ASC Report No. 13/2014

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Institute for Analysis and Scientific Computing Vienna University of Technology — TU Wien
www.asc.tuwien.ac.at ISBN 978-3-902627-05-6

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ISBN 978-3-902627-05-6

Simultaneous quasi-optimal convergence in FEM-BEM coupling

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We consider the symmetric FEM-BEM coupling that connects two linear elliptic second order partial differential equations posed in a bounded domain Ω and its complement, where the exterior problem is restated by an integral equation on the coupling boundary $\Gamma = \partial\Omega$. We assume that the corresponding transmission problem admits a shift theorem by more than $1/2$. We analyze the discretization by piecewise polynomials of degree k for the domain variable and piecewise polynomials of degree $k-1$ for the flux variable on the coupling boundary. Given sufficient regularity we show that (up to logarithmic factors) the optimal convergence $O(h^{k+1/2})$ in the $H^{-1/2}(\Gamma)$-norm is obtained for the flux variable, while classical arguments by Céa-type quasi-optimality and standard approximation results provide only $O(h^k)$ for the overall error in the natural product norm on $H^1(\Omega) \times H^{-1/2}(\Gamma)$.

Keywords: FEM-BEM coupling, *a priori* convergence analysis, transmission problem
MOS subject classification: 65N30; 65N38

1. Introduction.

The coupling of a linear differential equation in an exterior domain with an equation in a domain Ω arises frequently in numerical computations. One way to tackle such a problem is to use a FEM-BEM (finite element–boundary element) coupling procedure. Several techniques are available (see, e.g., [4] for an overview); typically, they involve the introduction of an extra unknown φ on the coupling boundary $\Gamma := \partial\Omega$. Since the recent contributions by Sayas [26], the stability and convergence analysis of the coupled problem is fairly well-understood: for many coupling procedures, one has quasi-optimality in (natural) norms that involve *both* the primal variable u on Ω and the variable φ on Γ, see again [4]. In many situations, the best approximation properties of the spaces used to approximate u and φ do not match. An example for such a mismatch is the setting of the present paper: here, u is approximated in $H^1(\Omega)$ by piecewise polynomials of degree k, and φ is approximated in $H^{-1/2}(\Gamma)$ by piecewise polynomials of degree $k-1$. The optimal rates are therefore $O(h^k)$ and $O(h^{k+1/2})$, respectively. The standard convergence theory, however, gives only $O(h^k)$ for both unknowns. In the present paper we show that (up to logarithmic terms) convergence $O(h^{k+1/2})$ is possible for the approximation of φ under suitable assumptions. We will show this for the symmetric FEM-BEM coupling procedure which has independently been proposed by Costabel [10] and Han [18]. We believe, however, that an extension to other techniques such as the Johnson-Nédélec coupling[20] is possible. Overall, this gives a first mathematical answer to observations in [7, 5], where (optimal) higher-order convergence of φ was noted for adaptive FEM-BEM computations.

The present work is closely related to our previous works [24, 23, 19], where the convergence of the Lagrange multiplier in mortar methods [24] and the convergence of surface fluxes [23] in mixed methods were studied. The unifying theme of these works is to obtain improved and even optimal convergence rates for the quantities associated with lower-dimensional manifolds. This entails a second link between all these works: they rely on the same analytical tools, namely, duality arguments that require the analysis of elliptic problems with right-hand sides that are supported by a thin tubular neighborhood of some lower-dimensional manifold. The basic mechanism that allows us to exploit, in a quantifiable way, that the support of the right-hand sides is small, is the same one in the works [24, 23, 19] and the present one. We mention that [19] is a refinement of [24] and uses a slightly different method of proof than the original [24]. The present work is closest to [19].

We close this introduction with some remarks on the techniques employed. As in our previous work, we employ regularity assertions in Besov spaces. A feature of this approach is that it allows for a clear formulation of the regularity properties of relevant dual problems and allows us to separate the question of elliptic regularity from FEM duality arguments as much as possible. Nevertheless, the use of Besov spaces is not essential and alternative approaches purely based on weighted Sobolev spaces are possible; we mention here [2, 1] and [21] as well as [15] in the context of mixed methods. Another alternative approach opens up when changing the regularity requirements of the solution: in the present paper, we require the solution u to be in

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the Besov space $B^{k+3/2}_{2,1}(\Omega)$; if instead $W^{k+1,\infty}(\Omega)$-regularity is assumed, then techniques from L^∞-estimates in FEM could be applied. We refer to [15, 30] for examples in this direction in the context of mixed methods.

The paper is structured as follows: in Section 2, we introduce the model problem (8). The variational formulation with the symmetric coupling is given in Section 2.4. Our numerical analysis will rely on duality arguments. The regularity theory for these dual problems, which turn out to be classical transmission problems, is the topic of Sections 2.6 and 2.7. Section 3 is devoted to the numerical analysis. The main result is Theorem 3.4: Estimate (38) gives an error bound for the variable u on a strip of width $O(h)$ near the coupling boundary Γ. Estimate (37) then employs this result to obtain the optimal convergence rate for the error in the variable φ. The variational crime associated with approximating the input data is assessed in Section 3.4.1. Up to logarithmic terms, Theorem 3.13 transfers the results of Theorem 3.4 also to this setting. Section 4 illustrates numerically the convergence results of Theorem 3.4 for several geometries.

2. Preliminaries and model problem.

2.1. Notation and spaces.

Let $\Omega \subset \mathbb{R}^d$, $d \in \{2,3\}$, be a bounded Lipschitz domain with boundary $\Gamma := \partial\Omega$. We assume that the boundary is polygonal/polyhedral domain with N_Γ edges/faces Γ_i, $i = 1,\dots,N_\Gamma$. *

For $s \in \mathbb{R}$, we employ standard notation for the Sobolev spaces $H^s(\Omega)$ and $H^s(\Gamma_i)$, $i \in \{1,\dots,N_\Gamma\}$, see, e.g., [27]. For $s > 0$, $s \notin \mathbb{N}_0$, we define the Besov spaces $B^s_{2,q}(\Omega)$ for $q \in [1,\infty]$ by interpolation (the "real" method, also known as "K-method", [27, 28]):

$$B^s_{2,q}(\Omega) = (H^\sigma(\Omega), H^{\sigma+1}(\Omega))_{\theta,q}, \qquad \sigma = \lfloor s \rfloor, \quad \theta = s - \sigma.$$

We recall that for Banach spaces $X_1 \subset X_0$, the interpolation space $X_{\theta,q} = (X_0, X_1)_{\theta,q}$ with $\theta \in (0,1)$ and $q \in [1,\infty]$ is defined by the norm $\|\cdot\|_{X_{\theta,q}}$ with

$$\|u\|_{X_{\theta,q}} := \left(\int_{t=0}^\infty \left|t^{-\theta}K(t,u)\right|^q \frac{dt}{t}\right)^{1/q}, \quad \text{for } q \in [1,\infty), \quad \text{and} \quad \|u\|_{X_{\theta,\infty}} := \sup_{t>0} t^{-\theta}K(t,u), \tag{1}$$

where $K(t,u) = \inf_{v \in X_1} \|u - v\|_{X_0} + t\|v\|_{X_1}$. We recall the interpolation estimate

$$\|x\|_{X_{\theta,q}} \lesssim \|x\|^{1-\theta}_{X_0}\|x\|^\theta_{X_1} \quad \forall x \in X_1. \tag{2}$$

For Γ, we also employ standard notation for the Sobolev spaces $H^s(\Gamma)$, $s \in [-1,1]$. For $s \geq 0$, we define the spaces $H^s_{pw}(\Gamma) \subset L^2(\Gamma)$ as broken spaces, i.e., we identify them with the product $\prod_i H^s(\Gamma_i)$. We introduce the nonstandard space

$$B^0_{2,\infty}(\Gamma) = (H^{-\varepsilon}(\Gamma), H^\varepsilon(\Gamma))_{1/2,\infty}, \qquad \varepsilon \in (0,1] \text{ arbitrary.}$$

(The precise choice of ε is immaterial due to the reiteration theorem [27, Thm. 26.3].) Important roles in our analysis are played by the distance function δ_Γ, the regularized distance function $\widetilde{\delta}_\Gamma$, and the strips S_h near Γ given by

$$\delta_\Gamma(x) := \operatorname{dist}(x,\Gamma), \qquad \widetilde{\delta}_\Gamma(x) := h + \delta_\Gamma(x), \tag{3}$$

$$S_h := \{x \in \Omega \,|\, \delta_\Gamma(x) < h\}, \qquad h > 0. \tag{4}$$

Naturally, properties of the trace operator $\gamma : H^1(\Omega) \to H^{1/2}(\Gamma) \subset L^2(\Gamma)$ feature prominently in coupling procedures. We recall that γ is also well-defined on $H^{1/2+\varepsilon}(\Omega)$ for all $\varepsilon > 0$ but not on $H^{1/2}(\Omega)$. It is, however, well-defined on the slightly smaller space $B^{1/2}_{2,1}(\Omega) \subset H^{1/2}(\Omega)$ (see, e.g., [28, Thm. 2.9.3], [27, Sec. 32]). We close this section with an embedding result that will be important to exploit additional regularity of the solution and to make use of the smallness of the support of the right-hand side of certain dual problems.

Lemma 2.1 *Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain. Recall the regularized distance function $\widetilde{\delta}_\Gamma := \delta_\Gamma + h$ from (3). Then, for all $z \in H^{1/2-\varepsilon}(\Omega)$ or $z \in B^{1/2}_{2,1}(\Omega)$, there holds*

$$\|\widetilde{\delta}_\Gamma^{-1/2+\varepsilon} z\|_{L^2(\Omega)} \leq \|\delta_\Gamma^{-1/2+\varepsilon} z\|_{L^2(\Omega)} \leq C\|z\|_{H^{1/2-\varepsilon}(\Omega)} \quad \forall\, 0 < \varepsilon \leq 1/2, \tag{5}$$

$$\|\widetilde{\delta}_\Gamma^{-1/2} z\|_{L^2(\Omega)} \leq C|\ln h|^{1/2}\|z\|_{B^{1/2}_{2,1}(\Omega)}, \tag{6}$$

$$\|\widetilde{\delta}_\Gamma^{-1/2-\varepsilon} z\|_{L^2(\Omega)} \leq Ch^{-\varepsilon}\|z\|_{B^{1/2}_{2,1}(\Omega)} \quad \forall \varepsilon > 0, \tag{7}$$

where $C > 0$ depends only on Ω and ε.

*The restriction to polygonal/polyhedral domains instead of smooth of piecewise smooth domains is not essential and due to our desire to use standard polynomial approximation results.

Proof: The estimate involving δ_Γ in (5) can be found, for example, in [17, Thm. 1.4.4.3]. The estimates (6), (7) follow from 1D Sobolev embedding theorems and locally flattening the boundary Γ in the same way as it is done in the proof of [22, Lemma 2.1]. □

2.2. Coupling model problem.

We denote by n the normal vector on Γ pointing into $\Omega^{\rm ext} := \mathbb{R}^d \setminus \overline{\Omega}$. For $d = 2$, we assume the scaling condition $\mathrm{diam}(\Omega) < 1$ so that the single layer operator (defined in (11) below) is a bijection and, in fact, $H^{-1/2}(\Gamma)$-elliptic. We emphasize that Ω is connected by definition.

Let $\mathfrak{A} \in L^\infty(\Omega; \mathbb{R}^{d\times d})$ be pointwise symmetric positive definite and satisfy $\mathfrak{A} \geq \alpha_0 > 0$ for some $\alpha_0 > 0$. We will require $\mathfrak{A} \in C^{0,1}(\overline{\Omega})$ for lowest order discretizations (the case $k = 1$ below) and $\mathfrak{A} \in C^{1,1}(\overline{\Omega})$ for higher order discretizations ($k > 1$ below). We mention in passing that the shift theorems Assumptions 2.5 and 2.9 also implicitly contain certain regularity requirements on $\mathfrak{A}$.

For given data $(f, u_0, \phi_0) \in L^2(\Omega) \times H^{1/2}(\Gamma) \times H^{-1/2}(\Gamma)$, we consider the linear interface problem

$$
\begin{aligned}
-\nabla\cdot(\mathfrak{A}\nabla u) &= f && \text{in } \Omega, && \text{(8a)}\\
-\Delta u^{\rm ext} &= 0 && \text{in } \Omega^{\rm ext}, && \text{(8b)}\\
u - u^{\rm ext} &= u_0 && \text{on } \Gamma, && \text{(8c)}\\
(\mathfrak{A}\nabla u - \nabla u^{\rm ext})\cdot n &= \phi_0 && \text{on } \Gamma, && \text{(8d)}\\
u^{\rm ext} &= O(|x|^{-1}) && \text{as } |x| \to \infty. && \text{(8e)}
\end{aligned}
$$

As usual, these equations are understood in the weak sense, i.e., we look for a solution $(u, u^{\rm ext}) \in H^1(\Omega) \times H^1_{\rm loc}(\Omega^{\rm ext})$, where $H^1_{\rm loc}(\Omega^{\rm ext}) = \{v \mid v \in H^1(K), K \subseteq \Omega^{\rm ext} \cup \Gamma \text{ compact}\}$, and (8c) and (8d) are understood in $H^{1/2}(\Gamma)$ and $H^{-1/2}(\Gamma)$, respectively. It is well-known (see also Lemma 2.3 and Section 2.5 below) that problem (8) admits a unique solution in 3D. In 2D, the given data have to fulfill the compatibility condition

$$\langle f, 1\rangle_\Omega + \langle \phi_0, 1\rangle_\Gamma = 0 \tag{9}$$

to ensure the behavior (8e) of the solution at infinity. Alternatively, one may relax the radiation condition (8e) to $u^{\rm ext} = O(\log|x|)$ as $|x| \to \infty$, see Lemma 2.4 below.

2.3. Operators.

Let G be the Green's function for the Laplacian, i.e.,

$$G(x,y) = -\frac{1}{2\pi}\log|x-y| \quad \text{if } d = 2, \qquad G(x,y) = \frac{1}{4\pi}\frac{1}{|x-y|} \quad \text{if } d = 3. \tag{10}$$

With G, we define the single layer and double layer potentials $\widetilde{\mathrm{V}}$ and $\widetilde{\mathrm{K}}$ by

$$(\widetilde{\mathrm{V}}\,\phi)(x) := \int_\Gamma G(x,y)\phi(y)\,dS(y), \qquad (\widetilde{\mathrm{K}}\,u)(x) := \int_\Gamma \partial^{\rm int}_{n(y)} G(x,y)u(y)\,dS(y) \quad \text{for } x \in \mathbb{R}^d \setminus \Gamma.$$

Recall that $\gamma = \gamma^{\rm int} : H^1(\Omega) \to H^{1/2}(\Gamma)$ denotes the (interior) trace operator. Similarly, $\gamma^{\rm ext} : H^1_{\rm loc}(\Omega^{\rm ext}) \to H^{1/2}(\Gamma)$ denotes the exterior trace operator. The *single layer operator* $\mathrm{V} : H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma)$, the *double layer operator* $\mathrm{K} : H^{1/2}(\Gamma) \to H^{1/2}(\Gamma)$, the *adjoint double layer operator* $\mathrm{K}' : H^{-1/2}(\Gamma) \to H^{-1/2}(\Gamma)$, and the *hypersingular operator* $\mathrm{D} : H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$ are defined by

$$
\begin{aligned}
\mathrm{V}\,\phi &:= \gamma^{\rm int}(\widetilde{\mathrm{V}}\,\phi) = \gamma^{\rm ext}(\widetilde{\mathrm{V}}\,\phi), & \mathrm{D}\,u &:= -\partial_n^{\rm int}(\widetilde{\mathrm{K}}\,u) = -\partial_n^{\rm ext}(\widetilde{\mathrm{K}}\,u),\\
\mathrm{K}'\,\phi &:= \partial_n^{\rm int}(\widetilde{\mathrm{V}}\,\phi) - 1/2\phi = \partial_n^{\rm ext}(\widetilde{\mathrm{V}}\,\phi) + 1/2\phi, & \mathrm{K}\,u &:= \gamma^{\rm int}(\widetilde{\mathrm{K}}\,u) + 1/2u = \gamma^{\rm ext}(\widetilde{\mathrm{K}}\,u) - 1/2u.
\end{aligned} \tag{11}
$$

Here and throughout, we define for sufficiently smooth v

$$\partial_n^{\rm ext} v = \gamma^{\rm ext}(\nabla v)\cdot n.$$

Lemma 2.2 *Let $u \in H^{1/2}(\Gamma)$, $\varphi \in H^{-1/2}(\Gamma)$. Define*

$$u^{\rm ext} := \widetilde{\mathrm{K}}\,u - \widetilde{\mathrm{V}}\,\varphi, \qquad \text{in } \Omega^{\rm ext}. \tag{12}$$

Then, the condition $\mathrm{V}\,\varphi + (1/2 - \mathrm{K})u = 0$ implies the following assertions (i)–(ii).

(i) $\gamma^{\rm ext} u^{\rm ext} = u$ and $\partial_n^{\rm ext} u^{\rm ext} = \varphi$.

(ii) u^{ext} *satisfies the exterior Calderón system:*

$$\gamma^{ext}u^{ext} = (1/2+\mathrm{K})(\gamma^{ext}u^{ext}) - \mathrm{V}(\partial_n^{ext}u^{ext}), \tag{13}$$
$$\partial_n^{ext}u^{ext} = -\mathrm{D}(\gamma^{ext}u^{ext}) + (1/2-\mathrm{K}')(\partial_n^{ext}u^{ext}). \tag{14}$$

Proof: Taking the exterior trace in (12), we obtain

$$\gamma^{ext}u^{ext} = (1/2+\mathrm{K})u - \mathrm{V}\,\varphi = u + (-1/2+\mathrm{K})u - \mathrm{V}\,\varphi = u.$$

A calculation (which is non-trivial for the 2D case) shows that u^{ext} satisfies the following, second representation formula (see, e.g., the proof of [6, Lemma 2.3] for details):

$$u^{ext} = \widetilde{\mathrm{K}}\,\gamma^{ext}u^{ext} - \widetilde{\mathrm{V}}\,\partial_n^{ext}u^{ext}.$$

The exterior Calderón system then follows from taking the trace and the trace of the (exterior) normal derivative. Then, the bijectivity of V implies $\partial_n^{ext}u^{ext} = \varphi$. □

2.4. Bilinear forms.

To state the FEM-BEM coupling (15) for the model problem (8), we define bilinear forms:

$$a(u,v) := \langle \mathfrak{A}\nabla u, \nabla v\rangle_\Omega, \qquad \widetilde{a}(u,v) := a(u,v) + \langle \mathrm{D}\,u, v\rangle_\Gamma,$$
$$c(\varphi,\psi) := \langle \mathrm{V}\,\varphi, \psi\rangle_\Gamma, \qquad b(u,\psi) := \langle (1/2-\mathrm{K})u, \psi\rangle_\Gamma.$$

The bilinear form $c(\cdot,\cdot)$ induces a norm that is equivalent to the $H^{-1/2}(\Gamma)$-norm: $\|\psi\|_V^2 := c(\psi,\psi) \sim \|\psi\|^2_{H^{-1/2}(\Gamma)}$. For bounded linear functionals $L_1 : H^1(\Omega) \to \mathbb{R}$ and $L_2 : H^{-1/2}(\Gamma) \to \mathbb{R}$, we consider the block system

$$\widetilde{a}(u,v) - b(v,\varphi) = L_1(v) \qquad \forall v \in H^1(\Omega), \tag{15a}$$
$$b(u,\psi) + c(\varphi,\psi) = L_2(\psi) \qquad \forall \psi \in H^{-1/2}(\Gamma). \tag{15b}$$

We set

$$X := H^1(\Omega) \times H^{-1/2}(\Gamma). \tag{15c}$$

The Galerkin formulation is obtained in the usual fashion: For a conforming subspace

$$X_h := V_h \times M_h \subset X, \tag{16a}$$

we define $(u_h, \varphi_h) \in X_h$ by requiring that (15) be satisfied with the spaces $H^1(\Omega)$ and $H^{-1/2}(\Gamma)$ replaced with V_h and M_h, i.e.,

$$\widetilde{a}(u_h,v) - b(v,\varphi_h) = L_1(v) \qquad \forall v \in V_h, \tag{16b}$$
$$b(u_h,\psi) + c(\varphi_h,\psi) = L_2(\psi) \qquad \forall \psi \in M_h. \tag{16c}$$

We note that both the system (15) and its discrete counterpart have unique solutions for any $(L_1, L_2) \in X'$ by coercivity properties of (15), which are collected in the following lemma.

Lemma 2.3 *The bilinear form* $A((u,\varphi);(v,\psi)) := \widetilde{a}(u,v) - b(v,\phi) + b(u,\psi) + c(\varphi,\psi)$ *satisfies:*

(i) $A(\cdot,\cdot)$ *is semi-definite:* $A((u,\varphi);(u,\varphi)) \geq C\left[\|\nabla u\|^2_{L^2(\Omega)} + \|\varphi\|_V^2\right]$, *where* $C > 0$ *depends only on* Γ *and the coercivity constant of* $\mathfrak{A}$.

(ii) $A(\cdot,\cdot)$ *satisfies a Gårding inequality.*

(iii) *The operator* $\mathbf{A} : X \to X'$ *induced by* $A(\cdot,\cdot)$ *is injective.*

(iv) *The operator* $\mathbf{A} : X \to X'$ *induced by* $A(\cdot,\cdot)$ *satisfies an inf-sup condition on* X.

(v) *Fix* $\xi \in H^{-1/2}(\Gamma)$ *with* $\langle \xi, 1\rangle_\Gamma \neq 0$ *(in particular,* $\xi \equiv 1$ *is admissible). For any conforming discretization* $X_h \subset X$ *that contains* $(0,\xi)$*, the discrete inf-sup condition is (uniformly) satisfied, i.e., the discrete inf-sup constant depends solely on* $\mathfrak{A}$ *and the choice of* ξ*, but is independent of* h*. In particular, (15) as well as (16) admit unique solutions* $(u,\phi) \in X$ *and* $(u_h,\phi_h) \in X_h$*. Moreover, there is a constant* $C > 0$ *depending only on* $\mathfrak{A}$ *and* ξ*, such that the following quasi-optimality result holds for the solutions* (u,ϕ) *of (15) and its discrete approximation* $(u_h,\phi_h) \in X_h$*:*

$$\|u-u_h\|_{H^1(\Omega)} + \|\varphi - \varphi_h\|_V \leq C \inf_{(v_h,\psi_h)\in X_h} \|u - v_h\|_{H^1(\Omega)} + \|\varphi - \psi_h\|_V. \tag{17}$$

Proof: Obviously, (i) is satisfied. The observation (ii) that **A** satisfies a Gårding inequality was first made in [9]. (It is also found in the seminal works [10, 18], where an additional Dirichlet boundary is assumed.) Together with the injectivity statement of (iii), the inf-sup condition (iv) for $A(\cdot,\cdot)$ holds. Item (v) is shown in [4]. The quasi-optimality assertion (17) is a consequence of the (uniform) discrete inf-sup condition.

Finally, let us discuss the injectivity (iii) of **A**. Starting from

$$A((u,\varphi);(v,\psi)) = 0 \qquad \forall (v,\psi) \in X,$$

we get from (i) that u is constant (Ω is connected!) and $\varphi = 0$. This implies in view of $b(u,\psi) = 0$ for all ψ and the well-known fact $(-1/2 + K)1 = -1$ that

$$0 = (1/2 - \mathrm{K})u \overset{u\ const}{=} -u.$$

□

2.5. Weak formulation and Galerkin approximation.

For the original problem (8), it is convenient to introduce the linear forms

$$L_1(v) := \langle f, v\rangle_\Omega + \langle \phi_0 + \mathrm{D}\, u_0, v\rangle_\Gamma, \tag{18a}$$

$$L_2(\psi) := \langle \psi, (1/2 - \mathrm{K})u_0\rangle_\Gamma. \tag{18b}$$

Our weak formulation of (8) is: Find $(u,\varphi) \in X$ such that

$$\widetilde{a}(u,v) - b(v,\varphi) = L_1(v) \qquad \forall v \in H^1(\Omega), \tag{19a}$$

$$b(u,\psi) + c(\varphi,\psi) = L_2(\psi) \qquad \forall \psi \in H^{1/2}(\Gamma). \tag{19b}$$

The Galerkin approximation is correspondingly given by

$$\widetilde{a}(u,v) - b(v,\varphi) = L_1(v) \qquad \forall v \in V_h, \tag{20a}$$

$$b(u,\psi) + c(\varphi,\psi) = L_2(\psi) \qquad \forall \psi \in M_h. \tag{20b}$$

By Lemma 2.3, we have unique solvability of (19). The following lemma clarifies in what sense it solve (8):

Lemma 2.4 *Let $(u,\varphi) \in X$ solve* (18)–(19). *Define the solution u^{ext} in Ω^{ext} by*

$$u^{\mathrm{ext}} := \widetilde{\mathrm{K}}(u - u_0) - \widetilde{\mathrm{V}}\,\varphi. \tag{21}$$

Then the conditions (8a)–(8d) *are satisfied (in the appropriate senses). Concerning the radiation condition at ∞, we have:*

- *For $d = 3$, the radiation condition* (8e) *is satisfied.*
- *For $d = 2$, the radiation condition* (8e) *is satisfied if the data f, ϕ_0 satisfy the condition*

$$a = 0 \quad \textit{with} \quad a := \langle f, 1\rangle_\Omega + \langle \phi_0, 1\rangle_\Gamma. \tag{22}$$

- *If $d = 2$ and the compatibility condition* (22) *is not fulfilled, then the solution u^{ext} satisfies with $a \in \mathbb{R}$ from* (22)

$$u^{\mathrm{ext}}(x) = a \log|x| + O(1/|x|), \qquad |x| \to \infty.$$

Proof: We first show (8a)–(8d). Integration by parts and varying the test functions yields

$$-\nabla\cdot(\mathfrak{A}\nabla u) = f \quad \text{in } \left(H^1_0(\Omega)\right)', \tag{23}$$

$$(\mathfrak{A}\nabla u)\cdot n + \mathrm{D}\,u - (1/2 - \mathrm{K}')\varphi = \phi_0 + \mathrm{D}\,u_0, \qquad \text{in } H^{-1/2}(\Gamma), \tag{24}$$

$$(1/2 - \mathrm{K})u + \mathrm{V}\,\varphi = (1/2 - \mathrm{K})u_0 \qquad \text{in } H^{1/2}(\Gamma). \tag{25}$$

From (25) and Lemma 2.2, we obtain that the function u^{ext} defined in (21) has the following traces on Γ:

$$\gamma^{\mathrm{ext}} u^{\mathrm{ext}} = u - u_0, \qquad \partial_n^{\mathrm{ext}} u^{\mathrm{ext}} = \varphi.$$

In particular, this proves (8c). Furthermore, Lemma 2.2 (ii) shows

$$\varphi = -\mathrm{D}\,\gamma^{\mathrm{ext}} u^{\mathrm{ext}} + (1/2 - \mathrm{K}')\varphi.$$

Upon insertion into (24), this gives

$$(\mathfrak{A}\nabla u)\cdot n = -\mathrm{D}\,u + (1/2 - \mathrm{K}')\varphi + \phi_0 + \mathrm{D}\,u_0 = -\mathrm{D}\,\gamma^{\mathrm{ext}} u^{\mathrm{ext}} + \varphi + \mathrm{D}\,\gamma^{\mathrm{ext}} u^{\mathrm{ext}} + \phi_0 = \partial_n^{\mathrm{ext}} u^{\mathrm{ext}} + \phi_0,$$

which is (8d).

In 3D, the radiation condition (8e) follows from the decay properties at ∞ of the potentials $\widetilde{\mathrm{V}}$ and $\widetilde{\mathrm{K}}$. In 2D, this decay property at ∞ follows from the properties of $\widetilde{\mathrm{V}}$ and $\widetilde{\mathrm{K}}$ *if* $\langle \varphi, 1\rangle_\Gamma = 0$. The compatibility condition (22) implies this with the test function $v \equiv 1$. Finally, for $d = 2$ and the case where (22) is not satisfied, then the leading order behavior of u^{ext} is clearly $a \log|x|$, and, in fact, $a \log|x| + O(1/|x|)$. □

2.6. The dual problem.

Our FEM analysis will rely on various dual problems. The first dual problem that we consider is: Find $(w, \lambda) \in X$ such that

$$\widetilde{a}(v, w) - b(v, \lambda) = f(v) \qquad \forall v \in H^1(\Omega), \tag{26a}$$

$$b(w, \psi) + c(\psi, \lambda) = 0 \qquad \forall \psi \in H^{-1/2}(\Gamma). \tag{26b}$$

Due to symmetry of $\widetilde{a}(\cdot,\cdot)$ and $c(\cdot,\cdot)$, Lemma 2.3 applies and proves existence and uniqueness of $(w, \lambda) \in X$. We denote the corresponding solution operator by

$$T^{\mathrm{dual}} : (H^1(\Omega))' \to X, \qquad f \mapsto (T^w f, T^\lambda f) := (w, \lambda). \tag{27}$$

If the right-hand side f has the form $f(v) = \langle f, v\rangle_\Omega$ for an $f \in L^2(\Omega)$, then the above developments of Lemma 2.4 show that (w, λ) satisfies the transmission problem (8) with $u_0 = 0$, $\phi_0 = 0$ and, in 2D, the radiation condition

$$u^{\mathrm{ext}} = a \log|x| + O(1/|x|), \qquad |x| \to \infty.$$

Hence, (26) it is a classical transmission problem for which we will make the following assumption:

Assumption 2.5 *There exists $s_0 \in (1/2, 1]$ such that the mapping $f \mapsto T^{\mathrm{dual}} f = (T^w f, T^\lambda f)$ from (27) satisfies*

$$\|T^w f\|_{H^{1+s_0}(\Omega)} + \|T^\lambda f\|_{H^{-1/2+s_0}_{pw}(\Gamma)} \le C\|f\|_{(H^{1-s_0}(\Omega))'}.$$

Remark 2.6 Assumption 2.5 is satisfied in the following simple cases:

1. The coefficient matrix $\mathfrak{A}$ is smooth and Γ is sufficiently smooth. Then, by classical regularity theory, $s_0 = 1$ is possible.
2. In 2D, Ω is a polygon and the matrix $\mathfrak{A}$ has the form $\mathfrak{A}(x) = a(x)\,\mathrm{Id}$ for a scalar-valued function a that is sufficiently smooth. See, e.g., [12], [11, Appendix], [25].
3. The discussion in [14, Rem. 5.1] shows that the shift theorem of Assumption 2.5 is in general false for piecewise smooth, pointwise SPD matrices $\mathfrak{A}$. ∎

Recall the definition of S_h from (3). We have the following

Lemma 2.7 *Let Assumption 2.5 be valid. Then the operator $T^{\mathrm{dual}} : (H^1(\Omega))' \to X$ from (27) satisfies*

$$\|T^w f\|_{B^{3/2}_{2,\infty}(\Omega)} + \|T^\lambda f\|_{B^0_{2,\infty}(\Gamma)} \le C\|f\|_{(B^{1/2}_{2,1}(\Omega))'}. \tag{28}$$

In particular, if $f \in L^2(\Omega)$ with $\operatorname{supp} f \subset \overline{S_h}$, then

$$\|T^w f\|_{B^{3/2}_{2,\infty}(\Omega)} + \|T^\lambda f\|_{B^0_{2,\infty}(\Gamma)} \le Ch^{1/2}\|f\|_{L^2(\Omega)}, \tag{29}$$

$$\|T^w f\|_{H^{3/2+\varepsilon}(\Omega)} + \|T^\lambda f\|_{H^\varepsilon_{pw}(\Gamma)} \le Ch^{1/2-\varepsilon}\|f\|_{L^2(\Omega)} \qquad \forall\, 0 < \varepsilon \le s_0 - 1/2. \tag{30}$$

The constant $C > 0$ in (28)–(29) depends only on Ω and Assumption 2.5, while that of (30) depends additionally on ε.

Proof: We follow the arguments of [24, Lemma 5.2]. The starting point for the proof of (28) is that interpolation and Assumption 2.5 yield with $\theta = 1/(2s_0) \in (0, 1)$ and hence $s_0\theta = 1/2$ well-posedness and stability of

$$T^w : ((H^1(\Omega))', (H^{1-s_0}(\Omega))')_{\theta,\infty} \to (H^1(\Omega), H^{1+s_0}(\Omega))_{\theta,\infty} = B^{1+s_0\theta}_{2,\infty}(\Omega) = B^{3/2}_{2,\infty}(\Omega).$$

The arguments for T^λ proceed along the same lines, but rely on the mapping properties $T^\lambda : (H^1(\Omega))' \to H^{-1/2}(\Gamma)$ as well as $T^\lambda : (H^{1-s_0}(\Omega))' \to H^{-1/2+s_0}_{pw}(\Gamma)$. Assuming, as we may, that $s_0 < 1$, we have $0 < -1/2 + s_0 < 1/2$. Hence, $H^{-1/2+s_0}(\Gamma)$ is isomorphic to the product space $\prod_{i=1}^{N_\Gamma} H^{-1/2+s_0}(\Gamma_i) \equiv H^{-1/2+s_0}_{pw}(\Gamma)$. Thus, interpolation proves well-posedness and stability of

$$T^\lambda : ((H^1(\Omega))', (H^{1-s_0}(\Omega))')_{\theta,\infty} \to (H^{-1/2}(\Gamma), H^{-1/2+s_0}(\Gamma))_{\theta,\infty} = B^{-1/2+s_0\theta}_{2,\infty}(\Gamma) = B^0_{2,\infty}(\Gamma).$$

As in [24, Lemma 5.2] (cf. [28, Thm. 1.11.2] or [27, Lemma 41.3]), we recognize that

$$((H^1(\Omega))', (H^{1-s_0}(\Omega))')_{\theta,\infty} = (B^{1-s_0\theta}_{2,1}(\Omega))' = (B^{1/2}_{2,1}(\Omega))'.$$

The combination of the last three observations proves (28). The proof of (29) follows by the same argument as in [24, Lemma 5.2]. To prove (30), we first note that the case $\varepsilon = s_0 - 1/2$ coincides with Assumption 2.5. For $0 < \varepsilon < s_0 - 1/2$, we argue as for (28). Interpolation with $0 < \theta < 1$ and $s_0\theta = 1/2 + \varepsilon$ yields

$$\|T^w f\|_{H^{3/2+\varepsilon}(\Omega)} + \|T^\lambda f\|_{H^\varepsilon_{pw}(\Gamma)} \lesssim \|f\|_{(H^{1/2-\varepsilon}(\Omega))'},$$

where we again used $H^\varepsilon(\Gamma) \equiv H^\varepsilon_{\rm pw}(\Gamma)$ as $0 < \varepsilon < s_0 - 1/2 \le 1/2$. Next, we use estimate (5) from Lemma 2.1 to see

$$\|f\|_{(H^{1/2-\varepsilon}(\Omega))'} = \sup_{v\in H^{1/2-\varepsilon}(\Omega)} \frac{\langle f, v\rangle}{\|v\|_{H^{1/2-\varepsilon}(\Omega)}} = \sup_{v\in H^{1/2-\varepsilon}(\Omega)} \frac{\langle \delta_\Gamma^{1/2-\varepsilon} f, \delta_\Gamma^{-(1/2-\varepsilon)} v\rangle}{\|v\|_{H^{1/2-\varepsilon}(\Omega)}} \le \|\delta_\Gamma^{1/2-\varepsilon} f\|_{L^2(\Omega)} \sup_{v\in H^{1/2-\varepsilon}(\Omega)} \frac{\|\delta_\Gamma^{-(1/2-\varepsilon)} v\|_{L^2(\Omega)}}{\|v\|_{H^{1/2-\varepsilon}(\Omega)}}$$
$$\lesssim h^{1/2-\varepsilon}\|f\|_{L^2(\Omega)}, \tag{31}$$

where we finally exploited the support property of f. □

The u-components of the solutions of (15) and of (26) solve classical elliptic problem that feature interior regularity. We formulate this in analogy to the corresponding result in [24, Lemma 5.4] and [19, Lemma 2.7]:

Lemma 2.8 *Let z solve*

$$-\nabla\cdot(\mathfrak{A}\nabla z) = v \quad \text{in } \Omega$$

for some $v \in L^2(\Omega)$ with $\operatorname{supp} v \subset \overline{S_h}$. Then there are constants c, $\tilde c$, $c' > 0$ that depend solely on Ω, such that for all sufficiently small $h > 0$ the following assertions (i)–(iv) hold:

(i) If $z \in B^{3/2}_{2,\infty}(\Omega)$, then $\|\delta_\Gamma^{1/2}\nabla^2 z\|_{L^2(\Omega\setminus S_{\tilde c h})} \le C_1\sqrt{|\ln h|}\|z\|_{B^{3/2}_{2,\infty}(\Omega)}$. The constant C_1 depends only on Ω and $\|\mathfrak{A}\|_{C^{0,1}(\overline{\Omega})}$.

(ii) For every $\alpha > 0$, there holds $\|\delta_\Gamma^\alpha\nabla^3 z\|_{L^2(\Omega\setminus S_{\tilde c h})} \le C_2\|\delta_\Gamma^{\alpha-1}\nabla^2 z\|_{L^2(\Omega\setminus S_{c'h})}$. The constant C_2 depends only on Ω and $\|\mathfrak{A}\|_{C^{1,1}(\overline{\Omega})}$.

(iii) If $z \in H^{3/2+\varepsilon}(\Omega)$ for some $\varepsilon \in (0, 1/2)$, then $\|\nabla^2 z\|_{L^2(\Omega\setminus S_{\tilde c h})} \le C_3\, h^{-1/2+\varepsilon}\|z\|_{H^{3/2+\varepsilon}(\Omega)}$. The constant $C_3 > 0$ depends only on Ω, $\|\mathfrak{A}\|_{C^{0,1}(\overline{\Omega})}$, and ε.

(iv) If Assumption 2.5 is valid and if $z = T^w v$ with T^w being the first component of the solution operator $T^{\rm dual}$ from (27), then $\|\nabla^2 z\|_{L^2(\Omega\setminus S_{ch})} \le C_4\|v\|_{L^2(\Omega)}$. The constant $C_4 > 0$ depends only on Ω and $\mathfrak{A}$ through the coercivity constant of $\mathfrak{A}$ and $\|\mathfrak{A}\|_{C^{0,1}(\overline{\Omega})}$.

Proof: *Proof of* (i), (ii)*:* [24, Lemma 5.4] is formulated for $-\Delta$. However, the essential property of the differential operator $-\Delta$ that is required, is just interior regularity. Hence, the result also stands for the present, more general elliptic operator $-\nabla\cdot(\mathfrak{A}\nabla)$. The precise dependence on the coefficient $\mathfrak{A}$ is taken from [16, Thm. 8.10].

Proof of (iii)*:* This follows again by local considerations similar to those employed in the proof of [24, Lemma 5.4] and the crude bound $\delta_\Gamma \gtrsim h$ on $\Omega\setminus S_{\tilde c h}$.

Proof of (iv)*:* In view of (iii), we have to estimate $\|z\|_{H^{3/2+\varepsilon}(\Omega)}$. By the support properties of v, the bound (30) yields $\|z\|_{H^{3/2+\varepsilon}(\Omega)} \le C\, h^{1/2-\varepsilon}\|v\|_{L^2(\Omega)}$. Inserting this in (iii) produces the result. □

2.7. The bidual problem.

Similar to the procedure in [19], the analysis of the discretization of the dual problem requires estimates in norms other than the standard energy-like norms. This analysis therefore requires a second class of problems, which we call the "bidual" problem. It is given as follows: Find $(w, \lambda) \in X$ such that

$$\widetilde{a}(w, v) - b(v, \lambda) = f(v) \qquad \forall v \in H^1(\Omega), \tag{32a}$$
$$b(w, \psi) + c(\lambda, \psi) = 0 \qquad \forall \psi \in H^{-1/2}(\Gamma), \tag{32b}$$

with solution operator $T^{\rm bidual} : f \mapsto (\widetilde{w}, \widetilde{\lambda})$. In view of the symmetry of the bilinear forms $\widetilde{a}(\cdot,\cdot)$ and $c(\cdot,\cdot)$, problem (32) is, of course, essentially the same as the dual problem (26). Thus, Assumption 2.5 holds for (32) if it does for (26). Nevertheless, in order to emphasize the structure of the regularity requirements of our convergence theory, we formulate this shift theorem as a separate assumption.

Assumption 2.9 *There exists $s_0 \in (1/2, 1]$ such that the mapping $f \mapsto T^{\rm bidual}(f) = (w, \lambda)$ given by* (32) *satisfies*

$$\|w\|_{H^{1+s_0}(\Omega)} + \|\lambda\|_{H^{-1/2+s_0}_{pw}(\Gamma)} \le C\|f\|_{(H^{1-s_0}(\Omega))'}.$$

Our analysis will require an understanding of the Galerkin error for certain dual problems. This in turn will lead to a bidual problem with right-hand sides in weighted spaces, which we now analyze:

Lemma 2.10 *Let Assumption 2.9 be valid. Recall the regularized distance function $\widetilde{\delta}_\Gamma := \delta_\Gamma + h$ from* (3). *Let $v \in L^2(\Omega)$ and $0 < \varepsilon \le s_0 - 1/2$. Then, the function $(w, \lambda) = T^{\rm bidual}(\widetilde{\delta}_\Gamma^{-1} v)$ satisfies*

$$\|w\|_{B^{3/2}_{2,\infty}(\Omega)} + \|\lambda\|_{B^0_{2,\infty}(\Gamma)} \le C|\ln h|^{1/2}\|\widetilde{\delta}_\Gamma^{-1/2} v\|_{L^2(\Omega)}, \tag{33}$$

$$\|w\|_{H^{3/2+\varepsilon}(\Omega)} + \|\lambda\|_{H^\varepsilon_{pw}(\Gamma)} \le C h^{-\varepsilon}\|\widetilde{\delta}_\Gamma^{-1/2} v\|_{L^2(\Omega)}. \tag{34}$$

Moreover, the function $(w,\lambda) = T^{\text{bidual}}(\widetilde{\delta}_\Gamma^{-1+2\varepsilon} v)$ *satisfies*

$$\|w\|_{H^{3/2+\varepsilon}(\Omega)} + \|\lambda\|_{H^{\varepsilon}_{pw}(\Gamma)} \leq C\|\widetilde{\delta}_\Gamma^{-1/2+\varepsilon} v\|_{L^2(\Omega)}. \tag{35}$$

The constant $C > 0$ *in* (33) *depends only on* Ω *and Assumption 2.9, while those of* (34)–(35) *depend additionally on* ε.

Proof: We proceed as in [24, Lemma 5.2]. In order to prove (33), we employ Assumption 2.9 and argue as in Lemma 2.7 to see

$$\|w\|_{B^{3/2}_{2,\infty}(\Omega)} + \|\lambda\|_{B^0_{2,\infty}(\Gamma)} \lesssim \|\widetilde{\delta}_\Gamma^{-1} v\|_{(B^{1/2}_{2,1}(\Omega))'}.$$

Then, we compute

$$\|\widetilde{\delta}_\Gamma^{-1} v\|_{(B^{1/2}_{2,1}(\Omega))'} = \sup_{z\in B^{1/2}_{2,1}(\Omega)} \frac{\langle \widetilde{\delta}_\Gamma^{-1} v, z\rangle}{\|z\|_{B^{1/2}_{2,1}(\Omega)}} = \sup_{z\in B^{1/2}_{2,1}(\Omega)} \frac{\langle \widetilde{\delta}_\Gamma^{-1/2} v, \widetilde{\delta}_\Gamma^{-1/2} z\rangle}{\|z\|_{B^{1/2}_{2,1}(\Omega)}} \lesssim \|\widetilde{\delta}_\Gamma^{-1/2} v\|_{L^2(\Omega)} \sup_{z\in B^{1/2}_{2,1}(\Omega)} \frac{\|\widetilde{\delta}_\Gamma^{-1/2} z\|_{L^2(\Omega)}}{\|z\|_{B^{1/2}_{2,1}(\Omega)}}.$$

The application of estimate (6) of Lemma 2.1 concludes the argument. For the estimate (35), we proceed similarly. First, Assumption 2.9 and interpolation yield

$$\|w\|_{H^{3/2+\varepsilon}(\Omega)} + \|\lambda\|_{H^{\varepsilon}_{pw}(\Gamma)} \lesssim \|\widetilde{\delta}_\Gamma^{-1+2\varepsilon} v\|_{(H^{1/2-\varepsilon}(\Omega))'}.$$

Second, we compute

$$\|\widetilde{\delta}_\Gamma^{-1+2\varepsilon} v\|_{(H^{1/2-\varepsilon}(\Omega))'} = \sup_{z\in H^{1/2-\varepsilon}(\Omega)} \frac{\langle \widetilde{\delta}_\Gamma^{-1+2\varepsilon} v, z\rangle}{\|z\|_{H^{1/2-\varepsilon}(\Omega)}} = \sup_{z\in H^{1/2-\varepsilon}(\Omega)} \frac{\langle \widetilde{\delta}_\Gamma^{-1/2+\varepsilon} v, \widetilde{\delta}_\Gamma^{-1/2+\varepsilon} z\rangle}{\|z\|_{H^{1/2-\varepsilon}(\Omega)}} \lesssim \|\widetilde{\delta}_\Gamma^{-1/2+\varepsilon} v\|_{L^2(\Omega)} \sup_{z\in H^{1/2-\varepsilon}(\Omega)} \frac{\|\widetilde{\delta}_\Gamma^{-1/2+\varepsilon} z\|_{L^2(\Omega)}}{\|z\|_{H^{1/2-\varepsilon}(\Omega)}}.$$

An application of Estimate (5) of Lemma 2.1 concludes the proof of (35). Finally, we show (34). First, Assumption 2.9 and interpolation yield

$$\|w\|_{H^{3/2+\varepsilon}(\Omega)} + \|\lambda\|_{H^{\varepsilon}_{pw}(\Gamma)} \lesssim \|\widetilde{\delta}^{-1} v\|_{H^{1/2-\varepsilon}(\Omega)}$$

Then, we compute

$$\|\widetilde{\delta}_\Gamma^{-1} v\|_{(H^{1/2-\varepsilon}(\Omega))'} = \sup_{z\in H^{1/2-\varepsilon}} \frac{\langle \widetilde{\delta}_\Gamma^{-1/2} v, \widetilde{\delta}_\Gamma^{-1/2} z\rangle}{\|z\|_{H^{1/2-\varepsilon}(\Omega)}} \leq \|\widetilde{\delta}_\Gamma^{-1/2} v\|_{L^2(\Omega)} h^{-\varepsilon} \sup_{z\in H^{1/2-\varepsilon}(\Omega)} \frac{\|\widetilde{\delta}_\Gamma^{-1/2+\varepsilon} z\|_{L^2(\Omega)}}{\|z\|_{H^{1/2-\varepsilon}(\Omega)}}.$$

Again, Estimate (5) of Lemma 2.1 finishes the proof. □

3. Numerical analysis.

3.1. Main results.

In the following, we assume that V_h, M_h of (20) are spaces of piecewise polynomials. For future reference, we formulate

Assumption 3.1 *Let* $\mathcal{T}_\Omega$ *and* $\mathcal{T}_\Gamma$ *be two (not necessarily matching) quasi-uniform, affine triangulations of* Ω *and* Γ *into volume and surfaces simplices (i.e., tetrahedrons and surface triangles for* $d = 3$*) both with mesh size* h*. For a fixed* $k \in \mathbb{N}$*, let* $V_h := S^{k,1}(\mathcal{T}_\Omega) := \{v \in H^1(\Omega) \,|\, v|_K \in \mathcal{P}_k \quad \forall K \in \mathcal{T}_\Omega\}$*,* $M_h := S^{k-1,0}(\mathcal{T}_\Gamma) := \{v \in L^2(\Gamma) \,|\, v|_K \in \mathcal{P}_{k-1} \quad \forall K \in \mathcal{T}_\Gamma\}$ *be spaces of piecewise polynomials of degree* k *and* $k-1$*, respectively. Set* $X_h := V_h \times M_h$*.*

Remark 3.2 Although Assumption 3.1 does not require the meshes $\mathcal{T}_\Omega$ and $\mathcal{T}_\Gamma$ to be matching, it is natural to do so in implementations. The analysis of the following Theorem 3.4 can be generalized to the case of two quasi-uniform meshes $\mathcal{T}_\Omega$, $\mathcal{T}_\Gamma$ with differing mesh sizes h_Ω, h_Γ. ■

Our starting point are the Galerkin orthogonalities satisfied by the exact solution $(u,\varphi) \in X$ and its Galerkin approximation $(u_h,\varphi_h) \in X_h$ that are obtained by subtracting (20) from (19); in order to be able to account for certain types of variational crimes we include additionally two linear forms $\varepsilon_1 : H^1(\Omega) \to \mathbb{R}$ and $\varepsilon_2 : H^{-1/2}(\Gamma) \to \mathbb{R}$ on the right-hand side:

$$\widetilde{a}(u-u_h, v) - b(v, \varphi-\varphi_h) = \varepsilon_1(v) \qquad \forall v \in V_h, \tag{36a}$$
$$b(u-u_h, \psi) + c(\varphi-\varphi_h, \psi) = \varepsilon_2(\psi) \qquad \forall \psi \in M_h. \tag{36b}$$

Remark 3.3 The exact Galerkin orthogonalities have the above form (36) with $\varepsilon_1 \equiv 0$ and $\varepsilon_2 \equiv 0$. The terms ε_1 and ε_2 are appropriate to control additional errors introduced by approximating the jumps u_0 and ϕ_0 (cf. (18)), e.g., by piecewise polynomial functions. Such approximations are practically unavoidable in view of the fact that the hypersingular operator appears on the right-hand side (18) of the coupling equations (19). This issue will be studied further in Section 3.4.1 and plays a role in the numerical examples in Section 4. ∎

The main result of this work is the following Theorem. We recall that the standard convergence theory yields $O(h^k)$ under the regularity assumption $(u, \varphi) \in H^{k+1}(\Omega) \times H^{k-1/2}_{pw}(\Gamma)$. Extra regularity of the solution (u, φ) allows us to improve this:

Theorem 3.4 *Let $\mathfrak{A} \in C^{0,1}(\overline{\Omega})$ if $k = 1$ and $\mathfrak{A} \in C^{1,1}(\overline{\Omega})$ if $k > 1$. Let Assumption 2.5 and 2.9 be valid*[†]*. Let $X_h = V_h \times M_h$ be given by Assumption 3.1. Let $(u, \varphi) \in X$ be the solution of (18)–(19) and $(u_h, \varphi_h) \in X_h$ be the solution of (20). Suppose extra regularity $u \in B^{k+3/2}_{2,1}(\Omega)$ and $\varphi \in H^k_{pw}(\Gamma)$. Then, we have*

$$\|\varphi - \varphi_h\|_{H^{-1/2}(\Gamma)} \le C h^{k+1/2}(1 + \delta_{k,1}|\ln h|)\|u\|_{B^{k+3/2}_{2,1}(\Omega)} + C h^{k+1/2}\|\varphi\|_{H^k_{pw}(\Gamma)}, \tag{37}$$

$$\|u - u_h\|_{L^2(S_h)} \le C h^{3/2+k}(1 + \delta_{k,1}|\ln h|)\|u\|_{B^{k+3/2}_{2,1}(\Omega)} + C h^{3/2+k}\|\varphi\|_{H^k_{pw}(\Gamma)}. \tag{38}$$

Here, $\delta_{k,1}$ denotes the Kronecker symbol, i.e., $\delta_{1,1} = 1$ and $\delta_{k,1} = 0$ for $k \ne 1$. The constant $C > 0$ depends only on Ω, the coefficient $\mathfrak{A}$, the approximation order k, Assumptions 2.5 and 2.9, as well as shape regularity of the quasi-uniform triangulations $\mathcal{T}_\Omega$ and $\mathcal{T}_\Gamma$. More precisely, the dependence on $\mathfrak{A}$ is–in addition to Assumptions 2.5, 2.9–in terms of the coercivity constant of $\mathfrak{A}$, the bound $\|\mathfrak{A}\|_{C^{0,1}(\overline{\Omega})}$ for $k = 1$ and $\|\mathfrak{A}\|_{C^{1,1}(\overline{\Omega})}$ for $k > 1$.

The proof of Estimate (38) in Theorem 3.4 is postponed to Section 3.4 since it requires several auxiliary results, which are provided in Section 3.2–3.3. The Estimate (37), however, is an immediate consequence of (38) as we now show.

Proof of Theorem 3.4, equation (37): We suppose that (38) is valid. The proof of (37) is based on the Galerkin orthogonality (36b) (with $\varepsilon_2 \equiv 0$ there). We have

$$\begin{aligned}
\|J_h\varphi - \varphi_h\|_V^2 &= c(J_h\varphi - \varphi_h, J_h\varphi - \varphi_h) \\
&= c(J_h\varphi - \varphi, J_h\varphi - \varphi_h) + c(\varphi - \varphi_h, J_h\varphi - \varphi_h) \\
&\overset{(36b)}{=} c(J_h\varphi - \varphi, J_h\varphi - \varphi_h) - b(u - u_h, J_h\varphi - \varphi_h) \\
&\lesssim \|J_h\varphi - \varphi\|_V\|J_h\varphi - \varphi_h\|_V + \|u - u_h\|_{H^{1/2}(\Gamma)}\|J_h\varphi - \varphi_h\|_V.
\end{aligned}$$

Hence, $\|J_h\varphi - \varphi_h\|_V \lesssim \|\varphi - J_h\varphi\|_V + \|u - u_h\|_{H^{1/2}(\Gamma)}$. The term $\|u - u_h\|_{H^{1/2}(\Gamma)}$ is estimated with an inverse estimate on Γ and a suitable norm equivalence as follows:

$$\begin{aligned}
\|u - u_h\|_{H^{1/2}(\Gamma)} &\le \|u - J_hu\|_{H^{1/2}(\Gamma)} + \|J_hu - u_h\|_{H^{1/2}(\Gamma)} \\
&\lesssim \|u - J_hu\|_{H^{1/2}(\Gamma)} + h^{-1/2}\|J_hu - u_h\|_{L^2(\Gamma)} \\
&\lesssim \|u - J_hu\|_{H^{1/2}(\Gamma)} + h^{-1}\|J_hu - u_h\|_{L^2(S_h)}.
\end{aligned}$$

Next, we use boundedness of the trace operator $\gamma : B^{1/2}_{2,1}(\Omega) \to L^2(\Gamma)$ (see, e.g., [28, Thm. 2.9.3]) to get boundedness of $\gamma : B^{k+3/2}_{2,1}(\Omega) \to H^{k+1}_{pw}(\Gamma) \cap H^1(\Gamma)$. For suitable $J_hu \in V_h$, we have $\|u - J_hu\|_{H^{1/2}(\Gamma)} + h^{-1}\|u - J_hu\|_{L^2(S_h)} \lesssim h^{k+1/2}\|\gamma u\|_{H^{k+1}_{pw}(\Gamma)}$. The approximation properties of M_h yield $\|\varphi - J_h\varphi\|_V \lesssim h^{k+1/2}\|\varphi\|_{H^k_{pw}(\Gamma)}$. Combining these estimates with (38) yields

$$\|\varphi - \varphi_h\|_V \lesssim \|\varphi - J_h\varphi\|_V + \|u - J_hu\|_{H^{1/2}(\Gamma)} + h^{-1}\|u - J_hu\|_{L^2(S_h)} + h^{-1}\|u - u_h\|_{L^2(S_h)} \lesssim h^{k+1/2}\Big[\|\varphi\|_{H^k_{pw}(\Gamma)} + \|u\|_{B^{k+3/2}_{2,1}(\Omega)}\Big].$$

This concludes the proof. □

3.2. Approximation estimates.

We recall that the spaces V_h and M_h have the following approximation properties:

Lemma 3.5 *(i) There is an elementwise defined (nodal) interpolation operator $J^k_h : C(\overline{\Omega}) \to V_h$ with*

$$\|\nabla^j(u - J^k_hu)\|_{L^2(K)} \le C\,\mathrm{diam}\,(K)^{k-j+1}\|\nabla^{k+1}u\|_{L^2(K)}$$

for $j \in \{0, 1, 2\}$ and all $K \in \mathcal{T}_\Omega$.

[†] Recall that these two assumptions coincide in the present case.

(ii) For $\varepsilon \geq 0$ and fixed $0 < D < D'$

$$C^{-1}\|\widetilde{\delta}_\Gamma^{-1/2-\varepsilon}\nabla(u - J_h^k u)\|_{L^2(S_D)} \leq h^k\|\widetilde{\delta}_\Gamma^{-1/2-\varepsilon}\nabla^{k+1}u\|_{L^2(S_{D+h})} \leq C\,h^k\|\nabla^{k+1}u\|_{B^{1/2}_{2,1}(S_{D'})}\begin{cases}|\ln h|^{1/2}, & \text{if } \varepsilon = 0,\\ h^{-\varepsilon}, & \text{if } \varepsilon > 0,\end{cases}$$

(iii) There are bounded linear operators $R_h : B^{3/2}_{2,\infty}(\Omega) \to V_h$ and $Q_h : B^0_{2,\infty}(\Gamma) \to M_h$ with

$$\|w - R_h w\|_{H^1(\Omega)} \leq Ch^{1/2}\|w\|_{B^{3/2}_{2,\infty}(\Omega)}, \qquad \|\psi - Q_h\psi\|_V \leq Ch^{1/2}\|\psi\|_{B^0_{2,\infty}(\Gamma)}.$$

The constant $C > 0$ depends only on k and shape regularity of $\mathcal{T}_\Omega$ resp. $\mathcal{T}_\Gamma$.

Proof: The assertion (i) is well-known. For (ii), we note that (i) yields the first inequality. The second inequality follows from Estimate (6)–(7) of Lemma 2.1. In item (iii), we only show the construction of Q_h. It suffices to consider the lowest order case $k = 1$, i.e., M_h consists of piecewise constant functions. For simplicity, let Q_h be the projection in the $H^{-1/2}(\Gamma)$-inner product. Then for (fixed) $\varepsilon \in (0, 1/2)$ by standard approximation properties $\|\,\mathrm{I} - Q_h\|_{H^{-1/2}(\Gamma)\leftarrow H^{-\varepsilon}(\Gamma)} \leq h^{1/2-\varepsilon}$ and $\|\,\mathrm{I} - Q_h\|_{H^{-1/2}(\Gamma)\leftarrow H^{\varepsilon}(\Gamma)} \leq h^{1/2+\varepsilon}$. The result follows by interpolation. □

3.2.1. Local estimates via duality arguments. For the solution $(u, \varphi) \in X$ of (19) and its Galerkin approximation $(u_h, \varphi_h) \in X_h$, which solves (20), we define the error

$$e := u - u_h. \tag{39}$$

Take the cut-off function χ_{S_h} to be the characteristic function of S_h. Let $(w, \lambda) = T^{\mathrm{dual}}(\chi_{S_h} e)$ be the solution of the dual problem

$$\widetilde{a}(v, w) - b(v, \lambda) = \langle v, \chi_{S_h} e\rangle_\Omega \qquad \forall v \in H^1(\Omega), \tag{40a}$$
$$b(w, \psi) + c(\psi, \lambda) = 0 \qquad \forall \psi \in H^{-1/2}(\Gamma). \tag{40b}$$

Its Galerkin approximation $(w_h, \lambda_h) \in X_h$ is given by

$$\widetilde{a}(v, w_h) - b(v, \lambda_h) = \langle v, \chi_{S_h} e\rangle_\Omega \qquad \forall v \in V_h, \tag{41a}$$
$$b(w_h, \psi) + c(\psi, \lambda_h) = 0 \qquad \forall \psi \in M_h. \tag{41b}$$

Subtracting (41) from (40) leads to the following Galerkin orthogonalities:

$$\widetilde{a}(v, w - w_h) - b(v, \lambda - \lambda_h) = 0 \qquad \forall v \in V_h, \tag{42a}$$
$$b(w - w_h, \psi) + c(\psi, \lambda - \lambda_h) = 0 \qquad \forall \psi \in M_h. \tag{42b}$$

Lemma 3.6 *For any approximation $(J_h u, J_h\varphi) \in X_h$, we have*

$$\|\chi_{S_h} e\|^2_{L^2(\Omega)} = \widetilde{a}(u - J_h u, w - w_h) - b(u - J_h u, \lambda - \lambda_h) - b(w - w_h, \varphi - J_h\varphi) - c(\varphi - J_h\varphi, \lambda - \lambda_h) + \varepsilon_1(w_h) - \varepsilon_2(\lambda_h).$$

Proof: The proof follows from simple manipulations with the Galerkin orthogonalities (36) and (42a) and the defining equations:

$$\begin{aligned}
\langle \chi_{S_h} e, e\rangle_\Omega &\overset{(40a)}{=} \widetilde{a}(e, w) - b(e, \lambda)\\
&= \widetilde{a}(e, w - w_h) + \widetilde{a}(e, w_h) - b(e, \lambda - \lambda_h) - b(e, \lambda_h)\\
&= \widetilde{a}(u - J_h u, w - w_h) + \widetilde{a}(J_h u - u_h, w - w_h) + \widetilde{a}(e, w_h) - b(u - J_h u, \lambda - \lambda_h) - b(J_h u - u_h, \lambda - \lambda_h) - b(e, \lambda_h)\\
&\overset{(42a)}{=} \widetilde{a}(u - J_h u, w - w_h) - b(u - J_h u, \lambda - \lambda_h) + \underbrace{\widetilde{a}(e, w_h)}_{=:I} - \underbrace{b(e, \lambda_h)}_{=:II}.
\end{aligned}$$

We rearrange the terms I and II.

$$\begin{aligned}
I = \widetilde{a}(e, w_h) &\overset{(36a)}{=} b(w_h, \varphi - \varphi_h) + \varepsilon_1(w_h)\\
&= b(w_h, \varphi - J_h\varphi) + b(w_h, J_h\varphi - \varphi_h) + \varepsilon_1(w_h)\\
&= b(w_h - w, \varphi - J_h\varphi) + b(w, \varphi - J_h\varphi) + b(w_h, J_h\varphi - \varphi_h) + \varepsilon_1(w_h)\\
&\overset{(40b),(41b)}{=} b(w_h - w, \varphi - J_h\varphi) - c(\varphi - J_h\varphi, \lambda) - c(J_h\varphi - \varphi_h, \lambda_h) + \varepsilon_1(w_h)\\
II = b(e, \lambda_h) &\overset{(36b)}{=} -c(\varphi - \varphi_h, \lambda_h) + \varepsilon_2(\lambda_h).
\end{aligned}$$

Hence, we obtain

$$\begin{aligned}\langle \chi_{S_h} e, e\rangle_\Omega &= \widetilde{a}(u - J_h u, w - w_h) - b(u - J_h u, \lambda - \lambda_h) + I - II \\ &= \widetilde{a}(u - J_h u, w - w_h) - b(u - J_h u, \lambda - \lambda_h) + b(w_h - w, \varphi - J_h\varphi) \\ &\quad - c(\varphi - J_h\varphi, \lambda) - c(J_h\varphi - \varphi_h, \lambda_h) + c(\varphi - \varphi_h, \lambda_h) + \varepsilon_1(w_h) - \varepsilon_2(\lambda_h) \\ &= \widetilde{a}(u - J_h u, w - w_h) - b(u - J_h u, \lambda - \lambda_h) + b(w_h - w, \varphi - J_h\varphi) - c(\varphi - J_h\varphi, \lambda - \lambda_h) + \varepsilon_1(w_h) - \varepsilon_2(\lambda_h),\end{aligned}$$

which is the desired equality. □

3.3. Analysis of the dual problems: estimating $w - w_h$ and $\lambda - \lambda_h$.

Lemma 3.6 shows that we can infer bounds for the error $u - u_h$ on a strip S_h near Γ from knowledge about the errors $w - w_h$ and $\lambda - \lambda_h$. The additional two terms $\varepsilon_1(w_h)$ and $\varepsilon_2(\lambda_h)$ that appear in Lemma 3.6 will be bounded in Corollary 3.13 below; we recall that they were introduced to treat certain types of variational crimes.

We will need the following regularity assertions for the solution $(w, \lambda) = T^{\mathrm{dual}}(\chi_{S_h} e)$ of the dual problem (26), which follow from Lemma 2.7:

$$\|w\|_{B^{3/2}_{2,\infty}(\Omega)} + \|\lambda\|_{B^0_{2,\infty}(\Gamma)} \lesssim h^{1/2}\|\chi_{S_h} e\|_{L^2(\Omega)}, \tag{43a}$$

$$\|w\|_{H^{3/2+\varepsilon}(\Omega)} + \|\lambda\|_{H^\varepsilon_{pw}(\Gamma)} \lesssim h^{1/2-\varepsilon}\|\chi_{S_h} e\|_{L^2(\Omega)} \quad \forall\, 0 < \varepsilon \leq s_0 - 1/2. \tag{43b}$$

3.3.1. Error analysis of $w - w_h$ and $\lambda - \lambda_h$ in the energy norms. The uniform inf-sup stability of the bilinear form $A(\cdot, \cdot)$ (cf. Lemma 2.3) provides the following *a priori* bound:

Lemma 3.7 *Let Assumption 2.5 be valid. Then, $\|w - w_h\|_{H^1(\Omega)} + \|\lambda - \lambda_h\|_{H^{-1/2}(\Gamma)} \leq Ch\|\chi_{S_h} e\|_{L^2(\Omega)}$. The constant $C > 0$ depends only on Ω, Assumption 2.5, the shape regularity of $\mathcal{T}_\Omega$, $\mathcal{T}_\Gamma$, and $\mathfrak{A}$ through the coercivity constant of $\mathfrak{A}$ and $\|\mathfrak{A}\|_{L^\infty(\Omega)}$.*

Proof: As observed in Section 2.6, the dual problem corresponds to a transmission problem and is hence covered by Lemma 2.3. By the uniform inf-sup stability ascertained in Lemma 2.3, (v), we have the quasi-optimality (17). Combining the regularity assertion (43a) with the approximation properties of Lemma 3.5 gives

$$\begin{aligned}\|w - w_h\|_{H^1(\Omega)} + \|\lambda - \lambda_h\|_{H^{-1/2}(\Gamma)} &\leq C \inf_{(v,\mu)\in X_h} \Big[\|w - v\|_{H^1(\Omega)} + \|\lambda - \mu\|_{H^{-1/2}(\Gamma)}\Big] \leq Ch^{1/2}\Big[\|w\|_{B^{3/2}_{2,\infty}(\Omega)} + \|\lambda\|_{B^0_{2,\infty}(\Gamma)}\Big] \\ &\leq Ch^{1/2}h^{1/2}\|\chi_{S_h} e\|_{L^2(\Omega)}.\end{aligned}$$

This concludes the proof. □

The estimates of Lemma 3.7 allow us to control the terms $b(u - J_h u, \lambda - \lambda_h)$, $b(w - w_h, \varphi - J_h\varphi)$, $c(\varphi - J_h\varphi, \lambda - \lambda_h)$ that appear in Lemma 3.6:

$$|b(u - J_h u, \lambda - \lambda_h)| + |b(w - w_h, \varphi - J_h\varphi)| + |c(\varphi - J_h\varphi, \lambda - \lambda_h)| \tag{44}$$

$$\leq Ch\|\chi_{S_h} e\|_{L^2(\Omega)}\Big[\|u - J_h u\|_{H^{1/2}(\Gamma)} + \|\varphi - J_h\varphi\|_{H^{-1/2}(\Gamma)}\Big]. \tag{45}$$

Moreover, Lemma 3.7 yields the estimate

$$|\langle \mathrm{D}(u - J_h u), w - w_h\rangle_\Gamma| \leq Ch\|\chi_{S_h} e\|_{L^2(\Omega)}\|u - J_h u\|_{H^{1/2}(\Gamma)}, \tag{46}$$

which is a part of the term $\widetilde{a}(u - J_h u, w - w_h)$ in Lemma 3.6. Thus, from the terms appearing in Lemma 3.6, only the term $a(u - J_h u, w - w_h)$ remains to be controlled. Its analysis is more elaborate and requires an analysis of $w - w_h$ in weighted norms. To see how this comes about, we fix $D > 0$ and write with a parameter $\varepsilon \geq 0$ that will be selected later

$$\begin{aligned}|a(u - J_h u, w - w_h)| &= \left|\int_{S_D} \mathfrak{A}\nabla(u - J_h u)\cdot\nabla(w - w_h) + \int_{\Omega\setminus S_D} \mathfrak{A}\nabla(u - J_h u)\cdot\nabla(w - w_h)\right| \\ &\lesssim \|\widetilde{\delta}_\Gamma^{-1/2-\varepsilon}\nabla(u - J_h u)\|_{L^2(S_D)}\|\widetilde{\delta}_\Gamma^{1/2+\varepsilon}\nabla(w - w_h)\|_{L^2(S_D)} + \|\nabla(u - J_h u)\|_{L^2(\Omega\setminus S_D)}\|\nabla(w - w_h)\|_{L^2(\Omega\setminus S_D)}.\end{aligned} \tag{47}$$

The choice $J_h u = J_h^k u$ with the nodal interpolant $J_h^k u$ of Lemma 3.5 puts us on familiar ground for the factors $\|\widetilde{\delta}_\Gamma^{-1/2-\varepsilon}\nabla(u - J_h u)\|_{L^2(S_D)}$ and $\|\nabla(u - J_h u)\|_{L^2(S_D)}$. Hence, we are left with estimating $\|\widetilde{\delta}_\Gamma^{1/2+\varepsilon}\nabla(w - w_h)\|_{L^2(S_D)}$ and $\|\nabla(w - w_h)\|_{L^2(\Omega\setminus S_D)}$, which is achieved in the subsequent Section 3.3.2. Although the parameter $\varepsilon \geq 0$ is arbitrary at this point, we mention that we will select $\varepsilon > 0$ arbitrary (but small) for the case $k > 1$ and $\varepsilon = 0$ for the lowest order case $k = 1$.

3.3.2. Error analysis of $w - w_h$ and $\lambda - \lambda_h$ in weighted norms. We estimate $\|\widetilde{\delta}_\Gamma^{1/2+\varepsilon}\nabla(w - w_h)\|_{L^2(S_D)}$ in a manner that is structurally similar to the procedure in [19] and also [24, Sec. 5.1.2]. Basically, we employ tools from local error analysis of FEM as described, for example, in [29, Sec. 5.3] to control $w - w_h$ in terms of a best approximation in a weighted H^1-norm and a lower-order term in a weighted L^2-norm. The best approximation in a weighted H^1-norm is estimated with a standard nodal interpolant; the lower-order term in a weighted L^2-norm requires more care and is handled in the following Lemma 3.8.

Lemma 3.8 *Let Assumptions 2.5 and 2.9 be valid. With the regularized distance function $\widetilde{\delta}_\Gamma = \delta_\Gamma + h$ from (3) we have for $0 < \varepsilon \le s_0 - 1/2$*

$$\|\widetilde{\delta}_\Gamma^{-1/2}(w - w_h)\|_{L^2(\Omega)} \le C_1 h^{3/2}|\ln h|^{1/2}\|\chi_{S_h} e\|_{L^2(\Omega)}, \tag{48}$$
$$\|\widetilde{\delta}_\Gamma^{-1/2+\varepsilon}(w - w_h)\|_{L^2(\Omega)} \le C_2 h^{3/2+\varepsilon}\|\chi_{S_h} e\|_{L^2(\Omega)}. \tag{49}$$

The constant $C_1 > 0$ depends on the same quantities as the constant in Lemma 3.7 and additionally on Assumption 2.9. The constant $C_2 > 0$ depends furthermore on ε.

Proof: Both estimates require yet another duality argument.
Proof of (48): Abbreviate $e_w := w - w_h$ and let $(z, \psi) := T^{\text{bidual}}(\widetilde{\delta}_\Gamma^{-1} e_w)$ denote the solution of the bidual problem

$$\widetilde{a}(z, v) - b(v, \psi) = \langle \widetilde{\delta}_\Gamma^{-1} e_w, v\rangle_\Omega \qquad \forall v \in H^1(\Omega), \tag{50a}$$
$$b(z, \mu) + c(\psi, \mu) = 0 \qquad \forall \mu \in H^{-1/2}(\Gamma), \tag{50b}$$

From this, we get with $v = e_w$ for arbitrary $J_h z \in V_h$ and $J_h\psi \in M_h$

$$\begin{aligned}
\|\widetilde{\delta}_\Gamma^{-1/2} e_w\|^2_{L^2(\Omega)} &= \widetilde{a}(z, e_w) - b(e_w, \psi) \\
&\overset{(50a)}{=} \widetilde{a}(z - J_h z, e_w) + \widetilde{a}(J_h z, e_w) - b(e_w, \psi - J_h\psi) - b(e_w, J_h\psi) \\
&\overset{(42a),(42b)}{=} \widetilde{a}(z - J_h z, e_w) + b(J_h z, \lambda - \lambda_h) - b(e_w, \psi - J_h\psi) + c(J_h\psi, \lambda - \lambda_h) \\
&= \widetilde{a}(z - J_h z, e_w) + b(J_h z - z, \lambda - \lambda_h) - b(e_w, \psi - J_h\psi) + c(J_h\psi - \psi, \lambda - \lambda_h) \\
&\quad + b(z, \lambda - \lambda_h) + c(\psi, \lambda - \lambda_h) \\
&\overset{(50b)}{=} \widetilde{a}(z - J_h z, e_w) + b(J_h z - z, \lambda - \lambda_h) - b(e_w, \psi - J_h\psi) + c(J_h\psi - \psi, \lambda - \lambda_h).
\end{aligned}$$

This yields

$$\|\widetilde{\delta}_\Gamma^{-1/2} e_w\|^2_{L^2(\Omega)} \lesssim \Big[\|z - J_h z\|_{H^1(\Omega)} + \|\psi - J_h\psi\|_{H^{-1/2}(\Gamma)}\Big]\Big[\|w - w_h\|_{H^1(\Omega)} + \|\lambda - \lambda_h\|_{H^{-1/2}(\Gamma)}\Big].$$

By virtue of Assumption 2.9 and hence the *a priori* estimate (33) of Lemma 2.10, we have

$$(z, \psi) = T^{\text{bidual}}(\widetilde{\delta}_\Gamma^{-1} e_w) \in B^{3/2}_{2,\infty}(\Omega) \times B^0_{2,\infty}(\Gamma) \quad \text{with} \quad \|z\|_{B^{3/2}_{2,\infty}(\Omega)} + \|\psi\|_{B^0_{2,\infty}(\Gamma)} \lesssim |\ln h|^{1/2}\|\widetilde{\delta}_\Gamma^{-1/2} e_w\|_{L^2(\Omega)}.$$

Lemma 3.5, (iii) yields

$$\inf_{(J_h z, J_h\psi)\in X_h}\Big[\|z - J_h z\|_{H^1(\Omega)} + \|\psi - J_h\psi\|_{H^{-1/2}(\Gamma)}\Big] \lesssim h^{1/2}\Big[\|z\|_{B^{3/2}_{2,\infty}(\Omega)} + \|\psi\|_{B^0_{2,\infty}(\Gamma)}\Big] \lesssim h^{1/2}|\ln h|^{1/2}\,\|\widetilde{\delta}_\Gamma^{-1/2} e_w\|_{L^2(\Omega)}.$$

By virtue of Assumption 2.5 and hence the *a priori* estimate (29) of Lemma 2.7, we have

$$(w, \lambda) = T^{\text{dual}}(\chi_{S_h} e) \in B^{3/2}_{2,\infty}(\Omega) \times B^0_{2,\infty}(\Gamma) \quad \text{with} \quad \|w\|_{B^{3/2}_{2,\infty}(\Omega)} + \|\lambda\|_{B^0_{2,\infty}(\Gamma)} \lesssim h^{1/2}\,\|\chi_{S_h} e\|_{L^2(\Omega)}.$$

With the quasi-optimality (17) of (w, λ), we infer

$$\begin{aligned}
\|w - w_h\|_{H^1(\Omega)} + \|\lambda - \lambda_h\|_{H^{-1/2}(\Gamma)} &\lesssim \inf_{(J_h w, J_h\lambda)\in X_h}\Big[\|w - J_h w\|_{H^1(\Omega)} + \|\lambda - J_h\lambda\|_{H^{-1/2}(\Gamma)}\Big] \\
&\lesssim h^{1/2}\Big[\|w\|_{B^{3/2}_{2,\infty}(\Omega)} + \|\lambda\|_{B^0_{2,\infty}(\Gamma)}\Big] \lesssim h\,\|\chi_{S_h} e\|_{L^2(\Omega)}.
\end{aligned}$$

Altogether, we arrive at

$$\|\widetilde{\delta}_\Gamma^{-1/2} e_w\|_{L^2(\Omega)} \lesssim h^{3/2}|\ln h|^{1/2}\|\chi_{S_h} e\|_{L^2(\Omega)}.$$

Proof of (49): Define the bidual problem as follows:

$$\widetilde{a}(z, v) - b(v, \psi) = \langle \widetilde{\delta}_\Gamma^{-1+2\varepsilon} e_w, v\rangle_\Omega \qquad \forall v \in H^1(\Omega), \tag{51a}$$
$$b(z, \mu) + c(\psi, \mu) = 0 \qquad \forall \mu \in H^{-1/2}(\Gamma). \tag{51b}$$

Then, we may proceed completely analogously as in the proof of (48) above. With the *a priori* estimate (35) of Lemma 2.10, we obtain the bound $\|z\|_{H^{3/2+\varepsilon}(\Omega)} + \|\psi\|_{H^\varepsilon_{pw}} \lesssim \|\widetilde{\delta}_\Gamma^{-1/2+\varepsilon} e_w\|_{L^2(\Omega)}$. Therefore, classical approximation estimates give

$$\inf_{(J_h z, J_h\psi)\in X_h} \Big[\|z - J_h z\|_{H^1(\Omega)} + \|\psi - J_h\psi\|_V\Big] \quad \lesssim h^{1/2+\varepsilon}\|\widetilde{\delta}_\Gamma^{-1/2+\varepsilon} e_w\|_{L^2(\Omega)}.$$

We employ the *a priori estimate* (30) of Lemma 2.7 and argue as above to see

$$\|\widetilde{\delta}_\Gamma^{-1/2+\varepsilon} e_w\|_{L^2(\Omega)} \lesssim h^{1/2+\varepsilon}\Big[\|w - w_h\|_{H^1(\Omega)} + \|\lambda - \lambda_h\|_V\Big] \le C h^{3/2+\varepsilon}\|\chi_{S_h} e\|_{L^2(\Omega)}.$$

This concludes the proof. □

We now turn to estimating $\|\widetilde{\delta}_\Gamma^{1/2+\varepsilon}\nabla(w - w_h)\|_{L^2(S_D)}$:

Lemma 3.9 *Let Assumptions 2.5 and 2.9 be valid.*

(i) There is a constants C_1, $C_2 > 0$ such that

$$\text{If } k = 1, \text{ then} \quad \|\widetilde{\delta}_\Gamma^{1/2}\nabla(w - w_h)\|_{L^2(\Omega)} \le C_1|\ln h|^{1/2} h^{3/2}\|\chi_{S_h} e\|_{L^2(\Omega)}.$$
$$\text{If } k > 1 \text{ and } \varepsilon \in (0, s_0 - 1/2] \text{ then} \quad \|\widetilde{\delta}_\Gamma^{1/2+\varepsilon}\nabla(w - w_h)\|_{L^2(\Omega)} \le C_2 h^{3/2+\varepsilon}\|\chi_{S_h} e\|_{L^2(\Omega)}.$$

The constant C_1 depends on the same quantities as the constant C_1 of Lemma 3.8 and $\|\mathfrak{A}\|_{C^{0,1}(\overline{\Omega})}$; the constant $C_2 > 0$ depends on the same quantities as the constant C_2 in Lemma 3.8 and additionally on $\|\mathfrak{A}\|_{C^{1,1}(\overline{\Omega})}$.

(ii) For any fixed $D' > 0$, we have

$$\|\nabla(w - w_h)\|_{L^2(\Omega\setminus S_{D'})} \le C_3\|\chi_{S_h} e\|_{L^2(\Omega)} \begin{cases} h^{3/2} & \text{if } k = 1 \\ h^{1+s_0} & \text{if } k > 1. \end{cases}$$

The constant $C_3 > 0$ depends on D' and on the same quantities as the constants C_1, C_2 in (i) for the cases $k = 1$ and $k > 1$, respectively.

Proof: *Proof of* (i)*:* The norm $\|\widetilde{\delta}_\Gamma^{1/2+\varepsilon}\nabla(w - w_h)\|_{L^2(\Omega)}$ is decomposed as

$$\|\widetilde{\delta}_\Gamma^{1/2+\varepsilon}\nabla(w - w_h)\|_{L^2(\Omega)} \le \|\delta_\Gamma^{1/2+\varepsilon}\nabla(w - w_h)\|_{L^2(S_{ch})} + \|\widetilde{\delta}_\Gamma^{1/2+\varepsilon}\nabla(w - w_h)\|_{L^2(\Omega\setminus S_{ch})} \tag{52}$$

for some fixed $c > 0$ (determined below in dependence on Ω and the shape regularity of the triangulation $\mathcal{T}_\Omega$) and each of these two contributions is estimated separately. We start with the simpler, first one:

$$\|\widetilde{\delta}_\Gamma^{1/2+\varepsilon}\nabla(w - w_h)\|_{L^2(S_{ch})} \le (ch + h)^{1/2+\varepsilon}\|\nabla(w - w_h)\|_{L^2(S_{ch})} \lesssim h^{1/2+\varepsilon}\|\nabla(w - w_h)\|_{L^2(\Omega)} \overset{Lem.\ 3.7}{\lesssim} h^{3/2+\varepsilon}\|\chi_{S_h} e\|_{L^2(\Omega)}. \tag{53}$$

The second term, $\|\widetilde{\delta}_\Gamma^{1/2+\varepsilon}\nabla(w - w_h)\|_{L^1(\Omega\setminus S_{ch})}$ requires tools from the local error analysis in FEM. The Galerkin orthogonality

$$a(w - w_h, v) = 0 \qquad \forall v \in V_h \cap H^1_0(\Omega)$$

allows us to use the techniques of the local error analysis of FEM as described in [29, Sec. 5.3]. This leads to the following estimate for arbitrary balls $B_r \subset B_{r'}$ with the same center (implicitly, $r' > r + O(h)$)

$$\|\nabla(w - w_h)\|_{L^2(B_r)} \lesssim \|\nabla(w - J_h^k w)\|_{L^2(B_{r'})} + \frac{1}{r' - r}\|w - w_h\|_{L^2(B_{r'})}, \tag{54}$$

where $J_h^k w$ is a local approximant such as the one of Lemma 3.5. By a covering argument, these local estimates can be combined into a global estimate of the following form, where for sufficiently small $c_1 \in (0, 1)$ (depending only on Ω and the shape regularity of $\mathcal{T}_\Omega$)

$$\|\delta_\Gamma^{1/2+\varepsilon}\nabla(w - w_h)\|_{L^2(\Omega\setminus S_{ch})} \lesssim \|\delta_\Gamma^{1/2+\varepsilon}\nabla(w - J_h^k w)\|_{L^2(\Omega\setminus S_{c_1 ch})} + \|\delta_\Gamma^{-1/2+\varepsilon}(w - w_h)\|_{L^2(\Omega\setminus S_{c_1 ch})}. \tag{55}$$

An implicit assumption is that $c_1 ch > 2h$. We emphasize that $\varepsilon = 0$ is admissible in (55).

We consider the cases $k = 1$ and $k > 1$ separately.

For $k > 1$, we assume $\varepsilon \in (0, s_0 - 1/2]$ in (55). Employing in (55) the bound (49) for the term $\|\widetilde{\delta}_\Gamma^{-1/2+\varepsilon}(w - w_h)\|_{L^2(\Omega\setminus S_{c_1 ch})}$ and the fact that $k > 1$ together with the approximation properties of J_h^k, we get for suitable $c_2 \in (0, 1)$ (again depending only on Ω and the shape regularity of $\mathcal{T}_\Omega$)

$$\begin{aligned}\|\widetilde{\delta}_\Gamma^{1/2+\varepsilon}\nabla(w - w_h)\|_{L^2(\Omega\setminus S_{ch})} &\overset{(49)}{\lesssim} \|\widetilde{\delta}_\Gamma^{1/2+\varepsilon}\nabla(w - J_h^k w)\|_{L^2(\Omega\setminus S_{c_1 ch})} + h^{3/2+\varepsilon}\|\chi_{S_h} e\|_{L^2(\Omega)} \\ &\lesssim h^2\|\delta_\Gamma^{1/2+\varepsilon}\nabla^3 w\|_{L^2(\Omega\setminus S_{c_2 c_1 ch})} + h^{3/2+\varepsilon}\|\chi_{S_h} e\|_{L^2(\Omega)}.\end{aligned} \tag{56}$$

In this estimate, we have implicitly assumed that $c > 0$ is sufficiently large so that $c_2 c_1 c h > 2h$. Combining Lemma 2.8, (ii), (iii), and the regularity assertion (43b) allows us to conclude with yet another constant $c_3 \in (0,1)$ (depending on Ω and the shape regularity of $\mathcal{T}_\Omega$)

$$\begin{aligned} \|\delta_\Gamma^{1/2+\varepsilon}\nabla^3 w\|_{L^2(\Omega\setminus S_{c_2c_1ch})} &\overset{\text{Lem. 2.8 (ii)}}{\lesssim} \|\delta_\Gamma^{-1/2+\varepsilon}\nabla^2 w\|_{L^2(\Omega\setminus c_3c_2c_1ch)} \\ &\lesssim h^{-1/2+\varepsilon}\|\nabla^2 w\|_{L^2(\Omega\setminus c_3c_2c_1ch)} && (57) \\ &\overset{\text{Lem. 2.8 (iii)}}{\lesssim} h^{-1+2\varepsilon}\|w\|_{H^{3/2+\varepsilon}(\Omega)} && (58) \\ &\overset{(43b)}{\lesssim} h^{-1/2+\varepsilon}\|\chi_{S_h} e\|_{L^2(\Omega)}. && (59) \end{aligned}$$

Again, the implicit assumption on c is that $c_3 c_2 c_1 c > \widetilde{c}$ with $\widetilde{c}$ given by Lemma 2.8. The final condition on c therefore is $c > \max\{\widetilde{c}/(c_1c_2c_3), 2/(c_1c_2)\}$. The above estimates, namely, the combination of (52), (53), (56), (59) shows for $k > 1$ and $\varepsilon \in (0, s_0 - 1/2]$ that $\|\widetilde{\delta}_\Gamma^{1/2+\varepsilon}\nabla(w - w_h)\|_{L^2(\Omega)} \lesssim h^{3/2+\varepsilon}\|\chi_{S_h} e\|_{L^2(\Omega)}$, which is the claimed estimate.

The case $k = 1$ corresponds to the limiting situation $\varepsilon = 0$ in (55). The procedure is analogous to that for the case $k > 1$ except that we use $\|\delta_\Gamma^{1/2}\nabla(w - J_h^k w)\|_{L^2(\Omega\setminus S_{c_1ch})} \lesssim h\|\delta_\Gamma^{1/2}\nabla^2 w\|_{L^2(\Omega\setminus S_{c_2c_1ch})}$ and then Lemma 2.8, (i) in conjunction with (43a). The contribution $\|\delta_\Gamma^{-1/2}(w - w_h)\|_{L^2(\Omega\setminus S_{c_1ch})}$ is controlled with the aid of (48) of Lemma 3.8.

Proof of (ii)*:* This follows more easily from (54). Since $D' > 0$ is fixed, (54) leads by a covering argument to

$$\|\nabla(w - w_h)\|_{L^2(\Omega\setminus S_{D'})} \lesssim \|\nabla(w - J_h^k w)\|_{L^2(\Omega\setminus S_{D''})} + \|\delta_\Gamma^{-1/2}(w - w_h)\|_{L^2(\Omega\setminus S_{D''})} \tag{60}$$

for some $D'' > 0$. Standard approximation properties of nodal interpolation

$$\|\nabla(w - J_h^k w)\|_{L^2(\Omega\setminus S_{D''})} \lesssim \begin{cases} h^1\|w\|_{H^2(\Omega\setminus S_{D'''})} & \text{if } k = 1 \\ h^2\|w\|_{H^3(\Omega\setminus S_{D'''})} & \text{if } k > 1. \end{cases} \tag{61}$$

Interior regularity (note: $-\nabla\cdot(\mathfrak{A}\nabla w) = 0$ on $\Omega \setminus S_h$) allows us to estimate

$$\|w\|_{H^2(\Omega\setminus S_{D'''})} \lesssim \|w\|_{H^1(\Omega)} \lesssim h^{1/2}\|\chi_{S_h} e\|_{L^2(\Omega)}, \quad \text{and} \quad \|w\|_{H^3(\Omega\setminus S_{D'''})} \lesssim \|w\|_{H^1(\Omega)} \lesssim h^{1/2}\|\chi_{S_h} e\|_{L^2(\Omega)} \tag{62}$$

where we employed (43). The term $\|\delta_\Gamma^{-1/2}(w - w_h)\|_{L^2(\Omega\setminus S_{D''})}$ is bounded by $h^{1+s_0}\|\chi_{S_h} e\|_{L^2(\Omega)}$ by (49) of Lemma 3.8. Inserting this and (61) in (60) yields the result. □

3.4. *Proof of Estimate* (38) *of Theorem 3.4.*

Proof of Theorem 3.4, Estimate (38): Theorem 3.4 disregards variational crimes. Therefore, Lemma 3.6 applies with $\varepsilon_1 \equiv 0$ and $\varepsilon_2 \equiv 0$. Together with (44)–(46), Lemma 3.6 yields for arbitrary $J_h u \in V_h$, $J_h\varphi \in M_h$:

$$\|\chi_{S_h} e\|_{L^2(\Omega)}^2 \lesssim h\|\chi_{S_h} e\|_{L^2(\Omega)}\Big[\|u - J_h u\|_{H^{1/2}(\Gamma)} + \|\varphi - J_h\varphi\|_{H^{-1/2}(\Gamma)}\Big] + a(u - J_h u, w - w_h). \tag{63}$$

Lemma 3.9, (i) provides

$$\|\widetilde{\delta}_\Gamma^{1/2+\varepsilon}\nabla(w - w_h)\|_{L^2(S_D)} \lesssim h^{3/2}\|\chi_{S_h} e\|_{L^2(\Omega)} \begin{cases} |\ln h|^{1/2} & \text{if } \varepsilon = 0 \\ h^\varepsilon & \text{if } \varepsilon \in (0, s_0 - 1/2] \text{ and } k > 1 \end{cases}$$

With (47), we hence get for the case $k > 1$ together with Lemma 3.9, (ii)

$$\begin{aligned} |a(u - J_h u, w - w_h)| &\overset{(47)}{\lesssim} \|\widetilde{\delta}_\Gamma^{-1/2-\varepsilon}\nabla(u - J_h u)\|_{L^2(S_D)}\|\widetilde{\delta}_\Gamma^{1/2+\varepsilon}\nabla(w - w_h)\|_{L^2(S_D)} + \|\nabla(u - J_h u)\|_{L^2(\Omega\setminus S_D)}\|\nabla(w - w_h)\|_{L^2(\Omega\setminus S_D)} \\ &\overset{\text{Lem. 3.9}}{\lesssim} \Big[\|\widetilde{\delta}_\Gamma^{-1/2-\varepsilon}\nabla(u - J_h u)\|_{L^2(S_D)}\, h^{3/2+\varepsilon} + \|\nabla(u - J_h u)\|_{L^2(\Omega\setminus S_D)}\, h^{1+s_0}\Big]\|\chi_{S_h} e\|_{L^2(\Omega)}. && (64) \end{aligned}$$

We choose $J_h u = J_h^k u$. From Lemma 3.5, we get

$$\|\widetilde{\delta}_\Gamma^{-1/2-\varepsilon}\nabla(u - J_h^k u)\|_{L^2(S_D)} \lesssim h^k\|\nabla^{k+1}u\|_{B^{1/2}_{2,1}(\Omega)} \begin{cases} |\ln h|^{1/2} & \text{if } \varepsilon = 0, \\ h^{-\varepsilon} & \text{if } \varepsilon > 0. \end{cases}$$

With this and standard approximation properties of J_h^k, we obtain

$$\begin{aligned} |a(u - J_h u, w - w_h)| &\overset{\text{Lem. 3.5}}{\lesssim} \Big[h^{k-\varepsilon}\|\nabla^{k+1}u\|_{B^{1/2}_{2,1}(\Omega)} h^{3/2+\varepsilon} + h^k\|\nabla^{k+1}u\|_{L^2(\Omega)} h^{1+s_0}\Big]\|\chi_{S_h} e\|_{L^2(\Omega)} \\ &\lesssim h^{k+3/2}\|u\|_{B^{k+3/2}_{2,1}(\Omega)}\|\chi_{S_h} e\|_{L^2(\Omega)}, && (65) \end{aligned}$$

where, in the last step, we employed the continuous embedding $B^{k+3/2}_{2,1}(\Omega) \subset H^{k+1}(\Omega)$ and the assumption $s_0 > 1/2$.

The corresponding estimate for the lowest order case $k = 1$ is

$$|a(u - J_h u, w - w_h)| \lesssim h^{k+3/2} |\ln h| \|u\|_{B^{k+3/2}_{2,1}(\Omega)} \|\chi_{S_h} e\|_{L^2(\Omega)}. \tag{66}$$

Taking for $J_h\varphi$ a suitable (nodal) interpolant in (63), we get with $k \geq 1$ and $s_0 > 1/2$ from (65)–(66)

$$\|\chi_{S_h} e\|_{L^2(\Omega)} \lesssim h^{k+3/2}(1 + \delta_{k,1}|\ln h|)\|u\|_{B^{k+3/2}_{2,1}(\Omega)} + h^{k+3/2}\|\varphi\|_{H^k_{pw}(\Gamma)}. \tag{67}$$

This concludes the proof. □

3.4.1. Extensions. Estimate (64) in the proof of Theorem 3.4 shows that we have actually obtained the following result:

Corollary 3.10 (best approximation property) *Assume the hypotheses of Theorem 3.4. Then, for arbitrary $J_h u \in V_h$ and $J_h\varphi \in M_h$ there holds:*

(i) If $k = 1$ then

$$\|u - u_h\|_{L^2(S_h)} \lesssim h^{3/2}|\ln h|^{1/2}\|\widetilde{\delta}_\Gamma^{-1/2}\nabla(u - J_h u)\|_{L^2(\Omega)} + h\|u - J_h u\|_{H^{1/2}(\Gamma)} + h\|\varphi - J_h\varphi\|_{H^{-1/2}(\Gamma)}.$$

(ii) If $k > 1$ then for arbitrary $\varepsilon \in (0, s_0 - 1/2]$ and $D > 0$

$$\|u - u_h\|_{L^2(S_h)} \lesssim h^{3/2+\varepsilon}\|\widetilde{\delta}_\Gamma^{-1/2-\varepsilon}\nabla(u - J_h u)\|_{L^2(S_D)} + h^{1+s_0}\|\nabla(u - J_h u)\|_{L^2(\Omega)} + h\|u - J_h u\|_{H^{1/2}(\Gamma)} + h\|\varphi - J_h\varphi\|_{H^{-1/2}(\Gamma)}.$$

The constant C depends on the same quantities as in Theorem 3.4 and additionally on D and ε in (ii). □

The statement of Corollary 3.10 (ii) for the case $k > 1$ suggests that the $B^{k+3/2}_{2,1}$-regularity of the solution is only required near the coupling boundary Γ, while away from Γ a weaker estimate is sufficient. The following result is meant to illustrate this point; it does not lay claim on sharpness of the regularity requirements (in fact, the presence of the small factor $h^{s_0-1/2}$ is a clear indication of a lack of sharpness):

Corollary 3.11 (reduced regularity away from Γ) *Assume the hypotheses of Theorem 3.4. Let $k \geq 2$. Let $u \in B^{k+3/2}_{2,1}(S_D) \cap H^{k+1}(\Omega)$ for some fixed $D > 0$. Let $\varphi \in H^k_{pw}(\Gamma)$. Then:*

$$\|u - u_h\|_{L^2(S_h)} \leq Ch^{3/2+k}\Big[\|u\|_{B^{k+3/2}_{2,1}(S_D)} + h^{s_0-1/2}\|u\|_{H^{k+1}(\Omega)} + \|\varphi\|_{H^k_{pw}(\Gamma)}\Big], \tag{68}$$

$$\|\varphi - \varphi_h\|_V \leq Ch^{1/2+k}\Big[\|u\|_{B^{k+3/2}_{2,1}(S_D)} + h^{s_0-1/2}\|u\|_{H^{k+1}(\Omega)} + \|\varphi\|_{H^k_{pw}(\Gamma)}\Big]. \tag{69}$$

The constant C depends on the same quantities as in Theorem 3.4 and additionally on D.

Proof: Follows from Corollary 3.10 (ii), the assumption $k \geq 2$, the special choice $J_h u = J^k_h u$ and Lemma 2.1. □

We recall that our proof of Theorem 3.4 does not assess the impact of variational crimes, i.e., it assumes $\varepsilon_1 \equiv 0$ and $\varepsilon_2 \equiv 0$ in the Galerkin orthogonalities (36). As mentioned above, the terms ε_1, ε_2 were introduced in Lemma 3.6 in order to be able to account for certain types of variational crimes. For their analysis, we need to address a technical issue, namely, the slight mismatch between the regularity of the dual solution w available to us and the regularity needed for the trace operator to be well-defined: we have $w \in B^{3/2}_{2,\infty}(\Omega)$ but for ∇w to have an $L^2(\Gamma)$-trace on Γ, we need $w \in B^{3/2}_{2,1}(\Omega)$, which is slightly stronger. The following lemma shows that for $w \in B^{n+3/2}_{2,\infty}(\Omega)$, one can construct a nearby function in the space $B^{n+1/2}_{2,1}(\Omega)$, which does have a trace.

Lemma 3.12 *Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain with boundary $\Gamma := \partial\Omega$ and $n \in \mathbb{N}_0$.*

(i) There is $C > 0$ such that for every $w \in B^{n+1/2}_{2,\infty}(\Omega)$ and every $\varepsilon \in (0, 1]$, one can find $w_\varepsilon \in H^{n+1}(\Omega)$ such that with $\theta = \frac{n+1/2}{n+1}$

$$\|w - w_\varepsilon\|_{L^2(\Omega)} + \varepsilon^{n/(n+1)}\|w - w_\varepsilon\|_{H^n(\Omega)} + \varepsilon\|w_\varepsilon\|_{H^{n+1}(\Omega)} \leq C\varepsilon^\theta\|w\|_{B^{n+1/2}_{2,\infty}(\Omega)}, \qquad \|w_\varepsilon\|_{B^{n+1/2}_{2,1}(\Omega)} \leq C(1 + |\ln\varepsilon|)\|w\|_{B^{n+1/2}_{2,\infty}(\Omega)}. \tag{70}$$

(ii) Fix $\delta \in (0, 1]$. There is $C > 0$ such that for every $\lambda \in B^0_{2,\infty}(\Gamma)$ and every $\varepsilon \in (0, 1]$, one can find $\lambda_\varepsilon \in H^\delta(\Gamma)$ such that

$$\|\lambda - \lambda_\varepsilon\|_{H^{-\delta}(\Gamma)} + \varepsilon^{1/2}\|\lambda - \lambda_\varepsilon\|_{B^0_{2,\infty}(\Gamma)} + \varepsilon\|\lambda_\varepsilon\|_{H^\delta(\Gamma)} \leq C\varepsilon^{1/2}\|\lambda\|_{B^0_{2,\infty}(\Gamma)}, \qquad \|\lambda_\varepsilon\|_{L^2(\Gamma)} \leq C\sqrt{1 + |\ln\varepsilon|}\|\lambda\|_{B^0_{2,\infty}(\Gamma)}.$$

The constant $C > 0$ depends only on Ω, n, and δ.

Proof: *Proof of* (i)*:* First, we recall that the reiteration theorem [27, Thm. 26.3] allows us to define the Besov spaces $B^s_{2,q}(\Omega)$ by interpolation between Sobolev spaces $H^{s_1}(\Omega)$ and $H^{s_2}(\Omega)$. We take specifically $s_1 = 0$ and $s_2 = n+1$. Then $B^{n+1/2}_{2,q}(\Omega) = (L^2(\Omega), H^{n+1}(\Omega))_{\theta,q}$ with $\theta = (n+1/2)/(n+1)$. We recall the definition of the pertinent K-functional $K(t,u) = \inf_{v\in H^{n+1}(\Omega)} \|u-v\|_{L^2(\Omega)} + t\|v\|_{H^{n+1}(\Omega)}$ and the corresponding interpolation norm from (1). By definition of the interpolation space $B^{n+1/2}_{2,\infty}(\Omega)$, one can find, for every $\varepsilon > 0$, a function $w_\varepsilon \in H^{n+1}(\Omega)$ such that

$$\|w - w_\varepsilon\|_{L^2(\Omega)} + \varepsilon\|w_\varepsilon\|_{H^{n+1}(\Omega)} \le C\varepsilon^\theta \|w\|_{B^{n+1/2}_{2,\infty}(\Omega)} \qquad \text{and} \qquad \|w - w_\varepsilon\|_{B^{n+1/2}_{2,\infty}(\Omega)} \le C\|w\|_{B^{n+1/2}_{2,\infty}(\Omega)}. \tag{71}$$

The first of these estimates follows from the definition, the second one is shown in [8, Lemma]. In order to complete the first bound in (70), we recall that the reiteration theorem [27, Thm. 26.3] yields $H^n(\Omega) = (L^2(\Omega), B^{n+1/2}_{2,\infty}(\Omega))_{\mu,2}$ with $\mu = n/(n+1/2)$. The interpolation yields inequality (2)

$$\|w - w_\varepsilon\|_{H^n(\Omega)} \lesssim \|w - w_\varepsilon\|^{1-n/(n+1/2)}_{L^2(\Omega)} \|w - w_\varepsilon\|^{n/(n+1/2)}_{B^{n+1/2}_{2,\infty}(\Omega)} \lesssim \varepsilon^{\theta(1-n/(n+1/2))}\|w\|_{B^{n+1/2}_{2,\infty}(\Omega)} = \varepsilon^{-n/(n+1)}\varepsilon^\theta\|w\|_{B^{n+1/2}_{2,\infty}(\Omega)}.$$

We turn to the second bound in (70). We first mention that, as it is shown in [13, Chap. 6, Sec. 7], we may replace the integral over $(0,\infty)$ in (1) by an integral over $(0,1)$. Next, we have the simple triangle inequality $K(t,w_\varepsilon) \le K(t,w) + \|w - w_\varepsilon\|_{L^2(\Omega)}$. A second bound for $K(t,w_\varepsilon)$ is obtained by taking $v = 0$ in the defining infimum: $K(t,w_\varepsilon) \le t\|w_\varepsilon\|_{H^{n+1}(\Omega)}$. Put together, we arrive at $K(t,w_\varepsilon) \le \min\{t\|w_\varepsilon\|_{H^{n+1}(\Omega)}, K(t,w) + \|w - w_\varepsilon\|_{L^2(\Omega)}\}$. We compute for $w_\varepsilon \in H^{n+1}(\Omega)$

$$\begin{aligned}\|w_\varepsilon\|_{B^{n+1/2}_{2,1}(\Omega)} &\simeq \int_{t=0}^{\varepsilon} t^{-\theta}K(t,w_\varepsilon)\frac{dt}{t} + \int_{t=\varepsilon}^{1} t^{-\theta}K(t,w_\varepsilon)\frac{dt}{t} \le \varepsilon^{1-\theta}\|w_\varepsilon\|_{H^{n+1}(\Omega)} + \varepsilon^{-\theta}\|w - w_\varepsilon\|_{L^2(\Omega)} + \int_{t=\varepsilon}^{1} t^{-\theta}K(t,w)\frac{dt}{t} \\ &\overset{(71)}{\lesssim} \varepsilon^{1-\theta}\varepsilon^{\theta-1}\|w_\varepsilon\|_{B^{n+1/2}_{2,\infty}(\Omega)} + \varepsilon^\theta\varepsilon^{-\theta}\|w\|_{B^{n+1/2}_{2,\infty}(\Omega)} + (1+|\ln\varepsilon|)\|w\|_{B^{n+1/2}_{2,\infty}(\Omega)} \lesssim (1+|\ln\varepsilon|)\|w\|_{B^{n+1/2}_{2,\infty}(\Omega)},\end{aligned}$$

where we finally used $\|w_\varepsilon\|_{B^{n+1/2}_{2,\infty}(\Omega)} \lesssim \|w\|_{B^{n+1/2}_{2,\infty}(\Omega)}$ from (71).

Proof of (ii)*:* We proceed by similar arguments as in the proof of (i): We exploit the characterizations $L^2(\Gamma) = (H^{-\delta}(\Gamma), H^\delta(\Gamma))_{1/2,2}$ and $B^0_{2,\infty}(\Gamma) = (H^{-\delta}(\Gamma), H^\delta(\Gamma))_{1/2,\infty}$. From the properties of the K-functional, we get the existence of λ_ε such that

$$\|\lambda - \lambda_\varepsilon\|_{H^{-\delta}(\Gamma)} + \varepsilon\|\lambda_\varepsilon\|_{H^\delta(\Gamma)} \le C\varepsilon^{1/2}\|\lambda\|_{B^0_{2,\infty}(\Gamma)}, \qquad \|\lambda - \lambda_\varepsilon\|_{B^0_{2,\infty}(\Gamma)} \le C\|\lambda\|_{B^0_{2,\infty}(\Gamma)},$$

where the second bound follows again from [8]. It remains to bound $\|\lambda_\varepsilon\|_{L^2(\Gamma)}$. As above, we observe $K(t,\lambda_\varepsilon) \le \min\{t\|\lambda_\varepsilon\|_{H^\delta(\Gamma)}, K(t,\lambda) + \|\lambda - \lambda_\varepsilon\|_{H^{-\delta}(\Gamma)}\}$. Hence,

$$\begin{aligned}\|\lambda_\varepsilon\|^2_{L^2(\Gamma)} &\simeq \int_{t=0}^{1} \left|t^{-1/2}K(t,\lambda_\varepsilon)\right|^2 \frac{dt}{t} \lesssim \int_{t=0}^{\varepsilon} \|\lambda_\varepsilon\|^2_{H^\delta(\Gamma)}\,dt + \int_{t=\varepsilon}^{1} \left|t^{-1/2}K(t,\lambda)\right|^2 \frac{dt}{t} + \int_{t=\varepsilon}^{1} t^{-2}\|\lambda - \lambda_\varepsilon\|^2_{H^{-\delta}(\Gamma)}\,dt \\ &\le \varepsilon\|\lambda_\varepsilon\|^2_{H^\delta(\Gamma)} + |\ln\varepsilon|\sup_{t>0}|t^{-1/2}K(t,\lambda)|^2 + \varepsilon^{-1}\|\lambda - \lambda_\varepsilon\|^2_{H^{-\delta}(\Gamma)} \lesssim (1+|\ln\varepsilon|)\|\lambda\|^2_{B^0_{2,\infty}(\Gamma)},\end{aligned}$$

from which the result follows. □

Theorem 3.13 (variational crimes) *Let $u_0 \in H^{k+1}_{pw}(\Gamma) \cap H^1(\Gamma)$ and $\phi_0 \in H^k_{pw}(\Gamma)$. Let $(u,\varphi) \in X$ be the solution of* (18)*–*(19)*. Let $(u_h,\varphi_h) \in X_h$ be the Galerkin solution of* (20) *with u_0 and ϕ_0 in* (18) *replaced by approximations $\Pi^k_h u_0$ and $\Pi^{k-1}_h \phi_0$ that have the following approximation properties:*

$$h^{-1/2}\|u_0 - \Pi^k_h u_0\|_{L^2(\Gamma)} + \|u_0 - \Pi^k_h u_0\|_{H^{1/2}(\Gamma)} \le C_{apx}h^{k+1/2}\|u_0\|_{H^{k+1}_{pw}(\Gamma)}, \tag{72}$$

$$h^{-1/2}\|\phi_0 - \Pi^{k-1}_h \phi_0\|_{H^{-1}(\Gamma)} + \|\phi_0 - \Pi^{k-1}_h \phi_0\|_{H^{-1/2}(\Gamma)} \le C_{apx}h^{k+1/2}\|\phi_0\|_{H^k_{pw}(\Gamma)}. \tag{73}$$

Under the assumptions of Theorem 3.4, it holds

$$\|\varphi - \varphi_h\|_{H^{-1/2}(\Gamma)} \le Ch^{k+1/2}|\ln h|\Big[\|u\|_{B^{k+3/2}_{2,1}(\Omega)} + \|u_0\|_{H^{k+1}_{pw}(\Gamma)} + \|\varphi\|_{H^k_{pw}(\Gamma)} + \|\phi_0\|_{H^k_{pw}(\Gamma)}\Big], \tag{74}$$

$$\|u - u_h\|_{L^2(S_h)} \le Ch^{k+3/2}|\ln h|\Big[\|u\|_{B^{k+3/2}_{2,1}(\Omega)} + \|u_0\|_{H^{k+1}_{pw}(\Gamma)} + \|\varphi\|_{H^k_{pw}(\Gamma)} + \|\phi_0\|_{H^k_{pw}(\Gamma)}\Big]. \tag{75}$$

The constant $C > 0$ in (74)*–*(75) *depends on the same quantities as the constant C in Theorem 3.4 and additionally on C_{apx}.*

Proof: The Galerkin orthogonality (36) for the primal problem now includes the linear functionals

$$\varepsilon_1(v) := \langle \phi_0 - \Pi_h^{k-1}\phi_0, v\rangle + \langle \mathrm{D}(u_0 - \Pi_h^k u_0), v\rangle,$$
$$\varepsilon_2(\psi) := \langle (\frac{1}{2} - \mathrm{K})(u_0 - \Pi_h^k u_0), \psi\rangle,$$

on the right-hand side. With Lemma 3.6, we observe that (63) then involves the summand $\varepsilon_1(w_h) - \varepsilon_2(\lambda_h)$ on the right-hand side. Recall that $e = u - u_h$ is the FEM-part of the primal error, and $(w, \lambda) \in X$ is the solution of the dual problem (40) with Galerkin approximation $(w_h, \lambda_h) \in X_h$. Arguing along the lines of the proof of Theorem 3.4, Estimate (38), we see that additional error terms

$$\frac{|\varepsilon_1(w_h)| + |\varepsilon_2(\lambda_h)|}{\|\chi_{S_h} e\|_{L^2(\Omega)}}$$

arise. We now claim the following two bounds:

$$|\varepsilon_1(w_h)| \lesssim h^{k+3/2} |\ln h| \Big[\|u_0\|_{H^{k+1}_{pw}(\Gamma)} + \|\phi_0\|_{H^k_{pw}(\Gamma)}\Big] \|\chi_{S_h} e\|_{L^2(\Omega)}, \tag{76}$$

$$|\varepsilon_2(\lambda_h)| \lesssim h^{k+3/2} |\ln h|^{1/2} \|u_0\|_{H^{k+1}_{pw}(\Gamma)} \|\chi_{S_h} e\|_{L^2(\Omega)}. \tag{77}$$

From the *a priori* estimate (29) of Lemma 2.7, we obtain $\|w\|_{B^{3/2}_{2,\infty}(\Omega)} \lesssim h^{1/2} \|\chi_{S_h} e\|_{L^2(\Omega)}$. Lemma 3.12 with $\varepsilon = h^2$ and $n = 1$ provides an approximation $w_\varepsilon \in H^2(\Omega)$ to w. (We still write w_ε to avoid confusion with the Galerkin approximation w_h.) The estimates (70) take the form

$$\|w - w_\varepsilon\|_{L^2(\Omega)} + h\,\|w - w_\varepsilon\|_{H^1(\Omega)} + h^2 \|w_\varepsilon\|_{H^2(\Omega)} \lesssim h^{3/2} \|w\|_{B^{3/2}_{2,\infty}(\Omega)} \lesssim h^2 \|\chi_{S_h} e\|_{L^2(\Omega)}, \tag{78a}$$

$$\|w_\varepsilon\|_{B^{3/2}_{2,1}(\Omega)} \lesssim |\ln h|\, \|w\|_{B^{3/2}_{2,\infty}(\Omega)} \lesssim h^{1/2}\, |\ln h|\, \|\chi_{S_h} e\|_{L^2(\Omega)}. \tag{78b}$$

Linearity of ε_1 yields

$$|\varepsilon_1(w_h)| \le |\varepsilon_1(w - w_h)| + |\varepsilon_1(w - w_\varepsilon)| + |\varepsilon_1(w_\varepsilon)|. \tag{79}$$

For the first summand in (79), stability of the hypersingular operator $\mathrm{D} : H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$, assumption (72)–(73), and Lemma 3.7 yield

$$\begin{aligned}
|\varepsilon_1(w - w_h)| &= |\langle \phi_0 - \Pi_h^{k-1}\phi_0, w - w_h\rangle + \langle \mathrm{D}(u_0 - \Pi_h^k u_0), w - w_h\rangle| \\
&\lesssim \Big[\|\phi_0 - \Pi_h^{k-1}\phi_0\|_{H^{-1/2}(\Gamma)} + \|u_0 - \Pi_h^k u_0\|_{H^{1/2}(\Gamma)}\Big] \|w - w_h\|_{H^{1/2}(\Gamma)} \\
&\overset{(72)-(73)}{\lesssim} \Big[h^{k+1/2} \|\phi_0\|_{H^k_{pw}(\Gamma)} + h^{k+1/2} \|u_0\|_{H^{k+1}_{pw}(\Gamma)}\Big]\, \|w - w_h\|_{H^1(\Omega)} \\
&\overset{\text{Lem. 3.7}}{\lesssim} h^{k+1/2} \Big[\|\phi_0\|_{H^k_{pw}(\Gamma)} + \|u_0\|_{H^{k+1}_{pw}(\Gamma)}\Big]\, h\, \|\chi_{S_h} e\|_{L^2(\Omega)}.
\end{aligned}$$

For the second summand in (79), we argue similarly, but rely on estimate (78a) to see

$$\begin{aligned}
|\varepsilon_1(w - w_\varepsilon)| &\overset{(72)-(73)}{\lesssim} \Big[h^{k+1/2} \|\phi_0\|_{H^k_{pw}(\Gamma)} + h^{k+1/2} \|u_0\|_{H^{k+1}_{pw}(\Gamma)}\Big]\, \|w - w_\varepsilon\|_{H^1(\Omega)} \\
&\overset{(78a)}{\lesssim} h^{k+1/2} \Big[\|\phi_0\|_{H^k_{pw}(\Gamma)} + \|u_0\|_{H^{k+1}_{pw}(\Gamma)}\Big]\, h\, \|\chi_{S_h} e\|_{L^2(\Omega)}.
\end{aligned}$$

For the third summand in (79), stability of $\mathrm{D} : L^2(\Gamma) \to H^{-1}(\Gamma)$ yields

$$\begin{aligned}
|\varepsilon_1(w_\varepsilon)| &= |\langle \phi_0 - \Pi_h^{k-1}\phi_0, w_\varepsilon\rangle + \langle \mathrm{D}(u_0 - \Pi_h^k u_0), w_\varepsilon\rangle| \\
&\le \Big[\|\phi_0 - \Pi_h^{k-1}\phi_0\|_{H^{-1}(\Gamma)} + \|u_0 - \Pi_h^k u_0\|_{L^2(\Gamma)}\Big]\, \|w_\varepsilon\|_{H^1(\Gamma)} \\
&\overset{(72)-(73)}{\lesssim} \Big[h^{k+1} \|\phi_0\|_{H^k_{pw}(\Gamma)} + h^{k+1} \|u_0\|_{H^{k+1}_{pw}(\Gamma)}\Big]\, \|w_\varepsilon\|_{H^1(\Gamma)}.
\end{aligned}$$

For the control of $\|w_\varepsilon\|_{H^1(\Gamma)}$, we use (78b) and the continuity of the trace operator $\gamma : B^{1/2}_{2,1}(\Omega) \to L^2(\Gamma)$ (see, [28, Thm. 2.9.1] for the present case of polygons/polyhedra or [19, Lemma 2.1] for the case of Lipschitz domains) to get

$$\|w_\varepsilon\|_{H^1(\Gamma)} \le \|w_\varepsilon\|_{L^2(\Gamma)} + \|\nabla w_\varepsilon\|_{L^2(\Gamma)} \lesssim \|w_\varepsilon\|_{B^{1/2}_{2,1}(\Omega)} + \|\nabla w_\varepsilon\|_{B^{1/2}_{2,1}(\Omega)} \lesssim \|w_\varepsilon\|_{B^{3/2}_{2,1}(\Omega)} \lesssim h^{1/2}\, |\ln h|\, \|\chi_{S_h} e\|_{L^2(\Omega)}.$$

Combining the last estimates, we arrive at

$$|\varepsilon_1(w_h)| \lesssim h^{k+3/2} (1 + |\ln h|) \Big[h^{k+1} \|\phi_0\|_{H^k_{pw}(\Gamma)} + h^{k+1} \|u_0\|_{H^{k+1}_{pw}(\Gamma)}\Big]\, \|\chi_{S_h} e\|_{L^2(\Omega)},$$

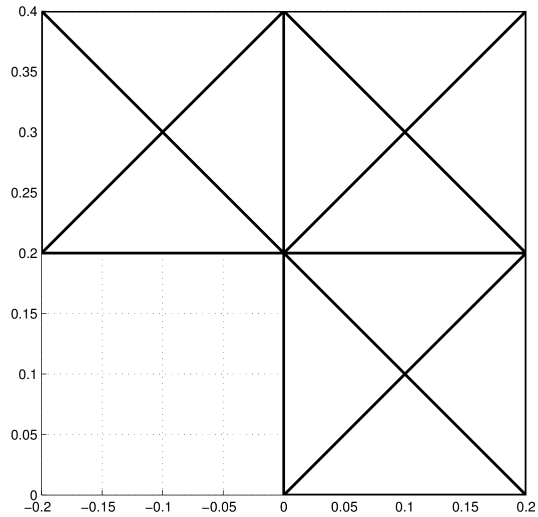

Figure 1. Domain $\Omega \subset \mathbb{R}^2$ and initial triangulation into 12 triangles and 8 boundary segments for the numerical experiments in Section 4.

which is (76).

We now indicate the proof of (77): From the a priori estimate (29) of Lemma 2.7, we obtain $\|\lambda\|_{B^0_{2,\infty}(\Gamma)} \lesssim h^{1/2}\|\chi_{S_h} e\|_{L^2(\Omega)}$. Lemma 3.12 with $\delta = 1/2$ and $\varepsilon = h$ provides $\lambda_\varepsilon \in H^{1/2}(\Gamma)$ with

$$\|\lambda - \lambda_\varepsilon\|_{H^{-1/2}(\Gamma)} + h^{1/2}\,\|\lambda - \lambda_\varepsilon\|_{B^0_{2,\infty}(\Gamma)} + h\,\|\lambda_\varepsilon\|_{H^{1/2}(\Gamma)} \lesssim h^{1/2}\,\|\lambda\|_{B^0_{2,\infty}(\Gamma)} \lesssim h\,\|\chi_{S_h} e\|_{L^2(\Omega)}, \tag{80a}$$

$$\|\lambda_\varepsilon\|_{L^2(\Gamma)} \lesssim |\ln h|^{1/2}\,\|\lambda\|_{B^0_{2,\infty}(\Gamma)} \lesssim h^{1/2}\,|\ln h|^{1/2}\,\|\chi_{S_h} e\|_{L^2(\Omega)}. \tag{80b}$$

Linearity of ε_2 yields

$$|\varepsilon_2(\lambda_h)| \le |\varepsilon_2(\lambda - \lambda_h)| + |\varepsilon_2(\lambda - \lambda_\varepsilon)| + |\varepsilon_2(\lambda_h)|.$$

We recall stability of $K : H^{1/2}(\Gamma) \to H^{1/2}(\Gamma)$ as well as $K : L^2(\Gamma) \to L^2(\Gamma)$. We use this, assumption (72), and estimates (80). Arguing as for $\varepsilon_1(w_h)$, we obtain

$$\begin{aligned}
|\varepsilon_2(\lambda_h)| &\overset{(72)}{\lesssim} h^{k+1/2}\|u_0\|_{H^{k+1}_{pw}(\Gamma)}\Big[\|\lambda - \lambda_h\|_{H^{-1/2}(\Gamma)} + \|\lambda - \lambda_\varepsilon\|_{H^{-1/2}(\Gamma)}\Big] + h^{k+1}\|u_0\|_{H^{k+1}_{pw}(\Gamma)}\,\|\lambda_\varepsilon\|_{L^2(\Gamma)} \\
&\overset{\text{Lem. 3.7 and (80),}}{\lesssim} h^{k+1/2}\,\|u_0\|_{H^{k+1}_{pw}(\Gamma)}\,h\,\|\chi_{S_h} e\|_{L^2(\Omega)} + h^{k+1}\,\|u_0\|_{H^{k+1}_{pw}(\Gamma)}\,h^{1/2}|\ln h|^{1/2}\,\|\chi_{S_h} e\|_{L^2(\Omega)} \\
&\le h^{k+3/2}(1 + |\ln h|^{1/2})\,\|u_0\|_{H^{k+1}_{pw}(\Gamma)}\,\|\chi_{S_h} e\|_{L^2(\Omega)}.
\end{aligned}$$

This proves (77) and completes the proof. □

Remark 3.14 1. The estimate (72) can be realized by standard (nodal) interpolation. The estimate (73) can be achieved by $L^2(\Gamma)$-projection. Given that the space M_h consists of discontinuous functions, this is again a local computation. Hence, Corollary 3.13 shows that interpolating u_0 does lead to a method that, up to a logarithmic factor, preserves the convergence rates of Theorem 3.4.

2. The proof of Theorem 3.13 shows that the logarithmic term can be removed if more regularity is required of u_0 and ϕ_0 and correspondingly higher order interpolants are employed. ■

4. Numerical results.

This section underlines the theoretical results of Theorem 3.4 and Theorem 3.13. Throughout, we consider the L-shaped domain

$$\Omega = [(-0.2, 0.2) \times (0, 0.4)] \backslash [(-0.2, 0) \times (0, 0.2)]$$

visualized in Figure 1. For an $\alpha > 0$, we prescribe the exact solution (u, u^{ext}) of (8) as

$$u(x, y) = 1000 \cdot \mathrm{Re}\big(z^\alpha\big) \qquad \text{with } z = x + iy, \tag{81a}$$

$$u^{\text{ext}}(x, y) = \mathrm{Re}\,\big(1/(z - v)\big) \qquad \text{with } z = x + iy \text{ and } v = (0.1, 0.1), \tag{81b}$$

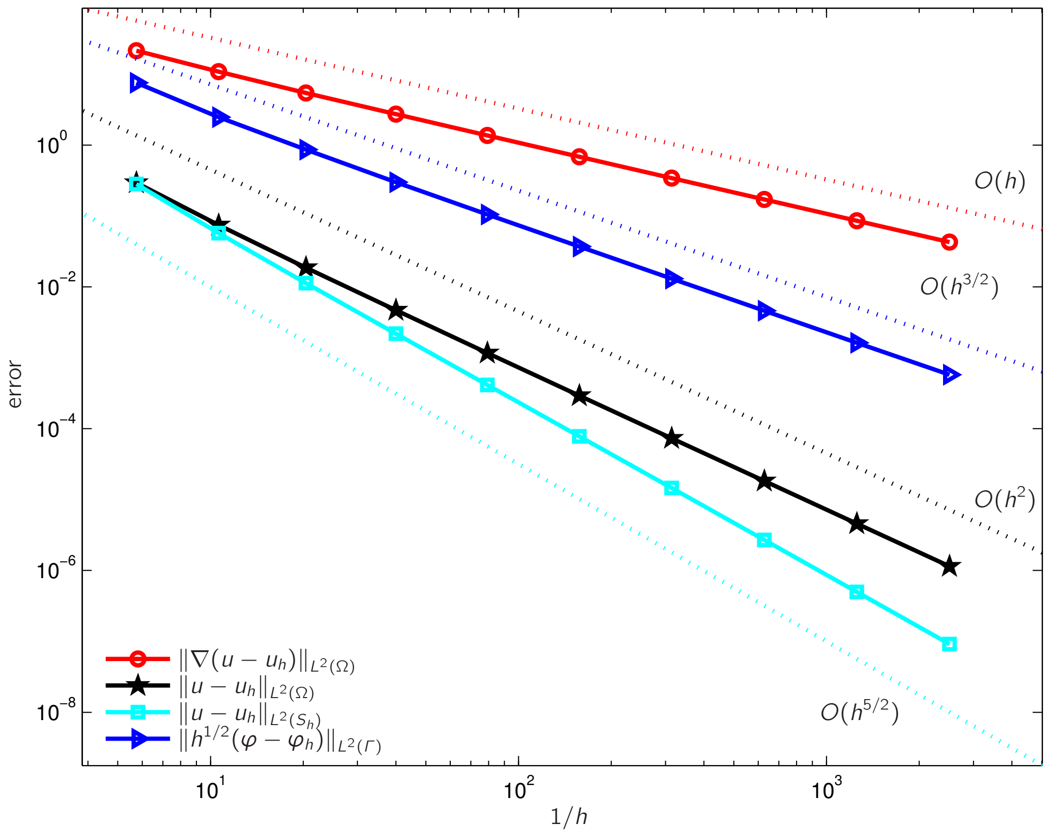

Figure 2. Performance of lowest-order FEM-BEM ($k = 1$) with $\alpha = 3/2$ in (81) for refinement levels $\ell = 0, \dots, 9$.

where $\mathfrak{A}$ is the identity, i.e., $-\nabla \cdot (\mathfrak{A}\nabla u) = -\Delta u$. We note that $f := -\Delta u = 0$ in Ω resp. $-\Delta u^{\mathrm{ext}} = 0$ in $\mathbb{R}\backslash\overline{\Omega}$. The data f, u_0, and ϕ_0 in (8) are calculated from the prescribed exact solutions. The exterior solution u^{ext} is smooth in $\mathbb{R}\backslash\overline{\Omega}$, while u has a singularity at the lower left corner $(0, 0)$ of Ω, whose strength is controlled by α. Away from this singularity, u is smooth, in particular, near the reentrant corner $(0, 0.2)$ of Ω, where the solutions of the dual problem (26) and the bidual problem (32) have a singularity.

With $\varphi = \partial_n^{\mathrm{ext}} u^{\mathrm{ext}}$, the pair (u, φ) is the unique solution of (18)–(19). By our choice of u^{ext}, the function φ is edgewise smooth and satisfies, in particular, all regularity requirements of the present work.

Remark 4.1 *1. For $\alpha \notin \mathbb{N}$, the solution u in Ω has the regularity $u \in B^{1+\alpha}_{2,\infty}(\Omega)$. We will use $\alpha = 3/2$ for the case $k = 1$ and $\alpha = 5/2$ with $k = 2$; that is, $k + 3/2 = 1 + \alpha$ and $u \in B^{k+3/2}_{2,\infty}(\Omega)$. This regularity is marginally lower than what is required in Theorem 3.4. Nevertheless, up to logarithmic terms (cf. Lemma 3.12 and the proof of Theorem 3.13) we expect the results of Theorem 3.4 to hold.*

2. The scaling factor 1000 *in our definition of the solution u is chosen to ensure that the approximation of u dominates also* preasymptotically *in the standard a priori estimate*

$$\|u-u_h\|_{H^1(\Omega)} + \|\varphi-\varphi_h\|_{H^{-1/2}(\Gamma)} \lesssim \inf_{v\in V_h} \|u-v\|_{H^1(\Omega)} + \inf_{\mu\in M_h} \|\varphi-\mu\|_{H^{-1/2}(\Gamma)},$$

in particular for the case $k = 2$. A calculation gives:

$$\alpha = 5/2: \qquad |u|_{H^3(\Omega)} \approx 2900, \qquad |\varphi|_{H^2(\Gamma)} \approx 867,$$
$$\alpha = 3/2: \qquad |u|_{H^2(\Omega)} \approx 828, \qquad |\varphi|_{H^1(\Gamma)} \approx 35.$$

■

Throughout, we consider a sequence of triangulations $\mathcal{T}_\Omega$ that are obtained by uniform red refinement of the initial triangulation $\mathcal{T}_\Omega$ depicted in Figure 1. The boundary mesh $\mathcal{T}_\Gamma$ is always induced by the volume mesh $\mathcal{T}_\Gamma = \mathcal{T}_\Omega|_\Gamma$. The computations consider meshes with $\#\mathcal{T}_\Omega = 12 \cdot 4^\ell$ triangles and $\#\mathcal{T}_\Gamma = 8 \cdot 2^\ell$ boundary segments, i.e., mesh-size $h = 0.2 \cdot 2^{-\ell}$ for refinement levels $\ell = 0, 1, 2, \dots$. Our computations are performed in MATLAB by means of the BEM library HILBERT [3]. The linear systems are solved with the MATLAB backslash operator. The matrix entries, in particular of the BEM matrices, are computed analytically.

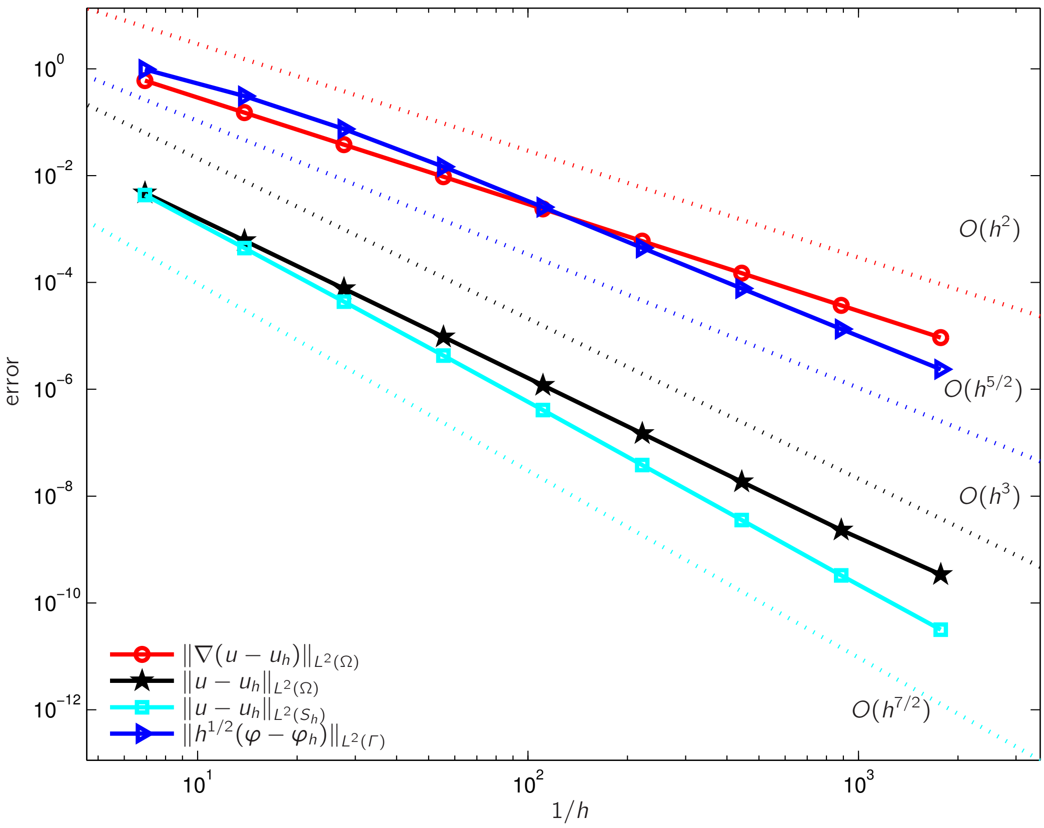

Figure 3. Performance of higher-order FEM-BEM ($k = 2$) with $\alpha = 5/2$ in (81) for refinement levels $\ell = 0, \ldots, 8$.

In our numerical experiments, we let $(u_h, \varphi_h) \in X_h = \mathcal{S}^{k,1}(\mathcal{T}_\Omega) \times \mathcal{S}^{k-1,0}(\mathcal{T}_\Gamma)$ be the Galerkin solution of

$$\widetilde{a}(u, v) - b(v, \varphi) = \langle \phi_0 + \mathrm{D}\,\Pi_h u_0, v\rangle_\Gamma \qquad \forall v \in V_h, \tag{82a}$$
$$b(u, \psi) + c(\varphi, \psi) = \langle \psi, (1/2 - \mathrm{K})\Pi_h u_0\rangle_\Gamma) \qquad \forall \psi \in M_h. \tag{82b}$$

Here, $\Pi_h : L^2(\Gamma) \to \mathcal{S}^{k,1}(\mathcal{T}_\Gamma)$ denotes the L^2-orthogonal projection. We note that $\mathcal{S}^{k,1}(\mathcal{T}_\Gamma) = \{v|_\Gamma \,:\, v \in \mathcal{S}^{k,1}(\mathcal{T}_\Omega)\}$ is the discrete trace space and that this choice satisfies the assumption (72) of Theorem 3.13 provided that $u_0 = u - u^{\mathrm{ext}}$ and $\phi_0 = (\nabla u - \nabla u^{\mathrm{ext}}) \cdot n$ are sufficiently smooth, i.e., $\alpha > k + 1/2$ in (81). Our actual choice $\alpha = k + 1/2$ corresponds to a limiting case, for which we still expect the convergence results to hold, up to logarithmic terms (cf. also Remark 4.1). The term $\langle \phi_0, v\rangle$ in (82a) is treated by a high order Gaussian quadrature rule.

In our experiments, we consider the lowest-order case $k = 1$ as well as $k = 2$ and plot the errors

- $\|u - u_h\|_{L^2(\Omega)}$,
- $\|\nabla(u - u_h)\|_{L^2(\Omega)}$,
- $\|h^{1/2}(\varphi - \varphi_h)\|_{L^2(\Gamma)}$,
- $\|u - u_h\|_{L^2(S_h)}$, where $S_h := \bigcup\{T \in \mathcal{T}_\Omega \,:\, \overline{T} \cap \Gamma \neq \emptyset\}$,

versus the mesh size $1/h$. In view of Theorem 3.13 we expect (up to logarithmic terms) $\|\nabla(u - u_h)\|_{L^2(\Omega)} = O(h^k)$, $\|h^{1/2}(\varphi - \varphi_h)\|_{L^2(\Gamma)} = O(h^{k+1/2})$, as well as $\|u - u_h\|_{L^2(S_h)} = O(h^{k+3/2})$. These rates are observed numerically in Figures 2 and 3 for the cases $k = 1, 2$, respectively.

Acknowledgements

The first two authors acknowledge support by the Austrian Science Fund (FWF) under grants W1245 (doctoral program "dissipation and dispersion in nonlinear PDEs"; JMM, DP) and P21732 (project "Adaptive Boundary Element Methods", DP).

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