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Analytical and numerical treatment of singular linear BVPs with unsmooth inhomogeneity

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Abstrakt. We investigate analytical and numerical properties of systems of linear ordinary differential equations with unsmooth nonintegrable inhomogeneities and a time singularity of the first kind. We are especially interested in specifying the structure of general linear two-point boundary conditions guaranteeing existence and uniqueness of solutions which are continuous on a closed interval including the singular point $t = 0$. Moreover, we study the convergence behaviour of collocation schemes applied to solve the problem numerically.

Keywords: linear systems of ODEs, singular boundary value problem, time singularity of the first kind, unsmooth inhomogeneity, existence and uniqueness, collocation method, convergence
PACS: 02.30.Hq, 02.60.Lj, 02.70.Jn

INTRODUCTION

Singular boundary value problems (BVPs) arise in numerous applications in natural sciences and engineering and therefore, for many years they have been in the focus of extensive research. We investigate analytical and numerical properties of linear singular boundary value problems of the form:

$$y'(t) = \frac{M}{t}y(t) + \frac{f(t)}{t},\ t \in (0,1], \tag{1}$$

$$B_0 y(0) + B_1 y(1) = \beta, \tag{2}$$

where $f \in C[0,1]$ but $f(t)/t$ may not be integrable on $[0,1]$, $M \in \mathbb{R}^{n\times n}, B_0, B_1 \in \mathbb{R}^{m\times n}$ and $\beta \in \mathbb{R}^m$, $m \le n$. We are mainly interested to find out under which circumstances the above problem has a solution $y \in C[0,1]$. It turns out that constant matrices B_0 and B_1 are subject to certain restrictions for a problem to be well–posed.

The BVPs of type (1), (2) arise in the modelling of the avalanche run up [9] and occur when the system of regular ordinary differential equations (ODEs) $u'(x) = Mu(x) + g(x)$, posed on the semi-infinite interval $x \in [0,\infty)$, is transformed by $x = -\ln t$ to a finite domain $t \in (0,1]$.

To compute the numerical solution of the following BVP:

$$y'(t) = \frac{M}{t}y(t) + g(t),\ t \in (0,1], \quad B_0 y(0) + B_1 y(1) = \beta, \tag{3}$$

where $g \in C[0,1]$, polynomial collocation was proposed in [3]. This was motivated by its advantageous convergence properties for (3), while in the presence of a singularity other high order methods show order reductions and become inefficient [5]. As an extension of previous results [11], we apply a piecewise collocation method to the more general BVP (1), (2), where the coefficient matrix is allowed to have eigenvalues with both positive and negative real parts. For this general problem no convergence analysis has not been carried out yet. However, according to numerical experiments, discussed at the end of the current paper, the classical stage convergence order of collocation is shown to hold.

ANALYTICAL PROPERTIES

The analytical properties of (1), (2) have been discussed in [11]. We recapitulate the most important results, where particular attention is paid to the structure of the most general boundary conditions which are necessary and sufficient for the existence of a unique continuous solution on the closed interval $[0,1]$. It turns out that the form of such conditions depends on the spectral properties of the coefficient matrix M. Therefore, we distinguish between three cases, where all eigenvalues of M have negative real parts, positive real parts, or they are zero.

Before discussing BVP (1), (2), we first consider the ODE system (1) and construct the general solution of (1). We denote by $J \in \mathbb{C}^{n\times n}$ the Jordan canonical form of M and by $E \in \mathbb{C}^{n\times n}$ the associated matrix of the generalized eigenvectors of M. Thus, $M = EJE^{-1}$. In the case that the matrix J consists of l Jordan boxes, $J_1, J_2, \ldots, J_l$, the fundamental solution matrix has the form of the block diagonal matrix, $t^J = \operatorname{diag}(t^{J_1}, t^{J_2}, \ldots, t^{J_l})$, where

$$t^{J_k} = t^{\lambda_k} \begin{pmatrix} 1 & \ln t & \frac{(\ln t)^2}{2} & \ldots & \frac{(\ln t)^{n_k-1}}{(n_k-1)!} \\ 0 & 1 & \ln t & \ldots & \frac{(\ln t)^{n_k-2}}{(n_k-2)!} \\ 0 & \ddots & 1 & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \ln t \\ 0 & \ldots & \ldots & 0 & 1 \end{pmatrix}, \quad t \in (0,1]. \tag{4}$$

Here, $\lambda_k = \sigma_k + i\rho_k \in \mathbb{C}$ is an eigenvalue of M and $\dim J_1 + \dim J_2 + \cdots + \dim J_l = n$. The general solution of equation (1) is then given by

$$y(t) = t^M c + t^M \int_1^t s^{-M-I} f(s)\, \mathrm{d}s, \quad t \in (0,1],$$

where $c = Ed \in \mathbb{C}^n$ and $t^M = Et^J E^{-1} \in \mathbb{C}^{n\times n}$. From the structure of the matrix t^{J_k} in (4), it is obvious that the solution contribution related to the k-th Jordan box may become unbounded for $t = 0$. Apparently, the asymptotic behaviour of the solution depends on the sign of the real part σ_k of the associated eigenvalue λ_k. Therefore, we have to distinguish between three cases, $\sigma_k < 0$, $\lambda_k = 0$, and $\sigma_k > 0$. We assume that M has no purely imaginary eigenvalues to exclude solutions of the form $t^{i\rho} = \cos(\rho \ln t) + i \sin(\rho \ln t)$.

In the case where all eigenvalues of M have negative real parts, it is necessary to prescribe initial conditions of a certain structure to guarantee that the solution is continuous on $[0,1]$. Moreover, this continuous solution of the associated initial value problem is shown to be unique. In the case where all eigenvalues of the matrix M have positive real parts, there exists a unique continuous solution of a terminal value problem. It turns out that its smoothness depends not only on the smoothness of an inhomogeneity f but also on the size of real parts of the eigenvalues of M. To provide the continuity of a solution in the case of zero eigenvalues, we need some more structure in f close to the singularity. For the detailed analysis, we refer to [11].

It is clear from the previous considerations that the form of the boundary conditions which guarantee the existence of a unique continuous solution of (1), (2) will depend on the spectral properties of the coefficient matrix M. Let S, R, and H denote projections onto the subspace spanned by the eigenvectors associated with eigenvalues with positive real parts, spanned by eigenvectors associated with zero eigenvalues, and spanned by principal eigenvectors associated with zero eigenvalues, respectively. Moreover, we define $Z := R + H$, $P := R + S$. We will also use $\tilde{P}$, $\tilde{R}$ to denote the matrices consisting of the maximal set of linearly independent columns of the respective projections. In order to obtain the main result for general BVP we have to assume that the inhomogeneity $f \in C[0,1]$ satisfies $Sf \in C^1[0,1]$ and $Zf(t) = O(t^\alpha h(t))$ for $t \to 0$, where h is continuous at zero. Furthermore, we have to assume that the $m \times m$ matrix $B_0\tilde{R} + B_1\tilde{P}$ is nonsingular, where $m = rankP$. Then, BVP (1), (2) has a unique continuous solution $y \in C[0,1]$. This solution satisfies two initial conditions, $Hy(0) = 0, MNy(0) = -Nf(0)$ which are necessary and sufficient for $y \in C[0,1]$.

NUMERICAL EXPERIMENTS

In this section we propose a polynomial collocation for the numerical treatment of BVP (1), (2), which we assume to be uniquely solvable. To obtain the collocation solution the interval of integration $[0,1]$ is partitioned via an equidistant mesh Δ,

$$\Delta := \{0 = t_0 < t_1 < \ldots < t_{I-1} < t_I = 1,\ t_j = jh,\ j = 0,\ldots,I = 1/h\},$$

and in each subinterval $[t_j, t_{j+1}]$, we introduce k collocation nodes

$$t_{jl} := t_j + u_l h,\ j = 0,\ldots,I-1,\ l = 1,\ldots,k,$$

where $0 < u_1 < u_2 < \ldots < u_k \leq 1$. We approximate the analytical solution by a globally continuous, piecewise polynomial function of a degree less or equal to k on each subinterval $[t_j, t_{j+1}]$ such that the numerical solution satisfies BVP (1), (2) at the collocation points. The convergence properties of the collocation applied to a certain subclass of (1), (2) was studied in the context of an initial value problems in [11] where we assume that the eigenvalues of the coefficient matrix M have no positive real parts. For an appropriately smooth solution, the polynomial collocation method executed with k arbitrary collocation points retains its classical stage order $O(h^k)$ uniformly in t. For Gaussian points the so-called small superconvergence order $O(h^{k+1})$ was also shown to hold uniformly in t. We expect the same convergence behaviour for polynomial collocation applied to BVPs (1), (2). The convergence order for this most general problem class, where the spectrum of M contains eigenvalues with both negative and positive real parts, has not been investigated yet. However, some new numerical simulations are already available. We have constructed model problems and run the collocation code `bvpsuite` [6] on coherently refined meshes to see the empirically estimated convergence orders of the scheme.

We consider the following boundary value problem:

$$y'(t) = \frac{M}{t} y(t) + \frac{f(t)}{t},\ t \in (0,1], \quad \begin{pmatrix} -2 & 3 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} y(0) + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 1 \end{pmatrix} y(1) = \begin{pmatrix} \frac{1}{2} \\ 0 \\ \frac{4}{5} \end{pmatrix}, \tag{5}$$

where

$$M = \begin{pmatrix} 1 & -3 & 0 \\ 2 & -4 & 0 \\ 2 & -14 & 10 \end{pmatrix}, \quad f(t) = \begin{pmatrix} \exp(t) + 3t^{\frac{21}{2}} \\ \exp(t) + 2t^{\frac{21}{2}} \\ \exp(t) + 2t^{\frac{21}{2}} + 2 \end{pmatrix}.$$

The matrix M has both positive and negative eigenvalues $\lambda_1 = 10, \lambda_2 = -2, \lambda_3 = -1$. The exact solution y of (5) has the form

$$y(t) = \begin{pmatrix} t^{-2} + t^{-1}\exp(t) - t^{-2}\exp(t) + \frac{6}{23} t^{\frac{21}{2}} \\ t^{-2} + t^{-1}\exp(t) - t^{-2}\exp(t) + \frac{4}{23} t^{\frac{21}{2}} \\ t^{-2} + t^{-1}\exp(t) - t^{-2}\exp(t) + \frac{4}{23} t^{\frac{21}{2}} + t^{10} - \frac{1}{5} \end{pmatrix}.$$

We see that $y \in C^{10}[0,1]$. In Tables 1 and 2, we illustrate the convergence behaviour for collocation executed at k equidistant and Gaussian collocation points, respectively. The number of the collocation points k was chosen to vary from 1 to 8. However, in the simulations shown here, we report only on the values 1 to 4 since the results for 5 to 8 are very similar. The maximal global error is computed in the mesh points, $\|Y_h - Y\|_\infty := \max_{0 \leq j \leq I} |p(t_j) - y(t_j)|$. The estimated order of convergence p and the error constant c are estimated using two consecutive meshes with the step sizes h and $h/2$.

According to the experiments, the empirical convergence orders support the following hypothesis: Polynomial collocation shows the classical convergence behaviour $O(h^k)$ for BVPs provided that their solutions are appropriately smooth. This hypothesis may become a subject of further studies. For Gaussian points, we observe the superconvergence order 2^k. However, in general, the superconvergence order $O(h^{2k})$ in the mesh points cannot be expected to hold. Counterexamples in [3] show that the superconvergence does not hold even for problems (3) with smooth inhomogeneity. According to results for problems with nonpositive real parts of eigenvalues [11], we do not expect better convergence behaviour than $O(h^{k+1})$.

Tabulka 1. BVP (5): Convergence of the collocation scheme, $k = 1, 2$

h	$k=1$						$k=2$					
	equidistant points			Gaussian points			equidistant points			Gaussian points		
	$\|Yh-Y\|_\infty$	c	p	$\|Yh-Y\|_\infty$	c	p	$\|Yh-Y\|_\infty$	c	p	$\|Yh-Y\|_\infty$	c	p
1/1	5.5e-01	–	–	5.5e-01	–	–	2.5e-01	–	–	1.7e-01	–	–
1/2	7.5e-02	4.1e+00	2.89	7.5e-02	4.1e+00	2.89	3.7e-02	1.7e+00	2.74	4.8e-03	5.6e+00	5.09
1/4	2.1e-02	9.7e-01	1.85	2.1e-02	9.7e-01	1.85	9.9e-03	5.2e-01	1.90	1.5e-04	5.2e+00	5.04
1/8	5.3e-03	1.2e+00	1.96	5.3e-03	1.2e+00	1.96	2.5e-03	6.0e-01	1.97	9.0e-06	6.6e-01	4.04
1/16	1.3e-03	1.3e+00	1.99	1.3e-03	1.3e+00	1.99	6.3e-04	6.4e-01	1.99	5.6e-07	5.8e-01	3.99
1/32	3.3e-04	1.4e+00	2.00	3.3e-04	1.4e+00	2.00	1.6e-04	6.5e-01	2.00	3.5e-08	5.9e-01	4.00
1/64	8.4e-05	1.4e+00	2.00	8.4e-05	1.4e+00	2.00	4.0e-05	6.5e-01	2.00	2.2e-09	5.9e-01	4.00
1/128	2.1e-05	1.4e+00	2.00	2.1e-05	1.4e+00	2.00	9.9e-06	6.5e-01	2.00	1.4e-10	5.9e-01	4.00
1/256	5.2e-06	1.4e+00	2.00	5.2e-06	1.4e+00	2.00	2.5e-06	6.5e-01	2.00	8.6e-12	5.9e-01	4.00
1/512	1.3e-06	1.4e+00	2.00	1.3e-06	1.4e+00	2.00	6.2e-07	6.5e-01	2.00	5.3e-13	6.2e-01	4.01

Tabulka 2. BVP (5): Convergence of the collocation scheme, $k = 3, 4$

h	$k=3$						$k=4$					
	equidistant points			Gaussian points			equidistant points			Gaussian points		
	$\|Yh-Y\|_\infty$	c	p	$\|Yh-Y\|_\infty$	c	p	$\|Yh-Y\|_\infty$	c	p	$\|Yh-Y\|_\infty$	c	p
1/1	9.4e-02	–	–	3.0e-02	–	–	2.7e-02	–	–	3.4e-03	–	–
1/2	2.7e-03	3.2e+00	5.10	2.8e-04	3.3e+00	6.75	1.1e-03	6.8e-01	4.64	1.3e-05	9.0e-01	8.02
1/4	1.8e-04	6.0e-01	3.89	3.7e-06	1.7e+00	6.27	7.3e-05	2.5e-01	3.91	5.2e-08	8.4e-01	7.98
1/8	1.2e-05	7.1e-01	3.97	5.4e-08	1.2e+00	6.09	4.6e-06	2.8e-01	3.98	2.1e-10	8.6e-01	7.99
1/16	7.4e-07	7.5e-01	3.99	8.3e-10	9.7e-01	6.02	2.9e-07	3.0e-01	3.99	8.1e-13	8.8e-01	8.00
1/32	4.6e-08	7.7e-01	4.00	1.3e-11	9.1e-01	6.01	1.8e-08	3.0e-01	4.00	3.6e-15	5.0e-01	7.83
1/64	2.9e-09	7.7e-01	4.00	2.0e-13	9.0e-01	6.00	1.1e-09	3.0e-01	4.00	6.7e-16	8.2e-11	2.42
1/128	1.8e-10	7.7e-01	4.00	2.8e-15	2.0e+00	6.17	7.1e-11	3.0e-01	4.00	1.1e-15	1.9e-17	-0.74
1/256	1.1e-11	7.7e-01	4.00	4.2e-15	9.7e-17	-0.60	4.4e-12	3.0e-01	4.00	4.2e-15	2.6e-20	-1.93
1/512	7.0e-13	7.8e-01	4.00	2.4e-15	5.8e-13	0.79	2.8e-13	3.1e-01	4.00	2.8e-15	1.8e-13	0.60

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