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On singular BVPs with unsmooth data. Part 1: Analysis of the linear case with variable coefficient matrix

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Abstract

In this paper, analytical properties of systems of singular linear ordinary differential equations with variable coefficient matrices and unsmooth inhomogeneities are investigated. The aim is to precisely formulate conditions which are necessary and/or sufficient for the existence and uniqueness of solutions which are at least continuous on the closed interval including the singular point. Smoothness properties of such solutions are also discussed. In the second part of the paper entitled - On singular BVPs with unsmooth data. Part 2: Convergence of the collocation schemes - the polynomial collocation is applied to solve the analytical problem and convergence properties are studied.

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1 Introduction

Singular boundary value problems (BVPs) arise in many relevant applications in natural sciences and engineering [1, 5, 8, 14] and therefore, numerous papers providing analytical results on their structural properties, stability and convergence of different numerical methods, and results of numerical simulations are available. A popular model class for the theoretical investigations is the following linear singular BVP,

$$y'(t) = \frac{M_0}{t} y(t) + f(t),\ t \in (0, 1], \quad B_0 y(0) + B_1 y(1) = \beta, \tag{1}$$

where y is a n-dimensional real function, M_0 is a $n \times n$ constant matrix and f is a n-dimensional function which is at least continuous, $f \in C[0,1]$. Here, B_0 and B_1 are constant matrices and it turns out that they are subject to certain restrictions for the problem with a unique continuous solution, cf. [10]. We say that BVP (1) has a time *singularity of the first kind* at $t = 0$.

Problems of type (1), where f may additionally depend on the space variable y and have a space singularity at $y = 0$, have been studied for example in [2, 16, 18]. Analytical properties of (1) have been discussed in [10, 20], where the attention was focused on the existence and uniqueness of solutions and their smoothness. In particular, the structure of the boundary conditions which are necessary and sufficient for (1) to have a unique continuous solution on $[0,1]$ was of special interest.

In [6], we generalized these analytical results to the problem

$$y'(t) = \frac{M_0}{t} y(t) + \frac{f(t)}{t},\ t \in (0,1], \quad B_0 y(0) + B_1 y(1) = \beta, \tag{2}$$

where $f \in C[0,1]$ but $f(t)/t$ may not be integrable on $[0,1]$. The BVPs of type (2) arise in the modelling of the avalanche run up [15] and occur when the regular system of ordinary differential equations (ODEs) $u'(x) = M_0 u(x) + g(x)$, posed on the semi-infinite interval $x \in [0,\infty)$, is transformed by $x = -\ln t$ to the finite domain $t \in (0,1]$.

In this paper, we are interested in extending results for the BVP (2) to the case, where the coefficient matrix M depends on t,

$$y'(t) = \frac{M(t)}{t} y(t) + \frac{f(t)}{t}, \quad B_0 y(0) + B_1 y(1) = \beta. \tag{3}$$

Here, $f : [0,1] \to \mathbb{R}^n$ and $M : [0,1] \to \mathbb{R}^{n \times n}$ have continuous components. Moreover, $B_0, B_1 \in \mathbb{R}^{m \times n}$ are constant matrices and $\beta \in \mathbb{R}^m$. Note that in general $m \le n$. We focus our attention on the existence and uniqueness of a solution $y \in C[0,1]$. This smoothness requirement results in general, in $n - m$ additional initial conditions the solution y has to satisfy. We also specify conditions for f and M which are sufficient for $y \in C^r[0,1]$, $r \in \mathbb{N}$.

The motivation for the above analysis of the variable coefficient case is twofold: First of all, in order to investigate the nonlinear case one can choose to study the properties of its linearization, see [10]. In this context a related linear BVP with a variable coefficient matrix has to be studied. More precisely, the technique applied in [10] is based on the *assumption that a solution to the nonlinear problem exists.* Next, the nonlinear problem is linearized at the exact solution and the well-posedness of this linearization is studied. We are not going to follow this technique however, and plan in an upcoming paper to *show the existence of the solution of the nonlinear BVP*, instead of assuming its existence. Secondly, for us, the investigation of the structural properties of (3) is necessary and interesting in its own right, as a prerequisite for the convergence theory of the collocation method described in Part 2 [7].

Before discussing the most general BVP (3), we first consider simpler problems consisting of the ODE system

$$y'(t) = \frac{M(t)}{t}y(t) + \frac{f(t)}{t}, \tag{4}$$

subject to initial/terminal conditions. This means that we deal with the initial value problem (IVP),

$$y'(t) = \frac{M(t)}{t}y(t) + \frac{f(t)}{t}, \quad B_0 y(0) = \beta, \tag{5}$$

where $B_0 \in \mathbb{R}^{m \times n}$, $\beta \in \mathbb{R}^m$, and $m \leq n$, or with the terminal value problem (TVP),

$$y'(t) = \frac{M(t)}{t}y(t) + \frac{f(t)}{t}, \quad B_1 y(1) = \beta, \tag{6}$$

where $B_1 \in \mathbb{R}^{n \times n}$, $\beta \in \mathbb{R}^n$, respectively.

As already mentioned, the analytical properties of (2) with a constant coefficient matrix have been discussed in [6]. Particular attention was paid to the structure of the most general two–point boundary conditions which are necessary and sufficient for the existence of a unique continuous solution on the closed interval $[0, 1]$. It turned out that the form of such conditions depends on the spectral properties of the coefficient matrix M_0. Motivated by the case with a constant coefficient matrix, we distinguish also in (4) between three cases, where all eigenvalues of $M(0)$ have negative real parts, positive real parts, or they are zero.

Moreover, we refer to papers [3, 4, 9, 13, 17], where the solvability of similar linear singular problems is discussed. Interesting results for linear BVPs with time singularities in weight-spaces can be found in [11, 12]. Although this framework is close to what we are aiming at here, it is not quite complete. So, in a way our results are closing the existing gaps.

The ODE system (4) was also investigated in [19], where the existence of a unique continuous solution y has been studied. Main results of [19] are formulated in [19, Theorem 1.1] and [19, Theorem 1.3]. In Theorem 1.1, f and M are assumed to be continuous and all eigenvalues of $M(0)$ to have negative real parts. In Theorem 1.3 a smoothness of higher derivatives of y up to order $r \geq 1$ has been characterized. It turns out that for $M, f \in C^r[0, 1]$ there exists a unique solution $y \in C^r[0, 1]$ provided that all real parts of the eigenvalues of $M(0)$ are smaller than r and different from natural numbers. The current paper completes the results of [19] for the variable matrix M. In contrast to [19], where only particular solutions without boundary conditions are considered, in this paper a general structure of linear two–point boundary conditions is of interest.

The paper is organized as follows: In Section 2, the necessary notation is introduced and in Section 3, the results for the constant coefficient matrix M_0 are recapitulated. In Sections 4, 5, and 6, three case studies are carried out, the case of only negative real parts of the eigenvalues of $M(0)$, positive real parts of the eigenvalues of $M(0)$, and zero eigenvalues of $M(0)$, respectively. Finally, the three case studies are used to formulate the results for the general IVPs, TVPs, and BVPs in Section 7. In Section 8, we summarize the most important results of the article.

2 Notation

Throughout the paper, the following notation is used. We denote by $\mathbb{R}^n$ and $\mathbb{C}^n$ the n-dimensional vector space of real-valued and complex-valued vectors, respectively, and denote the maximum vector norm by

$$|x| := |(x_1, \dots, x_n)^\top| = \max_{1 \le i \le n} |x_i|.$$

We denote by $C_n[0,1]$ the space of continuous real vector-valued functions on $[0,1]$. In this space, we use the maximum norm,

$$\|y\| := \max_{t \in [0,1]} |y(t)|,$$

and the norm restricted to the interval $[0,\delta]$, $\delta > 0$, is denoted by

$$\|y\|_\delta := \max_{t \in [0,\delta]} |y(t)|.$$

$C_n^p[0,\delta]$, $\delta > 0$, is the space of p times continuously differentiable real vector-valued functions on $[0,\delta]$ with the norm

$$\|y\|_{C_n^p[0,\delta]} := \sum_{k=0}^{p} \|y^{(k)}\|_\delta.$$

Furthermore, we denote by $\mathbb{R}^{m\times n}$, $\mathbb{C}^{m\times n}$ the $m \times n$-dimensional space of real-valued, complex-valued matrices, respectively, and denote the corresponding matrix norm by

$$|A| = \max_{1 \le i \le m} \sum_{j=1}^{n} |a_{ij}|.$$

Additionally, the space of p-times continuously differentiable real-valued matrix functions on $[0,\delta]$ is denoted by $C_{m\times n}^p[0,\delta]$, $\delta > 0$, $p \in \mathbb{N}$. This space is equipped with the norm

$$\|M\|_{C_{m\times n}^p[0,\delta]} := \sum_{k=1}^{p} \|M^{(k)}\|_\delta,$$

where

$$\|M\|_\delta := \max_{t \in [0,\delta]} |M(t)|.$$

If it cannot be confusing, we omit the subscripts m and n for simplicity of notation, and write $C[0,1] = C_n[0,1]$, $C^p[0,1] = C_n^p[0,1]$, $C^p[0,1] = C_{m\times n}^p[0,1]$, etc.

3 Linear problems with constant coefficient matrix

In this section, for reader's convenience, we recapitulate the most important results derived in [6] for the case of a constant coefficient matrix M_0. These results are necessary prerequisites for the investigation of problem (3) with a variable coefficient matrix M. More precisely, we collect analytical results for the ODE system

$$y'(t) = \frac{M_0}{t} y(t) + \frac{f(t)}{t}, \tag{7}$$

subject to initial/terminal conditions. This means that we deal with the IVP,

$$y'(t) = \frac{M_0}{t} y(t) + \frac{f(t)}{t}, \quad B_0 y(0) = \beta, \tag{8}$$

where $B_0 \in \mathbb{R}^{m \times n}$, $\beta \in \mathbb{R}^m$, and $m \le n$, and with the TVP,

$$y'(t) = \frac{M_0}{t} y(t) + \frac{f(t)}{t}, \quad B_1 y(1) = \beta, \tag{9}$$

where $B_1 \in \mathbb{R}^{n \times n}$, $\beta \in \mathbb{R}^n$, respectively.

In the first step of the analysis, we consider the ODE system (7) and construct its general solution. We denote by $J \in \mathbb{C}^{n \times n}$ the Jordan canonical form of M_0 and by $E \in \mathbb{C}^{n \times n}$ the associated matrix of the generalized eigenvectors of M_0. Thus, $M_0 = EJE^{-1}$. In the case that the matrix J consists of l Jordan boxes, $J_1, J_2, \dots, J_l$, the fundamental solution matrix has the form of the block diagonal matrix, $t^J = \operatorname{diag}(t^{J_1}, t^{J_2}, \dots, t^{J_l})$, where

$$J_k = \begin{pmatrix} \lambda_k & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_k \end{pmatrix} \in \mathbb{C}^{n_k \times n_k}, \quad k = 1, \dots, l,$$

and

$$t^{J_k} = t^{\lambda_k} \begin{pmatrix} 1 & \ln t & \frac{(\ln t)^2}{2} & \dots & \frac{(\ln t)^{n_k-1}}{(n_k-1)!} \\ 0 & 1 & \ln t & \dots & \frac{(\ln t)^{n_k-2}}{(n_k-2)!} \\ 0 & \ddots & 1 & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \ln t \\ 0 & \dots & \dots & 0 & 1 \end{pmatrix}, \quad t \in (0,1]. \tag{10}$$

Here, $\lambda_k = \sigma_k + i\rho_k \in \mathbb{C}$ is an eigenvalue of M_0 and $\dim J_1 + \dim J_2 + \dots + \dim J_l = n$. The general solution of equation (7) is then given by

$$y(t) = t^{M_0} c + t^{M_0} \int_1^t s^{-M_0 - I} f(s)\, \mathrm{d}s, \; t \in (0,1],$$

where $c \in \mathbb{C}^n$ and $t^{M_0} = Et^J E^{-1} \in \mathbb{C}^{n \times n}$. From the structure of the matrix t^{J_k} in (10), it is obvious that the solution contribution related to the k-th Jordan box may become unbounded for $t = 0$. Apparently, the asymptotic behaviour of the solution depends on the sign of the real part σ_k of the associated eigenvalue λ_k. Therefore, we have to distinguish between three cases, $\sigma_k < 0$, $\lambda_k = 0$, and $\sigma_k > 0$. We assume that M_0 has no purely imaginary eigenvalues to exclude solutions of the form $t^{i\rho} = \cos(\rho \ln t) + i \sin(\rho \ln t)$.

For the case where all eigenvalues of M_0 have negative real parts, it is necessary to prescribe initial conditions of a certain structure to guarantee that the solution is continuous on $[0, 1]$.

Theorem 1 (Theorem 5 in [6]). *Let us assume that all eigenvalues of M_0 have negative real parts. Then for any $f \in C[0, 1]$, IVP (8) with the initial condition $M_0 y(0) = -f(0)$ has a unique solution $y \in C[0, 1]$. This solution has the form*

$$y(t) = (\mathcal{L}_1 f)(t), \ t \in [0, 1],$$

where $\mathcal{L}_1 : C[0, 1] \to C[0, 1]$ is defined by

$$(\mathcal{L}_1 f)(t) := \int_0^1 s^{-M_0 - I} f(ts) \, \mathrm{d}s, \ t \in [0, 1].$$

The initial condition $M_0 y(0) = -f(0)$ is necessary and sufficient for y to be continuous on $[0, 1]$. Moreover, if $f \in C^r[0, 1]$, $r \geq 1$, then $y \in C^r[0, 1]$.

In the case that all eigenvalues of the matrix M_0 have positive real parts, there exists a unique continuous solution of a terminal value problem. It turns out that its smoothness depends not only on the smoothness of the inhomogeneity f but also on the size of real parts of the eigenvalues of M_0.

Theorem 2 (Theorem 8 in [6]). *Let us assume that all eigenvalues of M_0 have positive real parts and let the matrix $B_1 \in \mathbb{R}^{n \times n}$ in (9) be nonsingular. Then for any $f \in C^1[0, 1]$ and any $\beta \in \mathbb{R}^n$ there exists a unique solution $y \in C[0, 1]$ of TVP (9). This solution has the form*

$$y(t) = (\mathcal{L}_2 f)(t), \ t \in [0, 1],$$

where $\mathcal{L}_2 : C^1[0, 1] \to C[0, 1]$ is defined by

$$(\mathcal{L}_2 f)(t) := t^{M_0} B_1^{-1} \beta + t^{M_0} \int_1^t s^{-M_0 - I} f(s) \, \mathrm{d}s, \ t \in [0, 1]. \tag{11}$$

Moreover, this solution additionally satisfies the initial condition $M_0 y(0) = -f(0)$. Finally, if $f \in C^{r+1}[0, 1]$, $r \geq 0$ and if $\sigma_+ > r$, where σ_+ is the smallest positive real part of the eigenvalues of M_0, then $y \in C^r[0, 1]$.

We now construct $y \in C[0, 1]$ in the case of the inhomogeneity f which is not as smooth as in the previous theorem. This situation is explained in the following remark:

Remark 3 (Remark 9 in [6]). A continuous solution to (7) exists also in the case when f is not continuously differentiable on $[0, 1]$. However, in this case, we need some more structure in f close to the singularity. Let us assume that there exist a constant $\alpha > 0$ and a function $h \in C[0,\delta]$, $\delta > 0$ such that

$$f(t) = O(t^\alpha h(t)) \text{ for } t \to 0. \tag{12}$$

Let us denote

$$\Omega = \{f \in C[0,1] \text{ such that } f \text{ satisfies (12)}\}, \tag{13}$$

and let $\tilde{\mathcal{L}_2} : \Omega \to C[0,1]$ be defined by (11). Then the solution of (7) is still continuous on $[0,1]$ and has the form

$$y(t) = (\tilde{\mathcal{L}_2} f)(t),\ t \in [0,1].$$

Moreover, if the parameter α in (12) satisfies $\alpha > r+1$, $r \geq 0$, if $h \in C^{r+1}[0,\delta]$, and $\sigma_+ > r+1$, then $y \in C^{r+1}[0,1]$.

The special structure in f close to the singularity (12) is also required to cover the case of zero eigenvalues of M_0. Let us denote by R the projection matrix onto the space $X_0^{(e)}$ spanned by eigenvectors associated with zero eigenvalues and by $\tilde{R}$ the matrix consisting of the linearly independent columns of R.

Theorem 4 (Theorem 11 in [6])**.** *Let all eigenvalues of the matrix M_0 be zero, $m := \dim X_0^{(e)}$, and let the set Ω be given by (13). Then for any $B_0 \in \mathbb{R}^{m\times n}$ such that the matrix $B_0\tilde{R} \in \mathbb{R}^{m\times m}$ is nonsingular and for any $f \in \Omega$ and $\beta \in \mathbb{R}^m$, there exists a unique solution $y \in C[0,1]$ of IVP (8). This solution has the form*

$$y(t) = (\mathcal{L}_3 f)(t),\ t \in [0,1],$$

where $\mathcal{L}_3 : \Omega \to C[0,1]$ is defined by

$$(\mathcal{L}_3 f)(t) := \tilde{R}(B_0\tilde{R})^{-1}\beta + \int_0^1 s^{-M_0} s^{-1} f(st)\,\mathrm{d}s,\ t \in [0,1].$$

This solution satisfies also the initial condition $M_0 y(0) = 0$, which is necessary and sufficient for $y \in C[0,1]$. Moreover, if $\alpha \geq r+1$, $r \geq 0$, $f \in C^r[0,1]$, and $h \in C^r[0,\delta]$, then $y \in C^{r+1}[0,1]$.

4 Eigenvalues of $M(0)$ with negative real parts

Here, we investigate system (4), where all eigenvalues of $M(0)$ have negative real parts. Note that system (4) is equivalent to

$$y'(t) = \frac{M(0)}{t} y(t) + \frac{(M(t) - M(0))y(t) + f(t)}{t}.$$

In the proof of the existence and uniqueness of a continuous solution y of the ODE system (4), we use techniques developed in [6] for a constant coefficient matrix, see Theorem 1. No additional assumptions on the variable coefficient matrix M need to be made, provided that all eigenvalues of $M(0)$ have negative real parts. To show the smoothness of the solution $y \in C^r[0,1]$, $r \in \mathbb{N}$, condition (14) has to hold.

Theorem 5. *Let us assume that all eigenvalues of $M(0)$ have negative real parts and $M \in C[0,1]$. Then for any $f \in C[0,1]$ system (4) has a unique solution $y \in C[0,1]$. This solution satisfies the initial condition $M(0)y(0) = -f(0)$ which is necessary and sufficient for y to be continuous on $[0,1]$. Moreover, if $f \in C^r[0,1]$, $M \in C^r[0,1]$, $r \geq 1$ and M is such that*

$$M'(t) = t^\gamma D(t), \ \ t \in [0,1], \tag{14}$$

where $D \in C^{r-1}[0,1]$, $\gamma > r-1$, then $y \in C^r[0,1]$.

Proof: According to Theorem 1, any continuous solution of system (4) satisfies

$$y(t) = (\mathcal{L}_1 g)(t) = \int_0^1 s^{-M(0)-I} g(st, y(st))\,\mathrm{d}s, \ \ t \in [0,1],$$

where $g(t, y(t)) = (M(t) - M(0))\,y(t) + f(t)$. In order to show the existence and uniqueness of a continuous solution of (4), we choose $\delta \in (0,1]$ and study the fixed point equation

$$y = \mathcal{K}y, \ \ y \in C[0,\delta],$$

with an operator $\mathcal{K}$ defined by

$$\begin{aligned}(\mathcal{K}y)(t) &:= \int_0^1 s^{-M(0)-I} g(st, y(st))\,\mathrm{d}s \\ &= \int_0^1 s^{-M(0)-I} f(st)\,\mathrm{d}s + \int_0^1 s^{-M(0)-I}\,(M(st) - M(0))\,y(st)\,\mathrm{d}s, \ \ t \in [0,\delta].\end{aligned}$$

The proof is now carried out in two steps.

Step 1. Existence and uniqueness of a solution y.

We prove the existence and uniqueness of a solution $y = \mathcal{K}y$, $y \in C[0,\delta]$ by means of the Banach fixed point theorem. It follows immediately from Theorem 1 that the first contribution in $\mathcal{K}y$,

$$\int_0^1 s^{-M(0)-I} f(st)\,\mathrm{d}s,$$

is continuous on $[0,\delta]$. For any function $y \in C[0,\delta]$, the second contribution,

$$\int_0^1 s^{-M(0)-I}\,(M(st) - M(0))\,y(st)\,\mathrm{d}s,$$

is also continuous on $[0,1]$, cf. Theorem 1. Therefore, the operator $\mathcal{K}$ is a mapping from $C[0,\delta]$ to $C[0,\delta]$.

We now show that $\mathcal{K}$ is contracting. Let y_1, $y_2 \in C[0,\delta]$, then

$$\begin{aligned}\|\mathcal{K}y_1 - \mathcal{K}y_2\|_\delta &= \max_{t\in[0,\delta]} \left| \int_0^1 s^{-M(0)} s^{-1}\,(M(st) - M(0))\,(y_1(st) - y_2(st))\,\mathrm{d}s \right| \\ &\leq \max_{t\in[0,\delta]} \left\{ \int_0^1 \left| s^{-M(0)} \right| s^{-1}\,\mathrm{d}s \max_{s\in[0,1]} |M(st) - M(0)| \max_{s\in[0,1]} |y_1(st) - y_2(st)| \right\} \\ &\leq \max_{t\in[0,\delta]} \left\{ \int_0^1 \left| s^{-M(0)} \right| s^{-1}\,\mathrm{d}s \max_{s\in[0,t]} |M(s) - M(0)| \max_{s\in[0,t]} |y_1(s) - y_2(s)| \right\} \\ &\leq \ const.\, \|M(\cdot) - M(0)\|_\delta \|y_1 - y_2\|_\delta,\end{aligned}$$

where $\|M(\cdot)-M(0)\|_\delta = \max_{t\in[0,\delta]} |M(t)-M(0)|$. Note that by Lemma 3 [6], $\int_0^1 |s^{-M(0)}| s^{-1}\,\mathrm{d}s = const.$ and since M is continuous on $[0,1]$

$$\lim_{t\to 0} M(t) - M(0) = 0.$$

Therefore there exists a sufficiently small δ such that

$$const.\, \|M(\cdot)-M(0)\|_\delta =: L_N < 1, \tag{15}$$

and consequently, the operator $\mathcal{K}$ is a contraction. The Banach fixed point theorem yields the existence of a unique continuous solution y of (4) on $[0,\delta]$. Using the classical theory, this solution can be uniquely extended to $t=1$. The initial condition $M(0)y(0) = -f(0)$ follows from the form of $\mathcal{K}$ and Theorem 1.

Step 2. Smoothness of the solution.
Now, we deal with the smoothness of y. Let us assume that $r \in \mathbb{N}$ and $f,\, M \in C^r[0,1]$. The property $\mathcal{K} : C^r[0,\delta] \to C^r[0,\delta]$ follows by arguing as in the proof of Theorem 1. We now show that $\mathcal{K}$ is a contraction on $C^r[0,\delta]$ for a sufficiently small δ.

Let $r=1$ and $\gamma > 0$ in (14). Then for any $y_1,\, y_2 \in C^1[0,\delta]$,

$$\|\mathcal{K}y_1 - \mathcal{K}y_2\|_{C^1[0,\delta]} = \|\mathcal{K}y_1 - \mathcal{K}y_2\|_\delta + \|\,(\mathcal{K}y_1)' - (\mathcal{K}y_2)'\,\|_\delta,$$

where $(\mathcal{K}y)'$ is given by

$$\begin{aligned}(\mathcal{K}y)'(t) \;&=\; \int_0^1 s^{-M(0)} f'(st)\,\mathrm{d}s + \int_0^1 s^{-M(0)} M'(st) y(st)\,\mathrm{d}s \\ &\quad + \int_0^1 s^{-M(0)}\,(M(st)-M(0))\, y'(st)\,\mathrm{d}s,\;\; t\in[0,\delta],\end{aligned}$$

and according to (14), (15), the following estimate holds:

$$\begin{aligned}&\|\mathcal{K}y_1 - \mathcal{K}y_2\|_{C^1[0,\delta]} \le L_N \|y_1 - y_2\|_\delta \\ &\quad + \max_{t\in[0,\delta]} \left\{ t^\gamma \int_0^1 \left| s^{-M(0)} s^\gamma \right| \mathrm{d}s \max_{s\in[0,1]} |D(st)| \max_{s\in[0,1]} |y_1(st) - y_2(st)| \right\} \\ &\quad + \max_{t\in[0,\delta]} \left\{ \int_0^1 \left| s^{-M(0)} \right| \mathrm{d}s \max_{s\in[0,1]} |M(st)-M(0)| \max_{s\in[0,1]} |y_1'(st) - y_2'(st)| \right\} \\ &\le\; L_N \|y_1 - y_2\|_\delta + L_{N2}\|y_1 - y_2\|_\delta + L_{N3}\|y_1' - y_2'\|_\delta \le L\|y_1 - y_2\|_{C^1[0,\delta]},\end{aligned}$$

where

$$\begin{aligned} L \;&=\; \max\{L_N + L_{N2}, L_{N3}\}, \\ L_{N2} \;&=\; \delta^\gamma \int_0^1 \left| s^{-M(0)} \right| s^\gamma\,\mathrm{d}s \|D\|_\delta, \\ L_{N3} \;&=\; \int_0^1 \left| s^{-M(0)} \right| \mathrm{d}s \|M(\cdot) - M(0)\|_\delta. \end{aligned}$$

For a sufficiently small δ, the value of L is smaller than 1, and therefore, $\mathcal{K}$ is a contraction on $C^1[0,\delta]$.

Now, let $r = 2$ and $\gamma > 1$ in (14). Then for any $y_1,\, y_2 \in C^2[0,\delta]$,

$$\|\mathcal{K}y_1 - \mathcal{K}y_2\|_{C^2[0,\delta]} = \|\mathcal{K}y_1 - \mathcal{K}y_2\|_{C^1[0,\delta]} + \|\,(\mathcal{K}y_1)'' - (\mathcal{K}y_2)''\,\|_\delta,$$

where, by virtue of (14),

$$\begin{aligned}
(\mathcal{K}y)''(t) &= \int_0^1 s^{-M(0)} s f''(st)\,\mathrm{d}s + t^{\gamma-1}\int_0^1 s^{-M(0)} s^{\gamma} D(st)y(st)\,\mathrm{d}s \\
&\quad + t^{\gamma}\int_0^1 s^{-M(0)} s^{\gamma+1} D'(st)y(st)\,\mathrm{d}s + 2t^{\gamma}\int_0^1 s^{-M(0)} s^{\gamma+1} D(st)y'(st)\,\mathrm{d}s \\
&\quad + \int_0^1 s^{-M(0)} s\left(M(st) - M(0)\right) y''(st)\,\mathrm{d}s,\ t \in [0,\delta],
\end{aligned}$$

and the following estimate holds:

$$\begin{aligned}
\|\mathcal{K}y_1 - \mathcal{K}y_2\|_{C^2[0,\delta]} &\le L\|y_1 - y_2\|_{C^1[0,\delta]} + L_{N4}\|y_1 - y_2\|_\delta + L_{N5}\|y_1 - y_2\|_\delta \\
&\quad + L_{N6}\|y_1' - y_2'\|_\delta + L_{N7}\|y_1'' - y_2''\|_\delta \le L_2\|y_1 - y_2\|_{C^2[0,\delta]}.
\end{aligned}$$

Here,

$$\begin{aligned}
L_2 &= \max\{L, L_{N4} + L_{N5}, L_{N6}, L_{N7}\}, \\
L_{N4} &= \delta^{\gamma-1}\int_0^1 \left|s^{-M(0)}\right| s^{\gamma}\,\mathrm{d}s\|D\|_\delta, \\
L_{N5} &= \delta^{\gamma}\int_0^1 \left|s^{-M(0)}\right| s^{\gamma+1}\,\mathrm{d}s\|D'\|_\delta, \\
L_{N6} &= 2\delta^{\gamma}\int_0^1 \left|s^{-M(0)}\right| s^{\gamma+1}\,\mathrm{d}s\|D\|_\delta, \\
L_{N7} &= \int_0^1 \left|s^{-M(0)}\right| s\,\mathrm{d}s\|M(\cdot) - M(0)\|_\delta.
\end{aligned}$$

For a sufficiently small δ, the value of L_2 is smaller than 1, and thus, $\mathcal{K}$ is a contracting operator on $C^2[0,\delta]$.

Similarly, we can show that $\mathcal{K}$ is a contraction on $C^r[0,\delta]$ for $r > 2$ and $\gamma > r - 1$. This yields the existence of a unique solution $y \in C^r[0,\delta]$ such that $M(0)y(0) = -f(0)$. This solution can be uniquely extended to $t = 1$, so $y \in C[0,1] \cap C^r[0,\delta]$. Under the assumption $f,\, M \in C^r[0,1]$ the classical theory yields a unique solution $z \in C^r(0,1]$ of equation (4) satisfying $z(\delta) = y(\delta)$. Consequently, $z = y$ on $[0,1]$ and $y \in C^r[0,1]$.

□

5 Eigenvalues of $M(0)$ with positive real parts

In this section, we study system (4) when all eigenvalues of the matrix $M(0)$ have positive real parts. For this spectrum of $M(0)$, there exists a unique continuous solution of TVP (6).

It turns out that a special structure of M is required in order to successfully apply the Banach fixed point theorem, see Remark 3 which explains the situation with a continuous, not necessarily smooth, inhomogeneity. Consequently, we assume that a coefficient matrix M has the following form

$$M(t) = M(0) + t^{\gamma} D(t), \ \gamma > 0, \quad D \in C[0,1], \ t \in [0,1]. \tag{16}$$

Then, equation (4) can be equivalently rewritten,

$$y'(t) = \frac{M(0)}{t} y(t) + \frac{t^{\gamma} D(t) y(t) + f(t)}{t}.$$

Theorem 6. *Let us assume that all eigenvalues of $M(0)$ have positive real parts and M satisfies condition (16). Moreover, let $f \in C^1[0,1]$, the matrix $B_1 \in \mathbb{R}^{n \times n}$ be nonsingular, and $\beta \in \mathbb{R}^n$. Then there exists a unique solution $y \in C[0,1]$ of TVP (6). Moreover, if $f \in C^{r+1}[0,1]$, $r \in \mathbb{N}$, M satisfies condition (16) with $\gamma > r$, $D \in C^r[0,1]$, and the smallest positive real part of the eigenvalues of $M(0)$ satisfies $\sigma_+ > r$, then $y \in C^r[0,1]$.*

Proof: The existence and uniqueness of a solution $z \in C(0,1]$ of problem (6) follows from the classical theory because the interval $(0,1]$ does not contain the singular point $t = 0$. Now, we ask the question, if the solution z can be continuously extended to $t = 0$. In particular, we choose $\delta \in (0,1]$ and investigate the terminal value problem

$$y'(t) = \frac{M(t)}{t} y(t) + \frac{f(t)}{t}, \ t \in [0,\delta], \quad y(\delta) = z(\delta). \tag{17}$$

By a slight modification of Theorem 2, we easily see that any solution y of problem (17) satisfies

$$y(t) = \left(\frac{t}{\delta}\right)^{M(0)} z(\delta) + t^{M(0)} \int_{\delta}^{t} s^{-M(0)-I} g(s, y(s)) \, \mathrm{d}s, \ t \in [0,\delta],$$

where $g(t, y(t)) = f(t) + t^{\gamma} D(t) y(t)$. Therefore, the existence and uniqueness of a continuous solution to problem (17) is equivalent to the existence and uniqueness of a fixed point of the operator $\mathcal{K}$ defined on the space $C[0,\delta]$, where

$$\begin{aligned}(\mathcal{K}y)(t) \ &= \ \left(\frac{t}{\delta}\right)^{M(0)} z(\delta) + t^{M(0)} \int_{\delta}^{t} s^{-M(0)-I} f(s) \, \mathrm{d}s \\ &\quad + t^{M(0)} \int_{\delta}^{t} s^{-M(0)} s^{\gamma-1} D(s) y(s) \, \mathrm{d}s, \ t \in [0,\delta].\end{aligned}$$

Again, the proof is divided into two parts.

Step 1. Existence and uniqueness of a solution y.

In order to use the Banach fixed point theorem to solve

$$y = \mathcal{K}y, \ y \in C[0,\delta],$$

we first show that $\mathcal{K} : C[0,\delta] \to C[0,\delta]$. According to Theorem 2 the first contribution to $\mathcal{K}y$,

$$\left(\frac{t}{\delta}\right)^{M(0)} z(\delta) + t^{M(0)} \int_{\delta}^{t} s^{-M(0)-I} f(s) \, \mathrm{d}s,$$

is continuous. Moreover, for $y \in C[0,\delta]$ the function $t^\gamma D(t)y(t)$ belongs to Ω, cf. (13), and by Remark 3 we conclude that

$$t^{M(0)} \int_\delta^t s^{-M(0)} s^{\gamma-1} D(s) y(s)\,\mathrm{d}s \in C[0,\delta]$$

holds. Therefore, $\mathcal{K} : C[0,\delta] \to C[0,\delta]$. Also, the operator $\mathcal{K}$ is a contraction. To see this consider y_1, $y_2 \in C[0,\delta]$. Then,

$$\begin{aligned}
\|\mathcal{K}y_1 - \mathcal{K}y_2\|_\delta &= \max_{t\in[0,\delta]} \left| t^{M(0)} \int_\delta^t s^{-M(0)} s^{\gamma-1} D(s)(y_1(s) - y_2(s))\,\mathrm{d}s \right| \\
&\le \max_{t\in[0,\delta]} \left\{ t^{M(0)} \int_t^\delta \left| s^{-M(0)} \right| s^{\gamma-1}\,\mathrm{d}s \max_{s\in[t,\delta]} |D(s)| \max_{s\in[t,\delta]} |y_1(s) - y_2(s)| \right\} \\
&\le \max_{t\in[0,\delta]} \left\{ \int_t^1 \left| \left(\frac{t}{s}\right)^{M(0)} \right| s^{\gamma-1}\,\mathrm{d}s \right\} \|D\|_\delta \|y_1 - y_2\|_\delta.
\end{aligned}$$

According to Lemma 7 [6], the function

$$u(t) := \int_t^1 \left| \left(\frac{t}{s}\right)^{M(0)} \right| s^{\gamma-1}\,\mathrm{d}s$$

is bounded and for $\gamma > 0$, $\lim_{t\to 0} u(t) = 0$. Therefore, there exists a sufficiently small δ such that

$$\max_{t\in[0,\delta]} u(t)\|D\|_\delta =: L_S < 1, \tag{18}$$

and hence, the operator $\mathcal{K}$ is contracting for a sufficiently small δ. According to the Banach fixed point theorem, there exists a unique fixed point of $\mathcal{K}$ in $C[0,\delta]$. Thus, there exists a unique continuous solution y of problem (17). Since $y(\delta) = z(\delta)$, we have $y = z$ on $(0,\delta]$. If, we choose $z(0) := y(0)$, $y \in C[0,1]$ follows and this completes the proof of Step 1.

Step 2. Smoothness of the solution.
Let $f \in C^{r+1}[0,1]$, $r \ge 1$, let M satisfy condition (16) with $\gamma > r$, and let $D \in C^r[0,1]$. Finally, let $\sigma_+ > r$. By arguments similar to those used in Theorem 2 and Remark 3, it follows that $\mathcal{K} : C^r[0,\delta] \to C^r[0,\delta]$.

Let us first assume $r = 1$ and show that $\mathcal{K}$ is a contraction on $C^1[0,\delta]$. Choose y_1, $y_2 \in C^1[0,\delta]$. Then, we can write $(\mathcal{K}y_1 - \mathcal{K}y_2)(t)$ as shown below, after integration by parts was used. With the shorthand notation $N(t) := t^\gamma D(t)(y_1(t) - y_2(t))$, we have for any $t \in [0,\delta]$,

$$\begin{aligned}
(\mathcal{K}y_1 - \mathcal{K}y_2)(t) &= -t^{M(0)} M(0)^{-1} t^{-M(0)} N(t) \\
&\quad + t^{M(0)} \left(M(0)^{-1} \delta^{-M(0)} N(\delta) + M(0)^{-1} \int_\delta^t s^{-M(0)} N'(s)\,\mathrm{d}s \right) \\
&= M(0)^{-1} \left(-N(t) + t^{M(0)} \delta^{-M(0)} N(\delta) + t^{M(0)} \int_\delta^t s^{-M(0)} N'(s)\,\mathrm{d}s \right).
\end{aligned}$$

We now differentiate both sides of the above equality and use $N(\delta) = 0$ to obtain

$$
\begin{aligned}
&((\mathcal{K}y_1)' - (\mathcal{K}y_2)')(t) = \\
&= -M(0)^{-1}N'(t) + t^{M(0)-I}\delta^{-M(0)}N(\delta) + t^{M(0)-I}\int_\delta^t s^{-M(0)}N'(s)\,\mathrm{d}s \\
&\quad + M(0)^{-1}t^{M(0)}t^{-M(0)}N'(t) = t^{M(0)-I}\int_\delta^t s^{-M(0)}N'(s)\,\mathrm{d}s \\
&= \int_\delta^t \left(\frac{t}{s}\right)^{M(0)-I} s^{\gamma-2}\left(\gamma D(s) + sD'(s)\right)(y_1(s) - y_2(s))\,\mathrm{d}s \\
&\quad + \int_\delta^t \left(\frac{t}{s}\right)^{M(0)-I} s^{\gamma-1}D(s)(y_1'(s) - y_2'(s))\,\mathrm{d}s.
\end{aligned}
$$

Since $\gamma > 1$ and $\sigma_+ > 1$, Lemma 7 [6] implies that the function

$$
u(t) := \int_t^1 \left| \left(\frac{t}{s}\right)^{M(0)-I} \right| s^{\gamma-2}\,\mathrm{d}s
$$

is bounded and $\lim_{t\to 0} u(t) = 0$. Therefore, for $y_1,\, y_2 \in C^1[0,\delta]$ and for a sufficiently small $\delta > 0$, we conclude using (18),

$$
\begin{aligned}
\|\mathcal{K}y_1 - \mathcal{K}y_2\|_{C^1[0,\delta]} &= \|\mathcal{K}y_1 - \mathcal{K}y_2\|_\delta + \|(\mathcal{K}y_1)' - (\mathcal{K}y_2)'\|_\delta \le L_S\|y_1 - y_2\|_\delta \\
&\quad + \max_{t\in[0,\delta]} \left\{u(t)\left((\gamma\|D\|_\delta + \|D'\|_\delta)\,\|y_1 - y_2\|_\delta + \|D\|_\delta\|y_1' - y_2'\|_\delta\right)\right\}.
\end{aligned}
$$

Therefore $\mathcal{K}$ is a contraction on $C^1[0,\delta]$ for a sufficiently small δ. For $r > 1$, we can use similar arguments to show that $\mathcal{K}$ is a contraction on $C^r[0,\delta]$. Now the Banach fixed point theorem yields the existence and uniqueness of a solution $y \in C^r[0,\delta]$ of problem (17).

The classical theory implies that for the solution z of TVP (6), derived in Step 1, $z \in C^r(0,1]$ holds. Since $z(\delta) = y(\delta)$, $y = z$ on $[0,1]$ and $y \in C^r[0,1]$ follows.

□

6 Zero eigenvalues of $M(0)$

Finally, we consider the case, when all eigenvalues of $M(0)$ are zero. It turns out that some additional structure in the function f and in the variable coefficient matrix M is necessary for the solution y to be continuous. Let us recall the notation used in this section. By $\tilde{R}$, we denote the matrix consisting of the linearly independent columns of the projection matrix R onto $X_0^{(e)}$, which is the space spanned by the eigenvectors of $M(0)$ associated with zero eigenvalues. Finally, Ω is the set of continuous functions on $[0,1]$ satisfying condition (12).

Theorem 7. *Let all eigenvalues of the matrix $M(0)$ be zero. Let M satisfy condition* (16) *and $m := \dim X_0^{(e)}$. Assume that $f \in \Omega$, $B_0 \in \mathbb{R}^{m\times n}$ is such that the matrix $B_0\tilde{R} \in \mathbb{R}^{m\times m}$ is nonsingular, and $\beta \in \mathbb{R}^m$. Then there exists a unique solution $y \in C[0,1]$ of IVP* (5)*. This solution satisfies the initial* *condition* $M(0)y(0) = 0$*, which is necessary and sufficient for* $y \in C[0,1]$*. Moreover, if* $\alpha \geq r+1$*,* $\gamma \geq r+1$*,* $r \geq 1$*,* f*,* $D \in C^r[0,1]$*, and* $h \in C^r[0,\delta]$*, then* $y \in C^{r+1}[0,1]$*.*

Proof: By Theorem 4, any continuous solution of IVP (5) satisfies

$$y(t) = (\mathcal{L}_3 g)(t) = \tilde{R}(B_0\tilde{R})^{-1}\beta + \int_0^1 s^{-M(0)}s^{-1}g(st, y(st))\,\mathrm{d}s,\ t \in [0,1],$$

where $g(t,y(t)) = f(t) + t^\gamma D(t)y(t)$. Consequently, we have to study the following fixed point equation:

$$y = \mathcal{K}y,\ y \in C[0,\delta],$$

to prove the existence and uniqueness of solution of IVP (5). In particular, we choose $\delta \in (0,1]$ and define

$$\begin{aligned}(\mathcal{K}y)(t) &= \tilde{R}(B_0\tilde{R})^{-1}\beta + \int_0^1 s^{-M(0)}s^{-1}f(st)\,\mathrm{d}s \\ &\quad + t^\gamma \int_0^1 s^{-M(0)}s^{\gamma-1}D(st)y(st)\,\mathrm{d}s,\ t \in [0,\delta].\end{aligned}$$

Step 1. Existence and uniqueness of a solution y.

We use the Banach fixed point theorem in order to prove the first part of the statement. From Theorem 4, we see that

$$\tilde{R}(B_0\tilde{R})^{-1}\beta + \int_0^1 s^{-M(0)}s^{-1}f(st)\,\mathrm{d}s,$$

is continuous on $[0,\delta]$. For $y \in C[0,\delta]$ the function $t^\gamma D(t)y(t)$ belongs to Ω, cf. (13). Using Theorem 4 we see that

$$t^\gamma \int_0^1 s^{-M(0)}s^{\gamma-1}D(st)y(st)\,\mathrm{d}s \in C[0,1]$$

follows. Thus, $\mathcal{K} : C[0,\delta] \to C[0,\delta]$. Moreover, $\mathcal{K}$ is a contraction due to the following estimates. Let $y_1,\ y_2 \in C[0,\delta]$, then

$$\begin{aligned}\|\mathcal{K}y_1 - \mathcal{K}y_2\|_\delta &= \max_{t\in[0,\delta]} \left| t^\gamma \int_0^1 s^{-M(0)}s^{\gamma-1}D(st)\left(y_1(st) - y_2(st)\right)\mathrm{d}s \right| \\ &\leq \max_{t\in[0,\delta]} \left\{ t^\gamma \int_0^1 \left|s^{-M(0)}\right| s^{\gamma-1}\,\mathrm{d}s \max_{s\in[0,1]} |D(st)| \max_{s\in[0,1]} |y_1(st) - y_2(st)| \right\} \\ &\leq \max_{t\in[0,\delta]} \left\{ t^\gamma \int_0^1 \left|s^{-M(0)}\right| s^{\gamma-1}\,\mathrm{d}s \max_{s\in[0,t]} |D(s)| \max_{s\in[0,t]} |y_1(s) - y_2(s)| \right\} \\ &\leq \delta^\gamma const.\, \|D\|_\delta \|y_1 - y_2\|_\delta.\end{aligned}$$

Note that for $\gamma > 0$, $\int_0^1 |s^{-M(0)}|s^{\gamma-1}\,\mathrm{d}s = const.$ holds, see Lemma 3 [6]. Consequently, there exists a sufficiently small δ such that

$$\delta^\gamma const. \|D\|_\delta =: L_Z < 1, \tag{19}$$

and the operator $\mathcal{K}$ is a contraction. The Banach fixed point theorem yields the existence of a unique continuous solution of (5) on $[0,\delta]$. This solution can be uniquely extended to the point $t = 1$. The initial condition $M(0)y(0) = 0$ follows from the form of $\mathcal{K}$ and Theorem 4.

Step 2. Smoothness of the solution.

Let $\alpha \geq r+1$, $\gamma \geq r+1$, $r \geq 1$, $f,\, D \in C^r[0,1]$, and $h \in C^r[0,\delta]$. Then, in follows from Theorem 4 that $\mathcal{K} : C^r[0,\delta] \to C^r[0,\delta]$. We again use the Banach fixed point theorem, and thus we need to show that $\mathcal{K}$ is a contraction on $C^r[0,\delta]$ for a sufficiently small δ.

Let $r = 1$. Then for $y_1,\, y_2 \in C^1[0,\delta]$ we have

$$\|\mathcal{K}y_1 - \mathcal{K}y_2\|_{C^1[0,\delta]} = \|\mathcal{K}y_1 - \mathcal{K}y_2\|_{C[0,\delta]} + \|\,(\mathcal{K}y_1)' - (\mathcal{K}y_2)'\,\|_{C[0,\delta]},$$

where $(\mathcal{K}y)'$ is given by

$$(\mathcal{K}y)'(t) = \int_0^1 s^{-M(0)} f'(st)\,\mathrm{d}s + \gamma t^{\gamma-1}\int_0^1 s^{-M(0)} s^{\gamma-1} D(st)y(st)\,\mathrm{d}s$$
$$+\, t^{\gamma}\int_0^1 s^{-M(0)} s^{\gamma} D'(st)y(st)\,\mathrm{d}s + t^{\gamma}\int_0^1 s^{-M(0)} s^{\gamma} D(st)y'(st)\,\mathrm{d}s,\ t \in [0,\delta].$$

Moreover, by (19), the following estimate holds:

$$\begin{aligned}
&\|\mathcal{K}y_1 - \mathcal{K}y_2\|_{C^1[0,\delta]} = |\mathcal{K}y_1 - \mathcal{K}y_2\|_\delta + \|(\mathcal{K}y_1)' - (\mathcal{K}y_2)'\|_\delta \\
&\quad\leq\ L_Z\|y_1 - y_2\|_\delta \\
&\qquad + \max_{t\in[0,\delta]}\left\{\gamma t^{\gamma-1}\int_0^1 \left|s^{-M(0)}\right| s^{\gamma-1}\,\mathrm{d}s \max_{s\in[0,1]}|D(st)| \max_{s\in[0,1]}|y_1(st) - y_2(st)|\right\} \\
&\qquad + \max_{t\in[0,\delta]}\left\{t^{\gamma}\int_0^1 \left|s^{-M(0)}\right| s^{\gamma}\,\mathrm{d}s \max_{s\in[0,1]}|D'(st)| \max_{s\in[0,1]}|y_1(st) - y_2(st)|\right\} \\
&\qquad + \max_{t\in[0,\delta]}\left\{t^{\gamma}\int_0^1 \left|s^{-M(0)}\right| s^{\gamma}\,\mathrm{d}s \max_{s\in[0,1]}|D(st)| \max_{s\in[0,1]}|y_1'(st) - y_2'(st)|\right\} \\
&\quad\leq\ \left(L_Z + \gamma\delta^{\gamma-1} const.\,\|D\|_\delta + \delta^{\gamma} const.\,\|D'\|_\delta\right)\|y_1 - y_2\|_\delta \\
&\qquad + \delta^{\gamma} const.\,\|D\|_\delta\|y_1' - y_2'\|_\delta.
\end{aligned}$$

Therefore, for a sufficiently small δ, $\mathcal{K}$ is a contracting operator on $C^1[0,\delta]$. Now, let $r \geq 2$. By similar arguments we obtain a contraction on $C^r[0,\delta]$. This yields a unique solution $y \in C^r[0,\delta]$ of (5) on $[0,\delta]$, which can be uniquely extended to $t = 1$. For $f,\, D \in C^r[0,1]$, the classical theory implies the existence of a unique solution $z \in C^r(0,1]$ of system (4) subject to the initial condition $z(\delta) = y(\delta)$. Hence, $z = y$ on $[0,1]$ and $y \in C^r[0,1]$ which completes the proof.

□

Remark 8. In the case when $M(0) = 0$, $(m = n)$, we can also study the unique solvability of TVP (6). The existence and uniqueness of a solution $z \in C(0,1]$ of (6) follows from the classical theory. To obtain the respective result for $[0,1]$, we investigate equation (4) subject to the terminal condition $y(\delta) = z(\delta)$, where $\delta > 0$ is sufficiently small. The corresponding operator $\mathcal{K} : C[0,\delta] \to C[0,\delta]$ has in this case the form

$$(\mathcal{K}y)(t) = z(\delta) + \int_\delta^t s^{-1}f(s)\,\mathrm{d}s + \int_\delta^t s^{\gamma-1}D(s)y(s)\,\mathrm{d}s.$$

We can show that $\mathcal{K}$ is contractive in the way analogous to the case when $M(0) \neq 0$.

7 General IVPs, TVPs and BVPs

In this section, we discuss general IVPs (5) and TVPs (6), where all conditions which are necessary and sufficient to specify a unique solution $y \in C[0,1]$ are posed at only one point, either at $t = 0$ or at $t = 1$. According to the results derived above, restrictions on the spectrum of $M(0)$ need to be made.

A.1 For IVP (5) we assume that the matrix $M(0)$ has only eigenvalues with nonpositive real parts and if $\sigma = 0$ then $\lambda = 0$.

A.2 For TVP (6) we assume that the matrix $M(0)$ has only eigenvalues with nonnegative real parts and if $\sigma = 0$ then $\lambda = 0$. Additionally, if zero is an eigenvalue of $M(0)$, then the associated invariant subspace is assumed to be the eigenspace of $M(0)$.

For the subsequent discussion, we introduce the following notation:

X_+ is the invariant subspace associated with the eigenvalues with positive real parts;

$X_0^{(e)}$ is the space spanned by the eigenvectors associated with eigenvalues $\lambda = 0$;

X_- is the invariant subspace associated with the eigenvalues with negative real parts;

$X_0^{(h)}$ is the space spanned by the generalized eigenvectors associated with $\lambda = 0$;

S is the orthogonal projection onto X_+;

R is the orthogonal projection onto $X_0^{(e)}$;

$P := R + S$ is the projection onto $X_+ \oplus X_0^{(e)}$;

$Q := I - P$ is the projection onto $X_- \oplus X_0^{(h)}$;

Z is the orthogonal projection onto $X_0^{(e)} \oplus X_0^{(h)}$;

N is the orthogonal projection onto X_-;

H is the orthogonal projection onto $X_0^{(h)}$.

All projections are constructed using the generalized eigenbasis of $M(0)$.

Theorem 9. *Let us assume that A.1 holds, the $m \times m$ matrix $B_0\tilde{R}$ is nonsingular, and $\beta \in \mathbb{R}^m$. Then, for any $f \in C[0,1]$ such that Zf satisfies (12) and ZM satisfies (16), there exists a unique solution $y \in C[0,1]$ of IVP (5).*

Proof: The existence of a unique continuous solution follows from Theorems 5 and 7 by applying the Banach fixed point theorem to the linear operator $\mathcal{K} : C[0,\delta] \to C[0,\delta], \delta > 0$, defined by

$$(\mathcal{K}y)(t) = \tilde{R}(B_0\tilde{R})^{-1}\beta + \int_0^1 s^{-M(0)} s^{-1} f(st)\,\mathrm{d}s + t^\gamma \int_0^1 s^{-M(0)} s^{\gamma-1} D(st)y(st)\,\mathrm{d}s.$$

For a sufficiently small δ, the operator $\mathcal{K}$ is a contraction and the Banach fixed point theorem yields the existence of a unique continuous solution of (5) on the interval $[0,\delta]$. This solution can be uniquely extended to $t = 1$.

□

Theorem 10. *Let us assume that A.2 holds, $B_1 \in \mathbb{R}^{n\times n}$ is nonsingular, and $\beta \in \mathbb{R}^n$. Then, for any $f \in C[0,1]$ such that Rf satisfies (12), $Sf \in C^1[0,1]$, and M satisfies condition (16), there exists a unique solution $y \in C[0,1]$ of TVP (6).*

Proof: The existence and uniqueness of a solution $z \in C(0,1]$ of problem (6) follows from the classical theory. Therefore, we investigate (4) on $[0,\delta]$, $\delta \in (0,1]$, subject to $y(\delta) = z(\delta)$. The result follows from Theorem 6 and Remark 8, when the Banach fixed point theorem is applied to the operator $\mathcal{K} : C[0,\delta] \to C[0,\delta]$, $\delta \in (0,1]$, defined by

$$\begin{aligned}(\mathcal{K}y)(t) \;:=\; & t^{M(0)}\delta^{-M(0)}z(\delta) + t^{M(0)}\int_\delta^t s^{-M(0)-I}f(s)\,\mathrm{d}s \\ & + t^{M(0)}\int_\delta^t s^{-M(0)}s^{\gamma-1}D(s)y(s)\,\mathrm{d}s.\end{aligned}$$

Since for a sufficiently small δ the operator $\mathcal{K}$ is contracting, there exists a unique continuous solution y of system (4) on $[0,\delta]$. Since $y(\delta) = z(\delta)$, $y = z$ on $[0,1]$ follows and this completes the proof.

□

Finally, we study the general linear BVPs (3), where the matrix $M(0)$ may have an arbitrary spectrum,

$$y'(t) = \frac{M(t)}{t}y(t) + \frac{f(t)}{t}, \quad B_0 y(0) + B_1 y(1) = \beta. \tag{20}$$

Lemma 11. *Consider the following BVP:*

$$y'(t) = \frac{M(t)}{t}y(t) + \frac{f(t)}{t}, \tag{21}$$

$$Hy(0) = 0,\; M(0)Ny(0) = -Nf(0),\; Sy(1) = S\eta,\; Ry(0) = R\eta. \tag{22}$$

Let us assume that $f \in C[0,1]$ is such that Zf satisfies (12) and $Sf \in C^1[0,1]$. Moreover, let $M \in C[0,1]$ be given in such a way that the projections SM and ZM satisfy condition (16), and $\eta \in \mathbb{R}^n$. Then, there exists a unique solution $y \in C[0,1]$ of the BVP (21), (22).

Proof: According to the previous results, the solution y of (21), (22) consists of three contributions which depend on the real parts of eigenvalues of $M(0)$,

$$y = Ny + Sy + Zy.$$

First, we consider system (21) posed on an interval $[0,\delta]$, $\delta \in (0,1]$, subject to the following boundary conditions:

$$Hy(0) = 0,\; M(0)Ny(0) = -Nf(0),\; Sy(\delta) = S\zeta,\; Ry(0) = R\eta, \tag{23}$$

where $\zeta \in \mathbb{R}^n$ will be specified later. In order to prove the existence of a unique continuous solution of (21), (23), we apply the Banach fixed point theorem to

$$y = \mathcal{K}y,\; y \in C[0,\delta].$$

The operator $\mathcal{K} : C[0,\delta] \to C[0,\delta]$ is now defined as

$$(\mathcal{K}y)(t) := N(\mathcal{K}y)(t) + S(\mathcal{K}y)(t) + Z(\mathcal{K}y)(t),$$

where

$$
\begin{aligned}
N(\mathcal{K}y)(t) &:= N\int_0^1 s^{-M(0)}s^{-1}f(st)\,\mathrm{d}s + N\int_0^1 s^{-M(0)}s^{-1}\left(M(st)-M(0)\right)y(st)\,\mathrm{d}s,\\
S(\mathcal{K}y)(t) &:= \left(\frac{t}{\delta}\right)^{M(0)} S\zeta + t^{M(0)}S\int_\delta^t s^{-M(0)-I}f(s)\,\mathrm{d}s\\
&+ t^{M(0)}S\int_\delta^t s^{-M(0)}s^{\gamma-1}D(s)y(s)\,\mathrm{d}s,\\
Z(\mathcal{K}y)(t) &:= t^{M(0)}R\eta + Z\int_0^1 s^{-M(0)}s^{-1}f(st)\,\mathrm{d}s\\
&+ t^{\gamma}Z\int_0^1 s^{-M(0)}s^{\gamma-1}D(st)y(st)\,\mathrm{d}s.
\end{aligned}
$$

According to Theorems 5, 6, and 7, the operator $\mathcal{K}$ maps $C[0,\delta]$ into itself. Moreover, we can choose δ sufficiently small for

$$
L_N < \frac{1}{3},\ L_S < \frac{1}{3},\ L_Z < \frac{1}{3}
$$

to hold, cf. (15), (18) and (19). Therefore, $L := L_N + L_S + L_Z < 1$ and

$$
\begin{aligned}
\|\mathcal{K}y_1 - \mathcal{K}y_2\|_\delta &\le \|N\left(\mathcal{K}y_1 - \mathcal{K}y_2\right)\|_\delta + \|S\left(\mathcal{K}y_1 - \mathcal{K}y_2\right)\|_\delta + \|Z\left(\mathcal{K}y_1 - \mathcal{K}y_2\right)\|_\delta\\
&\le \left(L_N + L_S + L_Z\right)\|y_1 - y_2\|_\delta = L\|y_1 - y_2\|_\delta.
\end{aligned}
$$

Consequently, $\mathcal{K}$ is a contracting operator on $C[0,\delta]$. According to the Banach fixed point theorem, there exists a unique continuous solution y of problem (21), (23) on $[0,\delta]$. Since there is no singularity on $[\delta,1]$, we can use the classical theory to extend the solution contributions Ny and Zy to $[0,1]$. Moreover, the existence and uniqueness of a solution contribution $Sz \in C[\delta,1]$ of regular equation (21) subject to $Sz(1) = S\eta$ follows from the classical theory. Therefore, we choose ζ such that $S\zeta = Sz(\delta)$ and we put $Sy := Sz$ on $[\delta,1]$. Altogether, $Sy(\delta) = Sz(\delta)$ and Sy is the second solution contribution defined on $[0,1]$. This completes the proof.

□

In order to discuss the solvability of (20), we first use the superposition principle to provide an alternative representation of a general continuous solution of system (4),

$$
y(t) = \tilde{y}(t) + Y(t)\alpha,\ t \in [0,1],
$$

where $\alpha \in \mathbb{R}^m$, $\tilde{y}$ is the unique particular solution of (4) subject to boundary conditions

$$
H\tilde{y}(0) = 0, \quad M(0)N\tilde{y}(0) = -Nf(0), \quad P\tilde{y}(1) = 0,
$$

and Y is the unique continuous fundamental solution matrix of the homogeneous system

$$
Y'(t) = \frac{M(t)}{t}Y(t),\ t \in [0,1],
$$

with $Y(1) = \tilde{P}$, where $\tilde{P}$ is the $n \times m$ matrix consisting of the linearly independent columns of P, cf. [6].

To determine the vector α, we first provide another form of the solution of (21), (22). To this aim, we rewrite terms in $Zy(t)$, $t \in [0,1]$ and obtain

$$
\begin{aligned}
Zy(t) &= R\tilde{\eta} + t^{M(0)} R \int_1^t s^{-M(0)-I} f(s)\,\mathrm{d}s + t^{M(0)} H \int_0^t s^{-M(0)-I} f(s)\,\mathrm{d}s \\
&+ \; t^{M(0)} R \int_1^t s^{-M(0)-I} s^{\gamma} D(s) y(s)\,\mathrm{d}s + t^{M(0)} H \int_0^t s^{-M(0)-I} s^{\gamma} D(s) y(s)\,\mathrm{d}s,
\end{aligned}
$$

where

$$
\tilde{\eta} = \eta + R \int_0^1 s^{-M(0)-I} f(s)\,\mathrm{d}s + R \int_0^1 s^{-M(0)-I} s^{\gamma} D(s) y(s)\,\mathrm{d}s.
$$

Consequently, the solution $y \in C[0,1]$ of (21), (22) has the integral representation,

$$
\begin{aligned}
y(t) \;=\; & t^{M(0)} P\tilde{\eta} + t^{M(0)} P \int_1^t s^{-M(0)-I} f(s)\,\mathrm{d}s + t^{M(0)} Q \int_0^t s^{-M(0)-I} f(s)\,\mathrm{d}s \\
& + t^{M(0)} P \int_1^t s^{-M(0)-I} s^{\gamma} D(s) y(s)\,\mathrm{d}s \\
& + t^{M(0)} Q \int_0^t s^{-M(0)-I} \left(M(t) - M(0)\right) y(s)\,\mathrm{d}s,
\end{aligned}
$$

and satisfies the following boundary conditions (note that $S\tilde{\eta} = S\eta$):

$$
Hy(0) = 0, \;\; M(0)Ny(0) = -Nf(0), \;\; Py(1) = P\tilde{\eta}.
$$

Now we turn to the general boundary conditions specified in (20). From the above solution representation, for details see [6], we conclude

$$
y(0) = (P+N)\tilde{y}(0) + \tilde{R}\alpha, \quad y(1) = Q\tilde{y}(1) + \tilde{P}\alpha.
$$

Substituting $y(0)$ and $y(1)$ into the boundary conditions (20), we obtain

$$
B_0 y(0) + B_1 y(1) = B_0 \Big((P+N)\tilde{y}(0) + \tilde{R}\alpha \Big) + B_1 \Big(Q\tilde{y}(1) + \tilde{P}\alpha \Big) = \beta.
$$

Thus,

$$
\Big(B_0 \tilde{R} + B_1 \tilde{P} \Big) \alpha = \beta - B_0 (P\tilde{y}(0) + N\tilde{y}(0)) - B_1 Q\tilde{y}(1),
$$

and the unknown vector α can be uniquely determined if the $m \times m$ matrix

$$
B_0 \tilde{R} + B_1 \tilde{P}
$$

is nonsingular.

The following theorem stated without proof is a consequence of the above results.

Theorem 12. *Consider BVP* (20)*, where the inhomogeneity f is given in such a way such that $f \in C[0,1]$, Zf satisfies* (12)*, and $Sf \in C^1[0,1]$. Let the coefficient matrix $M \in C[0,1]$ be such that its projections PM, HM satisfy condition*

(16). *Moreover, let* $B_0, B_1 \in \mathbb{R}^{m\times n}$*,* $\beta \in \mathbb{R}^m$*,* $m = \operatorname{rank} P$*, and the* $m \times m$ *matrix* $B_0\tilde{R} + B_1\tilde{P}$ *be nonsingular. Then, BVP* (20) *has a unique continuous solution* $y \in C[0,1]$*. This solution satisfies two sets of initial conditions,*

$$Hy(0) = 0, \quad M(0)Ny(0) = -Nf(0)$$

which are necessary and sufficient for $y \in C[0,1]$*.*

Remark 13. Note that the smoothness result $y \in C^r[0,1]$ can be shown by applying results derived in Sections 4, 5, and 6 to the corresponding projections of the function f and the matrix M.

8 Conclusions

In the present paper, we investigated the analytical properties of the singular BVP with a variable coefficient matrix

$$y'(t) = \frac{M(t)}{t}y(t) + \frac{f(t)}{t}, \quad B_0y(0) + B_1y(1) = \beta.$$

The structure of the correctly posed boundary conditions which guarantee the existence of a unique solution $y \in C[0,1]$ depends on the spectral properties of the matrix $M(0)$. Therefore, we had to carry out in full detail the following three case studies, the case of only negative real parts of the eigenvalues of $M(0)$, positive real parts of the eigenvalues of $M(0)$, and zero eigenvalues of $M(0)$. The main technical tool used in the analysis is the Banach fixed point theorem which turned out to be very useful in mastering the difficulties caused by the singularity at $t = 0$. These case investigations were then used to generalize the results to the cases of general initial value, terminal value, and boundary value problems.

The study of the convergence properties of the collocation method applied to approximate the solution of the analytical problem has been postponed to Part 2 [7] of the paper.

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