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# Entropy-dissipating semi-discrete Runge-Kutta schemes for nonlinear diffusion equations

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# ENTROPY-DISSIPATING SEMI-DISCRETE RUNGE-KUTTA SCHEMES FOR NONLINEAR DIFFUSION EQUATIONS

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**ABSTRACT.** Semi-discrete Runge-Kutta schemes for nonlinear diffusion equations of parabolic type are analyzed. Conditions are determined under which the schemes dissipate the discrete entropy locally. The dissipation property is a consequence of the concavity of the difference of the entropies at two consecutive time steps. The concavity property is shown to be related to the Bakry-Emery approach and the geodesic convexity of the entropy. The abstract conditions are verified for quasilinear parabolic equations (including the porous-medium equation), a linear diffusion system, and the fourth-order quantum diffusion equation. Numerical experiments for various Runge-Kutta finite-difference discretizations of the one-dimensional porous-medium equation show that the entropy-dissipation property is in fact global.

## 1. INTRODUCTION

Evolution equations often contain some structural information reflecting inherent physical properties such as positivity of solutions, conservation laws, and entropy dissipation. Numerical schemes should be designed in such a way that these structural features are preserved on the discrete level in order to obtain accurate and stable algorithms. In the last decades, concepts of structure-preserving schemes, geometric integration, and compatible discretization have been developed [7], but much less is known about the preservation of entropy dissipation and large-time asymptotics.

Entropy-stable schemes were derived by Tadmor already in the 1980s [20] in the context of conservation laws, thus without (physical) diffusion. Later, entropy-dissipative schemes were developed for (finite-volume) discretizations of diffusion equations in [2, 10, 11]. In [5], a finite-volume scheme which preserves the gradient-flow structure and hence the entropy structure is proposed. All these schemes are based on the implicit Euler method and are of first order (in time) only. Higher-order time schemes with entropy-dissipating properties are investigated in very few papers. A second-order predictor-corrector approximation was suggested in [19], while higher-order semi-implicit Runge-Kutta (DIRK) methods, together with a spatial fourth-order central finite-difference discretization, were investigated in [3]. In [4, 17], multistep time approximations were employed, but they can be at most of

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second order and they dissipate only one entropy and not all functionals dissipated by the continuous equation. In this paper, we remove these restrictions by investigating time-discrete Runge-Kutta schemes of order  $p \geq 1$  for general diffusion equations.

We stress the fact that we are interested in the analysis of entropy-dissipating schemes by “translating” properties for the continuous equation to the semi-discrete level, i.e., we study the stability of the schemes. However, we will not investigate convergence, stiffness, or computational issues here (see e.g. [3]).

More precisely, we consider time discretizations of the abstract Cauchy problem

$$(1) \quad \partial_t u(t) + A[u(t)] = 0, \quad t > 0, \quad u(0) = u^0,$$

where  $A : D(A) \rightarrow X'$  is a (differential) operator defined on  $D(A) \subset X$  and  $X$  is a Banach space with dual  $X'$ . In this paper, we restrict ourselves to diffusion operators  $A[u]$  defined on some Sobolev space with solutions  $u : \Omega \times (0, \infty) \rightarrow \mathbb{R}^n$ , which may be vector-valued. A typical example is  $A[u] = \operatorname{div}(a(u)\nabla u)$  defined on  $X = L^2(\Omega)$  with domain  $D(A) = H^2(\Omega)$ , where  $a : \mathbb{R} \rightarrow \mathbb{R}$  is a smooth function (see section 3). Equation (1) often possesses a Lyapunov functional  $H[u] = \int_{\Omega} h(u)dx$  (here called *entropy*), where  $h : \mathbb{R}^n \rightarrow \mathbb{R}$ , such that

$$\frac{dH}{dt}[u] = \int_{\Omega} h'(u)\partial_t u dx = - \int_{\Omega} h'(u)A[u]dx \leq 0,$$

at least when the *entropy production*  $\int_{\Omega} h'(u)A[u]dx$  is nonnegative. Here,  $h'$  is the derivative of  $h$  and  $h'(u)A[u]$  is interpreted as the inner product of  $h'(u)$  and  $A[u]$  in  $\mathbb{R}^n$ . Furthermore, if  $h$  is convex, the convex Sobolev inequality  $\int_{\Omega} h'(u)A[u]dx \geq \kappa H[u]$  for some  $\kappa > 0$  may hold [6], which implies that  $dH/dt \leq -\kappa H$  and hence exponential convergence of  $H[u]$  to zero with rate  $\kappa$ . The aim is to design a higher-order time-discrete scheme which preserves this entropy-dissipation property.

To this end, we propose the following semi-discrete Runge-Kutta approximation of (1): Given  $u^{k-1} \in X$ , define

$$(2) \quad u^k = u^{k-1} + \tau \sum_{i=1}^s b_i K_i, \quad K_i = -A \left[ u^{k-1} + \tau \sum_{j=1}^s a_{ij} K_j \right], \quad i = 1, \dots, s,$$

where  $t^k$  are the time steps,  $\tau = t^k - t^{k-1} > 0$  is the uniform time step size,  $u^k$  approximates  $u(t^k)$ , and  $s \geq 1$  denotes the number of Runge-Kutta stages. Since the Cauchy problem is autonomous, the knots  $c_1, \dots, c_s$  are not needed here. In concrete examples (see below),  $u^k$  are functions from  $\Omega$  to  $\mathbb{R}^n$ . If  $a_{ij} = 0$  for  $j \geq i$ , the Runge-Kutta scheme is explicit, otherwise it is implicit and a nonlinear system of size  $s$  has to be solved to compute  $K_i$ . We assume that scheme (2) is solvable for  $u^k : \Omega \rightarrow \mathbb{R}^n$ .

Given  $h : \mathbb{R}^n \rightarrow \mathbb{R}$ , we wish to determine conditions under which the functional

$$(3) \quad H[u^k] = \int_{\Omega} h(u^k(x))dx$$

is dissipated by the numerical scheme (2),

$$(4) \quad H[u^k] + \tau \int_{\Omega} A[u^k] h'(u^k) dx \leq H[u^{k-1}], \quad k \in \mathbb{N}.$$

In many examples (see below),  $\int_{\Omega} A[u^k] h'(u^k) dx \geq 0$  and thus, the function  $k \mapsto H[u^k]$  is decreasing. Such a property is the first step in proving the preservation of the large-time asymptotics of the numerical scheme (see Remark 2).

Our main results, stated on an informal level, are as follows:

- (i) We determine an abstract condition under which the discrete entropy-dissipation inequality (4) holds for sufficiently small  $\tau^k > 0$ . This condition is made explicit for special choices of  $A$  and  $h$ , yielding entropy-dissipative implicit or explicit Runge-Kutta schemes of any order.
- (ii) Numerical experiments for the porous-medium equation indicate that  $\tau^k$  may be chosen independent of the time step  $k$ , thus yielding discrete entropy dissipation for all discrete times.
- (iii) We show that for Runge-Kutta schemes of order  $p \geq 2$ , the abstract condition in (i) is exactly the criterion of Liero and Mielke [18] to conclude geodesic 0-convexity of the entropy. In particular, it is related to the Bakry-Emery condition [1].

Let us describe the main results in more detail. We recall that the Runge-Kutta scheme (2) is consistent if  $\sum_{j=1}^s a_{ij} = c_i$  and  $\sum_{i=1}^s b_i = 1$ . Furthermore, if  $\sum_{i=1}^s b_i c_i = \frac{1}{2}$ , it is at least of order two [12, Chap. II]. We introduce the number

$$(5) \quad C_{\text{RK}} = 2 \sum_{i=1}^s b_i (1 - c_i),$$

which takes only three values:

$$\begin{aligned} C_{\text{RK}} &= 0 && \text{for the implicit Euler scheme,} \\ C_{\text{RK}} &= 1 && \text{for any Runge-Kutta scheme of order } p \geq 2, \\ C_{\text{RK}} &= 2 && \text{for the explicit Euler scheme.} \end{aligned}$$

The *first main result* is an abstract entropy-dissipation property of scheme (2) for entropies of type (3).

**Theorem 1** (Entropy-dissipation structure I). *Let  $h \in C^2(\mathbb{R}^n)$ , let  $A : D(A) \rightarrow X'$  be Fréchet differentiable with Fréchet derivative  $DA[u] : X \rightarrow X'$  at  $u \in D(A)$ , and let  $(u^k)$  be the Runge-Kutta solution to (2). Suppose that*

$$(6) \quad I_0^k := \int_{\Omega} (C_{\text{RK}} h'(u^k) DA[u^k](A[u^k]) + h''(u^k)(A[u^k])^2) dx > 0.$$

*Then there exists  $\tau^k > 0$  such that for all  $0 < \tau \leq \tau^k$ ,*

$$(7) \quad H[u^k] + \tau \int_{\Omega} A[u^k] h'(u^k) dx \leq H[u^{k-1}].$$

We assume that the solutions to (2) are sufficiently regular such that the integral (6) can be defined. In the vector-valued case,  $h''(u^k)$  is the Hessian matrix and we interpret  $h''(u^k)(A[u^k])^2$  as the product  $A[u^k]^\top h''(u^k)A[u^k]$ . For Runge-Kutta schemes of order  $p \geq 2$  (for which  $C_{\text{RK}} = 1$ ), the integral (6) corresponds exactly to the second-order time derivative of  $H[u(t)]$  for solutions  $u(t)$  to the *continuous* equation (1). Observe that the entropy-dissipation estimate (7) is only *local*, since the time step restriction depends on the time step  $k$ . For implicit Euler schemes (and convex entropies  $h$ ), it is known that  $\tau^k$  can be chosen independent of  $k$ . For general Runge-Kutta methods, we cannot prove rigorously that  $\tau^k$  stays bounded from below as  $k \rightarrow \infty$ . However, our numerical experiments in section 7 indicate that inequality (7) holds for sufficiently small  $\tau > 0$  uniformly in  $k$ .

**Remark 2** (Exponential decay of the discrete entropy). If the convex Sobolev inequality  $\int_{\Omega} A[u^k]h'(u^k)dx \geq \kappa H[u^k]$  holds for some constant  $\kappa > 0$  and if there exists  $\tau^* > 0$  such that  $\tau^k \geq \tau^* > 0$  for all  $k \in \mathbb{N}$ , we infer from (7) that for  $\tau := \tau^*$ ,

$$H[u^k] \leq (1 + \kappa\tau)^{-k} H[u^0] = \exp(-\eta\kappa t^k) H[u^0], \quad \text{where } \eta = \frac{\log(1 + \kappa\tau)}{\kappa\tau} < 1,$$

which implies exponential decay of the discrete entropy with rate  $\eta\kappa$ . This rate converges to the continuous rate  $\kappa$  as  $\tau \rightarrow 0$  and therefore, it is asymptotically sharp.  $\square$

Theorem 1 can be generalized to a larger class of entropies, namely to so-called *first-order entropies*

$$(8) \quad F[u^k] = \int_{\Omega} |\nabla f(u^k)|^2 dx,$$

where, for simplicity, we consider only the scalar case with  $f : \mathbb{R} \rightarrow \mathbb{R}$ . An important example is the Fisher information with  $f(u) = \sqrt{u}$ .

**Theorem 3** (Entropy-dissipating structure II). *Let  $f \in C^2(\mathbb{R})$ , let  $A : D(A) \rightarrow X'$  be Fréchet differentiable, and let  $(u^k)$  be the Runge-Kutta solution to (2). Assume that the boundary condition  $\nabla f(u^k) \cdot \nu = 0$  on  $\partial\Omega$  is satisfied. Furthermore, suppose that*

$$(9) \quad I_1^k := \int_{\Omega} \left( |\nabla(f'(u^k)A[u^k])|^2 - C_{\text{RK}} \Delta f(u^k) f'(u^k) DA[u^k](A[u^k]) \right. \\ \left. - \Delta f(u^k) f''(u^k) (A[u^k])^2 \right) dx > 0.$$

Then there exists  $\tau^k > 0$  such that for all  $0 < \tau \leq \tau^k$ ,

$$F[u^k] + \tau \int_{\Omega} A[u^k] f'(u^k) dx \leq F[u^{k-1}].$$

The key idea of the proof of Theorem 1 (and similarly for Theorem 3) is a concavity property of the difference of the entropies at two consecutive time steps with respect to the time step size  $\tau$ . To explain this idea, let  $u := u^k$  be fixed and introduce  $v(\tau) := u^{k-1}$ . Clearly,  $v(0) = u$ . A formal Taylor expansion of  $G(\tau) := H[u] - H[v(\tau)]$  yields

$$H[u^k] - H[u^{k-1}] = G(\tau) = G(0) + \tau G'(0) + \frac{\tau^2}{2} G''(\xi^k),$$

where  $0 < \xi^k < \tau$ . A computation, made explicit in section 2, shows that  $G'(0) = \int_{\Omega} A[u^k]h'(u^k)dx$  and  $G''(0) = -I_0^k$ . Now, if  $G''(0) < 0$ , there exists  $\tau^k > 0$  such that  $G''(\tau) \leq 0$  for  $\tau \in [0, \tau^k]$  and in particular  $G''(\xi^k) \leq 0$ . Consequently,  $G(\tau) \leq \tau G'(0)$ , which equals (4). The definition of  $v(\tau)$  assumes implicitly that (2) is *backward* solvable. We prove in Proposition 5 below that this property holds if the operator  $A$  is a smooth self-mapping on  $X$ .

**Remark 4** (Discussion of  $\tau^k$ ). Since  $(u^k)$  is expected to converge to the stationary solution,  $\lim_{k \rightarrow \infty} I_0^k = 0$ . Thus, in principle, for larger values of  $k$ , we expect that  $\tau^k$  becomes smaller and smaller, thus restricting the choice of time step sizes  $\tau$ . However, practically, the situation is better. For instance, for the implicit Euler scheme, if  $h$  is convex, we obtain

$$H[u^k] - H[u^{k-1}] \leq \int_{\Omega} h'(u^k)(u^k - u^{k-1})dx = -\tau \int_{\Omega} h'(u^k)A[u^k]dx$$

for *any* value of  $\tau > 0$ . Moreover, for other (higher-order) Runge-Kutta schemes, the numerical experiments in section 7 indicate that there exists  $\tau^* > 0$  such that  $G''(\tau) \leq 0$  holds for all  $\tau \in [0, \tau^*]$  uniformly in  $k \in \mathbb{N}$ . In this situation, inequality (7) holds for all  $0 < \tau \leq \tau^*$ , and thus our estimate is global. In fact, the function  $G''$  is numerically even nonincreasing in some interval  $[0, \tau^*]$  but we are not able to prove this analytically.  $\square$

The *second main result* is the specification of the abstract conditions (6) and (9) for a number of examples: a quasilinear diffusion equation, porous-medium or fast-diffusion equations, a linear diffusion system, and the fourth-order Derrida-Lebowitz-Speer-Spohn equation (see sections 3-6 for details). For instance, for the porous-medium equation

$$\partial_t u = \Delta(u^\beta) \text{ in } \Omega, \quad t > 0, \quad \nabla u^\beta \cdot \nu = 0 \text{ on } \partial\Omega, \quad u(0) = u^0,$$

we show that the Runge-Kutta scheme satisfies

$$H[u^k] + \tau\beta \int_{\Omega} (u^k)^{\alpha+\beta-2} |\nabla u^k|^2 dx \leq H[u^{k-1}], \quad \text{where } H[u] = \frac{1}{\alpha(\alpha+1)} \int_{\Omega} u^{\alpha+1} dx,$$

for  $0 < \tau \leq \tau^k$  and all  $(\alpha, \beta)$  belonging to some region in  $[0, \infty)^2$  (see Figure 1 below). For  $\alpha = 0$ , we write  $H[u] = \int_{\Omega} u(\log u - 1)dx$ . In one space dimension and for Runge-Kutta schemes of order  $p \geq 2$ , this region becomes  $-2 < \alpha - \beta < 1$ , which is the same condition as for the continuous equation (except the boundary values). Furthermore, the first-order entropy (8) is dissipated for Runge-Kutta schemes of order  $p \geq 2$ , in one space dimension,

$$F[u^k] + \tau C_{\alpha,\beta} \int_{\Omega} (u^k)^{\alpha+\beta-2} (u^k)_{xx}^2 dx \leq F[u^{k-1}], \quad \text{where } F[u] = \int_{\Omega} (u^{\alpha/2})_x^2 dx,$$

for  $0 < \tau \leq \tau^k$  and all  $(\alpha, \beta)$  belonging to the region shown in Figure 2 below, and  $C_{\alpha,\beta} > 0$  is some constant. This region is smaller than the region of admissible values  $(\alpha, \beta)$  for the continuous entropy. The borders of that region are indicated in the figure by dashed lines.

The proof of the above results, and namely of  $G''(0) < 0$ , is based on systematic integration by parts [14]. The idea of the method is to formulate integration by parts as

manipulations with polynomials and to conclude the inequality  $G''(0) < 0$  from a polynomial decision problem. This problem can be solved directly or by using computer algebra software.

Our *third main result* is the relation to geodesic 0-convexity of the entropy and the Bakry-Emery approach when  $C_{\text{RK}} = 1$  (Runge-Kutta scheme of order  $p \geq 2$ ). Liero and Mielke formulate in [18] the abstract Cauchy problem (1) as the gradient flow

$$\partial_t u = -K[u]DH[u], \quad t > 0, \quad u(0) = u^0,$$

where the Onsager operator  $K[u]$  describes the sum of diffusion and reaction terms. For instance, if  $A[u] = \operatorname{div}(a(u)\nabla u)$ , we can write  $A[u] = \operatorname{div}(a(u)h''(u)^{-1}\nabla h'(u))$  and thus, identifying  $h'(u)$  and  $DH[u]$ , we have  $K[u]\xi = \operatorname{div}(a(u)h''(u)^{-1}\nabla\xi)$ . It is shown in [18] that the entropy  $H$  is geodesic  $\lambda$ -convex if the inequality

$$(10) \quad M(u, \xi) := \langle \xi, DA[u]K[u]\xi \rangle - \frac{1}{2}\langle \xi, DK[u]A[u]\xi \rangle \geq \lambda\langle \xi, K[u]\xi \rangle$$

holds for all suitable  $u$  and  $\xi$ . We will prove in section 2 that

$$G''(0) = 2M(u^k, h'(u^k)).$$

Hence, if  $G''(0) \leq 0$  then (10) with  $\lambda = 0$  is satisfied for  $u = u^k$  and  $\xi = h'(u^k)$ , yielding geodesic 0-convexity for the semi-discrete entropy. Moreover, if  $G''(0) \leq -\lambda G'(0)$  then we obtain geodesic  $\lambda$ -convexity. Since  $G'(0) = -dH[u]/dt$  and  $G''(0) = -d^2H[u]/dt^2$  in the continuous setting, the inequality  $G''(0) \leq -\lambda G'(0)$  can be written as

$$\frac{d^2H}{dt^2}[u] \geq -\lambda \frac{dH}{dt}[u],$$

which corresponds to a variant of the Bakry-Emery condition [1], yielding exponential convergence of  $H[u]$  (if  $\tau^k \geq \tau^* > 0$  for all  $k$ ). Thus, our results constitute a first step towards a *discrete Bakry-Emery approach*.

The paper is organized as follows. The abstract method, i.e. the proof of backward solvability and of Theorems 1 and 3, is presented in section 2. The method is applied in the subsequent sections to a scalar diffusion equation (section 3), the porous-medium equation (section 4), a linear diffusion system (section 5), and the fourth-order Derrida-Lebowitz-Speer-Spohn equation (section 6). Finally, section 7 is devoted to some numerical experiments showing that  $G''$  is negative in some interval  $[0, \tau^*]$ .

## 2. THE ABSTRACT METHOD

In this section, we show that the Runge-Kutta scheme is backward solvable if  $A$  is a self-mapping and we prove Theorems 1 and 3.

**Proposition 5** (Backward solvability). *Let  $(\tau, u^k) \in [0, \infty) \times X$ , where  $X$  is some Banach space, and let  $A \in C^2(X, X)$  be a self-mapping. Then there exists  $\tau_0 > 0$ , a neighborhood  $V \subset X$  of  $u^k$ , and a function  $v \in C^2([0, \tau_0); X)$  such that (2) holds for  $u^{k-1} := v(\tau)$ . Moreover,*

$$(11) \quad v(0) = 0, \quad v'(0) = A[u], \quad \text{and} \quad v''(0) = C_{\text{RK}}DA[u](A[u]).$$



The self-mapping assumption is strong for differential operators  $A$  but it is somehow natural in the context of Runge-Kutta methods and valid for smooth solutions.

*Proof.* The idea of the proof is to apply the implicit function theorem in Banach spaces (see [8, Corollary 15.1]). To this end, we set  $u := u^k$  and define the mapping  $J = (J_0, \dots, J_s) : \mathbb{R} \times X^{s+1} \rightarrow X^{s+1}$  by

$$J_0(\tau, y) = v - u + \tau \sum_{i=1}^s b_i k_i, \quad \text{where } y = (k_1, \dots, k_s, v),$$

$$J_i(\tau, y) = k_i + A \left[ v + \tau \sum_{j=1}^s a_{ij} k_j \right], \quad i = 1, \dots, s.$$

The Fréchet derivative of  $J$  in the direction of  $(\tau_h, y_h)$ , where  $y_h = (k_{h1}, \dots, k_{hs}, v_h)$ , reads as

$$DJ_0(\tau, y)(\tau_h, y_h) = v_h + \tau_h \sum_{i=1}^s b_i k_i + \tau \sum_{i=1}^s b_i k_{hi},$$

$$DJ_i(\tau, y)(\tau_h, y_h) = k_{hi} + DA \left[ v + \tau \sum_{j=1}^s a_{ij} k_j \right] \left( v_h + \tau_h \sum_{j=1}^s a_{ij} k_j + \tau \sum_{j=1}^s a_{ij} k_{hj} \right),$$

where  $i = 1, \dots, s$ . Let  $\tau_0 = 0$  and  $y_0 = (-A[u], \dots, -A[u], u)$ . Then  $J(\tau_0, y_0) = 0$  and

$$DJ_0(\tau_0, y_0)(0, y_h) = v_h, \quad DJ_i(\tau_0, y_0)(0, y_h) = k_{ih} + DA[u](v_h), \quad i = 1, \dots, s.$$

The mapping  $y_h \mapsto DJ(\tau_0, y_0)(0, y_h)$  is clearly an isomorphism from  $X^{s+1}$  onto  $X^{s+1}$ . By the implicit function theorem, there exist an interval  $U \subset [0, \tau_0)$ , a neighborhood  $V \subset X^{s+1}$  of  $y_0$ , and a function  $(k, v) \in C^2([0, \tau_0); V)$  such that  $(k, v)(0) = (-A[u], \dots, -A[u], u)$  and  $J(\tau, k(\tau), v(\tau)) = 0$  for all  $\tau \in [0, \tau_0)$ .

Implicit differentiation of  $J(\tau, k(\tau), v(\tau)) = 0$  yields

$$0 = v'(\tau) + \sum_{i=1}^s b_i k_i'(\tau) + \tau \sum_{i=1}^s b_i k_i''(\tau),$$

$$0 = k_i'(\tau) + DA \left[ v + \tau \sum_{j=1}^s a_{ij} k_j(\tau) \right] \left( v'(\tau) + \sum_{j=1}^s a_{ij} k_j'(\tau) + \tau \sum_{j=1}^s a_{ij} k_j''(\tau) \right),$$

where  $i = 1, \dots, s$  and  $\tau \in [0, \tau_0)$ . Using  $\sum_{i=1}^s b_i = 1$  and  $\sum_{j=1}^s a_{ij} = c_i$ , we infer that

$$v'(0) = - \sum_{i=1}^s b_i k_i(0) = \sum_{i=1}^s b_i A[u] = A[u],$$

$$(12) \quad k_i'(0) = -DA[u] \left( A[u] - \sum_{j=1}^s a_{ij} A[u] \right) = -(1 - c_i) DA[u](A[u]).$$

Differentiating  $J_0(\tau, k(\tau), v(\tau)) = 0$  twice leads to

$$0 = v''(\tau) + 2 \sum_{i=1}^s b_i k_i'(\tau) + \tau \sum_{i=1}^s b_i k_i''(\tau).$$

Because of (12), this reads at  $\tau = 0$  as

$$v''(0) = -2 \sum_{i=1}^s b_i k_i'(0) = 2 \sum_{i=1}^s b_i (1 - c_i) DA[u](A[u]) = C_{\text{RK}} DA[u](A[u]).$$

This finishes the proof.  $\square$

We prove now Theorems 1 and 3.

*Proof of Theorem 1.* We set  $u := u^k$ . By Proposition 5, there exists a backward solution  $v \in C^2([0, \tau_0])$  such that  $v(0) = u$ ,  $v'(0) = A[u]$ , and  $v''(0) = C_{\text{RK}} DA[u](A[u])$ . Furthermore, the function  $G(\tau) = \int_{\Omega} (h(u) - h(v(\tau))) dx$  satisfies  $G(0) = 0$ ,

$$\begin{aligned} G'(0) &= - \int_{\Omega} h'(v(0)) v'(0) dx = - \int_{\Omega} h'(u) A[u] dx, \\ G''(0) &= - \int_{\Omega} (h'(v(0)) v''(0) + h''(v(0)) v'(0)^2) dx \\ &= - \int_{\Omega} (C_{\text{RK}} h'(u) DA[u](A[u]) + h''(u) (A[u])^2) dx = -I_0^k < 0, \end{aligned}$$

using the assumption. By continuity, there exists  $0 < \tau^k < \tau_0$  such that  $G'''(\xi) \leq 0$  for  $0 \leq \xi \leq \tau^k$ . Then the Taylor expansion  $G(\tau) = G(0) + G'(0)\tau + \frac{1}{2}G''(\xi)\tau^2 \leq G'(0)\tau$  concludes the proof.  $\square$

*Proof of Theorem 3.* Following the lines of the previous proof, it is sufficient to compute  $G'(0)$  and  $G''(0)$ , where now  $G(\tau) = \int_{\Omega} (|\nabla f(u)|^2 - |\nabla f(v(\tau))|^2) dx$ . Using integration by parts and the boundary condition  $\nabla f(v) \cdot \nu = 0$  on  $\partial\Omega$ , we compute

$$G'(0) = - \int_{\Omega} \nabla f(v(0)) \cdot \nabla (f'(v(0)) v'(0)) dx = \int_{\Omega} \Delta f(u) f'(v(\tau)) A[u] dx,$$

since  $v(0) = u$  and  $v'(0) = A[u]$ . Furthermore, again integrating by parts,

$$\begin{aligned} G''(\tau) &= - \int_{\Omega} \left( |\nabla (f'(v(\tau)) v'(\tau))|^2 + \nabla f(v(\tau)) \cdot \nabla (f''(v(\tau)) (v'(\tau))^2) \right. \\ &\quad \left. + \nabla f(v(\tau)) \cdot \nabla (f'(v(\tau)) v''(\tau)) \right) dx \\ &= - \int_{\Omega} \left( |\nabla (f'(v(\tau)) v'(\tau))|^2 - \Delta f(v(\tau)) f''(v(\tau)) (v'(\tau))^2 \right. \\ &\quad \left. - \Delta f(v(\tau)) f'(v(\tau)) v''(\tau) \right) dx. \end{aligned}$$

Since  $v''(0) = C_{\text{RK}}DA[u](A[u])$ , this reduces at  $\tau = 0$  to

$$G''(0) = - \int_{\Omega} \left( |\nabla(f'(u)A[u])|^2 - \Delta f(u)f''(u)(A[u])^2 - C_{\text{RK}}\Delta f(u)f'(u)DA[u](A[u]) \right) dx.$$

This expression equals  $-I_1^k$ , and the result follows.  $\square$

Finally, we show that  $G''(0)$  for entropies (3) is related to the geodesic convexity condition of [18].

**Lemma 6.** *Let  $A[u] = K(u)DH[u]$  for some symmetric operator  $K : D(A) \rightarrow X$  and Fréchet derivative  $DH[u]$ , let  $G$  be defined as in the proof of Theorem 1 for a solution  $u^k$  to the Runge-Kutta scheme (2) of order  $p \geq 2$ , and let  $M(u, \xi)$  be given by (10). Then*

$$G''(0) = -2M(u^k, DH[u^k]).$$

*Proof.* The proof is just a (formal) calculation. Recall that for Runge-Kutta schemes of order  $p \geq 2$ , we have  $C_{\text{RK}} = 1$ . Set  $u := u^k$  and identify  $DH[u]$  with  $\xi = h'(u)$ . Inserting the expression  $DA[u](v) = DK[u](v)h'(u) + K[u]h''(u)v$  into the definition of  $G''(0)$ , we find that

$$\begin{aligned} -G''(0) &= \langle \xi, DA[u](A[u]) \rangle + \langle A[u], h''(u)A[u] \rangle \\ &= \langle \xi, DK[u](A[u])\xi + K[u]h''(u)A[u] \rangle + \langle A[u], h''(u)A[u] \rangle \\ &= \langle \xi, DK[u](K[u]\xi)\xi \rangle + \langle \xi, K[u]h''(u)K[u]\xi \rangle + \langle K[u]\xi, h''(u)K[u]\xi \rangle \\ &= \langle \xi, DK[u](K[u]\xi)\xi \rangle + 2\langle \xi, K[u]h''(u)K[u]\xi \rangle, \end{aligned}$$

since  $K[u]$  is assumed to be symmetric. Rearranging the terms, we obtain

$$\begin{aligned} -G''(0) &= 2\langle \xi, DK[u](K[u]\xi)\xi \rangle + 2\langle \xi, K[u]h''(u)K[u]\xi \rangle - \langle \xi, DK[u](K[u]\xi) \rangle \\ &= 2\langle \xi, DA[u](K[u]\xi)\xi \rangle - \langle \xi, DK[u](A[u]) \rangle = 2M(u, \xi), \end{aligned}$$

which proves the claim.  $\square$

### 3. SCALAR DIFFUSION EQUATION

In this section, we analyze time-discrete Runge-Kutta schemes of the diffusion equation

$$(13) \quad \partial_t u = \operatorname{div}(a(u)\nabla u), \quad t > 0, \quad u(0) = u^0,$$

with periodic or homogeneous Neumann boundary conditions. This equation, also including a drift term, was analyzed in [18] in the context of geodesic convexity. Our results are similar to those in [18] but we consider the time-discrete and not the continuous equation and we employ systematic integration by parts [14].

Setting  $\mu(u) = a(u)/h''(u)$ , we can write the diffusion equation as a formal gradient flow:

$$\partial_t u = -A[u] := \operatorname{div}(\mu(u)\nabla h'(u)), \quad t > 0.$$

We prove that the Runge-Kutta scheme (2) dissipates all convex entropies subject to some conditions on the functions  $\mu$  and  $h$ .

**Theorem 7.** *Let  $\Omega \subset \mathbb{R}^d$  be convex with smooth boundary. Let  $(u^k)$  be a sequence of (smooth) solutions to the Runge-Kutta scheme (2) of the diffusion equation (13). Let  $k \in \mathbb{N}$  be fixed and  $u^k$  be not equal to the constant steady state of (13). We suppose that for all admissible  $u$ , it holds that  $a(u) \geq 0$ ,  $h''(u) \geq 0$ ,*

$$(14) \quad b(u) := \frac{2}{3}(C_{\text{RK}} + 1) \int_{u_0}^u \mu(v) \mu'(v) h''(v) dv \geq 0,$$

$$(15) \quad \frac{d-1}{d} b(u) \leq (C_{\text{RK}} + 1) h''(u) \mu(u)^2,$$

$$(16) \quad (C_{\text{RK}} + 2) \mu(u) \mu''(u) + (C_{\text{RK}} - 1) \mu'(u)^2 < 0.$$

Then there exists  $\tau^k > 0$  such that for all  $0 < \tau < \tau^k$ ,

$$H[u^k] + \tau \int_{\Omega} h''(u^k) a(u^k) |\nabla u^k|^2 dx \leq H[u^{k-1}].$$

Conditions (14)-(15) correspond to (4.12) in [18]. Condition (16) is satisfied for concave functions  $\mu$ , except for the explicit Euler scheme ( $C_{\text{RK}} = 2$ ) for which we need additionally  $4\mu\mu'' + (\mu')^2 < 0$ . For the implicit Euler scheme, we may allow even for nonconcave mobilities  $\mu$ , e.g.  $\mu(u) = u^\gamma$  for  $1 < \gamma < 2$ .

*Proof.* According to Theorem 1, we only need to show that  $I_0^k = -G''(0) > 0$ . To simplify, we set  $u := u^k$ . First, we observe that the boundary condition  $\nabla u \cdot \nu = 0$  on  $\Omega$  implies that  $0 = \partial_t \nabla u \cdot \nu = \nabla \partial_t u \cdot \nu = -\nabla A[u] \cdot \nu$  on  $\partial\Omega$ . Using  $DA[u](A[u]) = \text{div}(a'(u)A[u]\nabla u + a(u)\nabla A[u]) = \Delta(a(u)A[u])$ , the abbreviation  $\xi = h'(u)$ , and integration by parts, we compute

$$\begin{aligned} G''(0) &= - \int_{\Omega} \left( C_{\text{RK}} h'(u) \Delta(a(u)A[u]) + h''(u) (\text{div}(\mu(u)\nabla h'(u)))^2 \right) dx \\ &= \int_{\Omega} \left( C_{\text{RK}} \nabla h'(u) \cdot \nabla(a(u)A[u]) - h''(u) (\mu'(u)\nabla u \cdot \nabla h'(u) + \mu(u)\Delta h'(u))^2 \right) dx \\ &= - \int_{\Omega} \left( C_{\text{RK}} \Delta \xi a(u) A[u] + h''(u) \left( \frac{\mu'(u)}{h''(u)} |\nabla \xi|^2 + \mu(u) \Delta \xi \right)^2 \right) dx. \end{aligned}$$

The boundary integrals vanish since  $\nabla u \cdot \nu = \nabla A[u] \cdot \nu = 0$  on  $\partial\Omega$ . Replacing  $A[u]$  by  $\text{div}(\mu(u)\nabla \xi) = \mu(u)\Delta \xi + \mu'(u)|\nabla \xi|^2/h''(u)$  and expanding the square, we arrive at

$$\begin{aligned} (17) \quad G''(0) &= - \int_{\Omega} \left( (C_{\text{RK}} a(u) \mu(u) + h''(u) \mu(u)^2) (\Delta \xi)^2 \right. \\ &\quad \left. + \left( C_{\text{RK}} a(u) \frac{\mu'(u)}{h''(u)} + 2\mu(u) \mu'(u) \right) \Delta \xi |\nabla \xi|^2 + \frac{\mu'(u)^2}{h''(u)} |\nabla \xi|^4 \right) dx \\ &= - \int_{\Omega} \left( (C_{\text{RK}} + 1) h''(u) \mu(u)^2 \xi_L^2 + (C_{\text{RK}} + 2) \mu(u) \mu'(u) \xi_L \xi_G^2 + \mu'(u)^2 h''(u)^{-1} \xi_G^4 \right) dx, \end{aligned}$$

where we have employed the identity  $a(u) = \mu(u)h''(u)$  and the abbreviations  $\xi_G = |\nabla \xi|$  and  $\xi_L = \Delta \xi$ .

We apply now the method of systematic integration by parts [14]. The idea is to identify useful integration-by-parts formulas and to add them to  $G''(0)$  without changing the sign of  $G''(0)$ . The first formula is given by

$$(18) \quad \int_{\Omega} \operatorname{div} (\Gamma_1(u)(\nabla^2 \xi - \Delta \xi \mathbb{I}) \cdot \nabla \xi) dx = \int_{\partial \Omega} \Gamma_1(u) \nabla \xi^\top (\nabla^2 \xi - \Delta \xi \mathbb{I}) \nu ds,$$

where  $\Gamma_1(u) \leq 0$  is an arbitrary (smooth) scalar function which still needs to be chosen, and  $\mathbb{I}$  is the unit matrix in  $\mathbb{R}^{d \times d}$ . The left-hand side can be expanded as

$$\begin{aligned} & \int_{\Omega} \left( \frac{\Gamma_1'(u)}{h''(u)} \nabla \xi^\top (\nabla^2 \xi - \Delta \xi \mathbb{I}) \nabla \xi + \Gamma_1(u) \nabla^2 \xi : (\nabla^2 \xi - \Delta \xi \mathbb{I}) \right) dx \\ &= \int_{\Omega} \left( \frac{\Gamma_1(u)}{h''(u)} \xi_{GHG} - \frac{\Gamma_1'(u)}{h''(u)} \xi_L \xi_G^2 + \Gamma_1(u) \xi_H^2 - \Gamma_1(u) \xi_L^2 \right) dx, \end{aligned}$$

where we have set  $\xi_{GHG} = \nabla \xi^\top \nabla^2 \xi \nabla \xi$  and  $\xi_H = |\nabla^2 \xi|$ . The boundary integral in (18) becomes

$$\int_{\partial \Omega} \Gamma_1(u) \left( \frac{1}{2} \nabla (|\nabla \xi|^2) - \Delta \xi \nabla \xi \right) \cdot \nu ds = \frac{1}{2} \int_{\partial \Omega} \Gamma_1(u) \nabla (|\nabla \xi|^2) \cdot \nu ds \geq 0,$$

since  $\Gamma_1(u) \leq 0$ ,  $\nabla \xi \cdot \nu = 0$  on  $\partial \Omega$ , and it holds that  $\nabla (|\nabla \xi|^2) \cdot \nu \leq 0$  on  $\partial \Omega$  for all smooth functions satisfying  $\nabla \xi \cdot \nu = 0$  on  $\partial \Omega$  [18, Prop. 4.2]. Here we need the convexity of  $\Omega$ . Thus, the first integration-by-parts formula becomes

$$(19) \quad J_1 := \int_{\Omega} \left( \frac{\Gamma_1'(u)}{h''(u)} \xi_{GHG} - \frac{\Gamma_1'(u)}{h''(u)} \xi_L \xi_G^2 + \Gamma_1(u) \xi_H^2 - \Gamma_1(u) \xi_L^2 \right) dx \geq 0.$$

The second formula reads as

$$(20) \quad \begin{aligned} 0 &= \int_{\Omega} \operatorname{div} (\Gamma_2(u) |\nabla \xi|^2 \nabla \xi) dx \\ &= \int_{\Omega} \left( \frac{\Gamma_2'(u)}{h''(u)} \xi_G^4 + 2\Gamma_2(u) \xi_{GHG} + \Gamma_2(u) \xi_L \xi_G^2 \right) dx =: J_2, \end{aligned}$$

where  $\Gamma_2$  is an arbitrary scalar function. The goal is to find functions  $\Gamma_1(u) \leq 0$  and  $\Gamma_2(u)$  such that  $G''(0) \leq G''(0) + J_1 + J_2 < 0$ .

According to [15], the computations simplify if we introduce the variables  $\xi_R$  and  $\xi_S$  satisfying

$$(d-1) \xi_G^2 \xi_S = \xi_{GHG} - \frac{1}{d} \xi_L \xi_G^2, \quad \xi_H^2 = \frac{1}{d} \xi_L^2 + d(d-1) \xi_S^2 + \xi_R^2.$$

The existence of  $\xi_R$  follows from the inequality

$$\xi_H^2 = |\nabla^2 \xi|^2 \geq \frac{1}{d} (\Delta \xi)^2 + \frac{d}{d-1} \left( \frac{\nabla \xi^\top \nabla^2 \xi \nabla \xi}{\nabla \xi^2} - \frac{\Delta \xi}{d} \right)^2 = \frac{1}{d} \xi_L^2 + d(d-1) \xi_S^2,$$

which is proven in [15, Lemma 2.1]. Then

$$(21) \quad G''(0) \leq G''(0) + J_1 + J_2 = - \int_{\Omega} (a_1 \xi_L^2 + a_2 \xi_L \xi_G^2 + a_3 \xi_G^4 + a_4 \xi_S \xi_G^2 + a_5 \xi_R^2 + a_6 \xi_S^2) dx,$$

where

$$\begin{aligned}
(22) \quad a_1 &= (C_{\text{RK}} + 1)h''(u)\mu(u)^2 + \left(1 - \frac{1}{d}\right)\Gamma_1(u), \\
a_2 &= (C_{\text{RK}} + 2)\mu(u)\mu'(u) + \left(1 - \frac{1}{d}\right)\frac{\Gamma_1'(u)}{h''(u)} - \left(\frac{2}{d} + 1\right)\Gamma_2(u), \\
a_3 &= \frac{\mu'(u)^2 - \Gamma_2'(u)}{h''(u)}, \quad a_4 = -(d-1)\left(\frac{\Gamma_1'(u)}{h''(u)} + 2\Gamma_2(u)\right), \\
a_5 &= -\Gamma_1(u), \quad a_6 = -d(d-1)\Gamma_1(u).
\end{aligned}$$

The aim now is to determine conditions on  $a_1, \dots, a_6$  such that the polynomial  $P(\xi) = a_1\xi_L^2 + a_2\xi_L\xi_G^2 + a_3\xi_G^4 + a_4\xi_S\xi_G^2 + a_5\xi_R^2 + a_6\xi_S^2$  is nonnegative as this implies that  $G''(0) \leq 0$ . In the general case, this leads to nonlinear ordinary differential equations for  $\Gamma_1$  and  $\Gamma_2$  which cannot be easily solved. A possible solution is to require that the coefficients of the mixed terms vanish, i.e.  $a_2 = a_4 = 0$ , and that the remaining coefficients are nonnegative. The case  $d = 1$  being simpler than the general case (since  $J_1$  is not necessary), we assume that  $d > 1$ . Then  $a_4 = 0$  implies that  $\Gamma_1'(u)/h''(u) = -2\Gamma_2(u)$ . Replacing  $\Gamma_1'(u)/h''(u)$  by  $-2\Gamma_2(u)$  in  $a_2 = 0$  gives

$$\Gamma_2(u) = \frac{C_{\text{RK}} + 2}{3}\mu(u)\mu'(u).$$

On the other hand, replacing  $\Gamma_2(u)$  by  $-\Gamma_1'(u)/(2h''(u))$  in  $a_2 = 0$ , we find that

$$\Gamma_1'(u) = -\frac{2}{3}(C_{\text{RK}} + 2)\mu(u)\mu'(u)h''(u)$$

or, after integration,

$$\Gamma_1(u) = -\frac{2}{3}(C_{\text{RK}} + 2)\int_{u_0}^u \mu(v)\mu'(v)h''(v)dv.$$

These functions have to satisfy the conditions

$$\begin{aligned}
a_1 \geq 0 \quad &\text{or} \quad \frac{d-1}{d}\Gamma_1(u) \geq -(C_{\text{RK}} + 1)h''(u)\mu(u)^2, \\
a_3 \geq 0 \quad &\text{or} \quad (C_{\text{RK}} + 2)\mu(u)\mu''(u) + (C_{\text{RK}} - 1)\mu'(u)^2 \leq 0, \\
a_5 \geq 0 \quad &\text{or} \quad \Gamma_1(u) \leq 0 \quad \text{for all } u,
\end{aligned}$$

Note that  $a_1 \geq 0$  and  $a_5 \geq 0$  correspond to (15) and (14), respectively. This shows that  $P(\xi) \geq 0$  for all  $\xi \in \mathbb{R}^4$  and  $G''(0) \leq 0$ .

If  $G''(0) = 0$ , the nonnegative polynomial  $P$ , which depends on  $x \in \Omega$  via  $\xi$ , has to vanish. In particular,  $a_3\xi_G^4 = a_3|\nabla u|^4 = 0$  in  $\Omega$ . As  $a_3 > 0$  by assumption,  $u(x) = \text{const.}$  for  $x \in \Omega$ . This contradicts the hypothesis that  $u$  is not a steady state. Consequently,  $G''(0) < 0$ , and we finish the proof by setting  $b(u) = -\Gamma_1(u)$ .  $\square$

## 4. POROUS-MEDIUM EQUATION

The results of the previous section can be applied in principle to the Runge-Kutta scheme for the porous-medium or fast-diffusion equation

$$(23) \quad \partial_t u = \Delta(u^\beta) \quad \text{in } \Omega, \quad t > 0, \quad \nabla u^\beta \cdot \nu = 0 \quad \text{on } \partial\Omega, \quad u(0) = u^0,$$

where  $\beta > 0$ . It can be seen that conditions (14)-(16) are not optimal for particular entropies. This is not surprising since we have neglected the mixed terms in the polynomial in (21) (i.e.  $a_2 = a_4 = 0$ ) which is not optimal. In this section, we make a different approach by making an ansatz for the functions  $\Gamma_1$  and  $\Gamma_2$ , considering both zeroth-order and first-order entropies.

**4.1. Zeroth-order entropies.** We prove the following result.

**Theorem 8.** *Let  $\Omega \subset \mathbb{R}^d$  be convex with smooth boundary. Let  $(u^k)$  be a sequence of (smooth) solutions to the Runge-Kutta scheme (2) for (23). Let the entropy be given by  $H[u] = \alpha^{-1}(\alpha+1)^{-1} \int_\Omega u^{\alpha+1} dx$  with  $\alpha > 0$ , let  $k \in \mathbb{N}$ , and let  $u^k$  be not the constant steady state of (23). There exists a nonempty region  $R_0(d) \subset (0, \infty)^2$  and  $\tau^k > 0$  such that for all  $(\alpha, \beta) \in R_0(d)$  and  $0 < \tau \leq \tau^k$ ,*

$$H[u^k] + \tau\beta \int_\Omega (u^k)^{\alpha+\beta-2} |\nabla u^k|^2 dx \leq H[u^{k-1}], \quad k \in \mathbb{N}.$$

In one space dimension, we have

$$\begin{aligned} \text{implicit Euler:} & & R_0(1) &= (0, \infty)^2, \\ \text{Runge-Kutta of order } p \geq 2: & & R_0(1) &= \{(\alpha, \beta) \in (0, \infty)^2 : -2 < \alpha - \beta < 1\}, \\ \text{explicit Euler:} & & R_0(1) &= \{(\alpha, \beta) \in (0, \infty)^2 : -1 < \alpha - \beta < 1\}. \end{aligned}$$

For the implicit Euler scheme, the theorem shows that any positive values for  $(\alpha, \beta)$  is admissible which corresponds to the continuous situation. For the Runge-Kutta case with  $C_{\text{RK}} = 1$ , our condition is more restrictive. As expected, the explicit Euler scheme requires the most restrictive condition. The set  $R_0(d)$  is illustrated in Figure 1 for  $d = 2$  and  $d = 10$ .

*Proof.* Since  $k \in \mathbb{N}$  is fixed, we set  $u := u^k$ . We choose the functions

$$\Gamma_1(u) = c_1 \beta^2 u^{2\beta-\alpha-1}, \quad \Gamma_2(u) = c_2 \beta^2 u^{2\beta-2\alpha-1}.$$

It holds  $h''(u) = u^{\alpha-1}$  and  $\mu(u) = \beta u^{\beta-\alpha}$ . Then the coefficients in (22) are as follows:

$$\begin{aligned} a_1 &= \beta^2 \left( (C_{\text{RK}} + 1) + \left(1 - \frac{1}{d}\right) c_1 \right) u^{2\beta-\alpha-1}, \\ a_2 &= \beta^2 \left( (C_{\text{RK}} + 2)(\beta - \alpha) + \left(1 - \frac{1}{d}\right) (2\beta - \alpha - 1) c_1 - \left(\frac{2}{d} + 1\right) c_2 \right) u^{2\beta-2\alpha-1}, \\ a_3 &= \beta^2 \left( (\beta - \alpha)^2 - (2\beta - 2\alpha - 1) c_2 \right) u^{2\beta-3\alpha-2}, \\ a_4 &= -\beta^2 (d - 1) \left( (2\beta - \alpha - 1) c_1 + 2c_2 \right) u^{2\beta-2\alpha-1}, \\ a_5 &= -\beta^2 c_1 u^{2\beta-\alpha-1}, \quad a_6 = -\beta^2 d (d - 1) c_1 u^{2\beta-\alpha-1}. \end{aligned}$$

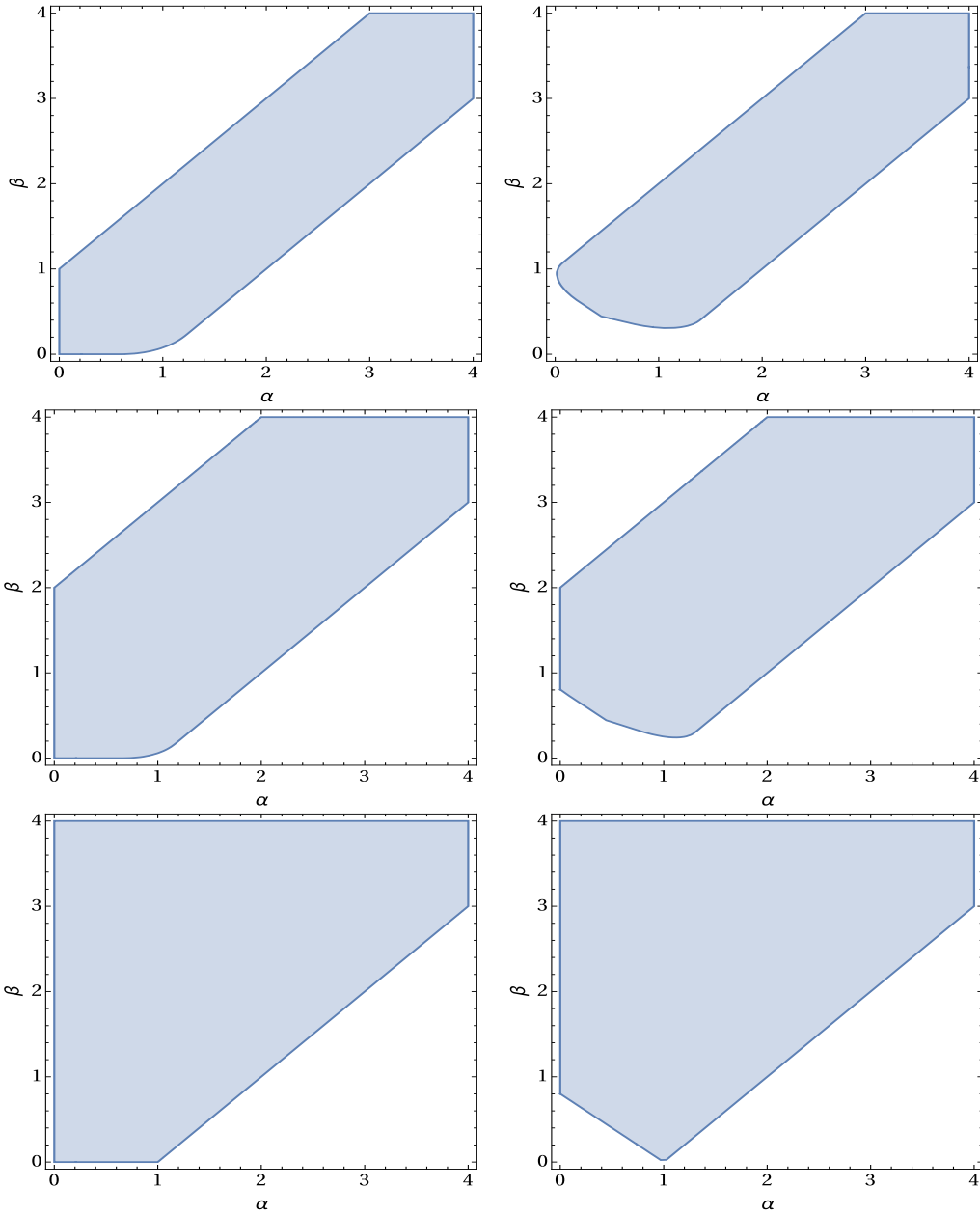


FIGURE 1. Set  $R_0(d)$  of all  $(\alpha, \beta)$  for which the zeroth-order entropy is dissipating. Left column:  $d = 2$ , right column:  $d = 10$ . Top row: explicit Euler scheme with  $C_{\text{RK}} = 2$ , middle row: implicit Euler scheme with  $C_{\text{RK}} = 1$ , bottom row: Runge-Kutta scheme of order  $p \geq 2$  with  $C_{\text{RK}} = 0$ .

Introducing the variables  $\eta_j = \xi_j/u^\alpha$  for  $j \in \{G, L, R, S\}$ , we can write (21) as

$$G''(0) \leq G'''(0) + J_1 + J_2 = -\beta^2 \int_{\Omega} u^{2\beta+\alpha-1} Q(\eta) dx,$$



where  $Q(\eta) = b_1\eta_L^2 + b_2\eta_L\eta_G^2 + b_3\eta_G^4 + b_4\eta_S\eta_G^2 + b_5\eta_R^2 + b_6\eta_S^2$

with coefficients

$$\begin{aligned} b_1 &= (C_{\text{RK}} + 1) + (1 - \frac{1}{d})c_1, \\ b_2 &= (C_{\text{RK}} + 2)(\beta - \alpha) + (1 - \frac{1}{d})(2\beta - \alpha - 1)c_1 - (\frac{2}{d} + 1)c_2, \\ b_3 &= (\beta - \alpha)^2 - (2\beta - 2\alpha - 1)c_2, \\ b_4 &= -(d - 1)((2\beta - \alpha - 1)c_1 + 2c_2), \\ b_5 &= -c_1, \quad b_6 = -d(d - 1)c_1. \end{aligned}$$

We need to determine all  $(\alpha, \beta)$  such that there exist  $c_1 \leq 0$ ,  $c_2 \in \mathbb{R}$  such that  $Q(\eta) \geq 0$  for all  $\eta = (\eta_G, \eta_L, \eta_R, \eta_S)$ . Without loss of generality, we exclude the cases  $b_1 = b_2 = 0$  and  $b_4 = b_6 = 0$  since they lead to parameters  $(\alpha, \beta)$  included in the region calculated below. Thus, let  $b_1 > 0$  and  $b_6 > 0$ . These inequalities give the bound  $-(C_{\text{RK}} + 1)/(1 - 1/d) < c_1 < 0$ . Thus, we may introduce the parameter  $\lambda \in (0, 1)$  by setting  $c_1 = -\lambda(C_{\text{RK}} + 1)/(1 - 1/d)$ . The polynomial  $Q(\eta)$  can be rewritten as

$$\begin{aligned} Q(\eta) &= b_1 \left( \eta_L + \frac{b_2}{2b_1} \eta_G^2 \right)^2 + b_6 \left( \eta_S + \frac{b_4}{2b_6} \eta_G^2 \right)^2 + b_5 \eta_R^2 + \eta_G^4 \left( b_3 - \frac{b_2^2}{4b_1} - \frac{b_4^2}{4b_6} \right) \\ &\geq \eta_G^4 \left( b_3 - \frac{b_4^2}{4b_6} - \frac{b_2^2}{4b_1} \right) =: \frac{\eta_G^4 (C_{\text{RK}} + 1)}{4b_1 b_6} R(c_2; \lambda, \alpha, \beta), \end{aligned}$$

where  $R(c_2; \lambda, \alpha, \beta)$  is a quadratic polynomial in  $c_2$  with the nonpositive leading term  $-d^2(4 - 3\lambda) + 4(2 - 3\lambda)d - 4$ . The polynomial  $R(c_2; \lambda, \alpha, \beta)$  is nonnegative for some  $c_2$  if and only if its discriminant  $4d^2\lambda(1 - \lambda)S(\lambda; \alpha, \beta)$  is nonnegative. Here,  $S(\lambda; \alpha, \beta)$  is a quadratic polynomial in  $\lambda$ . In order to derive the conditions on  $(\alpha, \beta)$  such that  $S(\lambda; \alpha, \beta) \geq 0$  for some  $\lambda \in (0, 1)$ , we employ the computer-algebra system `Mathematica`. The result of the command

```
Resolve[Exists[LAMBDA, S[LAMBDA] >= 0 && LAMBDA > 0
&& LAMBDA < 1], Reals]
```

gives all  $(\alpha, \beta) \in \mathbb{R}^2$  such that there exist  $c_1 \leq 0$ ,  $c_2 \in \mathbb{R}$  such that  $Q(\eta) \geq 0$ . The interior of this region equals the set  $R_0(d)$ , defined in the statement of the theorem. This shows that  $G''(0) \leq 0$  for all  $(\alpha, \beta) \in R_0(d)$ .

If  $G''(0) = 0$ , the nonnegative polynomial  $Q$  has to vanish. In particular,  $b_1\eta_L^2 = 0$ . If  $\eta_L = 0$  in  $\Omega$ , the boundary conditions imply that  $u$  is constant, which contradicts our assumption that  $u$  is not the steady state. Thus  $b_1 = 0$ . Similarly,  $b_2 = b_3 = b_4 = 0$ . This gives a system of four inhomogeneous linear equations for  $(c_1, c_2)$  which is unsolvable. Consequently,  $G''(0) < 0$ .

The set  $R_0(d)$  is nonempty since, e.g.,  $(1, 1) \in R_0(d)$ . Indeed, choosing  $c_1 = -1$  and  $c_2 = 0$ , we find that  $Q(\eta) = (C_{\text{RK}} + \frac{1}{d})\eta_L^2 + \eta_R^2 + d(d - 1)\eta_S^2 \geq 0$ .

In one space dimension, the situation simplifies since the Laplacian coincides with the Hessian and thus, the integration-by-parts formula (19) is not needed. Then (see (20))

$$G''(0) = G'''(0) + J_1 = -\beta^2 \int_{\Omega} u^{2\beta+\alpha-1} (a_1 \xi_L^2 + a_2 \xi_L \xi_G^2 + a_3 \xi_G^4) dx,$$

where

$$a_1 = C_{\text{RK}} + 1, \quad a_2 = (C_{\text{RK}} + 2)(\beta - \alpha) - 3c_2, \quad a_3 = (\beta - \alpha)^2 - (2\beta - 2\alpha - 1)c_2.$$

The polynomial  $P(\xi) = \xi_G^4(a_1 y^2 + a_2 y + a_3)$  with  $y = \xi_L/\xi_G^2$  is nonnegative if and only if  $a_1 \geq 0$  and  $4a_1 a_3 - a_2^2 \geq 0$ , which is equivalent to

$$(24) \quad -9c_2^2 + 2((C_{\text{RK}} - 2)(\alpha - \beta) + 2(C_{\text{RK}} + 1))c_2 - C_{\text{RK}}^2(\alpha - \beta)^2 \geq 0.$$

This inequality has a solution  $c_2 \in \mathbb{R}$  if and only if the quadratic polynomial has real roots, i.e. if its discriminant is nonnegative,

$$\begin{aligned} 0 &\leq ((C_{\text{RK}} - 2)(\alpha - \beta) + 2(C_{\text{RK}} + 1))^2 - 9C_{\text{RK}}^2(\alpha - \beta)^2 \\ &= 4(C_{\text{RK}} + 1) \left( -(2C_{\text{RK}} - 1)(\alpha - \beta)^2 + (C_{\text{RK}} - 2)(\alpha - \beta) + (C_{\text{RK}} + 1) \right). \end{aligned}$$

The polynomial  $-(2C_{\text{RK}} - 1)z^2 + (C_{\text{RK}} - 2)z + (C_{\text{RK}} + 1)$  with  $z = \alpha - \beta$  is always nonnegative if  $C_{\text{RK}} = 0$  (implicit Euler). For  $C_{\text{RK}} = 1$  and  $C_{\text{RK}} = 2$ , this property holds if and only if  $-(C_{\text{RK}} + 1)/(2C_{\text{RK}} - 1) \leq \alpha - \beta \leq 1$ . This concludes the proof.  $\square$

**4.2. First-order entropies.** We consider the one-dimensional case and first-order entropies with  $f(u) = u^{\alpha/2}$ ,  $\alpha > 0$ .

**Theorem 9.** *Let  $\Omega \subset \mathbb{R}$  be a bounded interval. Let  $(u^k)$  be a sequence of (smooth) solutions to the Runge-Kutta scheme (2) of order  $p \geq 2$  for (23) in one space dimension. Let the entropy be given by  $F[u] = \int_{\Omega} (u^{\alpha/2})_x^2 dx$  with  $\alpha > 0$ , let  $k \in \mathbb{N}$  be fixed, and let  $u^k$  be not the constant steady state of (23). There exists a nonempty region  $R_1 \in [0, \infty)^2$  and  $\tau^k > 0$  such that for all  $(\alpha, \beta) \in R_1$ , there is a constant  $C_{\alpha, \beta} > 0$  such that for all  $0 < \tau \leq \tau^k$ ,*

$$F[u^k] + \tau C_{\alpha, \beta} \int_{\Omega} (u^k)^{\alpha+\beta-3} (u_{xx}^k)^2 dx \leq F[u^{k-1}], \quad k \in \mathbb{N}.$$

Figure 2 illustrates the set  $R_1$ . The set of admissible values  $(\alpha, \beta)$  for the continuous equation is given by  $\{-2 \leq \alpha - 2\beta < 1\}$  (the borders of this set are depicted in the figure by the dashed lines).

*Proof.* First, we compute  $G'(0)$  according to Theorem 3:

$$G'(0) = -\alpha \int_{\Omega} u^{\alpha/2-1} (u^{\alpha/2})_{xx} (u^{\beta})_{xx} dx.$$

We show that  $G'(0)$  is nonpositive in a certain range of values  $(\alpha, \beta)$ . We formulate  $G'(0)$  as

$$G'(0) = -\frac{\alpha^2 \beta}{4} \int_{\Omega} u^{\alpha+\beta-1} ((\alpha - 2)(\beta - 1)\xi_1^4 + (\alpha + 2\beta - 4)\xi_1^2 \xi_2 + 2\xi_2^2) dx,$$

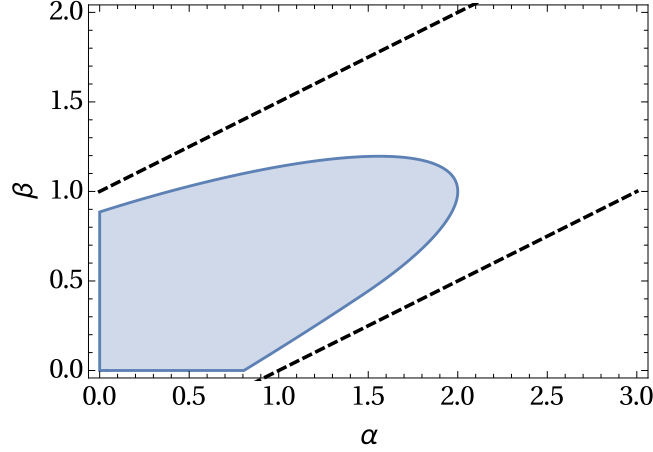


FIGURE 2. Set of all  $(\alpha, \beta)$  for which the discrete first-order entropy for solutions to the one-dimensional porous-medium equation is dissipating. The continuous first-order entropy is dissipated for  $-2 \leq \alpha - 2\beta < 1$ . The borders of this set is indicated in the figure by dashed lines.

where  $\xi_1 = u_x/u$ ,  $\xi_2 = u_{xx}/u$ . We employ the integration-by-parts formula

$$0 = \int_{\Omega} (u^{\alpha+\beta-4} u_x^3)_x dx = \int_{\Omega} u^{\alpha+\beta-1} ((\alpha + \beta - 4)\xi_1^4 + 3\xi_1^2 \xi_2) dx =: J.$$

Therefore,

$$G'(0) = G'(0) - \frac{\alpha^2 \beta}{4} c J = -\frac{\alpha^2 \beta}{4} \int_{\Omega} u^{\alpha+\beta-1} P(\xi) dx,$$

where

$$P(\xi) = ((\alpha - 2)(\beta - 1) + (\alpha + \beta - 4)c)\xi_1^4 + (\alpha + 2\beta - 4 + 3c)\xi_1^2 \xi_2 + 2\xi_2^2.$$

This polynomial is nonnegative if and only if

$$8((\alpha - 2)(\beta - 1) + (\alpha + \beta - 4)c) - (\alpha + 2\beta - 4 + 3c)^2 \geq 0,$$

which is equivalent to

$$g(c) := -9c^2 + 2(\alpha - 2\beta - 4)c - (\alpha - 2\beta)^2 \geq 0.$$

The maximizing value  $c^* = (\alpha - 2\beta - 4)/9$ , obtained from  $g'(c) = 0$ , yields

$$g(c^*) = -\frac{8}{9}(\alpha - 2\beta - 1)(\alpha - 2\beta + 2) \geq 0$$

and consequently  $G'(0) \leq 0$  if  $-2 \leq \alpha - 2\beta \leq 1$ . This condition is the same as in [6, Theorem 13] for the continuous equation.

Next, we turn to the proof of  $G''(0) < 0$ . The proof of Theorem 3 shows that

$$G''(0) = -\frac{\alpha}{2} \int_{\Omega} \left( \frac{\alpha}{2} (u^{\alpha/2-1} (u^\beta)_{xx})_x^2 - \left( \frac{\alpha}{2} - 1 \right) u^{\alpha/2-2} (u^{\alpha/2})_{xx} (u^\beta)_{xx}^2 \right)$$

$$- \beta C_{\text{RK}} u^{\alpha/2-1} (u^{\alpha/2})_{xx} (u^{\beta-1} (u^\beta)_{xx})_{xx} dx.$$

We integrate by parts in the last term and use  $(\beta u^{\beta-1} (u^\beta)_{xx})_x = 0$  on  $\partial\Omega$ :

$$G''(0) = -\frac{1}{8} \alpha^2 \beta^2 \int_{\Omega} u^{\alpha+2\beta-2} \times (a_1 \xi_1^6 + a_2 \xi_1^4 \xi_2 + a_3 \xi_1^3 \xi_3 + a_4 \xi_1^2 \xi_2^2 + a_5 \xi_1 \xi_2 \xi_3 + a_6 \xi_2^3 + a_7 \xi_3^2) dx,$$

where  $\xi_1 = u_x/u$ ,  $\xi_2 = u_{xx}/u$ ,  $\xi_3 = u_{xxx}/u$ , and

$$a_1 = (\beta - 1)(2C_{\text{RK}}\alpha^2\beta - 3C_{\text{RK}}\alpha^2 + 2\alpha\beta^2 - 2(5C_{\text{RK}} + 3)\alpha\beta + (15C_{\text{RK}} + 4)\alpha + 2\beta^3 - 14\beta^2 + 4(3C_{\text{RK}} + 7)\beta - 2(9C_{\text{RK}} + 8)),$$

$$a_2 = (\beta - 1)(4C_{\text{RK}}\alpha^2 + (8C_{\text{RK}} + 7)\alpha\beta - (32C_{\text{RK}} + 9)\alpha + 12\beta^2 - 2(8C_{\text{RK}} + 25)\beta + 6(8C_{\text{RK}} + 7)),$$

$$a_3 = C_{\text{RK}}\alpha^2 + 2\alpha\beta - (5C_{\text{RK}} + 2)\alpha + 4(C_{\text{RK}} + 1)\beta^2 - 2(5C_{\text{RK}} + 8)\beta + 12(C_{\text{RK}} + 1),$$

$$a_4 = 2(\beta - 1)(2(4C_{\text{RK}} + 1)\alpha + 9\beta - (16C_{\text{RK}} + 13)),$$

$$a_5 = 2(2C_{\text{RK}} + 1)\alpha + 4(2C_{\text{RK}} + 3)\beta - 16(C_{\text{RK}} + 1),$$

$$a_6 = 2 - \alpha, \quad a_7 = 2(C_{\text{RK}} + 1).$$

We employ three integration-by-parts formulas:

$$0 = \int_{\Omega} (u^{\alpha+2\beta-5} u_{xx}^2 u_x)_x dx = \int_{\Omega} u^{\alpha+2\beta-2} ((\alpha + 2\beta - 5)\xi_1^2 \xi_2^2 + 2\xi_1 \xi_2 \xi_3 + \xi_3^2) dx =: J_1,$$

$$0 = \int_{\Omega} (u^{\alpha+2\beta-6} u_{xx} u_x^3)_x dx = \int_{\Omega} u^{\alpha+2\beta-2} ((\alpha + 2\beta - 6)\xi_1^4 \xi_2 + \xi_1^3 \xi_3 + 3\xi_1^2 \xi_2^2) dx =: J_2,$$

$$0 = \int_{\Omega} (u^{\alpha+2\beta-7} u_x^5)_x dx = \int_{\Omega} u^{\alpha+2\beta-2} ((\alpha + 2\beta - 7)\xi_1^6 + 5\xi_1^4 \xi_2) dx =: J_3.$$

Then

$$G''(0) = G''(0) - \frac{1}{8} \alpha^2 \beta^2 (c_1 J_1 + c_2 J_2 + c_3 J_3) = -\frac{1}{8} \alpha^2 \beta^2 \int_{\Omega} u^{\alpha+2\beta-2} P(\xi) dx,$$

$$\text{where } P(\xi) = b_1 \xi_1^6 + b_2 \xi_1^4 \xi_2 + b_3 \xi_1^3 \xi_3 + b_4 \xi_1^2 \xi_2^2 + b_5 \xi_1 \xi_2 \xi_3 + b_6 \xi_2^3 + b_7 \xi_3^2,$$

and the coefficients are given by

$$b_1 = a_1 + (\alpha + 2\beta - 7)c_3,$$

$$b_2 = a_2 + (\alpha + 2\beta - 6)c_2 + 5c_3,$$

$$b_3 = a_3 + c_2,$$

$$b_4 = a_4 + (\alpha + 2\beta - 5)c_1 + 3c_2,$$

$$b_5 = a_5 + 2c_1,$$

$$b_6 = a_6 + c_1,$$

$$b_7 = a_7.$$

Choosing  $c_1 = -a_6$ , we eliminate the cubic term  $\xi_2^3$ . Furthermore, setting,  $x = \xi_2/\xi_1^2$  and  $y = \xi_3/\xi_1^3$ , we can write the polynomial  $P$  as a quadratic polynomial in  $(x, y)$ :

$$Q(x, y) = \xi_1^6 P(\xi) = b_1 + b_2 x + b_3 y + b_4 x^2 + b_5 xy + b_7 y^2.$$

The following lemma is a consequence of the proof of Lemma 2.2 in [16].

**Lemma 10.** *The polynomial  $p(x, y) = A + Bx + Cy + Dx^2 + Exy + Fy^2$  with  $F > 0$  is nonnegative for all  $(x, y) \in \mathbb{R}^2$  if and only if*

- (i)  $4DF - E^2 > 0$  and  $A(4DF - E^2) - B^2F - C^2D + BCE \geq 0$ , or
- (ii)  $4DF - E^2 = 0$  and  $2BF - CE = 0$  and  $4AF - C^2 \geq 0$ .

Note that in case  $4DF - E^2 = 0$  and  $E \neq 0$ , we may replace  $2BF - CE = 0$  by the condition  $2BEF = CE^2 = 4CDF$  or (since  $F > 0$ )  $BE = 2CD$ .

The first inequality in case (i),

$$0 < 4b_4b_7 - b_5^2 = -(C_{\text{RK}} + 1)(2C_{\text{RK}} + 1)\alpha^2 + (2C_{\text{RK}} + 2)(4C_{\text{RK}} - 3)\alpha\beta + (9C_{\text{RK}} + 9)\alpha - 2C_{\text{RK}}(4C_{\text{RK}} + 3)\beta^2 + (8C_{\text{RK}} + 12)\beta + (3C_{\text{RK}} + 3)c_2 - (12C_{\text{RK}} + 14),$$

is linear in  $c_2$  and provides a lower bound for  $c_2$ :

$$c_2 > \frac{1}{3(C_{\text{RK}} + 1)} \left( (C_{\text{RK}} + 1)(2C_{\text{RK}} + 1)\alpha^2 - (2C_{\text{RK}} + 2)(4C_{\text{RK}} - 3)\alpha\beta - (9C_{\text{RK}} + 9)\alpha + 2C_{\text{RK}}(4C_{\text{RK}} + 3)\beta^2 - (8C_{\text{RK}} + 12)\beta + (12C_{\text{RK}} + 14) \right) =: c_2^*.$$

The second inequality in case (i) becomes

$$0 \leq b_1(4b_4b_7 - b_5^2) - b_2^2b_7 - b_3^2b_4 + b_2b_3b_5 = -50(C_{\text{RK}} + 1)c_3^2 + p_1(\alpha, \beta, c_2)c_3 + p_2(\alpha, \beta, c_2),$$

where  $p_1$  and  $p_2$  are some polynomials in  $\alpha$ ,  $\beta$ , and  $c_2$ . This quadratic expression in  $c_3$  is nonnegative if and only if its discriminant is nonnegative,

$$\begin{aligned} 0 &\leq -200(C_{\text{RK}} + 1)p_2(\alpha, \beta, c_2) - p_1(\alpha, \beta, c_2)^2 \\ &= -8(4b_4b_7 - b_5^2)(25c_2^2 + p_3(\alpha, \beta)c_2 + p_4(\alpha, \beta)), \end{aligned}$$

where  $p_3(\alpha, \beta)$  and  $p_4(\alpha, \beta)$  are some polynomials in  $\alpha$  and  $\beta$ . The factor  $4b_4b_7 - b_5^2$  is positive, so we have to ensure that  $R_{\alpha, \beta}(c_2) = 25c_2^2 + p_3(\alpha, \beta)c_2 + p_4(\alpha, \beta) \leq 0$  for some  $c_2 > c_2^*$ . Therefore we must ensure that the rightmost root of  $R_{\alpha, \beta}(c_2)$  is larger or equal than the lower bound for  $c_2$ , i.e.,  $-p_3(\alpha, \beta) + \sqrt{p_3^2(\alpha, \beta) - 100p_4(\alpha, \beta)} \geq 50c_2^*$ . For  $C_{\text{RK}} = 1$ , the values  $(\alpha, \beta)$  for which there exists  $c_2 > c_2^*$  such that  $R_{\alpha, \beta}(c_2) \leq 0$  is depicted in Figure 2. In case (ii), we may immediately calculate  $c_2$  and  $c_3$  but this results in a region which is already contained in the first one. This shows that  $G''(0) \leq 0$ .

If  $G''(0) = 0$ , the polynomial  $Q$  vanishes. Thus, either  $u_x/u = \xi_1 = 0$  or  $P(\xi) = 0$  in  $\Omega$ . The first case is impossible since  $u$  is not constant in  $\Omega$ . As  $b_7 = a_7 = 2(C_{\text{RK}} + 1) > 0$ , the second case  $P(\xi) = 0$  implies that  $\xi_3 = 0$ . Hence,  $u$  is a quadratic polynomial. In view of the boundary conditions,  $u$  must be constant, but this contradicts our assumption. Hence,  $G''(0) < 0$ .  $\square$

## 5. LINEAR DIFFUSION SYSTEM

We consider the following linear diffusion system:

$$(25) \quad \partial_t u_1 - \rho_1 \Delta u_1 = \mu(u_2 - u_1), \quad \partial_t u_2 - \rho_2 \Delta u_2 = \mu(u_1 - u_2),$$

with initial and homogeneous Neumann boundary conditions,  $\rho_1, \rho_2, \mu > 0$ , and the entropy

$$(26) \quad H[u] = \int_{\Omega} h(u) dx = \int_{\Omega} \sum_{i=1}^2 u_i (\log u_i - 1) dx,$$

where  $u = (u_1, u_2)$ . If the initial data is nonnegative, the maximum principle shows that the solutions to (25) are nonnegative too.

**Theorem 11.** *Let  $(u^k)$  be a sequence of (smooth) nonnegative solutions to the Runge-Kutta scheme (2) for (25) with  $C_{\text{RK}} = 1$  and  $\rho := \rho_1 = \rho_2$ . Let the entropy  $H$  be given by (26). Let  $k \in \mathbb{N}$  be fixed and let  $u^k$  be not the steady state of (2). Then there exists  $\tau^k > 0$  such that for all  $0 < \tau < \tau^k$ ,*

$$H[u^k] + \tau \int_{\Omega} \left( \rho \sum_{i=1}^2 \frac{|\nabla u_i^k|^2}{u_i^k} + \mu (\log u_1^k - \log u_2^k) (u_1^k - u_2^k) \right) dx \leq H[u^{k-1}].$$

Note that we need equal diffusivities  $\rho_1 = \rho_2$  and higher-order schemes ( $C_{\text{RK}} = 1$ ). These conditions are in accordance of [18], where the continuous equation was studied. In order to highlight the step where these conditions are needed, the following proof is slightly more general than actually needed.

*Proof.* We fix  $k \in \mathbb{N}$  and set  $u := u^k$ . Let  $A[u] = (A_1[u], A_2[u]) = (\rho_1 \Delta u_1 + \mu(u_2 - u_1), \rho_2 \Delta u_2 + \mu(u_1 - u_2))$ . Since  $A$  is linear,  $DA[u](h) = A[h]$ . Thus,

$$G''(0) = - \int_{\Omega} (C_{\text{RK}} h'(u)^\top A[A[u]] + A[u]^\top h''(u) A[u]) dx = -G_1 - G_2.$$

In the following, we set  $\partial_i h = \partial h / \partial u_i$  for  $i = 1, 2$ . We integrate by parts twice, using the boundary conditions  $\nabla u_i \cdot \nu = 0$  and  $\nabla A_i[u] \cdot \nu = 0$  on  $\partial\Omega$ , and collect the terms:

$$\begin{aligned} G_1 &= C_{\text{RK}} \int_{\Omega} \left( \partial_1 h(u) (\rho_1 \Delta A_1[u] + \mu(A_2[u] - A_1[u])) \right. \\ &\quad \left. + \partial_2 h(u) (\rho_2 \Delta A_2[u] + \mu(A_1[u] - A_2[u])) \right) dx \\ &= C_{\text{RK}} \int_{\Omega} \left( \rho_1 \Delta \partial_1 h(u) A_1[u] + \rho_2 \Delta \partial_2 h(u) A_2[u] \right. \\ &\quad \left. + \mu (\partial_1 h(u) - \partial_2 h(u)) (A_2[u] - A_1[u]) \right) dx \\ &= C_{\text{RK}} \int_{\Omega} \left( \rho_1 (\partial_1^2 h(u) \Delta u_1 + \partial_1^3 h(u) |\nabla u_1|^2) (\rho_1 \Delta u_1 + \mu(u_2 - u_1)) \right. \\ &\quad \left. + \rho_2 (\partial_2^2 h(u) \Delta u_2 + \partial_2^3 h(u) |\nabla u_2|^2) (\rho_2 \Delta u_2 + \mu(u_1 - u_2)) \right) \end{aligned}$$

$$\begin{aligned}
& + \mu(\partial_2 h(u) - \partial_1 h(u))(\rho_1 \Delta u_1 - \rho_2 \Delta u_2 + 2\mu(u_2 - u_1)) dx \\
= & C_{\text{RK}} \int_{\Omega} \left( \rho_1^2 \partial_1^2 h(u) (\Delta u_1)^2 + \rho_2^2 \partial_2^2 h(u) (\Delta u_2)^2 + \rho_1^2 \partial_1^3 h(u) \Delta u_1 |\nabla u_1|^2 \right. \\
& + \rho_2^2 \partial_2^3 h(u) \Delta u_2 |\nabla u_2|^2 + \rho_1 \mu (\partial_1^2 h(u) (u_2 - u_1) + \partial_2 h(u) - \partial_1 h(u)) \Delta u_1 \\
& + \rho_2 \mu (\partial_2^2 h(u) (u_1 - u_2) + \partial_1 h(u) - \partial_2 h(u)) \Delta u_2 + \rho_1 \mu \partial_1^3 h(u) (u_2 - u_1) |\nabla u_1|^2 \\
& \left. + \rho_2 \mu \partial_2^3 h(u) (u_1 - u_2) |\nabla u_2|^2 + 2\mu^2 (\partial_2 h(u) - \partial_1 h(u)) (u_2 - u_1) \right) dx.
\end{aligned}$$

Furthermore,

$$\begin{aligned}
G_2 &= \int_{\Omega} \left( \partial_1^2 h(u) (\rho_1 \Delta u_1 + \mu(u_2 - u_1))^2 + \partial_2^2 h(u) (\rho_2 \Delta u_2 + \mu(u_1 - u_2))^2 \right) dx \\
&= \int_{\Omega} \left( \rho_1^2 \partial_1^2 h(u) (\Delta u_1)^2 + \rho_2^2 \partial_2^2 h(u) (\Delta u_2)^2 + 2\rho_1 \mu \partial_1^2 h(u) (u_2 - u_1) \Delta u_1 \right. \\
&\quad \left. + 2\rho_2 \mu \partial_2^2 h(u) (u_1 - u_2) \Delta u_2 + \mu^2 (\partial_1^2 h(u) + \partial_2^2 h(u)) (u_1 - u_2)^2 \right) dx.
\end{aligned}$$

Adding  $G_1$  and  $G_2$ , we arrive at

$$\begin{aligned}
G''(0) &= - \sum_{i=1}^2 \int_{\Omega} \left( \rho_i^2 (C_{\text{RK}} + 1) \partial_i^2 h(u) (\Delta u_i)^2 + \rho_i^2 C_{\text{RK}} \partial_i^3 h(u) \Delta u_i |\nabla u_i|^2 \right) dx \\
&\quad - \int_{\Omega} \left( \rho_1 \mu ((C_{\text{RK}} + 2) \partial_1^2 h(u) (u_2 - u_1) + C_{\text{RK}} (\partial_2 h(u) - \partial_1 h(u))) \Delta u_1 \right. \\
&\quad + \rho_2 \mu ((C_{\text{RK}} + 2) \partial_2^2 h(u) (u_1 - u_2) + C_{\text{RK}} (\partial_1 h(u) - \partial_2 h(u))) \Delta u_2 \\
&\quad + \rho_1 \mu C_{\text{RK}} \partial_1^3 h(u) (u_2 - u_1) |\nabla u_1|^2 + \rho_2 \mu C_{\text{RK}} \partial_2^3 h(u) (u_1 - u_2) |\nabla u_2|^2 \left. \right) dx \\
&\quad - \int_{\Omega} \mu^2 \left( 2(\partial_1 h(u) - \partial_2 h(u)) + (\partial_1^2 h(u) + \partial_2^2 h(u)) (u_1 - u_2) \right) (u_1 - u_2) dx \\
&= -I_2 - I_1 - I_0.
\end{aligned}$$

The idea of [18] is to show that each integral  $I_i$ , involving only derivatives of order  $i$ , is nonnegative. In contrast to [18], we employ systematic integration by parts, which allows for a simpler and more general proof in our context. For the term  $I_2$ , we use the following integration-by-parts formula:

$$0 = \int_{\Omega} \operatorname{div} (u_i^{-2} |\nabla u_i|^3) dx = \int_{\Omega} (-2u_i^{-3} |\nabla u_i|^4 + 3u_i^{-2} \Delta u_i |\nabla u_i|^2) dx =: J_i.$$

Then, for  $\varepsilon > 0$ ,

$$I_2 - c \sum_{i=1}^2 \rho_i^2 J_i - \varepsilon \sum_{i=1}^2 u_i^{-3} |\nabla u_i|^4 dx$$

$$= \sum_{i=1}^2 \rho_i^2 \int_{\Omega} \left( (C_{\text{RK}} + 1)u_i^{-1}(\Delta u_i)^2 - (3c + C_{\text{RK}})u_i^{-2}\Delta u_i|\nabla u_i|^2 + (2c - \varepsilon)u_i^{-3}|\nabla u_i|^4 \right) dx.$$

The integrand defines a quadratic polynomial in the variables  $\Delta u_i$  and  $|\nabla u_i|^2$  and is non-negative if its discriminant satisfies  $4(2c - \varepsilon)(C_{\text{RK}} + 1) - (3c + C_{\text{RK}})^2 \geq 0$ . It turns out that this inequality holds true for  $C_{\text{RK}} \in \{0, 1\}$  if we choose  $c = 2/3$  and  $\varepsilon > 0$  sufficiently small. When  $C_{\text{RK}} = 2$ , we can show only that  $I_2 \geq 0$  which is not sufficient to prove that  $G''(0) < 0$  (see below). We conclude that

$$(27) \quad I_2 \geq \varepsilon \sum_{i=1}^2 \int_{\Omega} u_i^{-3} |\nabla u_i|^4 dx.$$

Integrating by parts in  $I_1$  in order to obtain only first-order derivatives, we find after some rearrangements that

$$I_1 = \mu \int_{\Omega} (a_1 |\nabla \log u_1|^2 + a_2 \nabla \log u_1 \cdot \nabla \log u_2 + a_3 |\nabla \log u_2|^2) dx, \quad \text{where}$$

$$a_1 = 2\rho_1(C_{\text{RK}}u_1 + u_2), \quad a_3 = 2\rho_2(C_{\text{RK}}u_2 + u_1),$$

$$a_2 = -(C_{\text{RK}}(\rho_1 + \rho_2) + 2\rho_2)u_1 - (C_{\text{RK}}(\rho_1 + \rho_2) + 2\rho_1)u_2.$$

The integrand is nonnegative if and only if  $4a_1a_3 - a_2^2 \geq 0$  for all  $(u_1, u_2)$ . We compute:

$$C_{\text{RK}} = 0: \quad 4a_1a_3 - a_2^2 = -4(\rho_1u_2 - \rho_2u_1)^2,$$

$$C_{\text{RK}} = 1: \quad 4a_1a_3 - a_2^2 = (\rho_1 - \rho_2)(\rho_1(u_1^2 + 6u_1u_2 + 9u_2^2) - \rho_2(9u_1^2 + 6u_1u_2 + u_2^2)),$$

$$C_{\text{RK}} = 2: \quad 4a_1a_3 - a_2^2 = -4(\rho_1(u_1 + 2u_2) - \rho_2(2u_1 + u_2)).$$

Thus,  $4a_1a_3 - a_2^2 \geq 0$  is possible only if  $\rho_1 = \rho_2$  and  $C_{\text{RK}} = 1$ .

Finally, we see immediately that the remaining term

$$I_0 = \mu^2 \int_{\Omega} \left( 2(\log u_1 - \log u_2)(u_1 - u_2) + \left( \frac{1}{u_1} + \frac{1}{u_2} \right) (u_1 - u_2)^2 \right) dx$$

is nonnegative. This shows that  $G''(0) \leq 0$ . If  $G''(0) = 0$ , we infer from (27) that  $u_i = \text{const.}$ , but this contradicts our hypothesis that  $u_i$  is not a steady state.  $\square$

## 6. THE DERRIDA-LEBOWITH-SPEER-SPOHN EQUATION

Consider the one-dimensional fourth-order equation

$$(28) \quad \partial_t u = -(u(\log u)_{xx})_{xx} \quad \text{in } \Omega, \quad t > 0, \quad u(0) = u^0$$

with periodic boundary conditions. This equation appears as a scaling limit of the so-called (time-discrete) Toom model, which describes interface fluctuations in a two-dimensional spin system [9]. The variable  $u$  is the limit of a random variable related to the deviation of the spin interface from a straight line. The multi-dimensional version of (28) models the electron density  $u$  in a quantum semiconductor, and the equation is the zero-temperature, zero-field approximation of the quantum drift-diffusion model [13]. For existence results for (28), we refer to [15] and references therein.



To simplify our calculations, we analyze only the logarithmic entropy  $H[u] = \int_{\Omega} u(\log u - 1)dx$ . It is possible to verify condition (6) also for entropies of the form  $\int_{\Omega} u^{\alpha}dx$ , but it turns out that only sufficiently small  $\alpha > 0$  are admissible (about  $0 < \alpha < 0.15\dots$ ) and the computations are very tedious. Therefore, we restrict ourselves to the case  $\alpha = 0$ .

**Theorem 12.** *Let  $(u^k)$  be a sequence of (smooth) solutions to the Runge-Kutta scheme (2) with  $C_{\text{RK}} = 1$  for (28). Let the entropy be given by  $H[u] = \int_{\Omega} u(\log u - 1)dx$ , let  $k \in \mathbb{N}$  be fixed, and let  $u^k$  be not a steady state. Then there exists  $\tau^k > 0$  such that for all  $0 < \tau < \tau^k$ ,*

$$H[u^k] + \tau q \int_{\Omega} u(\log u)_x^8 dx + \tau \int_{\Omega} u(\log u)_{xx}^2 dx \leq H[u^{k-1}], \quad q \approx 0.0045.$$

*Proof.* First, we observe that  $G'(0) = -\int_{\Omega} (u(\log u)_{xx})_{xx} \log u dx = -\int_{\Omega} u(\log u)_{xx}^2 dx$ . With  $A[u] = (u(\log u)_{xx})_{xx}$  and  $DA[u](h) = (h_{xx} - 2(\log u)_x h_x + (\log u)_x^2 h)_{xx}$ , we can write  $G''(0) = -I_0^k$  according to (6) as

$$\begin{aligned} G''(0) &= -\int_{\Omega} \left( \log u (A[u]_{xx} - 2(\log u)_x A[u]_x + (\log u)_x^2 A[u])_{xx} + \frac{1}{u} A[u]^2 \right) dx \\ &= -\int_{\Omega} \left( (\log u)_{xx} (A[u]_{xx} - 2(\log u)_x A[u]_x + (\log u)_x^2 A[u]) + \frac{1}{u} A[u]^2 \right) dx \\ &= -\int_{\Omega} \left( (v_{xxxx} + 2(v_x v_{xx})_x + v_x^2 v_{xx}) A[u] + \frac{1}{u} A[u]^2 \right) dx, \end{aligned}$$

where we have integrated by parts several times and have set  $v = \log u$ . Then  $A[u] = u(v_x^2 v_{xx} + 2v_x v_{xxx} + v_{xx}^2 + v_{xxxx})$  and, with the abbreviations  $\xi_1 = v_x, \dots, \xi_4 = v_{xxxx}$ ,

$$\begin{aligned} G''(0) &= -\int_{\Omega} u \left( 2\xi_1^4 \xi_2^2 + 8\xi_1^3 \xi_2 \xi_3 + 5\xi_1^2 \xi_2^3 + 4\xi_1^2 \xi_2 \xi_4 + 8\xi_1^2 \xi_3^2 + 10\xi_1 \xi_2^2 \xi_3 \right. \\ &\quad \left. + 8\xi_1 \xi_3 \xi_4 + 3\xi_2^4 + 5\xi_2^2 \xi_4 + 2\xi_4^2 \right) dx. \end{aligned}$$

We employ the following integration-by-parts formulas:

$$\begin{aligned} 0 &= \int_{\Omega} (uv_x^7)_x dx = \int_{\Omega} u(\xi_1^8 + 7\xi_1^6 \xi_2) dx =: J_1, \\ 0 &= \int_{\Omega} (uv_{xx} v_x^5)_x dx = \int_{\Omega} u(\xi_1^6 \xi_2 + \xi_1^5 \xi_3 + 5\xi_1^4 \xi_2^2) dx =: J_2, \\ 0 &= \int_{\Omega} (uv_{xxx} v_x^4)_x dx = \int_{\Omega} u(\xi_1^5 \xi_3 + \xi_1^4 \xi_4 + 4\xi_1^3 \xi_2 \xi_3) dx =: J_3, \\ 0 &= \int_{\Omega} (uv_{xx}^2 v_x^3)_x dx = \int_{\Omega} u(\xi_1^4 \xi_2^2 + 2\xi_1^3 \xi_2 \xi_3 + 3\xi_1^2 \xi_2^3) dx =: J_4, \\ 0 &= \int_{\Omega} (uv_{xx} v_{xxx} v_x^2)_x dx = \int_{\Omega} u(\xi_1^3 \xi_2 \xi_3 + \xi_1^2 \xi_2 \xi_4 + \xi_1^2 \xi_3^2 + 2\xi_1 \xi_2^2 \xi_3) dx =: J_5, \\ 0 &= \int_{\Omega} (uv_{xxx}^2 v_x)_x dx = \int_{\Omega} u(\xi_1^2 \xi_3^2 + 2\xi_1 \xi_3 \xi_4 + \xi_2 \xi_3^2) dx =: J_6, \end{aligned}$$

$$0 = \int_{\Omega} (uv_{xx}^3 v_x)_x dx = \int_{\Omega} u(\xi_1^2 \xi_2^3 + 3\xi_1 \xi_2^2 \xi_3 + \xi_2^4) dx =: J_7,$$

$$0 = \int_{\Omega} (uv_{xxx} v_{xx}^2)_x dx = \int_{\Omega} u(\xi_1 \xi_2^2 \xi_3 + 2\xi_2 \xi_3^2 + \xi_2^2 \xi_4) dx =: J_8.$$

Then

$$\begin{aligned} G'''(0) = G''(0) - 4 \sum_{i=1}^8 c_i J_i = & - \int_{\Omega} u \left( a_1 \xi_1^8 + a_2 \xi_1^6 \xi_2 + a_3 \xi_1^5 \xi_3 + a_4 \xi_1^4 \xi_2^2 + a_5 \xi_1^4 \xi_4 \right. \\ & + a_6 \xi_1^3 \xi_2 \xi_3 + a_7 \xi_1^2 \xi_2^3 + a_8 \xi_1^2 \xi_2 \xi_4 + a_9 \xi_1^2 \xi_3^2 + a_{10} \xi_1 \xi_2^2 \xi_3 + a_{11} \xi_1 \xi_3 \xi_4 + a_{12} \xi_2^4 \\ & \left. + a_{13} \xi_2^2 \xi_4 + a_{14} \xi_2 \xi_3^2 + a_{15} \xi_4^2 \right) dx, \end{aligned}$$

where

$$\begin{aligned} a_1 &= 4c_1, & a_2 &= 28c_1 + 4c_2, & a_3 &= 4c_2 + 4c_3, \\ a_4 &= 2 + 20c_2 + 4c_4, & a_5 &= 4c_3, & a_6 &= 8 + 16c_3 + 8c_4 + 4c_5, \\ a_7 &= 5 + 12c_4 + 4c_7, & a_8 &= 4 + 4c_5, & a_9 &= 8 + 4c_5 + 4c_6, \\ a_{10} &= 10 + 8c_5 + 12c_7 + 4c_8, & a_{11} &= 8 + 8c_6, & a_{12} &= 3 + 4c_7, \\ a_{13} &= 5 + 4c_8, & a_{14} &= 4c_6 + 8c_8, & a_{15} &= 2. \end{aligned}$$

Next, we eliminate all terms involving  $\xi_4$  by formulating the following square:

$$\begin{aligned} G'''(0) = & - \int_{\Omega} u \left[ a_{15} \left( \xi_4 + \frac{a_5}{2a_{15}} \xi_1^4 + \frac{a_8}{2a_{15}} \xi_1^2 \xi_2 + \frac{a_{11}}{2a_{15}} \xi_1 \xi_3 + \frac{a_{13}}{2a_{15}} \xi_2^2 \right)^2 \right. \\ & + \left( a_1 - \frac{a_5^2}{4a_{15}} \right) \xi_1^8 + \left( a_2 - \frac{a_5 a_8}{2a_{15}} \right) \xi_1^6 \xi_2 + \left( a_3 - \frac{a_5 a_{11}}{2a_{15}} \right) \xi_1^5 \xi_3 \\ & + \left( a_4 - \frac{a_8^2}{4a_{15}} - \frac{a_5 a_{13}}{2a_{15}} \right) \xi_1^4 \xi_2^2 + \left( a_6 - \frac{a_8 a_{11}}{2a_{15}} \right) \xi_1^3 \xi_2 \xi_3 + \left( a_7 - \frac{a_8 a_{13}}{2a_{15}} \right) \xi_1^2 \xi_2^3 \\ & \left. + \left( a_9 - \frac{a_{11}^2}{4a_{15}} \right) \xi_1^2 \xi_3^2 + \left( a_{10} - \frac{a_{11} a_{13}}{2a_{15}} \right) \xi_1 \xi_2^2 \xi_3 + \left( a_{12} - \frac{a_{13}^2}{4a_{15}} \right) \xi_2^4 + a_{14} \xi_2 \xi_3^2 \right] dx. \end{aligned}$$

We eliminate all terms involving  $\xi_3$  and set the corresponding coefficients to zero. From  $a_{14} = 0$  we conclude that  $c_6 = -2c_8$ . Furthermore,

$$\begin{aligned} a_9 - \frac{a_{11}^2}{4a_{15}} = 0 & \quad \text{gives} \quad c_5 = 8c_8^2 - 6c_8, \\ a_{10} - \frac{a_{11} a_{13}}{2a_{15}} = 0 & \quad \text{gives} \quad c_7 = -\frac{20}{3}c_8^2 + \frac{8}{3}c_8, \\ a_6 - \frac{a_8 a_{11}}{2a_{15}} = 0 & \quad \text{gives} \quad c_4 = -2c_3 - 16c_8^3 + 16c_8^2 - 5c_8, \\ a_3 - \frac{a_5 a_{11}}{2a_{15}} = 0 & \quad \text{gives} \quad c_2 = c_3 - 4c_3 c_8. \end{aligned}$$

By these choices, we obtain

$$b_{12} := a_{12} - \frac{a_{11}^2}{4a_{15}} = -\frac{86}{3}c_8^2 + \frac{17}{3}c_8 - \frac{1}{8}.$$

This quadratic polynomial in  $c_8$  admits its maximal value at  $c_8^* = 17/172$  with value  $b_{12} = 20/129$ . The integral can now be written as

$$G''(0) \leq - \int_{\Omega} u(b_1 \xi_1^8 + b_2 \xi_1^6 \xi_2 + b_4 \xi_1^4 \xi_2^2 + b_7 \xi_1^2 \xi_2^3 + b_{12} \xi_2^4) dx,$$

where

$$b_1 = a_1 - \frac{a_5^2}{4a_{15}} = 4c_1 - 2c_3^2,$$

$$b_2 = a_2 - \frac{a_5 a_8}{2a_{15}} = 28c_1 - 32c_3 c_8^2 + 8c_3 c_8,$$

$$b_4 = a_4 - \frac{a_8^2}{4a_{15}} - \frac{a_5 a_{13}}{2a_{15}} = 7c_3 - 84c_3 c_8 - 128c_8^4 + 128c_8^3 - 40c_8^2 + 4c_8,$$

$$b_7 = a_7 - \frac{a_8 a_{13}}{2a_{15}} = -24c_3 - 244c_8^3 + \frac{448}{3}c_8^2 - \frac{70}{3}c_8.$$

If  $b_4 = 2b_2 b_{12}/b_7 + b_7^2/(4b_{12})$ , we can write the integral as the sum of two squares, noting that  $b_{12}$  is positive,

$$G''(0) \leq - \int_{\Omega} u \left( b_{12} \left( \xi_2^2 + \frac{b_7}{2b_{12}} \xi_1^2 \xi_2 + \frac{b_2}{b_7} \xi_1^4 \right)^2 + \left( b_1 - \frac{b_2^2 b_{12}}{b_7^2} \right) \xi_1^8 \right) dx.$$

The expression  $b_4 b_7 - 2b_2 b_{12} - b_7^3/(4b_{12}) = 0$  defines a polynomial in  $(c_1, c_3)$  which is linear in  $c_1$ . Solving it for  $c_1$  gives

$$c_1 = \frac{449307}{175}c_3^3 + \frac{741681}{2150}c_3^2 + \frac{35780649411}{2393160700}c_3 + \frac{34135130165539}{163091166664200}.$$

It remains to show that  $p(c_3) := b_1 - b_2^2 b_{12}/b_7^2$ , which is a polynomial of fourth order in  $c_3$ , is positive. Choosing  $c_3^* = -0.029$ , we find that  $p(c_3^*) \approx 0.0045 > 0$ . This shows that

$$G''(0) \leq -q(c_3^*) \int_{\Omega} u \xi_1^8 dx = -q(c_3^*) \int_{\Omega} u (\log u)_x^8 dx \leq 0.$$

Finally, if  $G''(0) = 0$ , we infer that  $u$  is constant which is excluded. Therefore,  $G''(0) < 0$ , which ends the proof.  $\square$

## 7. NUMERICAL EXAMPLES

The aim of this section is to explore the numerical behavior of the second-order derivative of the function  $G(\tau)$ , defined in the introduction, for the porous-medium equation (23) in one space dimension. The equation is discretized by standard finite differences, and we employ periodic boundary conditions. The discrete solution  $u_i^k$  approximates the solution

$u(x_i, t^k)$  to (23) with  $x_i = i\Delta x$ ,  $t^k = k\tau$ , and  $\Delta x$ ,  $\tau$  are the space and time step sizes, respectively. We choose the Barenblatt profile

$$(29) \quad u^0(x) = t_0^{-1/(\beta+1)} \max\left(0, C - \frac{\beta-1}{2\beta(\beta+1)} \frac{(x-1/2)^2}{t_0^{2/(\beta+1)}}\right)^{1/(\beta-1)}, \quad 0 \leq x \leq 1,$$

where

$$t_0 = 0.01, \quad C = \frac{\beta-1}{2\beta(\beta+1)} \frac{(x_R - 1/2)^2}{t_0^{2/(\beta+1)}}, \quad x_R = \frac{1}{4},$$

as the initial datum. Its support is contained in  $[\frac{1}{2} - x_R, \frac{1}{2} + x_R]$ ; see Figure 3 (left). We choose the exponent  $\beta = 2$ . The semi-logarithmic plot of the discrete entropy  $H_d[u^k] = \sum_{i=0}^N (u_i^k)^\alpha \Delta x$  with  $\alpha = 5$  versus time is illustrated in Figure 3 (right), using the implicit Euler scheme with parameters  $\tau = 10^{-4}$  and the number of grid points  $N = 1/\Delta x = 64$ . The decay is exponential for “large” times. The nonlinear discrete system is solved by Newton’s method with the tolerance  $\mathbf{tol} = 10^{-15}$ . We have highlighted four time steps  $t_i$  at which we will compute numerically the function  $G(\tau)$  for the following Runge-Kutta schemes:

$$\begin{aligned} \text{explicit Euler scheme:} & \quad u^k - u^{k-1} = -\tau A[u^{k-1}], \\ \text{implicit Euler scheme:} & \quad u^k - u^{k-1} = -\tau A[u^k], \\ \text{second-order trapezoidal rule:} & \quad u^k - u^{k-1} = -\frac{\tau}{2}(A[u^k] + A[u^{k-1}]), \\ \text{third-order Simpson rule:} & \quad u^k - u^{k-1} = -\frac{\tau}{6}(A[u^k] + 4A[(u^k + u^{k-1})/2] + A[u^{k-1}]). \end{aligned}$$

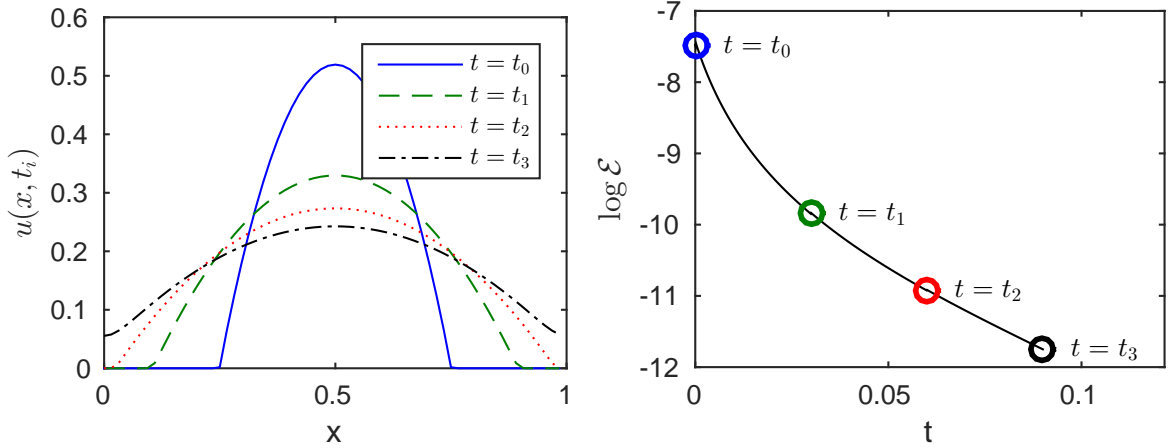


FIGURE 3. Left: Evolution of the initial datum (29) for  $\beta = 2$  at various time steps  $t_i$ ,  $i = 0, 1, 2, 3$ . Right: Semi-logarithmic plot of the discrete entropy  $H_d[u^k]$  versus time.

We set as before  $u := u^k$ ,  $v(\tau) := u^{k-1}$  and compute  $G(\tau) = H_d[u] - H_d[v(\tau)]$  and the discrete second-order derivative  $\partial^2 G$  of  $G$  (using central differences). The result is

presented in Figure 4. As expected, the discrete derivative  $\partial^2 G$  is negative on a (small) interval for all times  $t_i$ ,  $i = 1, 2, 3$ . We observe that  $\partial^2 G$  is even slightly decreasing, but we expect that it becomes positive for sufficiently large values of  $\tau$ . Clearly, the values for  $\partial^2 G$  tend to zero as we approach the steady state (see Remark 4). This experiment indicates that  $\tau^k$  from Theorem 1 is bounded from below by  $\tau^* = 3 \cdot 10^{-4}$ , for instance.

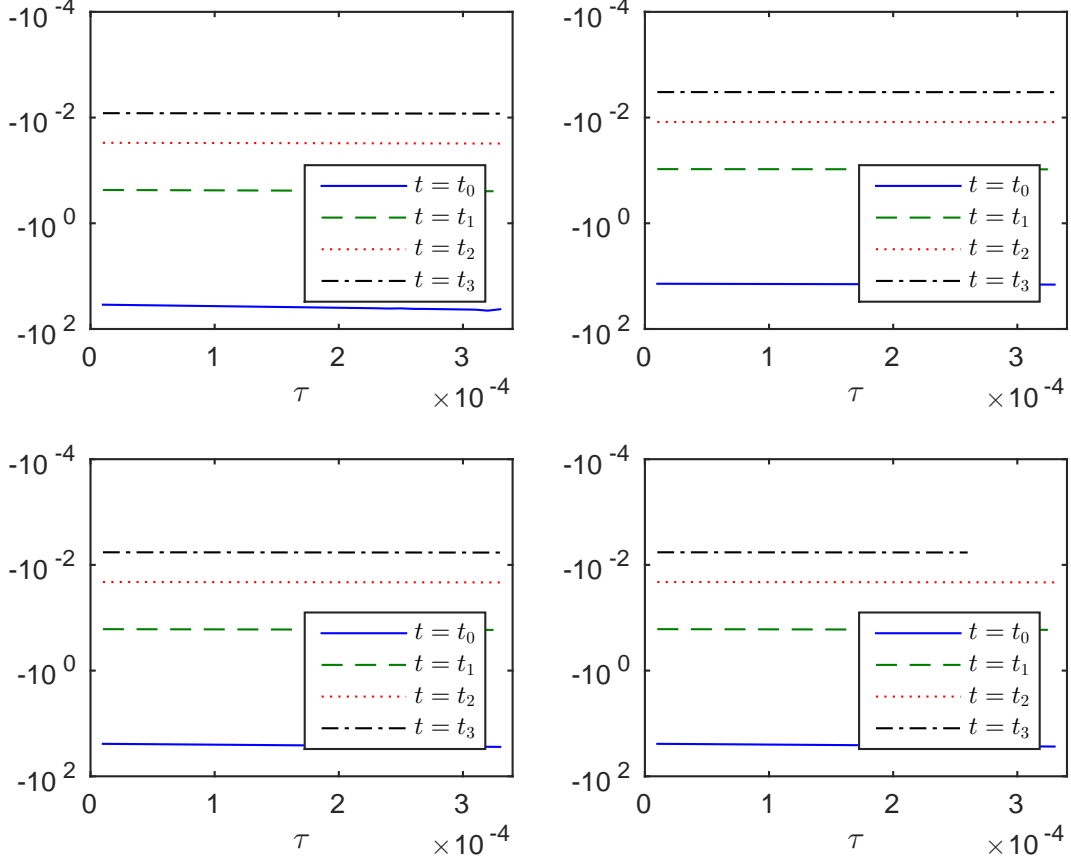


FIGURE 4. Numerical evaluation of the discrete version of  $G''(\tau)$  for various Runge-Kutta schemes at the time steps  $t_i$ . Top left: explicit Euler scheme; top right: implicit Euler scheme; bottom left: implicit trapezoidal rule; bottom right: Simpson rule.

In order to understand the behavior of  $G(\tau)$  in a better way, it is convenient to study the discrete version of the quotient

$$(30) \quad Q(\tau) := \frac{G''(\tau)}{\|u^{\alpha+2\beta-2}u_x^4\|_{L^1}}.$$

Indeed, the analysis in Section 4 gives an estimate of the type  $G''(0) \leq -C \int_{\Omega} u^{2\beta+\alpha-5}u_x^4 dx$  for some constant  $C > 0$ . Thus, we expect that for sufficiently small  $\tau > 0$ ,  $Q(\tau)$  is bounded

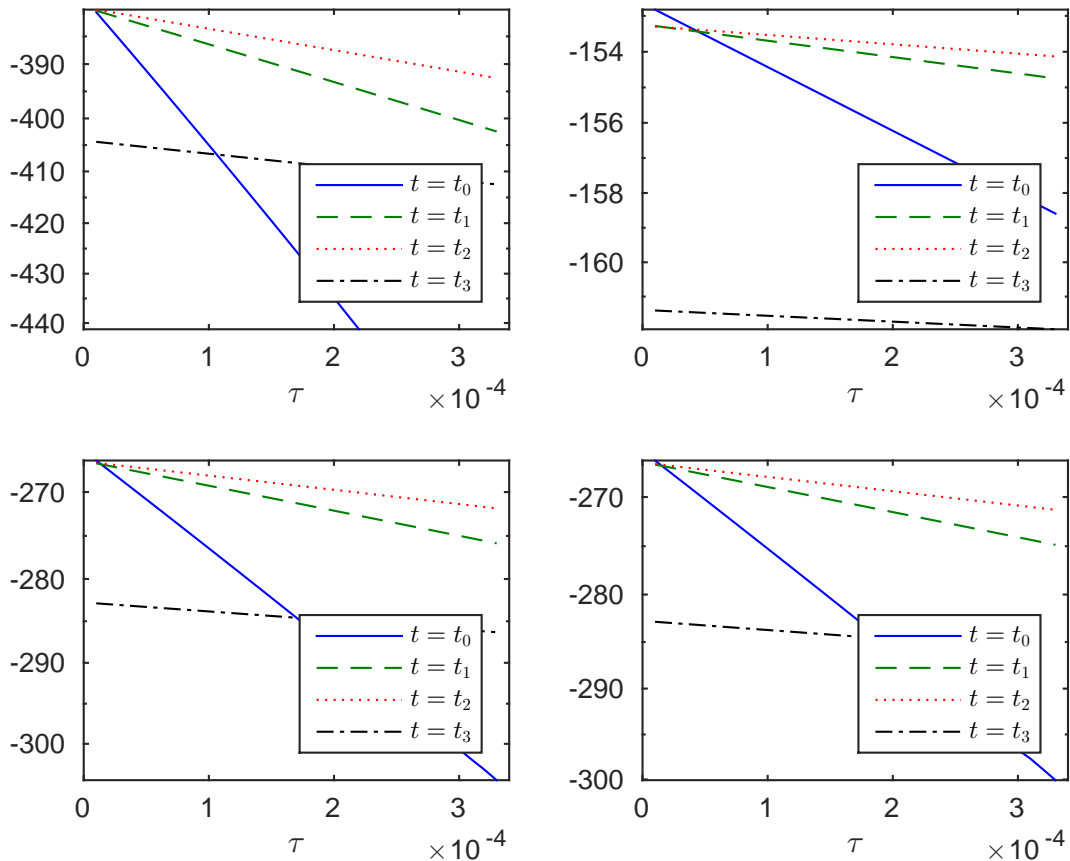


FIGURE 5. Numerical evaluation of the discrete version of  $Q(\tau)$ , defined in (30), for various Runge-Kutta schemes at the time steps  $t_i$ . Top left: explicit Euler scheme; top right: implicit Euler scheme; bottom left: implicit trapezoidal rule; bottom right: Simpson rule.

from above by some negative constant. This expectation is confirmed in Figure 5. In the examples,  $Q(\tau)$  is a decreasing function of  $\tau$ , and  $Q(0)$  is decreasing with increasing time.

All these results indicate that the threshold parameter  $\tau^k$  in Theorem 1 can be chosen independently of the time step  $k$ .

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