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Stability of an Euler-Bernoulli beam with a nonlinear dynamic feedback system

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Abstract

This paper is concerned with the stability analysis of a lossless Euler-Bernoulli beam that carries a tip payload which is coupled to a nonlinear dynamic feedback system. This setup comprises nonlinear dynamic boundary controllers satisfying the nonlinear KYP lemma as well as the interaction with a nonlinear passive environment. Global-in-time wellposedness and asymptotic stability is rigorously proven for the resulting closed-loop PDE–ODE system. The analysis is based on semigroup theory for the corresponding first order evolution problem. For the large-time analysis, precompactness of the trajectories is shown by deriving uniform-in-time bounds on the solution and its time derivatives.

1. Introduction

Let us consider a linear homogeneous Euler-Bernoulli beam, clamped at one end and with tip mass at the other free end. The state of the beam at time t is described by its transverse deflection $u(t,x)$ from the zero-state, where $x \in [0,L]$ is the longitudinal coordinate of the beam, see Figure 1. The well known PDE for the motion of the beam reads as

$$\rho u_{tt}(t,x) + \Lambda u^{\mathrm{IV}}(t,x) = 0, \tag{1.1}$$

with the mass per unit length ρ and the flexural rigidity Λ. The boundary conditions for the clamped end at $x = 0$ are given by

$$u(t,0) = u'(t,0) = 0, \tag{1.2}$$

and for the free end at $x = L$, we have

$$Ju'_{tt}(t,L) + \Lambda u''(t,L) = -\tau_e \tag{1.3a}$$

$$Mu_{tt}(t,L) - \Lambda u'''(t,L) = -f_e, \tag{1.3b}$$

where J and M denote the mass moment of inertia and the mass of the tip mass, respectively, and $-\tau_e$ and $-f_e$ describe the external torque and force acting on the tip mass. Here and in the following, the notation u_t is used for the derivative with respect to the time variable t, and u' for the x-derivative.

In literature, there exists a number of contributions dealing with the design of boundary controllers to stabilize this type of system. To mention but a few, in [12] the asymptotic stability was shown using semigroup formulation and applying the La Salle Invariance principle. To obtain stronger, exponential stability, frequency domain criteria [17], Riesz basis property [8], [9] or energy multiplier methods [23], [3] were employed. In contrast to these works, which are mainly based on linear static and dynamic boundary controllers, this paper is concerned with the interaction of the Euler-Bernoulli beam (1.1) - (1.3) with a dynamic nonlinear feedback system. In particular, it is assumed that this feedback system generates a reaction torque $\tau_e = \tau_{e,1} + \tau_{e,2}$ and a reaction force $f_e = f_{e,1} + f_{e,2}$, respectively. The reaction torque and force is composed of the response of a nonlinear spring- damper system

$$\tau_{e,1} = d_1(u_t'(t,L)) + k_1(u'(t,L)) \tag{1.4a}$$
$$f_{e,1} = d_2(u_t(t,L)) + k_2(u(t,L)) \tag{1.4b}$$

and the response of a nonlinear finite-dimensional system with state $z_j \in \mathbb{R}^{n_j}$, $j = 1, 2$,

$$(z_1)_t = a_1(z_1) + b_1(z_1)u_t'(t,L) \tag{1.5a}$$
$$\tau_{e,2} = c_1(z_1) \tag{1.5b}$$

and

$$(z_2)_t = a_2(z_2) + b_2(z_2)u_t(t,L) \tag{1.6a}$$
$$f_{e,2} = c_2(z_2), \tag{1.6b}$$

which constitutes a strictly passive map from the time derivative of the tip angle $u_t'(t,L)$ to the reaction torque $\tau_{e,2}$ and from the velocity of the tip position $u_t(t,L)$ to the reaction force $f_{e,2}$, respectively. The functions a_j, b_j, c_j, d_j, and k_j, $j = 1, 2$ as well as their mathematical properties will be specified in detail in the next section. Note that (1.4) - (1.6) represents a nonlinear dynamic boundary controller for the Euler-Bernoulli beam (1.1) - (1.3), a nonlinear dynamic environment, or a combination of both. The system (1.1) - (1.6) may be interpreted as a feedback interconnected system with the lossless Euler Bernoulli beam (1.1) - (1.3) in the forward path and the passive spring-damper system (1.4) as well as the strictly passive system (1.5), (1.6) in the feedback path, see Figure 2. It is well known that the feedback interconnection of passive systems preserves the passivity, see, e.g., [26]. This fact is often exploited in the controller design, see, e.g., [19], [20], for the finite-dimensional case. However, in the infinite-dimensional case the analysis is typically confined to linear systems, see, e.g., [14], [11], [27], or very recently [22]. Thus, with this work we want to take a first step towards an extension of the state of the art to the nonlinear case by still considering a linear PDE but allowing for a nonlinear ODE at the boundary.

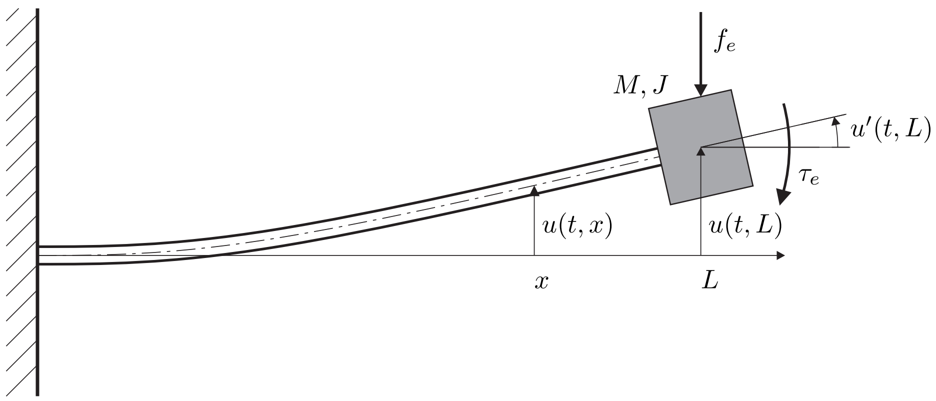

Figure 1: Euler-Bernoulli beam with tip mass.

The goal of this paper is to prove the global-in-time wellposedness and, most of all, the asymptotic stability of the feedback interconnected system (1.1) - (1.6) according to Figure 2. For both aspects, we have to deviate from the strategy employed in the analogous linear model (introduced and analyzed in [11, 15]): In the linear case, the generator of the evolution semigroup is dissipative, which readily yields large-time solutions. The nonlinear semigroup for (1.1) - (1.6) is *not* dissipative (in the sense of [4]). Hence, standard semigroup theory will first only yield local-in-time solutions, and the construction of an appropriate Lyapunov functional for (1.1) - (1.6) then shows their global existence.

Asymptotic stability of the linear counterpart model is based on the precompactness of the trajectories, which can be obtained from the compactness of the resolvent for the generator. For

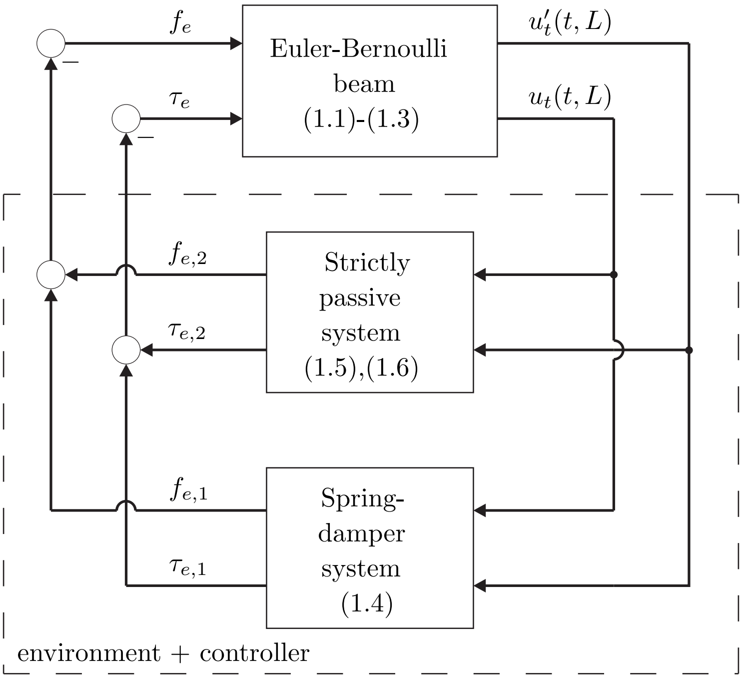

Figure 2: Interconnection of the Euler-Bernoulli beam system to a passive spring-damper system and a strictly passive system in the feedback path.

(1.1) - (1.6), we shall follow a strategy devised for a simpler systems in [16] (it consists of a beam with a nonlinear spring and damper at the free end). For the precompactness of the trajectories of (1.1) - (1.6), we shall here prove uniform C^1–bounds (w.r.t. time) on the solution, combined with compact Sobolev embeddings.

Note that the beam in (1.1) - (1.6) (and in its linear counterpart) is undamped. Damping of the complete feedback system is only introduced via the damper of (1.4) and the strictly passive systems (1.5), (1.6). This motivates that the linear model from [15] is asymptotically stable, but *not* exponentially stable. Hence, exponential stability also cannot be expected for our nonlinear system (1.1) - (1.6). Of course, exponential stability could be enforced by including damping terms into (1.1) (either a viscous damping of the form $+\alpha u_t$ or a Kelvin-Voigt damping of the form $+\alpha u_t^{\mathrm{IV}}$). While viscous damping would lead to a simple extension of the subsequent analysis, the higher order derivatives in the Kelvin-Voigt damping would require a rather different mathematical setup. Hence, we shall not elaborate on such dampings here.

This paper is organized as follows: In Section 2, the technical assumptions made for the coefficients and functions of the system (1.1) - (1.6) are specified, and in Section 3 the problem is formulated as a first order evolution equation. Using semigroup theory we prove in Section 4 that it has a unique global-in-time solution. In Section 5, the possible ω–limit set of this evolution is derived and analyzed. For proving the asymptotic stability of (1.1) - (1.6), we have to distinguish between two cases. For linear functions k_j, it is shown in Section 6 that asymptotic stability can be achieved for all mild solutions. For nonlinear k_j, it is much more involved to prove precompactness of the trajectories. In this case, asymptotic stability of classical solutions is shown in Section 7.

2. Preliminaries

In the following sections, we will give a rigorous mathematical analysis of the feedback interconnected system (1.1) - (1.6) according to Figure 2. For this, the assumptions on the parameters and functions appearing in (1.1) - (1.6) have to be specified. First of all, let us assume that the mass per unit length ρ, the flexural rigidity Λ, the mass moment of inertia J, and the mass M of the tip mass are constant and positive. For the spring-damper system (1.4), d_j and k_j are supposed to be C^2-regular and the following assumptions hold:

- For the scalar functions d_j there holds

$$d_j'(s) \geq 0, \quad \forall s \in \mathbb{R}, \tag{2.1a}$$

  and there exist constants $D_j > 0$ such that

$$\begin{aligned} d_j(s) &= D_j s + \delta_j(s), \quad \forall s \in \mathbb{R}, & (2.1b)\\ \delta_j(s) &= \mathcal{O}(s), \qquad \text{as } s \to 0. & (2.1c) \end{aligned}$$

- For the scalar functions k_j there exist constants $K_j > 0$ such that

$$\begin{aligned} k_j(s) &= K_j s + \kappa_j(s), \quad \forall s \in \mathbb{R}, & (2.2a)\\ \kappa_j(s) &= \mathcal{O}(s), \qquad \text{as } s \to 0, & (2.2b) \end{aligned}$$

  and furthermore

$$\begin{aligned} V_{k_j}(s) = \int_0^s k_j(\sigma)\,\mathrm{d}\sigma &> 0, \quad \forall s \in \mathbb{R}, & (2.2c)\\ \kappa_j(0) &= 0. & (2.2d) \end{aligned}$$

Based on these assumptions, it can be easily shown that the spring-damper system (1.4) is strictly passive from the inputs $u_t'(t, L)$ and $u_t(t, L)$ to the outputs $\tau_{e,1}$ and $f_{e,1}$, respectively, with the positive definite storage functions V_{k_j}, $j = 1, 2$, according to (2.2c).

The functions a_j, b_j, and c_j appearing in the description (1.5), (1.6) are also supposed to be C^2-regular. Since (1.5), (1.6) is strictly passive from $u_t'(t, L)$ to $\tau_{e,2}$ and from $u_t(t, L)$ to $f_{e,2}$, according to the Kalman-Yakubovich-Popov (KYP) lemma for nonlinear systems with affine input, see Lemma 4.4 in [13], there exist non-negative storage functions $V_j(z_j)$, $V_j(0) = 0$ such that

$$\begin{aligned} \nabla V_j(z_j) \cdot a_j(z_j) &< 0, \quad \forall z_j \neq 0, & (2.3a)\\ \nabla V_j(z_j) \cdot b_j(z_j) &= c_j(z_j). & (2.3b) \end{aligned}$$

The proofs in the upcoming sections demand for some technical assumptions on the storage functions and the coefficient functions a_j, b_j, and c_j, which are not restrictive at all[1]:

- The storage functions $V_j(z_j)$ are supposed to be element of $C^3(\mathbb{R}^{n_j}; \mathbb{R})$ with

$$\begin{aligned} \forall z_j \in \mathbb{R}^{n_j}: \quad V_j(z_j) \geq 0, \quad V_j(0) = 0, && (2.4a)\\ \lim_{|z_j| \to \infty} V_j(z_j) = \infty, && (2.4b)\\ P_j := \operatorname{Hess}(V_j)(0) > 0. && (2.4c) \end{aligned}$$

- There exist *regular* matrices $A_j \in \mathbb{R}^{n_j \times n_j}$ such that for all $z_j \in \mathbb{R}^{n_j}$

$$\begin{aligned} a_j(z_j) &= A_j z_j + \alpha_j(z_j), & (2.5a)\\ |\alpha_j(z_j)| &= \mathcal{O}(|z_j|^2) \quad \text{as } z_j \to 0. & (2.5b) \end{aligned}$$

  Note that (2.3a) implies

$$\begin{aligned} z_j^\top (P_j A_j) z_j &\leq 0, \quad \forall z_j \in \mathbb{R}^{n_j}, & (2.5c)\\ |\nabla V_j(z_j) \cdot a_j(z_j)| &\geq C|z_j|^2 \quad \text{as } z_j \to 0, & (2.5d) \end{aligned}$$

  for some positive constant C.

[1] Note that condition (2.4b) can be relaxed, see Remark 4.10.

- There exist vectors $B_j \in \mathbb{R}^{n_j}$ such that for all $z_j \in \mathbb{R}^{n_j}$

$$b_j(z_j) = B_j + \beta_j(z_j), \tag{2.6a}$$
$$\beta_j(0) = 0. \tag{2.6b}$$

- There exist vectors $C_j \in \mathbb{R}^{n_j}$ such that for all $z_j \in \mathbb{R}^{n_j}$

$$c_j(z_j) = C_j \cdot z_j + \gamma_j(z_j), \tag{2.7a}$$
$$|\gamma_j(z_j)| = \mathcal{O}(|z_j|^2) \quad \text{as } z_j \to 0. \tag{2.7b}$$

Note that (2.3b) implies

$$P_j B_j = C_j. \tag{2.7c}$$

3. Formulation as an Evolution Equation

System (1.1) - (1.6) is reformulated as an evolution equation in the Hilbert space

$$\mathcal{H} = \{y = [u, v, z_1, z_2, \xi, \psi]^\top : u \in \tilde{H}_0^2(0, L), v \in L^2(0, L), z_j \in \mathbb{R}^{n_j}, \xi, \psi \in \mathbb{R}\},$$

where

$$\tilde{H}_0^n(0, L) := \{f \in H^n(0, L) : f(0) = f'(0) = 0\}, \quad \text{for } n \geq 2.$$

The inner product is defined by

$$\begin{aligned}\langle y, \tilde{y}\rangle_{\mathcal{H}} = &\Lambda \int_0^L u''\tilde{u}''\,\mathrm{d}x + \rho \int_0^L v\tilde{v}\,\mathrm{d}x + \frac{1}{J}\xi\tilde{\xi} + \frac{1}{M}\psi\tilde{\psi} \\ &+ K_1 u'(L)\tilde{u}'(L) + K_2 u(L)\tilde{u}(L) + z_1^\top P_1 \tilde{z}_1 + z_2^\top P_2 \tilde{z}_2,\end{aligned} \tag{3.1}$$

where the positive definite matrices P_j are due to (2.4c). For the following, the operator

$$\mathcal{A} : \begin{bmatrix} u \\ v \\ z_1 \\ z_2 \\ \xi \\ \psi \end{bmatrix} \mapsto \begin{bmatrix} v \\ -\frac{\Lambda}{\rho} u^{\mathrm{IV}} \\ a_1(z_1) + \frac{1}{J} b_1(z_1)\xi \\ a_2(z_2) + \frac{1}{M} b_2(z_2)\psi \\ -\Lambda u''(L) - [c_1(z_1) + d_1(\frac{\xi}{J}) + k_1(u'(L))] \\ \Lambda u'''(L) - [c_2(z_2) + d_2(\frac{\psi}{M}) + k_2(u(L))] \end{bmatrix}$$

is introduced on the domain

$$D(\mathcal{A}) = \{y \in \mathcal{H} : u \in \tilde{H}_0^4(0, L), v \in \tilde{H}_0^2(0, L), \xi = Jv'(L), \psi = Mv(L)\}. \tag{3.2}$$

Based on the formulation of the coefficient functions, the operator $\mathcal{A}$ is decomposed into a linear and a nonlinear part:

Linear part: The linear part is denoted by A, which is the linearization of $\mathcal{A}$ around the origin:

$$A : \begin{bmatrix} u \\ v \\ z_1 \\ z_2 \\ \xi \\ \psi \end{bmatrix} \mapsto \begin{bmatrix} v \\ -\frac{\Lambda}{\rho} u^{\mathrm{IV}} \\ A_1 z_1 + \frac{1}{J} B_1 \xi \\ A_2 z_2 + \frac{1}{M} B_2 \psi \\ -\Lambda u''(L) - [C_1 z_1 + \frac{1}{J} D_1 \xi + K_1 u'(L)] \\ \Lambda u'''(L) - [C_2 z_2 + \frac{1}{M} D_2 \psi + K_2 u(L)] \end{bmatrix},$$

and the domain is $D(A) = D(\mathcal{A})$.

Nonlinear part: The nonlinear part will be referred to as $\mathcal{N}$, and it is defined as the difference $\mathcal{N} := \mathcal{A} - A$. i.e.

$$\mathcal{N} : \begin{bmatrix} u \\ v \\ z_1 \\ z_2 \\ \xi \\ \psi \end{bmatrix} \mapsto \begin{bmatrix} 0 \\ 0 \\ \alpha_1(z_1) + \frac{1}{J}\beta_1(z_1)\xi \\ \alpha_2(z_2) + \frac{1}{M}\beta_2(z_2)\psi \\ -\gamma_1(z_1) - \delta_1(\frac{\xi}{J}) - \kappa_1(u'(L)) \\ -\gamma_2(z_2) - \delta_2(\frac{\psi}{M}) - \kappa_2(u(L)) \end{bmatrix},$$

Theorem 3.1. *The linear operator A with domain $D(A)$ generates a C_0-semigroup of contractions in $\mathcal{H}$, denoted by $(\mathrm{e}^{tA})_{t\geq 0}$.*

Proof. This result has been shown in Section 4.2 in [11], we briefly sketch the main steps of the proof. A brief calculation yields for $y \in D(A)$, using (2.5c):

$$\langle Ay, y\rangle_{\mathcal{H}} = z_1^\top (P_1 A_1) z_1 + z_2^\top (P_2 A_2) z_2 - D_1 |v'(L)|^2 - D_2 |v(L)|^2 \leq 0.$$

Hence the operator A is dissipative in $\mathcal{H}$ with respect to the inner product (3.1). Furthermore, the inverse A^{-1} exists and is bounded (even compact). Now the statement immediately follows from the Lumer-Phillips theorem. $\square$

Remark 3.2. Since A is the infinitesimal generator of a C_0-semigroup of contractions, $\operatorname{ran}(\lambda - A) = \mathcal{H}$ for all $\lambda > 0$. In particular $\operatorname{ran}(I - A) = \mathcal{H}$. So A is *maximal dissipative* according to Theorem 2.2 in [4].

4. Existence of Solutions

We are interested in solutions of the following initial value problem in $\mathcal{H}$:

$$y_t(t) = \mathcal{A}y(t) = Ay(t) + \mathcal{N}y(t), \tag{4.1a}$$

$$y(0) = y_0 \in \mathcal{H}. \tag{4.1b}$$

Any (mild) solution $y(t) \in C([0,T];\mathcal{H})$, for $T > 0$, is known to satisfy Duhamel's formula:

$$y(t) = \mathrm{e}^{tA} y_0 + \int_0^t \mathrm{e}^{(t-s)A} \mathcal{N} y(s)\,\mathrm{d}s, \quad 0 \leq t < T. \tag{4.2}$$

Proposition 4.1. *For every $y_0 \in \mathcal{H}$, there exists some maximal $0 < T_{\max}(y_0) \leq \infty$ such that (4.1) has a unique mild solution $y(t)$ on $[0, T_{\max}(y_0))$. If $y_0 \in D(\mathcal{A})$, the corresponding mild solution $y(t)$ is a classical solution. If $T_{\max}(y_0) < \infty$, then $\lim_{t \nearrow T_{\max}(y_0)} \|y(t)\|_{\mathcal{H}} = \infty$.*

Proof. By assumption, the functions $\alpha_j, \beta_j, \gamma_j, \delta_j$ and κ_j are continuously differentiable, so $\mathcal{N} : \mathcal{H} \to \mathcal{H}$ is also continuously differentiable, and thus locally Lipschitz continuous. Furthermore, A is the generator of a C_0-semigroup, see Theorem 3.1. Now we may apply Theorem 6.1.4 in [21], which yields the existence of a unique mild solution, and the blowup at $T_{\max}(y_0)$. Moreover, Theorem 6.1.5 in [21] implies that for $y_0 \in D(\mathcal{A})$, any mild solution is a classical solution. $\square$

Next we introduce the functional $H : \mathcal{H} \to \mathbb{R}$, given by

$$H(y) := \frac{1}{2}\int_0^L \left(\Lambda |u''|^2 + \rho |v|^2\right) \mathrm{d}x + \frac{|\xi|^2}{2J} + \frac{|\psi|^2}{2M} + \underbrace{\int_0^{u'(L)} k_1(s)\,\mathrm{d}s}_{V_{k_1}(u'(L))} + \underbrace{\int_0^{u(L)} k_2(s)\,\mathrm{d}s}_{V_{k_2}(u(L))} + V_1(z_1) + V_2(z_2).$$

Note that the first integral term in $H(y)$ corresponds to the strain energy and kinetic energy of the Euler-Bernoulli beam, the next two summands are the translational and rotational part of the kinetic energy of the tip mass, $V_{k_j}, j = 1, 2$ is the potential energy stored in the nonlinear spring elements, see (1.4) and (2.2c), and $V_j, j = 1, 2$ are the non-negative storage functions of the strictly passive systems (1.5) and (1.6), respectively. Obviously $H(y) \geq 0$ for all $y \in \mathcal{H}$. Thus, $H(y)$ is exactly the sum of the storage functions of the lossless Euler-Bernoulli beam (1.1) - (1.3), the nonlinear spring-damper systems (1.4) and the strictly passive nonlinear dynamic feedback systems (1.5) and (1.6), cf. Figure 2. In the following, it will be shown that H qualifies as a Lyapunov function for the system (4.1).

Lemma 4.2. *The function H is continuous in $\mathcal{H}$.*

Proof. The continuity of the terms in H except for the k_j-terms is immediate. Due to the continuous embedding $H^2(0, L) \hookrightarrow C^1([0, L])$ the continuity of the remaining k_j-terms follows as well. □

Lemma 4.3. *Under the assumption* (2.4b) *we have for any sequence $\{y_n\}_{n\in\mathbb{N}} \subset \mathcal{H}$:*

$$\sup_{n\in\mathbb{N}} H(y_n) < \infty \quad \Leftrightarrow \quad \sup_{n\in\mathbb{N}} \|y_n\|_{\mathcal{H}} < \infty.$$

Proof. It suffices to notice that $\{V_j(z_{j,n})\}_{n\in\mathbb{N}}$ is unbounded iff $\{z_{j,n}\}_{n\in\mathbb{N}}$ is unbounded. □

We now define the generalized time derivative along the mild solution $y(t)$ of (4.1), i.e. for any initial value $y_0 \in \mathcal{H}$:

$$\dot{H}(y_0) := \limsup_{t\searrow 0} \frac{H(y(t)) - H(y_0)}{t},$$

which may take the value $-\infty$.

Lemma 4.4. *For $y_0 \in D(\mathcal{A})$ we have $\dot{H}(y_0) = \frac{\mathrm{d}^+}{\mathrm{d}t} H(y(t))|_{t=0} \leq 0$, i.e. H is monotonically decreasing along classical solutions.*

Proof. For $y_0 \in D(\mathcal{A})$ the corresponding solution $y(t)$ of (4.1) is classical, and therefore has a continuous right derivative on $[0, T_{\max}(y_0))$. So we can directly compute

$$\begin{aligned}\dot{H}(y_0) &= \frac{\mathrm{d}^+}{\mathrm{d}t} H(y(t))\big|_{t=0} \\ &= a_1(z_1) \cdot \nabla V_1(z_1) + a_2(z_2) \cdot \nabla V_2(z_2) - d_1(v'(L))v'(L) - d_2(v(L))v(L).\end{aligned}$$

Thereby we have used (2.3b). The non-positivity of the generalized time derivative of the storage function H can be directly concluded from (2.3a), (2.1a) and (2.1c). Clearly, this is also a consequence of the passivity property of the feedback interconnected system according to Figure 2. This concludes the proof. □

Corollary 4.5. *For $y_0 \in D(\mathcal{A})$ the corresponding classical solution $y(t)$ of* (4.1) *is global, i.e. it exists for all $t \in [0, \infty)$.*

Proof. According to Lemma 4.4, H is non-increasing along $y(t)$. Thus, according to Lemma 4.3 no blowup occurs in $y(t)$, and we have according to Proposition 4.1 that $T_{\max}(y_0) = \infty$. □

Since $\mathcal{N}$ is locally Lipschitz continuous and $D(\mathcal{A}) \subset \mathcal{H}$ is dense, we can apply Proposition 4.3.7 (ii) in [2] for the approximation of mild (non-classical) solutions:

Proposition 4.6. *Let $y_0 \in \mathcal{H}$ and $\{y_{0,n}\}_{n\in\mathbb{N}} \subset D(\mathcal{A})$ be such that $y_{0,n} \to y_0$ in $\mathcal{H}$. Denote by $y_n(t)$ the global classical solution of* (4.1) *to the initial value $y_{0,n}$ and by $y(t)$ the mild solution corresponding to the initial value y_0. Then $y_n(t) \to y(t)$ in $C([0, T]; \mathcal{H})$ for any $T \in (0, T_{\max}(y_0))$.*

Theorem 4.7. *For any $y_0 \in \mathcal{H}$ the corresponding solution $y(t)$ of the initial value problem* (4.1) *is global in time. Furthermore, $t \mapsto H(y(t))$ is non-increasing on $\mathbb{R}^+$ and $y(t)$ is uniformly bounded in $\mathcal{H}$ on $[0, \infty)$.*

Proof. Consider $y_0 \in \mathcal{H}$ and a sequence $\{y_{0,n}\} \subset D(\mathcal{A})$ with $y_{0,n} \to y_0$ in $\mathcal{H}$. Due to the convergence $y_n(t) \to y(t)$ for all $t \in [0, T_{\max}(y_0))$ shown in Proposition 4.6 and the continuity of H, we get $H(y_n(t)) \to H(y(t))$ for all $0 \le t < T_{\max}(y_0)$. Since H is non-increasing along every $y_n(t)$, this implies also that $t \mapsto H(y(t))$ is non-increasing on $[0, T_{\max}(y_0))$. Thus, according to Lemma 4.3 no blowup of $y(t)$ can occur at $t = T_{\max}(y_0)$. So, according to Proposition 4.1 the solution is global in time. Uniform boundedness of $y(t)$ now follows from Lemma 4.3. ☐

Corollary 4.8. *The function H is a Lyapunov function for the initial value problem* (4.1)*.*

Remark 4.9. Since all mild solutions are global, Proposition 4.6 holds for any $T \in (0, \infty)$.

For every $y_0 \in \mathcal{H}$ and the corresponding mild solution $y(t)$ we define $S(t)y_0 := y(t)$ for all $t \ge 0$. The family $S \equiv (S(t))_{t \ge 0}$ is a strongly continuous semigroup of nonlinear (bounded, continuous) operators in $\mathcal{H}$, cf. Theorem 9.3.2 in [2].

Remark 4.10. Since (2.4b) is only needed to show that no blowup of the solution occurs, we may replace it by the weaker assumption

$$\lim_{|z_j| \to \infty} V_j(z_j) > H(y_0), \tag{2.4b'}$$

depending on the initial condition y_0 for the problem (4.1). Thereby we argue as follows: According to Theorem 4.7 the function $t \mapsto H(y(t))$ is non-increasing (this is independent of (2.4b)), which ensures that no blowup can occur in any component of $y(t)$ except for z_j. If now $z_1(t)$ or $z_2(t)$ would blowup, we would get $\lim_{t\to\infty} H(y(t)) > H(y_0)$ according to (2.4b'). So $H(y(t))$ could not be monotonically decreasing, which is a contradiction. So (2.4b') is sufficient to show that no blowup occurs and that the solution is global in time.

5. ω-limit Set

In this section we investigate possible ω-limit sets of S. However, non-emptiness of the ω-limit sets will only be discussed in the subsequent sections. Again, S is the strongly continuous (nonlinear) semigroup generated by $\mathcal{A}$ on $\mathcal{H}$, defined at the end of the previous section. For $y_0 \in \mathcal{H}$ we define the trajectory $\gamma(y_0)$ by

$$\gamma(y_0) := \bigcup_{t \ge 0} S(t)y_0.$$

Definition 5.1 (ω-limit set)**.** Given the semigroup S, the ω-limit set for $y_0 \in \mathcal{H}$ is denoted by $\omega(y_0)$, and is the following set:

$$\omega(y_0) := \{y \in \mathcal{H} : \exists \{t_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^+, \lim_{n\to\infty} t_n = +\infty \wedge \lim_{n\to\infty} S(t_n)y_0 = y\}.$$

It is possible that $\omega(y_0) = \emptyset$.

According to Proposition 9.1.7 in [2] we have:

Lemma 5.2. *For $y_0 \in \mathcal{H}$ the set $\omega(y_0)$ is S-invariant, i.e. $S(t)\omega(y_0) \subseteq \omega(y_0)$ for all $t \ge 0$.*

Let us consider now some fixed $y_0 \in \mathcal{H}$. According to the results of Section 4, the function $t \mapsto H(S(t)y_0)$ is monotonically decreasing, and bounded from below by 0. Therefore, the following limit exists:

$$\mathfrak{H}(y_0) := \lim_{t\to\infty} H(S(t)y_0) \ge 0. \tag{5.1}$$

Lemma 5.3. *Suppose $\omega(y_0) \ne \emptyset$. Then there holds*

$$\forall y \in \omega(y_0) : \quad H(y) = \mathfrak{H}(y_0).$$

In particular, $\dot{H}(y) = 0$ for all $y \in \omega(y_0)$.

Proof. For every $y \in \omega(y_0)$ there exists a sequence $\{t_n\} \subset \mathbb{R}^+$ such that $S(t_n)y_0 \to y$. Since H is continuous, cf. Section 4, this implies that $H(y) = \lim_{n\to\infty} H(S(t_n)y_0)$. Due to (5.1) the right hand side equals $\mathfrak{H}(y_0)$, and the result follows. $\square$

We can use this lemma to identify the possible ω-limit sets by investigating trajectories along which the Lyapunov function H is constant.

Lemma 5.4. *Let $y \in \mathcal{H}$ such that $H(S(t)y) = \mathfrak{H}(y)$ for all $t \geq 0$, i.e. H is constant along $\gamma(y)$. Then $\gamma(y) \subset \{y \in \mathcal{H} : y = [u, v, 0, 0, 0, 0]^\top\}$.*

Proof. First, let $y \in D(\mathcal{A})$. We know from Lemma 4.4 and the corresponding proof that

$$\frac{\mathrm{d}}{\mathrm{d}t} H(S(t)y) = a_1(z_1)\cdot\nabla V_1(z_1) + a_2(z_2)\cdot\nabla V_2(z_2) - d_1(v'(L))v'(L) - d_2(v(L))v(L), \quad \forall t \geq 0, \tag{5.2}$$

where we omitted the dependence on t on the right hand side, i.e. $[u, v, z_j, z_2, Jv'(L), Mv(L)]^\top \equiv S(t)y$. Now (5.2) is required to be zero, and according to (2.3a) and (2.1c) this holds iff $\xi = \psi = z_1 = z_2 = 0$.

Now let $y \in \mathcal{H} \setminus D(\mathcal{A})$. Then there is a sequence $\{y_n\}_{n\in\mathbb{N}} \subset D(\mathcal{A})$ such that $y_n \to y$ as $n \to \infty$. According to Remark 4.9 we have $S(t)y_n \to S(t)y$ uniformly on $[0, T]$ for any $T > 0$. Therefore, we have also for the components

$$z_{j,n}(t) \to z_j(t), \quad \text{in } C([0,T];\mathbb{R}^{n_j}), \tag{5.3}$$
$$Mv_n(t, L) \to \psi(t), \quad \text{in } C([0,T];\mathbb{R}), \tag{5.4}$$
$$Jv_n'(t, L) \to \xi(t), \quad \text{in } C([0,T];\mathbb{R}). \tag{5.5}$$

Together with (5.2) this implies

$$\left\{\frac{\mathrm{d}}{\mathrm{d}t} H(S(t)y_n)\right\}_{n\in\mathbb{N}}$$

is a Cauchy sequence in $C([0,T];\mathbb{R})$. Since H is locally Lipschitz continuous in $\mathcal{H}$, we also have that $\{H(S(t)y_n)\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $C([0,T];\mathbb{R})$, so altogether $\{H(S(t)y_n)\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $C^1([0,T];\mathbb{R})$. So there exists a unique $h(t) \in C^1([0,T];\mathbb{R})$ such that

$$H(S(t)y_n) \to h(t) \quad \text{in } C^1([0,T];\mathbb{R}). \tag{5.6}$$

On the other hand we know that $\lim_{n\to\infty} H(S(t)y_n) = H(S(t)y) = \mathfrak{H}(y)$ for every $t \geq 0$, and hence $h(t) \equiv \mathfrak{H}(y)$. Together with (5.6) this implies $\frac{\mathrm{d}}{\mathrm{d}t}H(S(t)y_n) \to 0$ uniformly on $[0,T]$. By using (5.2) for every y_n this now yields that in (5.3) - (5.5) we obtain the limits $z_j(t) = \xi(t) = \psi(t) = 0$. So $S(t)y$ has to be of the form $S(t)y = [u(t), v(t), 0, 0, 0, 0]^\top$. $\square$

Before we prove that the ω-limit set consists only of the zero solution, we need the following technical lemma:

Lemma 5.5. *Let A generate a C_0-semigroup, and let $\mathcal{N} : \mathcal{H} \to \mathcal{H}$ be uniformly continuous and locally Lipschitz continuous. Furthermore, let every mild solution y of* (4.1) *be global. Then there holds for every $y_0 \in \mathcal{H}$ and for all $t > 0$:*

$$\int_0^t S(s)y_0 \,\mathrm{d}s \in D(\mathcal{A}), \tag{5.7}$$

and

$$S(t)y_0 - y_0 = A\int_0^t S(s)y \,\mathrm{d}s + \int_0^t \mathcal{N}S(s)y_0 \,\mathrm{d}s. \tag{5.8}$$

The proof is analogous to the proof of Lemma 5.4 in [16], see also [25] for a more general version of this lemma.

Theorem 5.6. *Let $\Omega \subset \mathcal{H}$ be an S-invariant set such that $H|_\Omega$ is constant. Then $\Omega = \{0\}$. In particular, for any $y_0 \in \mathcal{H}$ either $\omega(y_0) = \emptyset$ or $\omega(y_0) = \{0\}$.*

Proof. Take a fixed $y_0 \in \Omega$, and let $y(t)$ be the corresponding mild solution of (4.1). Clearly, $\gamma(y_0) \subset \Omega$, and according to Lemma 5.4 $y(t)$ is of the form $y(t) = [u(t), v(t), 0, 0, 0, 0]^\top$.

Step 1 (linear system for $u(t)$, $v(t)$): First we note that, according to (5.7), there holds for all $t \geq 0$:

$$0 = \int_0^t \psi(s)\,\mathrm{d}s = M\int_0^t v(s,L)\,\mathrm{d}s = M(u(t,L) - u_0(L)),$$
$$0 = \int_0^t \xi(s)\,\mathrm{d}s = J\Big(\int_0^t v(s,x)\,\mathrm{d}s\Big)'\Big|_{x=L} = J(u'(t,L) - u_0'(L)).$$

Thus $u(t,L)$ and $u'(t,L)$ are constant in time. According to (5.8) the (projected) mild solution $y_p(t) = [u(t), v(t)]^\top$ satisfies the following system (i.e. the first, second, fifth, and sixth component of (5.8)):

$$u(t) - u_0 = \int_0^t v(s,x)\,\mathrm{d}s, \tag{5.9a}$$
$$v(t) - v_0 = -\frac{\Lambda}{\rho}\Big(\int_0^t u(s,x)\,\mathrm{d}s\Big)^{\mathrm{IV}} \tag{5.9b}$$
$$0 = \Lambda\Big(\int_0^t u(s,x)\,\mathrm{d}s\Big)''\Big|_{x=L} + K_1\cdot\Big(\int_0^t u(s,x)\,\mathrm{d}s\Big)'\Big|_{x=L} + \int_0^t \kappa_1(u'(s,L))\,\mathrm{d}s, \tag{5.9c}$$
$$0 = -\Lambda\Big(\int_0^t u(s,x)\,\mathrm{d}s\Big)'''\Big|_{x=L} + K_2\cdot\Big(\int_0^t u(s,x)\,\mathrm{d}s\Big)\Big|_{x=L} + \int_0^t \kappa_2(u(s,L))\,\mathrm{d}s. \tag{5.9d}$$

Mild solutions satisfy $u \in C(\mathbb{R}^+; \tilde{H}_0^2(0,L))$. Hence, we can interchange the integration and differentiation in the last term of (5.9c). Using the fact that $u'(t,L)$ is constant, we have (for $u_0'(L) \neq 0$):

$$\int_0^t \kappa_1(u'(s,L))\,\mathrm{d}s = t\kappa_1(u_0'(L)) = \frac{\kappa_1(u_0'(L))}{u_0'(L)}\Big(\int_0^t u(s,x)\,\mathrm{d}s\Big)'\Big|_{x=L}.$$

Next we define the constants (since $\kappa_j(0) = 0$):

$$\begin{aligned}
\tilde{K}_1 &:= K_1 + \frac{\kappa_1(u_0'(L))}{u_0'(L)}, \quad \text{if } u_0'(L) \neq 0, \quad \text{else } \tilde{K}_1 := K_1,\\
\tilde{K}_2 &:= K_2 + \frac{\kappa_2(u_0(L))}{u_0(L)}, \quad \text{if } u_0(L) \neq 0, \quad \text{else } \tilde{K}_2 := K_2.
\end{aligned} \tag{5.10}$$

With this we may rewrite (5.9) as

$$u(t) - u_0 = \int_0^t v(s)\,\mathrm{d}s, \tag{5.11a}$$
$$v(t) - v_0 = -\frac{\Lambda}{\rho}\Big(\int_0^t u(s)\,\mathrm{d}s\Big)^{\mathrm{IV}} \tag{5.11b}$$
$$0 = \Lambda\Big(\int_0^t u(s,x)\,\mathrm{d}s\Big)''\Big|_{x=L} + \tilde{K}_1\Big(\int_0^t u(s,x)\,\mathrm{d}s\Big)'\Big|_{x=L}, \tag{5.11c}$$
$$0 = -\Lambda\Big(\int_0^t u(s,x)\,\mathrm{d}s\Big)'''\Big|_{x=L} + \tilde{K}_2\int_0^t u(s,x)\,\mathrm{d}s\Big|_{x=L}, \tag{5.11d}$$

making this system linear. Thus, the projected vector $y_p(t) = [u(t), v(t)]^\top$ is the unique mild solution of

$$(y_p)_t = A_p y_p, \tag{5.12a}$$
$$y_p(0) = [u_0, v_0]^\top, \tag{5.12b}$$

with the operator

$$A_p : \begin{bmatrix} u \\ v \end{bmatrix} \mapsto \begin{bmatrix} v \\ -\frac{\Lambda}{\rho} u^{\mathrm{IV}} \end{bmatrix}.$$

The equations (5.11c) and (5.11d) are incorporated into the domain $D(A_p)$. For further details on the operator A_p in the space $\mathcal{H}_p$ see the Appendix A.

Step 2 (proof of $u(t, L) = u'(t, L) = 0$): We now investigate solutions of the projected problem (5.12) with the additional property that $u(t, L)$ and $u'(t, L)$ are constant in time. Since the semigroup e^{tA_p} is unitary in $\mathcal{H}_p$, we know in particular that $\|v(t)\|_{L^2} \le C = \frac{1}{\rho}\|y_p(0)\|_{\mathcal{H}_p}$ for all $t \ge 0$ (cf. (A.1)). Applying the norm to (5.11b) this yields

$$\sup_{t \ge 0} \Big\| \Big(\int_0^t u(s)\,\mathrm{d}s \Big)^{\mathrm{IV}} \Big\|_{L^2(0,L)} < \infty. \tag{5.13}$$

Next we apply the following Gagliardo-Nirenberg inequalities (cf. [18]), which guarantee the existence of a $C > 0$ such that there holds for all $t \ge 0$:

$$\begin{aligned} \Big\| \int_0^t u(s)\,\mathrm{d}s \Big\|_{L^\infty(0,L)} &\le C \Big\| \Big(\int_0^t u(s)\,\mathrm{d}s \Big)^{\mathrm{IV}} \Big\|_{L^2(0,L)}^{\frac{1}{8}} \Big\| \int_0^t u(s)\,\mathrm{d}s \Big\|_{L^2(0,L)}^{\frac{7}{8}}, \\ \Big\| \int_0^t u'(s)\,\mathrm{d}s \Big\|_{L^\infty(0,L)} &\le C \Big\| \Big(\int_0^t u(s)\,\mathrm{d}s \Big)^{\mathrm{IV}} \Big\|_{L^2(0,L)}^{\frac{3}{8}} \Big\| \int_0^t u(s)\,\mathrm{d}s \Big\|_{L^2(0,L)}^{\frac{5}{8}}. \end{aligned} \tag{5.14}$$

The first factor on the right hand side in both inequalities is uniformly bounded (with respect to t) due to (5.13). For the second factor we observe that, according to Theorem 4.7, $t \mapsto \|u(t)\|_{L^2(0,L)}$ is uniformly bounded, and therefore $t \mapsto \|\int_0^t u(s)\,\mathrm{d}s\|_{L^2(0,L)}$ grows at most linearly. Hence, (5.14) implies that $t \mapsto \|\int_0^t u(s, L)\,\mathrm{d}s\|_{L^\infty(0,L)}$ grows at most like $t^{\frac{7}{8}}$ and $t \mapsto \|\int_0^t u'(s, L)\,\mathrm{d}s\|_{L^\infty(0,L)}$ at most like $t^{\frac{5}{8}}$ as $t \to \infty$. But this contradicts the fact that $u(t, L)$ and $u'(t, L)$ are constant, unless $u_0(L) = u_0'(L) = 0$. This shows that $u(t, L) = u'(t, L) = 0$ for all $t \ge 0$.

Step 3 (Holmgren's Theorem): By iterated t-integration we shall now construct C^4-solutions of (5.12a), for which we can apply the Holmgren Uniqueness Theorem [10, Section 3.5]. So we define $y_1(t) \equiv [u_1(t), v_1(t)]^\top := \int_0^t y_p(s)\,\mathrm{d}s + A_p^{-1}[u_0, v_0]^\top$. Due to Theorem 1.2.4 in [21] and Lemma A.1 we have $y_1(t) \in D(A_p)$ for all $t \ge 0$. So y_1 is a classical solution of (5.12a) to the initial condition $y_1(0) = A_p^{-1}[u_0, v_0]^\top$. Furthermore, because of $u(t, L) = u'(t, L) = 0$, again $u_1(t, L), u_1'(t, L)$ are constant in time. Completely analogous to the previous step we can show again that $u_1(t, L) = u_1'(t, L) = 0$.

Next we shall construct solutions of higher regularity. We iterate the previous step and define recursively $y_n(t) \equiv [u_n(t), v_n(t)]^\top := \int_0^t y_{n-1}(s)\,\mathrm{d}s + A_p^{-n}[u_0, v_0]^\top$, which solves (5.12a) classically with the initial condition $y_n(0) = A_p^{-n}[u_0, v_0]^\top$. Again we have $u_n(t, L) = u_n'(t, L) = 0$. Furthermore, by definition we have on the one hand $A_p y_n(t) = y_{n-1}(t)$. And on the other hand $A_p[u_n, v_n]^\top = [v_n, -\Lambda/\rho\, u_n^{\mathrm{IV}}]^\top$, so we can show inductively that $y_n \in C(\mathbb{R}^+, \tilde{H}_0^{2n+2}(0, L) \times \tilde{H}_0^{2n}(0, L))$. Now we consider the solution u_n for $n \ge 2$. It satisfies the following partial differential equation with boundary conditions:

$$(u_n)_{tt} = -\frac{\Lambda}{\rho} u_n^{\mathrm{IV}}, \tag{5.15a}$$

$$[u_n(0, x), (u_n)_t(0, x)]^\top = A_p^{-n}[u_0, v_0]^\top, \tag{5.15b}$$

$$u_n(t, 0) = u_n'(t, 0) = 0, \tag{5.15c}$$

$$u_n(t, L) = \ldots = u_n'''(t, L) = 0. \tag{5.15d}$$

By using (5.15a), $u_n \in C(\mathbb{R}^+; \tilde{H}_0^{2n+2}(0, L))$, and the fact that $(u_n)_t = v_n \in C(\mathbb{R}^+; \tilde{H}_0^{2n}(0, L))$, we obtain the following properties for the mixed fourth order space-time derivatives of u_n:

$$u_n^{\mathrm{IV}} \in C(\mathbb{R}^+, \tilde{H}_0^{2n-2}(0, L)),$$

$$\begin{aligned}
(u_n)'''_t &\in C(\mathbb{R}^+, \tilde{H}_0^{2n-3}(0,L)),\\
(u_n)''_{tt} &= -\frac{\Lambda}{\rho} u_n^{\mathrm{VI}} \in C(\mathbb{R}^+, \tilde{H}_0^{2n-4}(0,L)),\\
(u_n)'_{ttt} &= -\frac{\Lambda}{\rho} v_n^{\mathrm{V}} \in C(\mathbb{R}^+, \tilde{H}_0^{2n-5}(0,L)),\\
(u_n)_{tttt} &= \frac{\Lambda^2}{\rho^2} u_n^{\mathrm{VIII}} \in C(\mathbb{R}^+, \tilde{H}_0^{2n-6}(0,L)).
\end{aligned}$$

So for $n \geq 4$, all mixed derivatives of u_n of order four lie in $C(\mathbb{R}^+; \tilde{H}_0^2(0,L)) \subset C(\mathbb{R}^+ \times [0,L])$. Thus $u_n(t,x)$ is a C^4-solution of (5.15).

Now we can apply the Holmgren Uniqueness Theorem [10, Section 3.5] on the strip $\mathbb{R}^+ \times (0,L)$. Due to (5.15d) all partial derivatives up to order 3 of u_4 vanish on the line $\mathbb{R}^+ \times \{L\}$. Therefore, Holmgren's Uniqueness Theorem implies that $u_4 = 0$ has to hold everywhere in this strip. (See also the proof of Lemma 3 in [12] for a similar result – but without a detailed proof.) Therefore $A_p^{-4}[u_0, v_0]^\top = 0$ has to hold, and since A_p^{-1} is injective, this yields $[u_0, v_0]^\top = 0$. Since $y_p(t) = \mathrm{e}^{tA_p}[u_0, v_0]^\top$, we conclude that $u(t) = v(t) = 0$ for all $t \geq 0$. □

As a consequence we obtain convergence to zero for trajectories with $\omega(y_0) \neq \emptyset$:

Corollary 5.7. *If $\omega(y_0) \neq \emptyset$ for some $y_0 \in \mathcal{H}$, then*

$$\lim_{t\to\infty} \|S(t)y_0\| = 0.$$

Proof. If $\omega(y_0) \neq \emptyset$ then there exists a sequence $\{t_n\}_{n\in\mathbb{N}}$ with $t_n \to \infty$ such that $\lim_{n\to\infty} S(t_n)y_0 = 0$. Due to the continuity of the Lyapunov function H this implies that

$$\lim_{n\to\infty} H(S(t_n)y_0) = 0.$$

But since $t \mapsto H(S(t)y_0)$ is monotonically decreasing, this implies that even

$$\lim_{t\to\infty} H(S(t)y_0) = 0.$$

Due to the continuity of H this implies that $\|S(t)y_0\| \to 0$ as $t \to \infty$. □

6. Asymptotic Stability – Linear k_j

In the case where the k_j are linear, we are able to show precompactness for all trajectories, even for the mild, non-classical solutions. This will yield that the ω-limit set $\omega(y_0)$ is always non-empty, and hence the asymptotic stability of the nonlinear semigroup will follow.

Lemma 6.1. *Let $y_0 \in \mathcal{H}$, and $y(t)$ be the corresponding mild solution of* (4.1). *Let $\kappa_j = 0$. Then $\mathcal{N}y(t) \in L^1(\mathbb{R}^+; \mathcal{H})$.*

Proof. First, let us assume that $y_0 \in D(\mathcal{A})$, so $y(t)$ is a classical solution. We know from Theorem 4.7 that $H(y(t))$ is non-increasing. By integrating (5.2) with respect to time we obtain

$$H(y(T)) - H(y_0) = \int_0^T \left[-d_1\left(\frac{\xi}{J}\right)\frac{\xi}{J} - d_2\left(\frac{\psi}{M}\right)\frac{\psi}{M} + a_1(z_1)\cdot\nabla V_1(z_1) + a_2(z_2)\cdot\nabla V_2(z_2)\right] \mathrm{d}t =: I_T(y_0), \tag{6.1}$$

where all terms on the right hand side include elements of the vector $y(t)$, thus depend on t. If we let $T \to \infty$, we know that $H(y(T)) \to \mathfrak{H}(y_0)$, i.e. the limit exists and the integral $I_\infty(y_0)$ is finite.

Now we consider $y_0 \in \mathcal{H}$, and $y(t)$ is the corresponding mild solution of (4.1). Let $\{y_{0,n}\}_{n\in\mathbb{N}} \subset D(\mathcal{A})$ be a sequence with $y_{0,n} \to y_0$. According to Proposition 4.6 and Remark 4.9 the corresponding classical solutions $y_n(t)$ converge to $y(t)$ in $C([0,T]; \mathcal{H})$ for all $T > 0$. Therefore $I_T(y_{0,n}) \to I_T(y_0)$, cf. (6.1). Due to continuity of H, also $H(y_n(T)) - H(y_{0,n}) \to H(y(T)) - H(y_0)$ as $n \to \infty$.

Thus, (6.1) also holds for mild solutions for any $T > 0$. Since $H(y(T)) \to \mathfrak{H}(y_0) \in [0, H(y_0)]$ as $T \to \infty$, the integral $I_\infty(y_0)$ is finite.

Now we know that for any (mild) solution $y(t)$ the integral $I_\infty(y_0)$ from (6.1) is finite. Since all the terms in the integrand of (6.1) are non-positive, we conclude together with (2.5d) and (2.1b) - (2.1c) that

$$z_j(t), \psi(t), \xi(t) \in L^2(\mathbb{R}^+). \tag{6.2}$$

Under the assumptions we made in Section 2 for the functions occurring in the nonlinear operator $\mathcal{N}$, the properties (6.2) immediately imply $\mathcal{N}y(t) \in L^1(\mathbb{R}^+; \mathcal{H})$. Note that $|\beta_j(z_j)| = \mathcal{O}(|z_j|)$ as $z_j \to 0$. $\square$

Remark 6.2. To obtain $\mathcal{N}y(t) \in L^1(\mathbb{R}^+; \mathcal{H})$ in the above proof, we used in (6.1) that the nonlinear damping functions d_j include a non-vanishing linear part (i.e. $D_j > 0$). The same assumption will also be needed in Step 3 of the proof of Lemma 7.2 below. However, in the nonlinear spring-damper system of [16], a locally quadratic growth of the damper law was sufficient. From a practical point of view, this is not restrictive at all.

We note that (6.1) does not give any control on $u(t, L)$ and $u'(t, L)$ (in the sense of (6.2)). Hence, the linearity assumption $\kappa_j = 0$ was crucial for the above proof.

Theorem 6.3. *For any $y_0 \in \mathcal{H}$ there holds $\lim_{t\to\infty} S(t)y_0 = 0$, i.e. the semigroup S is asymptotically stable.*

Proof. According to Remark 3.2 the linear part A of $\mathcal{A}$ is a maximal dissipative operator on $\mathcal{H}$. Clearly $A(0) = 0$, and as seen in the proof of Theorem 3.1, A^{-1} exists and is compact. Since A generates a C_0-semigroup of contractions, $(\lambda - A)^{-1}$ exists and is compact for all $\lambda > 0$. Due to these facts, we can apply Theorem 4 in [5] with $f(t) := \mathcal{N}y(t)$. This shows that the ω-limit set $\omega(y_0)$ is non-empty (in fact the trajectory $\gamma(y_0)$ is precompact). Thus, due to Corollary 5.7 and Theorem 5.6, we conclude $\omega(y_0) = \{0\}$ and that the entire solution $y(t)$ converges to zero. $\square$

7. Asymptotic Stability – Nonlinear k_j

According to Corollary 5.7, any trajectory with a non-empty ω-limit set already is asymptotically stable. Thus, in order to complete the discussion we show in this section that (at least) any classical trajectory possesses a non-empty ω-limit. We do this by proving that every classical trajectory is precompact. To this end we follow a strategy introduced in [16]. We begin with the following preparatory result (which would be obvious for linear semigroups):

Lemma 7.1. *Let $y(t)$ be a (mild) solution of (4.1) and let $y_0 \in D(\mathcal{A}^2) := \{y \in D(\mathcal{A}) : \mathcal{A}y \in D(\mathcal{A})\}$. Then $y \in C^2([0,\infty), \mathcal{H})$ and $y_t(t) \in D(\mathcal{A})$ for all $t > 0$.*

Proof. If we already knew that $y \in C^2([0,\infty), \mathcal{H})$, it would follow that $\tilde{y} := y_t$ satisfies

$$\tilde{y}_t = A\tilde{y} + \begin{bmatrix} 0 \\ 0 \\ \alpha_1'(z_1)\tilde{z}_1 + \frac{1}{J}[\beta_1'(z_1)\tilde{z}_1\xi + \beta_1(z_1)\tilde{\xi}] \\ \alpha_2'(z_2)\tilde{z}_2 + \frac{1}{M}[\beta_2'(z_2)\tilde{z}_2\psi + \beta_2(z_2)\tilde{\psi}] \\ -\gamma_1'(z_1)\tilde{z}_1 - \frac{1}{J}\delta_1'\left(\frac{\xi}{J}\right)\tilde{\xi} - \kappa_1'(u'(L))\tilde{u}'(L) \\ -\gamma_2'(z_2)\tilde{z}_2 - \frac{1}{M}\delta_2'\left(\frac{\psi}{M}\right)\tilde{\psi} - \kappa_2'(u(L))\tilde{u}(L) \end{bmatrix}. \tag{7.1}$$

However, at this point we only know that $y(t) \in C^1([0,\infty); \mathcal{H})$, see Proposition 4.1. Motivated by (7.1) we define the following functions for this fixed $y(t) = [u, v, z_1, z_2, \xi, \psi]^\top(t)$:

$$G_1(t, Z) := \alpha_1'(z_1)\zeta_1 + \frac{1}{J}[\beta_1'(z_1)\zeta_1\xi + \beta_1(z_1)\Xi],$$

$$G_2(t, Z) := \alpha_2'(z_2)\zeta_2 + \frac{1}{M}[\beta_2'(z_2)\zeta_2\psi + \beta_2(z_2)\Psi],$$

$$G_3(t,Z) := -\gamma_1'(z_1)\zeta_1 - \frac{1}{J}\delta_1'\Big(\frac{\xi}{J}\Big)\Xi - \kappa_1'(u'(L))U'(L),$$
$$G_4(t,Z) := -\gamma_2'(z_2)\zeta_2 - \frac{1}{M}\delta_2'\Big(\frac{\psi}{M}\Big)\Psi - \kappa_2'(u(L))U(L),$$

where $Z := [U, V, \zeta_1, \zeta_2, \Xi, \Psi]^\top \in \mathcal{H}$. Since $y(t)$ is a classical solution, it follows that $t \mapsto G_j(t, Z)$ is continuously differentiable for all $j = 1, \dots, 4$. As a consequence the operator $\tilde{\mathcal{N}} : \mathbb{R}_0^+ \times \mathcal{H} \to \mathcal{H}$ defined by

$$\tilde{\mathcal{N}}(t,Z) := [0, 0, G_1(t,Z), G_2(t,Z), G_3(t,Z), G_4(t,Z)]^\top,$$

is also is differentiable with respect to t for every $Z \in \mathcal{H}$. Furthermore, $\tilde{\mathcal{N}}$ is linear with respect to $Z \in \mathcal{H}$, uniformly in $t \in [0, T]$ for every $T > 0$. Now the linear, non-autonomous initial value problem

$$Z_t = AZ + \tilde{\mathcal{N}}(t,Z), \tag{7.2a}$$
$$Z(0) = Z_0 \in \mathcal{H}, \tag{7.2b}$$

is considered. According to Theorem 6.1.2 in [21] there exists a unique global mild solution $Z(t)$ of (7.2) for every $Z_0 \in \mathcal{H}$. If $Z_0 \in D(A)$ this solution is classical, see Theorem 6.1.5 in [21].

Our next aim is to show that for the classical solution $y(t)$ fixed in the beginning, the (continuous) function $y_t(t)$ is indeed a mild solution of (7.2) for $Z_0 = \mathcal{A}y_0$: Since $y(t)$ satisfies the Duhamel formula (4.2) and is differentiable, we obtain after differentiating with respect to t

$$y_t(t) = \mathrm{e}^{tA}Ay_0 + \frac{\mathrm{d}}{\mathrm{d}t}\int_0^t \mathrm{e}^{(t-s)A}\mathcal{N}y(s)\,\mathrm{d}s. \tag{7.3}$$

According to the proof of Corollary 4.2.5 in [21] the following statement holds true

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t \mathrm{e}^{(t-s)A}\mathcal{N}y(s)\,\mathrm{d}s = \mathrm{e}^{tA}\mathcal{N}y_0 + \int_0^t \mathrm{e}^{(t-s)A}\frac{\mathrm{d}}{\mathrm{d}s}\mathcal{N}y(s)\,\mathrm{d}s. \tag{7.4}$$

Inserting (7.4) in (7.3) yields that $y_t(t)$ fulfills the Duhamel formula for (7.2). As a consequence $y_t(t)$ is the unique mild solution of (7.2) to the initial condition $Z_0 = \mathcal{A}y_0$. Moreover, we know that this mild solution $Z(t) = y_t(t)$ is a classical solution of (7.2) if $\mathcal{A}y_0 \in D(\mathcal{A})$, i.e. $y_0 \in D(\mathcal{A}^2)$. Hence $y_t \in C^1(\mathbb{R}^+; \mathcal{H})$ and $y \in C^2(\mathbb{R}^+; \mathcal{H})$. □

Lemma 7.2. *The trajectory $\gamma(y_0)$ is precompact in $\mathcal{H}$ for $y_0 \in D(\mathcal{A}^2)$. Moreover, there exists a constant $C > 0$ such that*

$$\|y_t(t)\|_{\mathcal{H}} \le C, \qquad \forall t \ge 0, \tag{7.5}$$

where C depends continuously on $\|y_0\|_{\mathcal{H}}$ and $\|y_t(0)\|_{\mathcal{H}}$.

Proof. In order to prove precompactness of the trajectory, it suffices to show that

$$\sup_{t>0} \|\mathcal{A}y(t)\|_{\mathcal{H}} < \infty,$$

due to the compact embeddings $H^4(0,L) \hookrightarrow\hookrightarrow H^2(0,L) \hookrightarrow\hookrightarrow L^2(0,L)$. However, this is equivalent to showing that y_t is uniformly bounded in $\mathcal{H}$, since $y_t = \mathcal{A}y$. Again, this is equivalent to

$$\begin{aligned} H(y_t) = \frac{\rho}{2}\int_0^L u_{tt}^2\,\mathrm{d}x + \frac{\Lambda}{2}\int_0^L \big(u_t''\big)^2\,\mathrm{d}x + \frac{J}{2}\big(u_{tt}'(L)\big)^2 + \frac{M}{2}\big(u_{tt}(L)\big)^2 \\ + \int_0^{u_t'(L)} k_1(s)\,\mathrm{d}s + \int_0^{u_t(L)} k_2(s)\,\mathrm{d}s + V_1((z_1)_t) + V_2((z_2)_t) \end{aligned}$$

being uniformly bounded. Since $y(t)$ is a classical solution, we have the following equalities

$$u_t(L) = \frac{\psi}{M}, \quad u_t'(L) = \frac{\xi}{J}.$$

According to Theorem 4.7 those terms are always uniformly bounded. Moreover, due to regularity of the functions a_j, b_j and Theorem 4.7 we see from (1.5a) and (1.6a) that $(z_j)_t \in L^\infty(\mathbb{R}^+)$ for $j = 1, 2$. Therefore, the boundedness of $H(y_t)$ is equivalent to the boundedness of the functional

$$\tilde{H}(y_t) := \frac{\rho}{2}\int_0^L u_{tt}^2 \,\mathrm{d}x + \frac{\Lambda}{2}\int_0^L \big(u_t''\big)^2 \,\mathrm{d}x + \frac{J}{2}\big(u_{tt}'(L)\big)^2 + \frac{M}{2}\big(u_{tt}(L)\big)^2.$$

Hence, our aim is to derive a system of equations satisfied by $y_t(t)$, and then to show that $\tilde{H}(y_t)$ is uniformly bounded.

Step 1 (Time derivative of the system): According to Lemma 7.1, $y(t) \in C^2([0, \infty); \mathcal{H})$. Differentiating (1.1) - (1.3) with respect to time hence shows that y_t is the classical solution to the following system:

$$\rho u_{ttt} + \Lambda u_t^{\mathrm{IV}} = 0, \tag{7.6a}$$

$$u_t(t, 0) = u_t'(t, 0) = 0, \tag{7.6b}$$

$$\Lambda u_t''(t, L) + J u_{ttt}'(t, L) + (\tau_e)_t(t) = 0, \tag{7.6c}$$

$$-\Lambda u_t'''(t, L) + M u_{ttt}(t, L) + (f_e)_t(t) = 0, \tag{7.6d}$$

where:

$$\begin{aligned} \tau_e &:= c_1(z_1) + d_1(u_t'(L)) + k_1(u'(L)), \\ f_e &:= c_2(z_2) + d_2(u_t(L)) + k_2(u(L)). \end{aligned} \tag{7.7}$$

Therefore, from (7.7) it follows:

$$\begin{aligned} (\tau_e)_t &= \nabla c_1(z_1) \cdot (z_1)_t + d_1'(u_t'(L))u_{tt}'(L) + k_1'(u'(L))u_t'(L), \\ (f_e)_t &= \nabla c_2(z_2) \cdot (z_2)_t + d_2'(u_t(L))u_{tt}(L) + k_2'(u(L))u_t(L), \end{aligned} \tag{7.8}$$

and from (1.5a) and (1.6a), we obtain

$$(z_1)_{tt} = [J_{a_1}(z_1) + u_t'(L)J_{b_1}(z_1)](z_1)_t + b_1(z_1)u_{tt}'(L), \tag{7.9a}$$

$$(z_2)_{tt} = [J_{a_2}(z_2) + u_t(L)J_{b_2}(z_2)](z_2)_t + b_2(z_2)u_{tt}(L), \tag{7.9b}$$

where J_{a_j}, J_{b_j} denote the Jacobian matrices of the functions a_j and b_j, respectively. Note that from Lemma 4.4 it follows that $z_j(\cdot), u_t(\cdot, L) = \frac{\psi}{M}, u_t'(\cdot, L) = \frac{\xi}{J} \in L^2(\mathbb{R}^+)$ (cf. (6.2) for a similar conclusion). Therefore (1.5a) and (1.6a) imply $(z_j)_t \in L^2(\mathbb{R}^+)$.

Step 2 (Time derivative of $\tilde{H}(y_t)$): We obtain

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\tilde{H}(y_t) &= \rho\int_0^L u_{ttt}u_{tt}\,\mathrm{d}x + \Lambda\int_0^L u_{tt}''u_t''\,\mathrm{d}x + J u_{ttt}'(L)u_{tt}'(L) + M u_{ttt}(L)u_{tt}(L) \\ &= u_{tt}(L)\big(M u_{ttt}(L) - \Lambda u_t'''(L)\big) + u_{tt}'(L)\big(\Lambda u_t''(L) + J u_{ttt}'(L)\big) \\ &= -u_{tt}(L)\big((z_2)_t^\top \nabla c_2(z_2) + k_2'(u(L))u_t(L)\big) \\ &\quad - u_{tt}'(L)\big((z_1)_t^\top \nabla c_1(z_1) + k_1'(u'(L))u_t'(L)\big) \\ &\quad - d_2'(u_t(L))\big(u_{tt}(L)\big)^2 - d_1'(u_t'(L))\big(u_{tt}'(L)\big)^2, \end{aligned} \tag{7.10}$$

where we have performed partial integration in x twice, and then used (7.6) and (7.8). Integrating (7.10) on the time interval $[0, t]$, for some arbitrary $t \in \mathbb{R}^+$, we get with (2.1a)

$$\tilde{H}(y_t(t)) \le \tilde{H}(y_t(0)) + I_1(t) + I_2(t), \tag{7.11}$$

where

$$I_1(t) := -\int_0^t u_{tt}'(L)\Big((z_1)_t^\top \nabla c_1(z_1) + k_1'(u'(L))u_t'(L)\Big)\,\mathrm{d}s,$$

$$I_2(t) := -\int_0^t u_{tt}(L)\Big((z_2)_t^\top \nabla c_2(z_2) + k_2'(u(L))u_t(L)\Big)\,\mathrm{d}s.$$

Step 3 (Boundedness of I_1 and I_2): Next, we show uniform boundedness for each component of I_2 by using partial integration in time:

$$\begin{aligned}
-\int_0^t u_{tt}(L)k_2'(u(L))u_t(L)\,\mathrm{d}s = -\frac{1}{2}\left(u_t(t,L)\right)^2 k_2'(u_t(t,L)) + \frac{1}{2}\left(u_t(0,L)\right)^2 k_2'(u_t(0,L))\\
+ \frac{1}{2}\int_0^t u_t(L)^3 k_2''(u(L))\,\mathrm{d}s \le C, \quad \forall t \ge 0.
\end{aligned}$$

Further, it holds that

$$\begin{aligned}
\int_0^t u_{tt}(L)(z_2)_t^\top \nabla c_2(z_2)\,\mathrm{d}s = u_t(t,L)(z_2)_t(t)^\top \nabla c_2(z_2(t)) - u_t(0,L)(z_2)_t(0)^\top \nabla c_2(z_2(0))\\
- \int_0^t u_t(L)[(z_2)_t^\top H_{c_2}(z_2)(z_2)_t + (z_2)_{tt}^\top \nabla c_2(z_2)]\,\mathrm{d}s.
\end{aligned}$$

Here, H_{c_2} denotes the Hessian of the function c_2. Since $c_2 \in C^2(\mathbb{R}^{n_2};\mathbb{R})$, it follows that

$$\int_0^t |u_t(L)(z_2)_t^\top H_{c_2}(z_2)(z_2)_t|\,\mathrm{d}s \le C\int_0^t |(z_2)_t|^2\,\mathrm{d}s,$$

and (with (7.9)):

$$\begin{aligned}
\int_0^t u_t(L)(z_2)_{tt}^\top \nabla c_2(z_2)\,\mathrm{d}s &= \int_0^t u_t(L)[J_{a_2}(z_2)(z_2)_t + u_t(L)J_{b_2}(z_2)(z_2)_t]^\top \nabla c_2(z_2)\,\mathrm{d}s\\
&\quad + \int_0^t \nabla c_2(z_2)^\top b_2(z_2)u_{tt}(L)u_t(L)\,\mathrm{d}s\\
&= \int_0^t u_t(L)[J_{a_2}(z_2)(z_2)_t + u_t(L)J_{b_2}(z_2)(z_2)_t]^\top \nabla c_2(z_2)\,\mathrm{d}s\\
&\quad + \frac{1}{2}\nabla c_2(z_2(t))^\top b_2(z_2(t))u_t(t,L)^2 - \frac{1}{2}\nabla c_2(z_2(0))^\top b_2(z_2(0))u_t(0,L)^2\\
&\quad - \frac{1}{2}\int_0^t u_t(L)^2(z_2)_t^\top\Big[J_{b_2}(z_2)^\top \nabla c_2(z_2) + H_{c_2}(z_2)b_2(z_2)\Big]\,\mathrm{d}s\\
&\le C\int_0^t |u_t(L)|^2 + |(z_2)_t|^2\,\mathrm{d}s + \frac{1}{2}\nabla c_2(z_2(t))^\top b_2(z_2(t))u_t(t,L)^2\\
&\quad - \frac{1}{2}\nabla c_2(z_2(0))^\top b_2(z_2(0))u_t(0,L)^2.
\end{aligned}$$

For the estimate of the second integral we have used the uniform boundedness of $(z_2)_t$, see the discussion before Step 1 of this proof. The uniform boundedness of I_1 follows analogously. Hence, $\tilde{H}(y_t(t))$ is uniformly bounded in time. Furthermore it can be seen that all the positive constants C appearing in the above calculations depend continuously on the initial conditions. This concludes the proof. □

In order to extend this result to all classical solutions, we need the following density argument.

Lemma 7.3. *For any $y \in D(\mathcal{A})$ there is a sequence $\{y_n\}_{n\in\mathbb{N}}$ in $D(\mathcal{A}^2)$ such that $\lim_{n\to\infty} y_n = y$ and $\lim_{n\to\infty} \mathcal{A}y_n = \mathcal{A}y$.*

Proof. Let an arbitrary $y \in D(\mathcal{A})$ be fixed. Notice that it suffices to show that there exists a sequence $\{y_n\}_{n\in\mathbb{N}}$ with $y_n = [u_n\, v_n\, z_{1n}\, z_{2n}\, \xi_n \psi_n]^\top$ in $D(\mathcal{A}^2)$ such that $\lim_{n\to\infty} y_n = y$ in the space $H^4(0,L)\times H^2(0,L)\times\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}\times\mathbb{R}\times\mathbb{R}$. The set $D(\mathcal{A}^2) = \{y \in D(\mathcal{A}) : \mathcal{A}y \in D(\mathcal{A})\}$ is equivalent to

$$v \in \tilde{H}_0^4(0,L), \tag{7.12}$$

$$u \in \tilde{H}_0^6(0,L) \wedge u^{\mathrm{IV}}(0) = u^{\mathrm{V}}(0) = 0, \tag{7.13}$$
$$\xi = Jv'(L), \tag{7.14}$$
$$\psi = Mv(L), \tag{7.15}$$
$$\Lambda u''(L) + \left[c_1(z_1) + d_1\Big(\frac{\xi}{J}\Big) + k_1(u'(L))\right] = \frac{\Lambda J}{\rho} u^{\mathrm{V}}(L), \tag{7.16}$$
$$-\Lambda u'''(L) + \left[c_2(z_2) + d_2\Big(\frac{\psi}{M}\Big) + k_2(u(L))\right] = \frac{\Lambda M}{\rho} u^{\mathrm{IV}}(L). \tag{7.17}$$

Since $\tilde{C}_0^\infty(0,L) := \{f \in C^\infty[0,L] \colon f^{(k)}(0) = 0, \forall k \in \mathbb{N}_0\}$ is dense in $\tilde{H}_0^2(0,L)$ (see Theorem 3.17 in [1]), there exists a sequence $\{v_n\}_{n\in\mathbb{N}} \subset \tilde{C}_0^\infty(0,L)$ such that $\lim_{n\to\infty} v_n = v$ in $H^2(0,L)$. Also, v_n satisfies (7.12), for all $n \in \mathbb{N}$. Defining $\xi_n := Jv_n'(L)$ and $\psi_n := Mv_n(L)$ ensures that y_n satisfies (7.14) and (7.15). Moreover, the Sobolev embedding $H^2(0,L) \hookrightarrow C^1[0,L]$ implies that $\lim_{n\to\infty} \xi_n = \xi$ and $\lim_{n\to\infty} \psi_n = \psi$ as well. Next, let $z_{1n} := z_1$ and $z_{2n} := z_2$ for all $n \in \mathbb{N}$.

Finally, the sequence $\{u_n\}_{n\in\mathbb{N}} \subset C^\infty[0,L]$ will be constructed such that u_n satisfies (7.13), (7.16), and (7.17) for all $n \in \mathbb{N}$, and $\lim_{n\to\infty} u_n = u$ in $H^4(0,L)$. To this end we introduce an auxiliary sequence of polynomial functions

$$h_n(x) := h_{2,n}x^2 + h_{3,n}x^3 + h_{6,n}x^6 + h_{7,n}x^7 + h_{8,n}x^8 + h_{9,n}x^9 + h_{10,n}x^{10} + h_{11,n}x^{11},$$

for all $n \in \mathbb{N}$, where $h_{2,n}, \ldots, h_{11,n} \in \mathbb{R}$ are to be determined. It immediately follows that

$$h_n(0) = h_n'(0) = h_n^{\mathrm{IV}}(0) = h_n^{\mathrm{V}}(0) = 0. \tag{7.18}$$

Let $h_{2,n} = \frac{u''(0)}{2}$ and $h_{3,n} = \frac{u'''(0)}{6}$, which is equivalent to

$$h_n''(0) = u''(0),\ h_n'''(0) = u'''(0). \tag{7.19}$$

Further conditions are imposed on h_n:

$$h_n^{(k)}(L) = u^{(k)}(L), \qquad k \in \{0,1,2,3\}.$$

This can equivalently be written in terms of coefficients[2]:

$$h_{6,n} + h_{7,n}L + h_{8,n}L^2 + h_{9,n}L^3 + h_{10,n}L^4 + h_{11,n}L^5 = r_1, \tag{7.20a}$$
$$6h_{6,n} + 7h_{7,n}L + 8h_{8,n}L^2 + 9h_{9,n}L^3 + 10h_{10,n}L^4 + 11h_{11,n}L^5 = r_2, \tag{7.20b}$$
$$6^{\underline{2}}h_{6,n} + 7^{\underline{2}}h_{7,n}L + 8^{\underline{2}}h_{8,n}L^2 + 9^{\underline{2}}h_{9,n}L^3 + 10^{\underline{2}}h_{10,n}L^4 + 11^{\underline{2}}h_{11,n}L^5 = r_3 \tag{7.20c}$$
$$6^{\underline{3}}h_{6,n} + 7^{\underline{3}}h_{7,n}L + 8^{\underline{3}}h_{8,n}L^2 + 9^{\underline{3}}h_{9,n}L^3 + 10^{\underline{3}}h_{10,n}L^4 + 11^{\underline{3}}h_{11,n}L^5 = r_4, \tag{7.20d}$$

with

$$r_1 = \frac{u(L)}{L^6} - \frac{u''(0)}{2L^4} - \frac{u'''(0)}{6L^3}, \quad r_2 = \frac{u'(L)}{L^5} - \frac{u''(0)}{L^4} - \frac{u'''(0)}{2L^3},$$
$$r_3 = \frac{u''(L)}{L^4} - \frac{u''(0)}{L^4} - \frac{u'''(0)}{L^3}, \quad r_4 = \frac{u'''(L)}{L^3} - \frac{u'''(0)}{L^3}.$$

We further require that h_n satisfies:

$$\frac{\Lambda M}{\rho} h_n^{\mathrm{IV}}(L) = -\Lambda u'''(L) + \left[c_2(z_2) + d_2\Big(\frac{\psi_n}{M}\Big) + k_2(u(L))\right] =: r_5, \tag{7.21}$$
$$\frac{\Lambda J}{\rho} h_n^{\mathrm{V}}(L) = \Lambda u''(L) + \left[c_1(z_1) + d_1\Big(\frac{\xi_n}{J}\Big) + k_1(u'(L))\right] := r_6, \tag{7.22}$$

[2] The coefficient $k^{\underline{l}}$ (the Pochhammer symbol, see [7]) for $k, l \in \mathbb{N}$, $l \le k$ is defined by $k^{\underline{l}} := k \cdot (k-1) \cdots (k-l+1)$.

where (7.21) and (7.22) are equivalent to:

$$6\underline{4}h_{6,n} + 7\underline{4}h_{7,n}L + 8\underline{4}h_{8,n}L^2 + 9\underline{4}h_{9,n}L^3 + 10\underline{4}h_{10,n}L^4 + 11\underline{4}h_{11,n}L^5 = r_5\frac{\rho}{\Lambda M L^2}, \tag{7.23a}$$

$$6\underline{5}h_{6,n} + 7\underline{5}h_{7,n}L + 8\underline{5}h_{8,n}L^2 + 9\underline{5}h_{9,n}L^3 + 10\underline{5}h_{10,n}L^4 + 11\underline{2}h_{11,n}L^5 = r_6\frac{\rho}{\Lambda J L}. \tag{7.23b}$$

Such h_n exists and is unique, due to the fact that linear system (7.20) and (7.23) has strictly positive determinant. Consequently, (7.18), (7.19), and (7.20) imply that $u - h_n \in H_0^4(0,L)$, for all $n \in \mathbb{N}$. Since $C_0^\infty(0,L)$ is dense in $H_0^4(0,L)$, there exists a sequence $\{\tilde{u}_n\}_{n\in\mathbb{N}} \subset C_0^\infty(0,L)$ such that $\|\tilde{u}_n - (u - h_n)\|_{H^4} < \frac{1}{n}$, $\forall n \in \mathbb{N}$. Now defining $u_n := \tilde{u}_n + h_n$, gives $\lim_{n\to\infty} u_n = u$ in $H^4(0,L)$. Obviously u_n satisfies (7.13) for all $n \in \mathbb{N}$. Also, due to (7.21) and (7.22), u_n satisfies (7.16) and (7.17), as well. Hence, the statement follows. □

Theorem 7.4. *For all $y_0 \in D(\mathcal{A})$ the trajectory $\gamma(y_0)$ is precompact in $\mathcal{H}$.*

Proof. Let $y_0 \in D(\mathcal{A})$ be chosen arbitrarily, and let $\{y_{n0}\}_{n\in\mathbb{N}} \subset D(\mathcal{A}^2)$ be an approximating sequence as in Lemma 7.3. Then there holds:

$$\lim_{n\to\infty} \mathcal{A}y_{n0} = \mathcal{A}y_0. \tag{7.24}$$

For an arbitrary $T > 0$, and by applying Proposition 4.6 it follows that the approximating solutions $y_n(t)$ converge to $y(t)$ in $C([0,T];\mathcal{H})$. Since $y_n(t) \in C^1([0,\infty);\mathcal{H})$ and solves (4.1) for all $n \in \mathbb{N}$, (7.24) yields

$$\lim_{n\to\infty} (y_n)_t(0) = \mathcal{A}y_0 \quad \text{in} \quad \mathcal{H}. \tag{7.25}$$

Hence, (7.5) and (7.25) imply that there exists a constant $C > 0$ such that for all $n \in \mathbb{N}$:

$$\sup_{t\geq 0} \|(y_n)_t(t)\|_{\mathcal{H}} \leq C(\|y_0\|_{\mathcal{H}}, \|\mathcal{A}y_0\|_{\mathcal{H}}),$$

where the constant C does not depend on n. From here it follows that $(y_n)_t$ is bounded in $L^\infty(\mathbb{R}^+;\mathcal{H})$. Hence, the Banach-Alaoglu Theorem (see Theorem I.3.15 in [24]) implies that there exists $w \in L^\infty(\mathbb{R}^+;\mathcal{H})$ and a subsequence $\{y_{n_k}\}_{k\in\mathbb{N}}$ such that

$$(y_{n_k})_t \overset{*}{\rightharpoonup} w \quad \text{in} \quad L^\infty(\mathbb{R}^+;\mathcal{H}).$$

For arbitrary $z \in \mathcal{H}$ and $t \geq 0$ there holds

$$\lim_{k\to\infty} \int_0^t \langle (y_{n_k})_t(\tau), z\rangle_{\mathcal{H}} \,\mathrm{d}\tau = \int_0^t \langle w(\tau), z\rangle_{\mathcal{H}} \,\mathrm{d}\tau,$$

which is equivalent to

$$\lim_{k\to\infty} \langle y_{n_k}(t) - y_{n_k}(0), z\rangle_{\mathcal{H}} = \left\langle \int_0^t w(\tau)\,\mathrm{d}\tau, z \right\rangle_{\mathcal{H}}.$$

Since $\lim_{n\to\infty} y_n(\tau) = y(\tau)$ (in $\mathcal{H}$) for all $\tau \in [0,\infty)$, it follows that

$$\langle y(t) - y(0), z\rangle_{\mathcal{H}} = \left\langle \int_0^t w(\tau)\,\mathrm{d}\tau, z \right\rangle_{\mathcal{H}}.$$

Since $z \in \mathcal{H}$ was arbitrary, we obtain

$$y(t) - y(0) = \int_0^t w(\tau)\,\mathrm{d}\tau, \quad \forall t \geq 0. \tag{7.26}$$

Due to continuous differentiability of y, the time derivative of (7.26) can be taken, which yields $y_t \equiv w$. This implies $y_t \in L^\infty(\mathbb{R}^+;\mathcal{H})$, i.e. $\|y_t(\cdot)\|_{\mathcal{H}}$ is uniformly bounded, which proves the theorem. □

Corollary 7.5. *For any $y_0 \in D(\mathcal{A})$ there holds $\lim_{t\to\infty} = S(t)y_0 = 0$.*

8. Conclusions

In this paper, we provide a rigorous stability proof of a lossless Euler-Bernoulli beam with tip mass which is feedback interconnected with a nonlinear spring-damper system and a strictly passive nonlinear dynamical system. Such a configuration comes into play if the tip payload is interacting with a nonlinear passive environment, if a strictly passive nonlinear boundary controller is employed, or for a combination of these cases. It is well known that the feedback interconnection of passive systems is passive with a storage function that is the sum of the storage functions of all subsystems. In the finite-dimensional case, this property is advantageously utilized for the controller design where the storage function usually qualifies as an appropriate Lyapunov function candidate. For the infinite-dimensional system under consideration, the passivity property still ensures that the storage functional is monotonically decreasing along classical solutions, however, it is well known that this does not directly entail asymptotic stability. In fact, a crucial step in the stability analysis is to prove the precompactness of the trajectories. For linear evolution problems this has been reported in many contributions in the literature, but when considering nonlinearities this is much more involved. Under rather mild conditions on the parameters and functions appearing in the resulting PDE–ODE model representing the overall closed-loop system, global-in-time wellposedness is proven by means of semigroup theory and the precompactness of the trajectories is shown by deriving uniform-in-time bounds on the solution and its time derivatives. With this, asymptotic stability of classical solutions can be guaranteed.

Appendix A. The Operator A_p

The system (5.11) is the mild formulation of the evolution problem $(y_p)_t = A_p y_p$ with $y_p = [u, v]^\top \in \mathcal{H}_p$. Thereby $\mathcal{H}_p := \tilde{H}_0^2(0,L) \times L^2(0,L)$, and

$$A_p : \begin{bmatrix} u \\ v \end{bmatrix} \mapsto \begin{bmatrix} v \\ -\frac{\Lambda}{\rho} u^{\mathrm{IV}} \end{bmatrix},$$

with the domain

$$D(A_p) = \{[u,v]^\top \in \mathcal{H}_p \colon u \in \tilde{H}_0^4(0,L), v \in \tilde{H}_0^2(0,L), \\ \Lambda u''(L) + \tilde{K}_1 u'(L) = 0, \Lambda u'''(L) - \tilde{K}_2 u(L) = 0\}.$$

The space $\mathcal{H}_p$ is equipped with the following inner product:

$$\langle y_p, \tilde{y}_p \rangle_p := \Lambda \int_0^L u'' \tilde{u}'' \,\mathrm{d}x + \rho \int_0^L v \tilde{v} \,\mathrm{d}x + \tilde{K}_1 u'(L) \tilde{u}'(L) + \tilde{K}_2 u(L) \tilde{u}(L). \tag{A.1}$$

The constants $\tilde{K}_1, \tilde{K}_2$ are defined in (5.10) and depend, at first glance, on the fixed $y_0 \in \Omega$ in the proof of Theorem 5.6. Hence, $D(\mathcal{A}_p)$ and the above inner product also depend on y_0. But this does not cause any problems. Anyhow, Step 2 in the proof of Theorem 5.6 shows that $u_0(L) = u_0'(L) = 0$. Hence, $\tilde{K}_j = K_j$.

We have the following results:

Lemma A.1. *The inverse $A_p : \mathcal{H}_p \to D(A_p)$ exists and is a bijection. Furthermore, A_p^{-1} is compact in $\mathcal{H}_p$.*

Proof. The proof is analogous to the proof of Theorem 3.1, see also Section 4.2 in [11]. □

Lemma A.2. *The operator A_p is skew-adjoint.*

Proof. First we show that A_p is skew-symmetric, i.e. for all $y, \tilde{y} \in D(A_p)$ there holds $\langle A_p y, \tilde{y} \rangle_p = -\langle y, A_p \tilde{y} \rangle_p$:

$$\langle A_p y, \tilde{y} \rangle_p = \Lambda \int_0^L v'' \tilde{u}'' \,\mathrm{d}x - \Lambda \int_0^L u^{\mathrm{IV}} \tilde{v} \,\mathrm{d}x + \tilde{K}_1 v'(L) \tilde{u}'(L) + \tilde{K}_2 v(L) \tilde{u}(L)$$

$$= \Lambda\Big(\int_0^L v\tilde{u}^{\mathrm{IV}}\,\mathrm{d}x + v'(L)\tilde{u}''(L) - v(L)\tilde{u}'''(L) - \int_0^L u''\tilde{v}''\,\mathrm{d}x - u'''(L)\tilde{v}(L) + u''(L)\tilde{v}'(L)\Big)$$
$$+ \tilde{K}_1 v'(L)\tilde{u}'(L) + \tilde{K}_2 v(L)\tilde{u}(L).$$

Using the boundary conditions $\Lambda u''(L) + \tilde{K}_1 u'(L) = 0$ and $\Lambda u'''(L) - \tilde{K}_2 u(L) = 0$ from $D(A_p)$ we obtain:

$$\begin{aligned}\langle A_p y, \tilde{y}\rangle_p &= \Lambda \int_0^L v\tilde{u}^{\mathrm{IV}}\,\mathrm{d}x - \tilde{K}_1 v'(L)\tilde{u}'(L) - \tilde{K}_2 v(L)\tilde{u}(L) - \Lambda\int_0^L u''\tilde{v}''\,\mathrm{d}x \\ &\quad - \tilde{K}_2 u(L)\tilde{v}(L) - \tilde{K}_1 u'(L)\tilde{v}'(L) + \tilde{K}_1 v'(L)\tilde{u}'(L) + \tilde{K}_2 v(L)\tilde{u}(L) \\ &= -\langle y, A_p\tilde{y}\rangle_p.\end{aligned}$$

So A_p is skew-symmetric. Furthermore, due to Lemma A.1 we know that $\operatorname{ran} A_p = \mathcal{H}_p$. So we can apply the Corollary of Theorem VII.3.1 in [28], which proves the skew-adjointness of A_p. □

Lemma A.3. *A_p generates a C_0-semigroup of unitary operators in $\mathcal{H}_p$.*

Proof. Since A_p is skew-adjoint, this follows from Stone's theorem [6, Theorem II.3.24]. □

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