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Convergence of Hardy Space Infinite Elements for Helmholtz Scattering and Resonance Problems

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Abstract

We perform a convergence analysis for discretization of Helmholtz scattering and resonance problems obtained by Hardy space infinite elements. Super-algebraic convergence with respect to the number of Hardy space degrees of freedom is achieved. As transparent boundary spheres and piecewise polytopes are considered. The analysis is based on a Gårding-type inequality and standard operator theoretical results.

1 Introduction

We consider Helmholtz scattering and resonance problems in two and three space dimensions in unbounded domains, which are complements of compact sets. Such problems can be solved with boundary elements, e.g. [17], or mesh based methods. For the latter the domain is truncated to a finite subdomain and boundary conditions have to be imposed at the new artificial boundary. There exist several approaches to construct such transparent boundary conditions, e.g. absorbing boundary conditions [4]. After the reintroduction of complex scaling techniques by Berenger [1] as perfectly matched layers (PML) the method became most popular. Reasons for this include the easy implementation into existing finite element codes and the preserved linear eigenvalue problem structure.

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In [10] it was shown that the radiation condition for scattering as well as for resonance problems is equivalent to a pole condition, which says that the Laplace transform of the solution in radial direction has an analytic extension to a specific half plane. Based on the pole condition a new family of (Hardy space) infinite elements was introduced for acoustic [8] and electromagnetic [15] scattering and resonance problems, which also preserves the linear eigenvalue problem structure. However, the outstanding potential of this method was pointed out in [6] for backward/left-handed structures and successfully applied for such cases in elastic scattering and resonance problems [7]. While the application of Hardy space infinite elements ranges already from acoustics over electromagnetics to elasticity, the theoretical understanding is still in its childhood. The only convergence result for problems in more than one space dimension is known for acoustic cylindrical waveguides [9]. This paper reports convergence of Hardy space infinite elements for acoustic scattering and resonance problems in outside a bounded set homogeneous two and three dimensional domains. It covers both cases of the artificial boundary to be either spherical or piecewise polytopial. Moreover, super-algebraic convergence speed with respect to the number of Hardy space degrees of freedom is achieved.

The outline of the paper is as follows. First we consider only three dimensional problems and show later in Subsection 3.1 how the results can be carried over to two dimensional problems. In Section 2 we recall basic ingredients of the infinite element method from [3, 8, 10, 15] and generalize several results from spherical boundaries to general boundaries. In particular, we derive variational formulations and show their equivalence to the classical strong formulations. In Section 3 we proof a main result of this paper, a Gårding-type inequality for the variational problem with spherical or piecewise polygonial transparent boundary. This allows us to apply standard operator theoretical results to show the convergence of the Galerkin method for the variational scattering and resonance problems. We further report super-algebraic approximation properties of the solution. At last we give in Section 4 a discussion on the choice of the method parameters.

2 Variational framework

In this section we recapitulate the variational framework of the pole condition and the Hardy space infinite element method introduced in [10, 8, 15]. We generalize several definitions and results from spherical boundaries to general boundaries. Finally, we derive variational formulations of scattering as well as of resonance problems and show their equivalence.

Let $\Omega := \mathbb{R}^3 \setminus K$ be a Lipschitz domain with compact scatterer $K \subset \mathbb{R}^3$ and ν the outward normal vector. We consider for given wave-number $\kappa \in \mathbb{R}^+$ and boundary datum $g \in H^{-1/2}(\partial K)$ the scattering problem to find $u \in H^1_{\text{loc}}(\Omega)$ such that

$$-\Delta u - \kappa^2 u = 0 \quad \text{in } \Omega, \tag{1a}$$

$$\partial_\nu u = g \quad \text{on } \partial K, \tag{1b}$$

$$u \text{ is outgoing,} \tag{1c}$$

and the resonance problem to find $(\kappa, u) \in \mathbb{C} \times H^1_{\text{loc}}(\Omega) \setminus \{0\}$ such that

$$-\Delta u = \kappa^2 u \quad \text{in } \Omega, \tag{2a}$$

$$\partial_\nu u = 0 \quad \text{on } \partial K, \tag{2b}$$

$$u \text{ is outgoing.} \tag{2c}$$

The radiation condition "u is outgoing" thereby means that outside a ball $B_R := \{x \in \mathbb{R}^3 \colon |x| < R\}$ containing K, u admits a representation

$$u(x) = \sum_{l=0}^{\infty} \sum_{m=0}^{M_l} \alpha_{l,m} h_l^{(1)}(\kappa |x|) Y_{l,m}\left(|x|^{-1} x\right) \tag{3}$$

where $h_l^{(1)}$ are spherical Hankel functions of the first kind, $Y_{l,m}$ are spherical harmonics, and $\alpha_{l,m} \in \mathbb{C}$.

2.1 Coordinate system

Let $\Omega = \Omega_{\text{int}} \cup \Gamma \cup \Omega_{\text{ext}}$ be a disjoint decomposition of Ω with bounded interior Lipschitz domain Ω_{int}, unbounded exterior Lipschitz domain Ω_{ext} and artificial boundary $\Gamma = \overline{\Omega_{\text{int}}} \cap \overline{\Omega_{\text{ext}}}$. We assume $\Omega_{\text{int}} \cup \overline{K}$ to be star-shaped with respect to $P_0 \in \Omega_{\text{int}} \cup \overline{K}$ and Γ to be piecewise C^1. We use a parametrization

$$T_{\text{polar}}(r, \hat{x}) := (r+1)(\hat{x} - P_0), \quad r \in \mathbb{R}^+, \hat{x} \in \Gamma. \tag{4}$$

of Ω_{ext} with focal point P_0. W.l.o.g. we assume in the following $P_0 = 0$. Let $\mathrm{D_x}$ be the standard Jacobian and $\mathrm{D_{\hat{x}}}, \nabla_{\hat{x}}$ the surface Jacobian and surface gradient on Γ, defined as

$$\mathrm{D_{\hat{x}}}\, u := \mathrm{D_x}(u \circ \gamma)\big((\mathrm{D_x}\,\gamma)^\top \mathrm{D_x}\,\gamma\big)^{-1}, \qquad \nabla_{\hat{x}} u := (\mathrm{D_{\hat{x}}}\, u)^\top \tag{5}$$

for any parametrization γ of Γ. Let

$$q(\hat{x}) := \frac{|\det(\hat{x}, \mathrm{D}_{\hat{\mathrm{x}}}\, \hat{x})|}{\sqrt{\det\left((\mathrm{D}_{\hat{\mathrm{x}}}\, \hat{x})^\top \mathrm{D}_{\hat{\mathrm{x}}}\, \hat{x}\right)}}, \tag{6a}$$

$$(Q_{nm})_{n,m=1,2,3}(\hat{x}) := q(\hat{x}) \begin{pmatrix} \hat{x}^\top \hat{x} & \hat{x}^\top \mathrm{D}_{\hat{\mathrm{x}}}\, \hat{x} \\ (\mathrm{D}_{\hat{\mathrm{x}}}\, \hat{x})^\top \hat{x} & (\mathrm{D}_{\hat{\mathrm{x}}}\, \hat{x})^\top \mathrm{D}_{\hat{\mathrm{x}}}\, \hat{x} \end{pmatrix}^{-1}. \tag{6b}$$

Straight forward computations yield

$$\int_{\Omega_{\mathrm{ext}}} uv \,\mathrm{d}x = \int_{\mathbb{R}^+ \times \Gamma} q(\hat{x})(r+1)^2 u_{\mathrm{ext}} v_{\mathrm{ext}} \,\mathrm{d}r\mathrm{d}\hat{x}, \tag{7a}$$

$$\int_{\Omega_{\mathrm{ext}}} \nabla u \cdot \nabla v \,\mathrm{d}x = \int_{\mathbb{R}^+ \times \Gamma} \begin{pmatrix} \partial_r u_{\mathrm{ext}} \\ \nabla_{\hat{x}} u_{\mathrm{ext}} \end{pmatrix} \cdot \begin{pmatrix} Q_{11}(\hat{x})(r+1)^2 & Q_{12}(\hat{x})(r+1) & Q_{13}(\hat{x})(r+1) \\ Q_{21}(\hat{x})(r+1) & Q_{22}(\hat{x}) & Q_{23}(\hat{x}) \\ Q_{31}(\hat{x})(r+1) & Q32(\hat{x}) & Q_{33}(\hat{x}) \end{pmatrix} \begin{pmatrix} \partial_r v_{\mathrm{ext}} \\ \nabla_{\hat{x}} v_{\mathrm{ext}} \end{pmatrix} \mathrm{d}r\mathrm{d}\hat{x}, \tag{7b}$$

where $u_{\mathrm{ext}} := u \circ T_{\mathrm{polar}}$ and $v_{\mathrm{ext}} := v \circ T_{\mathrm{polar}}$.

2.2 Hardy spaces and pole condition

The infinite element method used in this paper relies on a transformation of solution and test functions to a Hardy space. We therefore introduce the following spaces as in [3], where the stated auxiliary properties are also taken from therein.

Definition 2.1 ($H^+(S^1)$)**.** *The Hardy space $H^+(S^1)$ is the set of all functions $U \in L^2(S^1)$, $S^1 := \{z \in \mathbb{C} \colon |z| = 1\}$ that are L^2 boundary values of a function $\tilde{U}$, which is holomorphic in the unit disk $\mathbb{D} := \{z \in \mathbb{C} \colon |z| < 1\}$ and for which the integrals $\int_0^{2\pi} |\tilde{U}(re^{\mathrm{i}\theta})|^2 \,\mathrm{d}\theta$ are uniformly bounded for $r \in [0, 1)$.*

$H^+(S^1)$ equipped with the L^2-scalar product is a Hilbert space and a simple complete orthogonal system is given by the monomials $z^k, k \in \mathbb{N}_0$.

Definition 2.2 ($H^-(\kappa_0\mathbb{R})$)**.** *For $\kappa_0 \in \mathbb{C} \setminus \{0\}$ the Hardy space $H^-(\kappa_0\mathbb{R})$ is the set of all functions $U \in L^2(\kappa_0\mathbb{R})$, $\kappa_0\mathbb{R} := \{\kappa_0 x \colon x \in \mathbb{R}\}$, that are L^2 boundary values of a function $\tilde{U}$, which is analytic in $\{s \in \mathbb{C} \colon \Im(s\kappa_0^{-1}) < 0\}$ and for which the integrals $\int_{\kappa_0\mathbb{R}} |\tilde{U}(s - \kappa_0 \mathrm{i}\epsilon)|^2 \,\mathrm{d}|s|$ are uniformly bounded for $\epsilon > 0$. For $\kappa_0 = 1$ we shortly write $H^-(\mathbb{R}) := H^-(1\mathbb{R})$.*

$H^-(\kappa_0\mathbb{R})$ equipped with the L^2-scalar product is a Hilbert space. Moreover, by the Paley-Wiener theorem $H^-(\mathbb{R})$ is characterized by

$$H^-(\mathbb{R}) = \{u \in L^2(\mathbb{R}) \colon \mathcal{F}^{-1}\{u\}(-t) = 0 \text{ for all } t > 0\} \tag{8}$$

in terms of the inverse Fourier transform $\mathcal{F}^{-1}\{u\}(t) := \frac{-1}{2\pi}\int_{\mathbb{R}} u(s)e^{\mathrm{i}st}\,\mathrm{d}s$. For $\kappa_0 \neq 0$ let $\phi_{\kappa_0}(z) := \mathrm{i}\kappa_0\frac{z+1}{z-1}$ be the Möbius transformation, which maps $\mathbb{D}$ to $\{s \in \mathbb{C}\colon \Im(s\kappa_0^{-1}) < 0\}$. The spaces $H^+(S^1)$ and $H^-(\kappa_0\mathbb{R})$ can be identified with each other through

$$(\mathcal{M}_{\kappa_0}u)(z) := (u \circ \phi_{\kappa_0})(z)\frac{1}{z-1}, \tag{9}$$

where $\sqrt{-2\mathrm{i}\kappa_0}\mathcal{M}_{\kappa_0}\colon H^-(\kappa_0\mathbb{R}) \to H^+(S^1)$ is an unitary mapping. In the following we will use the symbol $\otimes$ for tensor products, see [11]. Similar to [8] we define

Definition 2.3 (Pole condition)**.** *Let $u\colon \Omega \to \mathbb{C}$ and assume that the Laplace transform*

$$\mathcal{L}\{v\}(s,\hat{x}) := \int_0^\infty v(r,\hat{x})e^{-sr}\,\mathrm{d}r \tag{10}$$

of $u|_{\Omega_{\mathrm{ext}}} \circ T_{\mathrm{polar}}$ is well defined and belongs to $L^2(\Gamma)$ for all s in some open region $D \subset \mathbb{C}$. We say that u satisfies the pole condition with respect to $\kappa_0 \in \mathbb{C}\setminus\{0\}$ if the function $D \to L^2(\Gamma), s \mapsto \mathcal{L}\{u|_{\Omega_{\mathrm{ext}}} \circ T_{\mathrm{polar}}\}(s,\bullet)$ has an analytic extension to $\{s \in \mathbb{C}\colon \Im(s\kappa_0^{-1}) < 0\}$ and $(\mathcal{M}_{\kappa_0} \otimes \mathrm{Id})\mathcal{L}|_{\kappa_0\mathbb{R}}\{u|_{\Omega_{\mathrm{ext}}} \circ T_{\mathrm{polar}}\} \in H^+(S^1) \otimes L^2(\Gamma)$, where $\mathcal{L}|_{\kappa_0\mathbb{R}}\{v\} := \mathcal{L}\{v\}|_{\kappa_0\mathbb{R}\times\Gamma}$.

In [10, 8] it was show that for solutions u to (1a)/(2a) and spherical Γ the pole condition with $\Re(\kappa\kappa_0^{-1}) > 0$ is equivalent to (1c)/(2c). We generalize this result in Lem. 2.4.

Lemma 2.4. *Let u be a solution to* (1a)/(2a) *and $\Re(\kappa\kappa_0^{-1}) > 0$. Then* (1c)/(2c) *is equivalent to the pole condition.*

Proof. Because u solves (1a), $u|_{\Omega_{\mathrm{ext}}} \circ T_{\mathrm{polar}}$ admits a representation

$$\sum_{l=0}^{\infty}\sum_{m=0}^{M_l} \alpha_{l,m}h_l^{(1)}(\kappa(r+1)|\hat{x}|)Y_{l,m}(|\hat{x}|^{-1}\hat{x})$$
$$+\sum_{l=0}^{\infty}\sum_{m=0}^{M_l} \beta_{l,m}h_l^{(2)}(\kappa(r+1)|\hat{x}|)Y_{l,m}(|\hat{x}|^{-1}\hat{x})$$

with $\alpha_{l,m}, \beta_{l,m} \in \mathbb{C}$. From [10] and a variable change $\kappa(r+1)|\hat{x}| \to r+a$ it follows, that(1c) is equivalent to that $\mathcal{L}\{u|_{\Omega_{\mathrm{ext}}} \circ T_{\mathrm{polar}}\}$ is well defined and admits an analytic extension to $\{s \in \mathbb{C}\colon \Im(s\kappa_0^{-1}) < 0\}$. It remains to show $\mathcal{L}|_{\kappa_0\mathbb{R}}\{u|_{\Omega_{\mathrm{ext}}} \circ T_{\mathrm{polar}}\} \in L^2(\kappa_0\mathbb{R}) \otimes L^2(\Gamma)$, which follows from Lem. 3.9. □

2.3 Hardy space operators

We define the bilinear forms

$$\mathcal{A}_d(U,V) := \pi^{-1}\int_\Gamma\int_{S^1} U(\bar{z},\hat{x})\cdot V(z,\hat{x})\,\mathrm{d}|z|\mathrm{d}\hat{x}, \qquad U,V\in[H^+(S^1)\otimes L^2(\Gamma)]^d,\ d=1,2,3. \tag{11}$$

From [8, Lem. A.1] we know

$$\int_\Gamma\int_0^\infty uv\,\mathrm{d}r\mathrm{d}\hat{x} = -\mathrm{i}\kappa_0\mathcal{A}_1((\mathcal{M}_{\kappa_0}\otimes\mathrm{Id})\mathcal{L}|_{\kappa_0\mathbb{R}}\{u\},(\mathcal{M}_{\kappa_0}\otimes\mathrm{Id})\mathcal{L}|_{\kappa_0\mathbb{R}}\{v\}), \tag{12}$$

if u,v fulfill certain conditions formulated in [8, Lem. A.1], which will always be satisfied in this paper. This allows us to transform a weak variational formulation in $H^1_{\mathrm{loc}}(\Omega)$ to a variational formulation in a subspace of $\mathfrak{X}^0 := L^2(\Omega_{\mathrm{int}})\oplus\big(H^+(S^1)\otimes L^2(\Gamma)\big)$. Henceforth we will write $\mathbf{U} = \binom{\mathrm{tr}_\Gamma u_{\mathrm{int}}}{U}$ for $\mathfrak{u} = u_{\mathrm{int}}\oplus U\in\mathfrak{X}^0$. Let

$$\mathcal{T}_\pm\binom{u_0}{U}(z) := \frac{1}{2}\big(u_0+(z\pm1)U(z)\big),\quad \binom{u_0}{U}\in\mathbb{C}\times H^+(S^1). \tag{13}$$

The operators $\mathcal{T}_\pm$ with their domain $\mathbb{C}\times H^+(S^1)$ are useful for two reasons. They allow to define a simple trace operator $\binom{u_0}{U}\mapsto u_0$ and avoid the unhandy operator $\mathcal{M}_{\kappa_0}\mathcal{L}\partial_r(\mathcal{M}_{\kappa_0}\mathcal{L})^{-1}$. Indeed [8] tells us that if $\Re\kappa_0>0$ and $u\in H^1_{\mathrm{loc}}(\Omega_{\mathrm{ext}})$ is either a test function of form (19) or a solution to either (1) or (2) and $\Re(\kappa\kappa_0^{-1})>0$, then

$$(\mathcal{M}_{\kappa_0}\otimes\mathrm{Id})\mathcal{L}|_{\kappa_0\mathbb{R}}\{\ u\circ T_{\mathrm{polar}}\} = -\mathrm{i}\kappa_0^{-1}(\mathcal{T}_-\otimes\mathrm{Id})\binom{\mathrm{tr}_\Gamma u}{U}, \tag{14a}$$

$$(\mathcal{M}_{\kappa_0}\otimes\mathrm{Id})\mathcal{L}|_{\kappa_0\mathbb{R}}\{\partial_r u\circ T_{\mathrm{polar}}\} = (\mathcal{T}_+\otimes\mathrm{Id})\binom{\mathrm{tr}_\Gamma u}{U}. \tag{14b}$$

with

$$U(z,\hat{x}) := \frac{\big(2\mathrm{i}\kappa_0(\mathcal{M}_{\kappa_0}\otimes\mathrm{Id})\mathcal{L}|_{\kappa_0\mathbb{R}}\{u\circ T_{\mathrm{polar}}\}\big)(z,\hat{x})-\mathrm{tr}_\Gamma u(\hat{x})}{z-1}, \tag{14c}$$

$U\in H^+(S^1)\otimes L^2(\Gamma)$. From [8, Lem. 4.1] we know that

$$\Big(\mathcal{Q}\binom{u_0}{U}\Big)(t) := \frac{-1}{2\pi}\int_\mathbb{R} e^{\mathrm{i}st}\Big(\mathcal{M}_{\kappa_0}^{-1}\mathcal{T}_-\binom{u_0}{U}\Big)(\kappa_0 s)\,\mathrm{d}s,\qquad t\geq 0 \tag{15a}$$

is a norm isomorphism $\mathcal{Q}\colon\mathbb{C}\times H^+(S^1)\to H^1(\mathbb{R}^+)$, such that

$$\Big(\mathcal{Q}\binom{u_0}{U}\Big)(0) = u_0, \tag{15b}$$

$$\Big(\mathcal{Q}\binom{u_0}{U}\Big)'(t) = \frac{-1}{2\pi}\int_\mathbb{R} e^{\mathrm{i}st}\Big(\mathcal{M}_{\kappa_0}^{-1}\mathcal{T}_+\binom{u_0}{U}\Big)(\kappa_0 s)\,\mathrm{d}s,\qquad t\geq 0. \tag{15c}$$

This shows that the operators $\mathcal{T}_\pm$ and the domain $\mathbb{C} \times H^+(S^1)$ are rather natural than mystical. We will also need to transform terms of the form $(r+1)u(r)$ and therefore introduce

$$(\hat{\mathcal{E}}\mathcal{L}\{u\})(s) := 2\mathrm{i}\kappa_0 \partial_s \mathcal{L}\{u\}(s) = -2\mathrm{i}\kappa_0 \mathcal{L}\{\bullet u\}(s), \tag{16a}$$

$$\mathcal{E} := \mathcal{M}_{\kappa_0} \hat{\mathcal{E}} \mathcal{M}_{\kappa_0}^{-1}, \quad \mathcal{D} := \mathcal{I} - \frac{1}{2\mathrm{i}\kappa_0}\mathcal{E}, \tag{16b}$$

where $\mathcal{I}$ denotes the identity operator on $H^+(S^1)$. Thus we can compute

$$(\mathcal{M}_{\kappa_0} \otimes \mathrm{Id})\mathcal{L}|_{\kappa_0\mathbb{R}}\{(r+1)u \circ T_{\mathrm{polar}}(r,\hat{x})\} = (\mathcal{D}\mathcal{M}_{\kappa_0} \otimes \mathrm{Id})\mathcal{L}|_{\kappa_0\mathbb{R}}\{u \circ T_{\mathrm{polar}}\}. \tag{16c}$$

From ([8, (3.9)]) we further know

$$(\mathcal{E}U)(z) = -(z-1)^2 U'(z) - (z-1)U(z), \quad U \in H^+(S^1). \tag{16d}$$

Let

$$(\mathcal{C}U)(z,\hat{x}) := \overline{U}(\bar{z},\hat{x}), \quad U \in H^+(S^1) \otimes L^2(\Gamma). \tag{17}$$

The conjugation $\mathcal{C}$ allows us to transform bilinear forms into sesquilinear forms:

$$\mathcal{A}_d(U, \mathcal{C}V) = \pi^{-1}\langle U, V\rangle_{[H^+(S^1)\otimes L^2(\Gamma)]^d}, \quad U, V \in [H^+(S^1) \otimes L^2(\Gamma)]^d, \tag{18}$$

for $d = 1, 2, 3$. Further for $\mathfrak{u} = u_{\mathrm{int}} \oplus U \in \mathfrak{X}^0$ we define the conjugation $\overline{\mathfrak{u}} := \overline{u_{\mathrm{int}}} \oplus \mathcal{C}U$, $\overline{\mathbf{U}} := \begin{pmatrix} \mathrm{tr}_\Gamma \overline{u_{\mathrm{int}}} \\ \mathcal{C}U \end{pmatrix}$.

2.4 Variational formulation

Let $\kappa_0 \in \mathbb{C}$, $\Re\kappa_0 > 0$ and $\Re(\kappa\kappa_0^{-1}) > 0$. Let $\Lambda \subset \mathbb{C}$ be an open set, such that $\Re(\lambda\kappa_0^{-1}), \Im(\lambda + \kappa) > 0$ for all $\lambda \in \Lambda$ and $\lambda_0 \in \Lambda$. Consider test functions $v \in H^1_{\mathrm{loc}}(\Omega)$ of the following form

$$\begin{aligned} v_{\mathrm{int}} &:= v|_{\Omega_{\mathrm{int}}} \in H^1(\Omega_{\mathrm{int}}), \\ v_{\mathrm{ext}}(r,\hat{x}) &:= v|_{\Omega_{\mathrm{ext}}} \circ T_{\mathrm{polar}}(r,\hat{x}) = \mathrm{tr}_\Gamma\, v_{\mathrm{int}}(\hat{x}) e^{\mathrm{i}\lambda_0 r} + v_1(\hat{x}) \frac{e^{\mathrm{i}\lambda_0 r} - e^{\mathrm{i}\lambda r}}{\mathrm{i}\lambda_0 - \mathrm{i}\lambda}, \end{aligned} \tag{19}$$

with $v_1 \in C^\infty(\Gamma)$ and $\lambda \in \Lambda \setminus \{\lambda_0\}$. Assume that u is a solution to (1) or (2). With (7) a straight forward computation yields

$$\begin{aligned} &\int_{\Omega_{\mathrm{int}}} \nabla u_{\mathrm{int}} \cdot \nabla v_{\mathrm{int}} - \kappa^2 u_{\mathrm{int}} v_{\mathrm{int}} \,\mathrm{d}x \\ &+ \int_\Gamma \int_0^\infty \begin{pmatrix} \partial_r u_{\mathrm{ext}} \\ \nabla_{\hat{x}} u_{\mathrm{ext}} \end{pmatrix} \cdot \begin{pmatrix} Q_{11}(\hat{x})(r+1)^2 & Q_{12}(\hat{x})(r+1) & Q_{13}(\hat{x})(r+1) \\ Q_{21}(\hat{x})(r+1) & Q_{22}(\hat{x}) & Q_{23}(\hat{x}) \\ Q_{31}(\hat{x})(r+1) & Q_{32}(\hat{x}) & Q_{33}(\hat{x}) \end{pmatrix} \begin{pmatrix} \partial_r v_{\mathrm{ext}} \\ \nabla_{\hat{x}} v_{\mathrm{ext}} \end{pmatrix} \mathrm{d}r\mathrm{d}\hat{x}, \\ &- \kappa^2 \int_\Gamma \int_0^\infty (r+1)^2 u_{\mathrm{ext}} v_{\mathrm{ext}} q \,\mathrm{d}r\mathrm{d}\hat{x} = \int_{\partial K} g v_{\mathrm{int}} \,\mathrm{d}s, \end{aligned} \tag{20}$$

where $u_{\text{int}} := u|_{\Omega_{\text{int}}}$, $u_{\text{ext}} := u|_{\Omega_{\text{ext}}} \circ T_{\text{polar}}$ and g is zero in the case of (2). Due to Lem. 2.4, (12), (14) and (16c) Equation (20) can be transformed to

$$\mathfrak{B}(\mathfrak{u}, \mathfrak{v}) = \int_{\partial K} g v_{\text{int}} \, \mathrm{d}s \tag{21}$$

with

$$\mathfrak{B} := \mathfrak{B}_A^{\text{int}} + \mathfrak{B}_A^{\text{ext}} - \kappa^2 (\mathfrak{B}_M^{\text{int}} + \mathfrak{B}_M^{\text{ext}}), \tag{22}$$

the interior bilinear forms

$$\mathfrak{B}_A^{\text{int}}(\mathfrak{u}, \mathfrak{v}) := \int_{\Omega_{\text{int}}} \nabla u_{\text{int}} \cdot \nabla v_{\text{int}} \, \mathrm{d}x, \tag{23a}$$

$$\mathfrak{B}_M^{\text{int}}(\mathfrak{u}, \mathfrak{v}) := \int_{\Omega_{\text{int}}} u_{\text{int}} v_{\text{int}} \, \mathrm{d}x, \tag{23b}$$

the multiplication operator $\mathrm{M_w} \colon L^2(\Gamma) \to L^2(\Gamma)$ with symbol $w \in L^\infty(\Gamma)$, the exterior bilinear forms

$$\mathfrak{B}_A^{\text{ext}}(\mathfrak{u}, \mathfrak{v}) := \mathrm{i}\kappa_0^{-1} \mathcal{A}_3 \Bigg(\begin{pmatrix} \mathcal{T}_+ \otimes \mathrm{Id} \\ \mathcal{T}_- \otimes \nabla_{\hat{x}} \end{pmatrix} \mathbf{U}, \begin{pmatrix} (\mathrm{i}\kappa_0 \mathcal{D})^2 \otimes \mathrm{M_{Q_{11}}} & \mathrm{i}\kappa_0 \mathcal{D} \otimes \mathrm{M_{Q_{12}}} & \mathrm{i}\kappa_0 \mathcal{D} \otimes \mathrm{M_{Q_{13}}} \\ \mathrm{i}\kappa_0 \mathcal{D} \otimes \mathrm{M_{Q_{21}}} & \mathcal{I} \otimes \mathrm{M_{Q_{22}}} & \mathcal{I} \otimes \mathrm{M_{Q_{23}}} \\ \mathrm{i}\kappa_0 \mathcal{D} \otimes \mathrm{M_{Q_{31}}} & \mathcal{I} \otimes \mathrm{M_{Q_{32}}} & \mathcal{I} \otimes \mathrm{M_{Q_{33}}} \end{pmatrix} \begin{pmatrix} \mathcal{T}_+ \otimes \mathrm{Id} \\ \mathcal{T}_- \otimes \nabla_{\hat{x}} \end{pmatrix} \mathbf{V} \Bigg), \tag{23c}$$

$$\mathfrak{B}_M^{\text{ext}}(\mathfrak{u}, \mathfrak{v}) := \mathrm{i}\kappa_0^{-1} \mathcal{A}_1 \big((\mathcal{D}\mathcal{T}_- \otimes \mathrm{Id}) \mathbf{U}, (\mathcal{D}\mathcal{T}_- \otimes \mathrm{M_q}) \mathbf{V} \big), \tag{23d}$$

$\mathfrak{u} := u_{\text{int}} \oplus U$, $\mathfrak{v} := v_{\text{int}} \oplus V$ and U, V as in (14c). Again g is zero in the case that u solves (2). Bilinear form $\mathfrak{B}$ suggests to introduce the space

$$\mathfrak{X}^1 := \{ \mathfrak{f} \in \mathfrak{X}_0 \colon \|\mathfrak{f}\|_{\mathfrak{X}^1} < \infty \} \tag{24}$$

with scalar product

$$\begin{aligned} \langle \mathfrak{f}, \mathfrak{g} \rangle_{\mathfrak{X}^1} := {} & \langle f_{\text{int}}, g_{\text{int}} \rangle^2_{H^1(\Omega_{\text{int}})} + \langle f_{\text{int}}, g_{\text{int}} \rangle^2_{L^2(\Gamma)} + \langle F, G \rangle^2_{L^2(\Gamma) \otimes L^2(S^1)} \\ & + \langle (\mathcal{E}\mathcal{T}_+ \otimes \mathrm{Id}) \mathbf{F}, (\mathcal{E}\mathcal{T}_+ \otimes \mathrm{Id}) \mathbf{G} \rangle^2_{L^2(S^1) \otimes L^2(\Gamma)} \\ & + \langle (\mathcal{E}\mathcal{T}_- \otimes \mathrm{Id}) \mathbf{F}, (\mathcal{E}\mathcal{T}_- \otimes \mathrm{Id}) \mathbf{G} \rangle^2_{L^2(S^1) \otimes L^2(\Gamma)} \\ & + \big\langle (\mathcal{T}_- \otimes \nabla_{\hat{x}}) \mathbf{F}, \big(\mathcal{T}_- \otimes ((\mathrm{D}_{\hat{\mathrm{x}}} \hat{x}^\top \mathrm{D}_{\hat{\mathrm{x}}} \hat{x})^{-1} \nabla_{\hat{x}}) \big) \mathbf{G} \big\rangle^2_{[L^2(S^1) \otimes L^2(\Gamma)]^2} . \end{aligned} \tag{25}$$

Finally we obtain

Theorem 2.5. *$\mathfrak{B}$ is bounded on $\mathfrak{X}^1 \times \mathfrak{X}^1$. Let $\Re\kappa_0 > 0$ and $\Re(\kappa\kappa_0^{-1}) > 0$. If u is a solution to* (1) *or* (2)*, then $\mathfrak{u} \in \mathfrak{X}^1$ with*

$$\mathfrak{u} := u|_{\Omega_{\text{int}}} \oplus U, \tag{26a}$$

$$U(z,\hat{x}) := \frac{\big(2\mathrm{i}\kappa_0(\mathcal{M}_{\kappa_0} \otimes \mathrm{Id})\mathcal{L}|_{\kappa_0\mathbb{R}}\{u \circ T_{\text{polar}}\}\big)(z,\hat{x}) - \operatorname{tr}_\Gamma u(\hat{x})}{z-1}, \tag{26b}$$

and

$$\mathfrak{B}(\mathfrak{u},\mathfrak{v}) = \int_{\partial K} g v_{\text{int}} \,\mathrm{d}s, \qquad \text{resp.} \qquad \mathfrak{B}(\mathfrak{u},\mathfrak{v}) = 0, \tag{27}$$

for all $\mathfrak{v} \in \mathfrak{X}^1$. Vice versa, if (27) *holds for an $\mathfrak{u} = u_{\text{int}} \oplus U \in \mathfrak{X}^1$ and all $\mathfrak{v} \in \mathfrak{X}^1$, then u_{int} is the restriction of a solution to* (1)*, resp.* (2)*.*

Proof. It is straight forward to check the boundedness of $\mathfrak{B}$. $\mathfrak{u} \in \mathfrak{X}^1$ holds due to Lem. 3.9. If u is a solution to (1) or (2), we showed in the preceding that (27) holds true for all transformations $\mathfrak{v}$ of test functions (19). Due to [8, Lem. A.2] $\operatorname{span}\{V\}$ is dense in $H^1(S^1) \otimes L^2(\Gamma)$. Hence $\operatorname{span}\{\mathfrak{v}\}$ is dense in $\mathfrak{X}^1$ and the first statement of the corollary is proven.

For the remaining part we proceed as in [8, Prop. 4.2]: Using $\mathcal{Q} \otimes \mathrm{Id}$ and [8, Lem. A.1] it can be shown that (27) is equivalent to the variational problem to find $u_{\text{int}} \oplus u_{\text{ext}} \in H^1(\Omega) \oplus H^1(\mathbb{R}^+ \times \Gamma)$ with $\operatorname{tr}_\Gamma u_{\text{int}} = u_{\text{ext}}(0,\bullet)$ such that

$$\begin{aligned}
&\int_{\Omega_{\text{int}}} \nabla u_{\text{int}} \cdot \nabla v_{\text{int}} - \kappa^2 u_{\text{int}} v_{\text{int}} \,\mathrm{d}x \\
&+ \int_\Gamma \int_0^\infty \begin{pmatrix} \partial_r u_{\text{ext}} \\ \nabla_{\hat{x}} u_{\text{ext}} \end{pmatrix} \cdot \\
&\qquad \begin{pmatrix} -\mathrm{i}\kappa_0 Q_{11}(\hat{x})(\mathrm{i}\kappa_0^{-1}r+1)^2 & Q_{12}(\hat{x})(\mathrm{i}\kappa_0^{-1}r+1) & Q_{13}(\hat{x})(\mathrm{i}\kappa_0^{-1}r+1) \\ Q_{21}(\hat{x})(\mathrm{i}\kappa_0^{-1}r+1) & \mathrm{i}\kappa_0^{-1} Q_{22}(\hat{x}) & \mathrm{i}\kappa_0^{-1} Q_{23}(\hat{x}) \\ Q_{31}(\hat{x})(\mathrm{i}\kappa_0^{-1}r+1) & \mathrm{i}\kappa_0^{-1} Q_{32}(\hat{x}) & \mathrm{i}\kappa_0^{-1} Q_{33}(\hat{x}) \end{pmatrix} \begin{pmatrix} \partial_r v_{\text{ext}} \\ \nabla_{\hat{x}} v_{\text{ext}} \end{pmatrix} \mathrm{d}r\mathrm{d}\hat{x}, \\
&- \kappa^2 \mathrm{i}\kappa_0^{-1} \int_\Gamma \int_0^\infty q(\hat{x})(\mathrm{i}\kappa_0^{-1}r+1)^2 u_{\text{ext}} v_{\text{ext}} \,\mathrm{d}r\mathrm{d}\hat{x} = \int_{\partial K} g v_{\text{int}} \,\mathrm{d}s,
\end{aligned}$$

for all $v_{\text{int}} \oplus v_{\text{ext}} \in H^1(\Omega) \oplus H^1(\mathbb{R}^+ \times \Gamma)$ with $\operatorname{tr}_\Gamma v_{\text{int}} = v_{\text{ext}}(0,\bullet)$ and $g = 0$ in the case of the resonance problem. Considering the strong form of this variational problem in the domain $\mathbb{R}^+ \times \Gamma$ shows that

$$\begin{aligned}
u_{\text{ext}}(r,\hat{x}) = &\sum_{l=0}^\infty \sum_{m=0}^{M_l} \alpha_{l,m} h_l^{(1)}(\kappa(\mathrm{i}\kappa_0^{-1}r+1)|\hat{x}|) Y_{l,m}(|\hat{x}|^{-1}\hat{x}) \\
&+ \sum_{l=0}^\infty \sum_{m=0}^{M_l} \beta_{l,m} h_l^{(2)}(\kappa(\mathrm{i}\kappa_0^{-1}r+1)|\hat{x}|) Y_{l,m}(|\hat{x}|^{-1}\hat{x}).
\end{aligned}$$

Because $u_{\text{ext}} \in H^1(\mathbb{R}^+ \times \Gamma)$, the coefficients $\beta_{l,m}$ have to vanish. It follows that $u \in H^1_{\text{loc}}(\Omega)$ with $u|_{\Omega_{\text{int}}} := u_{\text{int}}$ and $u|_{\Omega_{\text{ext}}} := \big(u_{\text{ext}}(-\mathrm{i}\kappa_0 \bullet, \bullet)\big) \circ T^{-1}_{\text{polar}}$ is a solution to (1), resp. (2). □

Remark 2.6. *In [8] only test functions of the form* $v_{\text{ext}}(r,\hat{x}) = \operatorname{tr}_\Gamma v_{\text{int}}(\hat{x}) e^{\mathrm{i}\lambda r}$ *were considered, which don't lead to a dense test space.*

Remark 2.7. *The variational problem derived in the proof of Cor. 2.5 is a PML formulation with complex scaling parameter* $\mathrm{i}\kappa_0^{-1}$*.*

3 Convergence results

In this section we show that bilinear form $\mathfrak{B}\colon \mathfrak{X}^1 \times \mathfrak{X}^1 \to \mathbb{C}$ can be written as $\mathfrak{B} = \mathfrak{B}_P + \mathfrak{B}_K$, such that $\mathfrak{B}_P$ is positive definite (Lem. 3.1) and $\mathfrak{B}_K\colon \mathfrak{Y} \times \mathfrak{Y} \to \mathbb{C}$, where the embedding $\mathfrak{X}^1 \hookrightarrow \mathfrak{Y}$ is compact (Lem. 3.4). This result is formulated in Prop. 3.5, which allows to apply certain literature. The consequences are presented in Thm. 3.7 and Thm. 3.8.

Let Γ be either a sphere with center P_0 or the boundary of a polyhedral, which is star-shaped with respect to P_0. W.l.o.g. we assume $P_0 = 0$. For every bilinear form $\mathfrak{B}_i\colon \mathfrak{X}^1 \times \mathfrak{X}^1 \to \mathbb{C}$ let $\mathfrak{S}_i(\bullet,\bullet) := \mathfrak{B}_i(\bullet,\overline{\bullet})$ be the associated sesquilinear form. We make the dependency of $\mathfrak{B}$ defined in (22) on κ explicit with the notation $\mathfrak{B}(\kappa)$.

Lemma 3.1. *Let* $\kappa_0 \in \mathbb{C}, \Re\kappa_0, \Im\kappa_0 > 0$*. For every* $\epsilon \in (0,\pi/2)$ *there exists* $C_\epsilon > 0$*, such that for*

$$\begin{aligned} \mathfrak{B}^\epsilon_K(\kappa)(\mathfrak{u},\mathfrak{v}) := &-(\kappa^2+1)\mathfrak{B}^{\text{int}}_M(\mathfrak{u},\mathfrak{v}) \\ &-(\kappa^2+C_\epsilon)\mathcal{A}_1\big((\mathcal{T}_- \otimes \mathrm{Id})\mathbf{U}, (\mathcal{T}_- \otimes \mathrm{M_q})\mathbf{V}\big) \end{aligned} \tag{28}$$

the sesquilinear form

$$\mathfrak{S}^\epsilon_P(\kappa) := \mathfrak{S}(\kappa) - \mathfrak{S}^\epsilon_K(\kappa) \tag{29}$$

is positive definite for all $\kappa \in \mathbb{K}_{\kappa_0}(\epsilon) := \{s \in \mathbb{C}\colon \Re(s\kappa_0^{-1}e^{-\mathrm{i}\epsilon}), -\Im(s\kappa_0^{-1}) > 0, \epsilon^{-1} > |s| > \epsilon\}$*.*

Proof. For spherical Γ it holds $Q_{12} = Q_{13} = 0$. Expanding $\mathcal{D}^2 = \mathcal{I} + \mathrm{i}\kappa_0^{-1}\mathcal{E} -$

$\kappa_0^{-2}/4\mathcal{E}^2$ yields

$$\begin{aligned}
-i\kappa_0 \mathfrak{S}_P^\epsilon(\kappa)(\mathfrak{u},\mathfrak{u}) = & -i\kappa_0 \|u_{\mathrm{int}}\|^2_{H^1(\Omega_{\mathrm{int}})} \\
& - \kappa_0^2 \pi^{-1} \|(\mathcal{T}_+ \otimes \mathrm{M}_{\sqrt{\mathrm{Q}_{11}}})\mathbf{U}\|^2_{H^+(S^1)\otimes L^2(\Gamma)} \\
& - i\kappa_0 \pi^{-1} \|(\mathcal{E}^{1/2}\mathcal{T}_+ \otimes \mathrm{M}_{\sqrt{\mathrm{Q}_{11}}})\mathbf{U}\|^2_{H^+(S^1)\otimes L^2(\Gamma)} \\
& + \pi^{-1}/4 \|(\mathcal{E}\mathcal{T}_+ \otimes \mathrm{M}_{\sqrt{\mathrm{Q}_{11}}})\mathbf{U}\|^2_{H^+(S^1)\otimes L^2(\Gamma)} \\
& + \pi^{-1} \Big\| \Big(\mathcal{T}_- \otimes \Big(\left(\begin{smallmatrix} Q_{22} & Q_{23} \\ Q_{32} & Q_{33} \end{smallmatrix}\right)^{1/2} \nabla_{\hat{x}}\Big)\Big)\mathbf{U}\Big\|^2_{[H^+(S^1)\otimes L^2(\Gamma)]^2} \\
& + C_\epsilon \pi^{-1} \|(\mathcal{T}_- \otimes \mathrm{M}_{\sqrt{\mathrm{q}}})\mathbf{U}\|^2_{H^+(S^1)\otimes L^2(\Gamma)} \\
& - \kappa^2 i\kappa_0^{-1} \pi^{-1} \|(\mathcal{E}^{1/2}\mathcal{T}_- \otimes \mathrm{M}_{\sqrt{\mathrm{q}}})\mathbf{U}\|^2_{H^+(S^1)\otimes L^2(\Gamma)} \\
& + \kappa^2 \kappa_0^{-2} \pi^{-1}/4 \|(\mathcal{E}\mathcal{T}_- \otimes \mathrm{M}_{\sqrt{\mathrm{q}}})\mathbf{U}\|^2_{H^+(S^1)\otimes L^2(\Gamma)}.
\end{aligned} \tag{30}$$

Apart from the term $\|(\mathcal{E}^{1/2}\mathcal{T}_- \otimes \mathrm{M}_{\sqrt{\mathrm{q}}})\mathbf{U}\|^2_{H^+(S^1)\otimes L^2(\Gamma)}$ the sign of each summand on the right hand side of (30) is contained in an open half-plane $\{s \in \mathbb{C} \colon \Im(s\tau^{-1}) < 0\}$, with $\Re\tau, \Im\tau > 0$ independent of κ. The term $\|(\mathcal{E}^{1/2}\mathcal{T}_- \otimes \mathrm{M}_{\sqrt{\mathrm{q}}})\mathbf{U}\|^2_{H^+(S^1)\otimes L^2(\Gamma)}$ can be estimated by $\delta\|(\mathcal{E}\mathcal{T}_- \otimes \mathrm{M}_{\sqrt{\mathrm{q}}})\mathbf{U}\|^2_{H^+(S^1)\otimes L^2(\Gamma)} + \frac{1}{4\delta}\|(\mathcal{T}_- \otimes \mathrm{Id})\mathbf{U}\|^2_{H^+(S^1)\otimes L^2(\Gamma)}$ for arbitrary $\delta > 0$. Since $\kappa^2\kappa_0^{-2}$ is contained in a proper circular subsector of the half-plane and $|\kappa| > \epsilon$, there exists $c > 0$, such that $-\Im(\tau^{-1}\kappa^2\kappa_0^{-2}\pi^{-1}/4) > c$. Let $\delta := c/2$. Because the metric tensor $(\mathrm{D}_{\hat{\mathrm{x}}}\, \hat{x}^\top\, \mathrm{D}_{\hat{\mathrm{x}}}\, \hat{x})^{-1}$ and all arising symbols of M are bounded from above and below and $\|(\mathcal{T}_+ \otimes \mathrm{Id})\mathbf{U}\|_{H^1(S^1)\otimes L^2(\Gamma)} + \|(\mathcal{T}_- \otimes \mathrm{Id})\mathbf{U}\|_{H^1(S^1)\otimes L^2(\Gamma)} \geq \|\operatorname{tr}_\Gamma u_{\mathrm{int}}\|_{L^2(\Gamma)} + \|U\|_{H^1(S^1)\otimes L^2(\Gamma)}$ the claim holds for $C_\epsilon^{-1} < 2c\epsilon^2|\kappa_0| \inf_{\hat{x}\in\Gamma} q(\hat{x})$.

In the case of piecewise polygonial Γ we have to work a bit more. Γ is the union of finitely many polygons $T_n, n = 1, \ldots, M, M \in \mathbb{N}$. Each polygon T_n can be parametrized as $a_n\nu_n + x_n\mathbf{t}_{n,1} + y_n\mathbf{t}_{n,2}$ with outer unit normal vector ν_n, $a_n \in \mathbb{R}^+$, tangential vectors $\mathbf{t}_{n,1} \perp \mathbf{t}_{n,2}$ and $(x_n, y_n) \in \hat{T}_n$ with a reference polygon $\hat{T}_n \subset \mathbb{R}^2$. As the following can be done for every polygon T_n, we drop the index n. A computation shows $q = a$ and

$$\begin{aligned}
(Q_{nm})_{n,m=1,2,3} &= a^{-3}\begin{pmatrix} 1 & -x & -y \\ -x & a^2+x^2 & xy \\ -y & yx & a^2+y^2 \end{pmatrix} \\
&= a^{-3}\begin{pmatrix} b & -x & -y \\ -x & a^2/2+x^2 & xy \\ -y & yx & a^2/2+y^2 \end{pmatrix} + a^{-3}\begin{pmatrix} 1-b & & \\ & a^2/2 & \\ & & a^2/2 \end{pmatrix},
\end{aligned}$$

with $b \in (\max_{(x,y)\in\hat{T}} \dfrac{x^2+y^2}{a^2/2+x^2+y^2}, 1)$. The condition on b ensures that both matrices in (33) are regular. Denote the spectrum of an operator with σ. If we can show that $\{\sigma(\mathcal{K}_{x,y}) \colon (x,y) \in \hat{T}\} \subset \{s \in \mathbb{C} \colon \Im(s\tau^{-1}) < 0\}$, where we consider

$$\mathcal{K}_{x,y} := \begin{pmatrix} b(\mathrm{i}\kappa_0\mathcal{D})^2 & -x\mathrm{i}\kappa_0\mathcal{D} & -y\mathrm{i}\kappa_0\mathcal{D} \\ -x\mathrm{i}\kappa_0\mathcal{D} & a^2/2+x^2 & xy \\ -y\mathrm{i}\kappa_0\mathcal{D} & yx & a^2/2+y^2 \end{pmatrix}$$

for fixed (x, y) as an unbounded operator on $[H^+(S^1)]^3$, the same arguments as for spheres yield the claim. The unitary transformation $\mathcal{M}_{\kappa_0}^{-1} \bullet \mathcal{M}_{\kappa_0}$ yields

$$\sigma(\mathcal{K}_{x,y}) = \sigma \begin{pmatrix} b(\mathrm{i}\kappa_0(1-\partial_s))^2 & -x\mathrm{i}\kappa_0(1-\partial_s) & -y\mathrm{i}\kappa_0(1-\partial_s) \\ -x\mathrm{i}\kappa_0(1-\partial_s) & a^2/2+x^2 & xy \\ -y\mathrm{i}\kappa_0(1-\partial_s) & yx & a^2/2+y^2 \end{pmatrix},$$

where the latter is an unbounded operator on $[H^-(\kappa_0\mathbb{R})]^3$. From the Fourier transformation and the Paley-Wiener Theorem it follows

$$\sigma(\mathcal{K}_{x,y}) = \left\{ \sigma \begin{pmatrix} b(\mathrm{i}\kappa_0-t)^2 & -x(\mathrm{i}\kappa_0-t) & -y(\mathrm{i}\kappa_0-t) \\ -x(\mathrm{i}\kappa_0-t) & a^2/2+x^2 & xy \\ -y(\mathrm{i}\kappa_0-t) & yx & a^2/2+y^2 \end{pmatrix} : t \geq 0 \right\}.$$

For $(x, y) = 0$ it holds $\sigma \begin{pmatrix} b(\mathrm{i}\kappa_0-t)^2 & -x(\mathrm{i}\kappa_0-t) & -y(\mathrm{i}\kappa_0-t) \\ -x(\mathrm{i}\kappa_0-t) & a^2/2+x^2 & xy \\ -y(\mathrm{i}\kappa_0-t) & yx & a^2/2+y^2 \end{pmatrix} = \{b(\mathrm{i}\kappa_0 - t)^2, a^2/2\}$. Hence let $(x, y) \neq 0$. A computation shows

$$\det \begin{pmatrix} b(\mathrm{i}\kappa_0-t)^2-\lambda & -x(\mathrm{i}\kappa_0-t) & -y(\mathrm{i}\kappa_0-t) \\ -x(\mathrm{i}\kappa_0-t) & a^2/2+x^2-\lambda & xy \\ -y(\mathrm{i}\kappa_0-t) & yx & a^2/2+y^2-\lambda \end{pmatrix} = (a^2/2 - \lambda)\cdot$$
$$\left((\mathrm{i}\kappa_0 - t)^2(a^2/2 - \lambda + (b-1)(x^2 + y^2)) - \lambda(a^2/2 - \lambda + x^2 + y^2)\right).$$

Let $\lambda_{1,2}(t, x^2+y^2)$ be the roots of the quadratic polynomial $(\mathrm{i}\kappa_0 - t)^2(a^2/2 - \lambda + (b-1)(x^2+y^2)) - \lambda(a^2/2 - \lambda + x^2 + y^2)$. Since $\Im\{(\mathrm{i}\kappa_0 - t)^2\} \neq 0$, it follows $\lambda_{1,2}(t, x^2 + y^2) \notin \mathbb{R} \cup (\mathrm{i}\kappa_0 - t)^2\mathbb{R}$. $\lambda_{1,2}$ are continuous functions of (t, x^2+y^2) due to definition. It remains to show that there exist (x_j, y_j) and $t_j \in \mathbb{R}_0^+$ such that $\lambda_j(t_j, x_j^2+y_j^2) \in \{r\exp(\mathrm{i}\phi) \colon r \in \mathbb{R}^+, \phi \in (\phi_2(t_j), 0)\}$, where $\phi_2(t) \in (-\pi, 0)$ is such that $(\mathrm{i}\kappa_0 - t)^2 = |\mathrm{i}\kappa_0 - t|^2 \exp(\mathrm{i}\phi_2(t))$, for $j = 1, 2$. A computation shows

$$\partial_{x^2+y^2}\lambda_1(0,0) = 1 - (\kappa_0^2 a^2/2 + b|\kappa_0|^4)\frac{b(2-b)}{|a^2/2 + b\kappa_0^2|^2},$$
$$\partial_{x^2+y^2}\lambda_2(0,0) = (\kappa_0^2 a^2/2 + b|\kappa_0|^4)\frac{b(2-b)}{|a^2/2 + b\kappa_0^2|^2},$$

where $\lambda_1(0,0) = a^2/2$ and $\lambda_2(0,0) = -\kappa_0^2 b$. Thus we can construct the demanded $(x_1, y_1), (x_2, y_2)$ and the claim is proven. □

Remark 3.2. *A reasonable question is if the domain $\mathbb{K}_{\kappa_0}(\epsilon)$ in Lem. 3.1 for which $\mathfrak{S}_P^\epsilon(\kappa)$ is positive definite can be extended over $\mathbb{K}_{\kappa_0}(0)$. The answer is* *no: The image of the normalized function* $F_\lambda := \frac{\sqrt{1-|\lambda|^2}}{1-\lambda\bullet} \in H^+(S^1)$, $|\lambda| < 1$ *under the operators* $\mathcal{T}_-, \mathcal{E}^{1/2}\mathcal{T}_-, \mathcal{E}\mathcal{T}_-, \mathcal{E}^{1/2}\mathcal{T}_+, \mathcal{E}\mathcal{T}_+$ *becomes arbitrary small for* $\lambda \to 1$ *and likewise under* $\mathcal{T}_+$ *for* $\lambda \to -1$. *Considering appropriate tensor product functions shows that* $\mathfrak{S}^\epsilon_P(\kappa)$ *is not positive definite.*

Remark 3.3. *As the minimum angle between* $\hat{x} - P_0$ *and* $\mathrm{D}_{\hat{\mathrm{x}}}\,\hat{x}$, $\hat{x} \in \Gamma$ *decreases, so does* $\min_{\hat{x}\in\Gamma} q(\hat{x})$. *Since this factor enters in the coerzivity constant of* $\mathfrak{S}^\epsilon_P$, *we expect the conditioning of* (27) *to worsen as* Γ *takes largely anisotropic shape.*

Lemma 3.4. *Let* $\mathfrak{Y}$ *be the closure of* $\mathfrak{X}^0$ *with respect to the scalar product*

$$\langle \mathfrak{u}, \mathfrak{v}\rangle_{\mathfrak{Y}} := \langle u_{\mathrm{int}}, v_{\mathrm{int}}\rangle_{L^2(\Omega_{\mathrm{int}})} + \langle (\mathcal{T}_- \otimes \mathrm{Id})\mathbf{U}, (\mathcal{T}_- \otimes \mathrm{Id})\mathbf{V}\rangle_{H^+(S^1)\otimes L^2(\Gamma)}. \quad (33)$$

The embedding $\mathfrak{X}^1 \hookrightarrow \mathfrak{Y}$ *is compact.*

Proof. Let $\mathfrak{u}^n, n \in \mathbb{N}$ be a sequence in $\mathfrak{X}^1$ bounded by C. We have to show that there exists $\mathfrak{u} \in \mathfrak{Y}$ and a subsequence converging to $\mathfrak{u}$. Because of the standard compact Sobolev embedding $H^1(\Omega_{\mathrm{int}}) \hookrightarrow L^2(\Omega_{\mathrm{int}})$, we already assume $u^n_{\mathrm{int}} \to u_{\mathrm{int}}$ for an $u_{\mathrm{int}} \in L^2(\Omega_{\mathrm{int}})$. In the following we construct a subsequence $n(m)$, such that $\mathbf{U}^{n(m)}$ is a Cauchy sequence. Let $\phi_l, l \in \mathbb{N}$ be a $L^2(\Gamma)$-orthogonal basis of eigenfunctions to the Laplace-Beltrami operator on Γ with nondecreasing eigenvalues $0 \le \lambda_l \to \infty, l \to \infty$. Define $\mathbf{U}^n_l := \langle \phi_l, \mathbf{U}^n\rangle_{L^2(\Gamma)}$. Then $\mathbf{U}^n = \sum_{l\in\mathbb{N}} \mathbf{U}^n_l \otimes \phi_l$ holds. For every $m \in \mathbb{N}$ let $L \in \mathbb{N}$ be such that $\lambda_l > m$ for all $l > L$. A computation shows

$$\begin{aligned}\|(\mathcal{T}_- \otimes \mathrm{Id})(\mathbf{U}^{n_1} - \mathbf{U}^{n_2})\|^2_{H^+(S^1)\otimes L^2(\Gamma)} &= \sum_{l\in\mathbb{N}} \|\mathcal{T}_-(\mathbf{U}^{n_1}_l - \mathbf{U}^{n_2}_l)\|^2_{H^+(S^1)} \\ &\le \frac{2C^2}{m} + \sum_{l\le L} \|\mathcal{T}_-(\mathbf{U}^{n_1}_l - \mathbf{U}^{n_2}_l)\|^2_{H^+(S^1)}.\end{aligned}$$

Further it holds

$$\begin{aligned}\sum_{l\le L} \|\mathcal{T}_-(\mathbf{U}^{n_1}_l - \mathbf{U}^{n_2}_l)\|^2_{H^+(S^1)} \le 2\sum_{l\le L} &\|(z-1)(U^{n_1}_l - U^{n_2}_l)\|^2_{H^+(S^1)} \\ &+ 2\|u^{n_1}_{\mathrm{int}} - u^{n_2}_{\mathrm{int}}\|^2_{L^2(\Gamma)}.\end{aligned}$$

We are able to choose a subsequence $n(m)$ of $n \in \mathbb{N}$, such that $\|u^{n(m_1)}_{\mathrm{int}} - u^{n(m_2)}_{\mathrm{int}}\|^2_{L^2(\Gamma)} \le \frac{1}{m}$ for all $m_1, m_2 \ge m$ and $m \in \mathbb{N}$, because of the continuous trace $H^1(\Omega_{\mathrm{int}}) \to H^{1/2}(\Gamma)$ and the compact embedding $H^{1/2}(\Gamma) \to L^2(\Gamma)$. Let $B_{1/m}(1)$ be the ball with center 1 and radius $1/m$. We compute

$$\sum_{l\le L} \|(z-1)(U^{n_1}_l - U^{n_2}_l)\|^2_{H^+(S^1)} \le \sum_{l\le L} \|(z-1)(U^{n_1}_l - U^{n_2}_l)\|^2_{L^2(S^1\setminus B_{1/m}(1))} + \frac{2C^2}{m^2}.$$

For any $\left(\begin{smallmatrix} f_0 \\ F \end{smallmatrix}\right) \in \mathbb{C} \times H^+(S^1)$ it holds

$$\begin{aligned}\|F'\|_{L^2(S^1\setminus B_{1/m}(1))} &\leq \frac{1}{(1-\frac{1}{m})^2}\|(z-1)^2F'(z)\|_{L^2(S^1\setminus B_{1/m}(1))}\\ &\leq \frac{1}{(1-\frac{1}{m})^2}\|(z-1)^2F'(z)+(z-1)F(z)\|_{L^2(S^1\setminus B_{1/m}(1))}\\ &+\frac{1}{(1-\frac{1}{m})^2}\|(z-1)F(z)\|_{L^2(S^1\setminus B_{1/m}(1))}.\end{aligned}$$

Due to (16d) this can be further estimated by

$$\begin{aligned}\frac{1}{(1-\frac{1}{m})^2}\big(\|\mathcal{E}\mathcal{T}_+\left(\begin{smallmatrix} f_0 \\ F \end{smallmatrix}\right)\|_{L^2(S^1\setminus B_{1/m}(1))}+\|\mathcal{E}\mathcal{T}_-\left(\begin{smallmatrix} f_0 \\ F \end{smallmatrix}\right)\|_{L^2(S^1\setminus B_{1/m}(1))}\\ +2\|F\|_{L^2(S^1\setminus B_{1/m}(1))}\big).\end{aligned}$$

Let $\iota U := U|_{S^1\setminus B_{1/m}(1)}$. We derive that each $\iota U_l^{n(m)}, l \leq L$ is bounded in $\iota H^+(S^1)\cap H^1(S^1\setminus B_{1/m}(1))$. Due to the compact embedding $\iota H^+(S^1)\cap H^1(S^1\setminus B_{1/m}(1)) \hookrightarrow \iota H^+(S^1)\cap L^2(S^1\setminus B_{1/m}(1))$, we are able to chose a subsequence $\tilde{n}(m)$ of $n(m)$, such that

$$2\sum_{l\leq L}\|(z-1)(U_l^{\tilde{n}(m_1)}-U_l^{\tilde{n}(m_2)})\|^2_{L^2(S^1\setminus B_{1/m}(1))} \leq \frac{1}{m},$$

for all $m_1, m_2 \geq m$ and $m \in \mathbb{N}$. All together we constructed a subsequence $\tilde{n}(m), m \in \mathbb{N}$, such that $\mathfrak{u}^{\tilde{n}(m)}$ is a Cauchy sequence in $\mathfrak{Y}$. Since $\mathfrak{Y}$ is a Hilbert space, the claim follows. □

From the two preceding lemmata it immediately follows

Proposition 3.5. *Variational Equations* (27) *can be written in operator form*

$$\mathfrak{A}(\kappa)\mathfrak{u} = \mathfrak{g}, \qquad \textit{resp.} \qquad \mathfrak{A}(\kappa)\mathfrak{u} = 0, \tag{34}$$

with $\mathfrak{g} \in \mathfrak{X}^1$, $\langle \mathfrak{g}, \mathfrak{v}\rangle_{\mathfrak{X}^1} = \int_{\partial K} \overline{v_{\mathrm{int}}} g \,\mathrm{d}s$ for all $\mathfrak{v} \in \mathfrak{X}^1$. For every $\epsilon \in (0, \pi/2)$ it holds $\mathfrak{A}(\kappa) = \mathfrak{A}_P^\epsilon(\kappa) + \mathfrak{A}_K^\epsilon(\kappa)$, the maps $\kappa \in \mathbb{K}(\epsilon)$, $\kappa \mapsto \mathfrak{A}_P^\epsilon(\kappa)$, $\kappa \mapsto \mathfrak{A}_K^\epsilon(\kappa)$, are analytic Fredholm operator functions, $\mathfrak{A}_K^\epsilon(\kappa)$ is compact and $\mathfrak{A}_P^\epsilon(\kappa)$ is positive definite for all $\kappa \in \mathbb{K}_{\kappa_0}(\epsilon)$.

Remark 3.6. *First order terms $b \cdot \nabla u, b \in \mathbb{R}^3$ in* (1a)/(1b) *would lead to additional terms*

$$\mathcal{A}_1\big((\mathcal{E}\mathcal{T}_+ \otimes M_{\tilde{b}_1})\mathbf{U}, (\mathcal{T}_- \otimes \mathrm{Id})\mathbf{V}\big) + \mathcal{A}_1\big((\mathcal{T}_- \otimes (\left(\begin{smallmatrix} \tilde{b}_2 \\ \tilde{b}_3 \end{smallmatrix}\right) \cdot \nabla_{\hat{x}}))\mathbf{U}, (\mathcal{T}_- \otimes \mathrm{Id})\mathbf{V}\big)$$

in $\mathfrak{B}_A^{\mathrm{ext}}$, which could be controlled by Youngs inequality. Thus Proposition 3.5 would still hold.

Theorem 3.7. *Let $\mathfrak{X}^1_n \subset \mathfrak{X}^1, n \in \mathbb{N}$ be a sequence of finite dimensional subspaces, such that the orthogonal projection $\mathfrak{P}_n\colon \mathfrak{X}^1 \to \mathfrak{X}^1_n$ onto $\mathfrak{X}^1_n$ converges to the identity $\mathfrak{I}\colon \mathfrak{X}^1 \to \mathfrak{X}^1$ in the strong operator norm. For $\kappa > 0$, $g \in H^{-1/2}(\partial K)$ and $\kappa_0 \in \mathbb{C}, \Re\kappa_0, \Im\kappa_0 > 0$ consider the problem: find $\mathfrak{u} \in \mathfrak{X}^1$, such that*

$$\mathfrak{B}(\kappa)(\mathfrak{u}, \mathfrak{v}) = \int_{\partial K} g v_{\text{int}} \, \mathrm{d}s, \tag{35}$$

for all $\mathfrak{v} \in \mathfrak{X}^1$ and the discrete problem: find $\mathfrak{u}_n \in \mathfrak{X}^1_n$, such that

$$\mathfrak{B}(\kappa)(\mathfrak{u}_n, \mathfrak{v}_n) = \int_{\partial K} g v_{\text{int},n} \, \mathrm{d}s, \tag{36}$$

for all $\mathfrak{v}_n \in \mathfrak{X}^1_n$. Then there exists a unique solution $\mathfrak{u}$ to (35), an index $N > 0$ and a constant $C > 0$, such that (36) admits a unique solution $\mathfrak{u}_n$ for all $n > N$ and $\mathfrak{u}_n$ converges to $\mathfrak{u}$ in $\mathfrak{X}^1$, with the estimate

$$\|u_{\text{int}} - u_{\text{int},n}\|_{H^1(\Omega_{\text{int}})} \leq \|\mathfrak{u} - \mathfrak{u}_n\|_{\mathfrak{X}^1} \leq C \inf_{\mathfrak{v}_n \in \mathfrak{X}^1_n} \|\mathfrak{u} - \mathfrak{v}_n\|_{\mathfrak{X}^1}. \tag{37}$$

Proof. From [2] it is known that (1) admits a unique solution. Due to Cor. 2.5 this also holds for (35). Prop. 3.5 and the Lemma of Lax-Milgram allow us to apply [14, Thm. 13.6, 13.7], which yield the claim. □

Theorem 3.8. *Let $\mathfrak{X}^1_n \subset \mathfrak{X}^1, n \in \mathbb{N}$ be a sequence of finite dimensional subspaces, such that the orthogonal projection $\mathfrak{P}_n\colon \mathfrak{X}^1 \to \mathfrak{X}^1_n$ onto $\mathfrak{X}^1_n$ converges to the identity $\mathfrak{I}\colon \mathfrak{X}^1 \to \mathfrak{X}^1$ in the strong operator norm. For $\kappa_0 \in \mathbb{C}, \Re\kappa_0, \Im\kappa_0 > 0$ let $\mathbb{K}_{\kappa_0} := \{s \in \mathbb{C}\colon \Re(\kappa_0^{-1}\kappa) > 0, \Im(\kappa_0^{-1}\kappa) < 0\}$. Consider the eigenvalue problem to find $(\kappa, \mathfrak{u}) \in \mathbb{K}_{\kappa_0} \times \mathfrak{X}^1$ such that*

$$\mathfrak{A}(\kappa)\mathfrak{u} = 0. \tag{38}$$

and its discretization to find $(\kappa \times \mathfrak{u}_n) \in \mathbb{K}_{\kappa_0} \times \mathfrak{X}^1_n$ such that

$$\mathfrak{P}_n \mathfrak{A}(\kappa)\mathfrak{u}_n = 0. \tag{39}$$

Then there hold the spectral properties of $\mathfrak{A}$:

1. *The spectrum $\sigma(\mathfrak{A}) = \mathbb{K}_{\kappa_0} \setminus \rho(\mathfrak{A})$ of $\mathfrak{A}(\bullet)$ has no cluster points in $\mathbb{K}_{\kappa_0}$,*
2. *every $\kappa \in \sigma(\mathfrak{A})$ is an eigenvalue of $\mathfrak{A}(\bullet)$,*
3. *the operator function $\mathfrak{A}^{-1}(\bullet)$ defined on $\rho(\mathfrak{A})$ by $\mathfrak{A}^{-1}(\kappa) = \mathfrak{A}(\kappa)^{-1}$ is analytic on $\rho(\mathfrak{A})$ and has poles of finite order at every point $\kappa \in \sigma(\mathfrak{A})$.*

For $\Lambda \subset \mathbb{K}_{\kappa_0}$ let $\mu(\Lambda, \mathfrak{A})$ be the sum of the algebraic multiplicities of all eigenvalues of $\mathfrak{A}$ in Λ. Then there hold the spectral convergence properties:

4. *For every eigenvalue κ of $\mathfrak{A}(\bullet)$ exists a sequence $\kappa_n, n \in \mathbb{N}$ converging to κ with κ_n being eigenvalues of $\mathfrak{A}_n(\bullet)$ for almost all $n \in \mathbb{N}$,*
5. *if $\kappa_n, n \in \mathbb{N}$ and $\mathfrak{u}_n, n \in \mathbb{N}$ are some sequences of eigenvalues κ_n of $\mathfrak{A}_n(\bullet)$ and normalized eigenelements $\mathfrak{u}_n$ of $\mathfrak{A}_n(\kappa_n)$ so that $\kappa_n \to \kappa \in \mathbb{K}_{\kappa_0}$, then*
   (a) κ is an eigenvalue of $\mathfrak{A}(\bullet)$,
   (b) $\mathfrak{u}_n, n \in \mathbb{N}$ is a compact sequence and its cluster points are normalized eigenelements of $\mathfrak{A}(\kappa)$,
6. *for every compact $\Lambda \subset \mathbb{K}_{\kappa_0}$ with boundary $\partial\Lambda \subset \rho(\mathfrak{A})$ exists an index $n(\Lambda)$, so that $n \geq n(\Lambda) \Rightarrow \mu(\Lambda, \mathfrak{A}_n) = \mu(\Lambda, \mathfrak{A})$.*

Denote $\eta(\kappa, \mathfrak{A})$ the order of the pole κ of the operator function $\mathfrak{A}^{-1}$ and $\mathfrak{G}(\kappa, \mathfrak{A})$ the generalized eigenspace of $\mathfrak{A}(\kappa)$. Let $\operatorname{dist}(\mathfrak{u}, \mathfrak{V}) := \inf_{\mathfrak{v}\in\mathfrak{V}} \|\mathfrak{u}-\mathfrak{v}\|_{\mathfrak{X}^1}$, $d_n := \max_{\mathfrak{u}\in\mathfrak{G}(\mathfrak{A},\kappa),\|\mathfrak{u}\|_{\mathfrak{X}^1}=1} \operatorname{dist}(\mathfrak{u}, \mathfrak{X}^1_n)$, $d^*_n := \max_{\mathfrak{u}\in\mathfrak{G}(\mathfrak{A}^*,\kappa),\|\mathfrak{u}\|_{\mathfrak{X}^1}=1} \operatorname{dist}(\mathfrak{u}, \mathfrak{X}^1_n)$, where $\mathfrak{A}^*$ denotes the adjoint operator of $\mathfrak{A}$. Let $\Lambda \subset \mathbb{K}_{\kappa_0}$ be compact with boundary $\partial\Lambda \subset \rho(\mathfrak{A})$, such that $\Lambda \cap \sigma(\mathfrak{A}) = \{\kappa_0\}$. Then the following convergence estimates hold: There exists $c > 0$ such that

7. *$|\kappa_n - \kappa_0| \leq c(d_n d^*_n)^{1/\eta(\kappa,\mathfrak{A})}$ for all $\kappa_n \in \sigma(\mathfrak{A}_n) \cap \Lambda$,*
8. *$|\overline{\kappa}_n - \kappa_0| \leq c d_n d^*_n$ where $\overline{\kappa}_n$ is the weighted mean $\overline{\kappa}_n := \sum_{\kappa\in\sigma(\mathfrak{A}_n)\cap\Lambda} \kappa \dfrac{\mu(\kappa, \mathfrak{A}_n)}{\mu(\kappa_0, \mathfrak{A})}$.*

Proof. The first three claims are standard properties of analytic Fredholm operator functions and hold due to Prop. 3.5 and $\mathbb{K}_{\kappa_0}(\epsilon) \to \mathbb{K}_{\kappa_0}$ for $\epsilon \to 0$. Prop. 3.5 and [5, (32)] enable us to apply [12, Thm. 2] for the second three claims and [13, Thm. 2,3] for the remaining ones. □

Lemma 3.9. *Let u be a solution to* (1) *or* (2)*, $\Re\kappa_0, \Re(\kappa\kappa_0^{-1}) > 0$ and U as in* (14c)*. Then $\operatorname{tr}_\Gamma u \in C^l(\Gamma)$ and $U \in H^k(S^1) \otimes C^l(\Gamma)$ for all $l, k \in \mathbb{N}$.*

Proof. A finer analysis shows that in [10, Cor. 9.2] σ can be chosen greater than $\log(1/2)$ for big enough N. Thus the condition "a sufficiently large" in [10, Thm. 9.3] is always satisfied for $a_p = 0$. Let u_∞, Ψ be the (unscaled) functions from [10]. A simple calculation shows

$$\mathcal{M}_{\kappa_0}^{-1}\mathcal{D}\mathcal{T}_- \begin{pmatrix} \operatorname{tr}_\Gamma u \\ U \end{pmatrix} (s, \hat{x}) = \frac{-e^{i\kappa|\hat{x}|}}{\kappa^2|\hat{x}|^2}\left(\frac{u_\infty(|\hat{x}|^{-1}\hat{x})}{\mathrm{i}\kappa - s} + \int_0^\infty \frac{\Psi_{\kappa|\hat{x}|}(t, |\hat{x}|^{-1}\hat{x})}{\mathrm{i}\kappa - t - s} dt\right)$$

and

$$\left(\mathcal{M}_{\kappa_0}^{-1}\mathcal{D}\mathcal{T}_+ + \mathcal{M}_{\kappa_0}^{-1}\mathcal{T}_-\right)\left(\begin{smallmatrix}\operatorname{tr}_\Gamma u\\ U\end{smallmatrix}\right)(s,\hat{x}) = \frac{-e^{i\kappa|\hat{x}|}}{\kappa^2|\hat{x}|^2}\left(\frac{\mathrm{i}u_\infty(|\hat{x}|^{-1}\hat{x})}{\mathrm{i}\kappa - s} + \int_0^\infty \frac{(\mathrm{i}\kappa - t)\Psi_{\kappa|\hat{x}|}(t,|\hat{x}|^{-1}\hat{x})}{\mathrm{i}\kappa - t - s}dt\right).$$

$\partial_a\Psi_a(t,|\hat{x}|^{-1}\hat{x}) = t\Psi_a(t,|\hat{x}|^{-1}\hat{x})$, [10, Thm. 9.3] and the arguments from the proof of [8, Lem. 4.4] show that $\mathcal{M}_{\kappa_0}$ applied to the right hand sides of the previous equations are in $H^k(S^1)\otimes C^l(\Gamma)$ for all $l,k\in\mathbb{N}$. Since $\mathcal{D}^{-1}\otimes \mathrm{Id}\colon H^k(S^1)\otimes C^l(\Gamma)\to H^k(S^1)\otimes C^l(\Gamma)$ is a bounded operator for all $l,k\in\mathbb{N}$, the claim follows. □

Theorem 3.10. *Let X_h^1 be a finite dimensional subspace of $H^1(\Omega_{\mathrm{int}})$, such that $\operatorname{tr}_\Gamma X_h^1 \subset H^1(\Gamma)$. For $N\in\mathbb{N}_0$ let further $H_N^+(S^1) := \operatorname{span}\{z^k, k = 0,\dots,N\}$ and $\mathfrak{X}_{h,N}^1 := X_h^1 \oplus (H_N^+\otimes \operatorname{tr}_\Gamma X_h^1)$. Let u be a solution to* (1) *or* (2)*, $\kappa_0\in\mathbb{C}, \Re\kappa_0>0$ such that $\kappa\in\mathbb{K}_{\kappa_0} := \{s\in\mathbb{C}\colon \Re(\kappa_0^{-1}\kappa)>0, \Im(\kappa_0^{-1}\kappa)<0\}$ and $\mathfrak{u} := u|_{\Omega_{\mathrm{int}}}\oplus U$ with U has in* (14c)*. Then there exists a constant $C>0$ independent of $\mathfrak{u},\kappa,\kappa_0,N,X_h^1$ and for every $k\in\mathbb{N}$ a constant $C_{k,U}$ which only depends on k and U, such that*

$$\begin{aligned}\inf_{\mathfrak{v}_{h,N}\in\mathfrak{X}_{h,N}^1}\|\mathfrak{u}-\mathfrak{v}_{h,N}\|_{\mathfrak{X}^1} \le{}& \inf_{v_h^{\mathrm{int}}\in X_h^1}\|u_{\mathrm{int}} - v_h^{\mathrm{int}}\|_{H^1(\Omega_{\mathrm{int}})} + C\|u_{\mathrm{int}} - v_h^{\mathrm{int}}\|_{H^1(\Gamma)}\\ &+ C\inf_{U_h\in H^+(S^1)\otimes\operatorname{tr}_\Gamma X_h^1}\|U-U_h\|_{H^1(S^1)\otimes L^2(\Gamma)} + \|U-U_h\|_{L^2(S^1)\otimes H^1(\Gamma)}\\ &+ C_{k,U}(N+1)^{-k}\end{aligned}\tag{40}$$

Proof. Follows from the definition of the norm $\|\bullet\|_{\mathfrak{X}^1}$ and Lem. 3.9. □

Thm. 3.10 is formulated for general discretizations X_h^1 of $H^1(\Omega_{\mathrm{int}})$. In practice one might chose polygonial Γ and X_h^1 as a high order finite element space. Since U is analytic with respect to $\hat{x}$, one can in particular obtain exponential convergence for the second line in (40), see [16]. The terms of the first line in (40) are standard. Their estimation might depend on the PDE for more general problems (see Sec. 4) and can be treated with [16].

3.1 Two space dimensions

The equivalent equations to (7) in two dimensions are

$$\int_{\Omega_{\text{ext}}} uv \,\mathrm{d}x = \int_{\mathbb{R}^+\times\Gamma} q(\hat{x})(r+1)u_{\text{ext}}v_{\text{ext}}\,\mathrm{d}r\mathrm{d}\hat{x}, \tag{41a}$$

$$\int_{\Omega_{\text{ext}}} \nabla u\cdot\nabla v\,\mathrm{d}x = \int_{\mathbb{R}^+\times\Gamma} \begin{pmatrix}\partial_r u_{\text{ext}}\\ \partial_{\hat{x}} u_{\text{ext}}\end{pmatrix}\cdot\begin{pmatrix} Q_{11}(\hat{x})(r+1) & Q_{12}(\hat{x})\\ Q_{21}(\hat{x}) & Q_{22}(\hat{x})(r+1)^{-1}\end{pmatrix}\begin{pmatrix}\partial_r v_{\text{ext}}\\ \partial_{\hat{x}} v_{\text{ext}}\end{pmatrix}\mathrm{d}r\mathrm{d}\hat{x}, \tag{41b}$$

with appropriate $q, (Q_{nm})_{n,m=1,2}$. Scaling $\tilde{u}_{\text{ext}}(r,\hat{x}) := (r+1)^{1/2}u_{\text{ext}}(r,\hat{x})$ and $\tilde{v}_{\text{ext}}(r,\hat{x}) := (r+1)^{1/2}v_{\text{ext}}(r,\hat{x})$ yields

$$\int_{\Omega_{\text{ext}}} uv \,\mathrm{d}x = \int_{\mathbb{R}^+\times\Gamma} q(\hat{x})(r+1)^2\tilde{u}_{\text{ext}}\tilde{v}_{\text{ext}}\,\mathrm{d}r\mathrm{d}\hat{x}, \tag{42a}$$

$$\begin{aligned}\int_{\Omega_{\text{ext}}} \nabla u\cdot\nabla v\,\mathrm{d}x = &\int_{\mathbb{R}^+\times\Gamma} \begin{pmatrix}\partial_r \tilde{u}_{\text{ext}}\\ \partial_{\hat{x}} \tilde{u}_{\text{ext}}\end{pmatrix}\cdot\begin{pmatrix} Q_{11}(\hat{x})(r+1)^2 & Q_{12}(\hat{x})(r+1)\\ Q_{21}(\hat{x})(r+1) & Q_{22}(\hat{x})\end{pmatrix}\begin{pmatrix}\partial_r \tilde{v}_{\text{ext}}\\ \partial_{\hat{x}} \tilde{v}_{\text{ext}}\end{pmatrix}\mathrm{d}r\mathrm{d}\hat{x}\\ &+\int_{\mathbb{R}^+\times\Gamma}\tilde{u}_{\text{ext}}\tilde{v}_{\text{ext}}Q_{11}(\hat{x})/4\,\mathrm{d}r\mathrm{d}\hat{x}.\end{aligned} \tag{42b}$$

Apart from the term $\int_{\mathbb{R}^+\times\Gamma}\tilde{u}_{\text{ext}}\tilde{v}_{\text{ext}}Q_{11}(\hat{x})/4\,\mathrm{d}r\mathrm{d}\hat{x}$, Equations (42b) and (42a) have the same structure as (7). Thus they lead to a bilinear form, which is aside from the space dimension $d=2$ instead of $d=3$ identical to $\mathfrak{B}$ in (22). The additional summand $\int_{\mathbb{R}^+\times\Gamma}\tilde{u}_{\text{ext}}\tilde{v}_{\text{ext}}Q_{11}(\hat{x})/4\,\mathrm{d}r\mathrm{d}\hat{x}$ converts to the bilinear form $\mathcal{A}_1\big((\mathcal{T}_-\otimes\mathrm{Id})\tilde{\mathbf{U}},(\mathcal{T}_-\otimes\mathrm{M}_{\frac{Q_{11}}{4}})\tilde{\mathbf{V}}\big)$, which is bounded by $\|Q_{11}\|_{L^\infty(\Gamma)}/4\|\tilde{\mathfrak{u}}\|_{\mathfrak{Y}}\|\tilde{\mathfrak{v}}\|_{\mathfrak{Y}}$. Hence the theory of Section 3 can be applied also in two space dimensions.

4 Conclusion

In this papers we considered Hardy space infinite element methods (HSIEM) for the Helmholtz scattering problem (1) and the Helmholtz resonance problem (2) in two and three space dimensions. Although we restricted ourselves for presentational purposes in this paper to purely homogeneous equations and solely Neumann data as right hand side, there is no difficulty at all to include non-constant coefficients for $-\Delta$ and κ^2, low order terms, source terms and other kinds of boundary conditions as long as they vanish all in Ω_{ext}. We also explained in Rem. 3.6 how to include first order terms with constant coefficients into our theory.

In Section 2 we presented an uniform framework for Hardy space variational formulations with general transparent boundary Γ, which covers both cases of spherical and piecewise polygonial Γ used in [8] and [15]. In Cor. 2.5 we showed the equivalence of the Hardy space variational formulations (27) and the strong formulations (1)/(2), which was previously only known for spherical Γ from [8]. In Rem. 2.7 we pointed out that the Hardy space variational formulation is equivalent to a PML variational formulation. Thus we can understand the HSIEM as a spectral PML infinite element method. Obviously it has the advantage that no domain truncation is necessary and the benefits of spectral methods. A main contribution of this paper are Lem. 3.1 and 3.4 for spherical and piecewise polyhedral Γ, which together result in a Gårding-type inequality. This allows us to apply standard theory [2, 14, 12, 13, 5], which yields the convergence of the HSIEM for scattering and resonance problems in Thm. 3.7 and Thm. 3.8. Further we showed in Thm. 3.10 the convergence rates for typical discretizations.

Disregarding the discretization of $H^1(\Omega_{\text{int}})$, the parameters for the HSIEM are the transparent boundary Γ, the focal point P_0, the parameter κ_0 and the degrees of freedom N for discretization of $H^+(S^1)$. The transparent boundary Γ needs to be star-shaped with respect to P_0. It is of course expedient to chose Γ close to K in order to reduce computational cost. It is also reasonable to chose these two in a way such that the minimum angle between Γ and $\hat{x} - P_0, \hat{x} \in \Gamma$ is not too small, see Rem. 3.3. Lem. 3.1 and Rem. 3.2 tell us in which sector to choose κ_0 to obtain stability. Due to $\mathcal{L}\{u(\kappa\bullet)\}(s) = \kappa^{-1}\mathcal{L}\{u\}(\kappa^{-1}s)$ and the factor κ_0 in the Möbius transformation ψ_{κ_0}, the choice $|\kappa_0| \approx |\kappa|$ is a wise in view of approximation properties. Thm. 3.10 tells us that the method converges super-algebraic with respect to N.

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