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# Spectral Theorem for definitizable normal linear operators on Krein spaces

MICHAEL KALTENBÄCK

**Abstract:**In the present note a spectral theorem for normal definitizable linear operators on Krein spaces is derived by developing a functional calculus  $\phi \mapsto \phi(N)$  which is the proper analogue of  $\phi \mapsto \int \phi dE$  in the Hilbert space situation.

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**Keywords:** Krein space, definitizable operators, normal operators, spectral theorem

## 1 Introduction

A bounded linear operator  $N$  on a Krein space  $(\mathcal{K}, [., .])$  is called normal, if  $N$  commutes with its Krein space adjoint  $N^*$ , i.e.  $NN^* = N^*N$ . This is equivalent to the fact that its real part  $A := \frac{N+N^*}{2}$  and its imaginary part  $B := \frac{N-N^*}{2i}$  commute. We call  $N$  definitizable whenever the selfadjoint operators  $A$  and  $B$  are both definitizable in classical sense, i.e. there exist so-called definitizing polynomials  $p(z)$  and  $q(z)$  such that  $[p(A)x, x] \geq 0$  and  $[q(B)x, x] \geq 0$  for all  $x \in \mathcal{K}$ ; see [L].

In the Hilbert space setting the spectral theorem for bounded linear, normal operators is a well-known functional analysis result. In fact, it is almost as as folklore as the older spectral theorem for bounded linear, selfadjoint operators.

In the Krein space world there exists no similar result for general selfadjoint operators. But assuming in addition definitizability a spectral theorem could be shown by Heinz Langer; cf. [L]. This theorem became an important starting point for various spectral results. The main difference to selfadjoint operators on Hilbert spaces is the appearance of (finitely many) critical points, where the spectral projections no longer behave like a measure.

Only a rather small number of publications dealt with the situation of a normal (definitizable) operators in a Krein space. The Pontryagin space case was studied up to a certain extent for example in [XiCh] and [LS]. Special normal operators on Krein spaces were considered for example in [AS] and [PST]. But until now no adequate version of a spectral theorem on normal definitizable operators in Krein spaces has been found.

In the present paper we present a spectral theorem for bounded linear, normal, definitizable operators formulated in terms of a functional calculus generalizing the functional calculus  $\phi \mapsto \int \phi dE$  in the Hilbert space case. In order to achieve this goal, we use the methods developed in [KP] for definitizable selfadjoint operators and extend them for two commuting definitizable selfadjoint operators.

Let us anticipate a little more explicitly what happens in this note. Denoting by  $p(z)$  and  $q(z)$  the definitizing real polynomials for  $A$  and  $B$ , respectively, we build a Hilbert space  $\mathcal{V}$  which is continuously and densely embedded in the given Krein space  $\mathcal{K}$  such that  $TT^* = p(A) + q(B)$ , where  $T : \mathcal{V} \rightarrow \mathcal{K}$  denotes that adjoint of the embedding mapping. Then we use the \*-homomorphism

$\Theta : (TT^*)' (\subseteq B(\mathcal{K})) \rightarrow (T^*T)' (\subseteq B(\mathcal{V}))$ ,  $C \mapsto (T \times T)^{-1}(C)$ , studied in [KP], in order to drag our normal operator  $N \in (TT^*)' \subseteq B(\mathcal{K})$  into  $(T^*T)' (\subseteq B(\mathcal{V}))$ . The resulting normal operator  $\Theta(N)$  acts in a Hilbert space, and therefore has a spectral measure  $E(\Delta)$ , where  $\Delta$  are Borel subsets of  $\mathbb{C}$ .

The proper family  $\mathcal{F}_N$  of functions suitable for the aimed functional calculus are bounded and measurable functions on

$$(\sigma(\Theta(N)) \cup (Z_p^{\mathbb{R}} + iZ_q^{\mathbb{R}})) \dot{\cup} Z^i (\subseteq \mathbb{C} \dot{\cup} \mathbb{C}^2).$$

Here  $Z_p^{\mathbb{R}} = p^{-1}\{0\} \cap \mathbb{R}$  and  $Z_q^{\mathbb{R}} = q^{-1}\{0\} \cap \mathbb{R}$  denote the real zeros of  $p(z)$  and  $q(z)$ , respectively, and  $Z^i = (p^{-1}\{0\} \times q^{-1}\{0\}) \setminus (\mathbb{R} \times \mathbb{R})$ . Moreover, the functions  $\phi \in \mathcal{F}_N$  assume values in  $\mathbb{C}$  on  $\sigma(\Theta(N)) \setminus (Z_p^{\mathbb{R}} + iZ_q^{\mathbb{R}})$ , values in  $\mathbb{C}^{\mathfrak{d}_p(\operatorname{Re} z) \cdot \mathfrak{d}_q(\operatorname{Im} z) + 2}$  at  $z \in Z_p^{\mathbb{R}} + iZ_q^{\mathbb{R}}$  and values in  $\mathbb{C}^{\mathfrak{d}_p(\xi) \cdot \mathfrak{d}_q(\eta)}$  at  $z = (\xi, \eta) \in Z^i$ . Here  $\mathfrak{d}_p(w)$  ( $\mathfrak{d}_q(w)$ ) denotes  $p$ 's ( $q$ 's) degree of zero at  $w$ . Finally,  $\phi \in \mathcal{F}_N$  satisfies a growth regularity condition at all points from  $Z_p^{\mathbb{R}} + iZ_q^{\mathbb{R}}$  which are not isolated in  $\sigma(\Theta(N)) \cup (Z_p^{\mathbb{R}} + iZ_q^{\mathbb{R}})$ .

Any polynomial  $s(z, w) \in \mathbb{C}[z, w]$  can be seen as a function  $s_N \in \mathcal{F}_N$ . The nice thing about these, somewhat tediously defined functions  $\phi \in \mathcal{F}_N$  is that

$$\phi(z) = s_N(z) + (p_N + q_N)(z) \cdot g(z), \quad z \in \sigma(\Theta(N)), \quad (1.1)$$

where  $s \in \mathbb{C}[z, w]$  is a suitable polynomial in two variables and  $g : \sigma(\Theta(N)) \setminus (Z_p^{\mathbb{R}} + iZ_q^{\mathbb{R}}) \rightarrow \mathbb{C}$  is bounded and measurable and  $g : \sigma(\Theta(N)) \cap (Z_p^{\mathbb{R}} + iZ_q^{\mathbb{R}}) \rightarrow \mathbb{C}^2$ .

We then define  $\phi(N) := s(A, B) + T \int_{\sigma(\Theta(N))}^{R_1, R_2} g dET^*$ , show that this operator does not depend on the actual decomposition (1.1) and that  $\phi \mapsto \phi(N)$  is indeed a \*-homomorphism. Here  $\int_{\sigma(\Theta(N))}^{R_1, R_2} g dE$  is the integral of  $g$  with respect to the spectral measure  $E$  taking into account the fact that  $g$  has values in  $\mathbb{C}^2$  on  $\sigma(\Theta(N)) \cap (Z_p^{\mathbb{R}} + iZ_q^{\mathbb{R}})$ .

If  $\phi$  stems from a characteristic function corresponding to a Borel subset  $\Delta$  of  $\mathbb{C}$  such that no point of  $Z_p^{\mathbb{R}} + iZ_q^{\mathbb{R}}$  belongs to the boundary of  $\Delta$ , then  $\phi(N)$  is a selfadjoint projection on  $\mathcal{K}$ . In fact, it can be seen as the corresponding special projection for  $N$ .

## 2 Multiple embeddings

For the present section we fix a Krein space  $(\mathcal{K}, [., .])$  and Hilbert spaces  $(\mathcal{V}, [., .])$ ,  $(\mathcal{V}_1, [., .])$  and  $(\mathcal{V}_2, [., .])$ . Moreover, let  $T_1 : \mathcal{V}_1 \rightarrow \mathcal{K}$ ,  $T_2 : \mathcal{V}_2 \rightarrow \mathcal{K}$  and  $T : \mathcal{V} \rightarrow \mathcal{K}$  be bounded linear, injective mappings such that

$$TT^* = T_1T_1^* + T_2T_2^*$$

holds true. Since for  $x \in \mathcal{K}$  we have

$$\begin{aligned} [T^*x, T^*x]_{\mathcal{V}} &= [TT^*x, x] = \\ &= [T_1T_1^*x, x] + [T_2T_2^*x, x] = [T_1^*x, T_1^*x]_{\mathcal{V}_1} + [T_2^*x, T_2^*x]_{\mathcal{V}_2}, \end{aligned}$$

one easily concludes that  $T^*x \mapsto T_j^*x$  constitutes a well-defined, contractive linear mapping from  $\operatorname{ran} T^*$  onto  $\operatorname{ran} T_j^*$  for  $j = 1, 2$ . By  $(\operatorname{ran} T^*)^{\perp} = \ker T = \{0\}$  and  $(\operatorname{ran} T_j^*)^{\perp} = \ker T_j = \{0\}$  these ranges are dense in the Hilbert spaces

$\mathcal{V}$  and  $\mathcal{V}_j$ . Hence, there is a unique bounded linear continuation of  $T^*x \mapsto T_j^*x$  to  $\mathcal{V}$ , which has dense range in  $\mathcal{V}_j$ .

Denoting by  $R_j$  for  $j = 1, 2$  the adjoint mapping of this continuation we clearly have  $T_j = TR_j$  and  $\ker R_j = (\text{ran } R_j^*)^\perp = \{0\}$ . From  $TT^* = T_1T_1^* + T_2T_2^*$  we conclude

$$T(I)T^* = TT^* = TR_1R_1^*T^* + TR_2R_2^*T^* = T(R_1R_1^* + R_2R_2^*)T^*.$$

$\ker T = \{0\}$  and the density of  $\text{ran } T^*$  yields  $R_1R_1^* + R_2R_2^* = I$ .

If  $T_1T_1^*$  and  $T_2T_2^*$  commute, then by  $TT^* = T_1T_1^* + T_2T_2^*$  also  $T_jT_j^*$  and  $TT^*$  commute. Moreover, in this case

$$T(T^*TR_jR_j^*)T^* = TT^*T_jT_j^* = T_jT_j^*TT^* = T(R_jR_j^*T^*T)T^*.$$

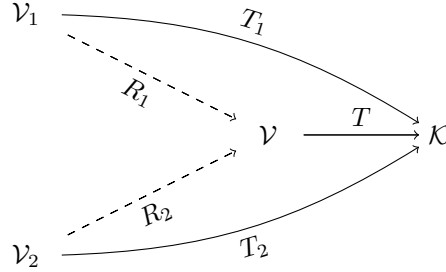
Employing again  $T$ 's injectivity and the density of  $\text{ran } T^*$  we see that  $R_jR_j^*$  and  $T^*T$  commute for  $j = 1, 2$ . From this we get

$$T_jT_jR_j^*R_j = R_j^*(T^*TR_jR_j^*)R_j = R_j^*(R_jR_j^*T^*T)R_j = R_j^*R_jT_j^*T_j.$$

Thus, we showed

**2.1 Lemma.** *With the above notations and assumptions there exist injective contractions  $R_1 : \mathcal{V}_1 \rightarrow \mathcal{V}$  and  $R_2 : \mathcal{V}_2 \rightarrow \mathcal{V}$  such that  $T_1 = TR_1$ ,  $T_2 = TR_2$  and  $R_1R_1^* + R_2R_2^* = I$ .*

*If  $T_1T_1^*$  and  $T_2T_2^*$  commute, then the operators  $R_jR_j^*$  and  $T^*T$  on  $\mathcal{V}$  commute as well as the operators  $R_j^*R_j$  and  $T_j^*T_j$  on  $\mathcal{V}_j$  for  $j = 1, 2$ .*



By  $\Theta_j : (T_jT_j^*)' (\subseteq B(\mathcal{K})) \rightarrow (T_j^*T_j)'$  ( $\subseteq B(\mathcal{V}_j)$ ),  $j = 1, 2$ , and by  $\Theta : (TT^*)' (\subseteq B(\mathcal{K})) \rightarrow (T^*T)'$  ( $\subseteq B(\mathcal{V})$ ) we shall denote the  $*$ -algebra homomorphisms mapping the identity operator to the identity operator as in Theorem 5.7 from [KP] corresponding to the mappings  $T_j$ ,  $j = 1, 2$ , and  $T$ :

$$\begin{aligned} \Theta_j(C_j) &= (T_j \times T_j)^{-1}(C_j) = T_j^{-1}C_jT_j, \quad C_j \in (T_jT_j^*)', \\ \Theta(C) &= (T \times T)^{-1}(C) = T^{-1}CT, \quad C \in (TT^*)'. \end{aligned} \quad (2.1)$$

We can apply Theorem 5.7 in [KP] also to the bounded linear, injective  $R_j : \mathcal{V}_j \rightarrow \mathcal{V}$ ,  $j = 1, 2$ , and denote the corresponding  $*$ -algebra homomorphisms by  $\Gamma_j : (R_jR_j^*)' (\subseteq B(\mathcal{V})) \rightarrow (R_j^*R_j)'$  ( $\subseteq B(\mathcal{V}_j)$ ):

$$\Gamma_j(D) = (R_j \times R_j)^{-1}(D) = R_j^{-1}DR_j, \quad D \in (R_jR_j^*)'.$$

**2.2 Proposition.** *With the above notations and assumptions we have  $(T_1T_1^*)' \cap (T_2T_2^*)' \subseteq (TT^*)'$  and  $\Theta((T_1T_1^*)' \cap (T_2T_2^*)') \subseteq (R_1R_1^*)' \cap (R_2R_2^*)' \cap (T^*T)'$ , where in fact  $(j = 1, 2)$*

$$\Theta(C)R_jR_j^* = R_j\Theta_j(C)R_j^* = R_jR_j^*\Theta(C), \quad C \in (T_1T_1^*)' \cap (T_2T_2^*)'. \quad (2.2)$$

Moreover,

$$\Theta_j(C) = \Gamma_j \circ \Theta(C), \quad C \in (T_1T_1^*)' \cap (T_2T_2^*)'. \quad (2.3)$$

*Proof.*  $(T_1T_1^*)' \cap (T_2T_2^*)' \subseteq (TT^*)'$  is clear from  $TT^* = T_1T_1^* + T_2T_2^*$ . According to Theorem 5.7 in [KP] we have  $\Theta_j(C)T_j^* = T_j^*C$  and  $T^*C = \Theta(C)T^*$  for  $C \in (T_1T_1^*)' \cap (T_2T_2^*)'$ . Therefore,

$$\begin{aligned} T(R_j\Theta_j(C)R_j^*)T^* &= T_j\Theta_j(C)T_j^* = T_jT_j^*C = \\ &= TR_jR_j^*T^*C = T(R_jR_j^*\Theta(C))T^*. \end{aligned}$$

$\ker T = \{0\}$  and the density of  $\text{ran } T^*$  yield  $R_j\Theta_j(C)R_j^* = R_jR_j^*\Theta(C)$  for  $j = 1, 2$ . Applying this equation to  $C^*$  and taking adjoints yields  $R_j\Theta_j(C)R_j^* = \Theta(C)R_jR_j^*$ . In particular,  $\Theta(C) \in (R_jR_j^*)'$ . Therefore, we can apply  $\Gamma_j$  to  $\Theta(C)$  and get

$$\Gamma_j \circ \Theta(C) = R_j^{-1}T^{-1}CTR_j = T_j^{-1}CT_j = \Theta_j(C).$$

□

For the following assertion note that by (2.3) and by the fact that  $\Gamma_j$  is a  $*$ -algebra homomorphism mapping the identity operator to the identity operator, we have  $(j = 1, 2)$

$$\sigma(\Theta(C)) \subseteq \sigma(\Theta_j(C)), \quad C \in (T_1T_1^*)' \cap (T_2T_2^*)'. \quad (2.4)$$

**2.3 Corollary.** *With the above notations and assumptions let  $N \in (T_1T_1^*)' \cap (T_2T_2^*)'$  be normal. Then  $\Theta(N), \Theta_1(N), \Theta_2(N)$  are all normal operators in the Hilbert spaces  $\mathcal{V}, \mathcal{V}_1, \mathcal{V}_2$ , respectively. If  $E(E_1, E_2)$  denotes the spectral measure for  $\Theta(N)$  ( $\Theta_1(N), \Theta_2(N)$ ), then  $E(\Delta) \in (R_1R_1^*)' \cap (R_2R_2^*)' \cap (T^*T)'$  and*

$$\Gamma_j(E(\Delta)) = E_j(\Delta), \quad j = 1, 2,$$

for all Borel subsets  $\Delta$  of  $\mathbb{C}$ , where  $E_j(\Delta) \in (R_j^*R_j) \cap (T_j^*T_j)'$ .

Moreover,  $\int h dE \in (R_1R_1^*)' \cap (R_2R_2^*)' \cap (T^*T)'$  and

$$\Gamma_j \left( \int h dE \right) = \int h dE_j$$

for any bounded and measurable  $h : \sigma(\Theta(N)) \rightarrow \mathbb{C}$ , where  $\int h dE_j$  belongs to  $(R_j^*R_j) \cap (T_j^*T_j)'$ .

*Proof.* The normality of  $\Theta(N), \Theta_1(N)$  and  $\Theta_2(N)$  is clear, since  $\Theta, \Theta_1, \Theta_2$  are  $*$ -homomorphisms. From Proposition 2.2 we know that  $\Theta(N) \in (R_1R_1^*)' \cap (R_2R_2^*)' \cap (T^*T)'$ . According to the well known properties of  $\Theta(N)$ 's spectral measure we obtain  $E(\Delta) \in (R_1R_1^*)' \cap (R_2R_2^*)' \cap (T^*T)'$  and, in turn,  $\int h dE \in (R_1R_1^*)' \cap (R_2R_2^*)' \cap (T^*T)'$ . In particular,  $\Gamma_j$  can be applied to  $E(\Delta)$  and  $\int h dE$ .

Similarly,  $\Theta_j(N) \in (T_j^* T_j)'$  implies  $E_j(\Delta), \int h dE_j \in (T_j^* T_j)'$  for a bounded and measurable  $h$ .

Recall from Theorem 5.7 in [KP] that  $\Gamma_j(D)R_j^*x = R_j^*D$  for  $D \in (T^*T)'$ . For  $x \in \mathcal{V}$  and  $y \in \mathcal{V}_j$  we therefore get

$$[\Gamma_j(E(\Delta))R_j^*x, y] = [R_j^*E(\Delta)x, y] = [E(\Delta)x, R_jy]$$

and, in turn,

$$\begin{aligned} \int_{\mathbb{C}} s(z, \bar{z}) d[\Gamma_j(E)R_j^*x, y] &= \int_{\mathbb{C}} s(z, \bar{z}) d[Ex, R_jy] = [s(\Theta(N), \Theta(N)^*)x, R_jy] = \\ &[R_j^*s(\Theta(N), \Theta(N)^*)x, y] = [\Gamma_j(s(\Theta(N), \Theta(N)^*))R_j^*x, y] \end{aligned}$$

for any trigonometric polynomial  $s(z, \bar{z}) \in \mathbb{C}[z, \bar{z}]$ . By (2.3) and the fact that  $\Gamma_j$  is a  $*$ -homomorphism we have  $\Gamma_j(s(\Theta(N), \Theta(N)^*)) = s(\Theta_j(N), \Theta_j(N)^*)$ . Consequently,

$$\int_{\mathbb{C}} s(z, \bar{z}) d[\Gamma_j(E)R_j^*x, y] = \int_{\mathbb{C}} s(z, \bar{z}) d[E_jR_j^*x, y].$$

Since  $E(\mathbb{C} \setminus K) = 0$  and  $E_j(\mathbb{C} \setminus K) = 0$  for a certain compact  $K \subseteq \mathbb{C}$  and since  $\mathbb{C}[z, \bar{z}]$  is densely contained in  $C(K)$ , we obtain from the uniqueness assertion of the Riesz Representation Theorem

$$[\Gamma_j(E(\Delta))R_j^*x, y] = [E_j(\Delta)R_j^*x, y], \quad x \in \mathcal{V}, y \in \mathcal{V}_j,$$

for all Borel subsets  $\Delta$  of  $\mathbb{C}$ . Due to the density of  $\text{ran } R_j^*$  in  $\mathcal{V}_j$  we even have  $[\Gamma_j(E(\Delta))z, y] = [E_j(\Delta)z, y]$ ,  $y, z \in \mathcal{V}_j$  and, in turn,  $\Gamma_j(E(\Delta)) = E_j(\Delta)$ . Since  $\Gamma_j$  maps into  $(R_j^*R_j)'$  we have  $E_j(\Delta) \in (R_j^*R_j)'$  and, in turn,  $\int h dE_j \in (R_j^*R_j)'$  for any bounded and measurable  $h$ .

If  $h : \sigma(\Theta(N)) \rightarrow \mathbb{C}$  is bounded and measurable, then, clearly, also its restriction to  $\sigma(\Theta_j(N)) = \sigma(\Gamma_j \circ \Theta(N))$  is bounded and measurable; see (2.4). Due to  $E_j(\Delta)R_j^* = \Gamma_j(E(\Delta))R_j^* = R_j^*E(\Delta)$  for  $x \in \mathcal{V}$  and  $y \in \mathcal{V}_j$  we have

$$\begin{aligned} [\Gamma_j \left( \int h dE \right) R_j^*x, y] &= [R_j^* \left( \int h dE \right) x, y] = \left[ \left( \int h dE \right) x, R_jy \right] = \\ &\int h d[Ex, R_jy] = \int h d[E_jR_j^*x, y] = \left[ \left( \int h dE_j \right) R_j^*x, y \right]. \end{aligned}$$

Again the density of  $\text{ran } R_j^*$  yields  $\Gamma_j \left( \int h dE \right) = \int h dE_j$ .  $\square$

Recall from Lemma 5.10 in [KP] the mappings ( $j = 1, 2$ )

$$\Xi_j : (T_j^* T_j)' (\subseteq B(\mathcal{V}_j)) \rightarrow (T_j T_j^*)' (\subseteq B(\mathcal{K})), \quad \Xi_j(D_j) = T_j D_j T_j^*, \quad (2.5)$$

and  $\Xi : (T^* T)' (\subseteq B(\mathcal{V})) \rightarrow (T T^*)' (\subseteq B(\mathcal{K})), \quad \Xi(D) = T D T^*$ . By ( $j = 1, 2$ )

$$\Lambda_j : (R_j^* R_j)' (\subseteq B(\mathcal{V}_j)) \rightarrow (R_j R_j^*)' (\subseteq B(\mathcal{V})), \quad \Lambda_j(D_j) = R_j D_j R_j^*,$$

we shall denote the corresponding mappings outgoing from the mappings  $R_j : \mathcal{V}_j \rightarrow \mathcal{V}$ . By Lemma 2.1 we have

$$\Xi_j(D_j) = T_j R_j D_j R_j^* T_j^* = \Xi \circ \Lambda_j(D_j) \quad \text{for } D_j \in (R_j^* R_j)' \cap (T_j^* T_j)'.$$

According to Lemma 5.10 in [KP],  $\Lambda_j \circ \Gamma_j(D) = DR_jR_j^*$ . Hence, using the notation from Corollary 2.3

$$\Xi_j\left(\int h dE_j\right) = \Xi \circ \Lambda_j \circ \Gamma_j\left(\int h dE\right) = \Xi(R_jR_j^* \int h dE). \quad (2.6)$$

Finally,  $T^{-1}T_jT_j^*T = T^{-1}TR_jR_j^*T^*T = R_jR_j^*T^*T$ . In case that  $T_1T_1^*$  and  $T_2T_2^*$  commute we have  $T_1T_1^*, T_2T_2^* \in (TT^*)'$  and the later equality can be expressed as ( $j = 1, 2$ )

$$\Theta(T_jT_j^*) = R_jR_j^*T^*T. \quad (2.7)$$

### 3 Normal definitizable operators

**3.1 Definition.** We will call a bounded linear and normal operator  $N$  on a Krein Space *definitizable* if its real part  $A := \frac{N+N^*}{2}$  and its imaginary part  $B := \frac{N-N^*}{2i}$  are both definitizable, i.e. there exist real polynomials  $p, q \in \mathbb{R}[z]$  such that  $p$  is definitizing for  $A$  ( $[p(A)x, x] \geq 0, x \in \mathcal{K}$ ) and such that  $q$  is definitizing for  $B$  ( $[q(B)x, x] \geq 0, x \in \mathcal{K}$ ); see [L].  $\diamond$

By Corollary 7.15 in [KP] the definitizability of  $A$  and  $B$  is equivalent to the concept of definitizability in [KP].

Also note that in Pontryagin spaces any bounded linear and normal operator is definitizable in the above sense; see Example 6.2 in [KP].

**3.2 Proposition.** *Let  $A$  and  $B$  be commuting, bounded linear, selfadjoint and definitizable operators on a Krein space  $(\mathcal{K}, [.,.])$  with definitizing polynomials  $p \in \mathbb{R}[z]$  for  $A$  and  $q \in \mathbb{R}[z]$  for  $B$ . Then there exist Hilbert spaces  $(\mathcal{V}_1, [.,.])$ ,  $(\mathcal{V}_2, [.,.])$ ,  $(\mathcal{V}, [.,.])$  and bounded linear and injective operators  $T_1 : \mathcal{V}_1 \rightarrow \mathcal{K}$ ,  $T_2 : \mathcal{V}_2 \rightarrow \mathcal{K}$ ,  $T : \mathcal{V} \rightarrow \mathcal{K}$  such that*

$$T_1T_1^* = p(A), \quad T_2T_2^* = q(B), \quad TT^* = p(A) + q(B) = T_1T_1^* + T_2T_2^*$$

with commuting  $T_1T_1^*$  and  $T_2T_2^*$ . Moreover, if  $\Theta : (TT^*)' (\subseteq B(\mathcal{K})) \rightarrow (T^*T)' (\subseteq B(\mathcal{V}))$  is as in (2.1) and  $R_j : \mathcal{V}_j \rightarrow \mathcal{V}$  ( $j = 1, 2$ ) are as in Lemma 2.1, then

$$\begin{aligned} p(\Theta(A)) &= R_1R_1^*(p(\Theta(A)) + q(\Theta(B))), \\ q(\Theta(B)) &= R_2R_2^*(p(\Theta(A)) + q(\Theta(B))), \end{aligned} \quad (3.1)$$

where  $R_1R_1^*$  and  $R_2R_2^*$  commute with  $p(\Theta(A)) + q(\Theta(B))$ .

*Proof.* Let  $(\mathcal{V}_1, [.,.])$  be the Hilbert space completion of  $\mathcal{K}/\ker p(A)$  with respect to  $[p(A).,.]$  and let  $T_1 : \mathcal{V}_1 \rightarrow \mathcal{K}$  be the adjoint of the embedding of  $\mathcal{K}$  into  $\mathcal{V}_1$ . Since  $T_1^*$  has dense range,  $T_1$  is injective. Analogously let  $(\mathcal{V}_2, [.,.])$  be the Hilbert space completion of  $\mathcal{K}/\ker q(B)$  with respect to  $[q(B).,.]$  and denote by  $T_2 : \mathcal{V}_2 \rightarrow \mathcal{K}$  the injective adjoint of the embedding of  $\mathcal{K}$  into  $\mathcal{V}_2$ . Finally, let  $(\mathcal{V}, [.,.])$  be the Hilbert space completion of  $\mathcal{K}/(\ker p(A) + q(B))$  with respect to  $[(p(A) + q(B)).,.]$  and let  $T : \mathcal{V} \rightarrow \mathcal{K}$  be the injective adjoint of the embedding of  $\mathcal{K}$  into  $\mathcal{V}$ .

From  $[T^*x, y] = [T^*x, T^*y]_{\mathcal{V}} = [x, y]_{\mathcal{V}} = [(p(A) + q(B))x, y]$ ,  $[T_1T_1^*x, y] = [T_1^*x, T_1^*y]_{\mathcal{V}_1} = [x, y]_{\mathcal{V}_1} = [p(A)x, y]$  and  $[T_2T_2^*x, y] = [q(B)x, y]$  for all  $x, y \in \mathcal{K}$  we conclude that

$$T_1T_1^* = p(A), \quad T_2T_2^* = q(B), \quad TT^* = p(A) + q(B),$$



where  $p(A) = T_1 T_1^*$  and  $q(B) = T_2 T_2^*$  commute, because  $A$  and  $B$  do.

From (2.7) and Theorem 5.7 in [KP] we get

$$\begin{aligned} p(\Theta(A)) &= \Theta(p(A)) = \Theta(T_1 T_1^*) = R_1 R_1^* T^* T = R_1 R_1^* \Theta(T T^*) = \\ &R_1 R_1^* \Theta(p(A) + q(B)) = R_1 R_1^* (p(\Theta(A)) + q(\Theta(B))). \end{aligned}$$

Similarly,  $q(\Theta(B)) = R_2 R_2^* (p(\Theta(A)) + q(\Theta(B)))$ . Finally,  $R_1 R_1^*$  and  $R_2 R_2^*$  commute with  $T^* T = p(\Theta(A)) + q(\Theta(B))$  by Lemma 2.1.  $\square$

The fact that a normal operator is definitizable implies certain spectral properties of  $\Theta(N)$ .

**3.3 Lemma.** *With the notion of Proposition 3.2 applied to the real part  $A := \frac{N+N^*}{2}$  and the imaginary part  $B := \frac{N-N^*}{2i}$  of a bounded linear, normal and definitizable operator  $N$  we have*

$$\{z \in \mathbb{C} : |p(\operatorname{Re} z)| > \|R_1 R_1^*\| \cdot |p(\operatorname{Re} z) + q(\operatorname{Im} z)|\} \subseteq \rho(\Theta(N)),$$

and

$$\{z \in \mathbb{C} : |q(\operatorname{Im} z)| > \|R_2 R_2^*\| \cdot |p(\operatorname{Re} z) + q(\operatorname{Im} z)|\} \subseteq \rho(\Theta(N)).$$

In particular, the zeros of  $p(\operatorname{Re} z) + q(\operatorname{Im} z)$  are contained in  $\rho(\Theta(N)) \cup \{z \in \mathbb{C} : p(\operatorname{Re} z) = 0 = q(\operatorname{Im} z)\}$ .

*Proof.* We are going to show the first inclusion. The second one is shown in the same manner. For this let  $n \in \mathbb{N}$  and set

$$\Delta_n := \{z \in \mathbb{C} : |p(\operatorname{Re} z)|^2 > \frac{1}{n} + \|R_1 R_1^*\|^2 \cdot |p(\operatorname{Re} z) + q(\operatorname{Im} z)|^2\}.$$

For  $x \in E(\Delta_n)(\mathcal{V})$  we then have

$$\begin{aligned} \|p(\Theta(A))x\|^2 &= \int_{\Delta_n} |p(\operatorname{Re} \zeta)|^2 d[E(\zeta)x, x] \geq \\ &\int_{\Delta_n} \frac{1}{n} d[E(\zeta)x, x] + \|R_1 R_1^*\|^2 \int_{\Delta_n} |p(\operatorname{Re} \zeta) + q(\operatorname{Im} \zeta)|^2 d[E(\zeta)x, x] \\ &\geq \frac{1}{n} \|x\|^2 + \|R_1 R_1^*\| (p(\Theta(A)) + q(\Theta(B)))x\|^2. \end{aligned}$$

By (3.1) this inequality can only hold for  $x = 0$ . Since  $\Delta_n$  is open, by the Spectral Theorem for normal operators on Hilbert spaces we have  $\Delta_n \subseteq \rho(N)$ . The asserted inclusion now follows from

$$\{z \in \mathbb{C} : |p(\operatorname{Re} z)| > \|R_1 R_1^*\| \cdot |p(\operatorname{Re} z) + q(\operatorname{Im} z)|\} = \bigcup_{n \in \mathbb{N}} \Delta_n.$$

$\square$

**3.4 Corollary.** *With the notation and assumptions from Lemma 3.3 we have*

$$\begin{aligned} R_1 R_1^* E\{z \in \mathbb{C} : p(\operatorname{Re} z) \neq 0 \text{ or } q(\operatorname{Im} z) \neq 0\} = \\ \int_{\{z \in \mathbb{C} : p(\operatorname{Re} z) \neq 0 \text{ or } q(\operatorname{Im} z) \neq 0\}} \frac{p(\operatorname{Re} z)}{p(\operatorname{Re} z) + q(\operatorname{Im} z)} dE(z) \end{aligned}$$

and

$$R_2 R_2^* E \{z \in \mathbb{C} : p(\operatorname{Re} z) \neq 0 \text{ or } q(\operatorname{Im} z) \neq 0\} = \int_{\{z \in \mathbb{C} : p(\operatorname{Re} z) \neq 0 \text{ or } q(\operatorname{Im} z) \neq 0\}} \frac{q(\operatorname{Re} z)}{p(\operatorname{Re} z) + q(\operatorname{Im} z)} dE(z)$$

*Proof.* First note that the integrals on the right hand sides exist as bounded operators, because by Lemma 3.3 we have  $|p(\operatorname{Re} z)| \leq \|R_1 R_1^*\| \cdot |p(\operatorname{Re} z) + q(\operatorname{Im} z)|$  and  $|q(\operatorname{Im} z)| \leq \|R_2 R_2^*\| \cdot |p(\operatorname{Re} z) + q(\operatorname{Im} z)|$  on  $\sigma(\Theta(N))$ .

Clearly, both sides vanish on the range of  $E\{z \in \mathbb{C} : p(\operatorname{Re} z) = 0 = q(\operatorname{Im} z)\}$ . Its orthogonal complement

$$\mathcal{H} := \operatorname{ran} E\{z \in \mathbb{C} : p(\operatorname{Re} z) = 0 = q(\operatorname{Im} z)\}^\perp = \operatorname{ran} E\{z \in \mathbb{C} : p(\operatorname{Re} z) \neq 0 \text{ or } q(\operatorname{Im} z) \neq 0\},$$

is invariant under  $\int (p(\operatorname{Re} z) + q(\operatorname{Im} z)) dE(z) = (p(\Theta(A)) + q(\Theta(B)))$ . By Lemma 3.3 the restriction of this operator to  $\mathcal{H}$  is injective, and hence, has dense range in  $\mathcal{H}$ . If  $x$  belongs to this dense range, i.e.  $x = (p(\Theta(A)) + q(\Theta(B)))y$  with  $y \in \mathcal{H}$ , then

$$\begin{aligned} \int_{\{z \in \mathbb{C} : p(\operatorname{Re} z) \neq 0 \text{ or } q(\operatorname{Im} z) \neq 0\}} \frac{p(\operatorname{Re} z)}{p(\operatorname{Re} z) + q(\operatorname{Im} z)} dE(z)x &= \\ \int_{\{z \in \mathbb{C} : p(\operatorname{Re} z) \neq 0 \text{ or } q(\operatorname{Im} z) \neq 0\}} p(\operatorname{Re} z) dE(z)y &= p(\Theta(A))y = \\ R_1 R_1^* (p(\Theta(A)) + q(\Theta(B)))y &= R_1 R_1^* x. \end{aligned}$$

By a density argument the first asserted equality of the present corollary holds true on  $\mathcal{H}$  and in turn on  $\mathcal{V}$ . The second equality is shown in the same manner.  $\square$

## 4 The proper function class

In order to introduce a functional calculus we have to introduce an algebra structure on  $\mathcal{A}_{m,n} := (\mathbb{C}^m \otimes \mathbb{C}^n) \times \mathbb{C}^2 \simeq \mathbb{C}^{m \cdot n + 2}$  and on  $\mathcal{B}_{m,n} := \mathbb{C}^m \otimes \mathbb{C}^n \simeq \mathbb{C}^{m \cdot n}$  for  $m, n \in \mathbb{N}$ . For notational convenience we also set  $\mathcal{A}_{0,0} := \mathbb{C}$ .

**4.1 Definition.** Firstly, let  $\mathcal{A}_{0,0} = \mathbb{C}$  be provided with the usual addition, scalar multiplication, multiplication and conjugation.

Secondly, in case that  $m, n \in \mathbb{N}$  we provide  $\mathcal{A}_{m,n}$  with the component-wise addition and scalar multiplication. Moreover, for  $a = (a_{k,l})_{(k,l) \in I_{m,n}}$ ,  $b = (b_{k,l})_{(k,l) \in I_{m,n}}$  with  $I_{m,n} := (\{0, \dots, m-1\} \times \{0, \dots, n-1\}) \cup \{(m, 0), (0, n)\}$  we set

$$a \cdot b := \left( \sum_{c=0}^k \sum_{d=0}^l a_{c,d} b_{k-c, l-d} \right)_{(k,l) \in I_{m,n}} \quad \text{and} \quad \bar{a} := (\bar{a}_{k,l})_{(k,l) \in I_{m,n}}.$$

On  $\mathcal{B}_{m,n}$  we define addition, scalar multiplication, multiplication and conjugation in the same way only neglecting the the entries with indices  $(m, 0)$  and  $(0, n)$ .

Finally, for  $m, n \in \mathbb{N}$  we introduce the projection  $\pi : \mathcal{A}_{m,n} \rightarrow \mathcal{B}_{m,n}$ ,  $(a_{k,l})_{(k,l) \in I_{m,n}} \mapsto (a_{k,l})_{\substack{0 \leq k \leq m-1 \\ 0 \leq l \leq n-1}}$ . On  $\mathcal{B}_{m,n}$  we assume  $\pi$  to be the identity.  $\diamond$

*4.2 Remark.* It is easy to check that  $\mathcal{A}_{m,n}$  and  $\mathcal{B}_{m,n}$  are commutative, unital  $*$ -algebras. Setting  $e_{0,0} = 1$  and  $e_{k,l} = 0$ ,  $(k,l) \neq (0,0)$ , it is easy to verify that  $(e_{k,l})_{(k,l) \in I_{m,n}}$  is the multiplicative unite in  $\mathcal{A}_{m,n}$  and  $(e_{k,l})_{\substack{0 \leq k \leq m-1 \\ 0 \leq l \leq n-1}}$  is the multiplicative unite in  $\mathcal{B}_{m,n}$ . We shall denote these unites by  $e$ .

Moreover, it is straight forward to check that an element  $(a_{k,l})$  of  $\mathcal{A}_{m,n}$  (of  $\mathcal{B}_{m,n}$ ) has a multiplicative inverse in  $\mathcal{A}_{m,n}$  (in  $\mathcal{B}_{m,n}$ ) if and only if  $a_{0,0} \neq 0$ .  $\diamond$

For the rest of the paper assume that  $N$  bounded linear, normal and definitizable operator in a Krein space  $\mathcal{K}$  with real part  $A$  and imaginary part  $B$ . Moreover, we fix definitizing polynomials  $p \in \mathbb{R}[z]$  for  $A$  and  $q \in \mathbb{R}[z]$  for  $B$ .

**4.3 Definition.** We define functions  $\mathfrak{d}_p, \mathfrak{d}_q : \mathbb{C} \rightarrow \mathbb{N} \cup \{0\}$  such that  $\mathfrak{d}_p(z)$  is  $p$ 's degrees of the zero at  $z$  and  $\mathfrak{d}_q(z)$  is  $q$ 's degrees of the zero at  $z$ . Moreover, we shall denote the set of their real zeros by  $Z_p^{\mathbb{R}}$  and  $Z_q^{\mathbb{R}}$ , i.e.

$$Z_p^{\mathbb{R}} := p^{-1}\{0\} \cap \mathbb{R}, Z_q^{\mathbb{R}} := q^{-1}\{0\} \cap \mathbb{R},$$

and we set  $Z^i := (p^{-1}\{0\} \times q^{-1}\{0\}) \setminus (\mathbb{R} \times \mathbb{R})$ .

Now we are going to introduce class of functions:

(i) By  $\mathcal{M}_N$  we denote the set of functions  $\phi$  defined on

$$(\sigma(\Theta(N)) \cup (Z_p^{\mathbb{R}} + iZ_q^{\mathbb{R}})) \dot{\cup} Z^i$$

with  $\phi(z) \in \mathfrak{C}(z)$ , where  $\mathfrak{C}(z) := \mathcal{B}_{\mathfrak{d}_p(\xi), \mathfrak{d}_q(\eta)}$  for  $z = (\xi, \eta) \in Z^i$  and where  $\mathfrak{C}(z) := \mathcal{A}_{\mathfrak{d}_p(\operatorname{Re} z), \mathfrak{d}_q(\operatorname{Im} z)}$  for  $z \in \sigma(\Theta(N)) \cup (Z_p^{\mathbb{R}} + iZ_q^{\mathbb{R}})$ .

(ii) We provide  $\mathcal{M}_N$  pointwise with scalar multiplication, addition and multiplication, where the operations on  $\mathcal{A}_{\mathfrak{d}_p(\operatorname{Re} z), \mathfrak{d}_q(\operatorname{Im} z)}$  or  $\mathcal{B}_{\mathfrak{d}_p(\xi), \mathfrak{d}_q(\eta)}$  are as in Definition 4.1. We also define a conjugate linear involution  $\#$  on  $\mathcal{M}_N$  by

$$\begin{aligned} \phi^\#(z) &= \overline{\phi(z)}, \quad z \in \sigma(\Theta(N)) \cup (Z_p^{\mathbb{R}} + iZ_q^{\mathbb{R}}), \\ \phi^\#(\xi, \eta) &= \overline{\phi(\bar{\xi}, \bar{\eta})}, \quad (\xi, \eta) \in Z^i. \end{aligned}$$

(iii) By  $\mathcal{R}$  we denote the set of all elements  $\phi \in \mathcal{M}_N$  such that  $\pi(\phi(z)) = 0$  for all  $z \in (Z_p^{\mathbb{R}} + iZ_q^{\mathbb{R}}) \dot{\cup} Z^i$ .  $\diamond$

With the operations introduced in Definition 4.3  $\mathcal{M}_N$  is a commutative  $*$ -algebra as can be verified in a straight forward manner. Moreover,  $\mathcal{R}$  is an ideal of  $\mathcal{M}_N$ .

**4.4 Definition.** Let  $f : \operatorname{dom} f \rightarrow \mathbb{C}$  be a function with  $\operatorname{dom} f \subseteq \mathbb{C}^2$  such that  $\tau(\sigma(\Theta(N)) \cup (Z_p^{\mathbb{R}} + iZ_q^{\mathbb{R}})) \subseteq \operatorname{dom} f$ , where  $\tau : \mathbb{C} \rightarrow \mathbb{C}^2$ ,  $(x + iy) \mapsto (x, y)$ , such that  $f \circ \tau$  is sufficiently smooth – more exactly, at least  $\max_{x,y \in \mathbb{R}} \mathfrak{d}_p(x) + \mathfrak{d}_q(y) - 1$

times continuously differentiable – on an open neighbourhood of  $Z_p^{\mathbb{R}} + iZ_q^{\mathbb{R}}$ , and such that  $f$  is holomorphic on an open neighbourhood of  $Z^i$ .

Then  $f$  can be considered as an element  $f_N$  of  $\mathcal{M}_N$  by setting  $f_N(z) := f \circ \tau(z)$  for  $z \in \sigma(\Theta(N)) \setminus (Z_p^{\mathbb{R}} + iZ_q^{\mathbb{R}})$ , by

$$f_N(z) := \left( \frac{1}{k!l!} \frac{\partial^{k+l}}{\partial x^k \partial y^l} f \circ \tau(z) \right)_{(k,l) \in I_{\mathfrak{d}_p(\operatorname{Re} z), \mathfrak{d}_q(\operatorname{Im} z)}}$$

for  $z \in Z_p^{\mathbb{R}} + iZ_q^{\mathbb{R}}$ , and by

$$f_N(\xi, \eta) := \left( \frac{1}{k!l!} \frac{\partial^{k+l}}{\partial z^k \partial w^l} f(\xi, \eta) \right)_{\substack{0 \leq k \leq \mathfrak{d}_p(\xi) - 1, \\ 0 \leq l \leq \mathfrak{d}_q(\eta) - 1}}$$

for  $(\xi, \eta) \in Z^i$ . ◇

*4.5 Remark.* By the Leibniz rule  $f \mapsto f_N$  is compatible with multiplication. Obviously, it is also compatible with addition and scalar multiplication. If we define for a function  $f$  as in Definition 4.4 the function  $f^\#$  by  $f^\#(z, w) = \overline{f(\bar{z}, \bar{w})}$ ,  $(z, w) \in \operatorname{dom} f$ , then we also have  $(f^\#)_{p,q} = (f_N)^\#$ . Note that in general  $(\bar{f})_{p,q} \neq (f_N)^\#$ .

Finally, note that  $\mathbf{1}_N(z)$  is the multiplicative unite in  $\mathfrak{C}(z)$  for all  $z \in (\sigma(\Theta(N)) \cup (Z_p^{\mathbb{R}} + iZ_q^{\mathbb{R}})) \dot{\cup} Z^i$ . ◇

*4.6 Example.* For the constant one function  $\mathbf{1}$  on  $\mathbb{C}^2$  we have  $\mathbf{1}_N(z) = e$  for all  $z \in (\sigma(\Theta(N)) \cup (Z_p^{\mathbb{R}} + iZ_q^{\mathbb{R}})) \dot{\cup} Z^i$ , where  $e$  is the multiplicative unite in  $\mathfrak{C}(z)$ ; see Remark 4.2. ◇

*4.7 Example.*  $p(z)$  considered as an element of  $\mathbb{C}[z, w]$  is clearly holomorphic on  $\mathbb{C}^2$ . Hence, we can consider  $p_N$  as defined in Definition 4.4. It satisfies  $p_N(z)_{k,l} = 0$ ,  $(k, l) \in I_{\mathfrak{d}_p(\operatorname{Re} z), \mathfrak{d}_q(\operatorname{Im} z)} \setminus \{(\mathfrak{d}_p(\operatorname{Re} z), 0)\}$ , and

$$p_N(z)_{\mathfrak{d}_p(\operatorname{Re} z), 0} = \frac{1}{\mathfrak{d}_p(\operatorname{Re} z)!} p^{(\mathfrak{d}_p(\operatorname{Re} z))}(\operatorname{Re} z)$$

for all  $z \in Z_p^{\mathbb{R}} + iZ_q^{\mathbb{R}}$ . Since  $\operatorname{Re} z$  is a zero of  $p$  of degree exactly  $\mathfrak{d}_p(\operatorname{Re} z)$  the entries with index  $(\mathfrak{d}_p(\operatorname{Re} z), 0)$  do not vanish. Moreover,  $p_N(\xi, \eta) = 0$  for all  $(\xi, \eta) \in Z^i$ . In particular,  $p_N \in \mathcal{R}$ .

Similarly, if  $q(w)$  is considered as an element of  $\mathbb{C}[z, w]$ , then  $q_N(z)_{k,l} = 0$ ,  $(k, l) \in I_{\mathfrak{d}_p(\operatorname{Re} z), \mathfrak{d}_q(\operatorname{Im} z)} \setminus \{(0, \mathfrak{d}_q(\operatorname{Im} z))\}$ , and

$$q_N(z)_{0, \mathfrak{d}_q(\operatorname{Im} z)} = \frac{1}{\mathfrak{d}_q(\operatorname{Im} z)!} q^{(\mathfrak{d}_q(\operatorname{Im} z))}(\operatorname{Im} z) \neq 0$$

for all  $z \in Z_p^{\mathbb{R}} + iZ_q^{\mathbb{R}}$ . Also here  $q_N(\xi, \eta) = 0$  for all  $(\xi, \eta) \in Z^i$  and, in turn,  $q_N \in \mathcal{R}$ . ◇

We need an easy algebraic lemma based in the Euclidean algorithm.

**4.8 Lemma.** For  $a(z), b(z) \in \mathbb{C}[z]$  we denote by  $a^{-1}\{0\}$  and  $b^{-1}\{0\}$  the set of all zeros of  $a$  and  $b$  in  $\mathbb{C}$ , and by  $\mathfrak{d}_a(z)$  ( $\mathfrak{d}_b(z)$ )  $a$ 's ( $b$ 's) degree of zero at  $z \in \mathbb{C}$ . Denote by  $m$  ( $n$ ) the degree of the polynomial  $a$  ( $b$ ). Then any  $s \in \mathbb{C}[z, w]$  can be written as

$$s(z, w) = a(z)u(z, w) + b(w)v(z, w) + r(z, w)$$

with  $u(z, w), v(z, w), r(z, w) \in \mathbb{C}[z, w]$  such that  $r$ 's  $z$ -degree is less than  $m$  and its  $w$ -degree is less than  $n$ . Here  $u(z, w), v(z, w), r(z, w)$  can be found in  $\mathbb{R}[z, w]$  if  $a(z), b(z) \in \mathbb{R}[z]$ ,  $s \in \mathbb{R}[z, w]$ .

If we define  $\varpi : \mathbb{C}[z, w] \rightarrow \mathbb{C}^{m \cdot n}$  by

$$\varpi(s) = \left( \left( \frac{1}{k!l!} \frac{\partial^{k+l}}{\partial z^k \partial w^l} s(z, w) \right)_{\substack{0 \leq k \leq \mathfrak{d}_a(z)-1 \\ 0 \leq l \leq \mathfrak{d}_b(w)-1}} \right)_{z \in a^{-1}\{0\}, w \in b^{-1}\{0\}},$$

then  $s \in \ker \varpi$  if and only if  $s(z, w) = a(z)u(z, w) + b(w)v(z, w)$  for some  $u(z, w), v(z, w) \in \mathbb{C}[z, w]$ . Moreover,  $\varpi$  restricted to the space of all polynomials from  $\mathbb{C}[z, w]$  with  $z$ -degree less than  $m$  and  $w$ -degree less than  $n$  is bijective.

*Proof.* Applying the Euclidean algorithm to  $s(z, w) \in \mathbb{C}[z, w]$  and  $a(z)$  we get  $s(z, w) = a(z)u(z, w) + t(z, w)$ , where  $u(z, w), t(z, w) \in \mathbb{C}[z, w]$  such that  $t$ 's  $z$ -degree is less than  $m$ . Applying the Euclidean algorithm to  $t(z, w)$  and  $b(w)$  we get

$$s(z, w) = a(z)u(z, w) + b(w)v(z, w) + r(z, w)$$

with  $v(z, w), r(z, w) \in \mathbb{C}[z, w]$  such that  $r$ 's  $z$ -degree is less than  $m$  and its  $w$ -degree is less than  $n$ . The resulting polynomials  $u(z, w), t(z, w), v(z, w), r(z, w)$  belong to  $\mathbb{R}[z, w]$  if  $a(z), b(z) \in \mathbb{R}[z]$ ,  $s(z, w) \in \mathbb{R}[z, w]$ .

In any case it is easy to check that then  $\varpi(s) = \varpi(r)$ . Hence,  $r(z, w) = 0$  yields  $s(z, w) \in \ker \varpi$ . On the other hand, if  $0 = \varpi(s) = \varpi(r)$ , then for each fixed  $\zeta \in a^{-1}\{0\}$  and  $k \in \{0, \dots, \mathfrak{d}_a(\zeta) - 1\}$  the function  $w \mapsto \frac{\partial^k}{\partial z^k} r(\zeta, w)$  has zeros at all  $w \in b^{-1}\{0\}$  with multiplicity at least  $\mathfrak{d}_b(w)$ . Since  $w \mapsto \frac{\partial^k}{\partial z^k} r(\zeta, w)$  is of  $w$ -degree less than  $n$ , it must be identically equal to zero.

This implies that for any  $\eta \in \mathbb{C}$  the polynomial  $z \mapsto r(z, \eta)$  has zeros at all  $\zeta \in a^{-1}\{0\}$  with multiplicity at least  $\mathfrak{d}_a(\zeta)$ . Since the degree of this polynomial in  $z$  is less than  $m$ , we obtain  $r(z, \eta) = 0$  for any  $z \in \mathbb{C}$ . Thus,  $r \equiv 0$ .

Our description of  $\ker \varpi$  shows in particular that  $\varpi$  restricted to the space of all polynomials from  $\mathbb{C}[z, w]$  with  $z$ -degree less than  $m$  and  $w$ -degree less than  $n$  is one-to-one. Comparing dimensions shows that this restriction of  $\varpi$  is also onto.  $\square$

**4.9 Corollary.** *With the notation from Definition 4.3 for any  $\phi \in \mathcal{M}_N$  we find an  $s \in \mathbb{C}[z, w]$  such that  $\phi - s_N \in \mathcal{R}$ .*

*Proof.* By Lemma 4.8 there exists an  $s \in \mathbb{C}[z, w]$  such that  $\varpi(s)_{\operatorname{Re} z, \operatorname{Im} z} = \pi(\phi(z))$  for all  $z \in Z_p^{\mathbb{R}} + iZ_q^{\mathbb{R}}$ , and such that  $\varpi(s)_{\xi, \eta} = \phi(\xi, \eta)$  for all  $(\xi, \eta) \in Z^i$ . According to  $\mathcal{R}$ 's definition we obtain  $\phi - s_N \in \mathcal{R}$ .  $\square$

**4.10 Remark.** Recall from Lemma 3.3 that  $p(\operatorname{Re} z) + q(\operatorname{Im} z) = 0$  with  $z \in \sigma(\Theta(N))$  implies  $p(\operatorname{Re} z) = 0 = q(\operatorname{Im} z)$ , i.e.  $z \in Z_p^{\mathbb{R}} + iZ_q^{\mathbb{R}}$ .

If  $\phi \in \mathcal{R}$ , then we find a function  $g$  on  $\sigma(\Theta(N))$  with  $g(z) \in \mathbb{C}$  for  $z \in \sigma(\Theta(N)) \setminus (Z_p^{\mathbb{R}} + iZ_q^{\mathbb{R}})$  and  $g(z) \in \mathbb{C}^2$  for  $z \in \sigma(\Theta(N)) \cap (Z_p^{\mathbb{R}} + iZ_q^{\mathbb{R}})$ , such that  $\phi(z) = (p_N + q_N)(z) \cdot g(z)$ ,  $z \in \sigma(\Theta(N))$ ; see Example 4.7. Here  $(p_N + q_N)(z) \cdot g(z)$  is the usual multiplication on  $\mathbb{C}$  for  $z \in \sigma(\Theta(N)) \setminus (Z_p^{\mathbb{R}} + iZ_q^{\mathbb{R}})$ , whereas

$$\left( (p_N + q_N)(z) \cdot g(z) \right)_{k,l} = 0, \quad k = 0, \dots, \mathfrak{d}_p(\operatorname{Re} z) - 1; l = 0, \dots, \mathfrak{d}_q(\operatorname{Im} z) - 1,$$

and

$$\left( (p_N + q_N)(z) \cdot g(z) \right)_{\mathfrak{d}_p(\operatorname{Re} z), 0} = (p_N + q_N)(z)_{\mathfrak{d}_p(\operatorname{Re} z), 0} \cdot g_1(z),$$

$$((p_N + q_N)(z) \cdot g(z))_{0, \mathfrak{d}_q(\operatorname{Im} z)} = (p_N + q_N)(z)(z)_{0, \mathfrak{d}_q(\operatorname{Im} z)} \cdot g_2(z).$$

for  $z \in \sigma(\Theta(N)) \cap (Z_p^{\mathbb{R}} + iZ_q^{\mathbb{R}})$ .

In fact, we simply set  $g(z) := \frac{\phi(z)}{p(\operatorname{Re} z) + q(\operatorname{Im} z)}$  for  $z \in \sigma(\Theta(N)) \setminus (Z_p^{\mathbb{R}} + iZ_q^{\mathbb{R}})$  and

$$g_1(z) := \frac{\mathfrak{d}_p(\operatorname{Re} z)! \phi(z)_{(\mathfrak{d}_p(\operatorname{Re} z), 0)}}{p^{(\mathfrak{d}_p(\operatorname{Re} z))}(\operatorname{Re} z)}, \quad g_1(z) := \frac{\mathfrak{d}_q(\operatorname{Im} z)! \phi(z)_{(0, \mathfrak{d}_q(\operatorname{Im} z))}}{q^{(\mathfrak{d}_q(\operatorname{Im} z))}(\operatorname{Im} z)}$$

for  $z \in \sigma(\Theta(N)) \cap (Z_p^{\mathbb{R}} + iZ_q^{\mathbb{R}})$ .  $\diamond$

We are going to introduce a subclass of  $\mathcal{M}_N$ , which will be the proper class, in order to build up our functional calculus.

**4.11 Definition.** With the notation from Definition 4.3 we denote by  $\mathcal{F}_N$  the set of all elements  $\phi \in \mathcal{M}_N$  such that  $z \mapsto \phi(z)$  is Borel measurable and bounded on  $\sigma(\Theta(N)) \setminus (Z_p^{\mathbb{R}} + iZ_q^{\mathbb{R}})$ , and such that for each  $w \in \sigma(\Theta(N)) \cap (Z_p^{\mathbb{R}} + iZ_q^{\mathbb{R}})$

$$\frac{\phi(z) - \sum_{k=0}^{\mathfrak{d}_p(\operatorname{Re} w)-1} \sum_{l=0}^{\mathfrak{d}_q(\operatorname{Im} w)-1} \phi(w)_{k,l} \operatorname{Re}(z-w)^k \operatorname{Im}(z-w)^l}{\max(|\operatorname{Re}(z-w)|^{\mathfrak{d}_p(\operatorname{Re} w)}, |\operatorname{Im}(z-w)|^{\mathfrak{d}_q(\operatorname{Im} w)})} \quad (4.1)$$

is bounded for  $z \in \sigma(\Theta(N)) \cap U(w) \setminus \{w\}$ , where  $U(w)$  is a sufficiently small neighbourhood of  $w$ .  $\diamond$

Note that (4.1) is immaterial if  $w$  is an isolated point of  $\sigma(\Theta(N))$ .

*4.12 Example.* For  $\zeta \in (Z_p^{\mathbb{R}} + iZ_q^{\mathbb{R}}) \dot{\cup} Z^i$  and  $a \in \mathfrak{C}(\zeta)$  consider the functions  $a\delta_\zeta \in \mathcal{M}_N$  which assumes the value  $a$  at  $\zeta$  and the value zero on the rest of  $(\sigma(\Theta(N)) \cup (Z_p^{\mathbb{R}} + iZ_q^{\mathbb{R}})) \dot{\cup} Z^i$

If  $\zeta$  belongs to  $Z^i$  or if  $\zeta$  is an isolated point of  $\sigma(\Theta(N)) \cup (Z_p^{\mathbb{R}} + iZ_q^{\mathbb{R}})$ , then  $a\delta_\zeta$  belongs to  $\mathcal{F}_N$ .  $\diamond$

*4.13 Remark.* Let  $h$  be defined on an open subset  $D$  of  $\mathbb{R}^2$  with values in  $\mathbb{C}$ . Moreover, assume that for given  $m, n \in \mathbb{N}$  the function  $h$  is  $m+n$  times continuously differentiable. Finally, fix  $w \in D$ .

The well-known Taylors Approximation Theorem from multidimensional calculus then yields

$$h(z) = \sum_{j=0}^{m+n-1} \sum_{\substack{k,l \in \mathbb{N}_0 \\ k+l=j}} \frac{1}{k!l!} \frac{\partial^{k+l} h}{\partial x^k \partial y^l}(w) \operatorname{Re}(z-w)^k \operatorname{Im}(z-w)^l + O(|z-w|^{m+n})$$

for  $z \rightarrow w$ . Since  $|z-w|^{m+n} \leq 2^{m+n} \max(|\operatorname{Re}(z-w)|^{m+n}, |\operatorname{Im}(z-w)|^{m+n}) = O(\max(|\operatorname{Re}(z-w)|^m, |\operatorname{Im}(z-w)|^n))$  and since  $\operatorname{Re}(z-w)^k \operatorname{Im}(z-w)^l = O(\max(|\operatorname{Re}(z-w)|^m, |\operatorname{Im}(z-w)|^n))$  for  $k \geq m$  or  $l \geq n$ , we also have

$$h(z) = \sum_{k=0}^{m-1} \sum_{l=0}^{n-1} \frac{1}{k!l!} \frac{\partial^{k+l} h}{\partial x^k \partial y^l}(w) \operatorname{Re}(z-w)^k \operatorname{Im}(z-w)^l + O(\max(|\operatorname{Re}(z-w)|^m, |\operatorname{Im}(z-w)|^n)).$$

$\diamond$

**4.14 Lemma.** Let  $f : \operatorname{dom} f (\subseteq \mathbb{C}^2) \rightarrow \mathbb{C}$  be a function with the properties mentioned in Definition 4.4. Then  $f_N$  belongs to  $\mathcal{F}_N$ .

*Proof.* For fixed  $w \in \sigma(\Theta(N)) \cap (Z_p^{\mathbb{R}} + iZ_q^{\mathbb{R}})$  and  $z \in \sigma(\Theta(N)) \setminus (Z_p^{\mathbb{R}} + iZ_q^{\mathbb{R}})$  by Remark 4.13 the expression

$$f_N(z) - \sum_{k=0}^{\mathfrak{d}_p(\operatorname{Re} w) - 1} \sum_{l=0}^{\mathfrak{d}_q(\operatorname{Im} w) - 1} f_N(w)_{k,l} \operatorname{Re}(z - w)^k \operatorname{Im}(z - w)^l =$$

$$f \circ \tau(z) - \sum_{k=0}^{\mathfrak{d}_p(\operatorname{Re} w) - 1} \sum_{l=0}^{\mathfrak{d}_q(\operatorname{Im} w) - 1} \frac{1}{k!l!} \frac{\partial^{k+l} f \circ \tau}{\partial x^k \partial y^l}(w) \operatorname{Re}(z - w)^k \operatorname{Im}(z - w)^l$$

is a  $O(\max(|\operatorname{Re}(z - w)|^{\mathfrak{d}_p(\operatorname{Re} w)}, |\operatorname{Im}(z - w)|^{\mathfrak{d}_q(\operatorname{Im} w)}))$  for  $z \rightarrow w$ . Therefore,  $f_N \in \mathcal{F}_N$ .  $\square$

In order to be able to prove spectral results for our functional calculus, we need that with  $\phi$  also  $z \mapsto \phi(z)^{-1}$  belongs to  $\mathcal{F}_N$  if  $\phi$  is bounded away from zero.

**4.15 Lemma.** *If  $\phi \in \mathcal{F}_N$  is such that  $\phi(z)$  is invertible in  $\mathfrak{C}(z)$  (see Remark 4.2) for all  $z \in (\sigma(\Theta(N)) \cup (Z_p^{\mathbb{R}} + iZ_q^{\mathbb{R}})) \dot{\cup} Z^i$  and such that 0 does not belong to the closure of  $\phi(\sigma(\Theta(N)) \setminus (Z_p^{\mathbb{R}} + iZ_q^{\mathbb{R}}))$ , then  $\phi^{-1} : z \mapsto \phi(z)^{-1}$  also belongs to  $\mathcal{F}_N$ .*

*Proof.* By the first assumption  $\phi^{-1}$  is a well-defined object belonging to  $\mathcal{M}_N$ . Clearly, with  $\phi$  also  $z \mapsto \phi(z)^{-1} = \frac{1}{\phi(z)}$  is measurable on  $\sigma(\Theta(N)) \setminus (Z_p^{\mathbb{R}} + iZ_q^{\mathbb{R}})$ . By the second assumption  $z \mapsto \phi(z)^{-1} = \frac{1}{\phi(z)}$  is bounded on this set.

It remains to verify the boundedness of (4.1) on a certain neighbourhood of  $w$  for each  $w \in \sigma(\Theta(N)) \cap (Z_p^{\mathbb{R}} + iZ_q^{\mathbb{R}})$  for  $\phi^{-1}$ . To do so, we calculate for  $z \in \sigma(\Theta(N)) \setminus (Z_p^{\mathbb{R}} + iZ_q^{\mathbb{R}})$

$$\phi^{-1}(z) - \sum_{k=0}^{\mathfrak{d}_p(\operatorname{Re} w) - 1} \sum_{l=0}^{\mathfrak{d}_q(\operatorname{Im} w) - 1} \phi^{-1}(w)_{k,l} \operatorname{Re}(z - w)^k \operatorname{Im}(z - w)^l = \quad (4.2)$$

$$\frac{1}{\phi(z)} - \frac{1}{\sum_{k=0}^{\mathfrak{d}_p(\operatorname{Re} w) - 1} \sum_{l=0}^{\mathfrak{d}_q(\operatorname{Im} w) - 1} \phi(w)_{k,l} \operatorname{Re}(z - w)^k \operatorname{Im}(z - w)^l} + \quad (4.3)$$

$$\frac{1}{\sum_{k=0}^{\mathfrak{d}_p(\operatorname{Re} w) - 1} \sum_{l=0}^{\mathfrak{d}_q(\operatorname{Im} w) - 1} \phi(w)_{k,l} \operatorname{Re}(z - w)^k \operatorname{Im}(z - w)^l} -$$

$$\sum_{k=0}^{\mathfrak{d}_p(\operatorname{Re} w) - 1} \sum_{l=0}^{\mathfrak{d}_q(\operatorname{Im} w) - 1} \phi^{-1}(w)_{k,l} \operatorname{Re}(z - w)^k \operatorname{Im}(z - w)^l \quad (4.4)$$

The expression in (4.3) can be written as

$$\frac{1}{\phi(z)} \cdot \frac{1}{\sum_{k=0}^{\mathfrak{d}_p(\operatorname{Re} w) - 1} \sum_{l=0}^{\mathfrak{d}_q(\operatorname{Im} w) - 1} \phi(w)_{k,l} \operatorname{Re}(z - w)^k \operatorname{Im}(z - w)^l}$$

$$\left( \phi(z) - \sum_{k=0}^{\mathfrak{d}_p(\operatorname{Re} w) - 1} \sum_{l=0}^{\mathfrak{d}_q(\operatorname{Im} w) - 1} \phi(w)_{k,l} \operatorname{Re}(z - w)^k \operatorname{Im}(z - w)^l \right)$$

Here  $\frac{1}{\phi(z)}$  is bounded by assumption. The assumed invertibility of  $\phi(w)$  means  $\phi(w)_{0,0} \neq 0$ . Hence,

$$\frac{1}{\sum_{k=0}^{\mathfrak{d}_p(\operatorname{Re} w)-1} \sum_{l=0}^{\mathfrak{d}_q(\operatorname{Im} w)-1} \phi(w)_{k,l} \operatorname{Re}(z-w)^k \operatorname{Im}(z-w)^l} = O(1)$$

for  $z \rightarrow w$ . From  $\phi \in \mathcal{F}_N$  we then conclude that (4.3) is a  $O(\max(|\operatorname{Re}(z-w)|^{\mathfrak{d}_p(\operatorname{Re} w)}, |\operatorname{Im}(z-w)|^{\mathfrak{d}_q(\operatorname{Im} w)}))$  for  $z \rightarrow w$ .

Because of  $\phi(w) \cdot \phi^{-1} = e$  (see Remark 4.2), (4.4) can be rewritten as

$$\begin{aligned} & - \frac{1}{\sum_{k=0}^{\mathfrak{d}_p(\operatorname{Re} w)-1} \sum_{l=0}^{\mathfrak{d}_q(\operatorname{Im} w)-1} \phi(w)_{k,l} \operatorname{Re}(z-w)^k \operatorname{Im}(z-w)^l} \\ & \left( \sum_{k=0}^{\mathfrak{d}_p(\operatorname{Re} w)-1} \sum_{l=0}^{\mathfrak{d}_q(\operatorname{Im} w)-1} \operatorname{Re}(z-w)^k \operatorname{Im}(z-w)^l \cdot \sum_{c=0}^k \sum_{d=0}^l \phi(w)_{c,d} \cdot \phi^{-1}(w)_{k-c,l-d} \right. \\ & \quad \left. + O(\max(|\operatorname{Re}(z-w)|^{\mathfrak{d}_p(\operatorname{Re} w)}, |\operatorname{Im}(z-w)|^{\mathfrak{d}_q(\operatorname{Im} w)})) - 1 \right) = \end{aligned}$$

$$\begin{aligned} & O(1) \cdot O(\max(|\operatorname{Re}(z-w)|^{\mathfrak{d}_p(\operatorname{Re} w)}, |\operatorname{Im}(z-w)|^{\mathfrak{d}_q(\operatorname{Im} w)})) = \\ & O(\max(|\operatorname{Re}(z-w)|^{\mathfrak{d}_p(\operatorname{Re} w)}, |\operatorname{Im}(z-w)|^{\mathfrak{d}_q(\operatorname{Im} w)})) \end{aligned}$$

for  $z \rightarrow w$ . Altogether (4.2) is a  $O(\max(|\operatorname{Re}(z-w)|^{\mathfrak{d}_p(\operatorname{Re} w)}, |\operatorname{Im}(z-w)|^{\mathfrak{d}_q(\operatorname{Im} w)}))$ . Therefore,  $\phi^{-1} \in \mathcal{F}_N$ .  $\square$

## 5 Functional Calculus

In this section we employ the same assumptions and notation as in the previous one.

**5.1 Lemma.** *For any  $\phi \in \mathcal{F}_N$  there exists a polynomial  $s \in \mathbb{C}[z, w]$  and a function  $g$  on  $\sigma(\Theta(N))$  with values in  $\mathbb{C}$  on  $\sigma(\Theta(N)) \setminus (Z_p^{\mathbb{R}} + iZ_q^{\mathbb{R}})$  and values in  $\mathbb{C}^2$  on  $\sigma(\Theta(N)) \cap (Z_p^{\mathbb{R}} + iZ_q^{\mathbb{R}})$  such that  $\phi - s_N \in \mathcal{R}$ , such that  $g$  is bounded and measurable on  $\sigma(\Theta(N)) \setminus (Z_p^{\mathbb{R}} + iZ_q^{\mathbb{R}})$ , and such that*

$$\phi(z) = s_N(z) + (p_N + q_N)(z) \cdot g(z), \quad z \in \sigma(\Theta(N)), \quad (5.1)$$

where the multiplication here has to be understood in the sense of Remark 4.10.

*Proof.* According to Corollary 4.9 there exists an  $s \in \mathbb{C}[z, w]$  such that  $\phi - s_N \in \mathcal{R}$ , and by Remark 4.10 we then find a function  $g$  such that (5.1) holds true. The measurability of

$$g(z) = \frac{\phi(z) - s(\operatorname{Re} z, \operatorname{Im} z)}{p(\operatorname{Re} z) + q(\operatorname{Im} z)} \quad \text{on } \sigma(\Theta(N)) \setminus (Z_p^{\mathbb{R}} + iZ_q^{\mathbb{R}})$$

follows from the assumption  $\phi \in \mathcal{F}_N$ ; see Definition 4.11.

In order to show  $g$ 's boundedness, first recall from Lemma 3.3 that

$$\max(|p(\operatorname{Re} z)|, |q(\operatorname{Im} z)|) \leq \max(\|R_1 R_1^*\|, \|R_2 R_2^*\|) |p(\operatorname{Re} z) + q(\operatorname{Im} z)|$$



for  $z \in \sigma(\Theta(N))$ . Hence,

$$\frac{\max(|p(\operatorname{Re} z)|, |q(\operatorname{Im} z)|)}{|p(\operatorname{Re} z) + q(\operatorname{Im} z)|} \leq \max(\|R_1 R_1^*\|, \|R_2 R_2^*\|),$$

$$z \in \sigma(\Theta(N)) \setminus (Z_p^{\mathbb{R}} + iZ_q^{\mathbb{R}}).$$

As  $\phi \in \mathcal{F}_N$  we find for each  $w \in \sigma(\Theta(N)) \cap (Z_p^{\mathbb{R}} + iZ_q^{\mathbb{R}})$  an open neighbourhood  $U(w)$  of  $w$  such that (4.1) is bounded for  $z \in U(w) \setminus \{w\}$ . Clearly, we can make the neighbourhoods  $U(w)$  smaller so that they are pairwise disjoint. Since for  $w \in \sigma(\Theta(N)) \cap (Z_p^{\mathbb{R}} + iZ_q^{\mathbb{R}})$  the real number  $\operatorname{Re} w$  ( $\operatorname{Im} w$ ) is a zero of  $p(\operatorname{Re} z)$  ( $q(\operatorname{Im} z)$ ) with multiplicity  $\mathfrak{d}_p(\operatorname{Re} w)$  ( $\mathfrak{d}_q(\operatorname{Im} w)$ ), we have

$$c|\operatorname{Re}(z - w)|^{\mathfrak{d}_p(\operatorname{Re} w)} \leq |p(\operatorname{Re} z)|, \quad d|\operatorname{Im}(z - w)|^{\mathfrak{d}_q(\operatorname{Im} w)} \leq |q(\operatorname{Im} z)|$$

for  $z \in U(w)$  with constants  $c, d > 0$ . Hence,

$$\frac{\max(|\operatorname{Re}(z - w)|^{\mathfrak{d}_p(\operatorname{Re} w)}, |\operatorname{Im}(z - w)|^{\mathfrak{d}_q(\operatorname{Im} w)})}{\max(|p(\operatorname{Re} z)|, |q(\operatorname{Im} z)|)} \leq C_w$$

on  $\sigma(\Theta(N)) \cap U(w) \setminus \{w\}$  for some  $C_w > 0$ . By what was said in Remark 4.13 and we also have

$$s(\operatorname{Re} z, \operatorname{Im} z) = \sum_{k=0}^{\mathfrak{d}_p(\operatorname{Re} w) - 1} \sum_{l=0}^{\mathfrak{d}_q(\operatorname{Im} w) - 1} \phi(w)_{k,l} \operatorname{Re}(z - w)^k \operatorname{Im}(z - w)^l +$$

$$O(\max(|\operatorname{Re}(z - w)|^{\mathfrak{d}_p(\operatorname{Re} w)}, |\operatorname{Im}(z - w)|^{\mathfrak{d}_q(\operatorname{Im} w)})),$$

because  $\phi - s_N \in \mathcal{R}$  implies  $\phi(w)_{k,l} = \frac{1}{k!l!} \frac{\partial^{k+l} s}{\partial x^k \partial y^l}(\operatorname{Re} w, \operatorname{Im} w)$ . Using the boundedness of (4.1) we altogether obtain the boundedness of

$$g(z) = \frac{\phi(z) - s(\operatorname{Re} z, \operatorname{Im} z)}{p(\operatorname{Re} z) + q(\operatorname{Im} z)} = \tag{5.2}$$

$$\frac{\max(|p(\operatorname{Re} z)|, |q(\operatorname{Im} z)|)}{p(\operatorname{Re} z) + q(\operatorname{Im} z)} \cdot$$

$$\frac{\max(|\operatorname{Re}(z - w)|^{\mathfrak{d}_p(\operatorname{Re} w)}, |\operatorname{Im}(z - w)|^{\mathfrak{d}_q(\operatorname{Im} w)})}{\max(|p(\operatorname{Re} z)|, |q(\operatorname{Im} z)|)}$$

$$\frac{\phi(z) - s(\operatorname{Re} z, \operatorname{Im} z)}{\max(|\operatorname{Re}(z - w)|^{\mathfrak{d}_p(\operatorname{Re} w)}, |\operatorname{Im}(z - w)|^{\mathfrak{d}_q(\operatorname{Im} w)})}$$

for  $z \in \sigma(\Theta(N)) \cap U(w) \setminus \{w\}$ . Since by Lemma 3.3 the function  $\frac{1}{p(\operatorname{Re} z) + q(\operatorname{Im} z)}$  is continuous, and hence bounded on  $\sigma(\Theta(N)) \setminus \bigcup_{w \in \sigma(\Theta(N)) \cap (Z_p^{\mathbb{R}} + iZ_q^{\mathbb{R}})} U(w)$ , we see that (5.2) is even bounded for  $z \in \sigma(\Theta(N)) \setminus (Z_p^{\mathbb{R}} + iZ_q^{\mathbb{R}})$ .  $\square$

**5.2 Definition.** For any  $\phi \in \mathcal{F}_N$  we define

$$\phi(N) := s(A, B) + \Xi \left( \int_{\sigma(\Theta(N))}^{R_1, R_2} g dE \right),$$

where  $s \in \mathbb{C}[z, w]$  and  $g$  is a function on  $\sigma(\Theta(N))$  with the properties mentioned in Lemma 5.1, and where

$$\int_{\sigma(\Theta(N))}^{R_1, R_2} g dE := \int_{\sigma(\Theta(N)) \setminus (Z_p^{\mathbb{R}} + iZ_q^{\mathbb{R}})} g dE + \sum_{w \in \sigma(\Theta(N)) \cap (Z_p^{\mathbb{R}} + iZ_q^{\mathbb{R}})} (g(w)_1 R_1 R_1^* E\{w\} + g(w)_2 R_2 R_2^* E\{w\}).$$

◇

First we shall show that  $\phi(N)$  is well defined.

**5.3 Theorem.** *Let  $\phi \in \mathcal{F}_N$ ,  $s, \tilde{s} \in \mathbb{C}[z, w]$  and functions  $g, \tilde{g}$  on  $\sigma(\Theta(N))$  be given, such that the assertion of Lemma 5.1 holds true for  $s, g$  as well as for  $\tilde{s}, \tilde{g}$ . Then*

$$s(A, B) + \Xi \left( \int_{\sigma(\Theta(N))}^{R_1, R_2} g dE \right) = \tilde{s}(A, B) + \Xi \left( \int_{\sigma(\Theta(N))}^{R_1, R_2} \tilde{g} dE \right).$$

*Proof.* By assumption we have  $\phi - s_N, \phi - \tilde{s}_N \in \mathcal{R}$ . Subtracting these functions yields  $\tilde{s}_N - s_N \in \mathcal{R}$ . Using the notation of Lemma 4.8 this gives  $\varpi(\tilde{s} - s)_{\xi, \eta} = 0$  for  $(\xi, \eta) \in p^{-1}\{0\} \times q^{-1}\{0\}$ . According to Lemma 4.8 we then get

$$\tilde{s}(z, w) - s(z, w) = p(z)u(z, w) + q(w)v(z, w) \quad (5.3)$$

for some  $u(z, w), v(z, w) \in \mathbb{C}[z, w]$ .

By Lemma 5.10 in [KP] we have

$$\Xi_1(u(\Theta_1(A), \Theta_1(B))) = \Xi_1(\Theta_1(u(A, B))) = p(A)u(A, B),$$

$$\Xi_2(v(\Theta_2(A), \Theta_2(B))) = \Xi_2(\Theta_2(v(A, B))) = q(B)v(A, B),$$

where  $\Xi_j$ ,  $j = 1, 2$ , are as defined in (2.5). Since  $u(\Theta_1(A), \Theta_1(B)) = \int u(\operatorname{Re} z, \operatorname{Im} z) dE_1(z)$ , we get from (2.6)

$$\Xi_1(u(\Theta_1(A), \Theta_1(B))) = \Xi(R_1 R_1^* \int u(\operatorname{Re} z, \operatorname{Im} z) dE(z)).$$

Similarly,  $\Xi_2(v(\Theta_2(A), \Theta_2(B))) = \Xi(R_2 R_2^* \int v(\operatorname{Re} z, \operatorname{Im} z) dE(z))$ . Therefore, employing Corollary 3.4 we get

$$\begin{aligned} \tilde{s}(A, B) - s(A, B) &= p(A)u(A, B) + q(B)v(A, B) = \\ &= \Xi(R_1 R_1^* \int u(\operatorname{Re} z, \operatorname{Im} z) dE(z) + R_2 R_2^* \int v(\operatorname{Re} z, \operatorname{Im} z) dE(z)) = \end{aligned} \quad (5.4)$$

$$\begin{aligned} &= \Xi \left( \int_{\sigma(\Theta(N)) \setminus (Z_p^{\mathbb{R}} + iZ_q^{\mathbb{R}})} \frac{p(\operatorname{Re} z)u(\operatorname{Re} z, \operatorname{Im} z) + q(\operatorname{Im} z)v(\operatorname{Re} z, \operatorname{Im} z)}{p(\operatorname{Re} z) + q(\operatorname{Im} z)} dE(z) + \right. \\ &\quad \left. \sum_{w \in \sigma(\Theta(N)) \cap (Z_p^{\mathbb{R}} + iZ_q^{\mathbb{R}})} (u(\operatorname{Re} w, \operatorname{Im} w) R_1 R_1^* E\{w\} + v(\operatorname{Re} w, \operatorname{Im} w) R_2 R_2^* E\{w\}) \right). \end{aligned}$$

On the other hand, since (5.1) holds true for  $s, g$  and  $\tilde{s}, \tilde{g}$ , we have

$$(\tilde{s}_N - s_N)(z) = (p_N + q_N)(z) \cdot (g(z) - \tilde{g}(z)), \quad z \in z \in \sigma(\Theta(N)). \quad (5.5)$$

For  $z \in \sigma(\Theta(N)) \setminus (Z_p^{\mathbb{R}} + iZ_q^{\mathbb{R}})$  by (5.3) this means

$$\begin{aligned} p(\operatorname{Re} z)u(\operatorname{Re} z, \operatorname{Im} z) + q(\operatorname{Im} z)v(\operatorname{Re} z, \operatorname{Im} z) = \\ \tilde{s}(\operatorname{Re} z, \operatorname{Im} z) - s(\operatorname{Re} z, \operatorname{Im} z) = (p(\operatorname{Re} z) + q(\operatorname{Im} z)) \cdot (g(z) - \tilde{g}(z)) \end{aligned}$$

and, in turn,

$$g(z) - \tilde{g}(z) = \frac{p(\operatorname{Re} z)u(\operatorname{Re} z, \operatorname{Im} z) + q(\operatorname{Im} z)v(\operatorname{Re} z, \operatorname{Im} z)}{p(\operatorname{Re} z) + q(\operatorname{Im} z)}.$$

Considering for  $z \in \sigma(\Theta(N)) \cap (Z_p^{\mathbb{R}} + iZ_q^{\mathbb{R}})$  the entries of (5.5) with indices  $(\mathfrak{d}_p(\operatorname{Re} z), 0)$  and  $(0, \mathfrak{d}_q(\operatorname{Im} z))$  together with (5.3) we get

$$\begin{aligned} \frac{1}{\mathfrak{d}_p(\operatorname{Re} z)!} p^{(\mathfrak{d}_p(\operatorname{Re} z))}(\operatorname{Re} z) u(\operatorname{Re} z, \operatorname{Im} z) = \\ \frac{1}{\mathfrak{d}_p(\operatorname{Re} z)!} \frac{\partial^{\mathfrak{d}_p(\operatorname{Re} z)}}{\partial x^{\mathfrak{d}_p(\operatorname{Re} z)}} (\tilde{s}(\operatorname{Re} z, \operatorname{Im} z) - s(\operatorname{Re} z, \operatorname{Im} z)) = \\ \frac{1}{\mathfrak{d}_p(\operatorname{Re} z)!} p^{(\mathfrak{d}_p(\operatorname{Re} z))}(\operatorname{Re} z) (g(z)_1 - \tilde{g}(z)_1) \end{aligned}$$

and

$$\begin{aligned} \frac{1}{\mathfrak{d}_q(\operatorname{Im} z)!} q^{(\mathfrak{d}_q(\operatorname{Im} z))}(\operatorname{Im} z) v(\operatorname{Re} z, \operatorname{Im} z) = \\ \frac{1}{\mathfrak{d}_q(\operatorname{Im} z)!} \frac{\partial^{\mathfrak{d}_q(\operatorname{Im} z)}}{\partial y^{\mathfrak{d}_q(\operatorname{Im} z)}} (\tilde{s}(\operatorname{Re} z, \operatorname{Im} z) - s(\operatorname{Re} z, \operatorname{Im} z)) = \\ \frac{1}{\mathfrak{d}_q(\operatorname{Im} z)!} q^{(\mathfrak{d}_q(\operatorname{Im} z))}(\operatorname{Im} z) (g(z)_2 - \tilde{g}(z)_2) \end{aligned}$$

where we employed the product rule and the fact that  $p^{(k)}(\operatorname{Re} z) = 0 = q^{(l)}(\operatorname{Im} z)$  for  $0 \leq k < \mathfrak{d}_p(\operatorname{Re} z)$ ,  $0 \leq l < \mathfrak{d}_q(\operatorname{Im} z)$ . Since  $p^{(\mathfrak{d}_p(\operatorname{Re} z))}(\operatorname{Re} z)$  and  $q^{(\mathfrak{d}_q(\operatorname{Im} z))}(\operatorname{Im} z)$  do not vanish for  $z \in \sigma(\Theta(N)) \cap (Z_p^{\mathbb{R}} + iZ_q^{\mathbb{R}})$ , we get  $u(\operatorname{Re} z, \operatorname{Im} z) = g(z)_1 - \tilde{g}(z)_1$  and  $v(\operatorname{Re} z, \operatorname{Im} z) = g(z)_2 - \tilde{g}(z)_2$ . Therefore, we can write (5.4) as

$$\tilde{s}(A, B) - s(A, B) = \Xi \left( \int_{\sigma(\Theta(N))}^{R_1, R_2} (g - \tilde{g}) dE \right),$$

showing the asserted equality.  $\square$

**5.4 Theorem.** *The mapping  $\phi \mapsto \phi(N)$  constitutes a  $*$ -homomorphism from  $\mathcal{F}_N$  into  $\{N, N^*\}''$  ( $\subseteq B(\mathcal{K})$ ) with  $s_N(N) = s(A, B)$  for all  $s \in \mathbb{C}[z, w]$ .*

*Proof.*  $s_N(N) = s(A, B)$  for all  $s \in \mathbb{C}[z, w]$  follows from Theorem 5.3 because we have  $s_N = s_N + (p_N + q_N)(z) \cdot 0$ ,  $z \in \sigma(\Theta(N))$ .

Assume that for  $\phi, \psi \in \mathcal{F}_N$  we have  $s, r \in \mathbb{C}[z, w]$  and functions  $g, h$  on  $\sigma(\Theta(N))$  such that  $\phi - s_N, \psi - r_N \in \mathcal{R}$ , such that  $g$  and  $h$  are bounded and measurable on  $\sigma(\Theta(N)) \setminus (Z_p^{\mathbb{R}} + iZ_q^{\mathbb{R}})$ , and such that (5.1) as well as

$$\psi(z) = r_N(z) + (p_N + q_N)(z) \cdot h(z), \quad z \in \sigma(\Theta(N)),$$

hold true; see Lemma 5.1. Then for  $\lambda, \mu \in \mathbb{C}$  we get from Remark 4.5

$$(\lambda\phi + \mu\psi)(z) = (\lambda s + \mu r)_N(z) + (p_N + q_N)(z) \cdot (\lambda g(z) + \mu h(z)), \quad z \in \sigma(\Theta(N)),$$

where  $\lambda\phi + \mu\psi - (\lambda s + \mu r)_N = \lambda(\phi - s_N) + \mu(\psi - r_N) \in \mathcal{R}$ , and where  $\lambda g + \mu h$  is bounded and measurable on  $\sigma(\Theta(N)) \setminus (Z_p^{\mathbb{R}} + iZ_q^{\mathbb{R}})$ . Since the definition of  $\phi(N)$  in Definition 5.2 depends linearly on  $s$  and  $g$ , we conclude from Theorem 5.3 that

$$(\lambda\phi + \mu\psi)(N) = \lambda\phi(N) + \mu\psi(N).$$

Similarly, we get  $\phi^\#(z) = (s^\#)_N(z) + (p_N + q_N)(z) \cdot \bar{g}(z)$ ,  $z \in \sigma(\Theta(N))$ ; see Remark 4.5. Thereby  $\phi^\# - (s^\#)_N = (\phi - s_N)^\# \in \mathcal{R}$  holds true due to the fact that  $\mathfrak{d}_p(\xi) = \mathfrak{d}_p(\bar{\xi})$  and  $\mathfrak{d}_q(\eta) = \mathfrak{d}_q(\bar{\eta})$  for all  $(\xi, \eta) \in Z^i$ . Since  $\bar{g}$  is bounded and measurable on  $\sigma(\Theta(N)) \setminus (Z_p^{\mathbb{R}} + iZ_q^{\mathbb{R}})$ , and since

$$\phi(N)^* = s^\#(A, B) + \Xi \left( \int_{\sigma(\Theta(N))}^{R_1, R_2} \bar{g} dE \right),$$

we again obtain from Theorem 5.3 that  $\phi^\#(N) = \phi(N)^*$ .

Concerning the compatibility with  $\cdot$ , first note that by Remark 4.5

$$\phi(z) \cdot \psi(z) = (s \cdot r)_N(z) + (p_N + q_N)(z) \cdot \omega(z), \quad z \in \sigma(\Theta(N)).$$

Here we have  $\omega(z) = s(z)h(z) + r(z)g(z) + g(z)h(z)(p(\operatorname{Re} z) + q(\operatorname{Im} z))$  for  $z \in \sigma(\Theta(N)) \setminus (Z_p^{\mathbb{R}} + iZ_q^{\mathbb{R}})$  and  $\omega(z)_j = s(z)g(z)_j + r(z)h(z)_j$ ,  $j = 1, 2$  for  $z \in \sigma(\Theta(N)) \cap (Z_p^{\mathbb{R}} + iZ_q^{\mathbb{R}})$  because  $a, b \in \ker \pi$  implies  $a \cdot b = 0$  and, in turn,  $(p_N + q_N)(z) \cdot (p_N + q_N)(z) = 0$  for  $z \in \sigma(\Theta(N)) \cap (Z_p^{\mathbb{R}} + iZ_q^{\mathbb{R}})$ .

On the other hand, by Lemma 5.10 in [KP] we have  $\Xi(D)C = \Xi(D\Theta(C))$ ,  $C\Xi(D) = \Xi(\Theta(C)D)$ , and  $\Xi(D_1)\Xi(D_2) = \Xi(D_1D_2T^*T)$ , where  $T^*T = p(A) + q(B)$ . Hence,

$$\begin{aligned} \phi(N) \psi(N) &= \\ s(A, B) r(A, B) + \Xi \left( \int_{\sigma(\Theta(N))}^{R_1, R_2} g dE \right) r(A, B) + s(A, B) \Xi \left( \int_{\sigma(\Theta(N))}^{R_1, R_2} h dE \right) + \\ &= \Xi \left( \int_{\sigma(\Theta(N))}^{R_1, R_2} g dE \right) \Xi \left( \int_{\sigma(\Theta(N))}^{R_1, R_2} h dE \right) = \\ (s \cdot r)(A, B) + \Xi \left( \int_{\sigma(\Theta(N))}^{R_1, R_2} (g \cdot r + h \cdot s) dE + \right. \\ &\quad \left. \int_{\sigma(\Theta(N)) \setminus (Z_p^{\mathbb{R}} + iZ_q^{\mathbb{R}})} (p(\operatorname{Re}(\cdot)) + q(\operatorname{Im}(\cdot))) \cdot h \cdot g dE \right) = \end{aligned}$$

$$(s \cdot r)(A, B) + \Xi \left( \int_{\sigma(\Theta(N))}^{R_1, R_2} \omega dE \right).$$

Here  $\omega$  is bounded and measurable on  $\sigma(\Theta(N)) \setminus (Z_p^{\mathbb{R}} + iZ_q^{\mathbb{R}})$  and, using the fact that  $\mathcal{R}$  is an ideal,

$$\phi \cdot \psi - (s \cdot r)_N = (\phi - s_N) \cdot \psi + (\psi - r_N) \cdot s_N \in \mathcal{R}.$$

Hence, we again obtain from Theorem 5.3 that  $\phi(N) \cdot \psi(N) = (\phi \cdot \psi)(N)$ .

Finally, we shall show that  $\phi(N) \in \{N, N^*\}''$ . Clearly,  $s(A, B) \in \{A, B\}'' = \{N, N^*\}''$ . If  $C \in \{A, B\}' \subseteq (p(A) + q(B))' = (TT^*)'$ , then  $\Theta(C) \in \{\Theta(A), \Theta(B)\}'$  because  $\Theta$  is a homomorphism. By the spectral theorem for normal operators  $\Theta(C)$  commutes with

$$D := \int_{\sigma(\Theta(N))}^{R_1, R_2} g dE.$$

According to Lemma 5.10 in [KP] we then get

$$\Xi(D)C = \Xi(D\Theta(C)) = \Xi(\Theta(C)D) = C\Xi(D).$$

Hence,  $\Xi(D) \in \{A, B\}'' = \{N, N^*\}''$ , and altogether  $\phi(N) \in \{A, B\}'' = \{N, N^*\}''$ .  $\square$

*5.5 Remark.* For  $\zeta \in Z^i$  or for an isolated  $\zeta \in \sigma(\Theta(N)) \cup (Z_p^{\mathbb{R}} + iZ_q^{\mathbb{R}})$  we saw in Example 4.12 that  $a\delta_\zeta \in \mathcal{F}_N$ . If  $a$  is the unite  $e \in \mathfrak{C}(\zeta)$  (see Remark 4.2), then  $(e\delta_\zeta) \cdot (e\delta_\zeta) = (e\delta_\zeta)$  together with Theorem 5.4 shows that  $(e\delta_\zeta)(N)$  is a projection. It is a kind of Riesz projection corresponding to  $\zeta$ .

We set  $\xi := \operatorname{Re} \zeta$ ,  $\eta := \operatorname{Im} \zeta$  if  $\zeta \in \sigma(\Theta(N)) \cup (Z_p^{\mathbb{R}} + iZ_q^{\mathbb{R}})$  and  $(\xi, \eta) := \zeta$  if  $\zeta \in Z^i$ . For  $\lambda \in \mathbb{C} \setminus \{\xi + i\eta\}$  and for  $s(z, w) := z + iw - \lambda$  we then have  $s_N \cdot (e\delta_\zeta) = (s_N(\zeta))\delta_\zeta$ , where the entry  $s(\xi, \eta)$  of  $s_N(\zeta)$  with index  $(0, 0)$  does not vanish. By Remark 4.2 it therefore has a multiplicative inverse  $b \in \mathfrak{C}(\zeta)$ . We then obtain

$$s_N \cdot (e\delta_\zeta) \cdot (b\delta_\zeta) = e\delta_\zeta.$$

From  $s_N(N) = N - \lambda$  we then get that  $N|_{\operatorname{ran}(e\delta_\zeta)(N)} - \lambda$  has  $(b\delta_\zeta)(N)|_{\operatorname{ran}(e\delta_\zeta)(N)}$  as its inverse operator. Thus,  $\sigma(N|_{\operatorname{ran}(e\delta_\zeta)(N)}) \subseteq \{\xi + i\eta\}$ .  $\diamond$

**5.6 Lemma.** *If for  $\phi \in \mathcal{F}_N$  we have  $\phi(z) = 0$  for all*

$$z \in (\sigma(\Theta(N)) \cup ((Z_p^{\mathbb{R}} + iZ_q^{\mathbb{R}}) \cap \sigma(N))) \dot{\cup} \{(\alpha, \beta) \in Z^i : \alpha + i\beta, \bar{\alpha} + i\bar{\beta} \in \sigma(N)\},$$

*then  $\phi(N) = 0$ .*

*Proof.* Since any  $\zeta \in (Z_p^{\mathbb{R}} + iZ_q^{\mathbb{R}}) \setminus \sigma(N)$  is isolated in  $\sigma(\Theta(N)) \cup (Z_p^{\mathbb{R}} + iZ_q^{\mathbb{R}})$ , we saw in Remark 5.5 that for

$$\zeta \in \underbrace{((Z_p^{\mathbb{R}} + iZ_q^{\mathbb{R}}) \setminus \sigma(N))}_{=: Z_1} \dot{\cup} \underbrace{\{(\alpha, \beta) \in Z^i : \alpha + i\beta \in \rho(N)\}}_{=: Z_2}$$

the expression  $(e\delta_\zeta)(N)$  is a bounded projection commuting with  $N$ . Hence,  $(e\delta_\zeta)(N)$  also commutes  $(N - (\xi + i\eta))^{-1}$ , where  $\xi := \operatorname{Re} \zeta$ ,  $\eta := \operatorname{Im} \zeta$  if  $\zeta \in Z_1$  and  $(\xi, \eta) := \zeta$  if  $\zeta \in Z_2$ .

Consequently,  $N|_{\text{ran}(e\delta_\zeta)(N)} - (\xi + i\eta)$  is invertible on  $\text{ran}(e\delta_\zeta)(N)$ , i.e.  $\xi + i\eta \notin \sigma(N|_{\text{ran}(e\delta_\zeta)(N)})$ . By Remark 5.5 we have  $\sigma(N|_{\text{ran}(e\delta_\zeta)(N)}) \subseteq \{\xi + i\eta\}$ . Hence,  $\sigma(N|_{\text{ran}(e\delta_\zeta)(N)}) = \emptyset$ , which is impossible for  $\text{ran}(e\delta_\zeta)(N) \neq \{0\}$ . Thus,  $(e\delta_\zeta)(N) = 0$ .

For  $(\xi, \eta) \in Z_3 := \{(\alpha, \beta) \in Z^i : \bar{\alpha} + i\bar{\beta} \in \rho(N)\}$  we get  $(\bar{\xi}, \bar{\eta}) \in Z_2$ . Hence,

$$0 = (e\delta_{(\bar{\xi}, \bar{\eta})})(N)^* = (e\delta_{(\xi, \eta)})(N).$$

By our assumption  $\phi$  is supported on  $Z_1 \cup Z_2 \cup Z_3$ . Hence,

$$\phi(N) = \left( \sum_{\zeta \in Z_1 \cup Z_2 \cup Z_3} \phi(\zeta)\delta_\zeta \right)(N) = \sum_{\zeta \in Z_1 \cup Z_2 \cup Z_3} \phi(\zeta)(e\delta_\zeta)(N) = 0.$$

□

*5.7 Remark.* As a consequence of Lemma 5.6 for  $\phi \in \mathcal{F}_N$  the operator  $\phi(N)$  only depends on  $\phi$ 's values on

$$\sigma_N := \left( \sigma(\Theta(N)) \cup ((Z_p^{\mathbb{R}} + iZ_q^{\mathbb{R}}) \cap \sigma(N)) \right) \dot{\cup} \{(\alpha, \beta) \in Z^i : \alpha + i\beta, \bar{\alpha} + i\bar{\beta} \in \sigma(N)\}$$

Thus, we can re-define the function class  $\mathcal{F}_N$  for our functional calculus so that the elements  $\phi$  of  $\mathcal{F}_N$  are functions on this set with  $\phi(z) \in \mathfrak{C}(z)$  such that  $z \mapsto \phi(z)$  is measurable and bounded on  $\sigma(\Theta(N) \setminus (Z_p^{\mathbb{R}} + iZ_q^{\mathbb{R}}))$  and such that (4.1) is bounded locally at  $w$  for all  $w \in \sigma(\Theta(N) \cap (Z_p^{\mathbb{R}} + iZ_q^{\mathbb{R}}))$ .  $\diamond$

**5.8 Lemma.** *If  $\phi \in \mathcal{F}_N$  is such that  $\phi(z)$  is invertible in  $\mathfrak{C}(z)$  (see Remark 4.2) for all  $z \in \sigma_N$  and such that 0 does not belong to the closure of  $\phi(\sigma(\Theta(N)) \setminus (Z_p^{\mathbb{R}} + iZ_q^{\mathbb{R}}))$ , then  $\phi(N)$  is a boundedly invertible operator on  $\mathcal{K}$ .*

*Proof.* We think of  $\phi$  as a function on  $(\sigma(\Theta(N)) \cup (Z_p^{\mathbb{R}} + iZ_q^{\mathbb{R}})) \dot{\cup} Z^i$  by setting  $\phi(z) = e$  (see Remark 4.2) for all  $z$  not belonging to  $\sigma_N$ . Then all assumptions of Lemma 4.15 are satisfied. Hence  $\phi^{-1} \in \mathcal{F}_N$ , and we conclude from Theorem 5.4 and Example 4.6 that

$$\phi^{-1}(N)\phi(N) = \phi(N)\phi^{-1}(N) = (\phi \cdot \phi^{-1})(N) = \mathbf{1}_N(N) = I.$$

□

**5.9 Corollary.** *If  $N$  is a definitizable normal operator on the Krein space  $\mathcal{K}$ , then  $\sigma(N)$  equals to*

$$\sigma(\Theta(N)) \cup ((Z_p^{\mathbb{R}} + iZ_q^{\mathbb{R}}) \cap \sigma(N)) \cup \{\alpha + i\beta : (\alpha, \beta) \in Z^i, \alpha + i\beta, \bar{\alpha} + i\bar{\beta} \in \sigma(N)\} \quad (5.6)$$

*Proof.* Since  $\Theta$  is a homomorphism, we have  $\sigma(\Theta(N)) \subseteq \sigma(N)$ . Hence, (5.6) is contained in  $\sigma(N)$ .

For the converse, consider the polynomial  $s(z, w) = z + iw - \lambda$  for a  $\lambda$  not belonging to (5.6). We conclude that for any  $z \in \sigma_N$  the first entry  $(s_N(z))_{0,0}$  of  $s_N(z) \in \mathfrak{C}(z)$  does not vanish, i.e. is invertible in  $\mathfrak{C}(z)$ .  $(s_N(\sigma(\Theta(N))))_{0,0} = \sigma(\Theta(N)) - \lambda$  being compact, 0 does not belong to the closure of  $s_N(\sigma(\Theta(N)) \setminus (Z_p^{\mathbb{R}} + iZ_q^{\mathbb{R}}))$ .

Applying Lemma 5.8 we see that  $s_N(N) = (N - \lambda)$  is invertible.  $\square$

**5.10 Corollary.** For  $\phi \in \mathcal{F}_N$  we have

$$\sigma(\phi(N)) \subseteq \overline{\phi(\sigma_N)_{0,0}}.$$

*Proof.* For  $\lambda \notin \overline{\phi(\sigma_N)_{0,0}}$  and any  $z \in \sigma_N$  we have  $(\phi(z) - \lambda \mathbf{1}_N(z))_{0,0} = \phi(z)_{0,0} - \lambda \neq 0$ . Hence  $\phi(z) - \lambda \mathbf{1}_N(z)$  is invertible in  $\mathfrak{C}(z)$ .

Moreover, 0 does not belong to the closure of  $\phi(\sigma(\Theta(N)) \setminus (Z_p^{\mathbb{R}} + iZ_q^{\mathbb{R}})) - \lambda = (\phi - \lambda \mathbf{1}_N)(\sigma(\Theta(N)) \setminus (Z_p^{\mathbb{R}} + iZ_q^{\mathbb{R}}))_{0,0}$ . Therefore, we can apply Lemma 5.8 to  $\phi - \lambda \mathbf{1}_N$ , and get  $\lambda \in \rho(\phi(N))$ .  $\square$

*5.11 Remark.* For any characteristic function  $\mathbf{1}_\Delta$  of a Borel subset  $\Delta \subseteq \mathbb{C}$  such that  $(Z_p^{\mathbb{R}} + iZ_q^{\mathbb{R}}) \cap \sigma(N) \cap \partial_{\mathbb{C}}\Delta = \emptyset$  the function  $(\mathbf{1}_{\tau(\Delta)})_N$  belongs to  $\mathcal{F}_N$ ; see Definition 4.4 and Lemma 4.14. Since this function is idempotent and satisfies  $(\mathbf{1}_{\tau(\Delta)})_N^\# = (\mathbf{1}_{\tau(\Delta)})_N$ ,  $(\mathbf{1}_{\tau(\Delta)})_N(N)$  is a bounded and self-adjoint projection on the Krein space  $\mathcal{K}$ . These projections constitute the family of spectral projections for  $N$ .  $\diamond$

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