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LOCALLY COMPACT NEAR ABELIAN GROUPS

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Abstract. A locally compact group G is near abelian if it contains a closed abelian normal subgroup A such that every closed topologically finitely generated subgroup of G/A is inductively monothetic and every closed subgroup of A is normal in G. Recent studies prove that projective limits of finite p-groups (p prime) are exactly those compact p-groups in which two closed subgroups commute (that is, are "quasihamiltonian" or "M-groups"). Such results are extended to locally compact near abelian groups and their structure is studied. This requires a review of the structure of locally compact periodic groups and thus of their automorphisms which leave all closed subgroups invariant. The new definition of locally compact near abelian groups necessitates to introuduce inductively monothetic locally compact groups. Their structure is completely classified. The global structure of locally compact near abelian groups is discussed. At the end of the paper we classify locally compact quasihamiltonian groups.

1. Introduction

All groups in this paper are locally compact and all homomorphisms continuous unless stated differently. The recent sources [7, 11] present studies of metabelian compact groups G which are rich in commuting pairs of closed subgroups. On the one hand, F. Kümmich defined in [14] a topological group C to be *topologically quasihamiltonian* (or briefly, *quasihamiltonian*), if $\overline{HK}$ is a closed subgroup of C for all closed subgroups H and K of C. On the other hand, we quote the following definition from [11]:

A compact group G is called *near abelian*, if it contains a closed normal abelian subgroup A such that G/A is monothetic and that all closed subgroups of A are normal in G.

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For a prime number p, a compact abelian group G is called a *compact p-group* if and only if its character group $\widehat{G}$ is a (discrete) p-group if and only if G is a pro-p group if and only if G is a topological $\mathbb{Z}_p$-module for the ring $\mathbb{Z}_p$ of p-adic integers. The structure of near abelian compact p-groups is explicitly described in [11, Theorems 4.21, 5.11, 6.7, 7.1, 7.2]. Complications arise in the case $p = 2$. One of the principal results for $p > 2$ is this:

A compact p-group is quasihamiltonian if and only if it is near abelian.

One of the tools of independent interest is the surprisingly nontrivial fact that

the class of compact near abelian p-groups is closed under the formation of projective limits.

The theory for near abelian compact p-groups developed in [11] summarizes results from [4, 5, 21, 22, 23], and it exhibits certain formal similarties to the structure theory of connected real Lie groups that have been called *diagonally metabelian Lie groups* (see [9]).

In the present article we generalize these results on compact near abelian p-groups in two ways: Staying all the while within the category of locally compact topological groups, we free ourselves from the restriction to compactness, and from the restriction to p-groups. For a preliminary indication of our main results, we need to agree on certain elements of notation:

We say a topological group G is *finitely generated* provided that there is a finite subset X of G and $G = \overline{\langle X \rangle}$. When $\mathcal{P}$ is a property of a topological group G (such as being *monothetic*, that is, having a dense cyclic subgroup), then we call G *inductively* $\mathcal{P}$, if every finitely generated subgroup of G is $\mathcal{P}$. Recall from [10, 7.44] that an element $g \in G$ is *compact* if $\overline{\langle g \rangle}$ is compact. The set of all compact elements is denoted by $\mathrm{comp}(G)$. During this article a group is *periodic* if $G = \mathrm{comp}(G)$ and totally disconnected. An element $x \in G$ is a *p-element* provided $\overline{\langle x \rangle}$ is a compact p-group. By $\pi(G)$ we denote the set of all primes p such that G has a nontrivial p-element.

We shall call a locally compact group G *near abelian* if it contains a closed normal abelian subgroup A such that every closed subgroup of A is normal in G and G/A is inductively monothetic, and we call such a subgroup like A a *base subgroup* of G. Such a situation clearly defines a representation $\rho\colon G/A \to \mathrm{Aut}(A)$ implemented by the inner automorphisms of G acting on A.

Here, $\mathrm{Aut}(A)$ denotes the group of continuous and continuously invertible automorphisms of a locally compact group A. In most situations, we consider it endowed with the compact open topology. However, if $\mathrm{Aut}(A)$ fails to be a topological group, then we consider it to be given a standard refinement, of the compact-open topology, often called *g-topology* (cf. [15], see also [1], §3, n^{o} 5).

Visualizing an abelian group A as additively written we call the automorphisms leaving all closed subgroups of A invariant the *scalar automorphisms* and denote the subgroup of scalar automorphisms $\mathrm{SAut}(A)$. We note that $\mathrm{SAut}(A)$ contains the two element subgroup $\{\mathrm{id}_A, -\mathrm{id}_A\}$ whose elements we call the *trivial* scalar automorphisms. A near abelian group G itself with be called *trivial* if $\rho(G/A) \subseteq \{\mathrm{id}_A, -\mathrm{id}_A\}$ for some base group A.

In an abelian group G the compact elements form a closed subgroup $\mathrm{comp}(G)$ of G (cf. [10], Definition 7.44). The significance of an abelian group being *periodic* for our discourse derives from Theorem 2.10 and Theorem 4.20 saying that the only case that A can fail to be periodic occurs when G is a trivial near abelian group. Theorem 4.20 provides insight into the theory of nonabelian trivial near abelian groups.

For understanding the base group A of a abelian group G, we therefore need a good understanding of periodic locally compact groups and their scalar automorphism groups. We survey these in Section 2, beginning with a quick review of Braconnier's concept of *local products* of families of locally compact groups with compact open subgroups applied to the primary decomposition of a locally compact abelian periodic group B (see 2.1), proceeding to the structure theory of its scalar automorphism group $\mathrm{SAut}(B)$ (see 2.3, 2.5, 2.6, and, notably, 2.9 and Theorems 2.10 and 2.11 (Mukhin)).

The second part of a near abelian group G is the factor group G/A modulo a suitable base group A. By definition, G/A is an inductively monothetic group, or IMG. This requires a detailed discussion of IMG-s which we provide in Section 3, all the way to a complete classification of IMG-s in Theorem 3.7 and its corollaries. For the structure theory of near abelian groups it is of course significant in which way an IMG can act by scalar automorphisms on a given periodic group. This is discussed in what we call the "Gamma Theorem" 3.11 in desirable detail. Since these theorems tell us that many IMG-s consist of discrete subgroups of $\mathbb{Q}$ or $\mathbb{Q}/\mathbb{Z}$, or indeed of the groups $\mathbb{Z}_p$ of p-adic integers as in the predecessor study [11], it is of interest to us to understand how the concept of divisibility is compatible with the structure of $\mathbb{Z}_p$, and we survey that question at the end of Section 3 in Proposition 3.12.

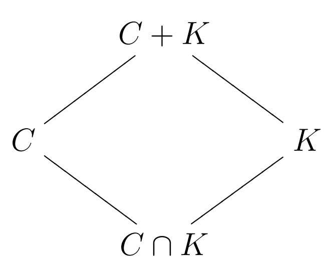
The information collected up to that point is then applied in Section 4 to present first global results on the structure of near abelian groups. After providing some typical examples we discuss the general structure of near abelian groups by addressing both the "nontrivial," and the "trivial" case by presenting, firstly, the "First Structure Theorem of Nontrivial Near Abelian Groups" 4.17 and, secondly, the "Structure Theorem of Trivial Near Abelian Groups" 4.20 which is the only place where nontrivial connected components can appear in the entire theory of near abelian groups.

2. Preliminaries

2.1. **Periodic abelian groups.** In order to understand where *periodic* abelian groups are positioned in the universe of *all* abelian groups we recommend that the reader might consult Theorems 7.56 and 7.67 in [10], p. 354, respectively, p.365. Indeed *every* abelian group G has a *periodic component* $\frac{\mathrm{comp}(G)}{\mathrm{comp}_0(G)}$ where $\mathrm{comp}(G)$ is the union of all compact subgroups and $\mathrm{comp}_0(G)$ is the unique largest compact connected subgroup; both of these are fully characteristic subgroups of G.

Turning now to periodic an abelian group G, we let G_p denote the union of all compact p-subgroups. Then G_p is a closed and fully characteristic subgroup of periodic group called the *p-primary component*.

If C and K are compact open subgroups of an abelian group, then so are $C + K$ and $C \cap K$, and $(G + K)/K \cong C/(C \cap K)$ is finite as are all other factor groups in the following diagram. In this sense, all compact open subgroups of a periodic group are close to each other; we keep this in mind:



Let us rephrase a well-known result due to J. Braconnier in [2]:

Theorem 2.1. (J. Braconnier) *Let G be a periodic abelian group and C any compact open subgroup of G. Then G is isomorphic to the local product*

$$\prod\nolimits_p^{\rm loc}(G_p, C_p).$$

Proof. Since G is topologically isomorphic to the dual group of $\widehat{G}$ and since G is periodic [8, (24.18) Corollary] implies that $\widehat{G}$ is zero dimensional. Therefore G and $\widehat{G}$ are both zero dimensional. Then [2, Théoréme 1, page 71] immediately yields the desired result. □

Remark 2.2. Theorem 2.1 implies that there is an exact sequence of abelian groups

$$1 \to C \to G \to D \to 1$$

where $C = \prod_p C_p$ is profinite and $D = \bigoplus_p D_p$ is a discrete torsion group with $D_p = G_p/C_p$. The p-primary subgroup of G gives rise to an exact sequence

$$1 \to C_p \to G_p \to D_p \to 1$$

If G is a periodic abelian group, then every automorphism (and indeed every endomorphism) α leaves G_p invariant. We write $\alpha_p = \alpha|G_p : G_p \to G_p$. If C is a compact open subgroup, let $\mathrm{Aut}(G, C)$ denote the subgroup of $\mathrm{Aut}(G)$ of all automorphisms leaving C invariant.

In view of Theorem 2.1 we may identify G with its canonical local product decomposition of the pair (G, C).

Corollary 2.3. *For a periodic abelian group G, the function*

$$\alpha \mapsto (\alpha_p)_p : \mathrm{Aut}(G, C) \to \prod_p \mathrm{Aut}(G_p, C_p)$$

is an isomorphism of groups, and $\alpha((g_p)_p) = (\alpha_p(g_p))_p$.

Proof. It is straightforward to verify that the function is an injective morphism of groups. If $(\phi_p)_p$ is any element of $\prod_p \mathrm{Aut}(G_p)$, then the morphism $\phi\colon \prod_p G_p \to \prod_p G_p$ defined by $\phi((g_p)_p) = (\phi_p(g_p))_p$ leaves $C = \prod_p C_p$ fixed as a whole and does the same with $\prod_p^{\rm loc}(G_p, C_p)$. Thus the function $\alpha \mapsto (\alpha_p)_p$ is surjective as well. □

2.2. **On scalar automorphisms.** We record some essential definitions which we have to keep in mind throughout.

Definition 2.4. An automorphism α of an abelian group G is called *scalar* if it leaves all closed subgroups invariant. The subgroup of scalar automorphisms of G is denoted $\mathrm{SAut}(G)$. The subgroup $\{\mathrm{id}_G, -\mathrm{id}_G\} \subseteq \mathrm{SAut}(G)$ is said to consist of *trivial scalar automorphisms*. All other scalar automorphisms are called *nontrivial*.

Every abelian group has these trivial automorphisms, but it happens that $-\mathrm{id}_G = \mathrm{id}_G$, namely, if and only if every element of G has order 2.

By duality, the isomorphism $\alpha \mapsto \widehat{\alpha} : \mathrm{Aut}(G) \to \mathrm{Aut}(\widehat{G})$ maps $\mathrm{SAut}(G)$ bijectively onto $\mathrm{SAut}(\widehat{G})$. The set $\mathrm{SAut}(G)$ is a closed subgroup of $\mathrm{Aut}(G)$ where the latter is equipped with the coarsest topology containing the compact open topology and turning $\mathrm{Aut}(G)$ into a topological group (the "g-topology" in [15]). If C is any compact open subgroup of G, then $\mathrm{SAut}(G) \subseteq \mathrm{Aut}(G, C)$.

Corollary 2.5. *For a periodic abelian group G and any compact open subgroup C, with the identification $G = \prod_p^{\mathrm{loc}}(G_p, C_p)$, the function*

$$\alpha \mapsto (\alpha_p)_p : \mathrm{SAut}(G) \to \prod_p \mathrm{SAut}(G_p)$$

is an isomorphism of groups.

Proof. Since every closed subgroup H of G satisfies $H_p = H \cap G_p$ and $H = \prod_p^{\mathrm{loc}}(H_p, H_p \cap C)$, the assertion is a consequence of Corollary 2.3. □

Thus we know $\mathrm{SAut}(G)$ if we know $\mathrm{SAut}(G_p)$ for each prime p. A abelian p-group G is a $\mathbb{Z}_p$-module. For any invertible scalar $r \in \mathbb{Z}_p^\times$ the function $\mu_r \colon G \to G$, given by $\mu_r(g) = r{\cdot}g$ is a scalar automorphism.

Lemma 2.6. *For an abelian p-group G the morphism*

$$r \mapsto \mu_r : \mathbb{Z}_p^\times \to \mathrm{SAut}(G)$$

is surjective for all *prime numbers p including $p = 2$.*

Proof. Let $\alpha \in \mathrm{SAut}(G)$. We must find an $r \in \mathbb{Z}_p^\times$ such that $\alpha = \mu_r$. If G is compact, this follows from Theorem 2.27 of [11]. Assume now that G is not compact. Then it is the union of its compact subgroups. If C is a nonsingleton compact subgroup then by Theorem 2.27 of [11] there is a unique scalar $r_C \in \mathbb{Z}_p^\times$ such that $\alpha|C = \mu_{r_C}|C$. If $K \supseteq C$ is a compact subgroup containing C, then $\mu_{r_K}|C = \mu_{r_C}|C$. Since $\mathbb{Z}_p^\times$ is compact, the net $(r_C)_C$, as C ranges through the directed set of compact subgroups of G, has a cluster point $r \in \mathbb{Z}_p^\times$; that is, there is a cofinal subset $C(j)$, $j \in J$ for some directed set J such that $r = \lim_j r_{C(j)}$. Since for every $g \in G$ there is a C with $g \in C$ and then a $j_0 \in J$ such that $j \geq j_0$ implies $C \subseteq C(j)$ we conclude that $\alpha(g) = r_C{\cdot}g = r_{C(j)}{\cdot}g$ for these j, and so $\alpha(g) = r{\cdot}g$. Thus $\alpha = \mu_r$ and the proof is complete. □

We extend [11, Definition 2.22] to not necessarily compact abelian p-groups.

Definition 2.7. For an abelian p-group G

$$R(G) := \begin{cases} \mathbb{Z}(e_p) & \text{if } G \text{ has finite exponent } e_p, \\ \mathbb{Z}_p & \text{otherwise} \end{cases}$$

is the *ring of scalars* and $R(G)^\times$ its multiplicative group of units.

There is an isomorphism $\sigma : \mathrm{SAut}(G) \to R(G)^\times$.

Remark 2.8. We use the opportunity to correct a typographical error in the the proof of [11], Theorem 2.27. The formula for $\mathrm{SAut}(A)$ (called $\mathrm{Aut}_{\mathrm{scal}}(A)$ there) should read $\mathrm{Aut}_{\mathrm{scal}}(A)$=

$$\begin{cases} \mathbb{Z}_p^\times \cong \mathbb{P}_p \times C_{p-1} & \text{if } \sigma_A \text{ is faithful, } p > 2, \\ \mathbb{Z}_2^\times \cong \mathbb{P}_4 \times C_2 & \text{if } \sigma_A \text{ is faithful, } p = 2, \\ \mathbb{Z}(p^n)^\times \cong (\mathbb{P}_p/\mathbb{P}_p^{[n]}) \times C_{p-1} & \text{if } \sigma_A \text{ is not faithful, } p{>}2,\ n{>}1, \\ \mathbb{Z}(2^n)^\times \cong (\mathbb{P}_4/\mathbb{P}_4^{[n]}) \times C_2 & \text{if } \sigma_A \text{ is not faithful, } p{=}2,\ n{>}2. \end{cases}$$

This correction does not affect the validity of the arguments in [11].

The universal zero dimensional compactification of the integers is denoted $\widetilde{\mathbb{Z}}$; we know that $\widetilde{\mathbb{Z}} \cong \prod_p \mathbb{Z}_p$, where p ranges through the set of all prime numbers. When, here and in the following, in formulae contexts like $\prod_p \mathbb{Z}_p$, $\bigoplus_p \mathbb{Z}(p^\infty)$, $(x_p)_p \in \prod_p \mathbb{Z}_p$, it will be understood that p ranges through the set of all primes whenever the context is clear.

Now every periodic p-group G is a $\widetilde{\mathbb{Z}}$-module in a natural fashion. Indeed, $\widetilde{\mathbb{Z}} = \prod_p \mathbb{Z}_p$ and $G = \prod_p^{\mathrm{loc}}(G_p, C_p)$ in a natural fashion so that for $z = (z_p)_p \in \widetilde{\mathbb{Z}}$ and $g = (g_p)_p$ we have $z{\cdot}g = (z_p{\cdot}g_p)_p$. Thus there is a morphism ζ of topological groups

$$z \mapsto (\mu(z_p))_p : \widetilde{\mathbb{Z}}^\times \to \prod_p \mathrm{SAut}(G_p) = \mathrm{SAut}(G).$$

In the next proposition we show that for an abelian periodic group G the group $\mathrm{SAut}(G)$ is a quotient group of $\widetilde{\mathbb{Z}}$, and we investigate the case when all scalar automorphisms are trivial: Information about this situation will be important.

Proposition 2.9. *Let G be a periodic abelian group. Then we have the following conclusions:*

(i) *The natural map $\zeta\colon \widetilde{\mathbb{Z}}^\times \to \mathrm{SAut}(G)$ is surjective. In particular, $\mathrm{SAut}(G)$ is a profinite group and a homomorphic image of $\widetilde{\mathbb{Z}}^\times$.*

(ii) *The subsequent two statements are equivalent:*

(a) $\mathrm{SAut}(G) = \{\mathrm{id}_G, -\mathrm{id}_G\}$.

(b) *The exponent of* G *is* 2, 3, *or* 4.

More precisely: The exponent of G *is* 2 *if and only if* $-\mathrm{id}_G = \mathrm{id}_G$.

Proof. (i) follows from the preceding Corollary 2.5 and Lemma 2.6.

(ii) If the exponent of G divides 4 or is 3 then, any $\alpha \in \mathrm{SAut}(G)$ satisfies $(\forall x \in G)\ \alpha(x) = x$ or $(\forall x \in G)\ \alpha(x) = -x$. So such an α is "trivial" in the sense of Definition 2.4.

On the other hand, if the exponent is not in $\{2, 3, 4\}$, then at least one of the following statements holds:

(α) There is a $p > 3$ such that $G_p \neq \{0\}$, or

(β) G_3 has exponent at least 9, or

(γ) G_2 has exponent at least 8.

In Case (α), C_{p-1} has order $p - 1 \geq 4$ and acts nontrivially on G_p as group of scalar automorphisms of G.

In Case (β), the map $x \mapsto 4x$ is a nontrivial scalar automorphism of G_3, and in Case (c), the function $x \mapsto 5x$ is a nontrivial scalar automorphisms of G_2.

Thus, since a nontrivial automorphism of any primary component G_q gives a nontrivial automorphism of G, if the exponent of G is not in $\{2, 3, 4\}$, then there are nontrivial scalar automorphisms. □

We note that for $p = 2$ the group $\mathrm{SAut}(G_2)$ is not monothetic if the exponent of G is at least 8. When $p = 2$ and the exponent is at least 8 then $\mathrm{SAut}(G_2)$ (and so also $\mathrm{SAut}(G)$) is not monothetic as follows from the arguments in [11], Theorem 2.7.

A proof of Part (i) of the preceding proposition can also be obtained as a corollary from a more general result by R. Winkler [24]:

Every (not necessarily continuous) endomorphism of an abelian group which leaves invariant every closed subgroup must act by scalars.

In the same line of structural results we point out a note by Moskalenko [17], which has an echo in the following sharper result.

Theorem 2.10. *For an abelian group* G*, we consider the following statements:*

(1) G *has nontrivial scalar automorphisms.*

(2) G *is periodic.*

Then (1) *implies* (2)*, and if* G *does not have exponent* 2, 3, *or* 4*, then both statements are equivalent.*

Proof. We just remarked in Proposition 2.9 that (2) implies (1) under the hypotheses stated there.

Now assume (1). If G has a discrete subgroup $\cong \mathbb{Z}$, then every scalar automorphism must be trivial. Hence Weil's Lemma (see e.g. [10], Proposition 7.34) implies that $G = \mathrm{comp}(G)$ and we have to argue that $G_0 = \{0\}$. But $(G_0)^\perp$ in $\widehat{G}$ is $\mathrm{comp}(\widehat{G})$ (see [10], Theorem 7.67). Since $\widehat{G}$ has no nontrivial scalar automorphisms by (1), we know $\widehat{G} = \mathrm{comp}(\widehat{G}) = (G_0)^\perp$. This means $G_0 = \{0\}$, as we had to show. $\square$

Theorem 2.11 below was proved by Yu. N. Mukhin in [19], but may now also be obtained quickly by combining our results above.

Theorem 2.11 (Yu. N. Mukhin, Theorem 2 in [19])**.** *Let G be an abelian group. If G is not periodic then* $\mathrm{SAut}(G) = \{\mathrm{id}, -\mathrm{id}\}$. *If, on the other hand, G is periodic, then* $\mathrm{SAut}(G) = \prod_p \mathrm{SAut}(G_p)$ *and* $\mathrm{SAut}(G_p)$ *identifies with the group of units of either the ring $\mathbb{Z}_p$ of p-adic integers or of the finite ring $\mathbb{Z}_p/p^m\mathbb{Z}_p \cong \mathbb{Z}(p^m)$ in case the exponent of G_p is p^m. In particular,* $\mathrm{SAut}(G)$ *is a homomorphic image of $\widetilde{\mathbb{Z}}^\times$.*

As a consequence, $\mathrm{SAut}(G)$ is a profinite central subgroup of $\mathrm{Aut}(G)$.

Remark 2.12. In order to avoid confusion let us note that the "usual scalar multiplication" in $\mathbb{R}$ does not yield a "scalar automorphism" of $\mathbb{R}$ in the present sense: if $0 \neq r$ is any real number, then multiplication by r is a scalar automorphism in our present sense if and only if $r = \pm 1$.

Remark 2.13. Every abelian group G has arbitrarily small compact subgroups N such that

(i) G/N is a Lie group.
(ii) N is $\mathrm{SAut}(G)$-invariant.

3. Inductively monothetic groups

We introduce and study a generalization of quasicyclic groups. Recall that if every finitely generated subgroup of a group G satisfies $\mathcal{P}$ then G is *inductively* $\mathcal{P}$.

Notation 3.1. When G is *inductively monothetic*, we say for short that G is an IMG.

Obviously, a closed subgroup of an IMG is an IMG.

For purely technical reasons we say that a topological group is

of class (I)	if it is discrete,
of class (II)	if it is infinite compact, and
of class (III)	if it is locally compact but is neither of class (I) nor of class (II).

Lemma 3.2. *If G is an* IMG *of class* (I)*, then it is isomorphic to a subgroup of $\mathbb{Q}$ or else of $\mathbb{Q}/\mathbb{Z}$, that is, if and only if it is either torsion-free of rank 1 or a torsion group of p-rank 1 for all primes.*

Proof. Exercise in view of the fact that a finitely generated abelian group is a direct sum of cyclic ones which then must be cyclic. □

If G is class (I) and is the directed union of subgroups which are IMG-s, then G is an IMG.

Lemma 3.3. *If G is an* IMG *of class* (II)*, then it is monothetic and is is either connected of dimension 1, or it is infinite profinite.*

Proof. By Proposition 2.42 of [10], a compact abelian group G is a Lie group iff it is of the form $\mathbb{T}^n \times F$ with a finite group F. For $n \geq 1$, the monothetic torus $\mathbb{T}^n$ contains the subgroup $\mathbb{Z}(2)^n$ which is an IMG only if $n = 1$. If $C \subseteq F$ is a nondegenerate cyclic subgroup of F and $n = 1$ then $\mathbb{T}$ contains an isomorphic copy C_* of C and so G contains $C_* \times C$ which fails to be an IMG. Thus G is a class (II) group and a Lie group iff it is isomorphic to $\mathbb{T}$ or is a finite class (I) group in which case its p-rank is 1 for all primes p dividing the order of F. Thus G is a Lie group of class (II) and an IMG iff $\widehat{G}$ is a finitely generated discrete IMG.

Now let G be an IMG of class (II). Then it is a strict projective limit of its Lie group quotients which are IMG-s of class (II). Hence $\widehat{G}$ is the directed union of its finitely generated subgroups which are IMG-s of class (I), and thus $\widehat{G}$ is an IMG. By Lemma 3.2, $\widehat{G}$ is then a subgroup of $\mathbb{Q}$ or of $\mathbb{Q}/\mathbb{Z}$. Now, by duality, G is of the kind asserted (cf. [10], 822.ff.). □

The proof has shown that an infinite compact abelian group G is an IMG of class (II) iff $\widehat{G}$ is an infinite IMG of class (I).

Lemma 3.4. *An* IMG *of class* (III) *is periodic.*

Proof. Suppose $\mathbb{R}$ were an IMG. But $\mathbb{R}$ is generated by 1 and $\sqrt{2}$; thus $\mathbb{R}$ would have to be monothetic which it is not. Thus an IMG of class (III) does not contain any vector subgroup and so the subgroup E in Theorem 7.57 of [10] is trivial and G has a compact open subgroup.

Since G is not discrete, there is an open infinite compact subgroup U which is an IMG. Suppose it were not 0-dimensional. Then by Lemma 3.3 it would be isomorphic to a 1-dimensional compact connected group. Being divisible and open, it would be an algebraic and topological direct summand; G being an IMG would imply $G = U$

which is impossible since G is of class (III). Thus U and hence G is 0-dimensional.

Suppose G is a class (III) IMG for which $G \neq \mathrm{comp}(G)$. Then by [10], Corollary 4.5, there would be an infinite cyclic discrete group $D \cong \mathbb{Z}$ and for every compact open subgroup U the open sum $U + D$ is direct. Now G being an IMG implies that U has to be trivial. Then G would be discrete, but it is of class (III). Therefore $G = \mathrm{comp}(G)$ and thus G is periodic. □

Now Theorem 2.1 reduces the classification of class (III) IMG-s to the case that G is a p-group.

Lemma 3.5. *Let G be a p-group which is a class* (III) IMG. *Then* $G \cong \mathbb{Q}_p$.

Proof. Let U be a compact open subgroup of G. By Lemma 3.3, U is a profinite monothetic group which cannot be finite since G is not discrete. Hence $U \cong \mathbb{Z}_p$. Similarly, the discrete IMG and p-group G/U is isomorphic to $\mathbb{Z}(p^\infty) = \left(\bigcup_{m=0}^{\infty} \frac{1}{p^m}\mathbb{Z}\right)/\mathbb{Z}$. Thus there is a surjective morphism

$$f\colon G \to \mathbb{Z}(p^\infty)$$

with kernel U. If we set

$$U_m = f^{-1}\left(\frac{\frac{1}{p^m}\mathbb{Z}}{\mathbb{Z}}\right)$$

then $G = \bigcup_{m=0}^{\infty} U_m$ and each U_m is generated by the set consisting of a generator of U and any element mapping by f onto a generator of $\frac{\frac{1}{p^m}\mathbb{Z}}{\mathbb{Z}}$ and thus is monothetic and hence isomorphic to $\mathbb{Z}_p$. Moreover, $U_m = p{\cdot}U_{m+1}$. Thus the sequence

$$U = U_0 \subseteq U_1 \subseteq U_2 \subseteq \cdots$$

is isomorphic to the sequence

$$\mathbb{Z}_p \subseteq \frac{1}{p}{\cdot}\mathbb{Z}_p \subseteq \frac{1}{p^2}{\cdot}\mathbb{Z}_p \subseteq \cdots ,$$

and thus $G = \mathbb{Q}_p$ follows. □

Lemma 3.6. *Let (G_p, C_p) be a pair consisting of inductively monothetic p-group G_p and C_p a compact open subgroup. Then we have the following possibilities for G_p, C_p, and G_p/C_p:*

$$\begin{array}{ccc} \mathbb{Z}(p^k) & p^{k-\ell}\mathbb{Z}(p^k) & \mathbb{Z}(p^\ell) \\ \mathbb{Z}(p^\infty) & \mathbb{Z}(p^k) & \mathbb{Z}(p^\infty) \\ & & \\ \mathbb{Z}_p & p^\ell\mathbb{Z}_p & \mathbb{Z}(p^\ell) \\ \mathbb{Q}_p & \mathbb{Z}_p & \mathbb{Z}(p^\infty) \end{array}$$

Proof. This lemma follows from the preceding lemmas 3.2, 3.3, 3.4, and 3.5 upon specializing to a p-group. □

Before we summarize our findings we draw attention to the following notation: we say that a discrete torsion group has p-rank 1 if its p-primary component is isomorphic to $\mathbb{Z}(p^n)$ for some $n \in \{1, 2, \dots, \infty\}$. (Cf. [10], Definition 2.12.)

Theorem 3.7. (Classification Theorem of Inductively Monothetic Groups) *Let G be an* IMG. *Then either G is a 1-dimensional compact connected abelian group, or it is totally disconnected and isomorphic to exactly one of the following groups:*

(1a) *a subgroup of the discrete group $\mathbb{Q}$,*
(1b) *a subgroup of the discrete group $\mathbb{Q}/\mathbb{Z}$,*
(2) *an infinite monothetic profinite, that is, infinite procyclic, group,*
(3) *a local product $M = \prod^{\mathrm{loc}}_{p\ \mathrm{prime}}(M_p, C_p)$ with the p-primary components M_p such that*
  (3a) *M_p/C_p is a p-group of p-rank 1, and*
  (3b) *both $C = \prod_p C_p$ and $M/C = \bigoplus_p(M_p/C_p)$ are infinite.*

In Case (3a), if $M_p/C_p \cong \mathbb{Z}(p^{n_p})$, then M_p is finite cyclic and C_p is a subgroup, or $M_p \cong \mathbb{Z}_p$ and C_p is a closed subgroup; and if $M_p/C_p \cong \mathbb{Z}(p^\infty)$, then $M_p \cong \mathbb{Q}_p$ and $C_p \cong \mathbb{Z}_p$, or $\cong \mathbb{Z}(p^\infty)$ and C_p is a finite subgroup.

The groups of connected type and type (2) of Theorem 3.7 are monothetic; other types may or may not be monothetic. An inspection of each or the five conditions yields at once the following

Corollary 3.8. *Let G be a locally compact inductively monothetic group. Then the following conditions are equivalent:*

(1) *$G = \mathrm{comp}(G)$.*
(2) *G is not isomorphic to a subgroup of the discrete group $\mathbb{Q}$.*

Also, the following two conditions are equivalent:

(3) *G is periodic.*
(4) *G is not connected and not isomorphic to a subgroup of the discrete group $\mathbb{Q}$.*

As a consequence of these remarks and of Weil's Lemma [10, Proposition 7.43] we record

Corollary 3.9. *Let G be inductively monothetic. Suppose that it contains a discrete subgroup $D \cong \mathbb{Z}$. Then G is discrete and torsion free of rank one.*

Examples of groups from the Classification Theorem 3.7 above are given by each of the groups $\mathbb{Q}_p$ or by local products of the type described in the following

Example 3.10. Define $G = \prod_p^{\text{loc}}(\mathbb{Z}_p, p\mathbb{Z}_p)$. Then $C = \prod_p p\mathbb{Z}_p$ is an open subgroup of G isomorphic to $\widetilde{\mathbb{Z}} = \prod_p \mathbb{Z}_p$, while its factor group G/C is isomorphic to the infinite discrete group $\bigoplus_p \mathbb{Z}(p)$. The inclusion map induces an injective continuous morphism $\varepsilon\colon G \to \widetilde{\mathbb{Z}} = \prod_p \mathbb{Z}_p$ with dense image which is not open onto its image.

Examples of this kind illustrate the considerable variety of groups of type (3) in Theorem 3.7.

The following theorem will be crucial for the later developments.

Theorem 3.11. (The Gamma Theorem) *Let Γ be a nonsingleton* IMG *acting nontrivially on an abelian group G as a group of scalar automorphisms and assume that* id_G *and* $-\mathrm{id}_G$ *are not the only actions of Γ on G.*

Then G is periodic and Γ is isomorphic to exactly one one of the following groups

(i) *a nontrivial discrete* proper *subgroup of $\mathbb{Q}$,*
(ii) *a subgroup of $\mathbb{Q}/\mathbb{Z}$ each of whose p-primary components is finite,*
(iii) *an infinite procyclic group,*
(iv) *a noncompact local product $\prod_p^{\text{loc}}(\Gamma_p, C_p)$ where each factor is either of the form $(\mathbb{Z}_p, p^k\mathbb{Z}_p)$ or of the form $(\mathbb{Z}(p^m), p^k\mathbb{Z}(p^m))$, and both $\prod_p C_p$ and $\bigoplus_p \Gamma_p/C_p$ are infinite.*

The group G has arbitrarily small Γ-invariant neighborhoods.

Proof. Let η denote the canonical morphism from Γ to $\mathrm{SAut}(G)$ that is guaranteed by the hypotheses and let $\Delta = \ker \eta$. Then $\eta(\Gamma) \cong \Gamma/\Delta$. Since $\eta(\Gamma) \not\leq \{\mathrm{id}_G, -\mathrm{id}_G\}$, by Theorem 2.10, G is periodic. So by Theorem 2.11 $\mathrm{SAut}(G)$ is a profinite group. Since a profinite group cannot contain any nontrivial divisible subgroups, Γ/Δ does not contain any divisible subgroup. In particular, since a homomorphic image of a divisible group is divisible, Γ itself cannot be divisible. Hence

Γ cannot be connected and so Γ is totally disconnected by Theorem 3.7. Likewise, since the group Γ cannot be divisible, it cannot be isomorphic to either $\mathbb{Q}$, or $\mathbb{Z}(p^\infty)$, or $\mathbb{Q}_p$ for any prime p. In view of the Classification Theorem 3.7, this completes the proof of assertions (i)–(iv).

The last statement of the Theorem is a consequence of Remark 2.13. □

Case (ii) of Theorem 3.11 includes the possibility that Γ is a finite cyclic group.

Note that G is the union of its compact (and therefore Γ-invariant) subgroups. The case that G is a compact p-group and Γ is a monothetic p-group was extensively discussed in [11].

The following proposition is added to show which examples are allowed typically by Theorem 3.11. Recall $\mathbb{Z}_p \subseteq \mathbb{Q}_p$ and $\mathbb{Q} \subseteq \mathbb{Q}_p$. We define $\mathbb{F}_p := \mathbb{Z}[\frac{1}{p}]$ to be the subring of the rationals consisting of fractions with numerator a power of p, and let ${}_p\mathbb{F}$ denote the subring of the rationals consisting of fractions $\frac{a}{b}$ where b is coprime to p. We note that, algebraically, $\mathbb{F}_p$ is the ring adjunction of $\frac{1}{p}$ to $\mathbb{Z}$, and that ${}_p\mathbb{F}$ is the p-localization of $\mathbb{Z}$.

Proposition 3.12. (i) *The additive group $\mathbb{Z}_p$ does not contain any nontrivial divisible subgroups.*

(ii) $\mathbb{Q} \cap \mathbb{Z}_p = {}_p\mathbb{F}$.

(iii) $\mathbb{Q} + \mathbb{Z}_p = \mathbb{Q}_p$.

(iv) $\mathbb{Z}_p/{}_p\mathbb{F} \cong \mathbb{R}$ *(as abstract groups).*

(v) $\mathbb{Z}_p/\mathbb{Z} \cong \mathbb{R} \times \frac{{}_p\mathbb{F}}{\mathbb{Z}}$, *where* $\frac{{}_p\mathbb{F}}{\mathbb{Z}} \cong \bigoplus_{p\neq q} \mathbb{Z}(q^\infty)$. *In particular, $\mathbb{Z}_p/\mathbb{Z}$ is divisible.*

Proof. (i) $\mathbb{Z}_p$ is not the direct sum of two proper subgroups (see [6], p. 122ff.). A divisible subgroup of an abelian group is a direct summand (see e.g. [10] Corollary A1.36(i)). So (i) follows.

(ii) A number $q \in \mathbb{Q}_p$ is of the form $q = p^n z$ with a unique smallest $n \in \mathbb{Z}$ and a unique $z \in \mathbb{Z}_p$. We have $q \in \mathbb{Z}_p$ iff $n \geq 0$. A rational number is of the form $\pm p^n z$ with a unique smallest $n \in \mathbb{Z}$ and a unique rational number z relatively prime to p. Thus $q \in \mathbb{Q} \cap \mathbb{Z}_p$ iff $n \geq 0$ and $z \in \mathbb{Q}$ relatively prime to p. iff $q \in {}_p\mathbb{F}$. It also follows from this that $\mathbb{F}_p \cap \mathbb{Z}_p = \mathbb{F}_p \cap {}_p\mathbb{F} = \mathbb{Z}$.

(iii) $\mathbb{F}_p + \mathbb{Z}_p$ is an additive subgroup of $\mathbb{Q}_p$ containing $\mathbb{Z}_p$ and such that $\mathbb{Z}(p^\infty) \cong \mathbb{Q}_p/\mathbb{Z}_p \supseteq (\mathbb{F}_p + \mathbb{Z}_p)/\mathbb{Z}_p \cong \mathbb{F}_p/(\mathbb{F}_p \cap \mathbb{Z}_p) = \mathbb{F}_p/\mathbb{Z} = \mathbb{Z}(p^\infty)$. Since $\mathbb{Z}(p^\infty)$ has no proper infinite subgroup, $\mathbb{Q}_p/\mathbb{Z}_p = (\mathbb{F}_p + \mathbb{Z}_p)/\mathbb{Z}_p$ follows, and this implies $\mathbb{F}_p + \mathbb{Z}_p = \mathbb{Q}_p$. A fortiori, $\mathbb{Q} + \mathbb{Z}_p = \mathbb{Q}_p$.

(iv) Using (ii) above we compute $\mathbb{Z}_p/{}_p\mathbb{F} = \frac{\mathbb{Z}_p}{\mathbb{Z}_p \cap \mathbb{Q}} \cong \frac{\mathbb{Q}+\mathbb{Z}_p}{\mathbb{Q}} = \mathbb{Q}_p/\mathbb{Q}$ in view of (iii) above. Now $\mathbb{Q}_p$ is a rational vector spaces of dimension $\mathfrak{c}$ (the cardinality of the continuum) and $\mathbb{Q}$ is a 1-dimensional vector subspace. Thus $\mathbb{Q}_p/\mathbb{Q}$ is a rational vector space of dimension $\mathfrak{c}$ and thus is isomorphic to $\mathbb{R}$ as a rational vector space.

(v) Firstly, we note that $\frac{\mathbb{Z}_p/\mathbb{Z}}{{}_p\mathbb{F}/\mathbb{Z}} \cong \frac{\mathbb{Z}_p}{{}_p\mathbb{F}} \cong \mathbb{R}$. Secondly we observe that ${}_p\mathbb{F}/\mathbb{Z} = \bigoplus_{p\neq q} \mathbb{Z}(q^\infty)$. Since this group is divisible and each divisible subgroup is a direct summand (see [2], A1.36(i)) conclude $\frac{\mathbb{Z}_p}{\mathbb{Z}} \cong \mathbb{R} \times \frac{{}_p\mathbb{F}}{\mathbb{Z}}$ as asserted.

□

If we write the circle group additively as $\mathbb{T} = \mathbb{R}/\mathbb{Z} \cong \mathbb{R} \times \frac{\mathbb{Q}}{\mathbb{Z}}$, then we notice that $\mathbb{Z}_p/\mathbb{Z} \cong \frac{\mathbb{T}}{\mathbb{Z}(p^\infty)}$ and $\mathbb{T} \cong \frac{\mathbb{Z}_p}{\mathbb{Z}} \times \mathbb{Z}(p^\infty)$.

As was observed in [11], the additive group G of the compact ring $\mathbb{Z}_p$ contains a copy of $\mathbb{Z}_p$ in its multipliative group of units whence $\mathrm{SAut}(G)$ contains a copy of $\mathbb{Z}_p$ and therefore $\mathbb{Z}_p$, any discrete subgroup of $\mathbb{Z}_p$ such as ${}_p\mathbb{F}$, or the noncompact group $\prod_p^{\mathrm{loc}}(\mathbb{Z}_p, p\mathbb{Z}_p)$ of Example 3.10 as Γ for $G = \mathbb{Z}_p$ in Theorem 3.11.

4. Near abelian groups

We extend to arbitrary locally compact groups the definition of *near abelian groups* [11], introduced in [11] for profinite groups.

4.1. **Definition and immediate consequences.** We begin with the basic definition.

Definition 4.1. A locally compact group G is *near abelian* provided it contains a closed normal abelian subgroup A such that

(1) G/A is inductively monothetic, and
(2) every closed subgroup of A is normal in G.

We shall call A a *base* of the near abelian group G. If there is a closed subgroup H such that $G = AH$, then we call H a *scaling subgroup*. It will be convenient to denote by Γ the quotient $G/C_G(A)$. Note that Γ is an IMG acting faithfully on A.

Some remarks are in order at once. By Theorem 3.7, G/A is abelian, and so $G' \subseteq A$. Every inner automorphism of G induces on A a scalar automorphism by the very definition of a scalar automorphism in Paragraph 2.2. Thus we have have a canonical morphism

$$\psi \colon G \to \mathrm{SAut}(A), \quad \psi(g)(a) = gag^{-1}$$

whose kernel is $\ker\psi$ is the centralizer $C_G(A)$ of A in G, containing A since A is abelian by Definition 4.1. So we note that the group $\Gamma := G/C_G(A)$ is a homomorphic image of G/A and therefore being inductively monothetic, acts faithfully via scalar automorphisms on A.

Theorem 3.11 immediately implies

Lemma 4.2. *If G/A acts nontrivially on A, then A is periodic.*

Lemma 4.3. *Let G be a near abelian group and A a base group.*

1. *If $f : G \to H$ is an epimorphism then H is near abelian and has base group $f(A)$.*
2. *Every closed subgroup L of G is near abelian with base group $L \cap A$.*

Proof. (1) Obvious.

(2) Let L be any closed subgroup of G. Then $L \cap A$ is an abelian normal subgroup of L such that every monothetic subgroup of $L \cap A$ is normal in L since it is normal in G. Pick a finite subset F in L. Then $\overline{\langle F\rangle}$ is σ-compact, $\overline{\langle F\rangle}A$ is locally compact and so [8, 5.33] implies that $\overline{\langle F\rangle}/\overline{\langle F\rangle} \cap A \cong \overline{\langle F\rangle}A/A$ is monothetic. Hence L is near abelian. □

Lemma 4.4. *Assume that G is a near abelian group with a base group A. Suppose that both A and G/A are periodic. Then G is periodic.*

Proof. There is no problem with zero-dimensionality. Fix an element $g \in G$. The coset gA in G/A generates a procyclic subgroup which in turn lifts to a near abelian subgroup $L = \overline{\langle g, A\rangle}$ of G with procyclic quotient L/A. In order to show that G is periodic we thus can restrict to G/A procyclic. We can lift a generator of G/A to an element $b \in G$ such that $G = AH$ for $H := \overline{\langle b\rangle}$. Setting in [8, (5.33) Theorem] $A := H$ and $H := A$ we find, in our situation, that AH/A is topologically isomorphic to $H/A \cap H$. By assumption the group AH/A is compact and therefore so is $H/A \cap H$. Since A is periodic and H is monothetic $A \cap H$ is compact. Hence H, being the extension of a compact group by a compact group, is compact.

Next let $g = ah \in G$ be arbitrary. Since h acts by scalars we find that $\overline{\langle ah\rangle} \subseteq \overline{\langle a\rangle}\,\overline{\langle h\rangle}$ and, since the group on the right hand side is compact, so is $\overline{\langle g\rangle}$. Hence G is periodic as has been claimed. □

Definition 4.5. If Γ acts by trivial scalar automorphisms, that is, by multiplication by ± 1, we say that G is a *trivial near abelian group* otherwise we call G a *nontrivial near abelian group.*

Note that an abelian near abelian group is trivial in this sense.

The following diagram may help to sort out this quantity of information:

$$
\begin{array}{lcl}
 & G & \\
\text{inductively monothetic } \left\{ \begin{array}{c} \\ \\ \\ \end{array} \right. & | & \Big\} \cong \Gamma \xrightarrow{\text{injected}} \mathrm{SAut}(A) \\
 & C_G(A) & \\
 & | & \Big\} \text{ inductively monothetic} \\
 & A & \\
 & | & \\
 & G' & \Bigg\} \text{ periodic if } G \text{ nontrivial} \\
 & | & \\
 & \{1\} &
\end{array}
$$

Lemma 4.2 and Lemma 4.4 yield at once

Corollary 4.6. *Assume that G is a nontrivial near abelian locally compact group with a base group A. Suppose that G/A is periodic. Then G is periodic.*

In the light of the above diagram, the following observation is of interest:

Proposition 4.7. *In a near abelian group G with any base group A, the centralizer $C_G(A)$ is an abelian normal subgroup of G containing A and is maximal with respect to this property.*

Proof. Let $x, y \in C_G(A)$. Since G/A and thus its subgroup $C_G(A)/A$ are inductively monothetic, there is a $z \in C_G(A)$ such that $x, y \in \overline{A\langle z\rangle}$. Since A and $\overline{\langle z\rangle}$ are both abelian and $\overline{\langle z\rangle}$ commutes elementwise with A, the elements x and y commute. Thus $C_G(A)$ is commutative and trivially contains A since A is abelian. If B is any commutative subgroup of G containing A, then $B \subseteq C_G(A)$, trivially. Hence $C_G(A)$ is the largest such subgroup B. □

In particular, a base group A of a near abelian group G need not be a maximal abelian normal subgroup in general.

This can be seen by writing $G = \mathbb{Z}(2) \times \mathbb{Z}(2) = AH$ and by letting H act trivially by scalars upon A, with either $A = G$ and $H = \{0\}$, or $A = \langle(1, 0)\rangle$ and $H := \langle(0, 1)\rangle$.

A distinguished situation of near abelian groups arises when G/A is procyclic (Case Theorem 3.7(3)), because in that case G has a scaling subgroup. For the ring $\widetilde{\mathbb{Z}}$ of universal scalars of a periodic abelian group see Proposition 2.9 and the paragraph preceding it.

Proposition 4.8. *Let G be a near abelian group and A a base group and assume that G/A is procyclic. Then there is a procyclic scaling subgroup $H = \overline{\langle b\rangle}$ in G such that $G = AH$ and, there is a scalar $r \in \widetilde{\mathbb{Z}}^\times$ in the group of units of $\widetilde{\mathbb{Z}}$ such that for all $a \in A$ we have $a^b = a^r$.*

If $\alpha\colon H \to \mathrm{SAut}(A)$ is the natural morphism implemented by inner automophisms, then the semidirect product $A \rtimes_\alpha H$ is a near abelian group and the function

$$\mu\colon A \rtimes_\alpha H \to G, \quad \mu(a,h) = ah,$$

is a quotient homomorphism with kernel $\{(h^{-1},h) : h \in A\cap H\}$ isomorphic to $D := A\cap H$, mapping both A and H faithfully.

The factor group G/D is a semidirect product of A/D and H/D and the composition

$$A \rtimes_\alpha H \to G \to G/D$$

is equivalent to the natural quotient morphism

$$A \rtimes_\alpha H \to \frac{A}{D} \rtimes \frac{H}{D}$$

with kernel $D \times D$.

Proof. Since G/A is monothetic there is an element $b \in G$ such that Ab is a generator of G/A. We set $H := \overline{\langle b\rangle}$. Then $p\colon H \to G/A$, $p(h) = Ah$, is a surjective morphism of compact groups and so $G = AH$. The inner automorphism $x \mapsto a^b = b^{-1}ab$ of A belongs to $\mathrm{SAut}(A)$ since G is near abelian. Now the existence of r and its properties follow from Proposition 2.9(i).

In view of these facts, the scaling group H acts on A by scalar endomorphisms under inner automorphisms and therefore leaves all closed subgroups of A invariant. Hence all closed subgroups of $A \times \{1\} \subseteq A \rtimes_\alpha H$ are normal and so $A \rtimes \alpha H$ is a near abelian group. The remainder is then a result of the standard Mayer-Vietoris formalism of [11], Lemma 2.11 in view of the fact that the morphism μ is proper due to the compactness of its kernel $\cong A \cap H$.

The last assertion is a straightforward conclusion for every Mayer-Vietoris formalism. ∎

What prevents the scaling group H from being a semidirect factor of G thus is the possible nondegenerate intersection $D = A\cap H$. Therefore, being a quotient of the semidirect product $A \rtimes_\alpha H$ and having the semidirect product $(A/D) \rtimes (H/D)$ as quotient is the next best thing for G to be a semidirect product itself.

4.2. **Examples of near abelian groups.** Some examples may serve as a first illustration of these concepts. The first example shows that *any* abelian group A can appear as the base group of a near abelian group.

Example 4.9. Let A be an arbitrary abelian group. Then the multiplicative group $C_2 = \{1, -1\} \subseteq \mathbb{Z}$ acts by trivial scalar automorphisms on A and $D(A) := A \rtimes C_2$, the so-called *dihedral extension of* A is a trivial near abelian group. It is abelian if and only if A has exponent 2, that is, is algebraically a vector space over GF(2). If H is an IMG with a morphism $\alpha\colon H \to C_2$, then $G = A \rtimes_\alpha H$ is a trivial abelian group which maps homomorphically into $D(A)$.

Compact near abelian p-groups in the sense of the definition in [11, page 2] are near abelian. Here is an illustrative near abelian p-group, a locally compact version of [11, Theorem 4.21 Case B]:

Example 4.10. Let p be a fixed prime. For every $n \in \mathbb{N}$ let G_n be either
Case A: a copy of $\mathbb{Q}_p$,
Case B: a copy of $\mathbb{Z}_p$,
and fix an open proper subgroup C_n of G_n. Then $C_n \cong \mathbb{Z}_p$. The multiplicative subgroup

$$H := (1 + p^k\mathbb{Z}_p, \times), \quad \begin{Bmatrix} k = 1 & \text{if } p > 2 \\ k = 2 & \text{if } p = 2 \end{Bmatrix} \cong (\mathbb{Z}_p, +)$$

of the ring $(\mathbb{Z}_p, +, \cdot)$ acts by scalar automorphisms on the direct product $\prod_{n\in\mathbb{N}} G_n$ (see Corollary 2.3). Therefore we can form the local product $A := \prod^{\text{loc}}_{n\in\mathbb{N}}(G_n, C_n)$ (see Theorem 2.1) and get H to act on A by scalar automorphisms (see Corollary 2.5). Therefore we obtain a periodic near abelian p-group $G := A \rtimes H$ with base group A and compact scaling group H; the base group A and thus the group G is noncompact in both Case A and Case B.

In the preceding examples, the base group A was noncompact while the scaling group was compact. The following example of a locally compact group illustrates the reverse situation:

Remark 4.11. Example 4.10 produces a near abelian p-group in which a base group A splits as a normal semidirect factor with a compact complement $H \cong \mathbb{Z}_p$; we shall encounter such groups in Proposition 4.27. If, in the same example, we pass to a subgroup of H abstractly isomorphic to $\mathbb{Z}$ and equip G with a topology that declares A to be an open subgroup, then the new example resulting in this fashion gives an example of a class of near abelian groups we shall discuss in Proposition 4.21.

Example 4.12. We let $A = \prod_p \mathbb{Z}_p$ with p ranging through all prime numbers. Let

$$H_p := (1 + p^k\mathbb{Z}_p, \times), \quad \begin{Bmatrix} k = 1 & \text{if } p > 2 \\ k = 2 & \text{if } p = 2 \end{Bmatrix} \cong (\mathbb{Z}_p, +),$$

Then each H_p acts by scalar automorphisms on $\mathbb{Z}_p$, and so $H := \prod_p H_p$ acts by scalar automorphisms on $A = \prod_p \mathbb{Z}_p$, componentwise. Then

$$A \rtimes H \cong \prod_p (\mathbb{Z}_p \rtimes H_p)$$

is a compact near abelian group. If $K_p = (1 + p^{k+1}\mathbb{Z}_p, \times)$, then $H_p/K_p = C_p$ is cyclic of order p, and $L := \prod_p^{\text{loc}}(H_p, K_p)$ as a noncompact IMG (see Example 3.10 above), and $\varepsilon\colon L \to H$ induced by the inclusion is an injective continuous morphism with dense image. Accordingly, $G = A \rtimes L$ is a noncompact near abelian group with a compact base group $\cong A$ and a noncompact scaling group $\cong L$ which is mapped injectively and continuously into the compact near abelian group $A \rtimes H$ with dense image by the morphism $\text{id}_A \times \varepsilon$.

Here is an example of a abelian group G with a base group A for which G/A is an infinite torsion group:

Example 4.13. Choose primes p_i and q_i such that $p_i \equiv 1 \pmod{q_i}$. Then there is $r_i \in \mathbb{Z}_{q_i}$ such that $r_i^{p_i} = 1$. Set $A := \prod_{i \in \mathbb{N}} \mathbb{Z}_{q_i}$ and let a generator b_i of $\mathbb{Z}(p_i)$ act trivially on $\mathbb{Z}_{q_j}$ if $j \neq i$ and as $z^{b_i} := z^{r_i}$. Forming the discrete direct sum $H := \bigoplus_{i \in \mathbb{N}} \mathbb{Z}(p_i)$ we can extend the actions of the b_i to a scalar action of H. The semidirect product $G := A \rtimes H$ has a torsion scaling group.

Finally we provide an example with G/A torsion free:

Example 4.14. Let p be an odd prime and recall that

$$\text{SAut}(\mathbb{Z}_p) \cong \mathbb{Z}_p \times \mathbb{Z}(p-1).$$

Let $H = {}_p\mathbb{F}^\times$ denote the set of all rational numbers $\frac{a}{b}$ with b relatively prime to p. By Proposition 3.12 (ii) every element in H is a unit in the ring $\mathbb{Z}_p$. Put $A := \mathbb{Z}_p$ and equip H with the discrete topology. Then $G := A \rtimes H$ has torsion free discrete countable scaling group.

Example 4.15. Consider the Prüfer group $\mathbb{Z}(2^\infty)$ as a discrete group and the profinite group of 2-adic integers $\mathbb{Z}_2$. We let the multiplicative group $1+4\mathbb{Z}_2$ of 2-adic integers act by scalar multiplication on $\mathbb{Z}(2^\infty)$. Namely, the topological generator $g = 1+2^n$ of $1+2^n\mathbb{Z}_2$ acts by multiplication by g on $\mathbb{Z}(2^\infty)$. Then the semidirect product $G := \mathbb{Z}(2^\infty) \rtimes (1+4\mathbb{Z}_2)$ is near abelian.

Since $\mathbb{Z}(2^\infty)$ is discrete the family of sets $1\times 2^n\mathbb{Z}_2$, $n=2,3,\dots$ forms a basis of identity neighborhoods of 1.

Now $(a,1)(0,g)(-a,1) = (a,g)(-a,1) = (a-(1+2^n)\cdot a, g) = (-2^n\cdot a, g)$, and thus the conjugacy class of g contains $\mathbb{Z}(2^\infty)\times\{g\}$. Since any identity neighborhood U, no matter how small, but contained in $\{0\}\times 4\mathbb{Z}_2$ contains an element $g = 1+2^n$ for sufficiently large n whose conjugacy class $\mathbb{Z}(2^\infty)\times\{g\}$ is not contained in $\{0\}\times(1+4\mathbb{Z}_2)$, the neighborhood U cannot be an invariant identity neighborhood. Hence G does not contain arbitrarily small invariant identity neighborhoods.

The following example of a compact group contains A on which G acts by scalars and G/A is monothetic but not inductively monothetic. So G is *not* near abelian; but the example shows that connectivity can be more tricky if, in the definition of a near abelian group one begins to vary the hypothesis on G/A.

Example 4.16. Let $\mathbb{Z}(9)$ be the natural $(1+3\mathbb{Z}(9),\times)\cong\mathbb{Z}(3)$-module and let $\Gamma = \mathbb{Z}(3)\times\mathbb{T}$ act on $\mathbb{Z}(9)$ by the scalar action of the first factor. Form the semidirect product $\mathbb{Z}(9)\rtimes(\mathbb{Z}(3)\times\mathbb{T})$. The subgroup $3\mathbb{Z}(9)\times\{0\}\times\frac{1}{3}\mathbb{Z}/\mathbb{Z}$ is central and contains a diagonal subgroup D generated by $(3+3\mathbb{Z}(9), \mathbf{0}, -1/3+\mathbb{Z})$. Then $G := (\mathbb{Z}(9)\rtimes(\mathbb{Z}(3)\times\mathbb{T}))/D$ has a finite base group of order 9 and a circle group as nontrivial identity component which intersects the base group in a subgroup of order 3 and which does not split over the base group. Note that Γ is monothetic but not inductively monothetic.

4.3. **Nontrivial near abelian groups.**

Our observations in the two preceding sections allow us to draw some conclusions in view of what we saw in Theorems 3.7 and 3.11 on the structure of inductively monothetic groups and their faithful actions on abelian groups. We keep in mind that we have a good understanding of abelian periodic groups due to Braconnier's Theorem 2.1 on their primary decomposition. It will be efficient to distinguish between the nontrivial and the trivial situation. *We keep in mind that the two situations are mutually exclusive.*

Theorem 4.17. (First Structure Theorem: Nontrivial Near Abelian Groups) *Let G be a* nontrivial *near abelian group with a base group A. Then the following statements hold:*

(1) *A is periodic and contains $G' \neq \{1\}$.*
(2) *G/A is isomorphic to one of the groups recorded in Theorem 3.11.*
(3) *G is totally disconnected.*
(4) *When $\Gamma = G/C_G(A)$ is compact or A is open, then G has arbitrarily small invariant neighborhoods.*

Proof. (1) Since the action of Γ is nontrivial, G is nonabelian, A is periodic by Lemma 4.2. Since G/A is abelian, $G' \subseteq A$.

(2) Since G/A is inductively monothetic by definition, Theorem 3.11 applies to it.

(3) Since A is totally disconnected by (1) and G/A is totally disconnected by (2) we conclude that G is totally disconnected.

(4) is a consequence of Theorem 3.11 and the fact, that A is periodic by (1) and thus has arbitrarily small compact open neighborhoods, and that all subgroups of A are normal. □

Example 4.15 shows us a periodic near abelian group without small invariant identity neighborhoods.

As an immediate consequence of Theorem 4.17 and Corollary 4.6, we have the following

Corollary 4.18. *For a nontrivial near abelian locally compact group G the following statements are equivalent:*
(1) *G is periodic.*
(2) *G/A is not isomorphic to a subgroup of the discrete group $\mathbb{Q}$.*

4.4. **Trivial near abelian groups.** The less sophisticated counterpart of Theorem 4.17 is the case of a *trivial* near abelian group G. The general structure of G/A is still given by Theorem 3.7. Recall that G_0 denotes the identity component of a topological group G and G' the commutator subgroup.

We should keep in mind the following simple fact:

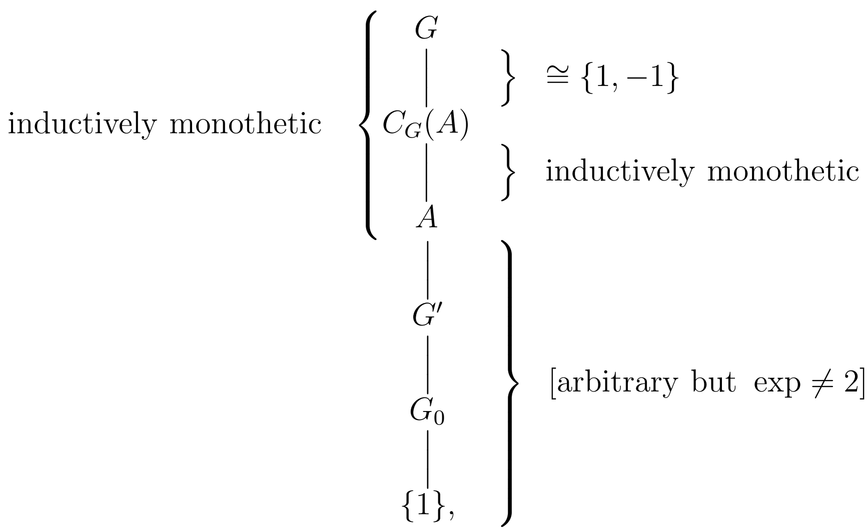
Lemma 4.19. *Let G be a topological group and H a subgroup. Assume that the identity component G_0 of G is contained in H. Then $H_0 = G_0$.*

Proof. Since G_0 contains all connected subspaces containing the identity, $H_0 \subseteq G_0$. Since $G_0 \subseteq H$ and H_0 contains all connected subspaces of H containing the identity, $G_0 \subseteq H_0$. □

Theorem 4.20. (Second Structure Theorem: Trivial Near Abelian Groups) *Let G be a* trivial *but nonabelian near abelian group with a base group A. Then the following statements hold*

(1) Γ *is of order 2.*
(2) $G_0 \subseteq G' \subseteq A$ *and* $G_0 = (G')_0 = A_0$.
(3) A *can be any abelian group except a group of exponent* 2.
(4) A *has arbitrarily small invariant identity neighborhoods.*

The following diagram summarizes the situation:



where the property in square brackets is invalid if the exponent of A *is* 3 *or* 4.

Proof. (1) is a direct consequence of the definition of a trivial near abelian group and the hypothesis, that G is not abelian.

(2) By Theorem 3.7, G/A is either zero-dimensional or compact connected. We claim the former holds: If G/A were connected, the canonical morphism $\psi\colon G \to \{1_A, -1_A\}$, which factors through G/A, would have to be trivial with the consequence that G would be abelian, contrary to the hypothesis. Thus G/A is totally disconnected, that is $G_0 \subseteq A$.

Since G/A is abelian by Theorem 3.7, we know that $G' \subseteq A$. In view of the action of G on A by scalar automorphisms implemented by multiplication with $\{1_A, -1_A\}$ when written additively, the subgroup $[G, A]$ generated by all commutators $gag^{-1}a^{-1}$, $a \in A$, $g \in G$, is in fact generated by the elements $a^{-1}a^{-1} = a^{-2}$, which themselves form a subgroup. Since the identity component G_0 of a abelian group is divisible (see [10], Corollary 7.58 and Corollary 8.5), every element in G_0 is a square and so $G_0 \subseteq [A, G] \subseteq G'$.

(3) The only additively written abelian group A failing to allow a nontrivial action of the scalar automorphism $-\operatorname{id}_A$ is one in which $-x = x$ for all $x \in A$, that is, $2x = 0$.

(4) This is an immediate consequence of the compactness of Γ and [10], p. 9, Proposition 1.11. □

Example 4.9 illustrates how trivial near abelian groups arise and Example 4.15 shows the existence of locally compact near abelian nontrivial groups having no small invariant identity neighborhoods.

Regarding the Second Structure Theorem 4.20 we add the following remark: The hypothesis that G is nonabelian has the consequence $(G : C_G(A)) = 2$, and thus A cannot have exponent 2. There is a subtle difference between this hypothesis and the hypothesis $\operatorname{SAut}(A) = \{\operatorname{id}_A, -\operatorname{id}_A\}$. Indeed the latter statement allows for the possibility that $-\operatorname{id}_A = \operatorname{id}_A$ in the case that the exponent of A is 2.

The constructions in Example 4.9 indicate that Theorem 4.20 is rather optimal. The fact that the two structure theorems cover two disjoint situations show, for instance, that a near abelian group with nondegenerate connected subspaces is necessarily trivial.

4.5. **Nonperiodic near abelian groups.** We turn our attention back to nontrivial near abelian groups G and consider those for which the factor group G/A contains a torsion free discrete subgroup. According to the First Structure Theorem 4.17, G/A is then isomorphic to a subgroup of the discrete group $\mathbb{Q}$, and G/A is as described in the Gamma Theorem 3.11(i). If $C_G(A) \neq A$, then $G/C_G(A) \cong (G/A)/(C_G(A)/A)$ is a torsion group all of whose p-primary components are finite, ruling out that $G/A \cong \mathbb{Q}$. If, on the other hand, $C_G(A) = A$, then $G/A = G/C_G(A) \cong \mathbb{Q}$ is ruled out by Theorem 3.11(i) directly. Thus our assumption is equivalent to saying that G/A is isomorphic to a *proper* subgroup of $\mathbb{Q}$. We now proceed by considering the following mutually exclusive cases

(a) G/A is cyclic.
(b) G/A is noncyclic and A is infinite compact.
(c) G/A is noncyclic and A is a discrete torsion group.
(d) G/A is noncyclic and A is nondiscrete and noncompact.

We deal with each case in the following statements; at each point we formulate the respective result in the greatest possible generality.

Proposition 4.21. *Let G be a locally compact group with a normal open subgroup A and G/A is infinite cyclic. Then G contains a discrete cyclic subgroup H such that G is the semidirect product AH.*

Proof. Let $h \in G$ be an element such that Ah is one of the two generators of G/A and set $H = \langle h \rangle$. Then

$$n \mapsto Ah^n = (Ah)^n : \mathbb{Z} \to G/A$$

is an isomorphism of discrete topological groups since A is open, G/A is infinite cyclic, and Ah is a generator of G/A. Its kernel $A \cap H$ is therefore singleton, and $G = AH$. The assertion follows. $\square$

Corollary 4.22. *Let G be a near abelian group with periodic A and $G/A \cong \mathbb{Z}$ discrete. Then $G = AH$ is a semidirect product and, for every $p \in \pi(A)$, there is a unit $r_p \in \mathbb{Z}_p$ such that for all $a_p \in A_p$ we have $a_p^b = a_p^{r_p}$.*

Proof. The desired splitting follows immediately from Proposition 4.21. The second statement is implied by H acting by scalars. $\square$

Proposition 4.23. *Let G be a locally compact group with a compact open normal subgroup A such that G/A is isomorphic to an infinite subgroup of the discrete group $\mathbb{Q}$. Then G contains a discrete subgroup $H \cong G/A$ such that G is a semidirect product AH.*

Proof. There is an infinite subgroup $D \subseteq \mathbb{Q}$ and an isomorphism $\phi\colon G/A \to D$. Since the case that D is cyclic is covered by Proposition 4.21, we assume that D is not cyclic. Then there exist a sequence $n_1, n_2, \ldots$ of natural numbers, each dividing the next such that the cyclic groups $D_m := (1/n_m)\mathbb{Z}$ form an ascending sequence of subgroups of D whose union is D. Let $\rho\colon G \to G/A$ be the quotient morphism. We claim that there exists a morphism $\sigma\colon D \to G$ such that $\phi \circ \rho \circ \sigma = \mathrm{id}_D$, that is, that the following diagram commutes.

$$\begin{array}{ccc} D & \xrightarrow{\ \sigma\ } & G \\ {\scriptstyle \mathrm{id}_D}\downarrow & & \downarrow{\scriptstyle \rho} \\ D & \xleftarrow[\ \phi\ ]{} & G/A \end{array}$$

Then the subgroup $H = \sigma(D)$ of G satisfies the requirements.

We begin the proof by selecting an arbitrary function $\delta\colon D \to G$ such that $A\delta(d) = \phi^{-1}(d)$ and so $\phi \circ \rho \circ \delta = \mathrm{id}_D$. For each $m \in \mathbb{N}$ we define a function $\sigma_m\colon D \to G$ by

$$\sigma_m(d) = \begin{cases} \delta(\frac{1}{n_m})^k & \text{if } d = \frac{k}{n_m}, \\ \delta(d) & \text{if } d \notin D_m. \end{cases}$$

Since A is compact so is $A\delta(d) = \phi^{-1}(d)$ for each $d \in D$, and so the space $F := \prod_{d \in D} A\delta(d)$ is a compact subspace of the function space G^D. We observe that

(i) $\sigma_m(d) \in A\delta(d)$, that is, $\sigma_m \in F$,
(ii) $\phi \circ \rho \circ \sigma_m = \mathrm{id}_D$, and
(iii) $\sigma_m|D_m\colon D_m \to G$ is a morphism.

In the compact space F the sequence $(\sigma_m)_{m\in\mathbb{N}}$ has a subnet $(\sigma_{m_j})_{j\in J}$ with a limit $\sigma = \lim_{j\in J} \sigma_{m_j} \in F$, $\sigma\colon D \to G$ with $\sigma(d) \in A\delta(d)$. Thus $\rho\circ\sigma = \rho\circ\delta$, and so $\phi\circ\rho\circ\sigma = \phi\circ\rho\circ\delta = \mathrm{id}_D$. Now let $d, e \in D$. Then there is an $m \in \mathbb{N}$ such that $d, e \in D_m$. By the definition of a subnet there is a $j_0 \in J$ such that for all $j \geq j_0$ we have $m_j \geq m$, and thus $d, e \in D_{m_j}$. Therefore by (iii) above, we have $\sigma_{m_j}(d+e) = \sigma_{m_j}(d)\sigma_{m_j}(e)$ for all $j \geq j_0$. Therefore

$$\sigma(d+e) = \lim_{j>j_0} \sigma_{m_j}(d+e) = \lim_{j>j_0} \sigma_{m_j}(d)\sigma_{m_j}(e)$$
$$= \lim_{j\geq j_0} \sigma_{m_j}(d) \cdot \lim_{j\geq j_0} \sigma_{m_j}(e) = \sigma(d)\sigma(e).$$

Thus $\sigma\colon D \to G$ is a morphism and satisfies the conditions we need to complete the proof. □

Lemma 4.24. *There exists an abelian group G with the following properties:*

(i) *The torsion subgroup $A := \mathrm{tor}\, G$ is isomorphic to*
$$\bigoplus_{p=2,3,5,\dots} \mathbb{Z}(p).$$
(ii) *The torsion free quotient G/A is isomorphic to*
$$\bigcup_{p=2,3,5,\dots} \frac{1}{2\cdot 3\cdot 5 \cdots p}\cdot\mathbb{Z} \subseteq \mathbb{Q}.$$
(iii) *The torsion subgroup A is not a direct summand.*

The group G is a subgroup of $\mathbb{R}/\mathbb{Z}$.

Proof. This is a recent construction due to D. Maier [16] □

Maier's construction was originally established to obtain the compact dual $\widehat{G}$, which is a compact metric monothetic group whose identity component is not a direct factor. For us it is significant that

the group G is a trivial near abelian group with a (discrete) periodic subgroup A such that G/A is isomorphic to a proper subgroup of the discrete group $\mathbb{Q}$ and that G does not possess a subgroup H such that G is a (semi-)direct product AH.

This is an indication that the chances of generalizing Propositions 4.21 and 4.23 in a reasonable fashion to nontrivial near abelian groups with torsion free discrete factor group G/A are slim. What remains for us to do is to describe in general terms for a noncyclic subgroup D of $\mathbb{Q}$ the extension

$$1 \to A \to G \to D \to 1.$$

We saw that each noncyclic subgroup $D \subseteq \mathbb{Q}$ can be represented as an ascending union $\bigcup_{m\in\mathbb{N}} \frac{1}{n_m}\cdot\mathbb{Z}$ of cyclic groups for a suitable sequence $n_1|n_2|n_3|\cdots$. There is no loss in generality to assume $n_1 = 1$, and since each quotient n_{m+1}/n_m is a product of prime powers, there is no loss to assume further that $n_m = p_1 \cdots p_{m-1}$ for $m = 2, 3, \dots$ for a sequence of prime numbers p_m. If $\phi\colon G/A \to D$ is any isomorphism, we recall that we can select, for each $m \in \mathbb{N}$ an element $b_m \in G$ such that $\phi(Ab_m) = \frac{1}{p_1\cdots p_m} \in D$. The cyclic subgroups $H_m = \langle b_m\rangle$ of G give rise to an ascending sequence of open subgroups $G_m = AH_m$, each being a semidirect product, whose union is G. Notice that $\phi(G_m/A) = \frac{1}{p_1\cdots p_m}\cdot\mathbb{Z} \subseteq D$ for $m \in \mathbb{N}$. Since the choice of the elements b_m was arbitrary in Ab_m and therefore the choice of the subgroups H_m making up the groups $G_m = AH_m$ is arbitrary to a large extent, we need a description of an apparatus to describe how the H_m are related. We describe one such in the following. We shall focus on the p-primary components A_p of A and the action of G/A on A_p by scalar automorphisms.

For this purpose we recall that A_p is a $\mathbb{Z}_p$-module. (Cf. Theorem 2.11, [11], notably 2.23ff.). If A_p has a finite exponent, we denote it by e_p. Recall Definition 2.7 where we defined

$$R(A_p) = \begin{cases} \mathbb{Z}(e_p) & \text{if } A_p \text{ has finite exponent,} \\ \mathbb{Z}_p & \text{otherwise} \end{cases}$$

to be the *ring of scalars* of A_p and noted that its group of units $R(A_p)^\times$ is naturally isomorphic to $\mathrm{SAut}(A_p)$. It may be useful to remind the reader that A_p, as an $R(A_p)$-module is usually written additively, while in the present context as a subgroup of of the near abelian group G is is written multiplicatively with the consequence that a scalar $r \in R(A)^\times$ acting on an element $a \in A_p$ is written as an exponent, like a^r.

We finally obtain a structure result comprising cases (b) and (d).

Proposition 4.25. *Suppose that G is a near abelian group with an open periodic base group A such that G/A is a discrete torsion free group isomorphic to a noncyclic subgroup of $\mathbb{Q}$. Then for each $m \in \mathbb{N}$, there are primes p_m, elements $b_m \in G \setminus A$, $a_m = (a_{mp})_p \in A$ and, for every fixed prime p, units $r_{mp} \in R(A_p)^\times$, such that for $H_m = \langle b_m\rangle$ and $G_m = AH_m$ we have*

(1) $(G_m : A) = p_1 \cdots p_m$, $m = 1, 2, \ldots$.
(2) $r_{m+1,p}^{p_m} = r_{mp}$ *in* $R(A_p)^\times$,

(i) $b_{m+1}^{p_m} = a_m b_m$,
(ii) $(a_m b_m)^{b_{m+1}} = a_m b_m$,
(iii) $(\forall a \in A_p)\, a^{b_m} = a^{r_{pm}}$.

For establishing a converse, assume that we have a sequence $(p_m)_{m\in\mathbb{N}}$ *of prime numbers, a sequence* $(a_m)_{m\in\mathbb{N}}$ *of elements of a periodic abelian group* A*, and a sequence of generators* $(b_m)_{m\in\mathbb{N}}$ *subject to the relations* (i), (ii)*, and* (iii)*.*

Then there is a group G *containing the elements* b_m *and* A *as an open subgroup such that* G *is a near abelian group with an open base group* A *such that* G/A *is a discrete rank 1 torsion free noncyclic abelian group.*

Proof. Certainly $A = \operatorname{comp}(A)$ by assumption, and since G/A is discrete torsion free, $A = \operatorname{comp}(G)$ follows. On the other hand, G/A being inductively monothetic, the structure theorem Theorem 3.11 shows that this can only be true if G/A is isomorphic to a discrete subgroup of $\mathbb{Q}$. Therefore $A = \operatorname{comp}(G)$ is abelian and $G/\operatorname{comp}(G)$ is a discrete rank one group.

Since G/A is discrete, torsion free, and, of rank one, there is a sequence of primes $(p_m)_m$ and a sequence $(Ab_m)_{m\in\mathbb{N}}$ of cosets in G/A such that

$$(*) \qquad (Ab_{m+1})^{p_m} = Ab_m,$$

and there is an ascending sequence of subgroups $G_m = A\langle b_m\rangle$ whose union is G and which satisfy $(G_m : G_{m+1}) = p_m$ for all $m \in \mathbb{N}$. We note that $(G_m : A) = \prod_{j=1}^{m-1}(G_{j+1} : G_j) = p_1 \cdots p_m$, yielding (1).

The relation (2) is equivalent to

$$(\forall m \in \mathbb{N})(\exists a_m \in A)\, b_{m+1}^{p_m} = a_m b_m.$$

This gives us the sequence $(a_m)_{m\in\mathbb{N}}$ satisfying (i). From (i) we see that $a_m b_m = b_{m+1}^{p_m}$ is fixed under the inner automorphism $x \mapsto x^{b_{m+1}}$, and this implies (ii). Now G/A acts on A via inner automorphisms by scalar multiplication since A is a base group of the near abelian group A. Hence

$$(\forall p \text{ prime})(\forall m \in \mathbb{N})(\exists r_{pm} \in R(A_p))(\forall a \in A_p)\ a^{b_m} = a^{r_{pm}}.$$

As a consequence, we have for all $a \in A_p$

$$a^{r_{p,m+1}^{p_m}} = a^{b_{m+1}^{p_m}} = a^{a_m b_m} = a^{r_{pm}}$$

and so $r_{p,m+1}^{p_m} = r_{p,m}$ in $R(A)$, that is, (iii) follows.

For proving the converse we construct an ascending sequence of groups $G_0 \subseteq G_1 \subseteq G_2 \subseteq \cdots$ recursively by beginning with $G_0 := A$.

Assume now that groups $G_m := A\langle b_1, \ldots, b_m\rangle$, have been constructed for some $m = 0, \ldots$.

If $m = 0$ then relation (iii) defines an automorphism of A to become an inner automorphism implemented by b_1. When $m \geq 1$ then (iii) and (ii) together define an automorphism of $G_m = A\langle a_m b_m\rangle$ to become an inner automorphism implemented by b_{m+1} that fixes $a_m b_m$ and whose p_m-th power, as (i) shows, is the inner automorphism induced by $a_m b_m$.

Thus there exists a topological group G_{m+1} containing G_m as an open subgroup and containing a cyclic subgroup $\langle b_{m+1}\rangle \cong \mathbb{Z}$ such that the relation $b_{m+1}^{p_m} = a_m b_m$ is satisfied.

The tower
$$A = G_0 \subseteq G_1 \subseteq G_2 \subseteq \cdots$$
has a union (a direct limit)
$$G := \bigcup_{m=0}^{\infty} G_m$$
which satisfies all requirements.

The additional statement in (2) is a consequence of Proposition 4.23, namely, that $G = AH$ is the product of the compact group A and a discrete group H isomorphic to a subgroup of $\mathbb{Q}$. □

Remark 4.26. If in Proposition 4.25 we assume, in addition, that A is compact, then the elements b_m can be found is such a way that statements (i), (ii), and (iii) hold with $a_m = 1$ for all $m \in \mathbb{N}$.

Proof. The remark is a consequence of Proposition 4.23, namely, that $G = AH$ is the product of the compact group A and a discrete group H isomorphic to a subgroup of $\mathbb{Q}$. □

4.6. **Near abelian p-groups.** The main result in [11] on compact near abelian p–groups extends as follows. Recall that every abelian p-group is a natural $\mathbb{Z}_p$-module and also recall the Definition 2.7 of the ring of scalars of a p-group.

Proposition 4.27. *Let G be a near abelian p-group and A a base group. Then G/A is monothetic. There is a monothetic scaling subgroup $H = \overline{\langle b\rangle}$ of G such that $G = AH$, and, there is a p-element $r \in R(A)^\times$ in the group of units of the ring of scalars of A such that for all $a \in A$ we have $a^b = a^r$.*

If G/A is infinite, then H is isomorphic to $\mathbb{Z}_p$ and $G = AH$ is a semidirect product.

Proof. Theorem 3.11 says that, as a p-group, G/A cannot be of type (i). If its of type (ii), then it is a finite cyclic group, and if it is of type (iii), then it is isomorphic to $\mathbb{Z}_p$, and that conclusion maintains to type (iv) as well. So G/A is monothetic and Proposition 4.8 applies. Since A and H are p-groups, we may assume that $r \in R(A)$.

If G/A is infinite, then the monothetic groups H and G/A are both isomorphic to $\mathbb{Z}_p$, and since $\mathbb{Z}_p$ has no proper infinite quotient groups, the morphism $h \mapsto Ah :: H \to G/A$ is an isomorphism, and so $A \cap H = \{1\}$. This shows that AH is a semidirect product. □

5. Topologically quasihamiltonian groups

Generalizations of the notion of *quasihamiltonian* to locally compact groups will be studied.

5.1. **Generalities.** Recall the definition of a topologically quasihamiltonian group from [14] and a stronger condition, introduced by Y. Mukhin in [18].

Definition 5.1. Let G be a topological group. Then G is a *topologically quasihamiltonian* group, for short a TQHG, if for any pair X and Y of closed subgroups we have $\overline{XY} = \overline{YX}$. It is a *strongly* TQHG if, in addition, XY is a closed subgroup.

As Kümmich further remarks, a group is a TQHG if and only if for any closed subgroups X and Y the set $\overline{XY}$ is a subgroup of G.

We collect some information about TQHG-s from the cited reference.

Proposition 5.2. *The class of* TQHG-*s is closed under forming subgroups, continuous homomorphic images and projective limits with open bonding maps and compact kernels. Moreover, a nonabelian* TQHG *is zero dimensional.*

Proof. This is the contents of Hilfsatz 2, 3 and 4 in [14]. □

The set of closed subgroups of a topological group G is a *lattice* when definining for closed subgroups X and Y the operations $X \vee Y := \overline{\langle X, Y \rangle}$ and $X \wedge Y := X \cap Y$. Recall from [22] that G is *Dedekind* if for all X, Y, and, Z in the lattice with $X \subseteq Y$ the modular law

$$X \vee (Y \wedge Z) = (X \vee Y) \wedge Z$$

holds.

Lemma 5.3. *Let G be nonabelian* TQHG*. Then the following properties hold*

(1) *If G is a strongly* TQHG *then G is Dedekind.*

(2) $\mathrm{comp}(G)$ *is an open characteristic subgroup of* G.
(3) *If* $G \neq \mathrm{comp}(G)$ *then* $\mathrm{comp}(G)$ *is abelian and* $G/\mathrm{comp}(G)$ *is torsion free of rank* 1. *In particular,* G *is near abelian.*

Proof. (1) The Dedekind property for subgroups $X \subseteq Z$ and Y of G is given as $X \vee (Y \cap Z) = (X \vee Y) \cap Z$ and becomes equivalent to $X(Y \cap Z) = (XY) \cap Z$.

(2) As G is TQHG, its compact elements form a subgroup of G which evidently is characteristic. Proposition 5.2 implies that $G_0 = 1$. Therefore [10, Theorem 10.89] implies the existence of an open compact subgroup G_1 of G showing that $\mathrm{comp}(G)$ is an open subgroup of G.

(3) Fix any $k \in \mathrm{comp}(G)$ and let us show that $\overline{\langle k \rangle}$ is normalized by any element $x \notin \mathrm{comp}(G)$. Since $\overline{\langle x, k \rangle} = \overline{\langle x \rangle}\,\overline{\langle k \rangle}$ is a group and as $\overline{\langle x \rangle} \cap \overline{\langle k \rangle} = 1$ conclude that $L := \overline{\langle x \rangle}\,\overline{\langle k \rangle}$ is closed. Therefore $L = \bigcup_{n \in \mathbb{Z}} x^n \overline{\langle k \rangle}$ is a coset decomposition. On the other hand $\langle x \rangle \cap \mathrm{comp}(L) = 1$ as well so that $\overline{\langle k \rangle} = \mathrm{comp}(L)$ follows. For showing that $\mathrm{comp}(G)$ is abelian, fix k and l in $\mathrm{comp}(G)$ and consider $T := \overline{\langle x, k, l \rangle} = \langle x \rangle \overline{\langle k, l \rangle}$. Since $\mathrm{comp}(T) = \overline{\langle k, l \rangle}$ is normal and finitely generated it has arbitrarily small open T-invariant normal subgroups. When N denotes such a subgroup we find that T/N is a quasihamiltonian discrete group and so $\mathrm{comp}(T)/N$ is torsion and hence, by [22, 2.4.8 Lemma] it is abelian. A standard projective limit argument implies that $\mathrm{comp}(T)$ is abelian and, since k and l were picked arbitrarily, $\mathrm{comp}(G)$ is abelian. For showing that $G/\mathrm{comp}(G)$ is torsion free of rank 1 it suffices to factor any open normal subgroup N of $\mathrm{comp}(G)$ and then to apply [22, 2.4.11]. ☐

5.2. **The structure of periodic** TQHG**-s.**

Lemma 5.4. *Let* G *be a periodic* TQHG *and* p *and* q *be distinct primes. Then in* G *every* p*-element* x *and* q*-element* y *commute.*

Proof. The finitely generated subgroup $L := \overline{\langle x, y \rangle}$ of G is a TQHG by Proposition 5.2 and therefore it is compact. Since any of its finite quotients is quasihamiltonian and hence nilpotent it follows that $\overline{\langle x, y \rangle} \cong \overline{\langle x \rangle} \times \overline{\langle y \rangle}$ is abelian, as claimed. ☐

Proposition 5.5. *Let* p *be a prime and* G *a near abelian* p*-group. Then* G *is a* TQHG *if and only if in the description of Proposition 4.27 the exponent* $r \in R(\mathbb{Z}_p)$ *can be taken to be* $r = 1 + p^s$ *for* $s \geq 1$ *if* p *is odd and* $s \geq 2$ *if* $p = 2$.

Proof. Replacing b by a suitable power (in $R(\mathbb{Z}_p)$) we can assume $r \in \mathbb{N}$. Then G contains a noncommutative compact subgroup $A_1\overline{\langle b \rangle}$ which must be a topologically quasihamiltonian. Now the result follows from passing to a noncommutative finite image and using Iwasawa's Theorem [22, 2.3.1]. □

Theorem 5.6. *Let G be a nonabelian periodic* TQHG*. Then G is a local product $\prod_p^{\mathrm{loc}}(G_p : C_p)$ where the G_p is a p-primary subgroup of G. In particular, $G = AH$ is near abelian where $H = \overline{\langle H_p \mid p \in \pi(G)\rangle}$ where H_p is any scaling group of $G_p = A_pH_p$. Conversely, every local product $G = \prod_p^{\mathrm{loc}}(G_p : C_p)$ of p-primary groups each a* TQHG*, is a* TQHG*.*

Proof. Lemma 5.4 implies that any primary groups commute. Since G is locally compact [10, Theorem 10.89] implies the existence of an open subgroup C of G. Thus $G = \prod_p^{\mathrm{loc}}(G_p : C_p)$ as claimed. Proposition 5.5 shows that $G_p = A_pH_p$ is near abelian. Put $A := \overline{\langle A_p \mid p \in \pi(G)\rangle}$ and $H := \overline{\langle H_p \mid p \in \pi(G)\rangle}$ then $G = AH$ is near abelian with base group A and scaling group H.

For proving the converse we first remark that every compact product $C = \prod_p C_p$ with C_p compact TQHG, is itself TQHG. Now consider closed subgroups X and Y of $G = \prod_p^{\mathrm{loc}}(G_p : C_p)$. Then, letting $(C_i)_i$ be any set of open compact subgroups of G we find

$$\begin{aligned}
\overline{\langle XY\rangle} &= \overline{\langle (\textstyle\bigcup_i (X\cap C_i))(\bigcup_j (Y\cap C_j))\rangle} \\
&= \overline{\langle \textstyle\bigcup_{i,j} \overline{\langle (X\cap C_i)(Y\cap C_j)\rangle}\rangle} \\
&= \overline{\langle \textstyle\bigcup_{i,j} \overline{\langle (Y\cap C_i)(X\cap C_j)\rangle}\rangle} \\
&= \overline{\langle \textstyle\bigcup_{i,j} (Y\cap C_i)(X\cap C_j)\rangle} \\
&= \overline{\langle YX\rangle}
\end{aligned}$$

and hence such G is a TQHG. □

5.3. **The structure of nonperiodic** TQHG**-s.**

Theorem 5.7. *Let G be a nonabelian topologically quasihamiltonian-group containing a discrete subgroup isomorphic to $\mathbb{Z}$. Then the structure of G is described in Proposition 4.25 with $r_2 \equiv 4 \pmod 1$ in situation (a) and $r_{m,2} \equiv 1 \pmod 4$ for (b).*

Proof. Lemma 5.3 shows that G is near abelian. According to Proposition 5.2 the projective limit $L = \varprojlim_K L/K$ with K running through a filtered system of normal compact subgroups and each L/K topologically quasihamiltonian, is itselft topologically quasihamiltonian. Therefore it suffices to restrict to discrete G since our group is the projective limit $\varprojlim_K G/K$ for K running through a system of open compact subgroups K of A. So assume that G is discrete. Then A being periodic is locally finite and the result follows at once from Iwasawa's Theorem [22, 2.4.11]. □

Remark 5.8. In [12] we shall show that indeed a TQHG is either of the form $Q_8 \times L$ for L having exponent 2 or is near abelian. Taking this into account we have a complete description of all TQHG-s.

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