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RESEARCH

On linear ODEs with a time singularity of the first kind and unsmooth inhomogeneity

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Abstract
In this paper we investigate analytical properties of systems of linear ordinary differential equations (ODEs) with unsmooth nonintegrable inhomogeneities and a time singularity of the first kind. We are especially interested in specifying the structure of general linear two-point boundary conditions guarantying existence and uniqueness of solutions which are continuous on the closed interval including the singular point. Moreover, we study the convergence behaviour of collocation schemes applied to solve the problem numerically. Our theoretical results are supported by numerical experiments.

Keywords: linear systems of ODEs; singular boundary value problem; time singularity of the first kind; unsmooth inhomogeneity; existence and uniqueness; collocation method; convergence

AMS Subject Classification: 34A12; 34A30; 34B05

1 Introduction

Singular boundary value problems (BVPs) arise in numerous applications in natural sciences and engineering and therefore, since many years, they are in focus of extensive investigations. An important class of linear singular problems takes the form of the following BVP:

$$y'(t) = \frac{M}{t^\alpha} y(t) + f(t),\ t \in (0,1], \quad B_0 y(0) + B_1 y(1) = \beta, \tag{1}$$

where $\alpha \geq 1$, y is an n-dimensional real function, M is $n \times n$ matrix and f is an n-dimensional function which is at least continuous $f \in C[0,1]$. We are mainly interested to find out under which circumstances the above problem has a solution $y \in C[0,1]$. B_0 and B_1 are constant matrices and it turns out that they are subject to certain restrictions for a well-posed problem with a unique continuous solution. If $\alpha = 1$, then we say that BVP (1) has a time *singularity of the first kind*, while for $\alpha > 1$ it has a time *singularity of the second kind*, commonly referred to as *essential singularity*.

Problems of type (1), where f may depend in addition on the space variable y and may have space singularity at $y = 0$, have been studied in [2, 21, 23, 24]. The analytical properties of (1) have been discussed in [9, 10], where the attention was focused on the existence and uniqueness of solutions and their smoothness. Especially, the structure of the boundary conditions which are necessary and sufficient for (1) to be well-posed was of special interest. Our aim is to generalize these analytical results to the problem

$$y'(t) = \frac{M}{t}y(t) + \frac{f(t)}{t},\ t \in (0,1], \quad B_0 y(0) + B_1 y(1) = \beta, \tag{2}$$

where $f \in C[0,1]$ but $f(t)/t$ may not be integrable on $[0,1]$. While for the BVP (1) and its applications, linear and nonlinear, comprehensive literature is available, this is not the case for problem (2). Let us add that the equation in (2) can be obtained from the regular equation $u'(x) = Mu(x) + g(x)$ considered on $[0,\infty)$ by the transformation $x = -\ln t$, and problems of type (2) arise in the modelling of the avalanche run up [20]. Further, we refer to papers [3, 4, 12, 16, 22], where the solvability of close linear singular problems is discussed. Interesting results about well-posedness for linear boundary value problems with time singularities in weight-spaces have been proved in [1, 13, 14, 15]. Although this framework is close to what we are aiming at here, it is not quite complete. So, in a way our results are closing the existing gaps. Note that analytical results which are focused on the unique solvability of equation in (2) can be found in [25, 26] where, in linear case, the main aim is to describe smooth particular solution of the equation and boundary conditions are not investigated there. The collocation scheme proposed to solve the problem numerically, is based on an approximation of the associated integral equation and is less accurate than the scheme proposed here.

To compute the numerical solution of (1) polynomial collocation was proposed in [8]. This was motivated by its advantageous convergence properties for (1), while in the presence of a singularity other high order methods show order reductions and become inefficient [11]. Consequently, we have implemented two MATLAB collocation based codes for singular BVPs [5, 17]. The code `sbvp` solves explicit first order ODEs [5], while `bvpsuite` can be applied to arbitrary order problems also in implicit formulation and to differential algebraic equations [17]. Over recent years, both codes were applied to simulate singular BVPs important for applications and proved to work dependably and efficiently. This was our motivation to also propose and analyse polynomial collocation for the approximation of the initial value problems (IVPs) (2).

The paper is organized as follows: In Section 2, we collect the preliminary results and introduce the necessary notation. In Sections 3, 4, and 5, three case studies are carried out, the case of only negative real parts of the eigenvalues of M, positive real parts of the eigenvalues of M, and zero eigenvalues of M, respectively. These results are summarized and compared with the case of smooth inhomogeneity in Section 6. Finally, the three case studies are used to formulate the respective results for the general IVPs and terminal value problems (TVPs) and BVPs in Section 7. We show convergence orders in context of general IVPs in Section 8 and illustrate the theoretical findings by experiments carried out using `bvpsuite` in Section 9. In Section 10, we recapitulate the most important results of the study.

2 Preliminaries

We are interested in analyzing the BVP

$$y'(t) = \frac{M}{t}y(t) + \frac{f(t)}{t},\ t \in (0,1], \quad y \in C[0,1], \quad B_0y(0) + B_1y(1) = \beta, \qquad (3)$$

where $M \in \mathbb{R}^{n\times n}, B_0, B_1 \in \mathbb{R}^{m\times n}, \beta \in \mathbb{R}^m$ and $f \in C[0,1]$. Note that in general $m \le n$ because the requirement $y \in C[0,1]$ results in additional $n-m$ conditions solution y has to satisfy [9].

Throughout the paper, the following notation is used. We denote by $\mathbb{R}^n$ and $\mathbb{C}^n$ the n-dimensional vector space of real-valued and complex-valued vectors, respectively, and denote the maximum vector norm by

$$|x| := |(x_1,\dots,x_n)^\top| = \max_{1\le i\le n} |x_i|.$$

We denote by $C_n[0,1]$ the space of continuous real vector-valued functions on $[0,1]$. In this space, we use the maximum norm,

$$\|y\| := \max_{t\in[0,1]} |y(t)|,$$

and the norm restricted to the interval $[0,\delta]$, $\delta > 0$, is denoted by

$$\|y\|_\delta := \max_{t\in[0,\delta]} |y(t)|.$$

Space $C_n^p[0,1]$ is the space of p times continuously differentiable real vector-valued functions on $[0,1]$. If there is no confusion, we omit the subscripts n and write $C[0,1] = C_n[0,1]$, $C^p[0,1] = C_n^p[0,1]$. For a matrix $A \in \mathbb{C}^{m\times n}$ we use the maximum norm,

$$|A| = \max_{1\le i\le m} \sum_{j=1}^{n} |a_{ij}|.$$

Before discussing the BVP (3), we first consider the easier problem consisting of the ODE system

$$y'(t) = \frac{M}{t}y(t) + \frac{f(t)}{t},\ t \in (0,1], \qquad (4)$$

subject to initial/terminal conditions, i.e. we deal with the initial value problem (IVP),

$$y'(t) = \frac{M}{t}y(t) + \frac{f(t)}{t},\ t \in (0,1], \quad y \in C[0,1], \quad B_0y(0) = \beta, \qquad (5)$$

where $B_0 \in \mathbb{R}^{m\times n}$, $\beta \in \mathbb{R}^m$, and $m \le n$, or with the terminal value problem (TVP),

$$y'(t) = \frac{M}{t}y(t) + \frac{f(t)}{t},\ t \in (0,1], \quad y \in C[0,1], \quad B_1y(1) = \beta, \qquad (6)$$

where $B_1 \in \mathbb{R}^{n\times n}$, $\beta \in \mathbb{R}^n$, respectively.

Particular attention is paid to the structure of initial/terminal and boundary conditions which are necessary and sufficient for the existence of a unique continuous solution on the closed interval $[0,1]$. It turns out that the form of such conditions depends on the spectral properties of the coefficient matrix M. Therefore, we distinguish between three cases, where all eigenvalues of M have negative real parts, positive real parts, or are zero.

In the first step, we construct the general solution of (4). Let us denote by $J \in \mathbb{C}^{n\times n}$ the Jordan canonical form of M and let $E \in \mathbb{C}^{n\times n}$ be the associated matrix of the generalized eigenvectors of M. Thus,

$$M = EJE^{-1}. \tag{7}$$

Moreover, let us introduce new variables, $v(t) := E^{-1}y(t)$ and $g(t) := E^{-1}f(t)$, then we can decouple the system (4) and obtain

$$v'(t) = \frac{J}{t}v(t) + \frac{g(t)}{t}. \tag{8}$$

By the variation of constant, any general solution of linear equation (8) is a complex-valued function of the form

$$v(t) = \Phi(t)d + \Phi(t)\int_1^t \Phi^{-1}(s)\frac{g(s)}{s}\,\mathrm{d}s = t^J d + t^J \int_1^t s^{-J-I}g(s)\,\mathrm{d}s,\ t \in (0,1], \tag{9}$$

where $d \in \mathbb{C}^n$ is an arbitrary vector and

$$\Phi(t) = t^J := \exp(J\ln(t)) = \sum_{j=0}^{\infty}\frac{J^j(\ln t)^j}{j!},$$

is the fundamental solution matrix satisfying

$$\Phi'(t) = \frac{J}{t}\Phi(t), \quad \Phi(1) = I, \quad t \in (0,1],$$

see [7, Chapter IV]. In case that J consists of m Jordan boxes, $J_1, J_2, \ldots, J_m$, the fundamental solution matrix has the form of the following block diagonal matrix, $t^J = \mathrm{diag}(t^{J_1}, t^{J_2}, \ldots, t^{J_m})$, where

$$J_k = \begin{pmatrix} \lambda_k & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_k \end{pmatrix}, \quad k = 1, \ldots, m, \tag{10}$$

and

$$t^{J_k} = t^{\lambda_k} \begin{pmatrix} 1 & \ln t & \frac{(\ln t)^2}{2} & \dots & \frac{(\ln t)^{n_k-1}}{(n_k-1)!} \\ 0 & 1 & \ln t & \dots & \frac{(\ln t)^{n_k-2}}{(n_k-2)!} \\ 0 & \ddots & 1 & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \ln t \\ 0 & \dots & \dots & 0 & 1 \end{pmatrix}, \quad t \in (0,1]. \tag{11}$$

Here, $\lambda_k = \sigma_k + i\rho_k \in \mathbb{C}$ is an eigenvalue of M and $\dim J_1 + \dim J_2 + \dots + \dim J_m = n$. The general solution of equation (4) is then given by

$$y(t) = t^M c + t^M \int_1^t s^{-M-I} f(s)\,\mathrm{d}s, \quad t \in (0,1], \tag{12}$$

where $c = Ed \in \mathbb{C}^n$ and $t^M = Et^J E^{-1} \in \mathbb{C}^{n \times n}$. Also,

$$\left(t^M\right)' = Mt^{M-I},\ t \in (0,1] \tag{13}$$

and

$$t^{-M} = \left(\frac{1}{t}\right)^M \Rightarrow \left(t^{-M}\right)' = -Mt^{-M-I}, \quad t \in (0,1]. \tag{14}$$

From the structure of matrix (11), it is obvious that the solution contribution related to the k-th Jordan box may become unbounded for $t = 0$. Apparently, the asymptotic behavior of the solution depends on the sign of the real part σ_k of the associated eigenvalue λ_k. Therefore, we have to distinguish between three cases, $\sigma_k < 0,\ \lambda_k = 0$, and $\sigma_k > 0$. We assume that M has no purely imaginary eigenvalues to exclude solutions of the form $t^{i\rho} = \cos(\rho \ln t) + i \sin(\rho \ln t)$.

We complete the preliminaries by two technical remarks, which will be frequently used in the following analysis.

Remark 1 The main focus of our investigations is on correctly posed initial/terminal conditions which guarantee the existence of continuously differentiable solutions of (4), $y \in C^1[0,1]$. Since logarithm terms occur in matrix (11), the relation

$$\lim_{t \to 0^+} t^\alpha (\ln t)^k = 0,\ \forall \alpha \in \mathbb{R}^+,\ \forall k \in \mathbb{N}, \tag{15}$$

is essential when discussing the smoothness of y.

Remark 2 By integrating (14) we obtain

$$M \int_t^1 s^{-M-I}\,\mathrm{d}s = -s^{-M}\big|_t^1 = t^{-M} - I,\ t \in (0,1]. \tag{16}$$

Moreover, if M has only eigenvalues with negative real parts, then $\lim_{s\to 0+} s^{-M} = 0$ due to Remark 1, and therefore

$$\int_0^1 s^{-M-I}\,\mathrm{d}s = (-M)^{-1}. \tag{17}$$

3 Eigenvalues of M with negative real parts

In this section, we consider system (4), such that all eigenvalues of M have negative real parts. It turns out that in this case, it is necessary to prescribe initial conditions of a certain structure to guarantee that the solution is continuous on $[0,1]$. Moreover, this continuous solution of the associated IVP (5) is shown to be unique and its form is provided in Theorem 5. In the proof of this theorem, we require the following lemmas.

Lemma 3 *Let $\gamma \ge 0$ and let the $n\times n$ matrix J be of the form*

$$J = \begin{pmatrix} \lambda & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & \lambda \end{pmatrix}, \quad \lambda = \sigma + i\rho, \tag{18}$$

where $\sigma \le 0$. For $\sigma = 0$, we assume $\lambda = 0$ and $\gamma > 0$. Then, for $t \in (0,1]$,

$$\int_0^t |s^{-J}| s^{\gamma-1}\,\mathrm{d}s = \sum_{j=0}^{n-1}\sum_{k=0}^{j} \frac{t^{\gamma-\sigma}(-\ln t)^k}{k!(\gamma-\sigma)^{j+1-k}}, \tag{19}$$

and in particular,

$$\int_0^1 |s^{-J}| s^{\gamma-1}\,\mathrm{d}s = \sum_{j=0}^{n-1} \frac{1}{(\gamma-\sigma)^{j+1}}. \tag{20}$$

Proof: Due to the form of J, the norm of s^{-J} for $s\in(0,1]$ is

$$|s^{-J}| = |s^{-\lambda}| \sum_{j=0}^{n-1} \frac{|\ln s|^j}{j!} = s^{-\sigma} \sum_{j=0}^{n-1} \frac{(-\ln s)^j}{j!}.$$

By repeated integration by parts, we obtain

$$\int \frac{(-\ln s)^j}{j!} s^{\gamma-\sigma-1}\,\mathrm{d}s = \frac{s^{\gamma-\sigma}}{\gamma-\sigma}\frac{(-\ln s)^j}{j!} + \int \frac{s^{\gamma-\sigma-1}}{\gamma-\sigma}\frac{(-\ln s)^{j-1}}{(j-1)!}\,\mathrm{d}s = \sum_{k=0}^{j} \frac{s^{\gamma-\sigma}(-\ln s)^k}{k!(\gamma-\sigma)^{j+1-k}}.$$

Therefore, due to (15),

$$\begin{aligned}\int_0^t |s^{-J}|s^{\gamma-1}\,\mathrm{d}s &= \int_0^t \sum_{j=0}^{n-1} \frac{(-\ln s)^j}{j!} s^{\gamma-\sigma-1}\,\mathrm{d}s = \left[\sum_{j=0}^{n-1}\sum_{k=0}^{j} \frac{s^{\gamma-\sigma}(-\ln s)^k}{k!(\gamma-\sigma)^{j+1-k}}\right]_0^t \\ &= \sum_{j=0}^{n-1}\sum_{k=0}^{j} \frac{t^{\gamma-\sigma}(-\ln t)^k}{k!(\gamma-\sigma)^{j+1-k}}.\end{aligned}$$

Clearly, for $t = 1$

$$\int_0^1 |s^{-J}|s^{\gamma-1}\,\mathrm{d}s = \sum_{j=0}^{n-1} \frac{1}{(\gamma-\sigma)^{j+1}}$$

which completes the proof. □

Lemma 4 *Assume that all eigenvalues of the matrix M have negative real parts. Then*

$$\lim_{t\to 0^+} \int_0^t \left|s^{-M-I}\right|\,\mathrm{d}s = 0. \tag{21}$$

Proof: Let $\lambda_k = \sigma_k + i\rho_k,\ k = 1,\dots,l$, be eigenvalues of matrix M and $J_k,\ k = 1,\dots,l$, the associated Jordan boxes of M. Then $s^{-M} = Es^{-J}E^{-1}$, where $s^{-J} = \operatorname{diag}\left(s^{-J_1}, s^{-J_2},\dots,s^{-J_l}\right)$. Therefore,

$$\lim_{t\to 0^+} \int_0^t \left|s^{-M-I}\right|\,\mathrm{d}s \le |E||E^{-1}| \lim_{t\to 0^+} \int_0^t \left|s^{-J}\right| s^{-1}\,\mathrm{d}s.$$

The result follows from (19) with $\gamma = 0$ and (15). □

Theorem 5 *Let us assume that all eigenvalues of M have negative real parts. Then for every $f \in C[0,1]$ system (4) has a unique solution $y \in C[0,1]$. This solution has the form*

$$y(t) = \int_0^1 s^{-M-I} f(ts)\,\mathrm{d}s,\ t \in [0,1]$$

and satisfies the initial condition

$$My(0) = -f(0).$$

This condition is necessary and sufficient for y to be continuous on $[0,1]$. Moreover, if $f \in C^r[0,1]$, $r \ge 0$, then $y \in C^r[0,1]$, and the following estimates hold for all $t \in [0,1]$:

$$|y^{(k)}(t)| \le const.\, \|f^{(k)}\|,\ k = 0,\dots,r.$$

Proof: The general solution of system (4) can be split into two parts

$$\begin{aligned} y(t) &= t^M c + t^M \int_1^t s^{-M-I} f(s)\,\mathrm{d}s \\ &= t^M \left(c - \int_0^1 s^{-M-I} f(s)\,\mathrm{d}s \right) + t^M \int_0^t s^{-M-I} f(s)\,\mathrm{d}s \\ &=: y_h(t) + y_p(t), \quad t \in (0,1]. \end{aligned} \tag{22}$$

First, we show that $y_p \in C[0,1]$. Change of variable, $u = s/t$, yields

$$y_p(t) = \int_0^1 u^{-M-I} f(ut)\,\mathrm{d}u,\ t \in (0,1].$$

Let us now introduce the functions,

$$z_m(t) := \int_{\frac{1}{m}}^1 s^{-M-I} f(st)\,\mathrm{d}s,\ m \in \mathbb{N}, \tag{23}$$

$$z_\infty(t) := \int_0^1 s^{-M-I} f(st)\,\mathrm{d}s. \tag{24}$$

Then, by (21),

$$\lim_{m\to\infty} |z_\infty(t) - z_m(t)| = \lim_{m\to\infty} \left| \int_0^{\frac{1}{m}} s^{-M-I} f(st)\,\mathrm{d}s \right| \le \|f\| \lim_{m\to\infty} \int_0^{\frac{1}{m}} \left| s^{-M-I} \right| \mathrm{d}s = 0.$$

Clearly $z_m(t) \in C[0,1]$, for $m \in \mathbb{N}$, and hence z_∞ is continuous as the uniform limit of continuous functions. Consequently, $y_p(t) \in C[0,1]$.

Since all real parts of eigenvalues are negative, y_h is not continuous at $t = 0$ and it is obvious that $y \in C[0,1]$ if and only if

$$\tilde{c} := c - \int_0^1 s^{-M-I} f(s)\,\mathrm{d}s = 0.$$

Thus the unique continuous solution satisfying (4) has the form

$$y(t) = \int_0^1 s^{-M-I} f(st)\,\mathrm{d}s,\ t \in [0,1], \tag{25}$$

and the estimate

$$|y(t)| \le const.\, \|f\|,\ t \in [0,1],$$

holds due to Lemma 4. This solution is uniquely determined by $\tilde{c} = 0$ and there are no additional conditions to be imposed. Note that $\tilde{c} = 0$ is equivalent to

$$My(0) = -f(0)$$

which follows from

$$y(0) = \int_0^1 s^{-M-I} f(0)\,\mathrm{d}s = (-M)^{-1}\left(1^{-M} - 0^{-M}\right) f(0) = -M^{-1} f(0),$$

according to Remark 2 and (17).

We now examine the smoothness of y. Let $f \in C^1[0,1]$. For the first derivative y', we have from (25)

$$y'(t) = \int_0^1 s^{-M} f'(ts)\,\mathrm{d}s, \quad |y'(t)| \le const.\,\|f'\|,\ t \in [0,1],$$

due to Lemma 3. Similarly, if $f \in C^2[0,1]$, it follows for the second derivative

$$y''(t) = \int_0^1 s^{I-M} f''(ts)\,\mathrm{d}s, \quad |y''(t)| \le const.\,\|f''\|,\ t \in [0,1].$$

Clearly, if $f \in C^r[0,1]$, then

$$y^{(r)}(t) = \int_0^1 s^{(r-1)I-M} f^{(r)}(ts)\,\mathrm{d}s, \quad |y^{(r)}(t)| \le const.\,\|f^{(r)}\|,\ t \in [0,1]$$

and the result follows. □

Theorem 5 shows that if all eigenvalues of M have negative real parts, then there exists a unique continuous solution y of IVP (5) if and only if $B_0 = M$, $\beta = -f(0)$, and $m = n$. Clearly, B_0 has to be nonsingular. Note that for this spectrum of M a terminal problem (6) cannot be set up in a reasonable way.

Remark 6 Interestingly, in the above case, the ODE system in (5) has a limit for $t \to 0$, which yields a representation for $y'(0)$ consistent with (26). Let us consider the solution y of (5) specified in (25). For $f \in C^1[0,1]$, $y' \in C[0,1]$ and the Taylor expansion at $t = 0$ yields

$$y'(t) = \frac{1}{t} M\left(y(0) + t y'(\xi)\right) + \frac{1}{t} f(t) = \left(\frac{1}{t} M y(0) + \frac{1}{t} f(t)\right) + M y'(\xi),$$

where $\xi \in (0,t)$. Letting $t \to 0$, results in

$$y'(0) = \lim_{t\to 0} \frac{1}{t}\left(M y(0) + f(t)\right) + M y'(0).$$

Since $My(0) = -f(0)$,

$$y'(0) = \lim_{t\to 0} \frac{1}{t}\left(f(t) - f(0)\right) + M y'(0)$$

and

$$y'(0) = (I - M)^{-1} f'(0)$$

which coincides with what we obtain by evaluating (26) at $t = 0$.

4 Eigenvalues of M with positive real parts

In this section we deal with system (4) whose matrix M has eigenvalues with positive real parts. It turns out that in this case there exists a unique continuous solution of problem (6). Its smoothness depends not only on the smoothness of f but also on the size of real parts of the eigenvalues of M. Before stating the main result of this section in Theorem 9, we show the following two lemmas.

Lemma 7 *Let $\gamma \geq 0$ and let the $n \times n$ matrix J be of the form (18),*

$$J = \begin{pmatrix} \lambda & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & \lambda \end{pmatrix}, \quad \lambda = \sigma + i\rho,$$

where $\sigma > 0$. Then for $t \in [0,1]$ the function

$$u(t) = \int_t^1 \left| \left(\frac{t}{s} \right)^J \right| s^{\gamma-1} \, \mathrm{d}s,$$

satisfies the following inequalities:

$$(i)\ u(t) \leq const.\, t^\gamma \ \ for\ \ \gamma < \sigma, \tag{26}$$

$$(ii)\ u(t) \leq const.\, t^\sigma \sum_{j=0}^{n-1} \frac{(-\ln t)^{j+1}}{j!} \ \ for\ \ \gamma = \sigma, \tag{27}$$

$$(iii)\ u(t) \leq const.\, t^\sigma \sum_{j=0}^{n-1} \frac{(-\ln t)^{j}}{j!} \ \ for\ \ \gamma > \sigma. \tag{28}$$

Proof: We discuss separately the cases $\gamma < \sigma$, $\gamma = \sigma$, and $\gamma > \sigma$.

(i) First, let $\gamma < \sigma$. Then there exists a constant $\varepsilon > 0$ such that $\sigma = \gamma + 2\varepsilon$. The term

$$\left(\frac{t}{s} \right)^\varepsilon \sum_{j=0}^{n-1} \frac{\left(-\ln\left(\frac{t}{s}\right)\right)^j}{j!}$$

is bounded on $[0,1]$ due to (15) and hence

$$\begin{aligned} \int_t^1 \left| \left(\frac{t}{s} \right)^J \right| s^{\gamma-1} \, \mathrm{d}s &= \int_t^1 \left(\frac{t}{s} \right)^\sigma \sum_{j=0}^{n-1} \frac{\left(-\ln\left(\frac{t}{s}\right)\right)^j}{j!} s^{\gamma-1} \, \mathrm{d}s \\ &\leq const.\, t^{\gamma+\varepsilon} \int_t^1 s^{-\varepsilon-1} \, \mathrm{d}s = const.\, t^\gamma. \end{aligned}$$

(ii) For $\gamma = \sigma$ function u can be estimated by

$$\begin{aligned}\int_t^1 \left|\left(\frac{t}{s}\right)^J\right| s^{\gamma-1}\,\mathrm{d}s &= \int_t^1 \left(\frac{t}{s}\right)^{\sigma} \sum_{j=0}^{n-1} \frac{\left(-\ln\left(\frac{t}{s}\right)\right)^j}{j!} s^{\gamma-1}\,\mathrm{d}s \\ &= t^{\sigma}\int_t^1 s^{-\sigma+\gamma-1} \sum_{j=0}^{n-1} \frac{\left(-\ln\left(\frac{t}{s}\right)\right)^j}{j!}\,\mathrm{d}s \\ &\le t^{\sigma}\sum_{j=0}^{n-1}\frac{(-\ln t)^j}{j!}\int_t^1 s^{-1}\,\mathrm{d}s \le const.\, t^{\sigma}\sum_{j=0}^{n-1}\frac{(-\ln t)^{j+1}}{j!}.\end{aligned}$$

(iii) Finally, for $\gamma > \sigma$, we have

$$\begin{aligned}\int_t^1 \left|\left(\frac{t}{s}\right)^J\right| s^{\gamma-1}\,\mathrm{d}s &= \int_t^1 \left(\frac{t}{s}\right)^{\sigma} \sum_{j=0}^{n-1} \frac{\left(-\ln\left(\frac{t}{s}\right)\right)^j}{j!} s^{\gamma-1}\,\mathrm{d}s \\ &\le t^{\sigma}\sum_{j=0}^{n-1}\frac{(-\ln t)^j}{j!}\int_t^1 s^{-\sigma+\gamma-1}\,\mathrm{d}s \le const.\, t^{\sigma}\sum_{j=0}^{n-1}\frac{(-\ln t)^{j}}{j!}.\end{aligned}$$

□

Lemma 8 *Let $\gamma \ge 0$ and let all eigenvalues of M have positive real parts. Then the function*

$$u(t) = \int_t^1 \left|\left(\frac{t}{s}\right)^M\right| s^{\gamma-1}\,\mathrm{d}s,\ t\in[0,1],$$

is bounded on $[0,1]$ and

$$\lim_{t\to 0^+} u(t) = 0 \ \ for\ \ \gamma > 0. \tag{29}$$

Proof: Let all eigenvalues of M have positive real parts. Then

$$u(t) = \int_t^1 \left|\left(\frac{t}{s}\right)^M\right| s^{\gamma-1}\,\mathrm{d}s \le |E||E^{-1}|\int_t^1 \left|\left(\frac{t}{s}\right)^J\right| s^{\gamma-1}\,\mathrm{d}s.$$

Estimates (26) to (28) and property (15) imply $u(t) \le const.\, t^{\sigma_0}$ for $t\in[0,1]$, where $\sigma_0 = \min\left\{\gamma, \frac{\sigma}{2}\right\} \ge 0$. This means that u is bounded in $[0,1]$. If $\gamma > 0$, then $\sigma_0 > 0$ and (29) follows. □

Theorem 9 *Let us assume that all eigenvalues of M have positive real parts. Then for every $f \in C^1[0,1]$ and every constant vector c, there exists a solution $y \in C[0,1]$ of (4). This solution has the form*

$$y(t) = \begin{cases} t^M c + t^M \int_1^t s^{-M-I} f(s)\,\mathrm{d}s \ \ for\ \ t\in(0,1], \\ -M^{-1}f(0)\ \ for\ \ t=0. \end{cases} \tag{30}$$

If the matrix $B_1 \in \mathbb{R}^{n\times n}$ in (6) is nonsingular, then for any $\beta \in \mathbb{R}^n$ there exists a unique solution of TVP (6). This solution is given by (30) with $c = B_1^{-1}\beta$.

Let $f \in C^{r+2}[0,1]$. Then the following statements hold:
(i) $y \in C^r[0,1] \cap C^{r+3}(0,1]$ for $0 \le r < \sigma_+ \le r+1$.
(ii) $y \in C^{r+1}[0,1] \cap C^{r+3}(0,1]$ for $\sigma_+ > r+1$.
Moreover, higher derivatives of y satisfy for $t \in [0,1]$
(i) $|y^{(k)}(t)| \le const.\ \left(t^{\sigma_+ - k}(1 + |\ln(t)|^{n_{max}-1}) + \|f^{(k)}\|\right)$ for $k = 0,1,\ldots,r$,
(ii) $|y^{(k)}(t)| \le const.\ \left(t^{\sigma_+ - k}(1 + |\ln(t)|^{n_{max}-1}) + \|f^{(k)}\|\right)$ for $k = 0,1,\ldots,r+1$,
where σ_+ is the smallest positive real part of the eigenvalues of M and n_{max} is the dimension of the largest Jordan box in J.

Proof: The general solution of equation (4) can be written in the following form:

$$y(t) = t^M c + t^M \int_1^t s^{-M-I} f(s)\,\mathrm{d}s = t^M c + \int_1^t \left(\frac{t}{s}\right)^M \frac{f(s)}{s}\,\mathrm{d}s =: y_h(t) + y_p(t). \quad (31)$$

Since all eigenvalues have positive real parts, it follows from (15) that $y_h(t) = t^M c$ is continuous on $[0,1]$.

Now, we show that $\lim_{t\to 0} y_p(t)$ exists and therefore $y \in C[0,1]$. Using the integration formula (16) we obtain

$$\int_1^t \left(\frac{t}{s}\right)^M \frac{f(0)}{s}\,\mathrm{d}s = M^{-1}(t^M - I)f(0),$$

and hence

$$-M^{-1}f(0) = \int_1^t \left(\frac{t}{s}\right)^M \frac{f(0)}{s}\,\mathrm{d}s - M^{-1}t^M f(0).$$

Therefore

$$\int_1^t \left(\frac{t}{s}\right)^M \frac{f(s)}{s}\,\mathrm{d}s - (-M)^{-1}f(0) = \int_1^t \left(\frac{t}{s}\right)^M \frac{f(s)-f(0)}{s}\,\mathrm{d}s + M^{-1}t^M f(0). \quad (32)$$

Since $f \in C^1[0,1]$, there exists $M_0 \in (0,\infty)$ such that

$$\left|\frac{f(s)-f(0)}{s}\right| \le M_0, \ \ s \in [0,1]. \quad (33)$$

Relation (32) together with (33) yield

$$\left|\int_1^t \left(\frac{t}{s}\right)^M \frac{f(s)}{s}\,\mathrm{d}s - (-M)^{-1}f(0)\right| \le M_0 \int_t^1 \left|\left(\frac{t}{s}\right)^M\right| \mathrm{d}s + \left|(-M)^{-1}t^M f(0)\right|.$$

Since all eigenvalues of M have positive real parts, (15) implies

$$\lim_{t\to 0^+} \left|(-M)^{-1}t^M f(0)\right| = 0.$$

Moreover, by Lemma 8 with $\gamma = 1$ property (29) holds and therefore,

$$\lim_{t\to 0^+} \left|\int_1^t \left(\frac{t}{s}\right)^M \frac{f(s)}{s}\,\mathrm{d}s - (-M)^{-1}f(0)\right| = 0.$$

Thus, $\lim_{t\to 0^+} y_p(t) = (-M)^{-1} f(0)$ and $y \in C[0,1]$.

It is clear from (31) that the solution y of (4) becomes unique if we specify the constant vector $c \in \mathbb{R}^n$. Note that at $t = 0$, $y(0)$ satisfies n linearly independent conditions $My(0) = -f(0)$ for any $c \in \mathbb{R}^n$. Therefore, we have to specify c via the terminal conditions given in (6). Let $\beta \in \mathbb{R}^n$ and let $B_1 \in \mathbb{R}^{n\times n}$ be nonsingular, then it follows from $B_1 y(1) = B_1 c = \beta$ that the unique solution of TVP (6) is given by (31), where $c = B_1^{-1}\beta$.

We now provide the estimate for y. To this aim, we utilize Lemma 8 with $\gamma = 0$ and the inequality

$$|t^M| = |Et^J E^{-1}| \le const.\, |t^J| \le const.\, t^{\sigma_+}(1 + |\ln(t)|^{n_{max}-1}).$$

Hence,

$$\begin{aligned}
|y(t)| &\le \left| t^M c + t^M \int_1^t s^{-M-I} f(s)\,\mathrm{d}s \right| \\
&\le \left| t^M B_1^{-1}\beta \right| + \left| \int_1^t \left(\frac{t}{s}\right)^M s^{-1} f(s)\,\mathrm{d}s \right| \\
&\le \left| t^M B_1^{-1}\beta \right| + ||f|| \int_1^t \left| \left(\frac{t}{s}\right)^M s^{-1} \right| \mathrm{d}s \\
&\le const.\, t^{\sigma_+}(1 + |\ln(t)|^{n_{max}-1}) \left| B_1^{-1}\beta \right| + const.\, \|f\|.
\end{aligned}$$

In order to discuss the smoothness of y, we first study the general solution of the homogeneous problem y_h. Assume that $0 \le r < \sigma_+ \le r+1$. Then, we have

$$\begin{aligned}
&y_h'(t) = \left(t^M c\right)' = M t^{M-I} c, \\
&y_h''(t) = \left(t^M c\right)'' = M(M-I) t^{M-2I} c, \\
&y_h^{(k)}(t) = \left(t^M c\right)^{(k)} = M(M-I)\cdots(M-(k-1)I) t^{M-kI} c, \;\; k = 1, \dots, r,
\end{aligned}$$

and it is easily seen that $y_h \in C^r[0,1] \cap C^\infty(0,1]$.

We now turn to the smoothness of the particular solution of the inhomogeneous problem y_p. First, we integrate by parts

$$\begin{aligned}
y_p(t) &= t^M \int_1^t s^{-M-I} f(s)\,\mathrm{d}s \\
&= t^M \left((-M)^{-1} t^{-M} f(t) - (-M)^{-1} I f(1) - (-M)^{-1} \int_1^t s^{-M} f'(s)\,\mathrm{d}s \right) \\
&= (M)^{-1} \left(t^M f(1) - f(t) + t^M \int_1^t s^{-M} f'(s)\,\mathrm{d}s \right).
\end{aligned}$$

Note that t^M and M^{-1} are commutative if t^M and M are commutative, since $t^M M^{-1} = (M t^{-M})^{-1}$. The later property will be shown in Lemma 19. Let us now assume that $f \in C^2[0,1]$. Then, we can differentiate the above equation and obtain

$$
\begin{aligned}
y_p'(t) &= (M)^{-1}\left(Mt^{M-I}f(1) - f'(t) + Mt^{M-I}\int_1^t s^{-M}f'(s)\,\mathrm{d}s + t^M t^{-M}f'(t)\right)\\
&= t^{M-I}f(1) + t^{M-I}\int_1^t s^{-M}f'(s)\,\mathrm{d}s.
\end{aligned}
$$

If $\sigma_+ > 1$, then we argue as at the beginning of the proof (in context of y and $\sigma_+ > 0$) and conclude that $y_p \in C^1[0,1]$. Moreover, the following estimate holds:

$$
\begin{aligned}
|y_p'(t)| &\le |f(1)|const.\, t^{\sigma_+-1}(1+|\ln(t)|^{n_{max}-1}) + \|f'\|t^{-1}\left|\int_1^t \left(\frac{t}{s}\right)^M \mathrm{d}s\right|\\
&\le |f(1)|const.\, t^{\sigma_+-1}(1+|\ln(t)|^{n_{max}-1}) + \|f'\|t^{-1}const.\, t\\
&\le const.\, \left(t^{\sigma_+-1}(1+|\ln(t)|^{n_{max}-1}) + \|f'\|\right), \quad t\in[0,1].
\end{aligned}
$$

This procedure can be iterated: Let $f \in C^3[0,1]$, then we integrate y_p' by parts and have

$$
y_p'(t) = (M-I)^{-1}\left(t^{M-I}f'(1) + (M-I)t^{M-I}f(1) - f'(t) + t^{M-I}\int_1^t s^{I-M}f''(s)\,\mathrm{d}s\right).
$$

Differentiating the above representation of y_p' yields

$$
y_p''(t) = t^{M-2I}f'(1) + (M-I)t^{M-2I}f(1) + t^{M-2I}\int_1^t s^{I-M}f''(s)\,\mathrm{d}s.
$$

If $\sigma_+ > 2$, then $y_p \in C^2[0,1]$ and the following estimate holds:

$$
\begin{aligned}
|y_p''(t)| &\le (|f(1)|+|f'(1)|)const.\, t^{\sigma_+-2}(1+|\ln(t)|^{n_{max}-1}) + \left|t^{-1}\int_1^t \left(\frac{t}{s}\right)^{M-I} f''(s)\right|\\
&\le const.\, \left(t^{\sigma_+-2}(1+|\ln(t)|^{n_{max}-1}) + \|f''\|\right), \quad t\in[0,1].
\end{aligned}
$$

Similarly, if $f \in C^{r+2}[0,1]$ and $\sigma_+ > r$, then $y_p \in C^r[0,1]$ and the following estimate holds:

$$
|y_p^{(r)}(t)| \le const.\, \left(t^{\sigma_+-r}(1+|\ln(t)|^{n_{max}-1}) + \|f^{(r)}\|\right), \quad t\in[0,1].
$$

For $\sigma_+ > r+1$, then $y_p \in C^{r+1}[0,1]$ and

$$
|y_p^{(r+1)}(t)| \le const.\, \left(t^{\sigma_+-r-1}(1+|\ln(t)|^{n_{max}-1}) + \|f^{(r+1)}\|\right), \quad t\in[0,1].
$$

It follows from (4) that if $f \in C^{r+2}[0,1]$ then $y_p \in C^{r+3}(0,1]$. Consequently, we have $y_p \in C^r[0,1]\cap C^{r+3}(0,1]$ for $r < \sigma_+ \le r+1$ and $y_p \in C^{r+1}[0,1]\cap C^{r+3}(0,1]$ for $\sigma_+ > r+1$.

We recall: if $r < \sigma_+ \le r+1$ then the solution $y_h \in C^r[0,1]\cap C^\infty(0,1]$ satisfies

$$
|y_h^{(r)}(t)| \le const.\, t^{\sigma_+-r}(1+|\ln(t)|^{n_{max}-1}), \quad t\in[0,1].
$$

For $\sigma_+ > r+1$, we have $y_h \in C^{r+1}[0,1] \cap C^\infty(0,1]$ and

$$|y_h^{(r+1)}(t)| \le const.\, t^{\sigma_+ - r-1}(1 + |\ln(t)|^{n_{max}-1}), \quad t \in [0,1].$$

The above smoothness results for y_p and y_h complete the proof. □

We recapitulate the case when all eigenvalues of M have positive real parts: For any $f \in C^1[0,1]$ and any vector $\beta \in \mathbb{R}^n$ there exists a unique continuous solution y of TVP (6) if and only if the matrix $B_1 \in R^{n\times n}$ is nonsingular. For each continuous solution y of (4), $My(0) = -f(0)$ holds independently on $c \in \mathbb{R}^n$ from (31). Consequently, in this case a well-posed initial problem (5) does not exist.

Remark 10 A continuous solution to (4) exists also in the case when f is not continuously differentiable in $[0,1]$. However, in this case, we need some more structure in f close to the singularity. Let us assume that

$$f(t) = O\left(t^\alpha h(t)\right) \text{ as } t \to 0, \tag{34}$$

for some constant $\alpha > 0$ and a function $h \in C[0,\delta_1]$, $\delta_1 > 0$. Then, the solution of (4) is still continuous on $[0,1]$. To see this note, that due to (34) there exists a $\delta_2 > 0$ such that $|f(t)| < const.\, t^\alpha |h(t)|$ for $t \in (0,\delta_2)$. Define $\delta := \min\{\delta_1,\delta_2\}$. Then

$$y_p(t) = \int_1^t \left(\frac{t}{s}\right)^M \frac{f(s)}{s}\,\mathrm{d}s = \int_1^\delta \left(\frac{t}{s}\right)^M \frac{f(s)}{s}\,\mathrm{d}s + \int_\delta^t \left(\frac{t}{s}\right)^M \frac{f(s)}{s}\,\mathrm{d}s.$$

Moreover by (29),

$$\lim_{t\to 0^+} \left| \int_\delta^t \left(\frac{t}{s}\right)^M \frac{f(s)}{s}\,\mathrm{d}s \right| < \lim_{t\to 0^+} const.\ \|h\|_{\delta_1} \int_1^t \left| \left(\frac{t}{s}\right)^M \right| s^{\alpha-1}\,\mathrm{d}s = 0.$$

Then, according to (15), we deduce

$$\lim_{t\to 0^+} y_p(t) = \lim_{t\to 0^+} \int_1^\delta \left(\frac{t}{s}\right)^M \frac{f(s)}{s}\,\mathrm{d}s = 0.$$

Therefore, $y \in C[0,1]$. Note that the function $f(t) = (t^{\alpha_1},\dots,t^{\alpha_n})^\top$, where $\alpha_i > 0$, $i = 1,\dots,n$, also satisfies condition (34).

5 Eigenvalues $\lambda = 0$

Finally, we consider the case when all eigenvalues of the matrix M are zero. We begin with a scalar equation (4) which for $M = \lambda = 0$ immediately reduces to

$$y'(t) = \frac{f(t)}{t}, \tag{35}$$

and show that additional structure in the function f is necessary to guarantee that the solution y is continuous on $[0,1]$. To see this, assume that f is a constant function, $f(t) \equiv 1$. Then, any solution y of equation (35) has the following form:

$$y(t) = y(1) + \int_1^t \frac{1}{s}\,\mathrm{d}s = y(1) + \ln t,\ \ t \in (0,1]$$

and, clearly, y is not continuous at $t = 0$.
Motivated by the scalar case, we require the inhomogeneity f to satisfy additional conditions providing the continuity of the associated solution.

Let us denote the eigenspace of M associated with the eigenvalues zero by X_0, the orthogonal projection onto X_0 by R, and the matrix, which consists of linearly independent columns of R by $\tilde{R}$. We also define the projection H as $H := I - R$.

Before formulating the main result of this section we show the following lemma.

Lemma 11 *Let us assume that all eigenvalues of the matrix M are zero. Then for $\alpha > 0$*

$$\lim_{t\to 0+} \int_0^t \left|s^{-M}\right| s^{\alpha-1}\,\mathrm{d}s = 0. \tag{36}$$

Proof: Let J_k, $k = 1,\ldots,l$, be the Jordan boxes of M. Then we can write $s^{-M} = Es^{-J}E^{-1},\ s^{-J} = \mathrm{diag}\left(s^{-J_1},\ldots,s^{-J_l}\right)$ and thus

$$\lim_{t\to 0+} \int_0^t \left|s^{-M}\right| s^{\alpha-1}\,\mathrm{d}s \le |E||E^{-1}| \int_0^t \left|s^{-J}\right| s^{\alpha-1}\,\mathrm{d}s.$$

Applying (19) and (15) we obtain (36). □

Theorem 12 *Let all eigenvalues of the matrix M be zero. Moreover, let us assume that there exist a constant $\alpha > 0$ and a function $h \in C[0,\delta]$, $\delta > 0$ such that*

$$f(t) = O(t^\alpha h(t)) \text{ for } t \to 0. \tag{37}$$

Then for any $f \in C[0,1]$, $\beta \in \mathbb{R}^m$, and a nonsingular $m \times m$ matrix $B_0\tilde{R}$, there exists a unique solution $y \in C[0,1]$ of IVP (5), where $m = \dim X_0$. This solution has the form

$$y(t) = \tilde{R}(B_0\tilde{R})^{-1}\beta + \int_0^1 s^{-M}s^{-1}f(st)\,\mathrm{d}s,\ \ t \in (0,1],$$

and satisfies the initial condition

$$My(0) = 0, \tag{38}$$

which is necessary and sufficient for $y \in C[0,1]$. Moreover,

$$|y(t)| \le |\tilde{R}(B_0\tilde{R})^{-1}\beta| + const.\,(\|f\| + t^\alpha\|h\|_\delta),\ \ t \in [0,1],$$

and if $\alpha \geq r+1$, $f \in C^r[0,1]$, *and* $h \in C^r[0,\delta]$, *then* $y \in C^{r+1}[0,1]$ *and the following estimates hold*

$$|y^{(k)}| \leq const. \sum_{j=0}^{k-1} \left(t^{\alpha-1}\right)^{(k-1-j)} \|h^{(j)}\|_\delta,\ t \in [0,\delta),$$
$$|y^{(k)}| \leq const. \sum_{j=0}^{k-1} \left(\left(t^{\alpha-1}\right)^{(k-1-j)} \|h^{(j)}\|_\delta + \left(t^{-1}\right)^{(k-1-j)} \|f^{(j)}\|\right),\ t \in [\delta,1],$$

where $k = 0,\dots,r+1$.

Proof: We split the general solution of (4) into two parts $y(t) = y_h(t) + y_p(t)$ as defined in (22). To prove that $y_p \in C[0,1]$, we again use the functions z_m with $m \in \mathbb{N}$ and z_∞ specified in (23) and (24). Due to (19), (36), and (37), we obtain

$$\begin{aligned}\lim_{m\to\infty} |z_\infty(t) - z_m(t)| &= \lim_{m\to\infty} \left| \int_0^{\frac{1}{m}} s^{-M} s^{-1} f(st)\,\mathrm{d}s \right| \\ &\leq \|h\|_\delta t^\alpha \lim_{m\to\infty} \int_0^{\frac{1}{m}} \left|s^{-M}\right| s^{\alpha-1}\,\mathrm{d}s = 0. \end{aligned} \tag{39}$$

Therefore, $y_p = z_\infty \in C[0,1]$ and $y_p(0) = 0$ since $f(0) = 0$ due to (37).

We now examine the continuity of

$$y_h(t) = t^M \left(c + \int_1^0 s^{-M} s^{-1} f(s)\,\mathrm{d}s \right) =: t^M \eta,$$

cf. (22). The fundamental solution matrix is given by $t^M = Et^J E^{-1}$, where t^J has the form $t^J = \mathrm{diag}(t^{J_1},\dots,t^{J_l})$ and

$$E = \left(v_1, h_1^{(1)}, h_1^{(2)}, \dots, h_1^{(n_1-1)}, v_2, h_2^{(1)}, \dots, h_2^{(n_2-1)}, \dots, v_l, h_l^{(1)}, \dots, h_l^{(n_l-1)}\right),$$

where for $k = 1,\dots,l$, v_k are the eigenvectors of M, $h_k^{(1)},\dots,h_k^{(n_k-1)}$ are the associated principal eigenvectors, and n_k are the dimensions of the Jordan boxes J_k. Clearly, because of the logarithmic terms occurring in t^J, see (11), y_h is not continuous at $t=0$ in general. Only when the contributions including the logarithmic terms vanish, y_h becomes continuous on $[0,1]$. It is clear from (11) that the only bounded contributions to y_h are linear combinations of the eigenvectors of M. Consequently, any linear combination of principal vectors has to vanish. This is the case when $\eta_i = 0, \forall i \neq 1, n_1+1, n_1+n_2+1, \dots, \sum_{k=1}^l n_k + 1$ and arbitrary η_i for all $i = 1, n_1+1, n_1+n_2+1, \dots, \sum_{k=1}^l n_k+1$. Thus, y_h is continuous on $[0,1]$ if and only if it is a constant linear combination of the eigenvectors of M. With other words, by setting $y_h(t) := \eta$, we have

$$y(t) \in C[0,1] \Leftrightarrow My(0) = M\eta = 0 \Leftrightarrow \eta \in \mathrm{Ker\,M}.$$

Consequently, $My(0) = 0$ is necessary and sufficient for the solution

$$y(t) = \eta + \int_0^1 s^{-M-I} f(ts)\,\mathrm{d}s,\ t \in [0,1] \tag{40}$$

to be continuous on $[0,1]$. The smoothness results for y follow from the smoothness of f and from Lemma 11.

Note that the regularity requirement $My(0)=0$ contains $n-l$ linearly independent conditions and can be equivalently expressed by $Hy(0)=0$, $y(0)=Ry(0)$ or $y(0)\in \operatorname{Ker} M$. The remaining l free constants have to be uniquely specified by appropriately prescribed initial conditions. Let us consider the initial conditions specified in (5), where $B_0\in\mathbb{R}^{m\times n}$ and $\beta\in\mathbb{R}^m$. Since $y_p(0)=0$ and $y_h(0)=\eta$, the initial condition $B_0y(0)=\beta$ is equivalent to $B_0\eta=\beta$. Due to the fact that $\eta\in\operatorname{Im} R$, there exists a unique l-dimensional vector d, $l=\dim X_0$, such that $\eta=\tilde{R}d$, where $\tilde{R}$ is the $n\times l$ matrix containing the linearly independent columns of R. Clearly, the problem is uniquely solvable if and only if $m=l=\dim X_0$ and the $m\times m$ matrix $B_0\tilde{R}$ is nonsingular. Hence,

$$B_0\eta=\beta \Leftrightarrow B_0\tilde{R}d=\beta \Rightarrow\ d=(B_0\tilde{R})^{-1}\beta \Rightarrow\ \eta=\tilde{R}(B_0\tilde{R})^{-1}\beta,$$

and the solution y has the form

$$y(t)=\tilde{R}(B_0\tilde{R})^{-1}\beta+\int_0^1 s^{-M}s^{-1}f(st)\,\mathrm{d}s,\ \ t\in[0,1].$$

This solution is bounded by

$$|y(t)|\le|\tilde{R}(B_0\tilde{R})^{-1}\beta|+\mathit{const.}\,(t^\alpha\|h\|_\delta+\|f\|),\ \ t\in[0,1],$$

due to

$$\begin{aligned}
|y_p(t)| &= \left|\int_0^1 s^{-M}s^{-1}f(ts)\,\mathrm{d}s\right|\\
&= \left|\int_\delta^1 s^{-M}s^{-1}f(ts)\,\mathrm{d}s\right|+\left|\int_0^\delta s^{-M}s^{-1}f(ts)\,\mathrm{d}s\right|\\
&\le \mathit{const.}\,\|f\|+\mathit{const.}\,t^\alpha\|h\|_\delta\int_0^\delta\left|s^{-M}\right|s^{\alpha-1}\,\mathrm{d}s,\ \ t\in[0,1]\\
&\le \mathit{const.}\,(\|f\|+t^\alpha\|h\|_\delta),\ \ t\in[0,1],
\end{aligned}$$

and (19).

In order to derive an estimate for the first derivative, we substitute the solution given by (40) into equation (4) and use the property $M\eta=0$ and Lemma 11. If $\alpha\ge 1$ then the first derivative is bounded by

$$\begin{aligned}
|y'(t)| &\le \mathit{const.}\,\frac{|M|}{t}t^\alpha\left|\int_0^1 s^{-M}s^{\alpha-1}h(st)\,\mathrm{d}s\right|+\mathit{const.}\,\frac{t^\alpha|h(t)|}{t}\\
&\le \mathit{const.}\,t^{\alpha-1}\|h\|_\delta,\ \ t\in[0,\delta),
\end{aligned}$$

$$\begin{aligned}
|y'(t)| &\le \mathit{const.}\,\frac{|M|}{t}t^\alpha\left|\int_0^\delta s^{-M}s^{\alpha-1}h(st)\,\mathrm{d}s\right|+\frac{|M|}{t}\left|\int_\delta^1 s^{-M}s^{-1}f(st)\,\mathrm{d}s\right|+\frac{|f(t)|}{t}\\
&\le \mathit{const.}\,t^{\alpha-1}\|h\|_\delta+\mathit{const.}\,t^{-1}\|f\|,\ \ t\in[\delta,1].
\end{aligned}$$

For $f \in C^1[0,1]$, $h \in C^1[0,\delta]$, and $\alpha \geq 2$, we now derive a bound of the second derivative,

$$\begin{aligned} y''(t) &= -\frac{M}{t^2}\int_0^1 s^{-M}s^{-1}f(st)\,\mathrm{d}s + \frac{M}{t}\int_0^1 s^{-M}f'(st)\,\mathrm{d}s - \frac{f(t)}{t^2} + \frac{f'(t)}{t}, \\ |y''(t)| &\leq const.\,(t^{\alpha-2}\|h\|_\delta + t^{\alpha-1}\|h'\|),\ t \in [0,\delta), \\ |y''(t)| &\leq const.\,(t^{\alpha-2}\|h\|_\delta + t^{\alpha-1}\|h'\| + t^{-2}\|f\| + t^{-1}\|f'\|),\ t \in [\delta,1]. \end{aligned}$$

Analogously, for $f \in C^r[0,1]$, $h \in C^r[0,\delta]$, $\alpha \geq r+1$, we have $y \in C^{r+1}[0,1]$ and

$$\begin{aligned} &|y^{(r+1)}| \leq const. \sum_{k=0}^{r} \left(t^{\alpha-1}\right)^{(r-k)} \|h^{(k)}\|_\delta,\ t \in [0,\delta), \\ &|y^{(r+1)}| \leq const. \sum_{k=0}^{r} \left(\left(t^{\alpha-1}\right)^{(r-k)} \|h^{(k)}\|_\delta + \left(t^{-1}\right)^{(r-k)} \|f^{(k)}\|\right),\ t \in [\delta,1]. \end{aligned}$$

□

Remark 13 Here, we deal with a purely polynomial inhomogeneity of the form

$$f(t) = (t^{\alpha_1},\dots,t^{\alpha_n})^\top,$$

where, $\alpha_i \in \mathbb{N}$, for $i = 1,\dots,n$. In this case, $y \in C^\infty[0,1]$. To see this, we consider the components of y,

$$y(t) = \gamma + \int_0^1 s^{-M}s^{-1}f(st)\,\mathrm{d}s,$$

given by

$$y_k(t) = \gamma_k + \int_0^1 \sum_{j=1}^{n} \left(s^{-M}\right)_{kj} s^{\alpha_j-1}t^{\alpha_j}\,\mathrm{d}s = \gamma_k + \sum_{j=1}^{n} w_j(t),$$

where $w_j(t) = t^{\alpha_j}\int_0^1 \left(s^{-M}\right)_{kj} s^{\alpha_j-1}\,\mathrm{d}s$. We now differentiate w_j, $j = 1\dots,n$, and obtain

$$\begin{aligned} &w_j'(t) = \alpha_j t^{\alpha_j-1}\int_0^1 s^{-M}s^{\alpha_j-1}\,\mathrm{d}s, \\ &w_j''(t) = \alpha_j(\alpha_j-1)t^{\alpha_j-2}\int_0^1 s^{-M}s^{\alpha_j-1}\,\mathrm{d}s, \\ &w_j^{(\alpha_j)}(t) = \alpha_j!\int_0^1 s^{-M}s^{\alpha_j-1}\,\mathrm{d}s, \\ &w_j^{(\alpha_j+1)}(t) = 0. \end{aligned}$$

Therefore $y(t) \in C^\infty[0,1]$.

In Theorem 12, we described the unique solvability of IVP (5) in case when all eigenvalues of M are zero. The dimension of the corresponding eigenspace X_0 was $m < n$ and it turned out that a regularity requirement $My(0) = 0$ has to be satisfied. If $m = n$, then $M = 0$ and the regularity condition holds. In this case we can also investigate if a well-posed TVP (6) exists. This question is dealt with in the next lemma.

Lemma 14 *Consider system (4) with the matrix $M = 0$. Let $f \in C[0,1]$ and assume that (37) is satisfied. Then, for any vector $\beta \in \mathbb{R}^n$ and a nonsingular matrix $B_1 \in \mathbb{R}^{n\times n}$ there exists a unique solution of (6),*

$$y(t) = B_1^{-1}\beta + \int_1^t \frac{f(s)}{s}\,\mathrm{d}s,$$

bounded by

$$|y(t)| \le |B_1^{-1}\beta| + const.\,(\|f\| + t^\alpha \|h\|_\delta).$$

Moreover, if $f \in C^r[0,1]$, $h \in C^r[0,\delta]$, and $\alpha \ge r+1$, then $y \in C^{r+1}[0,1]$ and the following estimates hold:

$$|y^{(k)}(t)| \le const. \sum_{j=0}^{k-1} \left(t^{\alpha-1}\right)^{(k-1-j)} \|h^{(j)}\|_\delta,\ t \in [0,\delta),$$
$$|y^{(k)}(t)| \le const. \sum_{j=0}^{k-1} \left(t^{-1}\right)^{(k-1-j)} \|f^{(j)}\|,\ t \in [\delta,1],$$

where $k = 0, \ldots, r+1$.

Proof: For $M = 0$ the system (4) reduces to

$$y'(t) = \frac{f(t)}{t},$$

and its solution is

$$y(t) = y(1) + \int_1^t \frac{f(s)}{s}\,\mathrm{d}s.$$

To show that $y \in C[0,1]$, we follow the arguments given in the proof of Theorem 12. The terminal condition $B_1 y(1) = \beta$ yields $y(1) = B_1^{-1}\beta$. Moreover,

$$\begin{aligned} |y(t)| &\le |B_1^{-1}\beta| + \int_1^\delta \frac{f(s)}{s}\,\mathrm{d}s + \int_\delta^t \frac{f(s)}{s}\,\mathrm{d}s \\ &\le |B_1^{-1}\beta| + \|f\||\ln(\delta)| + const.\,\|h\|_\delta \int_\delta^t s^{\alpha-1}\,\mathrm{d}s \\ &\le |B_1^{-1}\beta| + const.\,(\|f\| + \|h\|_\delta). \end{aligned}$$

Estimates for the higher derivatives of y follow in an analogous manner. □

6 Differences between linear systems with smooth and unsmooth inhomogeneity

Before discussing the case of an arbitrary spectrum of M which enables more general well-posed IVPs, TVPs, and BVPs, we summarize here the results from the previous sections and point out the differences when compared to the framework given in [9, 19], where linear systems with smooth inhomogeneity,

$$y'(t) = \frac{M}{t} y(t) + f(t), \ f \in C[0,1], \tag{41}$$

were studied.

6.1 Eigenvalues with negative real parts

Let us consider the ODE system (41) and assume that all eigenvalues of M have negative real parts. Then, according to [9, 19], $y \in C[0,1]$ if and only if $y(0) = 0$. Therefore, the following IVP has a unique solution:

$$y'(t) = \frac{M}{t} y(t) + f(t), \quad y(0) = 0.$$

Moreover, $y \in C^{r+1}[0,1]$ if $f \in C^r[0,1]$, $r \geq 0$.
According to Theorem 5, ODE system (4) has a solution $y \in C[0,1]$ if and only if $My(0) = -f(0)$. Consequently, the IVP specified below has a unique solution,

$$y'(t) = \frac{M}{t} y(t) + \frac{f(t)}{t}, \quad My(0) = -f(0).$$

Moreover, $y \in C^r[0,1]$ if $f \in C^r[0,1]$, $r \geq 0$.
The conditions $y(0) = 0$ and $My(0) = -f(0)$ are necessary and sufficient for the solution $y \in C[0,1]$ in case of the system (41) and (4), respectively.

6.2 Eigenvalues with positive real parts

For this spectrum of M neither for system (41) nor for (4) there exists a well-posed IVP. In both cases we need to specify the boundary conditions at $t = 1$ and solve the TVP. In particular, the TVP

$$y'(t) = \frac{M}{t} y(t) + f(t), \quad B_1 y(1) = \beta,$$

where $B_1 \in \mathbb{R}^{n\times n}$ is nonsingular and $\beta \in \mathbb{R}^n$, has a unique solution $y \in C[0,1]$. This solution satisfies $y(0) = 0$. If $f \in C^r[0,1]$ and $\sigma_+ > r+1$ then $y \in C^{r+1}[0,1]$, cf. [9]. In contrast to system (41), we need extra smoothness of the function f to obtain a unique continuous solution of the TVP

$$y'(t) = \frac{M}{t} y(t) + \frac{f(t)}{t}, \quad B_1 y(1) = \beta,$$

where $B_1 \in \mathbb{R}^{n\times n}$ is nonsingular and $\beta \in \mathbb{R}^n$. Theorem 9 states that $y \in C[0,1]$ if $f \in C^1[0,1]$. Additionally, if $f \in C^{r+2}[0,1]$ and $\sigma_+ > r+1$ then $y \in C^{r+1}[0,1]$, $r \geq 0$.

6.3 Eigenvalues $\lambda = 0$

If all eigenvalues of M are zero, then the well-posed IVP associated with (41) takes the form

$$y'(t) = \frac{M}{t}y(t) + f(t), \quad My(0) = 0, \quad B_0 y(0) = \beta,$$

where the $m \times m$ matrix $B_0\tilde{R}$ is nonsingular, $\beta \in \mathbb{R}^m$, and $m = \dim X_0$. The initial condition $My(0) = 0$ is necessary and sufficient for the solution to by continuous. The remaining m conditions necessary for its uniqueness are specified by $B_0 y(0) = \beta$. For $f \in C^r[0,1], r \geq 0$, $y \in C^{r+1}[0,1]$, see [9, 19].
In case of the unsmooth inhomogeneity in (4), f has to satisfy an additional requirement,

$$f(t) = O(t^\alpha h(t)) \text{ as } t \to 0,\ \alpha > 0,\ h \in C[0,\delta],\ \delta > 0,$$

to enable a continuous solution of the following IVP:

$$y'(t) = \frac{M}{t}y(t) + \frac{f(t)}{t}, \quad My(0) = 0, \quad B_0 y(0) = \beta,$$

where the $m \times m$ matrix $B_0\tilde{R}$ is nonsingular, $\beta \in \mathbb{R}^m$, and $m = \dim X_0$.
Finally, if $f \in C^r[0,1]$, $h \in C^r[0,\delta]$, and $\alpha \geq r+1$, then $y \in C^{r+1}[0,1]$.

7 General IVPs, TVPs and BVPs

In this section we study general IVPs, TVPs and BVPs. For the subsequent discussion, we have to introduce the following notation.

X_+ is the invariant subspace associated with the eigenvalues with positive real parts;

$X_0^{(e)}$ is the space of eigenvectors associated with eigenvalues $\lambda = 0$;

X_- is the invariant subspace associated with the eigenvalues with negative real parts;

$X_0^{(h)}$ is the space of generalized eigenvectors associated with the eigenvalue $\lambda = 0$;

S is the orthogonal projection onto X_+;

R is the orthogonal projection onto $X_0^{(e)}$;

$P := R + S$ is the projection onto $X_+ \oplus X_0^{(e)}$;

$Q := I - P$ is the projection onto $X_- \oplus X_0^{(h)}$;

Z is the orthogonal projection onto $X_0^{(e)} \oplus X_0^{(h)}$;

N is the orthogonal projection onto X_-;

H is the orthogonal projection onto $X_0^{(h)}$.

All projections are constructed using the eigenbasis of M.

Firstly, we discuss general IVPs (5) and TVPs (6), where all conditions which are necessary and sufficient to specify a unique solutions $y \in C[0,1]$ are posed at only one point, either at $t = 0$ or at $t = 1$. According to the results derived above, restrictions on the spectrum of M need to be made.

A.1 For IVP (5) we assume that the matrix M has only eigenvalues with nonpositive real parts and if $\sigma = 0$ then $\lambda = 0$.

A.2 For TVP (6) we assume that the matrix M has only eigenvalues with nonnegative real parts and if $\sigma = 0$ then $\lambda = 0$. Additionally, if zero is an eigenvalue of M, then the associated invariant subspace is assumed to be the eigenspace of M.

Results formulated below without proofs are simple consequences of Theorems 5, 9, 12, and Lemma 14.

Lemma 15 *Let us assume that $f \in C[0,1]$, $Sf \in C^1[0,1]$ and Zf satisfies condition (37).*

(i) Assume A.1 to hold. Let y be a continuous solution of IVP (5). Then

$$MNy(0) = -Nf(0), \quad Hy(0) = 0.$$

(ii) Assume A.2 to hold. Let y be a continuous solution of TVP (6). Then

$$MSy(0) = -Sf(0).$$

In both cases

$$My(0) = M(S+N)y(0) = -f(0).$$

The statement of Lemma 15 means that the conditions which are necessary for the solution of IVP (5) to be continuous are equivalent to

$$\operatorname{rank} M = \operatorname{rank} H + \operatorname{rank} N = \operatorname{rank} Q = n - \operatorname{rank} R$$

initial conditions, which the solution y has to satisfy. In case of TVP (6), where A.2 holds, any solution of (4) is continuous on $[0,1]$ and no regularity conditions have to be prescribed.

From Theorems 5 and 12 we obtain the following result for a general IVP (5).

Theorem 16 *Let us assume that A.1 holds, the $m \times m$ matrix $B_0\tilde{R}$ is nonsingular, where the matrix $\tilde{R}$ consists of the linear independent columns of the projection matrix R, and $\beta \in \mathbb{R}^m$. Then, for every $f \in C[0,1]$ such that Zf satisfies (37), there exists a unique solution $y \in C[0,1]$ of IVP (5),*

$$y(t) = \tilde{R}(B_0\tilde{R})^{-1}\beta + \int_0^1 s^{-M} s^{-1} f(st)\,\mathrm{d}s.$$

This solution is bounded by

$$|y(t)| \quad \leq \quad const.(t^\alpha \|h\|_\delta + \|f\|) + |\tilde{R}(B_0\tilde{R})^{-1}\beta|.$$

Let $Nf \in C^{r+1}[0,1]$ and $Zf \in C^r[0,1]$ satisfy condition (37) with $\alpha \geq r+1$. Then $y \in C^{r+1}[0,1]$.

The analogous result for a general TVP (6) follows from Theorems 9 and 12.

Theorem 17 *Let us assume that A.2 holds, $B_1 \in \mathbb{R}^{n\times n}$ is nonsingular, and $\beta \in \mathbb{R}^n$. Then, for every $f \in C[0,1]$ such that Rf satisfies (37) and $Sf \in C^1[0,1]$, there exists a unique solution $y \in C[0,1]$ of TVP (6),*

$$y(t) \quad = \quad t^M B_1^{-1}\beta + t^M \int_1^t s^{-M} s^{-1} f(s)\,\mathrm{d}s,\ \ t \in (0,1].$$

This solution satisfies $My(0) = -f(0)$ and is bounded by

$$|y(t)| \le const.\,(1 + t^{\sigma_+}(1 + |\ln(t)|^{n_{max}-1}))|B_1^{-1}\beta| + const.\,(\|f\| + t^{\alpha}\|h\|_\delta).$$

Let $r < \sigma_+ \le r+1$, $Sf \in C^{r+2}[0,1]$, and $Zf \in C^{r-1}[0,1]$ satisfy condition (37) with $\alpha \ge r$, then $y \in C^r[0,1]$. For $\sigma_+ > r+1$, $Sf \in C^{r+2}[0,1]$, and $Zf \in C^r[0,1]$ satisfying condition (37) with $\alpha \ge r+1$, we have $y \in C^{r+1}[0,1]$. Here, σ_+ denotes the smallest positive real part of the eigenvalues of M and n_{max} is the dimension of the largest Jordan box of M.

Further, we study the linear BVPs of the form

$$y'(t) = \frac{M}{t}y(t) + \frac{f(t)}{t},\ t \in (0,1], \quad y \in C[0,1], \quad B_0 y(0) + B_1 y(1) = \beta, \quad (42)$$

where the matrix M may have an arbitrary spectrum, $B_0, B_1 \in \mathbb{R}^{m\times n}$, $m \le n$, $\beta \in \mathbb{R}^m$, and $f \in C[0,1]$. It is clear from the previous considerations that the form of the boundary conditions which guarantee the existence of a unique continuous solution of (42) will depend on the spectral properties of the coefficient matrix M. Before proceeding with the analysis, we show the following auxiliary results.

Lemma 18 *Let R be a projection onto the eigenspace associated with eigenvalues $\lambda = 0$. Then*

$$t^M R = R,\ t \in [0,1].$$

Proof: Let M and R be represented using the eigenbasis of M, this means $M = EJE^{-1}$ and $R = E\hat{R}E^{-1}$, where $\hat{R}$ is a diagonal matrix with ones at the positions corresponding to the eigenvalues $\lambda = 0$ and zero entries elsewhere. It is sufficient to show $t^J\hat{R} = \hat{R}$ since this implies

$$t^M R = Et^J E^{-1} E\hat{R}E^{-1} = E\hat{R}E^{-1} = R.$$

The multiplication $t^J\hat{R}$ means an operation on columns of t^J and due to the structure of $\hat{R}$, the result of the matrix multiplication $t^J\hat{R}$ is a matrix whose columns corresponding to the eigenvectors associated with $\lambda = 0$ remain unchanged and all other columns vanish. Clearly, the nontrivial columns of $t^J\hat{R}$ are unit vectors, since $t^\lambda = 1$ for $\lambda = 0$, and the result, $t^J\hat{R} = \hat{R}$, follows. □

Lemma 19 *The projection matrices S, Z, and N and the matrix t^M commutate.*

Proof: We show the result for the projection S, for the other projections the proof is analogous. First, we prove that for the Jordan canonical form $t^J\hat{S} = \hat{S}t^J$ holds for $t \in [0,1]$, where $M = EJE^{-1}$ and $S = E\hat{S}E^{-1}$. The matrix $\hat{S}$ is a diagonal matrix with ones at the positions corresponding to the eigenvalues with positive real parts and zero entries elsewhere. The multiplication $\hat{S}t^J$ means operations on rows of t^J, while the multiplication $t^J\hat{S}$ represents operations on columns of t^J. Being aware of the block structure of matrix t^J we see that the result of $\hat{S}t^J$ or $t^J\hat{S}$ is a matrix containing Jordan boxes corresponding to eigenvalues with positive real parts and $t^J\hat{S} = \hat{S}t^J$ for $t \in [0,1]$. This implies

$$t^M S = Et^J E^{-1} E\hat{S}E^{-1} = E\hat{S}E^{-1}Et^J E^{-1} = St^M$$

and the statement follows. □

Note that the above projections S, Z, and N commutate with M.

To specify the boundary conditions which guarantee the well-posedness of BVP (42) the following lemma is required.

Lemma 20 *Consider the following BVP:*

$$y'(t) = \frac{M}{t}y(t) + \frac{f(t)}{t},\ t \in (0,1],$$
$$Hy(0) = 0,\ MNy(0) = -Nf(0),\ Sy(1) = S\gamma,\ Ry(0) = R\gamma.$$

Then, for every $f \in C[0,1]$, such that Zf satisfies (37) and $Sf \in C^1[0,1]$, and for any constant vector γ, there exist a unique continuous solution of the form

$$y(t) = t^M P\gamma + t^M S\int_1^t s^{-M-I} f(s)\,\mathrm{d}s + (Q+R)\int_0^1 s^{-M-I} f(st)\,\mathrm{d}s.$$

Proof: According to the previous results, the contributions to the solution y depend on the signs of the eigenvalues of M. For the eigenvalues with negative real parts the contribution has the form

$$y_-(t) = N\int_0^1 s^{-M-I} f(st)\,\mathrm{d}s,\ t \in [0,1].$$

For the eigenvalues with positive real parts the contribution is given by

$$y_+(t) = t^M S\gamma + t^M S\int_1^t s^{-M-I} f(s)\,\mathrm{d}s,\ t \in (0,1],$$

and can be continuously extended to $t = 0$. Finally, for the eigenvalues $\lambda = 0$, we have

$$y_0(t) = R\gamma + Z\int_0^1 s^{-M-I} f(st)\,\mathrm{d}s,\ t \in [0,1].$$

The solution y is the sum of all contributions, $y(t) = (N + S + Z)y(t)$. Therefore, we obtain

$$\begin{aligned} y(t) &= R\gamma + t^M S\gamma + (N + R + H)\int_0^1 s^{-M-I} f(st)\,\mathrm{d}s + t^M S\int_1^t s^{-M-I} f(s)\,\mathrm{d}s \\ &= t^M P\gamma + t^M S\int_1^t s^{-M-I} f(s)\,\mathrm{d}s + (Q + R)\int_0^1 s^{-M-I} f(st)\,\mathrm{d}s. \end{aligned}$$

We now evaluate y at the boundaries to show that the above boundary conditions are satisfied. According to (37), $Zf(0) = 0$ holds. This yields

$$(R + H)y(0) = Zy(0) = R\gamma + Z\int_0^1 s^{-M-I} f(0)\,\mathrm{d}s = R\gamma + \int_0^1 s^{-M-I}\,\mathrm{d}s Zf(0) = R\gamma.$$

Therefore $Hy(0) = 0$ and $Ry(0) = R\gamma$. Moreover,

$$Sy(t) = t^M S\gamma + t^M S\int_1^t s^{-M-I} f(s)\,\mathrm{d}s \Rightarrow Sy(1) = S\gamma.$$

Finally, we show that $MNy(0) = -Nf(0)$. First note that

$$MNy(0) = NM\int_0^1 s^{-M-I}\,\mathrm{d}s f(0).$$

According to (16),

$$M\int_t^1 s^{-M-I}\,\mathrm{d}s = t^{-M} - I$$

for $t \in (0, 1]$. Taking into account (17) and letting $t \to 0^+$, we obtain

$$NM\int_0^1 s^{-M-I}\,\mathrm{d}s = \lim_{t\to 0^+} Nt^{-M} - N = -N,$$

since the matrix Nt^{-M} consist only of Jordan boxes corresponding to eigenvalues with negative real parts. Therefore $MNy(0) = -Nf(0)$. □

We now turn to the general boundary conditions specified in (42). For the investigation of these condition, we have to rewrite the representation of the solution y, especially the term y_0,

$$\begin{aligned} y_0(t) &= R\gamma + Z\int_0^1 s^{-M-I} f(st)\,\mathrm{d}s = R\gamma + Zt^M\int_0^t s^{-M-I} f(s)\,\mathrm{d}s \\ &= R\gamma + t^M R\int_0^t s^{-M-I} f(s)\,\mathrm{d}s + t^M H\int_0^t s^{-M-I} f(s)\,\mathrm{d}s \\ &= R\tilde{\gamma} + t^M R\int_1^t s^{-M-I} f(s)\,\mathrm{d}s + t^M H\int_0^t s^{-M-I} f(s)\,\mathrm{d}s, \end{aligned}$$

where

$$\tilde{\gamma} := \gamma + \int_0^1 s^{-M-I} f(s)\,\mathrm{d}s.$$

Remark 21 Note that the function

$$t^M H \int_0^t s^{-M-I} f(s)\,\mathrm{d}s = t^M H t^{-M} \int_0^1 s^{-M-I} f(st)\,\mathrm{d}s,$$

is continuous on $[0,1]$. In order to see this, we again use functions z_m and z_∞ given by (23) and (24). Due to (15), (36), (37), and (39), we have

$$\lim_{m\to\infty} |(\ln t)^k z_\infty(t) - (\ln t)^k z_m(t)| \le \|h\|_\delta \|t^\alpha (\ln t)^k\| \lim_{m\to\infty} \int_0^{\frac{1}{m}} \left|s^{-M}\right| s^{\alpha-1}\,\mathrm{d}s = 0$$

for $k \in \mathbb{N}\cup\{0\}$. Since each entry of the matrix $t^M H t^{-M}$ is a sum of terms *const*. $(\ln t)^k$, $k \in \mathbb{N}\cup\{0\}$, the function

$$t^M H t^{-M} \int_0^1 s^{-M-I} f(st)\,\mathrm{d}s = t^M H t^{-M} z_\infty$$

is continuous on $[0,1]$.

Consequently, the general continuous solution of the ODE system given in (42) can be represented as

$$y(t) = t^M P\gamma + t^M P \int_1^t s^{-M-I} f(s)\,\mathrm{d}s + t^M Q \int_0^t s^{-M-I} f(s)\,\mathrm{d}s, \tag{43}$$

and satisfies the following boundary conditions:

$$Hy(0) = 0, \quad MNy(0) = -Nf(0), \quad Py(1) = P\gamma.$$

In the following lemma, we use the superposition principle to rewrite the solution (43) of (42) in a way convenient to discus the boundary conditions specified in (42).

Lemma 22 *Let us assume that the inhomogeneity $f \in C[0,1]$ is given in such a way that Zf satisfies (37) and $Sf \in C^1[0,1]$. Let the $n \times m$ matrix $\widetilde{P}$ be a matrix consisting of the linearly independent columns of P. Then the general continuous solution of (42) has the form*

$$y(t) = \widetilde{y}(t) + Y(t)\alpha,\ t \in [0,1], \tag{44}$$

where α is a constant m-dimensional vector and $\widetilde{y}$ is the unique solution of

$$\widetilde{y}'(y) = \frac{M}{t}\widetilde{y}(t) + \frac{f(t)}{t},\ t \in [0,1],\ H\widetilde{y}(0) = 0,\ MN\widetilde{y}(0) = -Nf(0),\ P\widetilde{y}(1) = 0,$$

and $Y(t)$ is the unique continuous fundamental solution matrix satisfying

$$Y'(t) = \frac{M}{t}Y(t),\ t \in [0,1],\ Y(1) = \widetilde{P}.$$

The case of the general boundary conditions (42) is covered by the following lemma.

Lemma 23 *Let $f \in C[0,1]$ be given in such a way that Zf satisfies (37) and $Sf \in C^1[0,1]$. Then, there exists a unique solution $y \in C[0,1]$ of BVP (42) if and only if the $m \times m$ matrix*

$$B_0\widetilde{R} + B_1\widetilde{P}$$

is nonsingular. Here, $B_0, B_1 \in \mathbb{R}^{m\times n}$, $\beta \in \mathbb{R}^m$, and $m = \operatorname{rank} P$.

Proof: We use (43) and (44) to calculate $y(0)$ and $y(1)$. Since $Hy(0) = 0$ and $\lim_{t\to 0} t^M S = 0$, we first deduce

$$\begin{aligned} y(0) &= (H+P+N)y(0) = (P+N)(\tilde{y}(0) + Y(0)\alpha) = (P+N)\tilde{y}(0) + PY(0)\alpha \\ &= (P+N)\tilde{y}(0) + (R+S)Y(0)\alpha = (P+N)\tilde{y}(0) + \widetilde{R}\alpha. \end{aligned}$$

Moreover, from $\tilde{P}y(1) = 0$ and $1^M S = S$, we have

$$\begin{aligned} y(1) &= (Q+P)y(1) = (Q+P)(\tilde{y}(1) + Y(1)\alpha) = Q\tilde{y}(1) + (QY(1) + \widetilde{P})\alpha \\ &= Q\tilde{y}(1) + \widetilde{P}\alpha. \end{aligned}$$

Finally, we substitute $y(0)$ and $y(1)$ into the boundary condition and obtain

$$B_0 y(0) + B_1 y(1) = B_0\left((P+N)\tilde{y}(0) + \widetilde{R}\alpha\right) + B_1\left(Q\tilde{y}(1) + \widetilde{P}\alpha\right) = \beta.$$

Thus,

$$\left(B_0\widetilde{R} + B_1\widetilde{P}\right)\alpha = \beta - B_0(P\tilde{y}(0) + N\tilde{y}(0)) - B_1 Q\tilde{y}(1),$$

and the unknown vector α can be uniquely determined if the $m \times m$ matrix

$$B_0\widetilde{R} + B_1\widetilde{P}$$

is nonsingular. This completes the proof. □

The following theorem stated without proof is a consequence of the above results.

Theorem 24 *Consider BVP (42), where the inhomogeneity f is given in such a way such that $f \in C[0,1]$, Zf satisfies (37), and $Sf \in C^1[0,1]$. Moreover, let $B_0, B_1 \in \mathbb{R}^{m\times n}$, $\beta \in \mathbb{R}^m$, and $m = \operatorname{rank} P$. Let us assume that the $m \times m$ matrix $B_0\widetilde{R} + B_1\widetilde{P}$ is nonsingular. Then, BVP (42) has a unique continuous solution $y \in C[0,1]$. This solution satisfies two initial conditions,*

$$Hy(0) = 0, \quad MNy(0) = -Nf(0)$$

which are necessary and sufficient for $y \in C[0,1]$.

8 Collocation method

In this section we propose and analyze the polynomial collocation, cf. [8], for the numerical treatment of IVP (5) which we assume to be well-posed,

$$y'(t) = \frac{M}{t}y(t) + \frac{f(t)}{t}, \quad B_0 y(0) = \beta,$$

where the matrix M has only eigenvalues with nonpositive real parts, and if $\sigma = 0$ then $\lambda = 0$. Moreover, $B_0 \in \mathbb{R}^{m\times n}$, $\beta \in \mathbb{R}^m$, where $\operatorname{rank} R = m \le n$. For the numerical treatment, we have to augment the m initial conditions specified by $B_0 y(0) = \beta$ by the $n-m$ linearly independent initial conditions singled out from the set

$$Hy(0) = 0, \quad MNy(0) = -Nf(0).$$

Consequently, we have to solve the initial value problem,

$$y'(t) = \frac{M}{t}y(t) + \frac{f(t)}{t}, \quad B_0 y(0) = \beta, \quad Hy(0) = 0, \quad MNy(0) = -Nf(0). \tag{45}$$

We first discretize the analytical problem (45). The interval of integration $[0,1]$ is partitioned by an equidistant mesh Δ,

$$\Delta := \{0 = t_0 < t_1 < \ldots < t_{I-1} < t_I = 1,\ \ t_j = jh,\ \ j = 0,\ldots, I = 1/h\},$$

and in each subinterval $[t_j, t_{j+1}]$, we introduce k collocation nodes $t_{jl} := t_j + u_l h,\ j = 0,\ldots, I-1,\ \ l = 1,\ldots,k$, where $0 < u_1 < u_2 < \ldots < u_k \le 1$. The computational grid including the mesh points and the collocation points is shown in Figure 1.

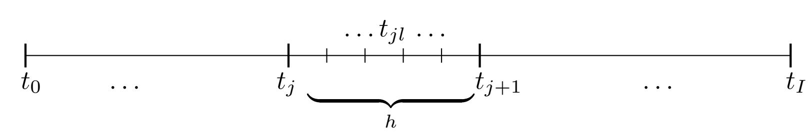

Figure 1 The computational grid

By $\mathcal{P}_{k,h}$ we denote the class of piecewise polynomial function of degree less or equal to k on each subinterval $[t_j, t_{j+1}]$. We approximate the analytical solution y by a piecewise polynomial function $p \in \mathcal{P}_{k,h} \cap C[0,1]$, $p(t) := p_j(t),\ t \in [t_j, t_{j+1}]$, $j = 0,\ldots, I-1$ such that p satisfies ODE system (4) at the collocation points,

$$p'(t_{jl}) - \frac{M}{t_{jl}}p(t_{jl}) = \frac{f(t_{jl})}{t_{jl}},\ \ l = 1,\ldots,k,\ \ j = 0,\ldots, I-1, \tag{46}$$

together with the continuity relations,

$$p_{j-1}(t_j) = p_j(t_j),\ \ j = 1,\ldots, I-1, \tag{47}$$

and p_0 satisfies the initial conditions

$$B_0 p(0) = \gamma, \quad Hp(0) = 0, \quad MNp(0) = -Nf(0). \tag{48}$$

Note, that $\operatorname{rank} B_0 + \operatorname{rank} H + \operatorname{rank} N = n$. Since in each subinterval $[t_j, t_{j+1}]$ $p(t) = p_j(t)$ is a polynomial of degree smaller or equal to k, the total number of unknowns, the coefficients in the ansatz function p, is $(k+1)In$. On the other hand, the system (46) consists of kIn equations, (47) provides $(I-1)n$, and (48) n conditions, which together add up to $(k+1)In$. This means that the collocation scheme (46), (47), and (48) is closed.

The collocation applied to solve (41) was studied in [8], where in particular, unique solvability of the collocation scheme and the convergence properties have been shown. For reader's convenience, we recapitulate in the next theorem an important auxiliary result from [8] required in the subsequent investigations. Note that since analytical problem (45) has a unique solution, its value $y(0)$ is known. Therefore, in Theorem 4.1 [8], a slightly simpler problem is considered, where instead of the initial conditions the correct value of $y(0) := \delta$ is prescribed.

Theorem 25 *(Theorem 4.1 in [8]) Let us consider the collocation scheme,*

$$p'(t_{jl}) - \frac{M}{t_{jl}} p(t_{jl}) = M^{\mu} \frac{c_{jl}}{t_{jl}^{\beta}}, \;\; l = 1, \ldots, k, \;\; j = 0, \ldots I-1, \quad p(0) = \delta, \tag{49}$$

where $\mu, \beta = 0, 1,$ *and* $p \in \mathcal{P}_{k,h} \cap C[0,1]$. *Then problem (49) has a unique solution provided that h is sufficiently small. This solution satisfies*

$$|p(t)| \leq const. \left(|\delta| + |\ln(h)|^d \, |M\delta| + |\ln(h)|^{(\beta(d-\mu))_+} C \right), \;\; t \in [0,1],$$

where $C = \max_{0 \leq j \leq I-1} \max_{1 \leq l \leq k} |c_{jl}|$, *$d$ is the dimension of the largest Jordan box of M associated to the eigenvalue $\lambda = 0$ and*

$$(x)_+ = \begin{cases} x & x \geq 0, \\ 0 & x < 0. \end{cases}$$

We are now in the position to formulate the convergence result for the collocation method.

Theorem 26 *Let us consider the initial value problem*

$$y'(t) - \frac{M}{t} y(t) = \frac{f(t)}{t}, \quad y(0) = \delta,$$

where $H\delta = 0$ *and* $MN\delta = -Nf(0)$. *Let us assume that the function f satisfies* $Nf \in C^{k+1}[0,1]$, $Zf = O(t^{\alpha} z(t))$, *with* $\alpha \geq k+1$, $Zf \in C^k[0,1]$ *and* $z \in C^k[0,1]$. *Let the function* $p \in \mathcal{P}_{k,h} \cap C[0,1]$ *satisfy the collocation scheme*

$$p'(t_{jl}) - \frac{M}{t_{jl}} p(t_{jl}) = \frac{f(t_{jl})}{t_{jl}}, \quad l = 1, \ldots k, \;\; j = 0, \ldots, I-1, \quad p(0) = \delta.$$

Then

$$|p(t) - y(t)| \le const.\, h^k,\ t \in [0,1].$$

Proof: The idea of the proof is to introduce an *error function* $e \in \mathcal{P}_{k,h} \cap C[0,1]$ and investigate how it is related to the global error $p - y$ of the scheme. Let e be defined as follows:

$$e'(t_{jl}) := y'(t_{jl}) - p'(t_{jl}), \quad l = 1, \dots, k,\ j = 0, \dots, I-1, \quad e(0) := 0.$$

Since on each subinterval $[t_j, t_{j+1}]$ the function $e'(t)$ is a polynomial of degree less or equal to $k-1$ it is uniquely determined by its values at k distinct points in this interval,

$$e'(t) = \sum_{i=1}^{k} l_i \left(\frac{t - t_j}{h} \right) y'(t_{ji}) - p'(t),\ t \in [t_j, t_{j+1}],$$

where

$$l_i(t) = w(t) / \left((t - u_i) w'(u_i) \right),\ i = 1, \dots, k, \quad w(t) = (t - u_1)(t - u_2) \cdots (t - u_k).$$

Since $y \in C^{k+1}[0,1]$ the interpolation error is $O(h^k)$ and hence,

$$e'(t) = y'(t) - p'(t) + O(h^k).$$

By integration in $[0, t]$, we obtain

$$e(t) = y(t) - p(t) + O(h^k t)$$

which means that e differs from $y - p$ by $O(h^k)$ terms. Moreover, we see that e satisfies the following collocation scheme:

$$\begin{aligned} e'(t_{jl}) - \frac{M}{t_{jl}} e(t_{jl}) &= y'(t_{jl}) - \frac{M}{t_{jl}} y(t_{jl}) - \left(p'(t_{jl}) - \frac{M}{t_{jl}} p(t_{jl}) \right) - \frac{M}{t_{jl}} O(t_{jl} h^k) \\ &= \frac{f(t_{jl})}{t_{jl}} - \frac{f(t_{jl})}{t_{jl}} - \frac{M}{t_{jl}} O(t_{jl} h^k) = O(h^k), \quad e(0) = 0. \end{aligned}$$

According to Theorem 25, we obtain $e(t) = O(h^k)$. This together with $e(t) = y(t) - p(t) + O(h^k)$ yields

$$|p(t) - y(t)| \le const.\ h^k$$

and the result follows. □

The especially attractive property of the collocation is the so called superconvergence. For regular ODEs and certain choices of the collocation points (Gaussian, Lobatto, Radau), the convergence order in the *mesh points* can be considerably higher than k, provided that the solution y is sufficiently smooth. For the Gaussian points the superconvergence order is $O(h^{2k})$. Since already for the problem

(41) counterexamples show that the superconvergence does not hold [8], we do not expect it for the problem at hand either. However, the so-called small superconvergence uniform in t can be shown, see next theorem. The main prerequisite for the proof is the property

$$\int_0^1 w(s)\,\mathrm{d}s = 0 \tag{50}$$

which holds for an appropriate choice of the collocation points.

Theorem 27 *Let $Nf \in C^{k+2}[0,1]$, $Zf \in C^{k+1}[0,1]$ and $Zf = O(t^\alpha z(t))$, where $\alpha \geq k+2$ and $z \in C^{k+1}[0,1]$. If (50) holds, then the estimate for the global error given in Theorem 26 can be replaced by*

$$|p(t) - y(t)| \leq const.\, h^{k+1} |\ln(h)|^{(d-1)_+}.$$

Proof: Consider again the error function e defined in Theorem 26. Due to the smoothness assumptions made for the problem data $y \in C^{k+2}[0,1]$ follows. Therefore,

$$\begin{aligned} e'(t) &= \sum_{i=1}^{k} l_i \left(\frac{t-t_j}{h}\right) y'(t_{ji}) - p'(t) \\ &= y'(t) - p'(t) + \frac{h^k}{k!} w\left(\frac{t-t_j}{h}\right) y^{(k+1)}(t_j) + O(h^{k+1}), \;\; t \in [t_j, t_{j+1}]. \end{aligned}$$

We integrate e' on $[0,1]$ and use (50) to obtain

$$\begin{aligned} e(t) &= y(t) - p(t) + \sum_{i=0}^{j-1} \frac{h^k}{k!} y^{(k+1)}(t_i) \int_{t_i}^{t_{i+1}} w\left(\frac{s-t_i}{h}\right) \mathrm{d}s \\ &+ \frac{h^k}{k!} y^{(k+1)}(t_j) \int_{t_j}^{t} w\left(\frac{s-t_j}{h}\right) \mathrm{d}s + O(th^{k+1}) = y(t) - p(t) + O(th^{k+1}). \end{aligned}$$

This implies

$$\begin{aligned} e'(t_{jl}) - \frac{M}{t_{jl}} e(t_{jl}) &= y'(t_{jl}) - \frac{M}{t_{jl}} y(t_{jl}) - \left(p'(t_{jl}) - \frac{M}{t_{jl}} p(t_{jl})\right) - \frac{M}{t_{jl}} O(h^{k+1}) \\ &= -\frac{M}{t_{jl}} O(h^{k+1}), \quad e(0) = 0. \end{aligned}$$

According to Theorem 25 we have

$$|e(t)| \leq const. \left(|\ln(h)|^{(d-1)_+} h^{k+1}\right),$$

and finally

$$|p(t) - y(t)| \leq const. \left(|\ln(h)|^{(d-1)_+} h^{k+1}\right).$$

This completes the proof. □

9 Numerical experiments

In order to illustrate the theoretical results derived in the previous section, we have constructed model problems and run the collocation code `bvpsuite` on coherently refined meshes to compare the empirically estimated convergence orders of the scheme with the theoretically predicted ones.

9.1 General IVP with smooth solution

We first deal with a linear system of ODEs,

$$y'(t) = \frac{1}{t}\begin{pmatrix} -4 & 2 & -1 \\ -8 & 4 & -2 \\ -12 & 8 & -4 \end{pmatrix} y(t) + \frac{f(t)}{t},\ t \in (0,1], \tag{51}$$

subject to initial conditions

$$B_0 y(0) = \begin{pmatrix} 3 & -2 & 1 \end{pmatrix} y(0) = 1, \quad \begin{pmatrix} -2 & 2 & -1 \\ -2 & 1 & 0 \end{pmatrix} y(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{52}$$

Here,

$$f(t) = \begin{pmatrix} t\exp(t) + 2\exp(t) + \sin(t)\cos(t) + t\cos^2(t) - t\sin^2(t) \\ 2t\exp(t) + 4\exp(t) + 2\sin(t)\cos(t) + 2t\cos^2(t) - 2t\sin^2(t) + 2t^2 \\ 2t\exp(t) + 4\exp(t) + t\cos^2(t) - t\sin^2(t) + 4t^2 \end{pmatrix},$$

where $f(0) = (2\ \ 4\ \ 4)^T$ and the exact solution $y \in C^\infty[0,1]$ of IVP (51), (52) reads:

$$y(t) = \begin{pmatrix} \exp(t) + \sin(t)\cos(t) \\ 2\exp(t) + 2\sin(t)\cos(t) + t^2 \\ 2\exp(t) + \sin(t)\cos(t) + 2t^2 \end{pmatrix}, \quad y(0) = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}.$$

First of all, note that $y(0)$ satisfies (52). The matrix M has a double eigenvalue $\lambda_1 = \lambda_2 = -2$, a single eigenvalue $\lambda_3 = 0$, and the Jordan canonical form is

$$J = \begin{pmatrix} -2 & 1 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Therefore, the second set of two linearly independent initial conditions in (52) are regularity conditions, necessary and sufficient for $y \in C[0,1]$. The remaining free constant has to be calculated from the first initial condition in (52). With the projection matrices

$$N = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 0 & 0 \\ 2 & -1 & 1 \end{pmatrix}, \quad R = \begin{pmatrix} 0 & 0 & 0 \\ -2 & 1 & 0 \\ -2 & 1 & 0 \end{pmatrix},$$

it is easily seen that the condition $MNy(0) = -Nf(0)$ is satisfied. Moreover, the 1×1 matrix

$$B_0 \tilde{R} = (3 \ -2 \ 1) \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = -1$$

is nonsingular, and therefore IVP (51), (52) is well-posed.

In Tables 1 to 4, we illustrate the convergence behaviour for the collocation executed with equidistant and Gaussian collocation points. The number of the collocation points k was chosen to vary from 1 to 8. However, in the simulations shown here, we report only on the values 1 to 4 since the results for 5 to 8 are very similar. The maximal global error is computed either in the mesh points,

$$\|Y_h - Y\|_\infty := \max_{0 \le j \le I} |p(t_j) - y(t_j)|,$$

or 'uniformly' in t,

$$\|Y_h - Y\|_\infty := \max_{0 \le i \le 1.000} |p(\tau_i) - y(\tau_i)|, \quad \tau_i = ih, \ h = 10^{-3}.$$

The estimated order of convergence p and the error constant c are estimated using two consecutive meshes with the step sizes h and $h/2$.

Since $\|Y_h - Y\| \approx ch^p$ for $h \to 0$, we have

$$\|Y_h - Y\|_\infty = ch^p, \quad \|Y_{h/2} - Y\|_\infty = c\left(\frac{h}{2}\right)^p \Rightarrow p = \ln\left(\frac{\|Y_h - Y\|_\infty}{\|Y_{h/2} - Y\|_\infty}\right)\frac{1}{\ln(2)}.$$

Having p, we calculate the error constant from $c = \|Y_{h/2} - Y\|_\infty / \left(\frac{h}{2}\right)^p$.

According to the experiments, the empirical convergence orders very well reflect the theoretical findings. For Gaussian points, we observe the small superconvergence order $k+1$ uniformly in t. The superconvergence order $2k$ in the mesh points does not hold in general, see case $k = 4$. For uniformly spaced equidistant collocation points we again observe the order $k+1$ which for this model is slightly better than we can show theoretically.

9.2 General IVP with 'unsmooth' solution

Next, we discuss an IVP whose solution is less smooth than in the previous model. The problem reads:

$$y'(t) = \frac{M}{t} y(t) + \frac{f(t)}{t}, \ (1 \ 0 \ 0) y(0) = \frac{1}{4}, \ \begin{pmatrix} -1 & 1 & 0 \\ 1 & -1 & 1 \end{pmatrix} y(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad (53)$$

where

$$M = \begin{pmatrix} -4 & 0 & 0 \\ -2 & -2 & 0 \\ 2 & -2 & 0 \end{pmatrix}, \quad f(t) = \begin{pmatrix} \exp(t) \\ \exp(t) + t \\ t + \frac{1}{2} t^{\frac{1}{2}} \sin(t) + t^{\frac{3}{2}} \cos(t) \end{pmatrix}.$$

The eigenvalues of M are $\lambda_1 = -4$, $\lambda_2 = -2$, and $\lambda_3 = 0$ and the initial conditions are designed in such a way that IVP (53) is well-posed with an analytical solution $y \in C^1[0,1]$ given by

$$y(t) = \begin{pmatrix} t^{-4}\big(6 - 6\exp(t) + 6t\exp(t) - 3t^2\exp(t) + t^3\exp(t)\big) \\ t^{-4}\big(6 - 6\exp(t) + 6t\exp(t) - 3t^2\exp(t) + t^3\exp(t)\big) + \frac{t}{3} \\ \frac{t}{3} + \sqrt{t}\sin(t) \end{pmatrix}.$$

The related numerical results are listed for $k = 4$ in Table 5. As expected, we observe an order reduction down to 1.5, not only for $k = 4$, but also for all other values of k.

9.3 General TVP with small positive eigenvalues

The case of the matrix M having eigenvalues with positive real parts has not been investigated yet, since the related theory is particularly tedious and involved, cf. [18]. However, some interesting numerical simulations are already available and therefore, the results of these experiments are shortly discussed here to complete the picture.

First we consider the following model problem

$$y'(t) = \frac{M}{t}y(t) + \frac{f(t)}{t}, \quad B_1 y(1) = \begin{pmatrix} 4 & -1 & 1 \\ 0 & 1 & 0 \\ 3 & -1 & 1 \end{pmatrix} y(1) = \begin{pmatrix} -1 \\ 7 \\ 2 \end{pmatrix}, \tag{54}$$

where

$$M = \begin{pmatrix} 3.5 & -1 & 1 \\ -14 & 5 & -4 \\ -24.5 & 8 & -7 \end{pmatrix}, \quad f(t) = \begin{pmatrix} 1 - 4t^2 \\ 4 + t^2\ln(t) \\ 1 + t^2\ln(t) + 14t^2 \end{pmatrix}.$$

The solution $y \in C[0,1]$ of TVP (54) is

$$y(t) = \begin{pmatrix} 3\sqrt{t} - 2t^2 - 4 \\ 12\sqrt{t} - 8 + 4t + t^2\ln(t) - t^2 \\ 3\sqrt{t} + 4t + 5 + 6t^2 + t^2\ln(t) \end{pmatrix}.$$

In Tables 6 and 7, we again see the order reduction down to 0.5, due to the fact that the first derivative y' is unbounded for $t \to 0$. Moreover, we see that the problem is hard to solve and the convergence is very slow – for $h \approx 2\cdot 10^{-3}$ the level of the global error is only $\|Y_h - Y\|_\infty \approx 10^{-1}$.

The remedy for this lack of smoothness due to the small size of the positive eigenvalues of M is to make a change of the independent variable [6], $t = \tau^\mu$ for some $\mu > 1$. Then $\tilde{y}(\tau) := y(\tau^\mu)$ satisfies the transformed ODE system

$$\tilde{y}'(\tau) = \frac{\widetilde{M}}{\tau}y(\tau) + \frac{\widetilde{f}(\tau)}{\tau}, \quad \tau \in (0,1], \tag{55}$$

where $\widetilde{M} = \mu M$ and $\widetilde{f}(\tau) = \mu f(\tau^{\mu})$. The eigenvalues of the matrix $\widetilde{M}$ become $\widetilde{\lambda} = \mu\lambda$ and therefore, the solution $\tilde{y}$ of the transformed equation is smoother than the solution y of the original one. One can also interpret the above smoothing in terms of the mesh adaptation – solving the ODE system (55) on an equidistant mesh, means solving the original ODE system on a mesh which is adequately refined close to the singularity, where the solution y and its derivatives rapidly change.

Consequently, we solve the TVP

$$\widetilde{y}'(\tau) = \frac{\widetilde{M}}{\tau}y(\tau) + \frac{\widetilde{f(\tau)}}{\tau}, \ \ \tau \in (0,1], \ B_1 y(1) = \begin{pmatrix} 4 & -1 & 1 \\ 0 & 1 & 0 \\ 3 & -1 & 1 \end{pmatrix} \widetilde{y}(1) = \begin{pmatrix} -1 \\ 7 \\ 2 \end{pmatrix}, \tag{56}$$

where for $\mu = 8$,

$$\widetilde{M} = \mu \begin{pmatrix} 3.5 & -1 & 1 \\ -14.0 & 5 & -4 \\ -24.5 & 8 & -7 \end{pmatrix} = \begin{pmatrix} 28 & -8 & 8 \\ -112 & 40 & -32 \\ -196 & 64 & -56 \end{pmatrix},$$

and

$$\widetilde{f(\tau)} = \begin{pmatrix} \mu - 4\mu\tau^{2\mu} \\ 4\mu + \mu\tau^{2\mu}\ln(\tau^{\mu}) \\ \mu + \mu\tau^{2\mu}\ln(\tau^{\mu}) + 14\mu\tau^{2\mu} \end{pmatrix} = \begin{pmatrix} 8 - 32\tau^{16} \\ 32 + 8\tau^{16}\ln(\tau^{8}) \\ 8 + 8\tau^{16}\ln(\tau^{8}) + 112\tau^{16} \end{pmatrix}.$$

The eigenvalues of M are $\lambda_1 = 0.5$, $\lambda_2 = 1$, and $\lambda_3 = 0$, and the eigenvalues of $\widetilde{M}$ become $\widetilde{\lambda}_1 = 4$, $\widetilde{\lambda}_2 = 8$, and $\widetilde{\lambda}_3 = 0$. The solution of (56) reads:

$$y(\tau) = \begin{pmatrix} 3\tau^{\frac{\mu}{2}} - 2\tau^{2\mu} - 4 \\ -8 + 12\tau^{\frac{\mu}{2}} + 4\tau^{\mu} + \tau^{2\mu}\ln\tau^{\mu} - \tau^{2\mu} \\ 5 + 6\tau^{2\mu} + 4\tau^{\mu} + \tau^{2\mu}\ln\tau^{\mu} + 3\tau^{\frac{\mu}{2}} \end{pmatrix} = \begin{pmatrix} 3\tau^{4} - 2\tau^{16} - 4 \\ -8 + 12\tau^{4} + 4\tau^{8} + \tau^{16}\ln\tau^{8} - \tau^{16} \\ 5 + 6\tau^{16} + 4\tau^{8} + \tau^{16}\ln\tau^{8} + 3\tau^{4} \end{pmatrix}.$$

Tables 8 and 9 show the desired effect. For $k = 2$ and equidistant collocation points, we observe the $O(h^k)$ behavior of the global error uniformly in t, as it was the case for a smooth IVP. For the Gaussian points we see the superconvergence $O(h^{2k})$, both, in the mesh points and uniformly in t which is better than expected. However, this very fast convergence for the Gaussian points is put into the right perspective by the data for $k = 3$ in Table 9. Here, only the expected order $O(h^{k+1})$ uniformly in t can be observed.

The above experiments in context of the TVPs suggest the following working hypothesis: The polynomial collocation shows the same convergence behaviour for the well-posed TVPs and IVPs, provided that their solutions are appropriately smooth. This hypothesis may become a subject of further studies.

10 Conclusions

In this paper, we investigated the analytical properties of the singular BVP

$$y'(t) = \frac{M}{t}y(t) + \frac{f(t)}{t},\ t \in (0,1], \quad y \in C[0,1], \quad B_0y(0) + B_1y(1) = \beta.$$

It turns out that the structure of the initial/terminal/boundary conditions to guarantee that the problem is well-posed and has a unique solution which is at least continuous on $[0,1]$ depend on the spectral properties of the matrix M. Also, the smoothness of higher derivatives of y depends on f and the spectrum of M. Interestingly, we can enlarge the positive real parts of the eigenvalues of M by a properly chosen transformation of the independent variable. In such a way only the smoothness of f influences the smoothness of y.

In context of an IVP with appropriately smooth solution, polynomial collocation method executed with k arbitrary collocation points retains the classical stage order $O(h^k)$ uniformly in t. For Gaussian points the small superconvergence order $O(h^{k+1})$ can be shown to hold uniformly in t. In general, the superconvergence order $O(h^{2k})$ in the mesh points cannot be expected.

Competing interests
The authors declare that they have no competing interests.

Author's contributions
All authors contributed to the analytical part of the paper. JV and EBW contributed to its numerical part. All authors read and approved the final version of the manuscript.

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References
1. Agarwal, R, Kiguradze, I: Two-point boundary value problems for higher-order linear differential equations with strong singularities. Boundary Value Problems 2006, Article ID 83910, 1-32.
2. Agarwal, R, O'Regan, D, Staněk, S: Dead Core Problems for Singular Equations with ϕ-Laplacian. Boundary Value Problems 2007, Article ID 18961, 16 pages.
3. Ashordia, M: On Boundary Value Problems for Systems of Linear Generalized Ordinary Differential Equations with Singularities. Differ. Uravn. **42**, 291–301 (2006).
4. Ashordia, M: On the Fredholm property of linear boundary value problems for systems of generalized ordinary differential equations with singularities. Mem. Differ. Equ. Math. Phys., **32**, (2004), 137–141.
5. Auzinger, W, Kneisl, G, Koch, O, Weinmüller, E: A collocation code for boundary value problems in ordinary differential equations. Numer. Algorithms **33**, 27–39 (2003).
6. Cash, J, Kitzhofer, G, Koch, O, Moore, G, Weinmüller E: Numerical Solution of Singular Two Point BVPs. JNAIAM J. Numer. Anal. Indust. and Appl. Math. **4**, 129–149 (2009).
7. Codington, E, Levison, N: Theory of Ordinary Differential Equations, McGraw- Hill, New York, 1955.
8. de Hoog, F, Weiss, R: Collocation Methods for Singular BVPs. SIAM J. Numer. Anal. **15**, 198–217 (1978).
9. de Hoog, F, Weiss, R: Difference methods for boundary value problems with a singularity of the first kind. SIAM J. Numer. Anal. **13**, 775–813 (1976).
10. de Hoog, F, Weiss, R: On the boundary value problem for systems of ordinary differential equations with a singularity of the second kind. SIAM J. Math. Anal. **11**, 41–60 (1980).
11. de Hoog, F, Weiss, R: The application of Runge-Kutta schemes to singular initial value problems. Math. Comp. **44**, 93–103 (1985).
12. Horishna, Y, Parasyuk, I, Protsak, L: Integral Representation of Solutions to Boundary-Value Problems on the Half-Line for Linear ODEs with Singularity of the First Kind. Electronic Journal of Differential Equations, Vol. 2008(2008), No. 137, pp. 1–18.
13. Kiguradze, I: On Boundary Value Problems for Linear Differential Systems with Singularities. Differ. Equ. **39**, 198–209 (2003).

14. Kiguradze, T: Estimates for the Cauchy Functions of Linear Singular Differential Equations and Some of Their Applications. Differ. Equ. **46**, 29–46 (2010).
15. Kiguradze, T: On Conditions for Linear Singular Boundary Value Problems to Be Well Posed. Differ. Equ. **62**, 183–190 (2010).
16. Kiguradze, I, Lomtatidze, A: On Certain Boundary Value Problems for Second-Order Linear Ordinary Differential Equations with Singularities. J. Math. Anal. Appl. **101**, 325-347 (1984).
17. Kitzhofer, G, Koch, O, Pulverer, G, Simon, C, Weinmüller, E: Numerical Treatment of Singular BVPs: The New Matlab Code `bvpsuite`. JNAIAM J. Numer. Anal. Indust. Appl. Math. **5**, 113–134 (2010).
18. Koch, O: Asymptotically Correct Error Estimation for Collocation Methods Applied to Singular Boundary Value Problems. Numer. Math. **101**, 143–164 (2005).
19. Koch, O, Kofler, P, Weinmüller, E: Initial Value Problems for Systems of Ordinary First and Second Order Differential Equations with a Singularity of the First Kind. Analysis **21**, 373–389 (2001).
20. Koch, O, Weinmüller, E: Analytical and Numerical Treatment of a Singular Initial Value Problem in Avalanche Modeling. Appl. Math. Comp. **148**, 561–570 (2004).
21. Malaguti, L, Marcelli, C, Matucci S: Continuous Dependence in Front Propagation of Convective Reaction-Diffusion Equations. Comm. Pure Appl. Anal. **9**, 1083–1098 (2010).
22. Pylypenko, V, Ronto, A: On slowly growing solutions of singular linear functional differential systems. Math. Nachr. **285**, No. 5-6, 727–743 (2012).
23. Staněk, S, Pulverer, G, Weinmüller, E: Analysis and numerical solution of positive and dead core solutions of singular two-point boundary value problems. Comput. Math. Appl. **56**, 1820–1837 (2008).
24. Staněk, S, Pulverer, G, Weinmüller, E: Analysis and numerical solution of positive and dead core solution of singular Sturm-Liouville problems. Adv. Differ. Equ. Volume 2010, Article ID 969536, 37 pages.
25. Vainikko, G: A Smooth Solution to a Linear Systems of Singular ODEs. J. Anal. Appl. (ZAA) **32**, 349–370 (2013).
26. Vainikko, G: A Smooth Solution to a Nonlinear Systems of Singular ODEs. AIP Conf. Proc. **1558**, 758–761 (2013).

Tables

	equidistant points			Gaussian points					
	uniform			mesh			uniform		
h	$\|Y_h - Y\|_\infty$	c	p	$\|Y_h - Y\|_\infty$	c	p	$\|Y_h - Y\|_\infty$	c	p
1/2	1.4e-01	–	–	9.2e-02	–	–	1.4e-01	–	–
1/4	3.5e-02	2.1e+00	1.96	2.3e-02	3.7e-01	2.01	3.5e-02	2.1e+00	1.96
1/8	9.0e-03	2.2e+00	1.98	5.8e-03	3.7e-01	2.00	9.0e-03	2.2e+00	1.98
1/16	2.3e-03	2.2e+00	1.99	1.4e-03	3.7e-01	2.00	2.3e-03	2.2e+00	1.99
1/32	5.7e-04	2.3e+00	2.00	3.6e-04	3.7e-01	2.00	5.7e-04	2.3e+00	2.00
1/64	1.4e-04	2.3e+00	2.00	9.0e-05	3.7e-01	2.00	1.4e-04	2.3e+00	2.00
1/128	3.6e-05	2.3e+00	2.00	2.2e-05	3.7e-01	2.00	3.6e-05	2.3e+00	2.00
1/256	8.9e-06	2.3e+00	2.00	5.6e-06	3.7e-01	2.00	8.9e-06	2.3e+00	2.00
1/512	2.2e-06	2.3e+00	2.00	1.4e-06	3.7e-01	2.00	2.2e-06	2.3e+00	2.00

Table 1 IVP (51), (52): Convergence of the collocation scheme, $k = 1$

	equidistant points			Gaussian points					
	uniform			mesh			uniform		
h	$\|Y_h - Y\|_\infty$	c	p	$\|Y_h - Y\|_\infty$	c	p	$\|Y_h - Y\|_\infty$	c	p
1/2	3.7e-03	–	–	3.7e-03	–	–	7.8e-03	–	–
1/4	5.1e-04	6.4e-01	2.87	1.6e-04	8.4e-02	4.50	1.2e-03	3.6e+00	2.73
1/8	6.6e-05	8.1e-01	2.96	8.1e-06	6.7e-02	4.34	1.5e-04	6.4e+00	2.93
1/16	8.3e-06	8.8e-01	2.99	4.4e-07	5.1e-02	4.21	2.0e-05	7.6e+00	2.98
1/32	1.0e-06	9.0e-01	3.00	2.5e-08	4.0e-02	4.12	2.5e-06	8.1e+00	2.99
1/64	1.3e-07	9.1e-01	3.00	1.5e-09	3.3e-02	4.06	3.1e-07	8.4e+00	3.00
1/128	1.6e-08	9.2e-01	3.00	9.3e-11	2.9e-02	4.03	3.8e-08	8.5e+00	3.00
1/256	2.0e-09	9.2e-01	3.00	5.7e-12	2.7e-02	4.02	4.8e-09	8.5e+00	3.00
1/512	2.6e-10	9.2e-01	3.00	3.5e-13	2.8e-02	4.03	6.0e-10	8.5e+00	3.00

Table 2 IVP (51), (52): Convergence of the collocation scheme, $k = 2$

h	equidistant points uniform			Gaussian points mesh			Gaussian points uniform		
	$\Vert Y_h - Y\Vert_\infty$	c	p	$\Vert Y_h - Y\Vert_\infty$	c	p	$\Vert Y_h - Y\Vert_\infty$	c	p
1/2	2.0e-04	–	–	3.6e-05	–	–	3.5e-04	–	–
1/4	7.4e-06	3.8e+00	4.75	5.1e-07	2.6e-03	6.15	1.3e-05	2.8e+02	4.73
1/8	3.0e-07	2.7e+00	4.63	7.6e-09	2.3e-03	6.07	5.4e-07	1.8e+02	4.61
1/16	1.4e-08	1.5e+00	4.46	1.2e-10	2.1e-03	6.03	2.5e-08	9.1e+01	4.44
1/32	6.9e-10	8.0e-01	4.30	1.8e-12	2.1e-03	6.02	1.3e-09	4.2e+01	4.29
1/64	3.8e-11	4.4e-01	4.18	2.0e-14	8.9e-03	6.44	7.1e-11	2.1e+01	4.17
1/128	2.2e-12	2.8e-01	4.10	3.4e-14	1.0e-15	-0.72	4.2e-12	1.3e+01	4.09
1/256	1.3e-13	2.0e-01	4.04	1.1e-14	1.1e-10	1.66	2.5e-13	8.5e+00	4.03
1/512	7.1e-15	8.0e-01	4.24	2.0e-14	5.9e-17	-0.94	1.6e-14	8.7e+00	4.03

Table 3 IVP (51), (52): Convergence of the collocation scheme, $k = 3$

h	equidistant points uniform			Gaussian points mesh			Gaussian points uniform		
	$\Vert Y_h - Y\Vert_\infty$	c	p	$\Vert Y_h - Y\Vert_\infty$	c	p	$\Vert Y_h - Y\Vert_\infty$	c	p
1/2	2.7e-05	–	–	6.9e-07	–	–	2.6e-05	–	–
1/4	9.1e-07	2.2e+00	4.91	5.8e-09	8.2e-05	6.90	8.6e-07	3.7e+02	4.90
1/8	2.9e-08	2.7e+00	4.98	4.7e-11	8.7e-05	6.95	2.7e-08	5.0e+02	4.98
1/16	9.0e-10	2.9e+00	5.00	3.7e-13	9.3e-05	6.97	8.6e-10	5.4e+02	5.00
1/32	2.8e-11	2.9e+00	5.00	6.2e-15	4.7e-06	5.90	2.7e-11	5.6e+02	5.00
1/64	8.8e-13	3.1e+00	5.01	1.5e-14	7.4e-17	-1.28	8.4e-13	5.8e+02	5.01
1/128	3.1e-14	1.2e+00	4.84	8.9e-15	3.6e-13	0.77	2.4e-14	1.1e+03	5.10
1/256	2.7e-15	2.4e-04	3.52	7.1e-15	4.2e-14	0.32	3.1e-15	1.3e-04	2.97
1/512	1.3e-15	3.4e-12	1.00	1.3e-14	4.7e-17	-0.91	3.6e-15	6.4e-16	-0.19

Table 4 IVP (51), (52): Convergence of the collocation scheme, $k = 4$

h	equidistant points uniform			Gaussian points mesh			Gaussian points uniform		
	$\Vert Y_h - Y\Vert_\infty$	c	p	$\Vert Y_h - Y\Vert_\infty$	c	p	$\Vert Y_h - Y\Vert_\infty$	c	p
1/2	3.2e-03	–	–	6.2e-04	–	–	1.2e-03	–	–
1/4	1.1e-03	1.0e-01	1.49	2.2e-04	1.7e-03	1.50	4.4e-04	1.9e-01	1.50
1/8	4.1e-04	1.0e-01	1.50	7.7e-05	1.7e-03	1.50	1.6e-04	1.9e-01	1.50
1/16	1.4e-04	1.0e-01	1.50	2.7e-05	1.7e-03	1.50	5.5e-05	1.9e-01	1.50
1/32	5.1e-05	1.0e-01	1.50	9.6e-06	1.7e-03	1.50	1.9e-05	1.9e-01	1.50
1/64	1.8e-05	1.0e-01	1.50	3.4e-06	1.7e-03	1.50	2.4e-04	3.6e-15	-3.65
1/128	8.0e-05	7.4e-11	-2.15	1.2e-06	1.7e-03	1.50	4.9e-03	3.4e-17	-4.34
1/356	7.9e-04	4.0e-14	-3.31	4.3e-07	1.7e-03	1.50	1.0e-01	2.1e-17	-4.40
1/512	2.5e-02	2.5e-19	-4.99	1.5e-07	1.7e-03	1.50	4.8e-01	1.5e-09	-2.20

Table 5 IVP (53): Convergence of the collocation scheme, $k = 4$

	equidistant points			Gaussian points					
	uniform			mesh			uniform		
h	$\Vert Y_h - Y\Vert_\infty$	c	p	$\Vert Y_h - Y\Vert_\infty$	c	p	$\Vert Y_h - Y\Vert_\infty$	c	p
1/2	2.2e+00	–	–	1.7e+00	–	–	1.7e+00	–	–
1/4	1.5e+00	5.4e+00	0.51	1.2e+00	2.4e+00	0.51	1.2e+00	5.4e+00	0.51
1/8	1.1e+00	5.3e+00	0.50	8.5e-01	2.4e+00	0.50	8.5e-01	5.3e+00	0.50
1/16	7.5e-01	5.3e+00	0.50	6.0e-01	2.4e+00	0.50	6.0e-01	5.2e+00	0.50
1/32	5.3e-01	5.2e+00	0.50	4.2e-01	2.4e+00	0.50	4.2e-01	5.2e+00	0.50
1/64	3.8e-01	5.2e+00	0.50	3.0e-01	2.4e+00	0.50	3.0e-01	5.2e+00	0.50
1/128	2.7e-01	5.2e+00	0.50	2.1e-01	2.4e+00	0.50	2.1e-01	5.2e+00	0.50
1/256	1.9e-01	5.2e+00	0.50	1.5e-01	2.4e+00	0.50	1.5e-01	5.2e+00	0.50
1/512	1.3e-01	5.2e+00	0.50	1.1e-01	2.4e+00	0.50	1.1e-01	5.2e+00	0.50

Table 6 TVP (54): Convergence of the collocation scheme, $k = 2$

	equidistant points			Gaussian points					
	uniform			mesh			uniform		
h	$\Vert Y_h - Y\Vert_\infty$	c	p	$\Vert Y_h - Y\Vert_\infty$	c	p	$\Vert Y_h - Y\Vert_\infty$	c	p
1/2	1.7e+00	–	–	1.2e+00	–	–	1.2e+00	–	–
1/4	1.2e+00	4.7e+00	0.50	8.6e-01	1.7e+00	0.50	8.6e-01	5.2e+00	0.50
1/8	8.3e-01	4.7e+00	0.50	6.1e-01	1.7e+00	0.50	6.1e-01	5.1e+00	0.50
1/16	5.8e-01	4.7e+00	0.50	4.3e-01	1.7e+00	0.50	4.3e-01	5.1e+00	0.50
1/32	4.1e-01	4.7e+00	0.50	3.0e-01	1.7e+00	0.50	3.0e-01	5.1e+00	0.50
1/64	2.9e-01	4.7e+00	0.50	2.1e-01	1.7e+00	0.50	2.1e-01	5.1e+00	0.50
1/128	2.1e-01	4.7e+00	0.50	1.5e-01	1.7e+00	0.50	1.5e-01	5.1e+00	0.50
1/256	1.5e-01	4.7e+00	0.50	1.1e-01	1.7e+00	0.50	1.1e-01	5.1e+00	0.50
1/512	1.0e-01	4.7e+00	0.50	7.6e-02	1.7e+00	0.50	7.6e-02	5.1e+00	0.50

Table 7 TVP (54): Convergence of the collocation scheme, $k = 3$

	equidistant points			Gaussian points					
	uniform			mesh			uniform		
h	$\Vert Y_h - Y\Vert_\infty$	c	p	$\Vert Y_h - Y\Vert_\infty$	c	p	$\Vert Y_h - Y\Vert_\infty$	c	p
1/2	5.4e+00	–	–	2.1e+00	–	–	1.9e+00	–	–
1/4	2.3e+00	5.2e+01	1.26	2.3e-01	1.8e+01	3.17	2.2e-01	2.2e+03	3.14
1/8	6.8e-01	1.7e+02	1.73	1.7e-02	4.2e+01	3.76	1.6e-02	1.3e+04	3.74
1/16	1.8e-01	3.1e+02	1.93	1.1e-03	6.1e+01	3.94	1.1e-03	2.6e+04	3.93
1/32	4.5e-02	3.8e+02	1.98	7.0e-05	6.9e+01	3.98	6.8e-05	3.3e+04	3.98
1/64	1.1e-02	4.1e+02	2.00	4.4e-06	7.2e+01	4.00	4.3e-06	3.5e+04	4.00
1/128	2.8e-03	4.2e+02	2.00	2.7e-07	7.3e+01	4.00	2.7e-07	3.6e+04	4.00
1/256	7.1e-04	4.2e+02	2.00	1.7e-08	7.3e+01	4.00	1.7e-08	3.6e+04	4.00
1/512	1.8e-04	4.2e+02	2.00	1.1e-09	7.4e+01	4.00	1.0e-09	3.6e+04	4.00

Table 8 TVP (56): Convergence of the collocation scheme, $k = 2$

h	equidistant points			Gaussian points					
	uniform			mesh			uniform		
	$\|Y_h - Y\|_\infty$	c	p	$\|Y_h - Y\|_\infty$	c	p	$\|Y_h - Y\|_\infty$	c	p
1/2	2.2e+00	–	–	1.5e-01	–	–	1.5e-01	–	–
1/4	2.8e-01	1.1e+03	2.98	3.7e-03	5.7e+00	5.29	4.1e-03	4.5e+05	5.18
1/8	2.1e-02	8.2e+03	3.71	8.3e-05	7.5e+00	5.49	8.3e-05	2.1e+06	5.62
1/16	1.4e-03	1.7e+04	3.93	5.2e-06	3.4e-01	4.00	5.2e-06	2.1e+03	4.00
1/32	8.8e-05	2.2e+04	3.98	3.3e-07	3.4e-01	4.00	3.3e-07	2.1e+03	4.00
1/64	5.5e-06	2.3e+04	4.00	2.0e-08	3.4e-01	4.00	2.0e-08	2.1e+03	4.00
1/128	3.5e-07	2.4e+04	4.00	1.3e-09	3.4e-01	4.00	1.3e-09	2.1e+03	4.00
1/256	2.2e-08	2.4e+04	4.00	8.0e-11	3.3e-01	3.99	8.0e-11	2.0e+03	3.99
1/512	1.4e-09	2.4e+04	4.00	7.0e-12	2.2e-02	3.50	7.0e-12	4.6e+01	3.50

Table 9 TVP (56): Convergence of the collocation scheme, $k = 3$