ for all $u, v \in H$. Then, $\Phi : H \to H$, $\Phi(u) := u - \frac{\alpha}{L^2} I^{-1}(Au - F)$ satisfies

$$\| \Phi(u) - \Phi(v) \| \leq q \| u - v \|$$
for all $u, v \in H$, where $q := (1 - \alpha^2/L^2)^{1/2} < 1$.

Hence, the Banach fixpoint theorem proves the existence and uniqueness of $u^* \in H$ with $\Phi(u^*) = u^*$ which is equivalent to (1). In particular, the Picard iteration $u^n := \Phi(u^{n-1})$ with arbitrary initial guess $u^0 \in H$ converges to $u^*$, and it holds

$$\| u^* - u^n \| \leq \frac{q}{1 - q} \| u^1 - u^n \|$$
for all $n \geq 1$.

With (O3), (1) (resp. (4) below) is equivalent to energy minimization, and $\| v - u^* \|^2$ is equivalent to the energy difference. This guarantees the quasi-orthogonality [4].
Applying (3) on the discrete level, we infer from [7] that there is a unique \( u^*_n \in X_n \) such that
\[
\langle Au^*_n, v_n \rangle = \langle F, v_n \rangle \quad \text{for all } v_n \in X_n.
\]
According to (O1)–(O2), it holds the Céa-type quasi-optimality \( \|u^* - u^*_n\| \leq \frac{L}{\alpha} \|u^* - v_n\| \) for all \( v_n \in X_n \). To solve the nonlinear system (4), we use the Picard iteration (applied in \( X_n \)): Given \( u^{n-1}_n \in X_n \), we compute \( u^n_n = pH(u^{n-1}_n) \) as follows:
- Solve the linear system \( (Au_n, v_n) = \langle Au^{n-1}_n - F, v_n \rangle \) for all \( v_n \in X_n \).
- Define \( u^T_n := u^{n-1}_n - \frac{L}{\alpha} v_n \).

Applying (3) on the discrete level, we infer from [7] that
\[
\|u^* - u^*_n\| \leq \|u^* - u^*_T\| + \frac{q}{1-q} \|u^*_n - u^*_{n-1}\| \leq \frac{L}{\alpha} \min_{v_n \in X_n} \|u^* - v_n\| + \frac{q^n}{1-q} \|u^*_1 - u^*_0\|.
\]

A posteriori error estimator. We suppose that all considered discrete spaces \( X_n \in \mathcal{H} \) are associated with a conforming triangulation \( T_n \) of a bounded Lipschitz domain \( \Omega \subset \mathbb{R}^d, d \geq 2 \). For all \( T \in T_n \) and all \( v_n \in X_n \), we suppose an a posteriori computable refinement indicator \( \eta_n(T, v_n) \geq 0 \). We then define
\[
\eta_n(v_n) := \eta_n(T_n, v_n) \quad \text{and} \quad \eta_n(U_n, v_n)^2 := \sum_{T \in U_n} \eta_n(T, v_n)^2 \quad \text{for all } U_n \subseteq T_n.
\]

We suppose that there exist constants \( C_{ax} > 0 \) and \( 0 < C_{ax} < 1 \) such that for all \( T_n \) and all refinements \( T_n \) of \( T_n \), the following properties (A1)–(A3) from [4] hold:
(A1) Stability on non-refined element domains:
\[
\eta_n(T_n \cap T, v_0) - \eta_n(T_n \cap T, v_n) \leq C_{ax} \|v_0 - v_n\| \quad \text{for all } v_n \in X_n, v_0 \in X_0.
\]
(A2) Reduction on refined element domains:
\[
\eta_n(T_n \setminus T, v_0) \leq C_{ax} \eta_n(T_n \setminus T, v_n) + C_{ax} \|v_0 - v_n\| \quad \text{for all } v_n \in X_n, v_0 \in X_0.
\]
(A3) Discrete reliability:
\[
\|u^*_n - u^*_n\| \leq C_{ax} \eta_n(T_n \setminus T, u^*_n).
\]

Note that (A1)–(A2) are required for all discrete functions (and follow from inverse estimates), while (A3) is only required for the discrete solutions \( u^*_n \) resp. \( u^*_T \) of (4).

Adaptive algorithm. With adaptivity parameters \( 0 < \theta \leq 1, \lambda > 0 \), and \( C_{mark} \geq 1 \), an initial conforming triangulation \( T_0 \), an initial guess \( u^0 \in X_0 \), our adaptive algorithm iterates the following steps (i)–(iii) for all \( \ell = 0, 1, 2, \ldots \):
(i) Repeat (a)–(b) for all \( n = 1, 2, 3, \ldots \), until \( \|u^T_n - u^T_{n-1}\| \leq \lambda \eta_n(u^T_n) \).
(a) Compute discrete Picard iterate \( u^n_T \in X_T \).
(b) Compute refinement indicators \( \eta(T, u^n_T) \) for all \( T \in T_T \).
(ii) Define \( u_T := u^n_T \) and determine a set \( M_T \subseteq T_T \) of minimal cardinality, up to the multiplicative factor \( C_{mark} \), such that \( \theta \eta_T(u_T) \leq \eta(M_T, u_T) \).
(iii) Employ newest vertex bisection [16] to generate the coarsest conforming refinement \( T_{T+1} \) of \( T_T \) such that \( M_T \subseteq T_T \setminus T_{T+1} \) (i.e., all marked elements have been refined) and define \( u^T_{T+1} := u_T \in X_{T_T} \subseteq X_{T_{T+1}} \).

In step (iii), we suppose that mesh-refinement leads to nested discrete spaces.
Lucky break-down of adaptive algorithm. First, if the repeat loop in step (i) does not terminate, it holds $u^* \in X_\ell$. Moreover, there exists $C > 0$ with
\[ \|u^* - u^*_\ell\| + \eta(\eta(u^*_\ell)) \leq C q^\infty \ell \rightarrow \infty 0. \]
Second, if the repeat loop in step (i) terminates with $M_\ell = \emptyset$ in step (ii), then $u^* = u_k$ as well as $M_k = \emptyset$ for all $k \geq \ell$. Overall, we may thus suppose that the repeat loop in step (i) terminates and that $\# T_\ell < \# T_{\ell+1}$ for all $\ell \geq 0$.

Bounded number of Picard iterations in step (i). There exists $C > 0$ such that nested iteration $u_0^\ell := u_{\ell-1}^\ell \in X_{\ell-1} \subseteq X_\ell$ guarantees
\[ u_\ell = u_n^\ell \quad \text{with} \quad n \leq C \left[ 1 + \log \left( \max \left\{ 1, \frac{\eta^{-1}(u_{\ell-1})}{\eta(u_\ell)} \right\} \right) \right] \quad \text{for all} \quad \ell \geq 1. \]

Linear convergence. For $0 < \theta \leq 1$ and all sufficiently small $\lambda > 0$, there exist constants $0 < \varrho < 1$ and $C > 0$ such that
\[ \eta(n(u_\ell+n)) \leq C \varrho^n \eta(u_\ell) \quad \text{for all} \quad \ell, n \geq 0. \]
In particular, there exists $C' > 0$ such that
\[ \|u^* - u_\ell\| \leq C' \eta(u_\ell) \leq C' C \varrho^{\ell} \eta_0(u_0) \ell \rightarrow \infty 0. \]

Optimal algebraic convergence rates. For sufficiently small $0 < \theta \ll 1$, sufficiently small $\lambda > 0$, and all $s > 0$, there exists $C > 0$ such that
\[ \eta(\lambda u_\ell) \leq C \left( \# T_\ell - \# T_0 + 1 \right)^{-s} \quad \text{for all} \quad \ell \geq 0, \]
provided that the rate $s$ is possible with respect to certain nonlinear approximation classes [4, 6].

Optimal computational complexity. Currently, the proof of optimal computational complexity is open, but it is observed in numerical experiments.

References