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An Application of Gröbner Bases to perturbed Polynomial Equations

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Abstract

The interplay of symbolic and numerical calculations in polynomial algebra is featured in the seminal monograph (4) by H.J. Stetter. We present a contribution to this field in form of an ideal-theoretic approach for determining solutions of an a perturbed polynomial system which are close to the zeros of the system subject to perturbation. Algorithmically, identifying these zeros is of special interest if the unperturbed system has a nontrivial solution manifold while the perturbed system admits only isolated zeros.

1. Introduction

H.J. Stetter, in (4) and (5), considers a polynomial system of equations with complex coefficients

$$p_i(x_1,\ldots,x_n) + \epsilon q_i(x_1,\ldots,x_n) = 0, \quad i = 1,\ldots,n,$$

and discusses how to retrieve information for the zeros of the polynomial system for small value(s) of ϵ from the zeros of the unperturbed system ($\epsilon = 0$). In (5), the case of particular interest is described: 'If the number of polynomials in a (non-redundant) polynomial system equals or exceeds the number of variables, the zero set is generally 0-dimensional or empty. [But:] *If such a system has a positive-dimensional zero manifold, this manifold disappears under almost all perturbations of the system.*'

In the latter case such a polynomial system is called singular. Solutions of the

*Dedicated to Hans J. Stetter on the occasion of his 85th birthday.

perturbed system for $\epsilon \approx 0$ close to a solution manifold Z of the unperturbed system are necessarily ill-conditioned and therefore difficult to approximate numerically. In (4) and (5), a method is proposed based on linearization about nearby points on Z, so-called 'd-points' in the terminology of (5, Definition 3.2).

In this paper we study an ideal-theoretic approach for finding zeros of an unperturbed system close to zeros of the perturbed system which is based on an elimination procedure using Gröbner basis computations.

Example 1.1: Given are the polynomials

$$f_1 := x_1^2 x_2 - x_2^2 + \epsilon,$$
$$f_2 := x_1^2 x_2 - x_1^4.$$

One is interested, for small nonzero ϵ, to compute the zeros of the system. For $\epsilon = 0$ one finds the solution set

$$Z = \{(t, t^2) \mid t \in \mathbb{C}\},$$

i.e., the given system admits a one-dimensional solution manifold; it is singular in the terminology of (4). On the other hand, for small nonzero ϵ one finds the solution set

$$Z' = Z'(\epsilon) = \{(0, \sqrt{\epsilon}) \mid \epsilon \text{ small}\}.$$

Only the origin $(0, 0)$ is a zero of the unperturbed system which has zeros of the perturbed system arbitrarily close.

In general, the zeros of the unperturbed system should approximate the perturbed ones. But then, as the example shows, it is indeed not necessarily true that every zero of the unperturbed system *does* approximate a zero of the unperturbed system. It is even so that the perturbation makes solutions of the unperturbed system disappear – as they cannot be approximated like points on the the parabola Z in Example 1.1 with exception of the origin. Therefore, also in this situation, one is in need to understand which zeros approximate zeros of the perturbed system.

Let us turn to a description of our problem in more generality.

Suppose that for $i = 1, \ldots, m$ there are given polynomials $p_i \in \mathbb{C}[x, y]$, where $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_s)$ are lists of indeterminates. Later we shall assume that the variables y_i will take small complex values w_i, i.e., they are *parameters* that encode *perturbation* of the system, while the coefficients of the p_i are considered to be *exact* (i.e., they are algebraic numbers and thus representable in any computer algebra system).

Let $J = \langle p_i \mid i = 1, \ldots, m \rangle$ be the ideal in $\mathbb{C}[x, y]$ generated by the polynomials p_i. We are interested in finding an ideal $J_0 \lhd \mathbb{C}[x]$ such that $\mathcal{Z}(J_0)$ (its zero set) coincides with the set

$$Z_0 := \{z \in \mathbb{C}^n \mid \forall\, \epsilon > 0 \;\; \exists\, (z_\epsilon, w_\epsilon) \in \mathbb{C}^n \times \mathbb{C}^s$$
$$\text{with } |w_\epsilon| < \epsilon \text{ and } p_i(z_\epsilon, w_\epsilon) = 0, \quad i = 1, \ldots, m\}.$$

Thus Z_0 is precisely the set of all zeros of the system for $w = 0$ which have arbitrarily close a zero of the perturbed system.

In Example 1.1 we have $x = (x_1, x_2)$ and $y = \epsilon$. The desired solution set Z_0 turned out to consist only of the origin, and the rest of the parabola Z needs to be *discarded.*

Observe that the set Z_0 defined above satisfies

$$Z_0 = \overline{\mathcal{Z}(J) \setminus (\mathbb{C}^n \times \{0\})} \cap (\mathbb{C}^n \times \{0\}).$$

Here the closure is taken with respect to the euclidean topology, which – we are dealing with zeros of polynomials having coefficients in $\mathbb{C}$ – coincides with the Zariski-closure of the same set. A set is Zariski closed if it is the zero set of a collection of polynomials. It should be noted that every Z-open set in $\mathbb{C}^m$ is open in the euclidean topologogy. Therefore we allow ourselves to consider in this note the following purely algebraic problem, which we shall solve by using Gröbner bases techniques as described e.g. in (2; 3):

PROBLEM 1.1: *Let k be an algebraically closed field, and $I \lhd k[x, y]$. Determine an ideal $J_0 \lhd k[x]$ such that for*

$$Z_0 := \overline{\mathcal{Z}(J) \setminus \mathcal{Z}(\langle y \rangle)} \cap \mathcal{Z}(\langle y \rangle)$$

one has $\mathcal{Z}(J_0) \times \{0\} = Z_0$.

The authors remember stimulating discussions with H.J. Stetter on this subject as well as on work of Cayley, Perron, and particularly for having shared his view of numerically approximating rather than dealing with exact arithmetic.

In the following we propose a symbolic method, based on Gröbner basis computations, for filtering out 'nonapproximating zeros' in the sense discussed above. Section 2 provides the theoretical setting. In Section 3 we realize some examples using computer algebra.

2. Ideal theoretic elimination of nonapproximating zeros

Let us fix notation. For any ring R, the ideal generated by a subset P of R will be denoted by $\langle P \rangle_R$. Furthermore, let $R[y, y^{-1}]$ denote the ring consisting of polynomials in y and inverses from y with coefficients in R. Elements of $R[y, y^{-1}]$ are the usual *Laurent polynomials.*

As before, let k be an algebraically closed field. Let $k[x, y]$ denote the polynomial ring over k in $n + s$ indeterminates $x = (x_1, \ldots, x_n) \in k^n$ and $y = (y_1, \ldots, y_s) \in k^s$. For a subset $P \subseteq k[x, y]$, let $\mathcal{Z}(P)$ denote the set of its common zeros, i.e.,

$$\mathcal{Z}(P) := \{(x, y) \in k^n \times k^s \mid p(x, y) = 0 \text{ for every } p \in P\}.$$

For a subset $S \subseteq k^n \times k^s$ let $\mathcal{I}(S)$ denote its vanishing ideal in $k[x,y]$, i.e.,

$$\mathcal{I}(S) := \{p \in k[x,y] \mid p(x,y) = 0 \text{ for every } (x,y) \in S\}.$$

For example, $\mathcal{I}(k^n \times \{0\}) = \langle y \rangle_{k[x,y]} = yk[x,y]$. Also observe that $k^n \times \{0\} = \mathcal{Z}(\langle y \rangle)$.

The *Zariski-closure* $\bar{S}$ of a subset S of $k^n \times k^s$ is defined to be $\bar{S} := \mathcal{Z}(\mathcal{I}(S))$. These sets are closed in the *Zariski topology* (see (1)). As mentioned above, this is the topology on $k^n \times k^s$ whose closed sets are exactly of the form $\mathcal{Z}(P)$ for some $P \subseteq k[x,y]$. It is helpful to recall that all polynomials in $k[x,y]$ are continuous w.r.t. this topology, i.e., whenever $p(S) = 0$ for a subset S of $k^n \times k^s$ then also $p(\bar{S}) = 0$.

When $s = 1$ we shall need to pass from $k[x,y]$ to a ring extension containing the inverse y^{-1} of y, namely $k[x,y,y^{-1}]$ which has elements

$$p = \sum_j p_j y^j \tag{1}$$

for j in a finite set of integers and $p_j \in k[x]$. Formally one may describe $k[x,y,y^{-1}]$ as the factor ring

$$k[x,y,z]/\langle yz - 1 \rangle_{k[x,y,z]},$$

that is, one computes with polynomials in $k[x,y,z]$ and observes the identity $yz = 1$ corresponding to passing to the factor ring. In other words, $p \in k[x,y,y^{-1}]$ is identified with the polynomial $r \in k[x,y,z]/\langle yz-1 \rangle_{k[x,y,z]}$ which is obtained as the remainder after dividing the expression $p(x,y,z)$ by $yz - 1$ formally with respect to the variable z; see Proposition 2.1 below. Then $k[x,y]$ appears naturally as a subring. Let us remark for later that every element $p \in k[x,y,y^{-1}]$ allows multiplication by some power y^r with positive r to become an element in $k[x,y]$.

For a given ideal $J \lhd k[x,y]$ set

$$J_y := \langle J \rangle_{k[x,y,y^{-1}]} \quad \text{and} \quad \tilde{J} := \langle J_y \cap k[x,y], y \rangle_{k[x,y]} \tag{2}$$

and note for later that

$$\tilde{J} = \langle J_y \cap k[x,y] \rangle_{k[x,y]} + yk[x,y] \tag{3a}$$

is a sum of ideals in $k[x,y]$, that is, every element $\tilde{p} \in \tilde{J}$ is of the form

$$\tilde{p} = p + yq, \quad \text{for suitable } p \in J_y \cap k[x,y] \text{ and } q \in k[x,y]. \tag{3b}$$

In order to provide a solution to Problem 1.1, see Corollary 2.1 below, we still need some preparations. From now on $s = 1$, i.e., $y = y_1$.

Lemma 2.1: *Let k be an algebraically closed field and $J \lhd k[x,y]$. Then, with respect to the Zariski closure,*

$$\overline{\mathcal{Z}(J) \setminus (k^n \times \{0\})} \cap (k^n \times \{0\}) = \mathcal{Z}(\langle J_y \cap k[x,y], y \rangle_{k[x,y]}).$$

Proof: '$\subseteq$': Pick $(a, 0)$ from the l.h.s. and let us show that it is contained in the r.h.s.. Recall from Eq. 2 that the r.h.s. is just $\mathcal{Z}(\tilde{J})$. Hence one needs to prove that for every $\tilde{p} \in \tilde{J}$ one has $p(a, 0) = 0$. From Eq. 3b we see that there exist $p \in J_y \cap k[x, y]$ and $q \in k[x, y]$ with

$$\tilde{p} = p + yq.$$

Since $(a, 0)$ certainly is a zero of yq it suffices to prove that it also is a zero of p. Now take, for the moment, an arbitrary zero $(b, \eta) \in \mathcal{Z}(J) \setminus (k^n \times \{0\})$, i.e., $(b, \eta) \in \mathcal{Z}(J)$ with $\eta \neq 0$, and observe that $y^r p \in J$ for some nonnegative r. Then certainly $\eta^r p(b, \eta) = 0$, and thus, since $\eta \neq 0$, deduce that $p(b, \eta) = 0$. But (b, η) has been chosen arbitrary in $\mathcal{Z}(J) \setminus (k^n \times \{0\})$ and therefore, by the Zariski continuity of p, every element in the Zariski closure

$$\overline{\mathcal{Z}(J) \setminus (k^n \times \{0\})}$$

is a zero of p as well. In particular $p(a, 0) = 0$ must hold. Hence the l.h.s. is a subset of the r.h.s..

'$\supseteq$': For proving the converse containment we first note that due to Eq. 3b every element from the r.h.s. $\mathcal{Z}(\tilde{J})$ is of the form $(a, 0)$. Picking such an arbitrary $(a, 0)$ we need to prove that every

$$p \in \mathcal{I}\big(\overline{\mathcal{Z}(J) \setminus (k^n \times \{0\})} \cap (k^n \times \{0\})\big)$$

satisfies $p(a, 0) = 0$. Making use of the elementary identity

$$\mathcal{I}\big(S \cap (k^n \times \{0\})\big) = \mathcal{I}(S) + yk[x, y]$$

for $S = \overline{\mathcal{Z}(J) \setminus (k^n \times \{0\})}$, we can decompose p as

$$p = p_0 + yq$$

for some element $p_0 \in \mathcal{I}(\mathcal{Z}(J) \setminus (k^n \times \{0\}))$ and $q \in k[x, y]$. Since $(a, 0)$ is a zero of yq it suffices to show that $p_0(a, 0) = 0$. As $p_0 \in k[x, y]$ there is a nonnegative integer r with $p_0 = y^r p_1$ and y not dividing p_1. For arbitrary $(b, \eta) \in \mathcal{Z}(J) \setminus (k^n \times \{0\})$ we have $p_0(b, \eta) = 0$, and from $\eta \neq 0$ we conclude $p_1(b, \eta) = 0$. Finally, for $b = a$ and due to the Zariski continuity of p_1 we conclude $p_1(a, 0) = 0$ and, in particular, $p_0(a, 0) = 0$. Thus we have established that $p(a, 0) = 0$, as required. □

We come to a statement concerning *elimination*. Recall that $k[x, y, y^{-1}]$ is just the ring extension $k[x, y, z]/\langle yz - 1 \rangle_{k[x,y,z]}$.

PROPOSITION 2.1: *For any ideal $J \lhd k[x,y]$ one has*

$$\langle J\rangle_{k[x,y,y^{-1}]} \,\cap\, k[x,y] = \langle J, yz-1\rangle_{k[x,y,z]} \cap k[x,y].$$

Proof: Any p from the l.h.s. has the form

$$p = \sum_j p_j y^j$$

for j in a finite set of integers and $p_j \in J$. Therefore

$$\tilde{p} := \sum_{j\geq 0} p_j y^j + \sum_{j<0} p_j z^{-j} \in k[x,y,z]$$

agrees modulo $yz-1$ with p and thus belongs to the r.h.s..

The converse containment is proved by a similar argument. □

As an immediate consequence one obtains:

COROLLARY 2.1: *For given $J \lhd k[x,y]$, the ideal*

$$\big(\langle J\rangle_{k[x,y,y^{-1}]} \cap k[x,y] + yk[x,y]\big) \cap k[x]$$

agrees with the ideal

$$\big(\langle J, yz-1\rangle_{k[x,y,z]} \cap k[x,y] + yk[x,y]\big) \cap k[x].$$

In particular, the latter ideal serves as a solution J_0 of Problem 1.1.

Suppose one is given an ideal $J \lhd k[x,y]$ and wants to derive an ideal basis of the *elimination ideal* $J \cap k[x]$. Such a situation arises in the preceding corollary and in the examples below. A method for finding an ideal basis of an elimination ideal consists of using a term ordering on the monomials in the variables x and y such that every monomial containing terms in y is larger than any one not containing some variable from y. E.g., one can use a lexicographic ordering with all variables in x preceding those in y. Compute a Gröbner basis G of J w.r.t. this term ordering. Then $G \cap k[x]$ turns out to be a Gröbner basis of the elimination ideal $J \cap k[x]$ (compare e.g. with (3, p.27)).

EXAMPLE 2.1: Let us explain how to apply the method to Example 1.1. Let $k = \mathbb{C}$, $x = (x_1, x_2)$, and $y = \epsilon$. We have

$$J = \langle f_1, f_2\rangle_{\mathbb{C}[x_1,x_2,\epsilon]} = \langle x_1^4 - x_2^2, x_1^2 x_2 - x_1^2 + \epsilon\rangle_{\mathbb{C}[x_1,x_2,\epsilon]}.$$

A way to express $k[\epsilon,\epsilon^{-1}]$ consists of representing it as the quotient $k[\epsilon,\epsilon^{-1}] = k[\epsilon,\eta]/\langle \epsilon\eta - 1\rangle_{k[\epsilon,\eta]}$. Then $\tilde{J} = \langle f_0, f_1, f_2\rangle_{\mathbb{C}[x_1,x_1,\epsilon,\eta]}$ with $f_0 = \epsilon\eta - 1$.

For computing the intersection $\tilde{J} \cap k[x,y] = \tilde{J} \cap \mathbb{C}[x_1,x_2,\epsilon]$ it is sufficient to find a Gröbner basis G of $\tilde{J}$ w.r.t. to the lexicographic ordering $x_1 < x_2 < \epsilon < \eta$ and look for the polynomials in $G \cap \mathbb{C}[x_1, x_2, \epsilon]$.

A reduced Gröbner basis turns out to be $G = \{\epsilon\eta - 1, x_1^2, \epsilon - x_2^2\}$. Therefore the ideal $\tilde{J} \cap k[x_1, x_2, \epsilon]$ in $k[x_1, x_2, \epsilon]$ has the Gröbner basis $G \cap k[x_1, x_2, \epsilon] = \{x_1^2, \epsilon - x_2^2\}$. Next, according to Corollary 2.1, put $J_0 = \langle x_1^2, \epsilon - x_2^2, \epsilon\rangle_{\mathbb{C}[x_1,x_2,\epsilon]} = \langle x_1^2, x_2^2, \epsilon\rangle_{\mathbb{C}[x_1,x_2,\epsilon]}$. It easily turns out that $J_0 = \langle x_1^2, x_2^2\rangle_{\mathbb{C}[x_1,x_2]}$ and certainly $\mathcal{Z}(J_0)$ consists of the origin $(0, 0)$ alone, as we also found by elementary analysis.

A description of an induction step for generalizing Lemma 2.1 to the situation where y is a list containing more than a single variable follows.

PROPOSITION 2.2: *Suppose that $y = (y', y'')$ is a decomposition of the list y into two sublists. Given $I \triangleleft k[x, y]$, suppose that $J \triangleleft k[x, y]$ is an ideal with*

$$\mathcal{Z}(J) = \overline{\mathcal{Z}(I) \setminus \mathcal{Z}(\langle y'\rangle)} \cap \mathcal{Z}(\langle y'\rangle).$$

Then

$$\overline{\mathcal{Z}(I) \setminus \mathcal{Z}(\langle y\rangle)} \cap \mathcal{Z}(\langle y\rangle) = \mathcal{Z}(\langle J, y\rangle) \cup \overline{\mathcal{Z}(\langle I, y'\rangle) \setminus \mathcal{Z}(\langle y''\rangle)} \cap \mathcal{Z}(\langle y''\rangle).$$

Proof: Claim 1:

$$\begin{aligned}&\overline{\mathcal{Z}(I) \setminus \mathcal{Z}(\langle y\rangle)} \cap \mathcal{Z}(\langle y\rangle)\\ &\quad = \overline{\mathcal{Z}(I) \setminus \mathcal{Z}(\langle y'\rangle)} \cap \mathcal{Z}(\langle y\rangle) \cup \overline{\mathcal{Z}(I) \cap \mathcal{Z}(\langle y'\rangle) \setminus \mathcal{Z}(\langle y''\rangle)} \cap \mathcal{Z}(\langle y''\rangle).\end{aligned}$$

Proof of the claim: First observe that $\mathcal{Z}(\langle y\rangle) = \mathcal{Z}(\langle y'\rangle) \cap \mathcal{Z}(y'')$ implies

$$l.h.s. = (\overline{\mathcal{Z}(I) \setminus \mathcal{Z}(\langle y'\rangle)} \cup \overline{\mathcal{Z}(I) \setminus \mathcal{Z}(\langle y''\rangle)}) \cap \mathcal{Z}(\langle y\rangle). \tag{4}$$

In order to show that $l.h.s. \subseteq r.h.s.$, it suffices to show that $\overline{\mathcal{Z}(I) \setminus \mathcal{Z}(\langle y''\rangle)} \cap \mathcal{Z}(\langle y\rangle) \subseteq r.h.s.$. This follows from the set-theoretic decomposition

$$\mathcal{Z}(I) = (\mathcal{Z}(I) \setminus \mathcal{Z}(\langle y'\rangle)) \cup (\mathcal{Z}(I) \cap \mathcal{Z}(\langle y'\rangle))$$

according to

$$\begin{aligned}&\overline{\mathcal{Z}(I)} \setminus \mathcal{Z}(\langle y''\rangle) \cap \mathcal{Z}(\langle y\rangle)\\ &\quad = (\overline{\mathcal{Z}(I) \setminus \mathcal{Z}(\langle y'\rangle) \setminus \mathcal{Z}(\langle y''\rangle)} \cup (\overline{\mathcal{Z}(I) \cap \mathcal{Z}(\langle y'\rangle) \setminus \mathcal{Z}(\langle y''\rangle)}) \cap \mathcal{Z}(\langle y\rangle)\\ &\quad \subseteq (\overline{\mathcal{Z}(I) \setminus \mathcal{Z}(\langle y'\rangle)} \cap \mathcal{Z}(\langle y\rangle)) \cup (\overline{\mathcal{Z}(I) \cap \mathcal{Z}(\langle y'\rangle) \setminus \mathcal{Z}(\langle y''\rangle)} \cap \mathcal{Z}(\langle y\rangle))\\ &\quad \subseteq r.h.s.\end{aligned}$$

The inclusion $r.h.s. \subseteq l.h.s.$ is obvious.

For proving Proposition 2.2 it now suffices to remark that $\mathcal{Z}(I) \cap \mathcal{Z}(\langle y'\rangle) = \mathcal{Z}(\langle I, y'\rangle)$ and, similarly, $\mathcal{Z}(J) \cap \mathcal{Z}(\langle y\rangle) = \mathcal{Z}(\langle J, y\rangle)$ hold. $\square$

3. Using computer algebra. A nontrivial example

We discuss the realization of three concrete examples in Maple 18[†], using the packages `Groebner` and `PolynomialIdeals`. In particular, `EliminationIdeal` eliminates variables from an ideal using a Gröbner basis computation. `Simplify` simplifies an ideal, i.e., it finds a minimal generating set of polynomials.

Example 3.1: We repeat the calculations from Example 2.1 in Maple.

$$\begin{aligned} f_0 &:= \epsilon\eta - 1, \\ f_1 &:= x_1^2 x_2 - x_2^2 + \epsilon, \\ f_2 &:= x_1^2 x_2 - x_1^4. \end{aligned}$$

We eliminate the variable η from the ideal generated by f_0, f_1, f_2:

```
J:=EliminationIdeal(<f[0],f[1],f[2]>,{x[1],x[2],epsilon});
```

$$\langle x_1^2,\, x_2^2 - \epsilon \rangle$$

Substituting $\epsilon = 0$ and simplifying results in

```
J0:=subs(epsilon=0,J);
```

$$\langle x_1^2,\, x_2^2 \rangle$$

The resulting polynomial system has exactly the solution set $Z_0 = Z'(0) = \{(0,0)\}$.

In the following examples we proceed in an analogous way.

Example 3.2: This is similar to the foregoing example, but for $\epsilon = 0$ some isolated solutions happily survive (these are not close to the solution manifold of the unperturbed system). Given are the polynomials

$$\begin{aligned} f_1 &:= x_1^2 x_2 - x_2^2 + \epsilon, \\ f_2 &:= x_1^2 x_2 - x_2^4. \end{aligned}$$

For $\epsilon = 0$ one finds the solution set

$$Z = \{(t,0) \,|\, t \in \mathbb{C}\} \cup \{(-1,1),(1,1),(-i,-1),(i,-1)\}.$$

On the other hand, for small nonzero ϵ one finds a solution set $Z'(\epsilon)$ consisting of 8 isolated solutions, namely of the type

$$\{(-1+\mathcal{O}(\epsilon), 1+\mathcal{O}(\epsilon)), (1+\mathcal{O}(\epsilon), 1+\mathcal{O}(\epsilon)), (-i+\mathcal{O}(\epsilon), -1+\mathcal{O}(\epsilon)), (i+\mathcal{O}(\epsilon), -1+\mathcal{O}(\epsilon))\}$$

[†]Maple is a Trademark of MapleSoft, Inc.

together with 4 solutions of the form $(\mathcal{O}(\epsilon^{3/4}), \mathcal{O}(\epsilon^{1/2}))$. For $\epsilon \to 0$ this solution set converges to

$$Z_0 := Z'(0) = \{(0,0)\} \cup \{(-1,1), (1,1), (-i,-1), (i,-1)\}.$$

The origin $(0,0)$ is the only point on the solution manifold $\{(t,0) \,|\, t \in \mathbb{C}\}$ of the unperturbed system which has zeros of the perturbed system arbitrarily close.

$$\begin{aligned} f_0 &:= \epsilon\eta - 1, \\ f_1 &:= x_1^2 x_2 - x_2^2 + \epsilon, \\ f_2 &:= x_1^2 x_2 - x_2^4. \end{aligned}$$

We proceed as before:

```
J:=EliminationIdeal(<f[0],f[1],f[2]>,{x[1],x[2],epsilon});
```

$$\langle x_2^3 - x_1^2,\; x_1^2 x_2 - x_2^2 + \epsilon,\; x_1^4 + \epsilon x_2^2 - x_2^2 + \epsilon \rangle$$

```
J0:=Simplify(subs(epsilon=0,J));
```

$$\langle x_2^4 - x_2^2,\; -x_2^3 + x_1^2 \rangle$$

The resulting polynomial system has exactly the solution set $Z_0 = Z'(0)$ indicated above.

EXAMPLE 3.3: This is Example 3.1 from (5), exhibiting a double limit point on the solution manifold for $\epsilon = 0$. Given are the polynomials

$$\begin{aligned} f_1 &:= x_1 x_3 + x_2^2 - x_3 + \epsilon, \\ f_2 &:= x_2 + x_3 + \epsilon, \\ f_3 &:= x_1 x_2 + x_3 + \epsilon. \end{aligned}$$

For $\epsilon = 0$ we find the x_1-axis is a solution manifold,

$$Z = \{(t,0,0) \,|\, t \in \mathbb{C}\}.$$

On the other hand, for small nonzero ϵ one finds a solution set $Z'(\epsilon)$ consisting of three isolated solutions,

$$Z'(\epsilon) = \{(2,0,-\epsilon),\; (1,\sqrt{-\epsilon},-\epsilon-\sqrt{-\epsilon}),\; (1,-\sqrt{-\epsilon},-\epsilon+\sqrt{-\epsilon})\}.$$

For $\epsilon \to 0$ this solution set converges to

$$Z_0 := Z'(0) = \{(2,0,0), (1,0,0)\},$$

where $(1,0,0)$ is a double limit point.

Again we augment the system by $f_0 := \epsilon\eta - 1$ and proceed as before:

```
J:=EliminationIdeal(<f[0],f[1],f[2],f[3]>,{x[1],x[2],x[3],epsilon});
```

$$\langle x_2 + x_3 + \epsilon,\, x_2^3 + \epsilon x_2,\, x_1 x_2 - x_2,\, x_1^2 - 3x_1 + 2,\, \epsilon x_1 - x_2^2 - 2\epsilon \rangle$$

```
J0:=Simplify(subs(epsilon=0,J));
```

$$\langle x_2^2,\, x_2 + x_3,\, x_1 x_2 - x_2,\, x_1^2 - 3x_1 + 2 \rangle$$

The resulting polynomial system has exactly the solution set $Z_0 = Z'(0)$ indicated above.

Example 3.4: This system is discussed in (4) and (5):

$$\begin{aligned} f_1 &:= a x_2^2 x_3^2 + \tfrac{1}{2} x_2^2 + 2 x_2 x_3 + \tfrac{1}{2} x_3^2 - a, \\ f_2 &:= a x_3^2 x_1^2 + \tfrac{1}{2} x_3^2 + 2 x_3 x_1 + \tfrac{1}{2} x_1^2 - a, \\ f_3 &:= a x_1^2 x_2^2 + \tfrac{1}{2} x_1^2 + 2 x_1 x_2 + \tfrac{1}{2} x_2^2 - a. \end{aligned}$$

For (e.g.) the special parameter value $a = a_0 := \frac{1}{2}\sqrt{3}$, the system is singular, i.e., it has a one-dimensional solution manifold Z, plus 4 isolated solutions. Generically, the system has 16 isolated solutions, and for $a \approx a_0$, 12 of these isolated solutions are close to Z.

Again we are interested in identifying those solutions for $a = a_0$ which are close to the solutions of the perturbed system, with $a = a_0 + \epsilon$. Therefore we consider

$$\begin{aligned} f_0 &:= \epsilon\eta - 1, \\ f_1 &:= a_0 x_2^2 x_3^2 + \tfrac{1}{2} x_2^2 + 2 x_2 x_3 + \tfrac{1}{2} x_3^2 - a_0 + \epsilon(x_2^2 x_3^2 - 1), \\ f_2 &:= a_0 x_3^2 x_1^2 + \tfrac{1}{2} x_3^2 + 2 x_3 x_1 + \tfrac{1}{2} x_1^2 - a_0 + \epsilon(x_3^2 x_1^2 - 1), \\ f_3 &:= a_0 x_1^2 x_2^2 + \tfrac{1}{2} x_1^2 + 2 x_1 x_2 + \tfrac{1}{2} x_2^2 - a_0 + \epsilon(x_1^2 x_2^2 - 1). \end{aligned}$$

We eliminate η, set ϵ to 0, and simplify:

```
J:=EliminationIdeal(<f[0],f[1],f[2],f[3]>,{x[1],x[2],x[3],epsilon}):
J0:=Simplify(subs(epsilon=0,J)):
```

After elimination of the auxiliary variable η we obtain the ideal J generated by 22 polynomials. Setting $\epsilon = 0$ this reduces to an ideal J_0 generated by 13 polynomials. For the resulting polynomial system we find 16 isolated solutions, 12 of them lying on the manifold Z, which are close to the solutions of the perturbed system.

The real-valued solutions, for $\epsilon = 0$ and $\epsilon \approx 0$, are displayed in Figure 1 together with the real branch of Z.

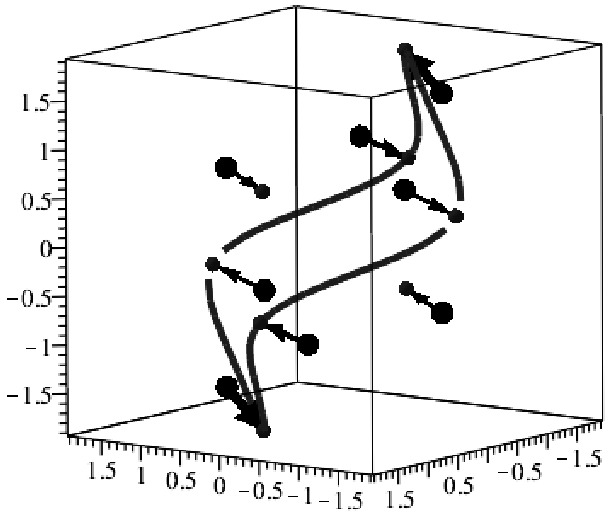

Figure 1: Visualization for Example 3.4: Real-valued solutions of the perturbed system, and associated nearby solutions of the unperturbed system admitting a one-dimensional solution manifold.

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