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Abstract We show that optimal L^2-convergence in the finite element method on quasi-uniform meshes can be achieved if the underlying boundary value problem admits a shift theorem by more than 1/2. For this, the lack of full elliptic regularity in the dual problem has to be compensated by additional regularity of the exact solution. Furthermore, we analyze for a Dirichlet problem the approximation of the normal derivative on the boundary without convexity assumption on the domain. We show that (up to logarithmic factors) the optimal rate is obtained.

Keywords L^2 *a priori* bounds · shift theorem · reentrant corners

1 Introduction

In the finite element method (FEM), the solution of a boundary value problem is approximated by piecewise polynomials of degree k. In the classical case of second order elliptic equations with an H^1-coercive bilinear form, the method is of optimal convergence order in the H^1-norm. An important tool for the convergence analysis in other norms such as the L^2-norm are duality arguments ("Nitsche trick"). The textbook procedure for optimal order convergence in L^2 is to exploit full elliptic regularity for the dual problem. Conversely, this procedure suggests a loss of the optimal convergence rate in L^2 if H^2-regularity fails to hold. This occurs, for example, in polygonal domains with reentrant corners. Nevertheless, it is possible to recover the optimal convergence rate in L^2, if the exact solution has additional regularity. More precisely: In this note, we consider a setting where an elliptic shift theorem holds with a shift $s_0 \in (1/2, 1]$ and show that if the solution is in the Sobolev space $H^{k+1+(1-s_0)}$, then the extra regularity $1 - s_0$ can be exploited

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to recover the optimal convergence rate in L^2 (up to a logarithmic factor in the lowest order case $k = 1$).

In the second part of this note, we consider the convergence in L^2 of the normal derivative on the boundary. We show that the optimal rate $O(h^k)$ (up to a logarithmic factor in the lowest order case) can be achieved, if the solution is sufficiently smooth. The key step for this result for the convergence of the flux is an optimal estimate for the FEM error on a strip of width $O(h)$ near the boundary. Although we present here the convergence of the flux for an H^1-conforming discretization, the techniques are applicable to mixed methods, [10], FEM-BEM coupling, [9], and mortar methods, [11]. In fact, the results of the present work lead to a sharpening of [11], where convexity of the domain was assumed to avoid the analysis of a suitable additional dual problem. The techniques employed here are, of course, similar to those developed in [11]. Nevertheless, they are significantly different since we have opted to forego the direct use of anisotropic norms and instead rely on weighted Sobolev norms and the embedding result of Lemma 2.1.

The analysis of the optimal convergence of fluxes has attracted some attention recently. Besides our own contributions [9–11], we mention the works [2,3,7,17] where similar results have been obtained by different methods.

1.1 Notation

For bounded Lipschitz domains $\Omega \subset \mathbb{R}^d$ with boundary $\Gamma := \partial\Omega$, we employ standard notation for Sobolev spaces $H^s(\Omega)$, [1,14]. We will formulate certain regularized results in terms of Besov space: for $s > 0$, $s \notin \mathbb{N}$, and $q \in [1,\infty]$ the Besov space $B^s_{2,q}(\Omega)$ is defined by interpolation (the "real" method, also known as K-method as described, e.g., in [14,15]) as

$$B^s_{2,q}(\Omega) = (H^{\lfloor s\rfloor}(\Omega), H^{\lceil s\rceil}(\Omega))_{\theta,q}, \qquad \theta = s - \lfloor s\rfloor.$$

Integer order Besov spaces $B^n_{2,q}(\Omega)$ with $n \in \mathbb{N}$ are also defined by interpolation:

$$B^n_{2,q}(\Omega) = (H^{n-1}(\Omega), H^{n+1}(\Omega))_{1/2,q}, \qquad n \in \mathbb{N}.$$

To give some indication of the relevance of the second parameter q in the definition of the Besov spaces, we recall the following (continuous) embeddings:

$$H^{s+\varepsilon}(\Omega) \subset B^s_{2,1}(\Omega) \subset H^s(\Omega) \subset B^s_{2,\infty}(\Omega) \subset H^{s-\varepsilon}(\Omega) \qquad \forall\varepsilon > 0.$$

Of importance will be the distance function δ_Γ and the regularized distance function $\widetilde{\delta}_\Gamma$ given by

$$\delta_\Gamma(x) := \operatorname{dist}(x,\Gamma), \qquad \widetilde{\delta}_\Gamma(x) := h + \operatorname{dist}(x,\Gamma). \tag{1.1}$$

Later on, the parameter $h > 0$ will be the mesh size of the quasi-uniform triangulation. Also of importance will be neighborhoods S_D of the boundary $\partial\Omega$ given by

$$S_D := \{x \in \Omega \,|\, \delta_\Gamma(x) < D\}, \tag{1.2}$$

with particular emphasis on the case $D = O(h)$.

1.2 Model problem

We let $\Omega \subset \mathbb{R}^d$, $d \in \{2,3\}$, be a bounded Lipschitz domain with a polygonal/polyhedral boundary and let (1.3) be our model problem:

$$-\nabla \cdot (\mathbf{A}(x)\nabla u) = f \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega. \tag{1.3}$$

We assume that $\mathbf{A}$ and f are sufficiently smooth. Moreover $\mathbf{A}$ is pointwise symmetric positive definite, and $\mathbf{A}(x) \geq \alpha_0\,\mathrm{I}$ for some $\alpha_0 > 0$ and all $x \in \Omega$. As usual, (1.3) is understood in a weak sense, i.e., for a right-hand side $f \in \left(H_0^1(\Omega)\right)'$ the boundary value problem (1.3) reads: Find $u \in H_0^1(\Omega)$ such that

$$a(u,v) := \int_\Omega \mathbf{A}\nabla u \cdot \nabla v = \langle f, v\rangle \qquad \forall v \in H_0^1(\Omega). \tag{1.4}$$

We denote by $T : (H_0^1(\Omega))' \to H_0^1(\Omega)$ the solution operator. We emphasize that the choice of boundary conditions (here: Dirichlet boundary conditions) is not essential for our purposes. Essential, however, is that we assume a shift theorem by more than 1/2:

Assumption 1.1 *There exists $s_0 \in (1/2, 1]$ such that the solution operator $f \mapsto Tf$ for (1.4) satisfies*

$$\|Tf\|_{H^{1+s_0}(\Omega)} \leq C\|f\|_{(H_0^{1-s_0}(\Omega))'} \leq C\|f\|_{L^2(\Omega)}.$$

Here and in the following $0 < c$, $C < \infty$ denote generic constants that do not depend on the mesh-size but possibly depend on s_0. We also use $\lesssim$ to abbreviate $\leq C$.

Remark 1.2 The present problem is symmetric. As a consequence certain dual problems that will be needed below coincide with the primal problem. This will simplify the presentation but is not essential. Inspection of the procedure below shows that we need the shift theorem for the dual problem as well as a form of the primal problem with weighted right-hand side. ∎

Let $\mathcal{T}$ be an affine quasi-uniform triangulation of Ω with mesh size h and $V_h := S_0^{k,1}(\mathcal{T}) \subset H_0^1(\Omega)$ the continuous space of piecewise polynomials of degree k. This space has the following well-known properties:

(i) Existence of a quasi-local approximation operator: The Scott-Zhang operator $I_h^k : H^1(\Omega) \to S^{k,1}(\mathcal{T})$ of [13] satisfies:
  - If $u \in H_0^1(\Omega)$ then $I_h^k u \in V_h = S_0^{k,1}(\mathcal{T})$.
  - I_h^k is quasi-local and stable: $\|\nabla I_h^k u\|_{L^2(K)} \lesssim \|\nabla u\|_{L^2(\omega_K)}$, where ω_K is the patch of elements sharing a node with K.
  - I_h^k has approximation properties:

$$\|\nabla^j(u - I_h^k u)\|_{L^2(K)} \lesssim h^{l+1-j}\|\nabla^{l+1}u\|_{L^2(\omega_K)}, \quad j \in \{0,1\}, \quad 0 \leq l \leq k. \tag{1.5}$$

(ii) For every $v \in B^{3/2}_{2,\infty}(\Omega) \cap H_0^1(\Omega)$ there holds

$$\inf_{z \in V_h} \|v - z\|_{H^1(\Omega)} \leq h^{1/2}\|v\|_{B^{3/2}_{2,\infty}(\Omega)}.$$

(iii) The space V_h satisfies standard elementwise inverse estimates: for integer $0 \leq j \leq m \leq k$

$$|v|_{H^m(K)} \leq Ch^{-(m-j)}|v|_{H^j(K)} \qquad \forall v \in V_h. \tag{1.6}$$

The Galerkin method for (1.4) is then: Find $u_h \in V_h$ such that

$$a(u_h, v) = \langle f, v\rangle \qquad \forall v \in V_h. \tag{1.7}$$

Remark 1.3 The restriction to affine triangulations is not essential. Our primary motivation for this restriction is that in this case the space V_h features the "superapproximation property" that underlies the local error analysis as presented in [16, Sec. 5.4]. ∎

2 Regularity

2.1 Preliminaries

A key mechanism in our arguments that will allow us to exploit additional regularity of a function is the following embedding theorem.

Lemma 2.1 *The following estimates hold, if $\Omega \subset \mathbb{R}^d$ is a bounded Lipschitz domain and z sufficiently regular.*

$$\|\widetilde{\delta}_\Gamma^{-1/2+\varepsilon} z\|_{L^2(\Omega)} \leq \|\delta_\Gamma^{-1/2+\varepsilon} z\|_{L^2(\Omega)} \leq C_\varepsilon \|z\|_{H^{1/2-\varepsilon}(\Omega)}, \quad \varepsilon \in (0, 1/2], \tag{2.1}$$

$$\|\widetilde{\delta}_\Gamma^{-1/2} z\|_{L^2(\Omega)} \leq C|\ln h|^{1/2}\|z\|_{B^{1/2}_{2,1}(\Omega)}, \tag{2.2}$$

$$\|\widetilde{\delta}_\Gamma^{-1/2-\varepsilon} z\|_{L^2(\Omega)} \leq C_\varepsilon h^{-\varepsilon}\|z\|_{B^{1/2}_{2,1}(\Omega)}, \qquad \varepsilon > 0, \tag{2.3}$$

$$\|z\|_{L^2(S_h)} \leq Ch^{1/2}\|z\|_{B^{1/2}_{2,1}(\Omega)}, \qquad h > 0, \tag{2.4}$$

$$\|z\|_{L^2(\Gamma)} \leq C\|z\|_{B^{1/2}_{2,1}(\Omega)}. \tag{2.5}$$

Proof The estimate involving δ_Γ in (2.1) can be found, e.g., in [4, Thm. 1.4.4.3] and (2.4) is shown in [8, Lemma 2.1]. The estimates (2.2), (2.3), (2.5) follow from 1D Sobolev embedding theorems for L^∞ and locally flattening the boundary Γ in the same way as it is done in the proof of [8, Lemma 2.1]. For example, for (2.5) we note that local flattening the boundary Γ and the 1D embedding $\|v\|^2_{L^\infty(0,1)} \lesssim \|v\|_{L^2(0,1)}\|v\|_{H^1(0,1)}$ implies $\|z\|^2_{L^2(\Gamma)} \lesssim \|z\|_{L^2(\Omega)}\|z\|_{H^1(\Omega)}$). This implies the estimate $\|z\|_{L^2(\Gamma)} \lesssim \|z\|_{B^{1/2}_{2,1}(\Omega)}$ by [14, Lemma 25.3]. □

One of several applications of Lemma 2.1 is that it allows us to transform negative norms into weighted L^2-estimates:

Lemma 2.2 *For $\varepsilon \in (0, 1/2]$ and sufficiently regular z there holds*

$$\|\delta_\Gamma^\beta z\|_{(H^{1/2-\varepsilon}(\Omega))'} \leq C_\varepsilon \|\delta_\Gamma^{\beta+1/2-\varepsilon} z\|_{L^2(\Omega)}, \quad -1 + 2\varepsilon \leq \beta \leq 0, \tag{2.6}$$

$$\|\widetilde{\delta}_\Gamma^{-1} z\|_{(B^{1/2}_{2,1}(\Omega))'} \leq C|\ln h|^{1/2}\|\widetilde{\delta}_\Gamma^{-1/2} z\|_{L^2(\Omega)}. \tag{2.7}$$

Proof Firstly, we show (2.6):

$$\|\delta_\Gamma^\beta z\|_{(H^{1/2-\varepsilon}(\Omega))'} = \sup_{v\in H^{1/2-\varepsilon}(\Omega)} \frac{\langle \delta_\Gamma^\beta z, v\rangle}{\|v\|_{H^{1/2-\varepsilon}(\Omega)}}$$
$$= \sup_{v\in H^{1/2-\varepsilon}(\Omega)} \frac{\langle \delta_\Gamma^{\beta+1/2-\varepsilon} z, \delta_\Gamma^{-1/2+\varepsilon} v\rangle}{\|v\|_{H^{1/2-\varepsilon}(\Omega)}} \le C_\varepsilon \|\delta_\Gamma^{\beta+1/2-\varepsilon} z\|_{L^2(\Omega)},$$

where, in the last step, we employed (2.1) of Lemma 2.1. Secondly, (2.7) follows by the same type of arguments, where the application of (2.1) is replaced with that of (2.2). □

2.2 Regularity

We recall the following variant of interior regularity of elliptic problems:

Lemma 2.3 *Let Ω be a bounded Lipschitz domain and $z \in H^{1+\beta}(\Omega)$, $\beta \in (0,1]$, solve*

$$-\nabla\cdot(\mathbf{A}\nabla z) = f \qquad \text{in } \Omega.$$

Then:

$$\|\delta_\Gamma^{1-\beta}\nabla^2 z\|_{L^2(\Omega)} \le C_\beta \left(\|\delta_\Gamma^{1-\beta} f\|_{L^2(\Omega)} + \|z\|_{H^{1+\beta}(\Omega)}\right).$$

Proof The upper bound follows from local interior regularity for elliptic problems (see [12, Lemma 5.7.2]) and a covering argument. See [6, Lemma A.3] where a closely related result is worked out in detail. □

2.2.1 Refined regularity for polygons and polyhedra

It is worth pointing out that neither the structure of the boundary Γ nor the kind of boundary conditions play a role in Lemma 2.3. One possible interpretation of Lemma 2.3 is that z could lose the H^2-regularity anywhere near Γ. For certain boundary conditions such as homogeneous Dirichlet conditions and piecewise smooth geometries Γ the solution fails to be in H^2 only near the points of nonsmoothness of the geometry. With methods similar to those of Lemma 2.3 one can show the following, stronger result:

Lemma 2.4 *Let Ω be a (curvilinear) polygon in 2D or a (curvilinear) polyhedron in 3D. Denote by $\mathcal{E}$ the set of all vertices of Ω in 2D and the set of all edges of Ω in 3D. Let $\delta_\mathcal{E}$ be the distance from $\mathcal{E}$. Let $z \in H^{1+\beta}(\Omega)$, $\beta \in (0,1]$, solve (1.3). Then*

$$\|\delta_\mathcal{E}^{1-\beta}\nabla^2 z\|_{L^2(\Omega)} \le C_\beta \left(\|\delta_\mathcal{E}^{1-\beta} f\|_{L^2(\Omega)} + \|z\|_{H^{1+\beta}(\Omega)}\right).$$

Proof Follows from local considerations as in Lemma 2.3. The novel aspect is the behavior near the boundary away from the vertices (in 2D) and the edges (in 3D). This is achieved with an adapted covering theorem of the type described in Theorems A.5, A.6. The key feature of these coverings is that they allow us to reduce the considerations to balls $B = B_r(x)$ and stretched balls $\widehat{B} = B_{(1+\varepsilon)r}(x)$ (with fixed $\varepsilon > 0$) with $r \sim \operatorname{dist}(x,\mathcal{E})$ and the following properties: either $x \in \Omega$ with $\widehat{B}_r(x) \subset \Omega$ or $x \in \Gamma$ and $\widehat{B} \cap \Omega$ is a half-ball. Local elliptic regularity assertions can then be employed for each ball B. □

Lemma 2.4 assumes that a loss of H^2-regularity occurs at any point of non-smoothness of Γ. However, the set of "singular" vertices or edges can be further reduced. For example, in 2D for $\mathbf{A} = \mathrm{Id}$, it is well-known that only the vertices of Ω with interior angle greater than π lead to a loss of full H^2-regularity. It will therefore be useful to introduce the closed set M_s of boundary points associated with a loss of H^2-regularity. Before introducing this set, we point out that this set is a subset of the vertices and edges:

Definition 2.5 **(H^2-regular part and singular part of the boundary)** *Let Ω be a polygon (in 2D) or a polyhedron (in 3D) with vertices $\mathcal{A}$ and edges $\mathcal{E}$.*

1. *A vertex $A \in \mathcal{A}$ of Ω is said to be H^2-regular, if there is a ball $B_\varepsilon(A)$ of radius $\epsilon > 0$ such that the solution u of (1.3) satisfies $u|_{B_\varepsilon(A)\cap\Omega} \in H^2(\Omega)$ whenever $f \in L^2(\Omega)$ together with the* a priori *estimate $\|u\|_{H^2(B_\varepsilon(A)\cap\Omega)} \leq C\|f\|_{L^2(\Omega)}$.*
2. *In 3D, an edge $e \in \mathcal{E}$ of Ω with endpoints A_1, A_2 is said to be H^2-regular if the following condition is satisfied: There is $c > 0$ such that for the neighborhood $S = \cup_{x\in e} B_{c\,\mathrm{dist}(x,\{A_1,A_2\})}(x)$ of the edge e we have the regularity assertion $u|_{S\cap\Omega} \in H^2$ for the solution u of (1.3) whenever $f \in L^2(\Omega)$ together with the* a priori *estimate $\|u\|_{H^2(S\cap\Omega)} \leq C\|f\|_{L^2(\Omega)}$.*

Denote by $\mathcal{A}_r \subset \mathcal{A}$ the set of H^2-regular vertices and by $\mathcal{E}_r \subset \mathcal{E}$ the set of H^2-regular edges. Correspondingly, let $\mathcal{A}_s := \mathcal{A} \setminus \mathcal{A}_r$ and $\mathcal{E}_s := \mathcal{E} \setminus \mathcal{E}_r$ be the vertices and edges associated with a loss of H^2-regularity. Define the singular set *M_s as*

$$M_s := \overline{\cup\mathcal{A}_s \bigcup \cup\mathcal{E}_s} \subset \Gamma. \tag{2.8}$$

With the notion of the singular set in hand, we can formulate the following regularity result:

Lemma 2.6 *Let Ω be a polygon or a polyhedron. Let M_s be the singular set as defined in Definition 2.5. Then the following is true for any solution $z \in H_0^1(\Omega)$ of (1.3): If $z \in H^{1+\beta}(\Omega)$ for some $\beta \in (0,1]$, then with $\delta_{M_s} := \mathrm{dist}(\cdot, M_s)$, there holds*

$$\|\delta_{M_s}^{1-\beta}\nabla^2 z\|_{L^2(\Omega)} \leq C_\beta \left(\|\delta_{M_s}^{1-\beta} f\|_{L^2(\Omega)} + \|z\|_{H^{1+\beta}(\Omega)}\right).$$

Proof Follows from local considerations as in Lemma 2.4. Not all vertices and edges (in 3D) are included in the singular set. This is accounted for by a further refinement of the covering employed. We only discuss the 3D situation. Using coverings provided by Theorem A.6, one may restrict the attention to balls $B_r = B_r(x)$ and stretched balls $\widehat{B} = B_{(1+\varepsilon)r}(x)$ (with fixed $\varepsilon > 0$) with $r \sim \mathrm{dist}(x, \mathcal{E})$ where one of the following additional properties is satisfied: a) $x \in \Omega$ with $\widehat{B}_r(x) \subset \Omega$; b) $x \in \mathcal{A}_r$ and $\widehat{B} \cap \Omega$ is a solid angle; c) $x \in \cup\mathcal{E}_r$ and $\widehat{B} \cap \Omega$ is a dihedral angle; d) x lies in the interior of a face and $\widehat{B} \cap \Omega$ is a half-ball. □

2.2.2 Shift theorems for locally supported right-hand sides

We have the following continuity results for the solution operator T for our model problem (1.3):

Lemma 2.7 *Let Assumption 1.1 be valid. Then* $T : (H_0^1(\Omega))' \to H_0^1(\Omega)$ *satisfies*

$$\|Tf\|_{B^{3/2}_{2,\infty}(\Omega)} \leq C\|f\|_{(B^{1/2}_{2,1}(\Omega))'}, \tag{2.9}$$

$$\|Tf\|_{H^{3/2+\varepsilon}(\Omega)} \leq C_\varepsilon \|\delta_\Gamma^{1/2-\varepsilon} f\|_{L^2(\Omega)}, \qquad 0 < \varepsilon \leq s_0 - 1/2. \tag{2.10}$$

In particular, if $f \in L^2(\Omega)$ *with* $\operatorname{supp} f \subset \overline{S_h}$, *then*

$$\|Tf\|_{B^{3/2}_{2,\infty}(\Omega)} \leq Ch^{1/2}\|f\|_{L^2(\Omega)}, \tag{2.11}$$

$$\|Tf\|_{H^{3/2+\varepsilon}(\Omega)} \leq C_\varepsilon h^{1/2-\varepsilon}\|f\|_{L^2(\Omega)}, \qquad 0 < \varepsilon \leq s_0 - 1/2. \tag{2.12}$$

Proof We follow the arguments of [11, Lemma 5.2]. The starting point for the proof of (2.9) is that interpolation and Assumption 1.1 yield with $\theta \in (0,1)$

$$T : ((H_0^{1-s_0}(\Omega))', (H_0^1(\Omega))')_{\theta,\infty} \to (H^{1+s_0}(\Omega), H^1(\Omega))_{\theta,\infty} = B^{1+s_0(1-\theta)}_{2,\infty}(\Omega).$$

Next, we recognize as in [11, Lemma 5.2] (cf. [15, Thm. 1.11.2] or [14, Lemma 41.3])

$$\begin{aligned}((H_0^{1-s_0}(\Omega))', (H_0^1(\Omega))')_{\theta,\infty} &= ((H_0^{1-s_0}(\Omega), H_0^1(\Omega))_{\theta,1})' \\ &\supset ((H^{1-s_0}(\Omega), H^1(\Omega))_{\theta,1})' = (B^{1-s_0(1-\theta)}_{2,1}(\Omega))'.\end{aligned}$$

Setting $\theta = 1 - 1/(2s_0) \in (0, 1/2]$, we get $(B^{1-s_0(1-\theta)}_{2,1}(\Omega))' = (B^{1/2}_{2,1}(\Omega))'$ and $B^{1+s_0(1-\theta)}_{2,\infty}(\Omega) = B^{3/2}_{2,\infty}(\Omega)$. The assertion (2.10) follows from the assumed shift theorem (Assumption 1.1) and (2.6) with $\beta = 0$. For the bound (2.11), we argue as in the proof of Lemma 2.2 and use (2.4), see also [11, Lemma 5.2]. Finally, the proof of (2.12) follows from (2.10) and the assumed support properties of f. □

We will also require mapping properties of the solution operator T in weighted spaces:

Lemma 2.8 *Let Assumption 1.1 be valid. Then for* $v \in L^2(\Omega)$

$$\|T(\widetilde{\delta}_\Gamma^{-1} v)\|_{B^{3/2}_{2,\infty}(\Omega)} \leq C|\ln h|^{1/2}\|\widetilde{\delta}_\Gamma^{-1/2} v\|_{L^2(\Omega)}, \tag{2.13}$$

$$\|T(\widetilde{\delta}_\Gamma^{-1} v)\|_{H^{3/2+\varepsilon}(\Omega)} \leq C_\varepsilon h^{-\varepsilon}\|\widetilde{\delta}_\Gamma^{-1/2} v\|_{L^2(\Omega)}, \quad \varepsilon \in (0, s_0 - 1/2], \tag{2.14}$$

$$\|T(\delta_\Gamma^{-1+2\varepsilon} v)\|_{H^{3/2+\varepsilon}(\Omega)} \leq C_\varepsilon \|\delta_\Gamma^{-1/2+\varepsilon} v\|_{L^2(\Omega)}, \quad \varepsilon \in (0, s_0 - 1/2]. \tag{2.15}$$

Proof The results follow by combining Lemmas 2.2 and 2.7. □

For the analysis of the FEM error on the neighborhood S_h, we need a refined version of interior regularity for elliptic problems. The following result is very similar to [11, Lemma 5.4] and closely related to Lemma 2.3:

Lemma 2.9 *Let* z *solve the equation*

$$-\nabla \cdot (\mathbf{A}\nabla z) = v \quad \text{in } \Omega.$$

Then there exist C, $c_1 > 0$ *such that for* $z \in B^{3/2}_{2,\infty}(\Omega)$, *we have*

$$\|\delta_\Gamma^{1/2}\nabla^2 z\|_{L^2(\Omega\setminus S_h)} \leq C\sqrt{|\ln h|}\|z\|_{B^{3/2}_{2,\infty}(\Omega)} + C\|\sqrt{\delta_\Gamma} v\|_{L^2(\Omega\setminus S_{c_1 h})}. \tag{2.16}$$

If the right-hand side v *is in* $L^2(\Omega)$ *and additionally satisfies* $\operatorname{supp} v \subset \overline{S_h}$ *and* $z = Tv$, *then there are constants* C, $c > 1$, $\tilde{c} > c' > 1$ *independent of* v *such that for all sufficiently small* $h > 0$:

(i) If $z \in B^{3/2}_{2,\infty}(\Omega)$ then $\|\delta_\Gamma^{1/2}\nabla^2 z\|_{L^2(\Omega\setminus S_{\tilde{c}h})} \leq C\sqrt{|\ln h|}\|z\|_{B^{3/2}_{2,\infty}(\Omega)}$.

(ii) For every $\alpha > 0$ there holds

$$\|\delta_\Gamma^\alpha \nabla^3 z\|_{L^2(\Omega\setminus S_{\tilde{c}h})} \leq C\left[\|\delta_\Gamma^{\alpha-1}\nabla^2 z\|_{L^2(\Omega\setminus S_{c'h})} + \|\nabla^2 \mathbf{A}\|_{L^\infty(\Omega)}\|\delta_\Gamma^\alpha \nabla z\|_{L^2(\Omega\setminus S_{c'h})}\right].$$

(iii) If $z \in H^{3/2+\varepsilon}(\Omega)$ for some $\varepsilon \in (0,1/2)$, then for some $C_\varepsilon > 0$ independent of z there holds $\|\nabla^2 z\|_{L^2(\Omega\setminus S_{\tilde{c}h})} \leq C_\varepsilon h^{-1/2+\varepsilon}\|z\|_{H^{3/2+\varepsilon}(\Omega)}$.

(iv) If Assumption 1.1 is valid, then $\|\nabla^2 z\|_{L^2(\Omega\setminus S_{ch})} \leq C\|v\|_{L^2(\Omega)}$.

Proof of (2.16), (i), (ii): [11, Lemma 5.4] is formulated for $-\Delta$. However, the essential property of the differential operator Δ that is required is just interior regularity. Hence, the result also stands for the present, more general elliptic operator (with the appropriate modifications due to the fact that the coefficient $\mathbf{A}$ is allowed to be non-constant). In the interest of generality, we have also tracked in (2.16) the dependence on the right-hand side v, which was not done in [11, Lemma 5.4]. A full proof can be found in [5, Appendix C].

Proof of (iii): This follows again by local considerations similar to those employed in the proof of [11, Lemma 5.4] and the obvious bound $\delta_\Gamma \geq h$ on $\Omega \setminus S_{\tilde{c}h}$. A full proof can be found in [5, Appendix C].

Proof of (iv): In view of (iii), we have to estimate $\|z\|_{H^{3/2+\varepsilon}(\Omega)}$. By the support properties of v, the bound (2.12) yields $\|z\|_{H^{3/2+\varepsilon}(\Omega)} \leq Ch^{1/2-\varepsilon}\|v\|_{L^2(\Omega)}$. Inserting this in (iii) gives the result. □

3 FEM L^2-error analysis

Let u_h be the FEM approximation and denote by $e = u - u_h$ the FEM error. The standard workhorse is the Galerkin orthogonality

$$a(e,v) = a(u - u_h, v) = 0 \qquad \forall v \in V_h. \tag{3.1}$$

We start with a weighted L^2-error:

Lemma 3.1 *Let Assumption 1.1 be valid. Assume that a function $z \in H^1_0(\Omega)$ satisfies the Galerkin orthogonality*

$$a(z,v) = 0 \qquad \forall v \in V_h.$$

Then

$$\|\delta_\Gamma^{-1/2+\varepsilon} z\|_{L^2(\Omega)} \leq C_\varepsilon h^{1/2+\varepsilon}\|z\|_{H^1(\Omega)}, \qquad \varepsilon \in (0, s_0 - 1/2], \tag{3.2}$$

$$\|\widetilde{\delta}_\Gamma^{-1/2} z\|_{L^2(\Omega)} \leq Ch^{1/2}|\ln h|^{1/2}\|z\|_{H^1(\Omega)}. \tag{3.3}$$

Proof The proof follows standard lines. Define $\psi = T(\delta_\Gamma^{-1+2\varepsilon} z)$, which solves

$$\langle v, \delta_\Gamma^{-1+2\varepsilon} z\rangle = a(v,\psi) \qquad \forall v \in H^1_0(\Omega).$$

Then we have by Galerkin orthogonality for arbitrary $I\psi \in V_h$

$$\|\delta_\Gamma^{-1/2+\varepsilon} z\|^2_{L^2(\Omega)} = a(z,\psi) = a(z, \psi - I\psi) \leq C\|z\|_{H^1(\Omega)}\|\psi - I\psi\|_{H^1(\Omega)}.$$

From (2.15) in Lemma 2.8, we have $\|\psi\|_{H^{3/2+\varepsilon}(\Omega)} \leq C_\varepsilon \|\delta_\Gamma^{-1/2+\varepsilon} z\|_{L^2(\Omega)}$ so that with the approximation properties of V_h we get

$$\inf_{I\psi \in V_h} \|\psi - I\psi\|_{H^1(\Omega)} \leq C_\varepsilon h^{1/2+\varepsilon} \|\delta_\Gamma^{-1/2+\varepsilon} z\|_{L^2(\Omega)}.$$

This shows (3.2). For (3.3), we proceed similarly using the regularity assertion (2.13) and the approximation property of V_h. □

Corollary 3.2 *Let Assumption 1.1 be valid and the solution u be in $H^s(\Omega)$, $s \geq 1$. Then the FEM error $e = u - u_h$ satisfies for $\varepsilon \in (0, s_0 - 1/2]$*

$$\|\delta_\Gamma^{-1/2+\varepsilon} e\|_{L^2(\Omega)} \leq C_\varepsilon h^{\mu - 1/2+\varepsilon} \|u\|_{H^\mu(\Omega)}, \qquad \mu := \min\{s, k+1\}.$$

The following Theorem 3.3 shows that the optimal rate of the L^2-convergence of the FEM can be achieved also for non-convex geometries if the solution has some additional regularity:

Theorem 3.3 *Let Assumption 1.1 be valid. Let the exact solution u satisfy the extra regularity $u \in H^{k+2-s_0}(\Omega)$. Then the FEM error $u - u_h$ satisfies*

$$\|u - u_h\|_{L^2(\Omega)} \lesssim h^{k+1} \|u\|_{H^{k+2-s_0}(\Omega)}. \tag{3.4}$$

More generally, if $u \in H^s(\Omega)$, $s \in [1, k+2-s_0]$, then

$$\|u - u_h\|_{L^2(\Omega)} \lesssim h^{s-1+s_0} \|u\|_{H^s(\Omega)}, \quad 1 \leq s \leq k+2-s_0. \tag{3.5}$$

Proof of (3.4): We proceed along a standard duality argument. To begin with, we note that the case $s_0 = 1$ is classical so that we may assume $s_0 < 1$ for the remainder of the proof. Set $\varepsilon := s_0 - 1/2 \in (0, 1/2)$ by our assumption $1/2 < s_0 < 1$. Let $w = Te$ and let $w_h \in V_h$ be its Galerkin approximation. Quasi-optimality gives us the following energy error estimate:

$$\begin{aligned} \|w - w_h\|_{H^1(\Omega)} &\lesssim \inf_{v \in V_h} \|w - v\|_{H^1(\Omega)} \lesssim h^{1/2+\varepsilon} \|w\|_{H^{3/2+\varepsilon}(\Omega)} \\ &\lesssim h^{1/2+\varepsilon} \|e\|_{(H^{1/2-\varepsilon}(\Omega))'} \lesssim h^{1/2+\varepsilon} \|e\|_{L^2(\Omega)}. \end{aligned} \tag{3.6}$$

The Galerkin orthogonalities satisfied by e and $w - w_h$ and a weighted Cauchy-Schwarz inequality yield for the Scott-Zhang interpolant $I_h^k u$

$$\|e\|^2_{L^2(\Omega)} = a(e, w) = a(e, w - w_h) = a(u - I_h^k u, w - w_h) \tag{3.7}$$

$$\leq C \|\widetilde{\delta}_\Gamma^{-1/2+\varepsilon} \nabla(u - Iu)\|_{L^2(\Omega)} \|\widetilde{\delta}_\Gamma^{1/2-\varepsilon} \nabla(w - w_h)\|_{L^2(\Omega)}. \tag{3.8}$$

We get by a covering argument and Lemma 2.1

$$\begin{aligned} \|\widetilde{\delta}_\Gamma^{-1/2+\varepsilon} \nabla(u - I_h^k u)\|_{L^2(\Omega)} &\lesssim h^k \|\widetilde{\delta}_\Gamma^{-1/2+\varepsilon} \nabla^{k+1} u\|_{L^2(\Omega)} \\ &\lesssim h^k \|\nabla^{k+1} u\|_{H^{1/2-\varepsilon}(\Omega)}. \end{aligned} \tag{3.9}$$

It should also be noted at this point that in (3.9), the weight $\widetilde{\delta}_\Gamma^{-1/2+\varepsilon}$ can be considered as constant in each element K. For the contribution $\|\widetilde{\delta}_\Gamma^{1/2-\varepsilon} \nabla(w - w_h)\|_{L^2(\Omega)}$ in (3.8), we have to analyze the Galerkin error $w - w_h$ in more detail, using the techniques from the local error analysis of the FEM. We split Ω into

$S_{ch} \cup \Omega \setminus S_{ch}$ where $c > 0$ will be selected sufficiently large below. For fixed $c > 0$, the L^2-norm on S_{ch} can easily be bounded with (3.6) by

$$\|\widetilde{\delta}_\Gamma^{1/2-\varepsilon}\nabla(w - w_h)\|_{L^2(S_{ch})} \lesssim h^{1/2-\varepsilon}\|\nabla(w - w_h)\|_{L^2(\Omega)} \lesssim h\|e\|_{L^2(\Omega)}. \tag{3.10}$$

The term $\|\widetilde{\delta}_\Gamma^{1/2-\varepsilon}\nabla(w - w_h)\|_{L^2(\Omega\setminus S_{ch})}$ requires more care. Obviously, $\widetilde{\delta}_\Gamma^{1/2-\varepsilon} \leq \delta_\Gamma^{1/2-\varepsilon}$ on $\Omega \setminus S_{ch}$. We have to employ the tools from the local error analysis in FEM. The Galerkin orthogonality satisfied by $w - w_h$ allows us to use techniques as described in [16, Sec. 5.3], which yields the following estimate for arbitrary balls $B_r \subset B_{r'}$ with the same center (implicitly, $r' > r + O(h)$)

$$\|\nabla(w - w_h)\|_{L^2(B_r)} \lesssim \|\nabla(w - I_h^k w)\|_{L^2(B_{r'})} + \frac{1}{r' - r}\|w - w_h\|_{L^2(B_{r'})}. \tag{3.11}$$

By a covering argument, these local estimates can be combined into a global estimate of the following form, where for sufficiently small $c_1 > 0$ (c_1 depends only on Ω and the shape regularity of the triangulation but is independent of h):

$$\begin{aligned}\|\delta_\Gamma^{1/2-\varepsilon}\nabla(w - w_h)\|_{L^2(\Omega\setminus S_{ch})} &\lesssim \\ \|\delta_\Gamma^{1/2-\varepsilon}\nabla(w - I_h^k w)\|_{L^2(\Omega\setminus S_{cc_1h})} &+ \|\delta_\Gamma^{-1/2-\varepsilon}(w - w_h)\|_{L^2(\Omega\setminus S_{cc_1h})}.\end{aligned} \tag{3.12}$$

This estimate implicitly assumed $c_1ch > 2h$, i.e., at least two layers of elements separate Γ from $\Omega \setminus S_{c_1ch}$. We now fix $c > 2/c_1$. The first term in (3.12) can easily be bounded by standard approximation properties of I_h^k, Lemma 2.3, and Assumption 1.1:

$$\begin{aligned}\|\delta_\Gamma^{1/2-\varepsilon}\nabla(w - I_h^k w)\|_{L^2(\Omega\setminus S_{cc_1h})} &\lesssim h\|\delta_\Gamma^{1/2-\varepsilon}\nabla^2 w\|_{L^2(\Omega)} \\ &\lesssim h\left[\|\delta_\Gamma^{1/2-\varepsilon}e\|_{L^2(\Omega)} + \|w\|_{H^{3/2+\varepsilon}(\Omega)}\right] \lesssim h\|e\|_{L^2(\Omega)}.\end{aligned}$$

In the last step, we have to deal with the term $\|\delta_\Gamma^{-1/2-\varepsilon}(w - w_h)\|_{L^2(\Omega\setminus S_{cc_1h})}$ of (3.12). Lemma 3.1 and (3.6) imply

$$\begin{aligned}\|\delta_\Gamma^{-1/2-\varepsilon}(w - w_h)\|_{L^2(\Omega\setminus S_{cc_1h})} &\lesssim h^{-2\varepsilon}\|\delta_\Gamma^{-1/2+\varepsilon}(w - w_h)\|_{L^2(\Omega)} \\ &\lesssim h^{-2\varepsilon}h^{1/2+\varepsilon}\|w - w_h\|_{H^1(\Omega)} \lesssim h\|e\|_{L^2(\Omega)}.\end{aligned} \tag{3.13}$$

Here we have used the quasi-optimality of the Galerkin approximation with respect to the H^1-norm.

Proof of (3.5): The above arguments show that the regularity of u enters in the bound (3.9). For $u \in H^1(\Omega)$, the stability properties of the Scott-Zhang operator I_h^k show

$$\|\widetilde{\delta}_\Gamma^{-1/2+\varepsilon}\nabla(u - I_h^k u)\|_{L^2(\Omega)} \lesssim h^{-1/2+\varepsilon}\|u\|_{H^1(\Omega)}. \tag{3.14}$$

Hence, a standard interpolation argument that combines (3.9) and (3.14) yields $\|\widetilde{\delta}_\Gamma^{-1/2+\varepsilon}\nabla(u - I_h^k u)\|_{L^2(\Omega)} \lesssim h^{-1/2+\varepsilon+s-1}\|u\|_{H^s(\Omega)}$ for $s \in [1, k+2-s_0]$. Combining this estimate with the above control of $\|\widetilde{\delta}_\Gamma^{1/2-\varepsilon}\nabla(w - w_h)\|_{L^2(\Omega)}$ yields the desired bound in the range $s \in [1, k+2-s_0]$. □

4 FEM L^2-error analysis on piecewise smooth geometries

The convergence analysis of Theorem 3.3 did not make *explicit* use of the fact that a piecewise smooth geometry is considered; the essential ingredient was the shift theorem of Assumption 1.1 (which, of course, is related to the geometry of the problem). This is reflected in our use of $\widetilde{\delta}_\Gamma$, which measures the distance from the boundary Γ. One interpretation of this procedure is that one assumes of the dual solution w (and, in fact, also of the solution of the "bidual" problem employed to estimate $\|\widetilde{\delta}_\Gamma^{-1/2+\varepsilon}(w - w_h)\|_{L^2(\Omega)}$ in Theorem 3.3) that it may lose H^2-regularity *anywhere* near Γ. However, for piecewise smooth geometries in conjunction with certain homogeneous boundary conditions (here: homogeneous Dirichlet conditions), this loss of H^2-regularity is restricted to a much smaller set, namely, a subset of vertices in 2D and a subset of the skeleton (i.e., the union of vertices and edges) in 3D. This set is given by M_s in Definition 2.5. For this set M_s, we introduce the distance function

$$\delta_{M_s} := \operatorname{dist}(\cdot, M_s), \qquad \widetilde{\delta}_{M_s} := h + \delta_{M_s}. \tag{4.1}$$

Theorem 4.1 *Let Ω be a polygon (in 2D) or a polyhedron (in 3D). Let M_s be the set of vertices (in 2D) or edges and vertices (in 3D) associated with a loss of H^2-regularity for (1.3) as given in Definition 2.5. Let Assumption 1.1 be valid. Let $Iu \in V_h$ be arbitrary. Then we have*

$$\|u - u_h\|_{L^2(\Omega)} \leq h\|\widetilde{\delta}_{M_s}^{s_0-1}\nabla(u - Iu)\|_{L^2(\Omega)}.$$

Proof We may assume $s_0 < 1$ since the case $s_0 = 1$ corresponds to the standard duality argument with full elliptic regularity and set $\varepsilon := s_0 - 1/2 \in (0, 1/2)$. The key observation is that, starting from the duality argument (3.7), one can replace the weight function $\widetilde{\delta}_\Gamma^{-1/2+\varepsilon}$ in (3.8) with *any* weight function. Taking as the weight function $\widetilde{\delta}_{M_s}^{-1/2+\varepsilon}$, we get

$$\|e\|^2_{L^2(\Omega)} \leq \|\widetilde{\delta}_{M_s}^{-1/2+\varepsilon}\nabla(u - Iu)\|_{L^2(\Omega)}\|\widetilde{\delta}_{M_s}^{1/2-\varepsilon}\nabla(w - w_h)\|_{L^2(\Omega)}. \tag{4.2}$$

The estimate of $w - w_h$ in the weighted norm is done similarly as in the proof of Theorem 3.3, taking into account the improved knowledge of the regularity of w. With $S_{M_s,ch} := \{x \in \Omega \,|\, \delta_{M_s}(x) < ch\}$ we have the trivial bound

$$\begin{aligned} &\|\widetilde{\delta}_{M_s}^{1/2-\varepsilon}\nabla(w - w_h)\|_{L^2(\Omega)} \\ &\quad\lesssim \|\widetilde{\delta}_{M_s}^{1/2-\varepsilon}\nabla(w - w_h)\|_{L^2(S_{M_s,ch})} + \|\widetilde{\delta}_{M_s}^{1/2-\varepsilon}\nabla(w - w_h)\|_{L^2(\Omega\setminus S_{M_s,ch})}, \end{aligned} \tag{4.3}$$

where the parameter c will be selected sufficiently large below. The first term in (4.3) is estimated in exactly the same way as in (3.10) and produces $\|\widetilde{\delta}_{M_s}^{1/2-\varepsilon}\nabla(w - w_h)\|_{L^2(S_{M_s,ch})} \leq Ch\|e\|_{L^2(\Omega)}$. The second term in (4.2) again requires the techniques from the local error analysis of the FEM, this time with the appropriate modifications to account for the boundary conditions. Inspection of the arguments in [16, Sec. 5.3] shows that the key estimate (3.11) extends up to the boundary in the following sense:

$$\|\nabla(w - w_h)\|_{L^2(B_r\cap\Omega)} \lesssim \|\nabla(w - I_h^k w)\|_{L^2(B_{r'}\cap\Omega)} + \frac{1}{r' - r}\|w - w_h\|_{L^2(B_{r'}\cap\Omega)}; \tag{4.4}$$

besides the implicit assumption $r' > r + O(h)$, the balls B_r and $B_{r'}$ are assumed to have the same center x and satisfy one of the following conditions:

1. $B_{r'} = B_{r'}(x) \subset \Omega$;
2. $x \in \partial\Omega$ and $B_{r'}(x) \cap \Omega$ is a half-disk;
3. x is a vertex of Ω;
4. (only for $d = 3$) x lies on an edge e and $B_{r'}(x) \cap \Omega$ is a dihedral angle (i.e., the intersection of $\partial(B_{r'}(x) \cap \Omega)$ with $\partial\Omega$ is contained in the two faces that share the edge e.

The reason for the restriction of the location of the centers of the balls is that the procedure presented in [16, Sec. 5.3] relies on Poincaré inequalities so that the number of possible shapes for the intersections $B_{r'} \cap \Omega$ should be finite. A covering argument (see Theorem A.5 for the 2D case and Theorem A.6 for the 3D situation) then leads to the following bound with an appropriate $c_1 > 0$ (here, $c > 0$ is implicitly assumed sufficiently large):

$$\|\widetilde{\delta}_{M_s}^{1/2-\varepsilon}\nabla(w - w_h)\|_{L^2(\Omega\setminus S_{M_s,ch})} \lesssim \|\widetilde{\delta}_{M_s}^{1/2-\varepsilon}\nabla(w - I_h^k w)\|_{L^2(\Omega\setminus S_{M_s,cc_1h})} + \|\widetilde{\delta}_{M_s}^{-1/2-\varepsilon}(w - w_h)\|_{L^2(\Omega\setminus S_{M_s,cc_1h})} \tag{4.5}$$

The first term in (4.5) can be estimated with the improved regularity assertion of Lemma 2.6 to produce (with appropriate $c_2 > 0$ and the implicit assumption on c that $cc_1c_2 > 2$)

$$\begin{aligned}\|\widetilde{\delta}_{M_s}^{1/2-\varepsilon}\nabla(w - I_h^k w)\|_{L^2(\Omega\setminus S_{M_s,cc_1h})} &\lesssim h\|\widetilde{\delta}_{M_s}^{1/2-\varepsilon}\nabla^2 w\|_{L^2(\Omega\setminus S_{M_s,cc_1c_2h})} \\ &\lesssim h\left[\|\widetilde{\delta}_{M_s}^{1/2-\varepsilon} e\|_{L^2(\Omega)} + \|w\|_{H^{3/2+\varepsilon}(\Omega)}\right] \lesssim h\|e\|_{L^2(\Omega)}.\end{aligned}$$

For the second term in (4.5) we note that $-1/2-\varepsilon < 0$ so that $\widetilde{\delta}_{M_s}^{-1/2-\varepsilon} \leq \widetilde{\delta}_{\Gamma}^{-1/2-\varepsilon}$. This leads to

$$\begin{aligned}\|\widetilde{\delta}_{M_s}^{-1/2-\varepsilon}(w - w_h)\|_{L^2(\Omega\setminus S_{M_s,cc_1h})} &\lesssim \|\widetilde{\delta}_{M_s}^{-1/2-\varepsilon}(w - w_h)\|_{L^2(\Omega)} \\ &\lesssim \|\widetilde{\delta}_{\Gamma}^{-1/2-\varepsilon}(w - w_h)\|_{L^2(\Omega)} \lesssim h^{-2\varepsilon}\|\widetilde{\delta}_{\Gamma}^{-1/2+\varepsilon}(w - w_h)\|_{L^2(\Omega)};\end{aligned}$$

the term $h^{-2\varepsilon}\|\widetilde{\delta}_{\Gamma}^{-1/2+\varepsilon}(w - w_h)\|_{L^2(\Omega)}$ has already been estimated in (3.13) in the desired form. □

The regularity requirements on the solution u can still be slightly weakened. As written, the exponent $s_0 - 1$ is related to the *global* regularity of the dual solution w. However, the developments above show that a *local* lack of full regularity of the dual solution w (and the bidual solution) needs to be offset by additional local regularity of the solution. To be more specific, we restrict our attention now to the 2D Laplacian, i.e., $\mathbf{A} = \mathrm{Id}$. In this case, the situation can be expressed as follows with the aid of the singular exponents $\alpha_j := \pi/\omega_j$, where $\omega_j \in (\pi, 2\pi)$ is the interior angle at the reentrant vertices A_j, $j = 1, \ldots, J$.

Corollary 4.2 *Let $\Omega \subset \mathbb{R}^2$ be a polygon and let $\mathbf{A} = \mathrm{Id}$. Let $\delta_j := \mathrm{dist}(\cdot, A_j)$, $j = 1, \ldots, J$, for the J reentrant corners. Set $\widetilde{\delta}_j := h + \delta_j$. Let ω_i and α_j be defined as described above. Fix $\beta_j > 1 - \alpha_j$ arbitrary. Then for any $Iu \in V_h$*

$$\|u - u_h\|_{L^2(\Omega)} \lesssim h \sum_{j=1}^{J} \|\widetilde{\delta}_j^{-\beta_j}\nabla(u - Iu)\|_{L^2(\Omega)}.$$

Proof The proof follows by an inspection of how the regularity of the solution $w = Te$ of the dual problem enters the proof of Theorem 4.1. By, e.g., [4] the solution $w = Te$ is in a weighted H^2-space with

$$\| \prod_{j=1}^{J} \delta_j^{\beta_j} \nabla^2 w\|_{L^2(\Omega)} \lesssim \|e\|_{L^2(\Omega)}, \tag{4.6}$$

and Assumption 1.1 holds with any $s_0 < \min_j \alpha_j$. The regularity assertion (4.6) suggests to choose $\prod_{i=1}^{J} \widetilde{\delta}_i^{\beta_i}$ as the weight in the proof of Theorem 3.3. Inspection of the procedure in the proof of Theorem 4.1 then leads to the result. □

We extract from this result another corollary that we will prove useful in the numerical results. We formulate it in terms of (standard, unweighted) Sobolev regularity in order to emphasize the difference in regularity requirements of the solution near the reentrant corners and away from them:

Corollary 4.3 *Assume the hypotheses of Corollary 4.2. Let $s > 1$ and $s_i > 1$, $i = 1, \ldots, J$. Let $\mathcal{U} := \Omega \setminus \cup \overline{\mathcal{U}}_i$, for some neighborhoods $\mathcal{U}_i$ of the reentrant vertices A_i. Let $u \in H^{s_i}(\mathcal{U}_i)$, $i = 1, \ldots, J$ and $u \in H^s(\mathcal{U})$. Then for arbitrary $\varepsilon > 0$*

$$\|u - u_h\|_{L^2(\Omega)} \le C_\varepsilon h^\tau, \quad \tau := \min(1 + k, s, \min_{j=1\ldots J}(-1 + \alpha_j + s_j - \varepsilon)).$$

Proof The approximant Iu in Corollary 4.2 may be taken as any standard nodal interpolant or the Scott-Zhang projection. Then standard estimates and Corollary 4.2 produce with the choice $\beta_j := 1 - \alpha_j + \varepsilon$ for arbitrary small but fixed $\varepsilon > 0$:

$$\|u - u_h\|_{L^2(\Omega)} \lesssim h \min_{j=1}^{J}\{h^{\min\{k,s-1\}}, h^{-\beta_j + s_j - 1}\} \lesssim \min_{j=1\ldots J}\{h^{\min\{k+1,s\}}, h^{\alpha_j + s_j - 1 - \varepsilon}\}.$$

□

5 Optimal $L^2(S_h)$-convergence

Additional regularity of the solution also allows us to prove that the error on the strip S_h of width $O(h)$ near Γ is of higher order:

Theorem 5.1 *Let Assumption 1.1 be valid. Then the FEM error $u - u_h$ satisfies*

$$\|u - u_h\|_{L^2(S_h)} \lesssim h^{k+3/2}(1 + \delta_{k,1}|\ln h|)\|u\|_{B^{k+3/2}_{2,1}(\Omega)},$$

$$\|u - u_h\|_{L^2(S_h)} \lesssim h^{s+3/2}(1 + \delta_{k,1}|\ln h|)\|u\|_{B^{s+3/2}_{2,\infty}(\Omega)}, \quad s \in (0, k),$$

$$\|u - u_h\|_{L^2(S_h)} \lesssim h^{3/2}(1 + \delta_{k,1}|\ln h|)\|u\|_{B^{3/2}_{2,1}(\Omega)},$$

where $\delta_{k,1}$ is the Kronecker symbol.

Remark 5.2 1. The regularity requirement $B^{k+3/2}_{2,1}(\Omega)$ can be weakened: it suffices that u be in $B^{k+3/2}_{2,1}(S_D)$ in a fixed neighborhood S_D of Γ and in $H^{k+1}(\Omega)$. See [9] for the details of a closely related problem.

2. Since $B^{s+3/2}_{2,\infty}(\Omega) \supset H^{s+3/2}(\Omega)$, the assertions for $s \in (0,k)$ can be weakened by replacing $\|u\|_{B^{s+3/2}_{2,\infty}(\Omega)}$ with $\|u\|_{H^{s+3/2}(\Omega)}$ on the right-hand side. Only for the limiting cases $s = 0$ and $s = k$, we require the stronger requirement $u \in B^{s+3/2}_{2,1}(\Omega) \subset H^{s+3/2}(\Omega)$. ■

Proof The structure of the proof is very similar to that of Theorem 3.3. The main difference arises from the fact that the right-hand side of the dual problem is supported by the thin neighborhood S_h, and this support property has to be exploited.

Let $e = u - u_h$. Let χ_{S_h} be the characteristic function of S_h. Let $w = T(\chi_{S_h} e)$ and $w_h \in V_h$ its Galerkin approximation. Again, Galerkin orthogonality for $u - u_h$ and $w - w_h$ implies

$$\begin{aligned}\|e\|^2_{L^2(S_h)} &= \langle e, \chi e\rangle = a(e, w) = a(e, w - w_h) = a(u - I^k_h u, w - w_h) \\ &\le C\|\widetilde{\delta}_\Gamma^{-1/2-\varepsilon}\nabla(u - I^k_h u)\|_{L^2(\Omega)}\|\widetilde{\delta}_\Gamma^{1/2+\varepsilon}\nabla(w - w_h)\|_{L^2(\Omega)}, \quad (5.1)\end{aligned}$$

where $\varepsilon \ge 0$ is arbitrary (in fact, $\varepsilon \in \mathbb{R}$ would be admissible). We flag at this point already that we will select $\varepsilon = 0$ for $k = 1$ and $\varepsilon > 0$ arbitrary (but sufficiently small) for $k > 1$. Each of the two factors in (5.1) is estimated separately.

1. step: For the first factor in (5.1) we use approximation properties of the Scott-Zhang operator I^k_h together with Lemma 2.1 to get for $j \in \{0, \dots, k\}$

$$\begin{aligned}\|\widetilde{\delta}_\Gamma^{-1/2-\varepsilon}\nabla(u - I^k_h u)\|_{L^2(\Omega)} &\lesssim h^j\|\widetilde{\delta}_\Gamma^{-1/2-\varepsilon}\nabla^{j+1}u\|_{L^2(\Omega)}, \\ &\lesssim h^j \begin{cases} |\ln h|^{1/2}\|\nabla^{j+1}u\|_{B^{1/2}_{2,1}(\Omega)} & \text{if } \varepsilon = 0 \\ h^{-\varepsilon}\|\nabla^{j+1}u\|_{B^{1/2}_{2,1}(\Omega)} & \text{if } \varepsilon > 0. \end{cases} \quad (5.2)\end{aligned}$$

With the Kronecker symbol $\delta_{0,\varepsilon}$, we have shown for $j \in \{0, 1, \dots, k\}$

$$\|\widetilde{\delta}_\Gamma^{-1/2-\varepsilon}\nabla(u - I^k_h u)\|_{L^2(\Omega)} \lesssim h^j h^{-\varepsilon}(1 + \delta_{0,\varepsilon}|\ln h|^{1/2})\|u\|_{B^{j+3/2}_{2,1}(\Omega)}. \quad (5.3)$$

Since the Scott-Zhang operator I^k_h is defined on $H^1(\Omega)$ irrespective of boundary conditions, we may use an interpolation argument to lift the restriction to integer values j. Specifically, the reiteration theorem (cf., e.g., [14, Thm. 23.6]) asserts that the Besov space $B^{s+3/2}_{2,\infty}(\Omega)$, which we have defined by interpolation between (integer order) Sobolev spaces, coincides with the interpolation space between Besov spaces, viz.,

$$B^{s+3/2}_{2,\infty}(\Omega) = (B^{3/2}_{2,1}(\Omega), B^{k+3/2}_{2,1}(\Omega))_{s/k,\infty} \qquad \text{(equivalent norms)}.$$

Hence, we may decompose for arbitrary $t > 0$ a function $u \in B^{s+3/2}_{2,\infty}(\Omega)$, $s \in (0,k)$, as $u = u - u_1 + u_1$ with $u_1 \in B^{k+3/2}_{2,1}(\Omega)$ and $u - u_1 \in B^{3/2}_{2,1}(\Omega)$ together with

$$\|u - u_1\|_{B^{3/2}_{2,1}(\Omega)} \le Ct^{s/k}\|u\|_{B^{s+3/2}_{2,\infty}(\Omega)}, \qquad \|u_1\|_{B^{k+3/2}_{2,1}(\Omega)} \le Ct^{s/k-1}\|u\|_{B^{s+3/2}_{2,\infty}(\Omega)}.$$

Writing $u - I^k_h u = \big[(u - u_1) - I^k_h(u - u_1)\big] + \big[(u - u_0) - I^k_h(u - u_0)\big]$ we can use (5.3) with $j = k$ for the first term in brackets and $j = 0$ for the second term in brackets to get with the choice $t = h^k$

$$\|\widetilde{\delta}_\Gamma^{-1/2-\varepsilon}\nabla(u - I^k_h u)\|_{L^2(\Omega)} \lesssim h^s h^{-\varepsilon}(1 + \delta_{0,\varepsilon}|\ln h|^{1/2})\|u\|_{B^{s+3/2}_{2,\infty}(\Omega)}. \quad (5.4)$$

Combining the estimates (5.3) with $j = 0$ and $j = k$ and (5.4) for $s \in (0, k)$ we arrive at

$$\|\widetilde{\delta}_\Gamma^{-1/2-\varepsilon}\nabla(u - I_h^k u)\|_{L^2(\Omega)} \lesssim (1 + \delta_{0,\varepsilon}|\ln h|^{1/2})h^{-\varepsilon}\begin{cases} h^s\|u\|_{B^{s+3/2}_{2,1}(\Omega)}, & s = 0, \\ h^s\|u\|_{B^{s+3/2}_{2,\infty}(\Omega)}, & s \in (0, k), \\ h^s\|u\|_{B^{k+3/2}_{2,1}(\Omega)}, & s = k. \end{cases} \tag{5.5}$$

2. step: The second factor in (5.1) requires more work. We start with a regularity assertion for w that exploits the support properties of $\chi_{S_h}e$ and follows from (2.11) and (2.12):

$$\|w\|_{B^{3/2}_{2,\infty}(\Omega)} \lesssim h^{1/2}\|\chi_{S_h}e\|_{L^2(\Omega)}, \tag{5.6}$$
$$\|w\|_{H^{3/2+\varepsilon}(\Omega)} \lesssim h^{1/2-\varepsilon}\|\chi_{S_h}e\|_{L^2(\Omega)}, \qquad \varepsilon \in (0, s_0 - 1/2]. \tag{5.7}$$

We obtain an energy error estimate for $w - w_h$ in the standard way by using quasi-optimality, the approximation properties of V_h, and the regularity assertion (5.6):

$$\|w - w_h\|_{H^1(\Omega)} \lesssim \inf_{v\in V_h}\|w - v\|_{H^1(\Omega)} \lesssim h^{1/2}\|w\|_{B^{3/2}_{2,\infty}(\Omega)} \lesssim h\|\chi_{S_h}e\|_{L^2(\Omega)}. \tag{5.8}$$

Lemma 3.1 is applicable with $z = w - w_h$; hence, obtain with (5.8)

$$\|\delta_\Gamma^{-1/2+\varepsilon}(w - w_h)\|_{L^2(\Omega)} \lesssim h^{3/2+\varepsilon}\|\chi_{S_h}e\|_{L^2(\Omega)}, \qquad \varepsilon \in (0, s_0 - 1/2], \tag{5.9}$$
$$\|\widetilde{\delta}_\Gamma^{-1/2}(w - w_h)\|_{L^2(\Omega)} \lesssim h^{3/2}|\ln h|^{1/2}\|\chi_{S_h}e\|_{L^2(\Omega)}. \tag{5.10}$$

The bound (5.1) informs us that control of $w - w_h$ in a weighted H^1-norm is required. In this direction, we first write for a constant $c > 0$ that will be determined later sufficiently large

$$\begin{aligned} \|\widetilde{\delta}_\Gamma^{1/2+\varepsilon}\nabla(w - w_h)\|_{L^2(\Omega)} &\le \|\widetilde{\delta}_\Gamma^{1/2+\varepsilon}\nabla(w - w_h)\|_{L^2(S_{ch})} + \|\widetilde{\delta}_\Gamma^{1/2+\varepsilon}\nabla(w - w_h)\|_{L^2(\Omega\setminus S_{ch})} \\ &\le Ch^{1/2+\varepsilon}\|\nabla(w - w_h)\|_{L^2(\Omega)} + \|\widetilde{\delta}_\Gamma^{1/2+\varepsilon}\nabla(w - w_h)\|_{L^2(\Omega\setminus S_{ch})} \\ &\overset{(5.8)}{\le} Ch^{3/2+\varepsilon}\|\chi_{S_h}e\|_{L^2(\Omega)} + \|\widetilde{\delta}_\Gamma^{1/2+\varepsilon}\nabla(w - w_h)\|_{L^2(\Omega\setminus S_{ch})}. \end{aligned} \tag{5.11}$$

We emphasize that $\varepsilon = 0$ is allowed in (5.11). It remains to control $\|\widetilde{\delta}_\Gamma^{1/2+\varepsilon}\nabla(w - w_h)\|_{L^2(\Omega\setminus S_{ch})}$. This is done again with the same arguments from the local error analysis as in the proof of Theorem 3.3. The estimate (3.12) holds verbatim, i.e.,

$$\|\delta_\Gamma^{1/2+\varepsilon}\nabla(w - w_h)\|_{L^2(\Omega\setminus S_{ch})} \lesssim \|\delta_\Gamma^{1/2+\varepsilon}\nabla(w - I_h^k w)\|_{L^2(\Omega\setminus S_{cc_1h})} + \|\delta_\Gamma^{-1/2+\varepsilon}(w - w_h)\|_{L^2(\Omega\setminus S_{cc_1h})}. \tag{5.12}$$

We emphasize that $\varepsilon = 0$ is admissible in (3.12). As in the proof of Theorem 3.3, the constant c will be selected in dependence of various inverse estimates that are applied. Combining (5.9), (5.10), (5.11), (5.12) we see that we have shown

$$\|\widetilde{\delta}_\Gamma^{1/2+\varepsilon}\nabla(w-w_h)\|_{L^2(\Omega)} \lesssim \begin{cases} h^{3/2+\varepsilon}\|\chi_{S_h}e\|_{L^2(\Omega)} + \|\widetilde{\delta}_\Gamma^{1/2+\varepsilon}\nabla(w-I_h^k w)\|_{L^2(\Omega\setminus S_{c_1ch})} & \text{if } \varepsilon>0,\\ h^{3/2}|\ln h|^{1/2}\|\chi_{S_h}e\|_{L^2(\Omega)} + \|\widetilde{\delta}_\Gamma^{1/2}\nabla(w-I_h^k w)\|_{L^2(\Omega\setminus S_{c_1ch})} & \text{if } \varepsilon=0.\end{cases} \tag{5.13}$$

3. step: We estimate the approximation error $\|\delta_\Gamma^{1/2+\varepsilon}\nabla(w-I_h^k w)\|_{L^2(\Omega\setminus S_{c_1ch})}$. At this point the cases $k=1$ and $k>1$ diverge: since w solves a *homogeneous* elliptic equation on $\Omega\setminus S_{c_1ch}$ (if $c_1c>1$), interior regularity is available so that higher order approximation can be brought to bear if $k>1$ in contrast to the case $k=1$. We start with the simpler case $k=1$.

The case $k=1$: From standard approximation results for I_h^k, the inverse estimate of Lemma 2.9, (i), and (5.6) we get for a constant $c_2\in(0,1)$ (implicitly, we assume that c is so large that $c_2c_1ch>2h$)

$$\begin{aligned}\|\widetilde{\delta}_\Gamma^{1/2}\nabla(w-I_h^k w)\|_{L^2(\Omega\setminus S_{c_1ch})} &\lesssim h^1\|\widetilde{\delta}_\Gamma^{1/2}\nabla^2 w\|_{L^2(\Omega\setminus S_{c_2c_1ch})}\\ &\lesssim h|\ln h|^{1/2}\|w\|_{B^{3/2}_{2,\infty}(\Omega)} \lesssim h^{3/2}|\ln h|^{1/2}\|\chi_{S_h}e\|_{L^2(\Omega)}.\end{aligned} \tag{5.14}$$

Inserting (5.5) (with $\varepsilon=0$) with the combination of (5.14) and (5.13) (again with $\varepsilon=0$) in (5.1) yields the desired final estimate for the case $k=1$ if we fix $c=2/(c_1c_2)$.

The case $k>1$: We fix an $\varepsilon\in(0,s_0-1/2]$ arbitrary. From standard approximation results for I_h^k, the inverse estimates of Lemma 2.9, and the regularity assertion (5.7) we get (again for suitable constants c_2, $c_3\in(0,1)$ and the implicit assumption that c is such that $c_3c_2c_1c$ is sufficiently large)

$$\begin{aligned}\|\widetilde{\delta}_\Gamma^{1/2+\varepsilon}\nabla(w-I_h^k w)\|_{L^2(\Omega\setminus S_{c_1ch})} &\lesssim h^2\|\widetilde{\delta}_\Gamma^{1/2+\varepsilon}\nabla^3 w\|_{L^2(\Omega\setminus S_{c_2c_1ch})}\\ &\overset{\text{Lem. 2.9,}(ii)}{\lesssim} h^2\left[\|\widetilde{\delta}_\Gamma^{-1/2+\varepsilon}\nabla^2 w\|_{L^2(\Omega\setminus S_{c_3c_2c_1ch})} + \|\widetilde{\delta}_\Gamma^{1/2+\varepsilon}\|\nabla w\|_{L^2(\Omega\setminus S_{c_3c_2c_1ch})}\right]\\ &\lesssim h^{2-1/2+\varepsilon}\left[\|\nabla^2 w\|_{L^2(\Omega\setminus S_{c_3c_2c_1ch})} + \|\nabla w\|_{L^2(\Omega\setminus S_{c_3c_2c_1ch})}\right]\\ &\overset{\text{Lem. 2.9,}(iv)}{\lesssim} h^{2-1/2+\varepsilon}\|\chi_{S_h}e\|_{L^2(\Omega)}.\end{aligned} \tag{5.15}$$

Combining this with (5.5) produces in (5.1) the desired final estimate for the case $k>1$. □

From Theorem 5.1 we can extract optimal convergence estimates for the flux error $\|\partial_n(u-u_h)\|_{L^2(\Gamma)}$:

Corollary 5.3 *Let Assumption 1.1 be valid. Then with the Kronecker symbol $\delta_{k,1}$*

$$\|\partial_n u - \partial_n u_h\|_{L^2(\Gamma)} \lesssim (1+\delta_{k,1}|\ln h|)\begin{cases} h^k\|u\|_{B^{k+3/2}_{2,1}(\Omega)},\\ h^s\|u\|_{B^{s+3/2}_{2,\infty}(\Omega)}, \quad s\in(0,k),\\ \|u\|_{B^{3/2}_{2,1}(\Omega)}.\end{cases}$$

Proof Structurally, the proof follows [11, Cor. 6.1] in that estimating the error on Γ is transferred to an estimate on the strip S_h. The triangle inequality gives

$$\|\partial_n(u-u_h)\|_{L^2(\Gamma)} \le \|\partial_n(u-I_h^k u)\|_{L^2(\Gamma)} + \|\partial_n(I_h^k u-u_h)\|_{L^2(\Gamma)}. \tag{5.16}$$

The two terms in (5.16) are estimated separately.

1. step: We claim that

$$\|\partial_n(u-I_h^k u)\|_{L^2(\Gamma)} \lesssim \begin{cases} h^k\|u\|_{B^{k+3/2}_{2,1}(\Omega)} \\ h^s\|u\|_{B^{s+3/2}_{2,\infty}(\Omega)}, & s\in(0,k), \\ \|u\|_{B^{3/2}_{2,1}(\Omega)}. \end{cases} \tag{5.17}$$

We will only show the limiting cases $u \in B^{k+3/2}_{2,1}(\Omega)$ and $u \in B^{3/2}_{2,1}(\Omega)$; the intermediate cases follows by an interpolation argument similar to the one used in the proof of Theorem 5.1. For the case of maximal regularity, we use an elementwise multiplicative trace inequality for the elements abutting Γ to get

$$\begin{aligned} &\|\partial_n(u-I_h^k u)\|_{L^2(\Gamma)} \lesssim h^{k/2}\sqrt{\|\nabla^{k+1}u\|_{L^2(S_{2h})}}\,h^{(k-1)/2}\sqrt{\|\nabla^{k+1}u\|_{L^2(S_{2h})}} \\ &\lesssim h^{k-1/2}\|\nabla^{k+1}u\|_{L^2(S_{2h})} \overset{(2.4)}{\lesssim} h^k\|\nabla^{k+1}u\|_{B^{1/2}_{2,1}(\Omega)} \lesssim h^k\|u\|_{B^{k+3/2}_{2,1}(\Omega)}. \end{aligned}$$

For the case of minimal regularity, $u\in B^{3/2}_{2,1}(\Omega)$ we first note that we obtain from (2.5) that $\|v\|_{L^2(\Gamma)} \lesssim \|v\|_{B^{1/2}_{2,1}(\Omega)}$. Using this and inverse estimates, we get

$$\begin{aligned} \|\partial_n(u-I_h^k u)\|_{L^2(\Gamma)} &\le \|\partial_n u\|_{L^2(\Gamma)} + \|\partial_n I_h^k u\|_{L^2(\Gamma)} \\ &\lesssim \|\nabla u\|_{B^{1/2}_{2,1}(\Omega)} + h^{-1/2}\|\nabla I_h^k u\|_{L^2(S_h)} \\ &\overset{I_h^k \text{stable}}{\lesssim} \|\nabla u\|_{B^{1/2}_{2,1}(\Omega)} + h^{-1/2}\|\nabla u\|_{L^2(S_{2h})} \overset{(2.4)}{\lesssim} \|\nabla u\|_{B^{1/2}_{2,1}(\Omega)}. \end{aligned}$$

2. step: The term $\|\partial_n(I_h^k u-u_h)\|_{L^2(\Gamma)}$ in (5.16) is controlled with inverse estimates and Theorem 5.1 as follows:

$$\begin{aligned} \|\partial_n(I_h^k u-u_h)\|_{L^2(\Gamma)} &\lesssim h^{-1/2}\|\nabla(I_h^k u-u_h)\|_{L^2(S_h)} \lesssim h^{-3/2}\|I_h^k u-u_h\|_{L^2(S_h)} \\ &\lesssim h^{-3/2}\|u-I_h^k u\|_{L^2(S_h)} + h^{-3/2}\|u-u_h\|_{L^2(S_h)}. \end{aligned}$$

The term $\|u-I_h^k u\|_{L^2(S_h)}$ can be controlled with the approximation properties of I_h^k in the desired fashion: $\|u-I_h^k u\|_{L^2(S_h)} \lesssim h\|\nabla u\|_{L^2(S_{2h})} \lesssim h^{3/2}\|\nabla u\|_{B^{1/2}_{2,1}(\Omega)}$. The contribution $\|u-u_h\|_{L^2(S_h)}$ is estimated with the aid of Theorem 5.1. □

6 Extension of the results of [11]

The arguments of the present paper are similar to those underlying [11], in spite of the fact that we did not employ the anisotropic norms that we introduced in [11] but instead worked with weighted Sobolev spaces. A feature of the analysis here that was not present in [11] is our FEM error analysis in Lemma 3.1 for a weighted L^2-estimate, which, in turn, relies on the regularity assertions of Lemma 2.8 for problems with data in weighted spaces. This additional technical issue was circumvented in [11] by assuming convexity of Ω so that optimal order L^2-estimates could be cited from the literature. The present analysis provides the necessary technical tools to remove this simplification in [11], where a more complex mortar setting is analyzed. It is possible to make use of weighted L^2-estimates similar to those of Lemma 3.1 in the setting of [11]. For that, regularity results of the type provided in Lemma 2.8 have to be used. The outcome of this refinement is that the main results of [11], namely, [11, Thm. 2.1], which provides L^2-estimates on strips of width $O(h)$ around the skeleton, and [11, Thm. 2.5], which provides optimal order approximations for the mortar variable, hold true if the geometry admits a shift theorem by more than 1/2. We will not provide the details of the arguments here and refer to [5, Appendix B] instead. Nevertheless, for future reference we record the end result:

Theorem 6.1 *In [11, Thms. 2.1, 2.5], the assumption of convexity of Ω can be replaced with [11, Assumption (5.2)], which is a shift theorem for the Dirichlet Laplacian by more than* $1/2$.

7 Numerical results

We consider the simple model equation $-\Delta u = f$ in $\Omega \subset \mathbb{R}$, $d \in \{2, 3\}$ with inhomogeneous Dirichlet boundary conditions. These are realized numerically by nodal interpolation of the prescribed exact solution u, and the data f is also computed from u. In the case of a non-smooth solution, we use a suitable quadrature formula on finer meshes to guarantee that the L^2-error is accurately evaluated.

7.1 Two-dimensional results

We use a sequence of uniformly refined triangular meshes, refined by quadrisection.

7.1.1 Lowest order discretization

We consider two typical domains for reentrant corners. We start with the L-shaped domain $(-1,1)^2 \setminus (0,1) \times (-1,0)$ and then consider a slit domain $(-1,1)^2 \setminus ((0,1) \times \{0\})$. In both cases, the prescribed solution is given in polar coordinates by $u(r,\phi) = r^\alpha \sin(a\phi)$ where α, a are given parameters. Moreover, we test the influence of the position (x_0, y_0) of the weak singularity at $r = 0$ by defining $r^2 := (x - x_0)^2 + (y - y_0)^2$. We note that irrespective of the location (x_0, y_0) of the singularity on the boundary Γ, we have $u \in B^{1+\alpha}_{2,\infty}(\Omega) \subset H^{1+\alpha-\varepsilon}(\Omega)$ for any $\varepsilon > 0$.

For the L-shaped domain, the shift parameter s_0 can be taken to be any $s_0 < 2/3$. From the theoretical results in Section 3, we therefore expect the error decay to have a rate of at least $\min(2, 1+\alpha-1/3)$ uniformly in the position (x_0, y_0) of the singularity. Table 7.1 shows the numerical results for $\alpha = 0.75$ and $a = 2/3\pi$, for which $\min(2, 1+\alpha-1/3) = 1.417$. As it can be seen for $(x_0, y_0) = (0, 0)$, we observe a good agreement with Theorem 3.3. However for the locations $(x_0, y_0) = (0.5, 0)$ and $(x_0, y_0) = (0, 1)$, the rates are substantially better. This can be explained by

$(x_0, y_0) = (0, 0)$ $a = 2/3\pi$		$(x_0, y_0) = (0.5, 0)$ $a = \pi$		$(x_0, y_0) = (0, 1)$ $a = \pi$	
L^2-error	rate	L^2-error	rate	L^2-error	rate
7.8969e-03	-	1.2069e-02	-	1.1063e-02	-
3.0011e-03	1.40	4.1395e-03	1.54	3.5229e-03	1.65
1.1566e-03	1.38	1.3543e-03	1.61	1.0832e-03	1.70
4.4413e-04	1.38	4.3196e-04	1.65	3.2888e-04	1.72
1.7025e-04	1.38	1.3569e-04	1.67	9.9264e-05	1.73
6.5055e-05	1.39	4.2198e-05	1.69	2.9859e-05	1.73
2.4769e-05	1.39	1.3030e-05	1.70	8.9609e-06	1.74
9.3974e-06	1.40	4.0022e-06	1.70	2.6847e-06	1.74

Table 7.1 L-shaped domain, $k = 1$: Influence of the position of singularity for $\alpha = 0.75$.

the more refined analysis of Section 4. Using Corollary 4.3, we expect an improved convergence rate of 1.75 for these cases.

$\alpha = 10/9$		$\alpha = 4/3$		$\alpha = 3/2$	
$L^2$2-error	rate	L^2-error	rate	L^2-error	rate
1.3295e-02	-	1.7801e-02	-	1.9678e-02	-
4.2329e-03	1.65	5.2865e-03	1.75	5.5836e-03	1.82
1.2489e-03	1.76	1.4561e-03	1.86	1.4800e-03	1.92
3.5939e-04	1.80	3.9056e-04	1.90	3.8327e-04	1.95
1.0148e-04	1.82	1.0294e-04	1.92	9.7928e-05	1.97
2.8339e-05	1.84	2.6855e-05	1.94	2.4843e-05	1.98
7.8663e-06	1.85	6.9612e-06	1.95	6.2773e-06	1.98
2.1779e-06	1.85	1.7972e-06	1.95	1.5823e-06	1.99

Table 7.2 L-shaped domain, $k = 1$: Influence of exponent α for $a = 2/3\pi$ and $(x_0, y_0) = (0, 0)$.

Table 7.2 shows the results for $(x_0, y_0) = (0, 0)$ and $\alpha \in \{10/9, 4/3, 3/2\}$. From Theorem 3.3, we expect convergence rates of 1.78, 2, and 2, respectively. The observed numerical rates of 1.85, 1.95, and 1.99 are quite close.

The situation is similar for the slit domain where the regularity of the dual problem is even further reduced. It corresponds to a limiting case of our theory, which, strictly speaking, we did not cover, since the parameter s_0 of the shift theorem may be taken to be any $s_0 < 1/2$. Nevertheless, one expects from Theorem 3.3 a convergence rate close to $\min\{2, 1 + \alpha - 1/2\}$. For $\alpha = 0.75$ this is 1.25, which is is visible in Table 7.3 for the case $(x_0, y_0) = (0, 0)$. Again, the better convergence behavior for $(x_0, y_0) = (0.5, 0)$ and $(x_0, y_0) = (0, 1)$ can be explained by the theory of Corollary 4.3, which predicts $1 + \alpha = 1.75$. Table 7.4 shows the results for $(x_0, y_0) = (0, 0)$ and $\alpha \in \{10/9, 4/3, 3/2\}$. From Theorem 3.3, we expect convergence rates of 1.61, 1.83 and 2, respectively. The observed numerical rates of 1.72, 1.9, and 1.98 are reasonably close to these predictions.

7.1.2 Second order finite elements

In this subsection, we test the performance of quadratic finite elements for the L-shaped domain. We use the same type of solution as before and vary the parameter α for $(x_0, y_0) = (0, 0)$, i.e., the re-entrant corner. Here we expect from our theory a

$(x_0, y_0) = (0,0)$ $a = \pi/2$		$(x_0, y_0) = (0.5,0)$ $a = \pi$		$(x_0, y_0) = (0,1)$ $a = \pi$	
L^2-error	rate	L^2-error	rate	L^2-error	rate
5.6427e-03	-	1.5479e-02	-	1.1518e-02	-
2.2076e-03	1.35	6.0053e-03	1.37	3.8977e-03	1.56
9.2506e-04	1.25	2.1881e-03	1.46	1.2592e-03	1.63
4.0400e-04	1.20	7.1529e-04	1.61	3.9846e-04	1.66
1.7782e-04	1.18	2.2878e-04	1.64	1.2446e-04	1.68
7.7725e-05	1.19	7.2999e-05	1.65	3.8538e-05	1.69
3.3635e-05	1.21	2.3250e-05	1.65	1.1858e-05	1.70
1.4428e-05	1.22	7.5457e-06	1.62	3.6315e-06	1.71

Table 7.3 Slit domain, $k = 1$: Influence of the position of singularity for $\alpha = 0.75$.

$\alpha = 10/9$		$\alpha = 4/3$		$\alpha = 3/2$	
L^2-error	rate	L^2-error	rate	L^2-error	rate
1.3303e-02	-	1.6823e-02	-	1.8589e-02	-
3.9682e-03	1.75	4.6200e-03	1.86	4.8854e-03	1.93
1.1798e-03	1.75	1.2515e-03	1.88	1.2627e-03	1.95
3.4835e-04	1.76	3.3569e-04	1.90	3.2327e-04	1.97
1.0291e-04	1.76	8.9685e-05	1.90	8.2403e-05	1.97
3.0594e-05	1.75	2.3944e-05	1.91	2.0960e-05	1.98
9.1844e-06	1.74	6.3997e-06	1.90	5.3258e-06	1.98
2.7892e-06	1.72	1.7142e-06	1.90	1.3524e-06	1.98

Table 7.4 Slit domain, $k = 1$: Influence of exponent α for $a = 1/2\pi$ and $(x_0, y_0) = (0, 0)$

convergence rate of $\min(3, \alpha+1-1/3)$. For $\alpha \in \{2.175, 2.275, 2.375\}$, the observed numerical rates, which are visible in Table 7.5, are very close to the theoretically predicted ones.

$\alpha = 2.175$		$\alpha = 2.275$		$\alpha = 2.375$	
L2 error	rate	L2 error	rate	L2 error	rate
5.6675e-04	-	5.4818e-04	-	5.4156e-04	-
7.8980e-05	2.84	7.3048e-05	2.91	6.9693e-05	2.96
1.1011e-05	2.84	9.7421e-06	2.91	8.9972e-06	2.95
1.5255e-06	2.85	1.2851e-06	2.92	1.1494e-06	2.97
2.0957e-07	2.86	1.6762e-07	2.94	1.4533e-07	2.98
2.8671e-08	2.87	2.1683e-08	2.95	1.8255e-08	2.99

Table 7.5 L-shaped domain, $k = 2$: Influence of α for $a = 2/3\pi$ and $(x_0, y_0) = (0, 0)$.

7.2 Three-dimensional results

In the three dimensional setting, we consider a Fichera corner $\Omega := (-1,1)^3 \setminus [0,1]^3$ and prescribe the smooth solution $u(x,y,z) := \sin((x+y)\pi)\cos(2\pi z)$. The inhomogeneous Dirichlet conditions are realized by nodal interpolation. The discretization is based on trilinear finite elements on hexahedra and uniform refinements. Although the dual problem lacks full regularity, which corresponds to a shift theorem with $s_0 < 1$, Theorem 3.3 asserts that this can be compensated by extra s_0 regularity of the primal solution to maintain full second order convergence in L^2.

DOF	L^2-error	rate
316	0.075444	-
3.032	0.017182	1.96
26.416	0.0039376	2.04
220.256	0.00094597	2.02
1.798.336	0.00023208	2.01
14.532.992	5.7491e-05	2.00

Table 7.6 Fichera corner, $k = 1$: L^2-error for a smooth solution.

Table 7.6 shows that we observe numerically already for coarse discretizations the predicted convergence order two, and the theoretical results are confirmed.

A Coverings

In this appendix, the distance $\operatorname{dist}(x, M)$ for some set M appears frequently. For notational convenience, we set $\operatorname{dist}(x, \emptyset) = 1$ to include the degenerate case $M = \emptyset$.

We quote from [11, Lemma A.1]:

Lemma A.1 *Let $\Omega \subset \mathbb{R}^d$ be bounded open and $M = \overline{M}$ be a closed set. Fix $c \in (0,1)$ and $\varepsilon \in (0,1)$ such that*

$$1 - c(1+\varepsilon) =: c_0 > 0. \tag{A.1}$$

For each $x \in \Omega$, let $B_x := \overline{B}_{c \operatorname{dist}(x,M)}(x)$ be the closed ball of radius $c \operatorname{dist}(x, M)$ centered at x, and let $\widehat{B}_x := \overline{B}_{(1+\varepsilon)c \operatorname{dist}(x,M)}(x)$ denote the stretched (closed) ball of radius $(1+\varepsilon)c \operatorname{dist}(x, M)$ also centered at x.

Then there exists a countable set $x_i \in \Omega$, $i \in \mathbb{N}$, and a constant $N \in \mathbb{N}$ depending solely on the spatial dimension d with the following properties:

1. *(covering property) $\cup_{i \in \mathbb{N}} B_{x_i} \supset \Omega$*
2. *(finite overlap on Ω) for each $x \in \Omega$, there holds $\operatorname{card}\{i \,|\, x \in \widehat{B}_{x_i}\} \leq N$.*

Proof [11, Lemma A.1] assumed that $M \subset \overline{\Omega}$. However, an inspection of the proof shows that this is not necessary.

Before we proceed with variants of the covering result of Lemma A.1, we introduce the notation of sectorial neighborhoods relative a singular set M:

Definition A.2 (sectorial neighborhood) Let e, $M \subset \mathbb{R}^d$ and $\widetilde{c} > 0$. Then

$$S_{e,M,\widetilde{c}} := \cup_{x \in e} B_{\widetilde{c} \operatorname{dist}(x,M)}(x)$$

is a *sectorial neighborhood of the set e relative to the singular set M.*

We are interested in coverings of lower-dimensional manifolds by balls whose centers are located on these manifolds:

Lemma A.3 *Let $d \in \mathbb{N}$ and $1 \leq d' < d$. Let $\omega \subset \mathbb{R}^{d'}$ and let $\Omega \subset \mathbb{R}^d$ be the canonical embedding of ω into $\mathbb{R}^d$, i.e., $\Omega := \omega \times \{0\} \times \cdots \times \{0\} \subset \mathbb{R}^d$. Assume the hypotheses and notation of Lemma A.1. Then there are $\widetilde{c} > 0$, $N > 0$, and a collection of balls B_{x_i}, $i \in \mathbb{N}$, as described in Lemma A.1 such that*

- *(i) (covering property for Ω) $\cup_{i \in \mathbb{N}} B_{x_i} \supset \Omega$.*
- *(ii) (covering property for a sectorial neighborhood of Ω) $\cup_{i \in \mathbb{N}} B_{x_i} \supset S_{\Omega,M,\widetilde{c}}$.*
- *(iii) (finite overlap property on $\mathbb{R}^d$) for each $x \in \mathbb{R}^d$, there holds $\operatorname{card}\{i \,|\, x \in \widetilde{B}_{x_i}\} \leq N$.*

Proof We employ the result of Lemma A.1 for the lower-dimensional manifold ω noting that $B_x \cap \omega$ is a ball in $\mathbb{R}^{d'}$. In order to be able to ensure the covering condition for the sectorial neighborhood of Ω stated in (iii), we introduce the auxiliary balls $B'_x := \overline{B}_{c/2 \operatorname{dist}(x,M)}(x)$ of half the radius. Applying Lemma A.1 with these balls B'_x and the stretched balls $\widehat{B}_x$ therefore produces a collection of centers $x_i \in \Omega$, $i \in \mathbb{N}$, such that

1. $B'_{x_i} \cap \Omega$ covers Ω;

2. for the stretched balls $\widehat{B}_{x_i}$, we have a finite overlap property on Ω:

$$\forall x \in \Omega: \quad \operatorname{card}\{i \,|\, x \in \widehat{B}_{x_i}\} \leq N. \tag{A.2}$$

We next see that the balls $\widehat{B}_{x_i}$ even have the following, stronger finite overlap property:

$$\forall x \in \mathbb{R}^d: \quad \operatorname{card}\{i \,|\, x \in \widehat{B}_{x_i}\} \leq N. \tag{A.3}$$

To see this, define the infinite cylinders $\widehat{C}_{x_i} := \{x \,|\, \pi_{d'}(x) \in \widehat{B}_{x_i} \cap \Omega\}$, where $\pi_{d'}$ is the canonical projection onto the hyperplane $\{x = (x_1, \ldots, x_d) \in \mathbb{R}^d \,|\, x_{d'+1} = \cdots = x_d = 0\}$. Clearly, $\widehat{B}_{x_i} \subset \widehat{C}_{x_i}$. These infinite cylinders have a finite overlap property by (A.2) as can be seen by writing any $x \in \mathbb{R}^d$ in the form $x = (\pi_{d'}(x), x')$ for some $x' \in \mathbb{R}^{d-d'}$ and then noting that $x \in \widehat{C}_{x_i}$ implies $\pi_{d'}(x) \in \widehat{B}_{x_i} \cap \Omega$.

Is remains to see that the balls B_{x_i} cover a sectorial neighborhood of Ω. To that end, we note that the balls B'_{x_i} cover Ω. Furthermore, for each $x \in \Omega$, we pick x_i such that $x \in B'_{x_i} \subset B_{x_i}$. Since the radius of B_{x_i} is twice that of B'_{x_i}, we even have $B_{c/2\,\operatorname{dist}(x_i,M)}(x) \subset B_{x_i}$. Furthermore, by $c \in (0,1)$, we have $0 < (1-c)\operatorname{dist}(x_i, M) \leq \operatorname{dist}(x, M) \leq (1+c)\operatorname{dist}(x_i, M)$. Therefore, there is $c' > 0$ such that $B_{\widetilde{c}\,\operatorname{dist}(x,M)}(x) \subset B_{x_i}$ and thus

$$\cup_{x\in\Omega} B_{\widetilde{c}\,\operatorname{dist}(x,M)}(x) \subset \cup_i B_{x_i}.$$

□

We next show covering theorems for polygons and polyhedra. In the interest of clarity of presentation, we formulate two separate results. Before doing so, we point out that balls with center located on the boundary of the polygon/polyhedron Ω will feature importantly so that the intersection of this ball with Ω will be of interest. We therefore introduce the following notions:

Definition A.4 (solid angles and dihedral angles)

1. Let $\Omega \subset \mathbb{R}^2$ be a Lipschitz polygon. Let A be a vertex where the edges e_1, e_2 meet. We say that the set $B_\varepsilon(A) \cap \Omega$ is a solid angle, if $\partial(B_\varepsilon(A) \cap \Omega) \cap \partial\Omega$ is contained in $\{A\} \cup e_1 \cup e_2$.
2. Let $\Omega \subset \mathbb{R}^3$ be a Lipschitz polyhedron. Let A be a vertex of Ω. We say that the set $B_\varepsilon(A) \cap \Omega$ is a *solid angle*, if $\partial(B_\varepsilon(A) \cap \Omega) \cap \partial\Omega$ is contained the union of $\{A\}$ and the edges and faces meeting at A.
3. Let $\Omega \subset \mathbb{R}^3$ be a Lipschitz polyhedron. Let e be an edge of Ω, which is shared by the faces f_1, f_2. Let $x \in e$. We say that the set $B_\varepsilon(x) \cap \Omega$ is a *dihedral angle*, if $\partial(B_\varepsilon(x) \cap \Omega) \cap \partial\Omega$ is contained in $e \cup f_1 \cup f_2$.

Theorem A.5 *Let $\Omega \subset \mathbb{R}^2$ be a bounded Lipschitz polygon with vertices A_j, $j = 1, \ldots, J$, and edges $\mathcal{E}$. Let $M \subset \{A_1, \ldots, A_J\}$. Set $\mathcal{A}' := \{A_1, \ldots, A_J\} \setminus M$ and fix $\varepsilon \in (0,1)$.*

(i) There is a sectorial neighborhood $S_{\mathcal{A}',M,\widetilde{c}}$ of the vertices $\mathcal{A}'$ and a constant $c \in (0,1)$ such that $S_{\mathcal{A}',M,\widetilde{c}}$ is covered by balls $B_i := \overline{B}_{c\,\operatorname{dist}(x_i,M)}(x_i)$ with centers $x_i \in \mathcal{A}'$. Furthermore, the stretched balls $\widehat{B}_i := \overline{B}_{(1+\varepsilon)c\,\operatorname{dist}(x_i,M)}(x_i)$ are solid angles and satisfy a finite overlap property on $\mathbb{R}^2$.

(ii) *Fix a sectorial neighborhood* $\mathcal{U} := S_{\mathcal{A}',M,c'}$ *of the vertices* $\mathcal{A}'$*. For each edge* $e \in \mathcal{E}$*, there is a sectorial neighborhood* $S_{e,M,\widetilde{c}}$ *and a constant* $c \in (0,1)$ *such that* $S_{e,M,\widetilde{c}} \setminus \mathcal{U}$ *is covered by balls* $B_i = \overline{B}_{c\,\mathrm{dist}(x_i,M)}(x_i)$ *whose centers* x_i *are located on* e*. Furthermore, the stretched balls* $\widehat{B}_i = \overline{B}_{(1+\varepsilon)c\,\mathrm{dist}(x_i,M)}(x_i)$ *satisfy a finite overlap property on* $\mathbb{R}^2$ *and are such that each* $\widehat{B}_i \cap \Omega$ *is a half-disk.*

(iii) *Fix a sectorial neighbood* $\mathcal{U} := S_{\mathcal{E},M,c'}$ *of the edges* $\mathcal{E}$*. There is* $c \in (0,1)$ *such that* $\Omega \setminus \mathcal{U}$ *is covered by balls* $B_i = \overline{B}_{c\,\mathrm{dist}(x_i,M)}(x_i)$ *such that the stretched balls* $\widehat{B}_i = \overline{B}_{(1+\varepsilon)c\,\mathrm{dist}(x_i,M)}(x_i)$ *are completely contained in* Ω *and satisfy a finite overlap property on* $\mathbb{R}^2$*.*

Proof The assertion (i) is almost trivial and only included to emphasize the structure of the arguments. Assertions (ii), (iii) follow from suitable applications of Lemmas A.3 and A.1. □

The 3D variant of Theorem A.5 is formulated in Theorem A.6. We emphasize that the "singular" set M need not be the union of *all* edges and vertices but can be just a subset. We also emphasize that it is not necessarily related to the notion of "singular set" in Definition 2.5, although it is used in this way. The key property of the covering balls is again such that the centers are either a) in Ω (in which case the stretched ball is contained in Ω); or b) on a face (in which case the stretched ball $\widehat{B}_i$ is such that $\widehat{B}_i \cap \Omega$ is a half-ball); or c) on an edge in which case $\widehat{B}_i \cap \Omega$ is a dihedral angle (see Definition A.4); or d) in a vertex in which case $\widehat{B}_i \cap \Omega$ is a solid angle (see Definition A.4).

Theorem A.6 *Let* $\Omega \subset \mathbb{R}^3$ *be a Lipschitz polyhedron with faces* $\mathcal{F}$*, edges* $\mathcal{E}$*, and vertices* $\mathcal{A}$*. Let* $M_{\mathcal{A}} \subset \mathcal{A}$ *and* $M_{\mathcal{E}} \subset \mathcal{E}$*. Let* $M = \overline{M} = \overline{M_{\mathcal{A}} \cup M_{\mathcal{E}}}$ *and fix* $\varepsilon \in (0,1)$*. Let* $\mathcal{A}' := \{A \in \mathcal{A} \,|\, A \notin M\}$ *be the vertices not in* M *and* $\mathcal{E}' := \{e \in \mathcal{E} \,|\, \overline{e} \cap M = \emptyset\}$ *be the edges not abutting* M*. Then:*

(i) *(non-singular vertices) There is a sectorial neighborhood* $S_{\mathcal{A}',M,\widetilde{c}}$ *of the vertices in* $\mathcal{A}'$ *and a constant* $c \in (0,1)$ *such that* $S_{\mathcal{A}',M,\widetilde{c}}$ *is covered by balls* $B_i := \overline{B}_{c\,\mathrm{dist}(x_i,M)}(x_i)$ *with centers* $x_i \in \mathcal{A}'$*. Furthermore, the stretched balls* $\widehat{B}_i := \overline{B}_{(1+\varepsilon)c\,\mathrm{dist}(x_i,M)}(x_i)$ *are solid angles and satisfy a finite overlap property on* $\mathbb{R}^3$*.*

(ii) *(non-singular edges) Fix a sectorial neighborhood* $\mathcal{U} := S_{\mathcal{A}',M,c'}$ *of* $\mathcal{A}'$*. For each edge* $e \in \mathcal{E}'$*, there is a sectorial neighborhood* $S_{e,M,\widetilde{c}}$ *and a constant* $c \in (0,1)$ *such that* $S_{e,M,\widetilde{c}} \setminus \mathcal{U}$ *is covered by balls* $B_i = \overline{B}_{c\,\mathrm{dist}(x_i,M)}(x_i)$ *whose centers* x_i *are located on* e*. Furthermore, the stretched balls* $\widehat{B}_i = \overline{B}_{(1+\varepsilon)c\,\mathrm{dist}(x_i,M)}(x_i)$ *satisfy a finite overlap property on* $\mathbb{R}^3$ *and* $\widehat{B}_i \cap \Omega$ *is a dihedral angle.*

(iii) *(faces) Fix a sectorial neighbood* $\mathcal{U} := S_{\mathcal{E},M,c'}$ *of* $\mathcal{E}$*. There is a sectorial neighborhood* $S_{\mathcal{F},M,\widetilde{c}}$ *and a constant* $c \in (0,1)$ *such that* $S_{\mathcal{F},M,\widetilde{c}} \setminus \mathcal{U}$ *is covered by balls* $B_i = \overline{B}_{c\,\mathrm{dist}(x_i,M)}(x_i)$ *with centers* $x_i \in \partial\Omega$*. Furthermore, the stretched balls* $\widehat{B}_i = \overline{B}_{(1+\varepsilon)c\,\mathrm{dist}(x_i,M)}(x_i)$ *satisfy a finite overlap property on* $\mathbb{R}^3$ *and* $\widehat{B}_i \cap \Omega$ *is a half-ball.*

(iv) *(interior) Fix a sectorial neighbood* $\mathcal{U} := S_{\mathcal{F},M,c'}$ *of* $\mathcal{F}$*, where* $\mathcal{F}$ *is the set of faces. Then there is* $c \in (0,1)$ *such that* $\Omega \setminus \mathcal{U}$ *is covered by balls* $B_i =$

$\overline{B}_{c\,\mathrm{dist}(x_i,M)}(x_i)$ *with centers* $x_i \in \Omega$. *Furthermore, the stretched balls* $\widehat{B}_i = \overline{B}_{(1+\varepsilon)c\,\mathrm{dist}(x_i,M)}(x_i)$ *satisfy a finite overlap property on* $\mathbb{R}^3$ *and* $\widehat{B}_i \subset \Omega$.

Proof Follows from Lemmas A.3 and A.1. $\square$

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