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Rate optimal adaptive FEM with inexact solver for strongly monotone operators

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(joint work with Gregor Gantner, Alexander Haberl, and Bernhard Stiftner)

Meanwhile, the mathematical understanding of adaptive FEM has reached a mature state; see [9, 14, 3, 15, 6, 11] for some milestones for linear elliptic PDEs, [17, 8, 2, 13] for non-linear problems, and [4] for some general framework. Optimal adaptive FEM with inexact solvers has already been addressed in [15, 1, 4] for linear PDEs and in [5] for eigenvalue problems. However, for problems involving nonlinear operators, optimal adaptive FEM with inexact solvers has not been analyzed yet. Our work [12] aims to close the gap between convergence analysis (e.g. [4]) and empirical evidence (e.g. [10]) by analyzing an algorithm from [7].

Model problem. We follow [4] and present our results from [12] in an abstract framework, while precise examples for our setting are given, e.g., in [2, 13]. Let $\mathcal{H}$ be a separable Hilbert space over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ with norm $\|\cdot\|$ and scalar product $(\cdot,\cdot)_{\mathcal{H}}$. With the duality pairing $\langle \cdot, \cdot \rangle$ between $\mathcal{H}$ and its dual $\mathcal{H}^*$, let $A : \mathcal{H} \to \mathcal{H}^*$ be a nonlinear operator which satisfies the following assumptions:

(O1) ***A* is strongly monotone:** There exists $\alpha > 0$ such that

$$\alpha \|u - v\|^2 \le \operatorname{Re} \langle Au - Av, u - v \rangle \quad \text{for all } u, v \in \mathcal{H}.$$

(O2) ***A* is Lipschitz continuous:** There exists $L > 0$ such that

$$\|Au - Av\|_* := \sup_{w \in \mathcal{H} \setminus \{0\}} \frac{\langle Au - Av, w \rangle}{\|w\|} \le L \|u - v\| \quad \text{for all } u, v \in \mathcal{H}.$$

(O3) ***A* has a potential:** There exists a Gâteaux differentiable function $P : \mathcal{H} \to \mathbb{K}$ with Gâteaux derivative $\mathrm{d}P = A$, i.e.,

$$\langle Au, v \rangle = \lim_{r \to 0} \frac{P(u + rv) - P(u)}{r} \quad \text{for all } u, v \in \mathcal{H}.$$

Let $F \in \mathcal{H}^*$. According to the main theorem on strongly monotone operators [18], (O1)–(O2) imply the existence and uniqueness of $u^\star \in \mathcal{H}$ such that

$$\langle Au^\star, v \rangle = \langle F, v \rangle \quad \text{for all } v \in \mathcal{H}. \tag{1}$$

To sketch the proof, let $\mathcal{I} : \mathcal{H} \to \mathcal{H}^*$ denote the Riesz mapping defined by $\langle \mathcal{I}u, v \rangle = (u, v)_{\mathcal{H}}$ for all $u, v \in \mathcal{H}$. Then, $\Phi : \mathcal{H} \to \mathcal{H}$, $\Phi(u) := u - \frac{\alpha}{L^2} \mathcal{I}^{-1}(Au - F)$ satisfies

$$\|\Phi(u) - \Phi(v)\| \le q \|u - v\| \text{ for all } u, v \in \mathcal{H}, \text{ where } q := (1 - \alpha^2/L^2)^{1/2} < 1. \tag{2}$$

Hence, the Banach fixpoint theorem proves the existence and uniqueness of $u^\star \in \mathcal{H}$ with $\Phi(u^\star) = u^\star$ which is equivalent to (1). In particular, the Picard iteration $u^n := \Phi(u^{n-1})$ with arbitrary initial guess $u^0 \in \mathcal{H}$ converges to $u^\star$, and it holds

$$\|u^\star - u^n\| \le \frac{q}{1-q} \|u^n - u^{n-1}\| \le \frac{q^n}{1-q} \|u^1 - u^0\| \quad \text{for all } n \ge 1. \tag{3}$$

With (O3), (1) (resp. (4) below) is equivalent to energy minimization, and $\|v - u^\star\|^2$ is equivalent to the energy difference. This guarantees the quasi-orthogonality [4].

FEM with iterative solver based on Picard iteration. Let $\mathcal{X}_\bullet \subset \mathcal{H}$ be a discrete subspace. As in the continuous case, there is a unique $u_\bullet^\star \in \mathcal{X}_\bullet$ such that

$$\langle A u_\bullet^\star, v_\bullet \rangle = \langle F, v_\bullet \rangle \quad \text{for all } v_\bullet \in \mathcal{X}_\bullet. \tag{4}$$

According to (O1)–(O2), it holds the Céa-type quasi-optimality $\|u^\star - u_\bullet^\star\| \leq \frac{L}{\alpha} \|u^\star - v_\bullet\|$ for all $v_\bullet \in \mathcal{X}_\bullet$. To solve the nonlinear system (4), we use the Picard iteration (applied in $\mathcal{X}_\bullet$): Given $u_\bullet^{n-1} \in \mathcal{X}_\bullet$, we compute $u_\bullet^n = \Phi_\bullet(u_\bullet^{n-1})$ as follows:

- Solve the linear system $(w_\bullet, v_\bullet)_{\mathcal{H}} = \langle A u_\bullet^{n-1} - F, v_\bullet \rangle$ for all $v_\bullet \in \mathcal{X}_\bullet$.
- Define $u_\bullet^n := u_\bullet^{n-1} - \frac{\alpha}{L^2} w_\bullet$.

Applying (3) on the discrete level, we infer from [7] that

$$\|u^\star - u_\bullet^n\| \leq \|u^\star - u_\bullet^\star\| + \frac{q}{1-q} \|u_\bullet^n - u_\bullet^{n-1}\| \leq \frac{L}{\alpha} \min_{v_\bullet \in \mathcal{X}_\bullet} \|u^\star - v_\bullet\| + \frac{q^n}{1-q} \|u_\bullet^1 - u_\bullet^0\|. \tag{5}$$

***A posteriori* error estimator.** We suppose that all considered discrete spaces $\mathcal{X}_\bullet \subset \mathcal{H}$ are associated with a conforming triangulation $\mathcal{T}_\bullet$ of a bounded Lipschitz domain $\Omega \subset \mathbb{R}^d$, $d \geq 2$. For all $T \in \mathcal{T}_\bullet$ and all $v_\bullet \in \mathcal{X}_\bullet$, we suppose an *a posteriori* computable refinement indicator $\eta_\bullet(T, v_\bullet) \geq 0$. We then define

$$\eta_\bullet(v_\bullet) := \eta_\bullet(\mathcal{T}_\bullet, v_\bullet) \quad \text{and} \quad \eta_\bullet(\mathcal{U}_\bullet, v_\bullet)^2 := \sum_{T \in \mathcal{U}_\bullet} \eta_\bullet(T, v_\bullet)^2 \text{ for } \mathcal{U}_\bullet \subseteq \mathcal{T}_\bullet. \tag{6}$$

We suppose that there exist constants $C_{\mathrm{ax}} > 0$ and $0 < \varrho_{\mathrm{ax}} < 1$ such that for all $\mathcal{T}_\bullet$ and all refinements $\mathcal{T}_\circ$ of $\mathcal{T}_\bullet$, the following properties (A1)–(A3) from [4] hold:

(A1) Stability on non-refined element domains:

$$|\eta_\circ(\mathcal{T}_\bullet \cap \mathcal{T}_\circ, v_\circ) - \eta_\bullet(\mathcal{T}_\bullet \cap \mathcal{T}_\circ, v_\bullet)| \leq C_{\mathrm{ax}} \|v_\circ - v_\bullet\| \text{ for all } v_\bullet \in \mathcal{X}_\bullet, v_\circ \in \mathcal{X}_\circ.$$

(A2) Reduction on refined element domains:

$$\eta_\circ(\mathcal{T}_\circ \backslash \mathcal{T}_\bullet, v_\circ) \leq \varrho_{\mathrm{ax}} \, \eta_\bullet(\mathcal{T}_\bullet \backslash \mathcal{T}_\circ, v_\bullet) + C_{\mathrm{ax}} \|v_\circ - v_\bullet\| \text{ for all } v_\bullet \in \mathcal{X}_\bullet, v_\circ \in \mathcal{X}_\circ.$$

(A3) Discrete reliability:

$$\|u_\circ^\star - u_\bullet^\star\| \leq C_{\mathrm{ax}} \, \eta_\bullet(\mathcal{T}_\bullet \backslash \mathcal{T}_\circ, u_\bullet^\star).$$

Note that (A1)–(A2) are required for all discrete functions (and follow from inverse estimates), while (A3) is only required for the discrete solutions $u_\circ^\star$ resp. $u_\bullet^\star$ of (4).

Adaptive algorithm. With adaptivity parameters $0 < \theta \leq 1$, $\lambda > 0$, and $C_{\mathrm{mark}} \geq 1$, an initial conforming triangulation $\mathcal{T}_0$, and an initial guess $u_0^0 \in \mathcal{X}_0$, our adaptive algorithm iterates the following steps (i)–(iii) for all $\ell = 0, 1, 2, \ldots$

(i) Repeat (a)–(b) for all $n = 1, 2, 3, \ldots$, until $\|u_\ell^n - u_\ell^{n-1}\| \leq \lambda \, \eta_\ell(u_\ell^n)$.
  (a) Compute discrete Picard iterate $u_\ell^n \in \mathcal{X}_\ell$.
  (b) Compute refinement indicators $\eta_\ell(T, u_\ell^n)$ for all $T \in \mathcal{T}_\ell$

(ii) Define $u_\ell := u_\ell^n$ and determine a set $\mathcal{M}_\ell \subseteq \mathcal{T}_\ell$ of minimal cardinality, up to the multiplicative factor C_{mark}, such that $\theta \, \eta_\ell(u_\ell) \leq \eta_\ell(\mathcal{M}_\ell, u_\ell)$.

(iii) Employ newest vertex bisection [16] to generate the coarsest conforming refinement $\mathcal{T}_{\ell+1}$ of $\mathcal{T}_\ell$ such that $\mathcal{M}_\ell \subseteq \mathcal{T}_\ell \backslash \mathcal{T}_{\ell+1}$ (i.e., all marked elements have been refined) and define $u_{\ell+1}^0 := u_\ell \in \mathcal{X}_\ell \subseteq \mathcal{X}_{\ell+1}$.

In step (iii), we suppose that mesh-refinement leads to nested discrete spaces.

Lucky break-down of adaptive algorithm. First, if the repeat loop in step (i) does not terminate, it holds $u^\star \in \mathcal{X}_\ell$. Moreover, there exists $C > 0$ with

$$\|u^\star - u_\ell^n\| + \eta_\ell(u_\ell^n) \le C\, q^n \xrightarrow{n\to\infty} 0. \tag{7}$$

Second, if the repeat loop in step (i) terminates with $\mathcal{M}_\ell = \emptyset$ in step (ii), then $u^\star = u_k$ as well as $\mathcal{M}_k = \emptyset$ for all $k \ge \ell$. Overall, we may thus suppose that the repeat loop in step (i) terminates and that $\#\mathcal{T}_\ell < \#\mathcal{T}_{\ell+1}$ for all $\ell \ge 0$.

Bounded number of Picard iterations in step (i). There exists $C > 0$ such that nested iteration $u_\ell^0 := u_{\ell-1} \in \mathcal{X}_{\ell-1} \subseteq \mathcal{X}_\ell$ guarantees

$$u_\ell = u_\ell^n \quad \text{with} \quad n \le C\Big[1 + \log\Big(\max\Big\{1, \frac{\eta_{\ell-1}(u_{\ell-1})}{\eta_\ell(u_\ell)}\Big\}\Big)\Big] \quad \text{for all } \ell \ge 1. \tag{8}$$

Linear convergence. For $0 < \theta \le 1$ and all sufficiently small $\lambda > 0$, there exist constants $0 < \varrho < 1$ and $C > 0$ such that

$$\eta_{\ell+n}(u_{\ell+n}) \le C\varrho^n\, \eta_\ell(u_\ell) \quad \text{for all } \ell, n \ge 0. \tag{9}$$

In particular, there exists $C' > 0$ such that

$$\|u^\star - u_\ell\| \le C'\, \eta_\ell(u_\ell) \le C' C \varrho^\ell \eta_0(u_0) \xrightarrow{\ell\to\infty} 0. \tag{10}$$

Optimal algebraic convergence rates. For sufficiently small $0 < \theta \ll 1$, sufficiently small $\lambda > 0$, and all $s > 0$, there exists $C > 0$ such that

$$\eta_\ell(u_\ell) \le C\,\big(\#\mathcal{T}_\ell - \#\mathcal{T}_0 + 1\big)^{-s} \quad \text{for all } \ell \ge 0, \tag{11}$$

provided that the rate s is possible with respect to certain nonlinear approximation classes [4, 6].

Optimal computational complexity. Currently, the proof of optimal computational complexity is open, but it is observed in numerical experiments.

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