ASC Report No. 36/2013

Optimal preconditioning for the symmetric and non-symmetric coupling of adaptive finite elements and boundary elements

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www.asc.tuwien.ac.at ISBN 978-3-902627-05-6

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ISBN 978-3-902627-05-6

ASC TU WIEN

OPTIMAL PRECONDITIONING FOR THE SYMMETRIC AND NON-SYMMETRIC COUPLING OF ADAPTIVE FINITE ELEMENTS AND BOUNDARY ELEMENTS

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ABSTRACT. We analyze a multilevel diagonal additive Schwarz preconditioner for the adaptive coupling of FEM and BEM for a linear 2D Laplace transmission problem. We rigorously prove that the condition number of the preconditioned system stays uniformly bounded, independently of the refinement level and the local mesh-size of the underlying adaptively refined triangulations. Although the focus is on the non-symmetric Johnson-Nédélec one-equation coupling, the principle ideas also apply to other formulations like the symmetric FEM-BEM coupling. Numerical experiments underline our theoretical findings.

1. INTRODUCTION

There exist plenty of works on preconditioning of FEM-BEM coupling equations, covering mainly the symmetric coupling with quasi-uniform meshes, see [CKL98, FS09, HPPS03, HMS99, HS98, KS02, MS98] and the references therein. In contrast to that, only little is known on preconditioning of the non-symmetric Johnson-Nédélec coupling, see e.g. [Med98], and also on preconditioning of *adaptive* FEM-BEM couplings. It is the main goal of this paper to close this gap and to extend the existing analysis to the case of the (adaptive) symmetric as well as non-symmetric Johnson-Nédélec coupling [JN80]. For the symmetric coupling [Cos88, Han90], the approach of Bramble & Pasciak [BP88] applies, which guarantees positive definiteness and symmetry of the Galerkin matrix with respect to a special inner product [CKL98, HPPS03, KS02]. Therefore, efficient iterative solvers designed for symmetric and positive definite matrices are applicable. However, due to the non-symmetry, such an approach may not work for the Johnson-Nédélec coupling in general. In [Med98], it was assumed that the coupling boundary is smooth. Hence, the double-layer integral operator $\mathcal{K}$ is compact. The system matrix can therefore be split into a symmetric part plus a compact perturbation part $\mathbf{K}$ (the Galerkin matrix of the double-layer integral operator). Preconditioning is done only on the symmetric part with the theory of [BP88], and convergence results for iterative solvers can then be obtained by compact perturbation theory assuming that the mesh-size of the coarsest mesh is sufficiently small. In general, however,

Date: November 22, 2013.

2010 *Mathematics Subject Classification.* 65N30, 65N38, 65F08.

Key words and phrases. FEM-BEM coupling, preconditioner, multilevel additive Schwarz, adaptivity.

Acknowledgment: The research of the authors MF, TF, and DP is supported through the FWF research project*Adaptive Boundary Element Method*, see `http://www.asc.tuwien.ac.at/abem/`, funded by the Austrian Science Fund (FWF) under grant P21732, as well as through the *Innovative Projects Initiative* of Vienna University of Technology. Moreover, MF and DP are partially funded through the FWF doctoral school *Dissipation and Dispersion in Nonlinear PDEs*, funded under grant W1245. This support is thankfully acknowledged.

the coupling boundary is not smooth. Therefore, the preconditioner theory must not rely on compactness of $\mathcal{K}$.

To state the contributions of the current work, we consider the non-symmetric stiffness matrix of the (stabilized) Johnson-Nédélec coupling, which reads in block-form

$$\mathbf{A}_{\mathcal{L}} := \begin{pmatrix} \mathbf{A}_A & -\mathbf{M}^T \\ \frac{1}{2}\mathbf{M} - \mathbf{K} & \mathbf{A}_{\mathcal{V}} \end{pmatrix} + \mathbf{S}\mathbf{S}^T \in \mathbb{R}^{(N+M)\times(N+M)}, \tag{1}$$

see Section 2 below. Here, $\mathbf{S} \in \mathbb{R}^{N+M}$ denotes an appropriate stabilization vector, which ensures positive definiteness of $\mathbf{A}_{\mathcal{L}}$. The $N\times N$ matrix block $\mathbf{A}_A$ is the (positive semi-definite) Galerkin matrix of the FEM part, and the $M \times M$ matrix block $\mathbf{A}_{\mathcal{V}}$ is the Galerkin matrix of the simple-layer integral operator $\mathcal{V}$. As in [FS09, MS98], we deal with block-diagonal preconditioners of the form

$$\mathbf{P}_{\mathcal{L}} = \begin{pmatrix} \mathbf{P}_{\mathcal{A}} & \mathbf{0} \\ \mathbf{0} & \mathbf{P}_{\mathcal{V}} \end{pmatrix}. \tag{2}$$

Here, the appropriate operator $\mathcal{A} : H^1(\Omega) \to (H^1(\Omega))^*$ induces a coercive, symmetric, and bounded bilinear form $\langle \mathcal{A}(\cdot)\,,\,(\cdot)\rangle \simeq \|\cdot\|^2_{H^1(\Omega)}$. The symmetric and positive definite matrices $\mathbf{P}_{\mathcal{A}}$ resp. $\mathbf{P}_{\mathcal{V}}$ are spectrally equivalent to the symmetric and positive definite Galerkin matrices $\mathbf{A}_{\mathcal{A}}$ resp. $\mathbf{A}_{\mathcal{V}}$, i.e.

$$d_{\mathcal{A}}\langle \mathbf{P}_{\mathcal{A}}\boldsymbol{X}\,,\,\boldsymbol{X}\rangle_2 \le \langle \mathbf{A}_{\mathcal{A}}\boldsymbol{X}\,,\,\boldsymbol{X}\rangle_2 \le D_{\mathcal{A}}\langle \mathbf{P}_{\mathcal{A}}\boldsymbol{X}\,,\,\boldsymbol{X}\rangle_2 \quad \text{for all } \boldsymbol{X} \in \mathbb{R}^N, \tag{3a}$$

$$d_{\mathcal{V}}\langle \mathbf{P}_{\mathcal{V}}\mathbf{\Phi}\,,\,\mathbf{\Phi}\rangle_2 \le \langle \mathbf{A}_{\mathcal{V}}\mathbf{\Phi}\,,\,\mathbf{\Phi}\rangle_2 \le D_{\mathcal{V}}\langle \mathbf{P}_{\mathcal{V}}\mathbf{\Phi}\,,\,\mathbf{\Phi}\rangle_2 \quad \text{for all } \mathbf{\Phi} \in \mathbb{R}^M, \tag{3b}$$

where $\mathbf{A}_{\mathcal{A}}$ is strongly related to the FEM block $\mathbf{A}_A$ of the FEM-BEM system (1), see (23)–(25) below. Inspired by [MS98], we prove that the condition number of $\mathbf{P}_{\mathcal{L}}^{-1}\mathbf{A}_{\mathcal{L}}$ as well as the number of iterations to reduce the relative residual by a factor τ in the preconditioned GMRES algorithm with inner product $\langle\cdot\,,\,\cdot\rangle_{\mathbf{P}_{\mathcal{L}}}$ depends only on $\max\{D_{\mathcal{A}}, D_{\mathcal{V}}\}/\min\{d_{\mathcal{A}}, d_{\mathcal{V}}\}$.

Usually, the condition number of Galerkin matrices $\mathbf{A}_{\mathcal{A}}$ and $\mathbf{A}_{\mathcal{V}}$ on adaptively refined meshes hinges on the global mesh-ratio as well as on the number of degrees of freedom. Therefore, the construction of optimal preconditioners for iterative solvers is a necessary task. Here, optimality is understood in the sense that the condition number of the preconditioned matrix is independent of the mesh-size and the degrees of freedom.

Very recently, it was proven in [XCH10] for 2D FEM with energy space H^1 that a local multilevel additive Schwarz preconditioner $\mathbf{P}_{\mathcal{A}}$ is optimal, i.e. the constants $d_{\mathcal{A}}, D_{\mathcal{A}}$ in (3) are independent of the mesh-size, the number of degrees of freedom, and the number of levels. Here, "local" means, that scaling at each level is done only on newly created nodes plus neighbouring nodes, where the associated basis functions have changed. An analogous result for 2D and 3D hypersingular integral equations with energy space $H^{1/2}$ has been derived by the authors [FFPS13]. In [FFPS13, XCH10], the proofs rely on a stable space decomposition of the discrete subspaces in H^1 resp. $H^{1/2}$. Alternatively, [XCN09] provides stable subspace decompositions in H^1 for higher order elements in any dimension on bisection grids.

In the present work, we prove the optimality of some local multilevel additive Schwarz preconditioner $\mathbf{P}_{\mathcal{V}}$ for 2D weakly-singular integral equations with energy space $H^{-1/2}$. The proof is derived by postprocessing of the corresponding result for the hypersingular integral equation [FFPS13]. Combining this with the result of [XCH10], we prove optimality of $\mathbf{P}_{\mathcal{L}}$ for the FEM-BEM coupling.

Notation. Throughout the work, we explicitly state all constants and their dependencies in all statements of results. In proofs, however, we use the notation $a \lesssim b$ to abbreviate $a \le Cb$ with a constant $C > 0$ which is clear from the context. Moreover, $a \simeq b$ abbreviates $a \lesssim b \lesssim a$. Furthermore, the entries of a vector $\mathbf{b}$ or a matrix $\mathbf{A}$ are denoted by $(\mathbf{b})_j$ resp. $(\mathbf{A})_{jk}$. By $\langle \cdot\,,\,\cdot \rangle$, we denote the duality brackets between a Hilbert space $\mathcal{H}$ and its dual $\mathcal{H}^*$.

Outline. The remainder of this work is organized as follows: In Section 2, we recall the basic facts on the Johnson-Nédélec coupling and define the admissible mesh-refinement strategies. We also formulate the preconditioned GMRES Algorithm 3, which is required to state the main result (Theorem 4). In Section 3, we prove the spectral estimates (3). Section 4 adapts the analysis for the symmetric coupling [MS98] and contains the proof of the main result (Theorem 4). The short Section 5 deals with extensions of the developed theory to the symmetric coupling and the one-equation Bielak-MacCamy coupling. Numerical examples from Section 6 underline our theoretical predictions and conclude the work.

2. Johnson-Nédélec coupling and main result

2.1. Model problem and analytical setting. Let $\Omega \subset \mathbb{R}^2$ be a polygonal and simply connected domain with boundary $\Gamma = \partial\Omega$. We consider the following Laplace transmission problem in free space:

$$-\operatorname{div}(A\nabla u) = f \qquad \text{in } \Omega, \tag{4a}$$

$$-\Delta u^{\text{ext}} = 0 \qquad \text{in } \Omega^{\text{ext}} := \mathbb{R}^2 \backslash \overline{\Omega}, \tag{4b}$$

$$u - u^{\text{ext}} = u_0 \qquad \text{on } \Gamma, \tag{4c}$$

$$(A\nabla u - \nabla u^{\text{ext}}) \cdot \boldsymbol{n} = \phi_0 \qquad \text{on } \Gamma, \tag{4d}$$

$$|u^{\text{ext}}(x)| = \mathcal{O}(1/|x|) \qquad \text{as } |x| \to \infty. \tag{4e}$$

Here, $\boldsymbol{n}$ denotes the outer normal on Γ, and $A \in L^\infty(\Omega)$ satisfies $A(x) \in \mathbb{R}^{2\times 2}_{\text{sym}}$ with uniform bounds on the maximal resp. minimal eigenvalue

$$0 < c_A := \operatorname*{ess\,inf}_{x\in\Omega} \lambda_{\min}(A(x)) \le \operatorname*{ess\,sup}_{x\in\Omega} \lambda_{\max}(A(x)) =: C_A < \infty. \tag{5}$$

With $H^1(\Omega)$ resp. $H^{1/2}(\Gamma)$ and its dual $H^{-1/2}(\Gamma) = H^{1/2}(\Gamma)^*$, we denote the usual Sobolev spaces on Ω resp. Γ. For given data $f \in L^2(\Omega)$, $u_0 \in H^{1/2}(\Gamma)$, and $\phi_0 \in H^{-1/2}(\Gamma)$, it is well-known that the model problem (4) admits a unique solution $u \in H^1(\Omega)$, $u^{\text{ext}} \in H^1_{\text{loc}}(\Omega^{\text{ext}})$ with finite energy $\nabla u^{\text{ext}} \in L^2(\Omega^{\text{ext}})$, if we impose the compatibility condition

$$\langle f\,,\,1\rangle_\Omega + \langle \phi_0\,,\,1\rangle_\Gamma = 0 \tag{6}$$

to ensure the radiation condition (4e). Here, $\langle \cdot\,,\,\cdot\rangle_\Omega$ stands for the $L^2(\Omega)$ inner product, whereas $\langle \cdot\,,\,\cdot\rangle_\Gamma$ denotes the extended $L^2(\Gamma)$ inner product.

2.2. Johnson-Nédélec coupling. For the formulation of the Johnson-Nédélec coupling [JN80], the exterior solution (4b) is formulated by Green's third formula. The latter gives rise to the simple-layer and double-layer integral operator

$$\mathcal{V} \in L(H^{-1/2+s}(\Gamma); H^{1/2+s}(\Gamma)), \quad \mathcal{V}\phi(x) := \int_\Gamma G(x,y)\phi(y)\,dy, \tag{7a}$$

$$\mathcal{K} \in L(H^{1/2+s}(\Gamma); H^{1/2+s}(\Gamma)), \quad \mathcal{K}v(x) := \int_\Gamma \partial_{\boldsymbol{n}(y)} G(x,y)\,v(y)\,dy, \tag{7b}$$

where $G(x,y) := -\frac{1}{2\pi}\log|x-y|$ denotes the fundamental solution of the 2D Laplacian and $\partial_{\boldsymbol{n}}(\cdot)$ is the normal derivative. Note that boundedness holds for all $-1/2 \le s \le 1/2$. In addition, our analysis requires the hypersingular integral operator

$$\mathcal{W} \in L(H^{1/2+s}(\Gamma); H^{-1/2+s}(\Gamma)), \quad \mathcal{W}v(x) := -\partial_{\boldsymbol{n}(x)} \int_\Gamma \partial_{\boldsymbol{n}(y)} G(x,y)\,v(y)\,dy, \tag{7c}$$

where the integral is understood as finite part integral. It is known that $\mathcal{V}$ and $\mathcal{W}$ are symmetric in the sense of

$$\langle \phi\,,\,\mathcal{V}\psi\rangle_\Gamma = \langle \psi\,,\,\mathcal{V}\phi\rangle_\Gamma \quad \text{and} \quad \langle \mathcal{W}v\,,\,w\rangle_\Gamma = \langle \mathcal{W}w\,,\,v\rangle_\Gamma \tag{8}$$

for all $\phi, \psi \in H^{-1/2}(\Gamma)$ and $v, w \in H^{1/2}(\Gamma)$. Moreover, the assumption $\mathrm{diam}(\Omega) < 1$ ensures $H^{-1/2}(\Gamma)$-ellipticity of $\mathcal{V}$,

$$c_{\mathcal{V}}\,\|\psi\|^2_{H^{-1/2}(\Gamma)} \le \langle \psi\,,\,\mathcal{V}\psi\rangle_\Gamma \quad \text{for all } \psi \in H^{-1/2}(\Gamma). \tag{9}$$

Finally, $\mathcal{W}$ is semi-elliptic with kernel being the constant functions,

$$\mathcal{W}1 = 0 \quad \text{and} \quad c_{\mathcal{W}}\,\|v\|^2_{H^{1/2}(\Gamma)} \le \langle \mathcal{W}v\,,\,v\rangle_\Gamma \quad \text{for all } v \in H^{1/2}(\Gamma) \text{ with } \int_\Gamma v\,dx = 0. \tag{10}$$

The constants $c_{\mathcal{V}}, c_{\mathcal{W}} > 0$ depend only on Ω.

With the definitions (7) of the layer integral operators, the model problem (4) is equivalently recast by the Johnson-Nédélec coupling: Given $f \in L^2(\Omega)$, $u_0 \in H^{1/2}(\Gamma)$, and $\phi_0 \in H^{-1/2}(\Gamma)$, find $(u,\phi) \in \mathcal{H} := H^1(\Omega) \times H^{-1/2}(\Gamma)$ such that

$$\langle A\nabla u\,,\,\nabla v\rangle_\Omega - \langle \phi\,,\,v\rangle_\Gamma = \langle f\,,\,v\rangle_\Omega + \langle \phi_0\,,\,v\rangle_\Gamma, \tag{11a}$$

$$\langle \psi\,,\,(\tfrac{1}{2}-\mathcal{K})u + \mathcal{V}\phi\rangle_\Gamma = \langle \psi\,,\,(\tfrac{1}{2}-\mathcal{K})u_0\rangle_\Gamma \tag{11b}$$

for all $(v,\psi) \in \mathcal{H}$. Setting $(u,\phi) = (1,0) = (v,\psi)$ in (11), we see that the (non-stabilized) linear operator associated to the left-hand side of (11) is indefinite. However, let $1/2 \le c_{\mathcal{K}} < 1$ denote the contraction constant of the double-layer potential [SW01]. Following the analysis in [OS13, Say09, Ste11], also (stabilized) Galerkin formulations of (11) admit unique solutions, if the ellipticity constant c_A from (5) satisfies

$$c_A \ge c_{\mathcal{K}}/4 \tag{12}$$

and if the equation is either *explicitly stabilized* [OS13, Ste11] or if the discrete subspace of $H^{-1/2}(\Gamma)$ contains the constant functions [Say09]. In [AFF+13a], the result of [Say09] is reproduced with a new proof. Introducing the notion of *implicit stabilization*, an equivalent elliptic operator equation of (11) is derived which fits in the frame of the Lax-Milgram lemma and thus leads to non-symmetric, but positive definite Galerkin matrices. The main result of [AFF+13a] reads as follows (and also holds for a strongly monotone, but nonlinear material tensor A):

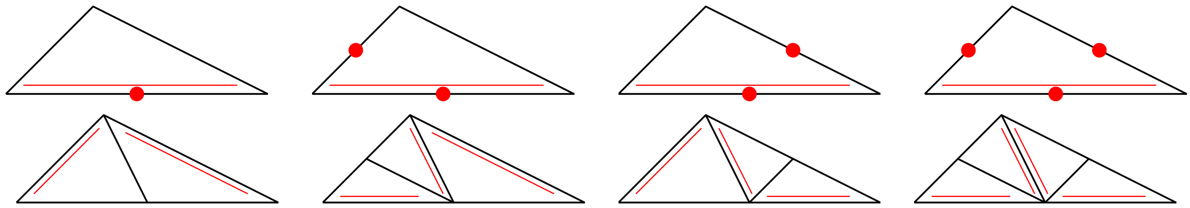

FIGURE 1. For each triangle $T \in \mathcal{T}_\ell^\Omega$, there is one fixed *reference edge*, indicated by the double line (left, top). Refinement of T is done by bisecting the reference edge, where its midpoint becomes a new node. The reference edges of the son triangles are opposite to this newest vertex (left, bottom). To avoid hanging nodes, one proceeds as follows: We assume that certain edges of T, but at least the reference edge, are marked for refinement (top). Using iterated newest vertex bisection, the element is then split into 2, 3, or 4 son triangles (bottom).

Lemma 1. *Let* (12) *be satisfied. For* $(u,\phi),(v,\psi)\in\mathcal{H}=\mathcal{X}\times\mathcal{Y}$, *define*

$$\begin{aligned}\langle\mathcal{L}(u,\phi)\,,\,(v,\psi)\rangle := \langle A\nabla u\,,\,\nabla v\rangle_\Omega - \langle\phi\,,\,v\rangle_\Gamma + \langle\psi\,,\,(\tfrac12-\mathcal{K})u+\mathcal{V}\phi\rangle_\Gamma \\ + \langle 1\,,\,(\tfrac12-\mathcal{K})u+\mathcal{V}\phi\rangle_\Gamma\langle 1\,,\,(\tfrac12-\mathcal{K})v+\mathcal{V}\psi\rangle_\Gamma\end{aligned} \tag{13a}$$

as well as

$$\begin{aligned}\langle F\,,\,(v,\psi)\rangle := \langle f\,,\,v\rangle_\Omega + \langle\phi_0\,,\,v\rangle_\Gamma + \langle\psi\,,\,(\tfrac12-\mathcal{K})u_0\rangle_\Gamma \\ + \langle 1\,,\,(\tfrac12-\mathcal{K})u_0\rangle_\Gamma\langle 1\,,\,(\tfrac12-\mathcal{K})v+\mathcal{V}\psi\rangle_\Gamma.\end{aligned} \tag{13b}$$

Let $\mathcal{X}^\ell\subseteq\mathcal{X}$ *and* $\mathcal{Y}^\ell\subseteq\mathcal{Y}$ *be arbitrary closed subspaces with* $1\in\mathcal{Y}^\ell$. *Then, the pair* $(u_\ell,\phi_\ell)\in\mathcal{H}^\ell:=\mathcal{X}^\ell\times\mathcal{Y}^\ell$ *solves the Galerkin formulation of the Johnson-Nédélec coupling* (11)

$$\langle A\nabla u_\ell\,,\,\nabla v_\ell\rangle_\Omega - \langle\phi_\ell\,,\,v_\ell\rangle_\Gamma = \langle f\,,\,v_\ell\rangle_\Omega + \langle\phi_0\,,\,v_\ell\rangle_\Gamma, \tag{14a}$$

$$\langle\psi_\ell\,,\,(\tfrac12-\mathcal{K})u_\ell+\mathcal{V}\phi_\ell\rangle_\Gamma = \langle\psi_\ell\,,\,(\tfrac12-\mathcal{K})u_0\rangle_\Gamma \tag{14b}$$

for all $(v_\ell,\psi_\ell)\in\mathcal{H}^\ell$, *if and only if it solves the operator formulation*

$$\langle\mathcal{L}(u_\ell,\phi_\ell)\,,\,(v_\ell,\psi_\ell)\rangle = \langle F\,,\,(v_\ell,\psi_\ell)\rangle \tag{15}$$

for all $(v_\ell,\psi_\ell)\in\mathcal{H}^\ell$. *The operator* $\mathcal{L}:\mathcal{H}\to\mathcal{H}^*$ *is non-symmetric, but linear, continuous, and elliptic, and the constants*

$$c_\mathcal{L} := \inf_{\mathbf{0}\neq(u,\phi)\in\mathcal{H}}\frac{\langle\mathcal{L}(u,\phi)\,,\,(u,\phi)\rangle}{\|(u,\phi)\|_\mathcal{H}^2} \quad\text{and}\quad C_\mathcal{L} := \sup_{\mathbf{0}\neq(u,\phi)\in\mathcal{H}}\frac{\|\mathcal{L}(u,\phi)\|_{\mathcal{H}^*}}{\|(u,\phi)\|_\mathcal{H}} \tag{16}$$

satisfy $0<c_\mathcal{L}\le C_\mathcal{L}<\infty$ *and depend only on* c_A *and* C_A *from* (5) *as well as on* Ω. *Moreover,* $F\in\mathcal{H}^*$ *is a continuous linear functional on* $\mathcal{H}$. *In particular,* (15) *(and hence also* (14)*) admits a unique solution* $(u_\ell,\phi_\ell)\in\mathcal{H}^\ell$, *and there holds the Céa-type estimate*

$$\|(u,\phi)-(u_\ell,\phi_\ell)\|_\mathcal{H} \le \frac{C_\mathcal{L}}{c_\mathcal{L}}\inf_{(v_\ell,\psi_\ell)\in\mathcal{H}^\ell}\|(u,\phi)-(v_\ell,\psi_\ell)\|_\mathcal{H}, \tag{17}$$

where $(u,\phi)\in\mathcal{H}$ *denotes the unique solution of the Johnson-Nédélec coupling* (11). □

2.3. Adaptive mesh-refinement and discrete spaces. Let $\mathcal{T}_0^\Omega$ be a given conforming initial triangulation of Ω into compact and non-degenerate triangles. We suppose that a sequence $\mathcal{T}_{\ell+1}^\Omega = \text{refine}(\mathcal{T}_\ell^\Omega, \mathcal{M}_\ell^\Omega)$ of refined triangulations is obtained by newest vertex bisection, see Figure 1, where $\mathcal{T}_{\ell+1}^\Omega$ is the coarsest conforming mesh such that all marked elements $T \in \mathcal{M}_\ell^\Omega \subseteq \mathcal{T}_\ell^\Omega$ have been bisected. For our analysis, the set $\emptyset \neq \mathcal{M}_\ell^\Omega \subseteq \mathcal{T}_\ell^\Omega$ of marked elements is arbitrary, but in practice obtained from local a posteriori refinement indicators, see e.g. [AFF$^+$13a]. We note that newest vertex bisection guarantees uniform shape regularity in the sense that

$$\sup_{T \in \mathcal{T}_\ell^\Omega} \frac{\text{diam}(T)}{|T|^{1/2}} \leq \gamma < \infty \quad \text{for all } \ell \in \mathbb{N}_0, \tag{18}$$

where γ depends only on the initial mesh $\mathcal{T}_0^\Omega$, see e.g. [Ver13, KPP13] and the references therein.

Let $\mathcal{T}_0^\Gamma$ be a given initial partition of the coupling boundary Γ into compact line segments. We suppose that a sequence $\mathcal{T}_{\ell+1}^\Gamma = \text{refine}(\mathcal{T}_\ell^\Gamma, \mathcal{M}_\ell^\Gamma)$ of refined partitions is obtained by bisection, where the refined elements $T \in \mathcal{T}_\ell^\Gamma \backslash \mathcal{T}_{\ell+1}^\Gamma$ are refined into two sons $T', T'' \in \mathcal{T}_{\ell+1}^\Gamma \backslash \mathcal{T}_\ell^\Gamma$ of half length, i.e. $T = T' \cup T''$ with $\text{diam}(T') = \text{diam}(T'') = \text{diam}(T)/2$ and where at least the marked elements $T \in \mathcal{M}_\ell^\Gamma \subseteq \mathcal{T}_\ell^\Gamma$ are refined, i.e. $\mathcal{M}_\ell^\Gamma \subseteq \mathcal{T}_\ell^\Gamma \backslash \mathcal{T}_{\ell+1}^\Gamma$. In addition, we suppose that the meshes are uniformly γ-shape regular in the sense that

$$\frac{\text{diam}(T)}{\text{diam}(T')} \leq \gamma < \infty \quad \text{for all } T, T' \in \mathcal{T}_\ell^\Gamma \text{ with } T \cap T' \neq \emptyset \text{ and all } \ell \in \mathbb{N}_0, \tag{19}$$

where γ depends only on the initial partition $\mathcal{T}_0^\Gamma$. Possible choices include the 1D bisection algorithms from [AFF$^+$13b]. A further choice is to consider the partition $\mathcal{T}_\ell^\Gamma := \mathcal{T}_\ell^\Omega|_\Gamma$ of Γ which is induced by the triangulation $\mathcal{T}_\ell^\Omega$ of Ω. Formally, such a coupling of $\mathcal{T}_\ell^\Omega$ and $\mathcal{T}_\ell^\Gamma$ is not required for the analysis, but simplifies the implementation and is therefore used in the numerical experiments of Section 6.

In this work, we consider lowest-order Galerkin elements. We approximate functions $u \in H^1(\Omega)$ by functions $u_\ell \in \mathcal{X}^\ell$ and functions $\phi \in H^{-1/2}(\Gamma)$ by functions $\phi_\ell \in \mathcal{Y}^\ell$, where

$$\mathcal{X}^\ell := \big\{ v \in C(\Omega) \,:\, v|_T \text{ is affine for all } T \in \mathcal{T}_\ell^\Omega \big\}, \tag{20}$$

$$\mathcal{Y}^\ell := \big\{ \psi \in L^2(\Gamma) \,:\, \psi|_T \text{ is constant for all } T \in \mathcal{T}_\ell^\Gamma \big\}. \tag{21}$$

Let $\mathcal{N}_\ell^\Omega$ denote the set of nodes of the triangulation $\mathcal{T}_\ell^\Omega$, and let $\mathcal{N}_\ell^\Gamma$ denote the set of nodes of the triangulation $\mathcal{T}_\ell^\Gamma$. For $z \in \mathcal{N}_\ell^\Omega$ resp. $z \in \mathcal{N}_\ell^\Gamma$, we define the patch $\omega_\ell^\Omega(z) := \big\{ T \in \mathcal{T}_\ell^\Omega \,:\, z \in T \big\}$ resp. $\omega_\ell^\Gamma(z) := \big\{ T \in \mathcal{T}_\ell^\Gamma \,:\, z \in T \big\}$. For the construction of optimal multilevel preconditioners on adaptively refined triangulations, we need the following subsets of $\mathcal{N}_\ell^\Omega$ resp. $\mathcal{N}_\ell^\Gamma$:

$$\widetilde{\mathcal{N}}_0^\Omega := \mathcal{N}_0^\Omega, \qquad \widetilde{\mathcal{N}}_\ell^\Omega := \mathcal{N}_\ell^\Omega \backslash \mathcal{N}_{\ell-1}^\Omega \cup \big\{ z \in \mathcal{N}_{\ell-1}^\Omega \,:\, \omega_\ell^\Omega(z) \subsetneqq \omega_{\ell-1}^\Omega(z) \big\} \quad \text{for } \ell \geq 1, \tag{22a}$$

$$\widetilde{\mathcal{N}}_0^\Gamma := \mathcal{N}_0^\Gamma \qquad \widetilde{\mathcal{N}}_\ell^\Gamma := \mathcal{N}_\ell^\Gamma \backslash \mathcal{N}_{\ell-1}^\Gamma \cup \big\{ z \in \mathcal{N}_{\ell-1}^\Gamma \,:\, \omega_\ell^\Gamma(z) \subsetneqq \omega_{\ell-1}^\Gamma(z) \big\} \quad \text{for } \ell \geq 1. \tag{22b}$$

The sets $\widetilde{\mathcal{N}}_\ell^\Omega$ resp. $\widetilde{\mathcal{N}}_\ell^\Gamma$ thus consist of the new nodes plus the old nodes, where the corresponding patches have changed. A visualization of $\widetilde{\mathcal{N}}_\ell^\Omega$ is given in Figure 2. For each node $z \in \mathcal{N}_\ell^\Omega$, $\eta_z^\ell \in \mathcal{X}^\ell$ denotes the associated hat-function with $\eta_z^\ell(z') = \delta_{zz'}$ for all $z' \in \mathcal{N}_\ell^\Omega$, where $\delta_{zz'}$ is Kronecker's delta.

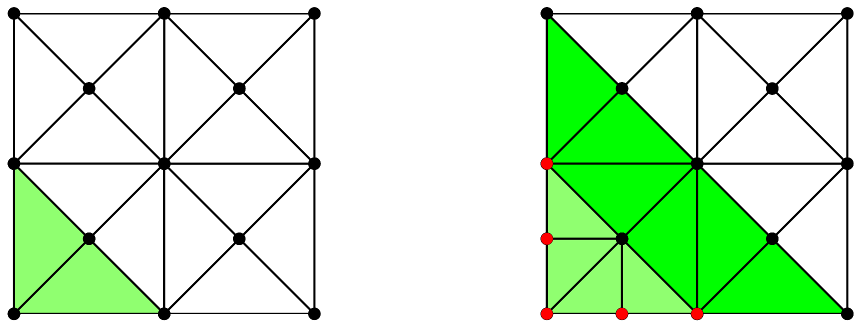

FIGURE 2. The left figure shows a FEM mesh $\mathcal{T}_{\ell-1}^{\Omega}$, where the two elements (green) are marked for refinement. Bisection of these two elements provides the mesh $\mathcal{T}_{\ell}^{\Omega}$ (right), where two *new nodes* are created. The set $\widetilde{\mathcal{N}}_{\ell}^{\Omega}$ consists of these new nodes plus their *immediate neighbours* (red), where the corresponding patches have changed. The union of the support of the basis functions associated to nodes in $\widetilde{\mathcal{N}}_{\ell}^{\Omega}$ is given by the light- and dark-green areas in the right figure.

2.4. Galerkin system and block-diagonal preconditioning. Let $\{\eta_{z_j}^{\ell}\}_{j=1}^{N_\ell}$ with $N_\ell := \#\mathcal{N}_\ell^{\Omega}$ denote the nodal basis of $\mathcal{X}^\ell$, and let $\{\psi_{T_j}\}_{j=1}^{M_\ell}$ with $M_\ell := \#\mathcal{T}_\ell^{\Gamma}$ denote a basis of $\mathcal{Y}^\ell$, where ψ_{T_j} denotes the characteristic function on $T_j \in \mathcal{T}_\ell^{\Gamma}$. The Galerkin matrix $\mathbf{A}_{\mathcal{L}} = \mathbf{A}_{\mathcal{L}}^{\ell} \in \mathbb{R}^{(N_\ell+M_\ell)\times(N_\ell+M_\ell)}$ of the operator $\mathcal{L}$ has the form

$$\mathbf{A}_{\mathcal{L}} = \begin{pmatrix} \mathbf{A}_A & -\mathbf{M}^T \\ \frac{1}{2}\mathbf{M} - \mathbf{K} & \mathbf{A}_{\mathcal{V}} \end{pmatrix} + \mathbf{S}\mathbf{S}^T,$$

where the block matrices $\mathbf{A}_A \in \mathbb{R}^{N_\ell\times N_\ell}$, $\mathbf{A}_{\mathcal{V}} \in \mathbb{R}^{M_\ell\times M_\ell}$, $\mathbf{M} \in \mathbb{R}^{M_\ell\times N_\ell}$, and $\mathbf{K} \in \mathbb{R}^{M_\ell\times N_\ell}$ as well as the stabilization (column) vector $\mathbf{S} \in \mathbb{R}^{N_\ell+M_\ell}$ are given by

$$(\mathbf{A}_A)_{jk} = \langle A\nabla\eta_{z_k}^{\ell}\,,\,\nabla\eta_{z_j}^{\ell}\rangle_\Omega \quad \text{for } j,k = 1,\dots,N_\ell, \tag{23a}$$

$$(\mathbf{A}_{\mathcal{V}})_{jk} = \langle \psi_{T_j}\,,\,\mathcal{V}\psi_{T_k}\rangle_\Gamma \quad \text{for } j,k = 1,\dots,M_\ell, \tag{23b}$$

$$(\mathbf{M})_{jk} = \langle \psi_{T_j}\,,\,(\eta_{z_k}^{\ell})|_\Gamma\rangle_\Gamma \quad \text{for } j = 1,\dots,M_\ell, k = 1,\dots,N_\ell, \tag{23c}$$

$$(\mathbf{K})_{jk} = \langle \psi_{T_j}\,,\,\mathcal{K}(\eta_{z_k}^{\ell})|_\Gamma\rangle_\Gamma \quad \text{for } j = 1,\dots,M_\ell, k = 1,\dots,N_\ell, \tag{23d}$$

$$\begin{aligned} (\mathbf{S})_j &= \langle 1\,,\,(\tfrac{1}{2}-\mathcal{K})(\eta_{z_j}^{\ell})|_\Gamma\rangle_\Gamma \quad \text{for } j = 1,\dots,N_\ell, \\ (\mathbf{S})_{j+N_\ell} &= \langle 1\,,\,\mathcal{V}\psi_{T_j}\rangle_\Gamma \quad \text{for } j = 1,\dots,M_\ell. \end{aligned} \tag{23e}$$

We stress that $\mathbf{A}_A$ as well as $\mathbf{M}$ are sparse, whereas $\mathbf{A}_{\mathcal{V}}$ is dense. Note that the number of non-zeros in the matrix $\mathbf{K}$ is bounded by $\#(\mathcal{N}_\ell^{\Omega}\cap\Gamma)\cdot M_\ell$. Moreover, the application of the rank-1 stabilization matrix $\mathbf{S}\mathbf{S}^T$ can be implemented efficiently with complexity $\mathcal{O}(N_\ell + M_\ell)$ for use with an iterative solver.

The discrete variational formulation (15) is equivalent to solving the following linear system of equations: Find $\mathbf{U} \in \mathbb{R}^{N_\ell+M_\ell}$ such that

$$\mathbf{A}_{\mathcal{L}}\mathbf{U} = \mathbf{F} \in \mathbb{R}^{N_\ell+M_\ell}, \tag{24}$$

where the right-hand side vector $\mathbf{F} \in \mathbb{R}^{N_\ell+M_\ell}$ reads

$$(\mathbf{F})_j = \langle F\,,\,(\eta^\ell_{z_j},0)\rangle \quad \text{for } j=1,\dots,N_\ell,$$
$$(\mathbf{F})_{j+N_\ell} = \langle F\,,\,(0,\psi_{T_j})\rangle \quad \text{for } j=1,\dots,M_\ell.$$

To formulate our block-diagonal preconditioner, we require an appropriate operator $\mathcal{A}$ which is related to the FEM-domain part of $\mathcal{L}$. The next lemma follows from a Rellich compactness argument, since $\mathcal{K}c = -c/2$ for all constants $c \in \mathbb{R}$. Details are analogous to, e.g., [AFF⁺13a, Lemma 10] and therefore left to the reader.

Lemma 2. *For $u, v \in H^1(\Omega)$, define*

$$\langle \mathcal{A}u\,,\,v\rangle := \langle A\nabla u\,,\,\nabla v\rangle_\Omega + \langle 1\,,\,(\tfrac12-\mathcal{K})u\rangle_\Gamma\langle 1\,,\,(\tfrac12-\mathcal{K})v\rangle_\Gamma. \tag{25a}$$

Then, the operator $\mathcal{A} : H^1(\Omega) \to (H^1(\Omega))^*$ is linear, symmetric, continuous, and elliptic, and the constants

$$c_\mathcal{A} := \inf_{0\neq u\in H^1(\Omega)} \frac{\langle \mathcal{A}u\,,\,u\rangle}{\|u\|^2_{H^1(\Omega)}} \quad \text{and} \quad C_\mathcal{A} := \sup_{0\neq u\in H^1(\Omega)} \frac{\|\mathcal{A}u\|_{(H^1(\Omega))^*}}{\|u\|_{H^1(\Omega)}} \tag{25b}$$

satisfy $0 < c_\mathcal{A} \le C_\mathcal{A} < \infty$ and depend only on c_A and C_A from (5) *as well as on Ω.* □

In this work, we investigate block-diagonal preconditioners of the form

$$\mathbf{P}_\mathcal{L} = \begin{pmatrix} \mathbf{P}_\mathcal{A} & \mathbf{0} \\ \mathbf{0} & \mathbf{P}_\mathcal{V} \end{pmatrix}, \tag{26}$$

where $\mathbf{P}_\mathcal{A}$ is a "good" approximation of the Galerkin matrix $\mathbf{A}_\mathcal{A}$ corresponding to the operator $\mathcal{A}$ from Lemma 2 with respect to the nodal basis of $\mathcal{X}^\ell$, and $\mathbf{P}_\mathcal{V}$ is a "good" approximation of the Galerkin matrix $\mathbf{A}_\mathcal{V}$. Our construction below ensures that $\mathbf{P}_\mathcal{A}, \mathbf{P}_\mathcal{V}$ and hence $\mathbf{P}$ are symmetric and positive definite.

Instead of (24), we solve the preconditioned system

$$\mathbf{P}_\mathcal{L}^{-1}\mathbf{A}_\mathcal{L}\mathbf{U} = \mathbf{P}_\mathcal{L}^{-1}\mathbf{F} \in \mathbb{R}^{N_\ell+M_\ell}. \tag{27}$$

For this non-symmetric system of linear equations, we use a preconditioned GMRES algorithm [SS86], which will be discussed in the following subsection. The preconditioned Galerkin matrix reads in block form

$$\mathbf{P}_\mathcal{L}^{-1}\mathbf{A}_\mathcal{L} = \begin{pmatrix} \mathbf{P}_\mathcal{A}^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{P}_\mathcal{V}^{-1} \end{pmatrix} \left[\begin{pmatrix} \mathbf{A}_A & -\mathbf{M}^T \\ \frac12\mathbf{M}-\mathbf{K} & \mathbf{A}_\mathcal{V} \end{pmatrix} + \mathbf{S}\mathbf{S}^T \right]. \tag{28}$$

2.5. Preconditioned GMRES algorithm. Let $\mathbf{P} \in \mathbb{R}^{N\times N}$ denote a symmetric and positive definite matrix and let $\mathbf{A} \in \mathbb{R}^{N\times N}$ denote a (possibly) non-symmetric, but positive definite matrix. Let, $\mathbf{E}^k \in \mathbb{R}^N$ denote the standard unit vector with entries $(\mathbf{E}^k)_j = \delta_{kj}$. The preconditioned GMRES algorithm reads as follows.

Algorithm 3 (GMRES)**. Input:** *Matrices $\mathbf{P}, \mathbf{A} \in \mathbb{R}^{N\times N}$, right-hand side vector $\mathbf{F} \in \mathbb{R}^N$, initial guess $\mathbf{U}^0 \in \mathbb{R}^N$, relative tolerance $\tau > 0$, and maximum number of iterations $K \in \mathbb{N}$ with $K \le N$.*

(a) *Allocate memory for the matrix $\mathbf{H} \in \mathbb{R}^{(K+1)\times K}$, the vectors $\mathbf{V}^i \in \mathbb{R}^N$, $i = 1,\dots,K$, $\mathbf{W} \in \mathbb{R}^N$, $\mathbf{U} \in \mathbb{R}^N$, $\mathbf{R}^0 \in \mathbb{R}^N$ and $\mathbf{R} \in \mathbb{R}^N$.*
(b) *Compute initial residual $\mathbf{R}^0 \leftarrow \mathbf{P}^{-1}(\mathbf{F}-\mathbf{A}\mathbf{U}^0)$ and $\mathbf{V}^1 \leftarrow \mathbf{R}^0/\|\mathbf{R}^0\|_\mathbf{P}$.*
(c) *Set counter $k \leftarrow 1$, and initialize $(\mathbf{H})_{ij} \leftarrow 0$ for all $i = 1,\dots,K+1, j = 1,\dots,K$.*

Iterate the following steps (i)–(vii)*:*

(i) *Compute* $\mathbf{W} \leftarrow \mathbf{P}^{-1}\mathbf{A}\mathbf{V}^k$.

(ii) *For all* $i = 1, \dots, k$ *compute*

$$(\mathbf{H})_{ik} \leftarrow \langle \mathbf{W}\,,\, \mathbf{V}^i \rangle_{\mathbf{P}} \quad \textit{and} \quad \mathbf{W} \leftarrow \mathbf{W} - \mathbf{V}^i(\mathbf{H})_{ik}.$$

(iii) *Compute* $(\mathbf{H})_{k+1,k} \leftarrow \|\mathbf{W}\|_{\mathbf{P}}$.

(iv) *Define the sub-matrix* $\overline{\mathbf{H}}^k \in \mathbb{R}^{(k+1)\times k}$ *with entries* $(\overline{\mathbf{H}}^k)_{ij} := (\mathbf{H})_{ij}$ *for* $i = 1, \dots, k+1, j = 1, \dots, k$ *and compute*

$$\mathbf{Y}^k \leftarrow \operatorname*{arg\,min}_{\mathbf{Y}\in\mathbb{R}^k} \| \|\mathbf{R}_0\|_{\mathbf{P}}\mathbf{E}^1 - \overline{\mathbf{H}}^k\mathbf{Y}\|_2.$$

(v) *Compute* $\mathbf{U} \leftarrow \mathbf{U}^0 + (\mathbf{V}^1 \cdots \mathbf{V}^k)\mathbf{Y}^k$ *and* $\mathbf{R} \leftarrow \mathbf{P}^{-1}(\mathbf{F} - \mathbf{A}\mathbf{U})$.

(vi) *Stop iteration if* $\|\mathbf{R}\|_{\mathbf{P}} \le \tau\|\mathbf{R}^0\|_{\mathbf{P}}$ *or* $k \ge K$.

(vii) *Otherwise, compute* $\mathbf{V}^{k+1} \leftarrow \mathbf{W}/(\mathbf{H})_{k+1,k}$*, update counter* $k \leftarrow k+1$*, and goto (iv).*

Output: $\mathbf{U}$*,* k*, and* $\|\mathbf{R}\|_{\mathbf{P}}/\|\mathbf{R}^0\|_{\mathbf{P}}$.

Note that for $\mathbf{P}$ being the identity matrix, Algorithm 3 is the usual GMRES algorithm with inner product $\langle\cdot\,,\,\cdot\rangle_2$, see e.g. [SS86]. The main memory consumption is given by the vectors $\mathbf{V}^i \in \mathbb{R}^N$, while the matrix $\overline{\mathbf{H}}^k \in \mathbb{R}^{(k+1)\times k}$ in step (iv) is a sub-block of the matrix $\mathbf{H} \in \mathbb{R}^{(K+1)\times K}$ and thus does not need to be stored explicitly.

As is often the case for multilevel preconditioners, the application of $\mathbf{P}^{-1}$ on a vector is known, whereas the application of $\mathbf{P}$ is unknown. We therefore note that the GMRES Algorithm 3 can be implemented without using $\mathbf{P}$ to compute the inner products $\langle\cdot\,,\,\cdot\rangle_{\mathbf{P}}$ and the norms $\|\cdot\|_{\mathbf{P}}$. To this end, one replaces the computation of $\mathbf{V}^1$ by $\widetilde{\mathbf{V}}^1 \leftarrow (\mathbf{F}-\mathbf{A}\mathbf{U}^0)/\|\mathbf{R}^0\|_{\mathbf{P}}$, where $\|\mathbf{R}^0\|_{\mathbf{P}}^2 = \langle \mathbf{P}^{-1}(\mathbf{F} - \mathbf{A}\mathbf{U}^0)\,,\, \mathbf{F} - \mathbf{A}\mathbf{U}^0\rangle_2$. Then, $\mathbf{V}^1 \leftarrow \mathbf{P}^{-1}\widetilde{\mathbf{V}}^1$. In step (i), we compute $\widetilde{\mathbf{W}} \leftarrow \mathbf{A}\mathbf{V}^k$ instead of $\mathbf{W}$. Step (ii) is replaced by $(\mathbf{H})_{ik} \leftarrow \langle \mathbf{P}^{-1}\widetilde{\mathbf{W}}\,,\, \mathbf{V}^i\rangle_{\mathbf{P}} = \langle \widetilde{\mathbf{W}}\,,\, \mathbf{V}^i\rangle_2$ and $\widetilde{\mathbf{W}} \leftarrow \widetilde{\mathbf{W}} - \widetilde{\mathbf{V}}^i(\mathbf{H})_{ik}$. Instead of step (iii), we then compute $(\mathbf{H})_{k+1,k} \leftarrow \|\mathbf{P}^{-1}\widetilde{\mathbf{W}}\|_{\mathbf{P}} = \sqrt{\langle \mathbf{P}^{-1}\widetilde{\mathbf{W}}\,,\, \widetilde{\mathbf{W}}\rangle_2}$. Note that $\|\mathbf{R}\|_{\mathbf{P}}^2 = \langle \mathbf{P}^{-1}(\mathbf{F} - \mathbf{A}\mathbf{U})\,,\, \mathbf{F} - \mathbf{A}\mathbf{U}\rangle_2$ in step (vi). Lastly, in step (vii) we replace the computation of $\mathbf{V}^{k+1}$ by $\widetilde{\mathbf{V}}^{k+1} \leftarrow \widetilde{\mathbf{W}}/(\mathbf{H})_{k+1,k}$ and $\mathbf{V}^{k+1} \leftarrow \mathbf{P}^{-1}\widetilde{\mathbf{V}}^{k+1}$.

2.6. Local multilevel preconditioner and main result. For both the FEM part $\mathbf{P}_{\mathcal{A}}$ and BEM part $\mathbf{P}_{\mathcal{V}}$ in (26), we will use local multilevel preconditioners which are optimal in the sense that the condition numbers of the preconditioned systems are independent of the number of levels L and the mesh-size h_L.

For the preconditioner $\mathbf{P}_{\mathcal{A}}$, we use an additive Schwarz multilevel diagonal preconditioner similar to the one in [XCH10]. Recall $\widetilde{\mathcal{N}}_\ell^\Omega \subseteq \mathcal{N}_\ell^\Omega$ from (22) Define

$$\widetilde{\mathcal{X}}^\ell := \operatorname{span}\big\{\eta_z^\ell \,:\, z \in \widetilde{\mathcal{N}}_\ell^\Omega\big\} \subseteq \mathcal{X}^\ell. \tag{29}$$

Let $\widetilde{\mathcal{I}}^\ell : \widetilde{\mathcal{X}}^\ell \to \mathcal{X}^L$ denote the canonical embedding with matrix representation $\widetilde{\mathbf{I}}^\ell \in \mathbb{R}^{\widetilde{N}_\ell^\Omega \times N_L^\Omega}$ and $\widetilde{N}_\ell^\Omega := \#\widetilde{\mathcal{N}}_\ell^\Omega$. Furthermore, let $\widetilde{\mathbf{D}}_{\mathcal{A}}^\ell$ denote the diagonal of the Galerkin matrix $\widetilde{\mathbf{A}}_{\mathcal{A}}^\ell$ with respect to the local set of nodes $\widetilde{\mathcal{N}}_\ell^\Omega$, i.e. $(\widetilde{\mathbf{A}}_{\mathcal{A}}^\ell)_{jk} = \langle \mathcal{A}\eta_{z_k}^\ell\,,\,\eta_{z_j}^\ell\rangle$ for $j, k = 1, \dots, \widetilde{N}_\ell^\Omega$ and

$(\widetilde{\mathbf{D}}^\ell_{\mathcal{A}})_{jk} := \delta_{jk}(\widetilde{\mathbf{A}}^\ell_{\mathcal{A}})_{jj}$. Then, our local multilevel diagonal preconditioner $\mathbf{P}_{\mathcal{A}}$ is defined via

$$\mathbf{P}_{\mathcal{A}}^{-1} := (\mathbf{P}_{\mathcal{A}}^L)^{-1} := \sum_{\ell=0}^{L} \widetilde{\mathbf{I}}^\ell (\widetilde{\mathbf{D}}^\ell_{\mathcal{A}})^{-1} (\widetilde{\mathbf{I}}^\ell)^T. \tag{30}$$

From the definition, we see that this preconditioner corresponds to a diagonal scaling on each level, where scaling is done on the local subset $\widetilde{\mathcal{N}}_\ell^\Omega$ only.

For all boundary nodes $z \in \mathcal{N}_\ell^\Gamma$, let

$$\zeta_z^\ell \in \mathcal{Z}^\ell := \big\{ v \in C(\Gamma) \,:\, v|_T \text{ is affine for all } T \in \mathcal{T}_\ell^\Gamma \big\} \tag{31}$$

denote the boundary hat-function with $\zeta_z^\ell(z') = \delta_{zz'}$ for all $z' \in \mathcal{N}_\ell^\Gamma$. To construct an efficient preconditioner $\mathbf{P}_{\mathcal{V}}$ for the weakly-singular integral operator $\mathcal{V}$ in 2D, we use the Haar-basis functions $\chi_z^\ell := (\zeta_z^\ell)'$ for all $z \in \mathcal{N}_\ell^\Gamma$. Recall $\widetilde{\mathcal{N}}_\ell^\Gamma \subseteq \mathcal{N}_\ell^\Gamma$ from (22). Let $\widetilde{N}_\ell^\Gamma := \#\widetilde{\mathcal{N}}_\ell^\Gamma$. Define the local subspaces

$$\widetilde{\mathcal{Y}}^\ell := \operatorname{span}\big\{ \chi_z^\ell \,:\, z \in \widetilde{\mathcal{N}}_\ell^\Gamma \big\} \subsetneqq \mathcal{Y}^\ell, \qquad \widetilde{\mathcal{Z}}^\ell := \operatorname{span}\big\{ \zeta_z^\ell \,:\, z \in \widetilde{\mathcal{N}}_\ell^\Gamma \big\} \subseteq \mathcal{Z}^\ell, \tag{32}$$

and the matrix $\widetilde{\mathbf{H}}^\ell \in \mathbb{R}^{M_\ell \times \widetilde{N}_\ell^\Gamma}$ which represents the Haar-basis functions with respect to the canonical basis $\big\{ \psi_T \,:\, \psi_T \text{ is characteristic function on } T \in \mathcal{T}_\ell^\Gamma \big\}$ of $\mathcal{Y}^\ell$, i.e.

$$\chi_{z_k}^\ell = \sum_{j=1}^{M_\ell} (\widetilde{\mathbf{H}}^\ell)_{jk} \psi_{T_j} \quad \text{for } j = 1, \dots, M_\ell, \quad k = 1, \dots, \widetilde{N}_\ell^\Gamma.$$

Maue's formula [Mau49] states the relation

$$\langle \mathcal{W} v \,,\, w \rangle_\Gamma = \langle \mathcal{V} v' \,,\, w' \rangle_\Gamma \quad \text{for all } v, w \in H^1(\Gamma) \tag{33}$$

and thus reveals the identity

$$\Big((\widetilde{\mathbf{H}}^\ell)^T \mathbf{A}^\ell_{\mathcal{V}} \widetilde{\mathbf{H}}^\ell \Big)_{jk} = \langle \mathcal{V} \chi_{z_k}^\ell \,,\, \chi_{z_j}^\ell \rangle_\Gamma = \langle \mathcal{W} \zeta_{z_k}^\ell \,,\, \zeta_{z_j}^\ell \rangle_\Gamma =: (\widetilde{\mathbf{A}}^\ell_{\mathcal{W}})_{jk} \quad \text{for } j, k = 1, \dots, \widetilde{N}_\ell^\Gamma$$

for the Galerkin matrix $\widetilde{\mathbf{A}}^\ell_{\mathcal{W}}$ of the hypersingular integral operator with respect to the nodes $\widetilde{\mathcal{N}}_\ell^\Gamma$. Let $\widetilde{\mathbf{D}}^\ell_{\mathcal{W}}$ denote the diagonal of $\widetilde{\mathbf{A}}^\ell_{\mathcal{W}}$, and let $\mathcal{J}^\ell : \mathcal{Y}^\ell \to \mathcal{Y}^L$ denote the canonical embedding with matrix representation $\mathbf{J}^\ell$. Moreover, define $D := \langle 1 \,,\, \mathcal{V} 1 \rangle_\Gamma$ and let $\mathbf{1} \in \mathbb{R}^{M_L}$ denote the vector with constant entries $(\mathbf{1})_j = 1$ for all $j = 1, \dots M_L$. Then, our multilevel diagonal preconditioner $\mathbf{P}_{\mathcal{V}}$ for the weakly-singular integral operator reads

$$\mathbf{P}_{\mathcal{V}}^{-1} := (\mathbf{P}_{\mathcal{V}}^L)^{-1} := \mathbf{1} D^{-1} \mathbf{1}^T + \sum_{\ell=0}^{L} \mathbf{J}^\ell \widetilde{\mathbf{H}}^\ell (\widetilde{\mathbf{D}}^\ell_{\mathcal{W}})^{-1} (\widetilde{\mathbf{H}}^\ell)^T (\mathbf{J}^\ell)^T. \tag{34}$$

The following theorem is the main result of this work. Let $\operatorname{cond}_{\mathbf{C}}(\mathbf{A}) = \|\mathbf{A}\|_{\mathbf{C}} \|\mathbf{A}^{-1}\|_{\mathbf{C}}$ denote the condition number of the matrix $\mathbf{A}$ with respect to the norm $\|\cdot\|_{\mathbf{C}}$ induced by the symmetric and positive definite matrix $\mathbf{C}$.

Theorem 4. *Let $\mathbf{P}_{\mathcal{A}}$ resp. $\mathbf{P}_{\mathcal{V}}$ denote the multilevel preconditioners defined in* (30) *resp.* (34)*. Then, the condition number*

$$\operatorname{cond}_{\mathbf{P}_{\mathcal{L}}}(\mathbf{P}_{\mathcal{L}}^{-1} \mathbf{A}_{\mathcal{L}}) \le C < \infty \tag{35}$$

is uniformly bounded. Moreover, the j-th residual $\mathbf{R}^j$ from the preconditioned GMRES Algorithm 3 with $\mathbf{P} = \mathbf{P}_\mathcal{L}$ from (26) satisfies

$$\|\mathbf{R}^j\|_{\mathbf{P}_\mathcal{L}} \le q_{\mathrm{GMRES}}^j \|\mathbf{R}^0\|_{\mathbf{P}_\mathcal{L}}. \tag{36}$$

The constants $C > 0$ and $0 < q_{\mathrm{GMRES}} < 1$ depend only on Ω, the ellipticity and continuity constants of the material tensor A from (5), the initial triangulations $\mathcal{T}_0^\Omega$ and $\mathcal{T}_0^\Gamma$, as well as on the mesh-refinement strategy chosen.

3. Spectral estimates for $\mathbf{A}_\mathcal{A}$ and $\mathbf{A}_\mathcal{V}$

In this section, we provide spectral estimates for the matrices $\mathbf{A}_\mathcal{A}$, $\mathbf{A}_\mathcal{V}$. In particular, the equivalences $\langle \mathbf{A}_\mathcal{A}\boldsymbol{X}\,,\,\boldsymbol{X}\rangle_2 \simeq \langle \mathbf{P}_\mathcal{A}\boldsymbol{X}\,,\,\boldsymbol{X}\rangle_2$ for $\boldsymbol{X} \in \mathbb{R}^{N_L}$ as well as $\langle \mathbf{A}_\mathcal{V}\boldsymbol{\Phi}\,,\,\boldsymbol{\Phi}\rangle_2 \simeq \langle \mathbf{P}_\mathcal{V}\boldsymbol{\Phi}\,,\,\boldsymbol{\Phi}\rangle_2$ for $\boldsymbol{\Phi} \in \mathbb{R}^{M_L}$ are optimal in the sense, that the involved constants are independent of L and h_L.

The remainder of this section is organized as follows: In Section 3.1, we focus on the optimality of the preconditioner $\mathbf{P}_\mathcal{A}$, which follows from [WC06, XCH10]. In Section 3.2, we analyze the preconditioner $\mathbf{P}_\mathcal{V}$ and prove optimality thereof. Note that optimality of $\mathbf{P}_\mathcal{V}$ for uniform meshes has already been proved in [TS96], where $\widetilde{\mathcal{N}}_\ell^\Omega = \mathcal{N}_\ell^\Omega$ and $\widetilde{\mathcal{N}}_\ell^\Gamma = \mathcal{N}_\ell^\Gamma$, while optimality on adaptive meshes is a particular contribution of the present work.

3.1. Optimality of the multilevel preconditioner $\mathbf{P}_\mathcal{A}$. Define the local projection operators $\mathfrak{A}_z^\ell : \mathcal{X}^L \to \mathcal{X}_z^\ell := \mathrm{span}\{\eta_z^\ell\}$ by

$$\langle \mathfrak{A}_z^\ell v\,,\,w_z^\ell\rangle_\mathcal{A} = \langle v\,,\,w_z^\ell\rangle_\mathcal{A} \quad \text{for all } w_z^\ell \in \mathcal{X}_z^\ell, \tag{37}$$

and the multilevel additive Schwarz preconditioner

$$\mathfrak{A}_{\mathrm{AS}} := \sum_{\ell=0}^{L}\sum_{z\in\widetilde{\mathcal{N}}_\ell} \mathfrak{A}_z^\ell \,:\, \mathcal{X}^L \to \mathcal{X}^L. \tag{38}$$

A straightforward calculation shows the identity

$$\langle \mathbf{P}_\mathcal{A}^{-1}\mathbf{A}_\mathcal{A}\boldsymbol{X}\,,\,\boldsymbol{X}\rangle_{\mathbf{A}_\mathcal{A}} = \langle \mathfrak{A}_{\mathrm{AS}} v\,,\,v\rangle_\mathcal{A}, \tag{39}$$

where $v \in \mathcal{H}^L$ is given by $v = \sum_{j=1}^{N_L}(\boldsymbol{X})_j\eta_{z_j}^L$, and $N_L := \#\mathcal{N}_L^\Omega$ denotes the number of nodes in the FEM domain. Thus, bounds for the extremal eigenvalues of the operator $\mathfrak{A}_{\mathrm{AS}}$ provide bounds for the extremal eigenvalues of the preconditioned system.

Theorem 5. *The preconditioner matrix $\mathbf{P}_\mathcal{A}$ is symmetric and positive definite. There holds*

$$d_\mathcal{A}\langle \mathbf{P}_\mathcal{A}\boldsymbol{X}\,,\,\boldsymbol{X}\rangle_2 \le \langle \mathbf{A}_\mathcal{A}\boldsymbol{X}\,,\,\boldsymbol{X}\rangle_2 \le D_\mathcal{A}\langle \mathbf{P}_\mathcal{A}\boldsymbol{X}\,,\,\boldsymbol{X}\rangle_2 \quad \text{for all } \boldsymbol{X} \in \mathbb{R}^{N_L}. \tag{40}$$

The constants $d_\mathcal{A}, D_\mathcal{A}$ depend only on Ω, the initial triangulation $\mathcal{T}_0^\Omega$, as well as the use of newest vertex bisection for mesh-refinement.

Proof. There holds a similar result to [XCH10, Theorem 4.2] for the additive Schwarz operator $\mathfrak{A}_{\mathrm{AS}}$, i.e., $\mathfrak{A}_{\mathrm{AS}}$ is $\langle\cdot\,,\,\cdot\rangle_\mathcal{A}$-symmetric and there holds for all $v \in \mathcal{X}^L$

$$\langle \mathfrak{A}_{\mathrm{AS}} v\,,\,v\rangle_\mathcal{A} \simeq \langle \mathcal{A}v\,,\,v\rangle \tag{41}$$

where the hidden constants depend only on Ω, the initial triangulation $\mathcal{T}_0^\Omega$, as well as on the mesh-refinement chosen. In particular, the proof of the equivalence (41) follows the lines of [XCH10, Section 4.1]. The two key ingredients of the proof are the results [WC06, Lemma 3.2–3.3], which also hold for the problem considered in this work. ☐

3.2. Optimality of the multilevel preconditioner $\mathbf{P}_\mathcal{V}$. In this section we prove that the optimal additive Schwarz preconditioner for the hypersingular integral operator provided in [FFPS13], which is based on a space decomposition of lowest-order hat-functions, induces optimality of the additive Schwarz operator for the weakly-singular integral operator. The key ingredient of the proof is Maue's formula (33), which allows us, roughly speaking, to change between the $H^{1/2}$ and $H^{-1/2}$ norms. For uniform meshes, a similar approach, which uses a generalised antiderivative operator [HS96], is considered in [TS96]. The remainder of this section can be seen as an alternate proof of the results from [TS96, Section 3] as well as an extension to locally refined meshes.

Theorem 6. *The preconditioner matrix $\mathbf{P}_\mathcal{V}$ is symmetric and positive definite. There holds*

$$d_\mathcal{V}\langle \mathbf{P}_\mathcal{V}\mathbf{\Phi}\,,\,\mathbf{\Phi}\rangle_2 \le \langle \mathbf{A}_\mathcal{V}\mathbf{\Phi}\,,\,\mathbf{\Phi}\rangle_2 \le D_\mathcal{V}\langle \mathbf{P}_\mathcal{V}\mathbf{\Phi}\,,\,\mathbf{\Phi}\rangle_2 \quad \textit{for all } \mathbf{\Phi}\in\mathbb{R}^{M_L}. \tag{42}$$

The constants $d_\mathcal{V}, D_\mathcal{V}$ depend only on Γ, the initial triangulation $\mathcal{T}_0^\Gamma$, and the chosen mesh-refinement. Moreover, the eigenvalues of the preconditioned matrix $\mathbf{P}_\mathcal{V}^{-1}\mathbf{A}_\mathcal{V}$ are bounded by

$$d_\mathcal{V} \le \lambda_{\min}(\mathbf{P}_\mathcal{V}^{-1}\mathbf{A}_\mathcal{V}) \le \lambda_{\max}(\mathbf{P}_\mathcal{V}^{-1}\mathbf{A}_\mathcal{V}) \le D_\mathcal{V}. \tag{43}$$

According to (7a), and (8)–(9), $\langle \phi\,,\,\psi\rangle_\mathcal{V} := \langle \mathcal{V}\phi\,,\,\psi\rangle_\Gamma$ defines a scalar product with equivalent norm $\|\phi\|_\mathcal{V}^2 := \langle \phi\,,\,\phi\rangle_\mathcal{V}$ on $H^{-1/2}(\Gamma)$. According to (7c), (8), (10), and the Rellich compactness theorem, $\langle u\,,\,v\rangle_\mathcal{W} := \langle \mathcal{W}u\,,\,v\rangle_\Gamma + \langle u\,,\,1\rangle_\Gamma\langle v\,,\,1\rangle_\Gamma$ defines a scalar product with equivalent norm $\|u\|_\mathcal{W}^2 := \langle u\,,\,u\rangle_\mathcal{W}$ on $H^{1/2}(\Gamma)$.

We need to define a subspace decomposition of the space $\mathcal{Y}^L$. To that end, we make use of the Haar basis functions $\chi_z^\ell = (\zeta_z^\ell)'$, which are the arclength derivatives of the hat functions ζ_z^ℓ. Then, $\mathcal{Z}^L = \operatorname{span}\{\zeta_z^L \,:\, z\in\mathcal{N}^L\}$ can be decomposed into

$$\mathcal{Z}^L = \sum_{\ell=0}^{L}\sum_{z\in\widetilde{\mathcal{N}}_\ell^\Gamma}\mathcal{Z}_z^\ell \quad \text{with } \mathcal{Z}_z^\ell := \operatorname{span}\{\zeta_z^\ell\}.$$

Moreover, simple calculations with $\langle \chi_z^\ell\,,\,1\rangle_\Gamma = 0$ show that

$$\mathcal{Y}^L = \mathcal{Y}^{00} \oplus \sum_{\ell=0}^{L}\sum_{z\in\widetilde{\mathcal{N}}_\ell^\Gamma}\mathcal{Y}_z^\ell \quad \text{with } \mathcal{Y}^{00} := \operatorname{span}\{1\} \text{ and } \mathcal{Y}_z^\ell := \operatorname{span}\{\chi_z^\ell\}$$

defines a (direct sum) decomposition of $\mathcal{Y}^L$ into $\mathcal{Y}^{00}$ and an additive Schwarz space. With this, we define the additive Schwarz operator

$$\mathfrak{V}_{\mathrm{AS}} = \mathfrak{V}^{00} + \sum_{\ell=0}^{L}\sum_{z\in\widetilde{\mathcal{N}}_\ell^\Gamma}\mathfrak{V}_z^\ell, \tag{44}$$

where $\mathfrak{V}^{00}$ resp. $\mathfrak{V}_z^\ell$ are defined for all $\phi\in\mathcal{Y}^L$ via

$$\langle \mathfrak{V}^{00}\phi\,,\,\phi_0\rangle_\mathcal{V} = \langle \phi\,,\,\phi_0\rangle_\mathcal{V} \quad \text{for all } \phi_0\in\mathcal{Y}^{00} \quad \text{resp.} \tag{45a}$$

$$\langle \mathfrak{V}_z^\ell\phi\,,\,\phi_z^\ell\rangle_\mathcal{V} = \langle \phi\,,\,\phi_z^\ell\rangle_\mathcal{V} \quad \text{for all } \phi_z^\ell\in\mathcal{Y}_z^\ell. \tag{45b}$$

We note that the symmetry of the orthogonal projectors $\mathfrak{V}^{00}$ resp. $\mathfrak{V}_z^\ell$ implies that also $\mathfrak{V}_{\mathrm{AS}}$ is symmetric

$$\langle \mathfrak{V}_{\mathrm{AS}}\phi\,,\,\psi\rangle_{\mathcal{V}} = \langle \phi\,,\,\mathfrak{V}_{\mathrm{AS}}\phi\rangle_{\mathcal{V}} \quad \text{for all } \phi,\psi \in \mathcal{Y}^L. \tag{46}$$

Our analysis of $\mathfrak{V}_{\mathrm{AS}}$ builds on own results [FFPS13] on the additive Schwarz operator associated to the hypersingular integral equation,

$$\mathfrak{W}_{\mathrm{AS}} = \sum_{\ell=0}^{L}\sum_{z\in\widetilde{\mathcal{N}}_\ell^\Gamma} \mathfrak{W}_z^\ell, \quad \text{where } \langle \mathfrak{W}_z^\ell v\,,\,v_z^\ell\rangle_{\mathcal{W}} = \langle v\,,\,v_z^\ell\rangle_{\mathcal{W}} \quad \text{for all } v_z^\ell \in \mathcal{X}_z^\ell. \tag{47}$$

The analysis of [FFPS13] provides the following result.

Lemma 7 ([FFPS13, Proposition 4])**.** *The operator* $\mathfrak{W}_{\mathrm{AS}} : \mathcal{Z}^L \to \mathcal{Z}^L$ *is symmetric and satisfies*

$$C_1\|v\|_{\mathcal{W}}^2 \le \langle \mathfrak{W}_{\mathrm{AS}} v\,,\,v\rangle_{\mathcal{W}} \le C_2\|v\|_{\mathcal{W}}^2 \quad \textit{for all } v \in \mathcal{Z}^L,$$

where the constants $C_1, C_2 > 0$ *depend only on* Γ*, the initial triangulation* $\mathcal{T}_0^\Gamma$*, as well as on the chosen mesh-refinement.* □

For each $\phi \in \mathcal{Y}^L$, we follow [TS96] and split

$$\phi = \phi_0 + \widetilde{\phi} \quad \text{with unique } \phi_0 = \langle \phi\,,\,1\rangle_\Gamma/|\Gamma| \in \mathcal{Y}^{00} \text{ and } \widetilde{\phi} \in \sum_{\ell=0}^{L}\sum_{z\in\widetilde{\mathcal{N}}_\ell^\Gamma} \mathcal{Y}_z^\ell. \tag{48}$$

Let us introduce a mechanism to switch between the $H^{1/2}$ and $H^{-1/2}$ norms. For $\widetilde{\phi} \in \mathcal{Y}_*^L := \{\phi \in \mathcal{Y}^L \,:\, \langle \phi\,,\,1\rangle_\Gamma = 0\}$, there exists a unique function $\widetilde{v} \in \mathcal{Z}_*^L := \{v \in \mathcal{Z}^L \,:\, \langle v\,,\,1\rangle_\Gamma = 0\}$ such that

$$\widetilde{\phi} = (\widetilde{v})'. \tag{49}$$

To see this, let $\widetilde{\phi} \in \mathcal{Y}_*^L$ with

$$\widetilde{\phi} = \sum_{\ell=0}^{L}\sum_{z\in\widetilde{\mathcal{N}}_\ell^\Gamma} \alpha_z^\ell \chi_z^\ell \quad \text{and define} \quad v := \sum_{\ell=0}^{L}\sum_{z\in\widetilde{\mathcal{N}}_\ell^\Gamma} \alpha_z^\ell \zeta_z^\ell \in \mathcal{Z}^L. \tag{50}$$

Then, $\widetilde{v} := v - \langle v\,,\,1\rangle_\Gamma/|\Gamma| \in \mathcal{Z}_*^L$, and it holds $(\widetilde{v})' = v' = \widetilde{\phi}$ as $\chi_z^\ell = (\zeta_z^\ell)'$. Maue's formula (33) provides the important identities

$$\langle \mathcal{W}\zeta_z^\ell\,,\,\zeta_z^\ell\rangle_\Gamma = \langle \mathcal{V}\chi_z^\ell\,,\,\chi_z^\ell\rangle_\Gamma \quad \text{as well as} \quad \langle \mathcal{W}\widetilde{v}\,,\,\widetilde{v}\rangle_\Gamma = \langle \mathcal{V}\widetilde{\phi}\,,\,\widetilde{\phi}\rangle_\Gamma. \tag{51}$$

We stress that (51) allows to switch between the spaces $H^{1/2}$ and $H^{-1/2}$. This is the heart of the proof of the following proposition.

Proposition 8. *The operator* $\mathfrak{V}_{\mathrm{AS}} : \mathcal{Y}^L \to \mathcal{Y}^L$ *is symmetric* (46)*, and it holds*

$$C_3\|\psi\|_{\mathcal{V}}^2 \le \langle \mathfrak{V}_{\mathrm{AS}}\psi\,,\,\psi\rangle_{\mathcal{V}} \le C_4\|\psi\|_{\mathcal{V}}^2 \quad \textit{for all } \psi \in \mathcal{Y}^L. \tag{52}$$

The constants $C_3, C_4 > 0$ *depend only on* Γ*, the initial triangulation* $\mathcal{T}_0^\Gamma$*, as well as on the bisection algorithm from Section 2.3.*

For the proof of Proposition 8 we need the following result, see e.g. [Zha92], where the first part is known as Lions' lemma.

Lemma 9. *(i) Let $c > 0$ and $\psi \in \mathcal{Y}^L$. Suppose that there exists a decomposition $\psi = \psi_0 + \sum_{\ell=0}^{L} \sum_{z\in\widetilde{\mathcal{N}}_\ell^\Gamma} \psi_z^\ell$ with $\psi_0 \in \mathcal{Y}^{00}, \psi_z^\ell \in \mathcal{Y}_z^\ell$ such that*

$$\|\psi_0\|_{\mathcal{V}}^2 + \sum_{\ell=0}^{L} \sum_{z\in\widetilde{\mathcal{N}}_\ell^\Gamma} \|\psi_z^\ell\|_{\mathcal{V}}^2 \le c^{-1}\|\psi\|_{\mathcal{V}}^2. \tag{53}$$

Then it follows, $c\|\psi\|_{\mathcal{V}}^2 \le \langle \mathfrak{V}_{\mathrm{AS}}\psi \,,\, \psi\rangle_{\mathcal{V}}$.

(ii) Let $C > 0$ and $\psi \in \mathcal{Y}^L$. Suppose that for all decompositions $\psi = \psi_0 + \sum_{\ell=0}^{L} \sum_{z\in\widetilde{\mathcal{N}}_\ell^\Gamma} \psi_z^\ell$ with $\psi_0 \in \mathcal{Y}^{00}$ and $\psi_z^\ell \in \mathcal{Y}_z^\ell$ holds

$$\|\psi\|_{\mathcal{V}}^2 \le C\big(\|\psi_0\|_{\mathcal{V}}^2 + \sum_{\ell=0}^{L} \sum_{z\in\widetilde{\mathcal{N}}_\ell^\Gamma} \|\psi_z^\ell\|_{\mathcal{V}}^2\big). \tag{54}$$

Then, it follows $\langle \mathfrak{V}_{\mathrm{AS}}\psi \,,\, \psi\rangle_{\mathcal{V}} \le C\|\psi\|_{\mathcal{V}}^2$.

Proof of Proposition 8, lower bound in (52). By means of Lemma 9, we have to provide a stable subspace decomposition. For $\phi \in \mathcal{Y}^L$, we consider the unique decomposition $\phi = \phi_0 + \widetilde{\phi}$ from (48). With $\phi_0 = \langle \phi \,,\, 1\rangle_\Gamma/|\Gamma|$, we infer

$$\|\phi_0\|_{\mathcal{V}} = \frac{\langle \phi \,,\, 1\rangle_\Gamma}{|\Gamma|}\|1\|_{\mathcal{V}} \le \frac{\|\phi\|_{H^{-1/2}(\Gamma)}\|1\|_{H^{1/2}(\Gamma)}}{|\Gamma|}\|1\|_{\mathcal{V}} \lesssim \|\phi\|_{\mathcal{V}}. \tag{55}$$

Moreover, there exists $\widetilde{v} \in \mathcal{Z}_*^L$, such that $(\widetilde{v})' = \widetilde{\phi}$. The abstract result [Zha92, Lemma 3.1] states

$$\lambda_{\min}(\mathfrak{W}_{\mathrm{AS}}) = \min_{v\in\mathcal{Z}^L} \frac{\|v\|_{\mathcal{W}}^2}{\min\limits_{\sum_{\ell=0}^{L}\sum_{z\in\widetilde{\mathcal{N}}_\ell^\Gamma} v_z^\ell = v} \sum_{\ell=0}^{L} \sum_{z\in\widetilde{\mathcal{N}}_\ell^\Gamma} \|v_z^\ell\|_{\mathcal{W}}^2}, \tag{56}$$

since $\mathfrak{W}_{\mathrm{AS}}$ is a finite sum of symmetric projectors. Lemma 7 provides uniform boundedness of the Rayleigh quotient $\langle \mathfrak{W}_{\mathrm{AS}} v \,,\, v\rangle_{\mathcal{W}}/\|v\|_{\mathcal{W}}^2 \ge C_1 > 0$. Thus, $\lambda_{\min}(\mathfrak{W}_{\mathrm{AS}}) \ge C_1 > 0$ is uniformly bounded, and we infer from (56) the existence of a decomposition $\widetilde{v} = \sum_{\ell=0}^{L} \sum_{z\in\widetilde{\mathcal{N}}_\ell^\Gamma} v_z^\ell$ with $v_z^\ell \in \mathcal{Z}_z^\ell$ such that

$$\sum_{\ell=0}^{L} \sum_{z\in\widetilde{\mathcal{N}}_\ell^\Gamma} \|v_z^\ell\|_{\mathcal{W}}^2 \lesssim \|\widetilde{v}\|_{\mathcal{W}}^2. \tag{57}$$

This provides a decomposition of $\widetilde{\phi}$ into functions $\phi_z^\ell = (v_z^\ell)' \in \mathcal{Y}_z^\ell$ by

$$\widetilde{\phi} = (\widetilde{v})' =: \sum_{\ell=0}^{L} \sum_{z\in\widetilde{\mathcal{N}}_\ell^\Gamma} \phi_z^\ell. \tag{58}$$

The identities from (51) imply

$$\sum_{\ell=0}^{L} \sum_{z\in\widetilde{\mathcal{N}}_\ell^\Gamma} \|\phi_z^\ell\|_{\mathcal{V}}^2 = \sum_{\ell=0}^{L} \sum_{z\in\widetilde{\mathcal{N}}_\ell^\Gamma} \langle \mathcal{V}\phi_z^\ell \,,\, \phi_z^\ell\rangle_\Gamma = \sum_{\ell=0}^{L} \sum_{z\in\widetilde{\mathcal{N}}_\ell^\Gamma} \langle \mathcal{W}v_z^\ell \,,\, v_z^\ell\rangle_\Gamma \tag{59}$$

The estimate $\langle \mathcal{W} v_z^\ell \,,\, v_z^\ell \rangle_\Gamma \le \|v_z^\ell\|_{\mathcal{W}}^2$ and (57) prove

$$\sum_{\ell=0}^{L} \sum_{z\in\widetilde{\mathcal{N}}_\ell^\Gamma} \langle \mathcal{W} v_z^\ell \,,\, v_z^\ell \rangle_\Gamma \le \sum_{\ell=0}^{L} \sum_{z\in\widetilde{\mathcal{N}}_\ell^\Gamma} \|v_z^\ell\|_{\mathcal{W}}^2 \lesssim \|\widetilde{v}\|_{\mathcal{W}}^2. \tag{60}$$

Since $\langle \widetilde{v} \,,\, 1\rangle_\Gamma = 0$, there holds $\|\widetilde{v}\|_{\mathcal{W}}^2 = \langle \mathcal{W}\widetilde{v} \,,\, \widetilde{v}\rangle_\Gamma$. This, (59)–(60), and Maue's formula (51) yield

$$\sum_{\ell=0}^{L} \sum_{z\in\widetilde{\mathcal{N}}_\ell^\Gamma} \|\phi_z^\ell\|_{\mathcal{V}}^2 \lesssim \langle \mathcal{W}\widetilde{v} \,,\, \widetilde{v}\rangle_\Gamma = \langle \mathcal{V}\widetilde{\phi} \,,\, \widetilde{\phi}\rangle_\Gamma = \|\widetilde{\phi}\|_{\mathcal{V}}^2. \tag{61}$$

Recall that $\widetilde{\phi} = \phi - \phi_0$. With (55), the triangle inequality yields

$$\|\phi_0\|_{\mathcal{V}}^2 + \sum_{\ell=0}^{L} \sum_{z\in\widetilde{\mathcal{N}}_\ell^\Gamma} \|\phi_z^\ell\|_{\mathcal{V}}^2 \lesssim \|\phi_0\|_{\mathcal{V}}^2 + \|\widetilde{\phi}\|_{\mathcal{V}}^2 \lesssim \|\phi_0\|_{\mathcal{V}}^2 + \|\phi\|_{\mathcal{V}}^2 \lesssim \|\phi\|_{\mathcal{V}}^2, \tag{62}$$

where the hidden constants depend only on Γ, the initial triangulation $\mathcal{T}_0^\Gamma$, as well as the chosen mesh-refinement strategy. By means of Lemma 9 (i), this proves the lower bound in (52). □

Proof of Proposition 8, upper bound in (52). Recall the unique splitting $\phi = \phi_0 + \widetilde{\phi}$ from (48). Let $\sum_{\ell=0}^{L} \sum_{z\in\widetilde{\mathcal{N}}_\ell^\Gamma} \phi_z^\ell = \widetilde{\phi}$ denote an arbitrary splitting of $\widetilde{\phi} \in \sum_{\ell=0}^{L} \sum_{z\in\widetilde{\mathcal{N}}_\ell^\Gamma} \mathcal{Y}_z^\ell$. Note that $\phi_z^\ell = \alpha_z^\ell \chi_z^\ell$ for some $\alpha_z^\ell \in \mathbb{R}$. We define $\widetilde{v}$ as

$$\widetilde{v} := \sum_{\ell=0} \sum_{z\in\widetilde{\mathcal{N}}_\ell^\Gamma} v_z^\ell \quad \text{with } v_z^\ell := \alpha_z^\ell \zeta_z^\ell \tag{63}$$

and stress that $\widetilde{\phi} = (\widetilde{v})'$ as well as $\phi_z^\ell = (v_z^\ell)'$. The abstract result [Zha92, Lemma 3.1] states

$$\lambda_{\max}(\mathfrak{W}_{\rm AS}) = \max_{v\in\mathcal{Z}^L} \frac{\|v\|_{\mathcal{W}}^2}{\min\limits_{\sum_{\ell=0}^{L}\sum_{z\in\widetilde{\mathcal{N}}_\ell^\Gamma} v_z^\ell = v} \sum_{\ell=0}^{L} \sum_{z\in\widetilde{\mathcal{N}}_\ell^\Gamma} \|v_z^\ell\|_{\mathcal{W}}^2}, \tag{64}$$

since $\mathfrak{W}_{\rm AS}$ is a finite sum of symmetric projections. From Lemma 7, we get uniform boundedness of the Rayleigh quotient $\langle \mathfrak{W}_{\rm AS} v \,,\, v\rangle_{\mathcal{W}} / \|v\|_{\mathcal{W}}^2 \le C_2$. Thus, $\lambda_{\max}(\mathfrak{W}_{\rm AS}) \le C_2 < \infty$ is uniformly bounded. Then, (64) yields

$$\|\widetilde{v}\|_{\mathcal{W}}^2 \lesssim \sum_{\ell=0}^{L} \sum_{z\in\widetilde{\mathcal{N}}_\ell^\Gamma} \|v_z^\ell\|_{\mathcal{W}}^2. \tag{65}$$

Together with Maue's formula (51), the definition (63), and the norm equivalence

$$\|v_z^\ell\|_{\mathcal{W}}^2 \simeq \|v_z^\ell\|_{H^{1/2}(\Gamma)}^2 = \|v_z^\ell\|_{\widetilde{H}^{1/2}(\omega_\ell(z))}^2 \simeq \langle \mathcal{W} v_z^\ell \,,\, v_z^\ell\rangle_\Gamma = \langle \mathcal{V}\phi_z^\ell \,,\, \phi_z^\ell\rangle_\Gamma,$$

we infer

$$\|\widetilde{\phi}\|_{\mathcal{V}}^2 = \langle \mathcal{V}\widetilde{\phi} \,,\, \widetilde{\phi}\rangle_\Gamma = \langle \mathcal{W}\widetilde{v} \,,\, \widetilde{v}\rangle_\Gamma \le \|\widetilde{v}\|_{\mathcal{W}}^2 \lesssim \sum_{\ell=0}^{L} \sum_{z\in\widetilde{\mathcal{N}}_\ell^\Gamma} \|v_z^\ell\|_{\mathcal{W}}^2 \simeq \sum_{\ell=0}^{L} \sum_{z\in\widetilde{\mathcal{N}}_\ell^\Gamma} \|\phi_z^\ell\|_{\mathcal{V}}^2.$$

Finally, $\phi = \phi_0 + \widetilde{\phi}$ yields

$$\|\phi\|_{\mathcal{V}}^2 \lesssim \|\phi_0\|_{\mathcal{V}}^2 + \|\widetilde{\phi}\|_{\mathcal{V}}^2 \lesssim \|\phi_0\|_{\mathcal{V}}^2 + \sum_{\ell=0}^{L} \sum_{z\in\widetilde{\mathcal{N}}_\ell^\Gamma} \|\phi_z^\ell\|_{\mathcal{V}}^2, \tag{66}$$

where the hidden constants depend only on Γ, the initial triangulation $\mathcal{T}_0^\Gamma$, and the chosen mesh-refinement strategy. Lemma 9 (ii) and (66) prove the upper bound in (52). □

Proof of Theorem 6. Recall that $\psi_{T_m}^\ell \in \mathcal{Y}^\ell$ denotes the characteristic function of $T_m^\ell \in \mathcal{T}_\ell^\Gamma$. First, we prove the relation

$$\mathfrak{V}_{\rm AS}\phi = \sum_{m=1}^{M_L} (\mathbf{P}_{\mathcal{V}}^{-1}\mathbf{A}_{\mathcal{V}}\mathbf{\Phi})_m \psi_{T_m^L} \quad \text{for all } \phi = \sum_{j=1}^{M_L} (\mathbf{\Phi})_j \psi_{T_j^L} \in \mathcal{Y}^L.$$

Recall $\mathfrak{V}_{\rm AS} = \mathfrak{V}^{00} + \sum_{\ell=0}^{L}\sum_{z\in\widetilde{\mathcal{N}}_\ell^\Gamma} \mathfrak{V}_z^\ell$ from (44) and $\mathbf{1} \in \mathbb{R}^{M_L}$ with $(\mathbf{1})_j = 1$ for all $j = 1,\dots,M_L$. By definition (45) of $\mathfrak{V}^{00}$, it follows with $D = \|1\|_{\mathcal{V}}^2$

$$\mathfrak{V}^{00}\phi = D^{-1}\langle \phi\,,\, 1\rangle_{\mathcal{V}}\, 1 = D^{-1}\langle \mathbf{\Phi}\,,\, \mathbf{1}\rangle_{\mathbf{A}_{\mathcal{V}}} = \sum_{m=1}^{M_L} (\mathbf{1}D^{-1}\mathbf{1}^T\mathbf{A}_{\mathcal{V}}\mathbf{\Phi})_m \psi_{T_m^L}. \tag{67}$$

Moreover, from (45) we also infer

$$\mathfrak{V}_z^\ell \phi = \|\chi_z^\ell\|_{\mathcal{V}}^{-2} \langle \phi\,,\, \chi_z^\ell\rangle_{\mathcal{V}}\, \chi_z^\ell \quad \text{for all } z \in \widetilde{\mathcal{N}}_\ell^\Gamma. \tag{68}$$

With the definition of $\widetilde{\mathbf{H}}^\ell$ and $\mathbf{J}^\ell$ from Section 2.6, each Haar basis function $\chi_z^\ell \in \widetilde{\mathcal{Z}}^\ell$ can be represented as

$$\chi_{z_k}^\ell = \sum_{j=1}^{M_\ell} (\widetilde{\mathbf{H}}^\ell)_{jk}\psi_{T_j^\ell} = \sum_{j=1}^{M_\ell}\sum_{m=1}^{M_L} (\widetilde{\mathbf{H}}^\ell)_{jk}(\mathbf{J}^\ell)_{mj}\psi_{T_m^L} = \sum_{m=1}^{M_L} (\mathbf{J}^\ell\widetilde{\mathbf{H}}^\ell)_{mk}\psi_{T_m^L}. \tag{69}$$

Thus,

$$\begin{aligned}\langle \phi\,,\, \chi_{z_k}^\ell\rangle_{\mathcal{V}} &= \sum_{m=1}^{M_L} (\mathbf{J}^\ell\widetilde{\mathbf{H}}^\ell)_{mk}\langle \phi\,,\, \psi_{T_m^L}\rangle_{\mathcal{V}} = \sum_{m=1}^{M_L} ((\widetilde{\mathbf{H}}^\ell)^T(\mathbf{J}^\ell)^T)_{km}(\mathbf{A}_{\mathcal{V}}\mathbf{\Phi})_m\\ &= ((\widetilde{\mathbf{H}}^\ell)^T(\mathbf{J}^\ell)^T\mathbf{A}_{\mathcal{V}}\mathbf{\Phi})_k.\end{aligned}$$

Furthermore, the last identity together with (68), (69), and $\|\chi_{z_k}^\ell\|_{\mathcal{V}}^2 = (\widetilde{\mathbf{D}}_{\mathcal{W}}^\ell)_{kk}$ show

$$\mathfrak{V}_{z_k}^\ell \phi = \sum_{m=1}^{M_L} (\mathbf{J}^\ell\widetilde{\mathbf{H}}^\ell)_{mk}(\widetilde{\mathbf{D}}_{\mathcal{W}}^\ell)_{kk}^{-1}((\widetilde{\mathbf{H}}^\ell)^T(\mathbf{J}^\ell)^T\mathbf{A}_{\mathcal{V}}\mathbf{\Phi})_k\psi_{T_m^L}. \tag{70}$$

Summing the last terms over all $k = 1,\dots,\widetilde{N}_\ell^\Gamma := \#\widetilde{\mathcal{N}}_\ell^\Gamma$ and $\ell = 0,\dots,L$ yields with (67)

$$\begin{aligned}\mathfrak{V}_{\rm AS}\phi &= \sum_{m=1}^{M_L} (\mathbf{1}D^{-1}\mathbf{1}^T\mathbf{A}_{\mathcal{V}}\mathbf{\Phi})_m\psi_{T_m^L} + \sum_{\ell=0}^{L}\sum_{m=1}^{M_L} (\mathbf{J}^\ell\widetilde{\mathbf{H}}^\ell(\widetilde{\mathbf{D}}_{\mathcal{W}}^\ell)^{-1}(\widetilde{\mathbf{H}}^\ell)^T(\mathbf{J}^\ell)^T\mathbf{A}_{\mathcal{V}}\mathbf{\Phi})_m\psi_{T_m^L}\\ &= \sum_{m=1}^{M_L} (\mathbf{P}_{\mathcal{V}}^{-1}\mathbf{A}_{\mathcal{V}}\mathbf{\Phi})_m\psi_{T_m^L}.\end{aligned} \tag{71}$$

The last identity together with Proposition 8 implies

$$\langle \mathbf{P}_{\mathcal{V}}^{-1}\mathbf{A}_{\mathcal{V}}\mathbf{\Phi}\,,\,\mathbf{\Phi}\rangle_{\mathbf{A}_{\mathcal{V}}} = \langle \mathfrak{V}_{\mathrm{AS}}\phi\,,\,\phi\rangle_{\mathcal{V}} \simeq \|\phi\|_{\mathcal{V}}^2 = \|\mathbf{\Phi}\|_{\mathbf{A}_{\mathcal{V}}}^2, \tag{72}$$

where the hidden constants depend only on Γ, the initial triangulation $\mathcal{T}_0^\Gamma$, as well as on the chosen mesh-refinement. Finally, by setting $\mathbf{\Phi} = \mathbf{A}_{\mathcal{V}}^{-1/2}(\mathbf{A}_{\mathcal{V}}^{-1/2}\mathbf{P}_{\mathcal{V}}\mathbf{A}_{\mathcal{V}}^{-1/2})^{1/2}\mathbf{A}_{\mathcal{V}}^{1/2}\mathbf{\Psi}$ in (72), we get

$$\langle \mathbf{P}_{\mathcal{V}}\mathbf{\Psi}\,,\,\mathbf{\Psi}\rangle_2 \simeq \langle \mathbf{A}_{\mathcal{V}}\mathbf{\Psi}\,,\,\mathbf{\Psi}\rangle_2 \quad \text{for all } \mathbf{\Psi} \in \mathbb{R}^{M_L},$$

which concludes the proof. □

4. Proof of Theorem 4

Basically, we follow the lines of the proof of [MS98, Theorem 5.2], which was stated for the symmetric coupling and a block-diagonal preconditioner based on a hierarchical basis decomposition of the underlying discrete spaces. Here, we adapt the proof to the non-symmetric Johnson-Nédélec coupling.

For the analysis of the proposed block-diagonal preconditioner, we define the operator $\mathcal{B}$, which can be interpreted as a preconditioning form of the operator $\mathcal{L}$. The next result directly follows from the properties of the operator $\mathcal{A}$ from Lemma 2 and the properties of the simple-layer integral operator $\mathcal{V}$.

Lemma 10. *For $(u,\phi),(v,\psi) \in \mathcal{H}$, define*

$$\langle \mathcal{B}(u,\phi)\,,\,(v,\psi)\rangle := \langle \mathcal{A}u\,,\,v\rangle + \langle \psi\,,\,\mathcal{V}\phi\rangle_\Gamma.$$

Then, the operator $\mathcal{B} : \mathcal{H} \to \mathcal{H}^*$ is linear, symmetric, continuous, and elliptic, and the constants.

$$c_{\mathcal{B}} := \inf_{\mathbf{0}\neq(u,\phi)\in\mathcal{H}} \frac{\langle \mathcal{B}(u,\phi)\,,\,(u,\phi)\rangle}{\|(u,\phi)\|_{\mathcal{H}}^2} \quad \text{and} \quad C_{\mathcal{B}} := \sup_{\mathbf{0}\neq(u,\phi)\in\mathcal{H}} \frac{\|\mathcal{B}(u,\phi)\|_{\mathcal{H}^*}}{\|(u,\phi)\|_{\mathcal{H}}}$$

satisfy $0 < c_{\mathcal{B}} \le C_{\mathcal{B}} < \infty$ and depend only on $c_{\mathcal{A}}$ and $C_{\mathcal{A}}$ from (25b) *as well as on Ω.* □

The following auxiliary result is explicitly stated for the Johnson-Nédélec coupling and also used in [MS98] for the symmetric coupling accordingly.

Lemma 11. *Let $\mathcal{H}^L \subset \mathcal{H} = H^1(\Omega) \times H^{-1/2}(\Gamma)$ be a finite dimensional subspace. Let $\mathcal{L}_L, \mathcal{B}_L : \mathcal{H}^L \to (\mathcal{H}^L)^*$ denote the operators $\mathcal{L}, \mathcal{B}$ restricted to the discrete space $\mathcal{H}^L$, i.e.*

$$\langle \mathcal{L}_L\boldsymbol{u}_L\,,\,\boldsymbol{v}_L\rangle := \langle \mathcal{L}\boldsymbol{u}_L\,,\,\boldsymbol{v}_L\rangle$$
$$\langle \mathcal{B}_L\boldsymbol{u}_L\,,\,\boldsymbol{v}_L\rangle := \langle \mathcal{B}\boldsymbol{u}_L\,,\,\boldsymbol{v}_L\rangle$$

for all $\boldsymbol{u}_L, \boldsymbol{v}_L \in \mathcal{H}^L$. Define $\mathcal{Q}_L := \mathcal{L}_L^*\mathcal{B}_L^{-1}\mathcal{L}_L : \mathcal{H}^L \to (\mathcal{H}^L)^*$. Then, the Galerkin matrix $\mathbf{Q}$ of $\mathcal{Q}_L$ with respect to the basis $\{\varphi_j\}_{j=1}^{N_L+M_L}$ of $\mathcal{H}^L$, i.e. $(\mathbf{Q})_{jk} = \langle \mathcal{Q}_L\varphi_k\,,\,\varphi_j\rangle$ for $j,k = 1,\ldots,N_L+M_L$, satisfies

$$\mathbf{Q} = \mathbf{A}_{\mathcal{L}}^T\mathbf{A}_{\mathcal{B}}^{-1}\mathbf{A}_{\mathcal{L}}. \tag{73}$$

Proof. Let $\{\mathbf{E}^j\}_{j=1}^{N_L+M_L}$ denote the canonical basis of $\mathbb{R}^{N_L+M_L}$ with $\langle \mathbf{E}^j\,,\,\mathbf{E}^k\rangle_2 = \delta_{jk}$. The matrix entry of the j-th row and k-th column is given by

$$(\mathbf{Q})_{jk} = \langle \mathcal{L}_L^*\mathcal{B}_L^{-1}\mathcal{L}_L\varphi_k\,,\,\varphi_j\rangle = \langle \mathcal{B}_L^{-1}\mathcal{L}_L\varphi_k\,,\,\mathcal{L}_L\varphi_j\rangle.$$

Let $\boldsymbol{w}_k := \mathcal{B}_L^{-1}\mathcal{L}_L\varphi_k$ and note that by definition of the inverse of the *discrete* operator $\mathcal{B}_L$, $\boldsymbol{w}_k$ satisfies

$$\langle \mathcal{B}_L \boldsymbol{w}_k \,,\, \varphi_m\rangle = \langle \mathcal{L}_L\varphi_k \,,\, \varphi_m\rangle \quad m = 1,\dots,N_L+M_L. \tag{74}$$

Note that $\langle \mathcal{L}_L\varphi_k \,,\, \varphi_m\rangle = (\mathbf{A}_\mathcal{L})_{mk} = (\mathbf{A}_\mathcal{L}\mathbf{E}^k)_m$. Together with the basis representation $\boldsymbol{w}_k = \sum_{j=1}^{N_L+M_L}(\mathbf{W})_j\varphi_j$, the Galerkin formulation (74) of $\boldsymbol{w}_k$ is thus equivalent to

$$\mathbf{A}_\mathcal{B}\mathbf{W} = \mathbf{A}_\mathcal{L}\mathbf{E}^k.$$

By choice of $\boldsymbol{w}_k$, we get

$$\begin{aligned}(\mathbf{Q})_{jk} = \langle \boldsymbol{w}_k \,,\, \mathcal{L}_L\varphi_j\rangle &= \sum_{m=1}^{N_L+M_L}(\mathbf{W})_m\langle\varphi_m \,,\, \mathcal{L}_L\varphi_j\rangle = \sum_{m=1}^{N_L+M_L}\big(\mathbf{A}_\mathcal{B}^{-1}\mathbf{A}_\mathcal{L}\mathbf{E}^k\big)_m\big(\mathbf{A}_\mathcal{L}\mathbf{E}^j\big)_m\\ &= \langle \mathbf{A}_\mathcal{B}^{-1}\mathbf{A}_\mathcal{L}\mathbf{E}^k \,,\, \mathbf{A}_\mathcal{L}\mathbf{E}^j\rangle_2 = \langle \mathbf{A}_\mathcal{L}^T\mathbf{A}_\mathcal{B}^{-1}\mathbf{A}_\mathcal{L}\mathbf{E}^k \,,\, \mathbf{E}^j\rangle_2 \quad \text{for all } j,k = 1,\dots,N_L+M_L.\end{aligned}$$

This concludes the proof. □

Proof of Theorem 4. We use the following result on the reduction of the relative residual in the preconditioned GMRES Algorithm 3, which can be found, e.g., in [HS98, Section 3]: Due to [EES83, SS86], the j-th residuum from the preconditioned GMRES Algorithm 3 is bounded by

$$\|\mathbf{R}_j\|_{\mathbf{P}_\mathcal{L}} \le \big(1-\alpha^2/\beta^2\big)^{j/2}\|\mathbf{R}_0\|_{\mathbf{P}_\mathcal{L}}, \tag{75a}$$

with constants

$$\alpha := \inf_{\mathbf{U}\neq\mathbf{0}}\frac{\langle \mathbf{P}_\mathcal{L}^{-1}\mathbf{A}_\mathcal{L}\mathbf{U} \,,\, \mathbf{U}\rangle_{\mathbf{P}_\mathcal{L}}}{\|\mathbf{U}\|_{\mathbf{P}_\mathcal{L}}^2} \le \inf_{\mathbf{U}\neq\mathbf{0}}\frac{\|\mathbf{P}_\mathcal{L}^{-1}\mathbf{A}_\mathcal{L}\mathbf{U}\|_{\mathbf{P}_\mathcal{L}}}{\|\mathbf{U}\|_{\mathbf{P}_\mathcal{L}}} = \|(\mathbf{P}_\mathcal{L}^{-1}\mathbf{A}_\mathcal{L})^{-1}\|_{\mathbf{P}_\mathcal{L}}^{-1}, \tag{75b}$$

$$\beta := \sup_{\mathbf{U}\neq\mathbf{0}}\frac{\|\mathbf{P}_\mathcal{L}^{-1}\mathbf{A}_\mathcal{L}\mathbf{U}\|_{\mathbf{P}_\mathcal{L}}}{\|\mathbf{U}\|_{\mathbf{P}_\mathcal{L}}} = \|\mathbf{P}_\mathcal{L}^{-1}\mathbf{A}_\mathcal{L}\|_{\mathbf{P}_\mathcal{L}}, \tag{75c}$$

when $\langle\cdot\,,\,\cdot\rangle_{\mathbf{P}_\mathcal{L}} := \langle \mathbf{P}_\mathcal{L}(\cdot)\,,\,\cdot\rangle_2$ is used as inner product in the preconditioned GMRES algorithm. We also refer to [SS07] for a discussion on preconditioned GMRES methods using different inner products.

Due to (75), $\mathrm{cond}_{\mathbf{P}_\mathcal{L}}(\mathbf{P}_\mathcal{L}^{-1}\mathbf{A}_\mathcal{L}) \le \beta/\alpha$ and we have to provide a lower bound for (75b) and an upper bound for (75c). Recall that the preconditioner matrix $\mathbf{P}_\mathcal{L}$ and the Galerkin matrix $\mathbf{A}_\mathcal{B}$ of $\mathcal{B}$ have the form

$$\mathbf{P}_\mathcal{L} = \begin{pmatrix}\mathbf{P}_\mathcal{A} & \mathbf{0}\\ \mathbf{0} & \mathbf{P}_\mathcal{V}\end{pmatrix}, \qquad \mathbf{A}_\mathcal{B} = \begin{pmatrix}\mathbf{A}_\mathcal{A} & \mathbf{0}\\ \mathbf{0} & \mathbf{A}_\mathcal{V}\end{pmatrix}.$$

Define $d_\mathcal{B} := \min\{d_\mathcal{A}, d_\mathcal{V}\}$, $D_\mathcal{B} := \max\{D_\mathcal{A}, D_\mathcal{V}\}$. From Theorem 5 and Theorem 6, it follows

$$d_\mathcal{B}\langle \mathbf{P}_\mathcal{L}\mathbf{V} \,,\, \mathbf{V}\rangle_2 \le \langle \mathbf{A}_\mathcal{B}\mathbf{V} \,,\, \mathbf{V}\rangle_2 \le D_\mathcal{B}\langle \mathbf{P}_\mathcal{L}\mathbf{V} \,,\, \mathbf{V}\rangle_2 \quad \text{for all } \mathbf{V}\in\mathbb{R}^{N_L+M_L}, \tag{76}$$

which is equivalent to

$$d_\mathcal{B}\langle \mathbf{A}_\mathcal{B}^{-1}\mathbf{U} \,,\, \mathbf{U}\rangle_2 \le \langle \mathbf{P}_\mathcal{L}^{-1}\mathbf{U} \,,\, \mathbf{U}\rangle_2 \le D_\mathcal{B}\langle \mathbf{A}_\mathcal{B}^{-1}\mathbf{U} \,,\, \mathbf{U}\rangle_2 \quad \text{for all } \mathbf{U}\in\mathbb{R}^{N_L+M_L}. \tag{77}$$

Here, the equivalence of (76)–(77) follows from the choice $\mathbf{U} = \mathbf{A}_\mathcal{B}^{1/2}(\mathbf{A}_\mathcal{B}^{-1/2}\mathbf{P}_\mathcal{L}\mathbf{A}_\mathcal{B}^{-1/2})^{1/2}\mathbf{A}_\mathcal{B}^{1/2}\mathbf{V}$ resp. $\mathbf{V} = \mathbf{A}_\mathcal{B}^{-1/2}(\mathbf{A}_\mathcal{B}^{-1/2}\mathbf{P}_\mathcal{L}\mathbf{A}_\mathcal{B}^{-1/2})^{-1/2}\mathbf{A}_\mathcal{B}^{-1/2}\mathbf{U}$ and elementary calculations. Setting $\mathbf{U} =$

$\mathbf{A}_\mathcal{B}\mathbf{W}$ in (77) and since $\mathbf{P}_\mathcal{L}^{-1}\mathbf{A}_\mathcal{B}$ is symmetric with respect to $\langle\cdot\,,\,\cdot\rangle_{\mathbf{A}_\mathcal{B}}$ yields

$$d_\mathcal{B} \le \lambda_{\min}(\mathbf{P}_\mathcal{L}^{-1}\mathbf{A}_\mathcal{B}) \le \langle \mathbf{P}_\mathcal{L}^{-1}\mathbf{A}_\mathcal{B}\mathbf{W}\,,\,\mathbf{W}\rangle_{\mathbf{A}_\mathcal{B}}/\|\mathbf{W}\|_{\mathbf{A}_\mathcal{B}}^2 \le \lambda_{\max}(\mathbf{P}_\mathcal{L}^{-1}\mathbf{A}_\mathcal{B}) \le D_\mathcal{B}. \tag{78}$$

For given $\mathbf{U} = (\boldsymbol{X}, \boldsymbol{\Phi})^T \in \mathbb{R}^{N_L+M_L}$, let $\mathbf{u}_L = (u_L, \phi_L) \in \mathcal{H}^L$ denote the corresponding function. We start to prove a lower bound for (75b). Lemma 2, Lemma 10, and (76) yield

$$\begin{aligned}\langle \mathbf{P}_\mathcal{L}^{-1}\mathbf{A}_\mathcal{L}\mathbf{U}\,,\,\mathbf{U}\rangle_{\mathbf{P}_\mathcal{L}} &= \langle \mathbf{A}_\mathcal{L}\mathbf{U}\,,\,\mathbf{U}\rangle_2 = \langle \mathcal{L}\mathbf{u}_L\,,\,\mathbf{u}_L\rangle \ge c_\mathcal{L}\|\mathbf{u}_L\|_\mathcal{H}^2 \ge c_\mathcal{L}C_\mathcal{B}^{-1}\langle \mathcal{B}\mathbf{u}_L\,,\,\mathbf{u}_L\rangle \\ &= c_\mathcal{L}C_\mathcal{B}^{-1}\langle \mathbf{A}_\mathcal{B}\mathbf{U}\,,\,\mathbf{U}\rangle_2 = c_\mathcal{L}C_\mathcal{B}^{-1}d_\mathcal{B}\langle \mathbf{P}_\mathcal{L}\mathbf{U}\,,\,\mathbf{U}\rangle_2 = c_\mathcal{L}C_\mathcal{B}^{-1}d_\mathcal{B}\|\mathbf{U}\|_{\mathbf{P}_\mathcal{L}}^2.\end{aligned}$$

Thus, $\widetilde{\alpha} := c_\mathcal{L}C_\mathcal{B}^{-1}d_\mathcal{B} = c_\mathcal{L}C_\mathcal{B}^{-1}\min\{d_\mathcal{A}, d_\mathcal{V}\}$ is a lower bound for α from (75b).

It remains to prove an upper bound for β from (75c). With (77) and the discrete operator $\mathcal{Q}_L = \mathcal{L}_L^*\mathcal{B}_L^{-1}\mathcal{L}_L$ with Galerkin matrix $\mathbf{Q}$ from Lemma 11, we infer

$$\begin{aligned}\|\mathbf{P}_\mathcal{L}^{-1}\mathbf{A}_\mathcal{L}\mathbf{U}\|_{\mathbf{P}_\mathcal{L}}^2 = \langle \mathbf{A}_\mathcal{L}\mathbf{U}\,,\,\mathbf{P}_\mathcal{L}^{-1}\mathbf{A}_\mathcal{L}\mathbf{U}\rangle_2 \le D_\mathcal{B}\langle \mathbf{A}_\mathcal{B}^{-1}\mathbf{A}_\mathcal{L}\mathbf{U}\,,\,\mathbf{A}_\mathcal{L}\mathbf{U}\rangle_2 &= D_\mathcal{B}\langle \mathbf{Q}\mathbf{U}\,,\,\mathbf{U}\rangle_2 \\ &= D_\mathcal{B}\langle \mathcal{Q}_L\mathbf{u}_L\,,\,\mathbf{u}_L\rangle.\end{aligned}$$

Moreover, it holds

$$\begin{aligned}\langle \mathcal{Q}_L\mathbf{u}_L\,,\,\mathbf{u}_L\rangle = \langle \mathcal{L}_L^*\mathcal{B}_L^{-1}\mathcal{L}_L\mathbf{u}_L\,,\,\mathbf{u}_L\rangle &= \langle \mathcal{B}_L^{-1}\mathcal{L}_L\mathbf{u}_L\,,\,\mathcal{L}_L\mathbf{u}_L\rangle = \langle \mathcal{B}_L^{-1}\mathcal{L}_L\mathbf{u}_L\,,\,\mathcal{L}\mathbf{u}_L\rangle \\ &\le \|\mathcal{L}\mathbf{u}_L\|_{\mathcal{H}^*}\|\mathcal{B}_L^{-1}\mathcal{L}_L\mathbf{u}_L\|_\mathcal{H} \le C_\mathcal{L}\|\mathbf{u}_L\|_\mathcal{H}\|\mathcal{B}_L^{-1}\mathcal{L}_L\mathbf{u}_L\|_\mathcal{H}.\end{aligned}$$

Note that $\boldsymbol{w}_L := \mathcal{B}_L^{-1}\mathcal{L}_L\mathbf{u}_L$ is the Galerkin solution of

$$\langle \mathcal{B}\boldsymbol{w}_L\,,\,\mathbf{v}_L\rangle = \langle \mathcal{L}\mathbf{u}_L\,,\,\mathbf{v}_L\rangle \quad \text{for all } \mathbf{v}_L \in \mathcal{H}^L.$$

Therefore, we can estimate the norm of $\boldsymbol{w}_L$ by

$$c_\mathcal{B}\,\|\boldsymbol{w}_L\|_\mathcal{H}^2 \le \langle \mathcal{B}\boldsymbol{w}_L\,,\,\boldsymbol{w}_L\rangle = \langle \mathcal{L}\mathbf{u}_L\,,\,\boldsymbol{w}_L\rangle \le \|\mathcal{L}\mathbf{u}_L\|_{\mathcal{H}^*}\|\boldsymbol{w}_L\|_\mathcal{H} \le C_\mathcal{L}\|\mathbf{u}_L\|_\mathcal{H}\|\boldsymbol{w}_L\|_\mathcal{H}.$$

With (76), this altogether gives

$$\begin{aligned}\|\mathbf{P}_\mathcal{L}^{-1}\mathbf{A}_\mathcal{L}\mathbf{U}\|_{\mathbf{P}_\mathcal{L}}^2 &\le D_\mathcal{B}c_\mathcal{B}^{-1}C_\mathcal{L}^2\|\mathbf{u}_L\|_\mathcal{H}^2 \le D_\mathcal{B}c_\mathcal{B}^{-2}C_\mathcal{L}^2\langle \mathcal{B}\mathbf{u}_L\,,\,\mathbf{u}_L\rangle = D_\mathcal{B}c_\mathcal{B}^{-2}C_\mathcal{L}^2\langle \mathbf{A}_\mathcal{B}\mathbf{U}\,,\,\mathbf{U}\rangle_2 \\ &\le D_\mathcal{B}^2c_\mathcal{B}^{-2}C_\mathcal{L}^2\langle \mathbf{P}_\mathcal{L}\mathbf{U}\,,\,\mathbf{U}\rangle_2 = D_\mathcal{B}^2c_\mathcal{B}^{-2}C_\mathcal{L}^2\|\mathbf{U}\|_{\mathbf{P}_\mathcal{L}}^2.\end{aligned}$$

Then, $\widetilde{\beta} := D_\mathcal{B}c_\mathcal{B}^{-1}C_\mathcal{L} = \max\{D_\mathcal{A}, D_\mathcal{V}\}c_\mathcal{B}^{-1}C_\mathcal{L}$ is an upper bound for β from (75c).

Finally, the definition

$$q_{\mathrm{gmres}} := (1 - \widetilde{\alpha}^2/\widetilde{\beta}^2)^{1/2} = \Big(1 - \Big(\frac{c_\mathcal{L}c_\mathcal{B}\min\{d_\mathcal{A}, d_\mathcal{V}\}}{C_\mathcal{L}C_\mathcal{B}\max\{D_\mathcal{A}, D_\mathcal{V}\}}\Big)^2\Big)^{1/2}$$

concludes the proof. □

Remark 12. *Note that the last proof unveils*

$$\mathrm{cond}_{\mathbf{P}_\mathcal{L}}(\mathbf{P}_\mathcal{L}^{-1}\mathbf{A}_\mathcal{L}) \lesssim \frac{\lambda_{\max}(\mathbf{P}_\mathcal{L}^{-1}\mathbf{A}_\mathcal{B})}{\lambda_{\min}(\mathbf{P}_\mathcal{L}^{-1}\mathbf{A}_\mathcal{B})} = \mathrm{cond}_{\mathbf{P}_\mathcal{L}}(\mathbf{P}_\mathcal{L}^{-1}\mathbf{A}_\mathcal{B}) = \mathrm{cond}_{\mathbf{A}_\mathcal{B}}(\mathbf{P}_\mathcal{L}^{-1}\mathbf{A}_\mathcal{B}). \tag{79}$$

This result can be obtained by replacing $d_\mathcal{B}$ with $\lambda_{\min}(\mathbf{P}_\mathcal{L}^{-1}\mathbf{A}_\mathcal{B})$ resp. $D_\mathcal{B}$ with $\lambda_{\max}(\mathbf{P}_\mathcal{L}^{-1}\mathbf{A}_\mathcal{B})$ in the proof of Theorem 4.

5. Extension to other coupling methods and further remarks

5.1. Symmetric coupling. The model problem (4) can equivalently be reformulated by means of the symmetric coupling [Cos88, Han90]: Find $(u, \phi) \in \mathcal{H} := H^1(\Omega) \times H^{-1/2}(\Gamma)$ such that

$$\begin{aligned} \langle A\nabla u\,,\,\nabla v\rangle_\Omega + \langle \mathcal{W}u\,,\,v\rangle_\Gamma + \langle (\mathcal{K}' - \tfrac{1}{2})\phi\,,\,v\rangle_\Gamma &= \langle f\,,\,v\rangle_\Omega + \langle \phi_0 + \mathcal{W}u_0\,,\,v\rangle_\Gamma, \\ \langle \psi\,,\,(\tfrac{1}{2} - \mathcal{K})u + \mathcal{V}\phi\rangle_\Gamma &= \langle \psi\,,\,(\tfrac{1}{2} - \mathcal{K})u_0\rangle_\Gamma. \end{aligned} \tag{80}$$

for all $(v, \psi) \in \mathcal{H}$. Analogously to the Johnson-Nédélec coupling (13)–(14), we define the operator $\widetilde{\mathcal{L}} : \mathcal{H} \to \mathcal{H}^*$ resp. the linear functional $\widetilde{F} \in \mathcal{H}^*$ for an equivalent operator formulation

$$\widetilde{\mathcal{L}}(u, \phi) = \widetilde{F}$$

of the symmetric coupling (80) by

$$\begin{aligned} \langle \widetilde{\mathcal{L}}(u, \phi)\,,\,(v, \psi)\rangle &:= \langle A\nabla u\,,\,\nabla v\rangle_\Omega + \langle \mathcal{W}u\,,\,v\rangle_\Gamma + \langle (\mathcal{K}' - \tfrac{1}{2})\phi\,,\,v\rangle_\Gamma + \langle \psi\,,\,(\tfrac{1}{2} - \mathcal{K})u + \mathcal{V}\phi\rangle_\Gamma \\ &\quad + \langle 1\,,\,(\tfrac{1}{2} - \mathcal{K})u + \mathcal{V}\phi\rangle_\Gamma \langle 1\,,\,(\tfrac{1}{2} - \mathcal{K})v + \mathcal{V}\psi\rangle_\Gamma, \\ \langle \widetilde{F}\,,\,(u, \phi)\rangle &:= \langle f\,,\,v\rangle_\Gamma + \langle \phi_0 + \mathcal{W}u_0\,,\,v\rangle_\Gamma + \langle \psi\,,\,(\tfrac{1}{2} - \mathcal{K})u_0\rangle_\Gamma \\ &\quad + \langle 1\,,\,(\tfrac{1}{2} - \mathcal{K})u_0\rangle_\Gamma \langle 1\,,\,(\tfrac{1}{2} - \mathcal{K})v + \mathcal{V}\psi\rangle_\Gamma, \end{aligned}$$

for all $(u, \phi), (v, \psi) \in \mathcal{H}$. We stress that Lemma 1 also holds for the symmetric coupling with $(\mathcal{L}, F)$ replaced by $(\widetilde{\mathcal{L}}, \widetilde{F})$. The following result can be found in [AFF+13a, Section 5].

Lemma 13. *Lemma 1 holds accordingly for the symmetric coupling, where* (12) *is replaced by* $c_A > 0$. □

Let $\mathbf{A}_\mathcal{W}$ denote the Galerkin matrix of the hypersingular integral operator with respect to the nodal basis of $\mathcal{X}^L$, i.e. $(\mathbf{A}_\mathcal{W})_{jk} = \langle \mathcal{W}\eta^L_{z_k}\,,\,\eta^L_{z_j}\rangle_\Gamma$ for all $j, k = 1, \dots, N_L$. The Galerkin matrix $\mathbf{A}_{\widetilde{\mathcal{L}}}$ of the operator $\widetilde{\mathcal{L}}$ reads in matrix block form

$$\mathbf{A}_{\widetilde{\mathcal{L}}} = \begin{pmatrix} \mathbf{A}_A + \mathbf{A}_\mathcal{W} & \mathbf{K}^T - \frac{1}{2}\mathbf{M}^T \\ \frac{1}{2}\mathbf{M} - \mathbf{K} & \mathbf{A}_\mathcal{V} \end{pmatrix} + \mathbf{S}\mathbf{S}^T.$$

We use the block-diagonal preconditioner

$$\mathbf{P}_{\widetilde{\mathcal{L}}} := \begin{pmatrix} \mathbf{P}_{\widetilde{\mathcal{A}}} & \mathbf{0} \\ \mathbf{0} & \mathbf{P}_\mathcal{V} \end{pmatrix},$$

which is similar to the one for the Johnson-Nédélec coupling. Here, $\widetilde{\mathcal{A}} : H^1(\Omega) \to (H^1(\Omega))^*$ is defined as $\langle \widetilde{\mathcal{A}}u\,,\,v\rangle := \langle \mathcal{A}u\,,\,v\rangle + \langle \mathcal{W}u\,,\,v\rangle_\Gamma$ and $\mathbf{P}_{\widetilde{\mathcal{A}}}$ is defined as $\mathbf{P}_\mathcal{A}$ with the diagonals of $\mathbf{A}_\mathcal{A}$ replaced by the diagonals of the Galerkin matrix of $\widetilde{\mathcal{A}}$. We seek for a solution of the preconditioned system

$$\mathbf{P}_{\widetilde{\mathcal{L}}}^{-1}\mathbf{A}_{\widetilde{\mathcal{L}}}\mathbf{U} = \mathbf{P}_{\widetilde{\mathcal{L}}}^{-1}\widetilde{\mathbf{F}}, \tag{81}$$

where $\widetilde{\mathbf{F}}$ denotes the discretization of the right-hand side $\widetilde{F}$. The following theorem is proved along the lines of Section 4 with the obvious modifications.

Theorem 14. *Theorem 4 holds accordingly for the symmetric coupling.* □

5.2. One-equation Bielak-MacCamy coupling. The model problem (4) can equivalently be rewritten by means of the one-equation Bielak-MacCamy coupling [BM84] which can be seen as the "transposed" Johnson-Nédélec coupling: Find $(u, \phi) \in \mathcal{H}$ such that

$$
\begin{aligned}
\langle A\nabla u \,,\, \nabla v\rangle_\Omega + \langle(\tfrac{1}{2} - \mathcal{K}')\phi \,,\, v\rangle_\Gamma &= \langle f \,,\, v\rangle_\Omega + \langle \phi_0 \,,\, v\rangle_\Gamma, \\
\langle \psi \,,\, \mathcal{V}\phi - u\rangle_\Gamma &= -\langle \psi \,,\, u_0\rangle_\Gamma,
\end{aligned} \tag{82}
$$

for all $(v, \psi) \in \mathcal{H}$. Analogously to the Johnson-Nédélec coupling (13)–(14), we define the operator $\widetilde{\mathcal{L}} : \mathcal{H} \to \mathcal{H}^*$ and the linear functional $\widetilde{F} \in \mathcal{H}^*$ for an equivalent operator formulation

$$\widetilde{\mathcal{L}}(u, \phi) = \widetilde{F}$$

of the Bielak-MacCamy coupling (82) by

$$
\begin{aligned}
\langle \widetilde{\mathcal{L}}(u, \phi) \,,\, (v, \psi)\rangle &:= \langle A\nabla u \,,\, \nabla v\rangle_\Omega + \langle(\tfrac{1}{2} - \mathcal{K}')\phi \,,\, v\rangle_\Gamma + \langle \psi \,,\, \mathcal{V}\phi - u\rangle_\Gamma \\
&\quad + \langle 1 \,,\, \mathcal{V}\phi - u\rangle_\Gamma \langle 1 \,,\, \mathcal{V}\psi - v\rangle_\Gamma, \\
\langle \widetilde{F} \,,\, (v, \psi)\rangle &:= \langle f \,,\, v\rangle_\Omega + \langle \phi_0 \,,\, v\rangle_\Gamma - \langle \psi \,,\, u_0\rangle_\Gamma - \langle 1 \,,\, u_0\rangle_\Gamma \langle 1 \,,\, \mathcal{V}\psi - v\rangle_\Gamma,
\end{aligned}
$$

for all $(u, \phi), (v, \psi) \in \mathcal{H}$. The following result is found in [AFF$^+$13a, Section 3].

Lemma 15. *Provided* (12), *Lemma 1 holds accordingly for the Bielak-MacCamy coupling.* □

The Galerkin matrix $\mathbf{A}_{\widetilde{\mathcal{L}}}$ of the operator $\widetilde{\mathcal{L}}$ reads in matrix block form

$$\mathbf{A}_{\widetilde{\mathcal{L}}} = \begin{pmatrix} \mathbf{A}_A & \frac{1}{2}\mathbf{M}^T - \mathbf{K}^T \\ -\mathbf{M} & \mathbf{A}_{\mathcal{V}} \end{pmatrix} + \widetilde{\mathbf{S}}\widetilde{\mathbf{S}}^T,$$

where the (column) vector $\widetilde{\mathbf{S}}$ is defined componentwise by $(\widetilde{\mathbf{S}})_j := -\langle 1 \,,\, \eta^L_{z_j}\rangle_\Gamma$ for $j = 1, \dots, N_L$ and $(\widetilde{\mathbf{S}})_{j+N_L} := \langle 1 \,,\, \mathcal{V}\psi_{T_j}\rangle_\Gamma$ for $j = 1, \dots, M_L$. We use the same block-diagonal preconditioner (26) as for the Johnson-Nédélec coupling. The following theorem is proved along the lines of Section 4 with the obvious modifications.

Theorem 16. *Theorem 4 holds accordingly for the Bielak-MacCamy coupling.* □

5.3. Further remarks. The analysis in Section 4 depends only on the spectral estimates (40) and (42). Therefore, the multilevel additive Schwarz preconditioners $\mathbf{P}_\mathcal{A}$ and $\mathbf{P}_\mathcal{V}$ can be replaced by any preconditioners $\widetilde{\mathbf{P}}_\mathcal{A}$ and $\widetilde{\mathbf{P}}_\mathcal{V}$ such that

$$\langle \widetilde{\mathbf{P}}_\mathcal{A}\boldsymbol{X} \,,\, \boldsymbol{X}\rangle_2 \simeq \langle \mathbf{A}_\mathcal{A}\boldsymbol{X} \,,\, \boldsymbol{X}\rangle_2 \quad \text{and} \quad \langle \widetilde{\mathbf{P}}_\mathcal{V}\boldsymbol{\Phi} \,,\, \boldsymbol{\Phi}\rangle_2 \simeq \langle \mathbf{A}_\mathcal{V}\boldsymbol{\Phi} \,,\, \boldsymbol{\Phi}\rangle_2 \tag{83}$$

holds for all $\boldsymbol{X} \in \mathbb{R}^{N_L}, \boldsymbol{\Phi} \in \mathbb{R}^{M_L}$. The reduction constant q_{GMRES} from Theorem 4 then depends on the equivalence constants in (83). Preferably, the preconditioners $\widetilde{\mathbf{P}}_\mathcal{A}$ and $\widetilde{\mathbf{P}}_\mathcal{V}$ should be chosen such that these constants are independent of mesh-related quantities as is the case for the local multilevel additive Schwarz preconditioners considered here.

The techniques presented in this work may also apply for the (quasi-)symmetric Bielak-MacCamy coupling [BM84]. A stability analysis of this coupling method can be found in [GHS12].

It is also possible to apply our analysis to other model problems, e.g., transmission problems for linear elasticity. We stress that our approach requires a (possibly non-symmetric)

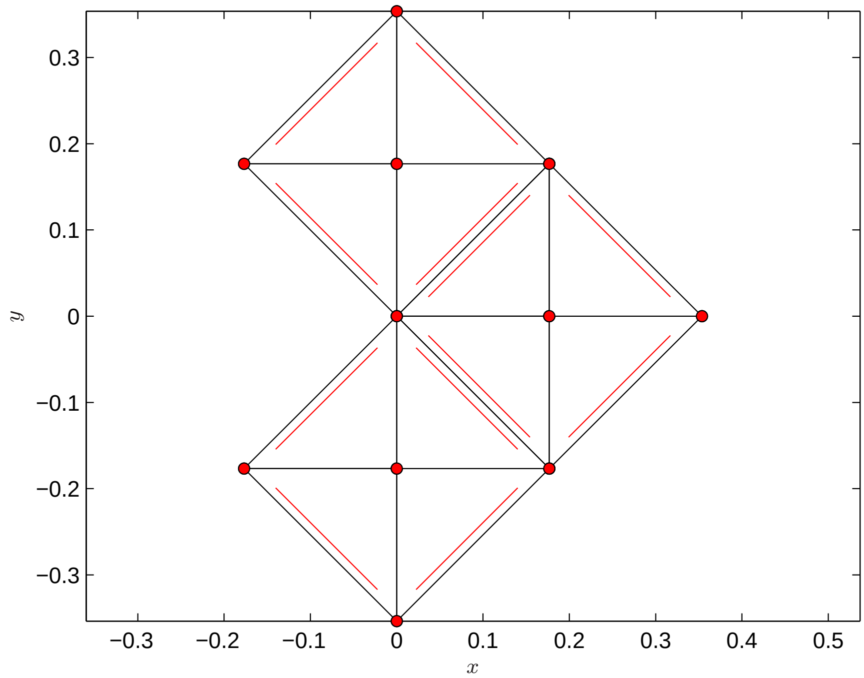

FIGURE 3. L-shaped domain Ω with $\mathrm{diam}(\Omega) < 1$ and initial volume triangulation $\mathcal{T}_0^\Omega$. The initial triangulation $\mathcal{T}_0^\Gamma$ of the boundary is given by the restriction $\mathcal{T}_0^\Omega|_\Gamma$ of the volume triangulation on the boundary. The initial triangulations consist of $\#\mathcal{T}_0^\Omega = 12$ resp. $\#\mathcal{T}_0^\Gamma = 8$ elements. Red lines indicate the reference edges for the newest vertex bisection of the initial volume triangulation.

positive definite Galerkin matrix, associated to the coupling method. For Lamé-type problems, this can be ensured by stabilization, where a result analogously to Lemma 1 remains valid [FFKP12].

6. Numerical examples

6.1. Weakly-singular integral equation with adaptive mesh-refinement. In our first experiment, we underline the result of Theorem 6, which states the uniform boundedness of the condition number of the preconditioned simple-layer operator. We consider the homogeneous Laplace equation

$$-\Delta u = 0 \quad \text{in } \Omega, \tag{84a}$$

$$u = g \quad \text{on } \Gamma := \partial\Omega \tag{84b}$$

with given Dirichlet data $g \in H^{1/2}(\Gamma)$ and the L-shaped domain Ω from Figure 3. We note that $\mathrm{diam}(\Omega) = 2/3 < 1$. Problem (84) is equivalent to the weakly-singular integral equation

$$\mathcal{V}\phi = (1/2 + \mathcal{K})g, \tag{85}$$

where $\phi = \partial_{\boldsymbol{n}} u$ and $(\mathcal{K} - 1/2) : H^{1/2}(\Gamma) \to H^{1/2}(\Gamma)$ denotes the trace of the double layer potential

$$\widetilde{\mathcal{K}}g(x) = \int_\Gamma \partial_{\boldsymbol{n}(y)} G(x-y) g(y) \, d\Gamma_y.$$

Equation (85) reads in the variational formulation: Find $\phi \in \mathcal{Y} := H^{-1/2}(\Gamma)$ such that

$$\langle \psi \, , \, \phi \rangle_{\mathcal{V}} = \langle \psi \, , \, (1/2 + \mathcal{K}) g \rangle_\Gamma \quad \text{for all } \psi \in \mathcal{Y}. \tag{86}$$

We prescribe the exact solution

$$u(x, y) = r^{2/3} \cos(2\varphi/3)$$

with $(x, y) = (r \cos\varphi, r \sin\varphi)$ given in 2D polar coordinates. Then, $g := u|_\Gamma$ and $\phi = \partial_{\boldsymbol{n}} u$. The exact solution of (86) exhibits a generic singularity at the reentrant corner $(0, 0) \in \mathbb{R}^2$. We use the local ZZ-type error indicators developed in [FFKP14] to steer the mesh-adaptation and to resolve this singularity effectively.

The discrete version of (86) reads in matrix notation: Find $\mathbf{\Phi} \in \mathbb{R}^{M_L}$ such that

$$\mathbf{A}_{\mathcal{V}} \mathbf{\Phi} = \mathbf{G} \in \mathbb{R}^{M_L}, \tag{87}$$

where $(\mathbf{G})_j = \langle \psi_{T_j} \, , \, (1/2 + \mathcal{K}) g \rangle_\Gamma$ for all $j = 1, \ldots, M_L$. Due to [AMT99], the ℓ_2-condition number of the Galerkin matrix $\mathbf{A}_{\mathcal{V}} \in \mathbb{R}^{M_L \times M_L}$ is bounded by

$$\mathrm{cond}_2(\mathbf{A}_{\mathcal{V}}) \lesssim M_L \Big(\frac{h_{\max}}{h_{\min}} \Big)^2 (1 + |\log(M_L h_{\max})|) =: \alpha_L \tag{88}$$

and, thus, can become bad on adaptively refined meshes. Therefore, we consider the preconditioned system

$$\mathbf{P}^{-1} \mathbf{A}_{\mathcal{V}} \mathbf{\Phi} = \mathbf{P}^{-1} \mathbf{G} \tag{89}$$

where the preconditioner matrix $\mathbf{P} \in \mathbb{R}^{M_L \times M_L}$ is either the local multilevel preconditioner $\mathbf{P}_{\mathcal{V}}$ proposed in Section 2.6 or the simple diagonal scaling $\mathbf{P}_{\mathrm{diag}} := \mathrm{diag}(\mathbf{A}_{\mathcal{V}})$ proposed in [AMT99]. According to Theorem 6, the eigenvalues of $\mathbf{P}_{\mathcal{V}}^{-1} \mathbf{A}_{\mathcal{V}}$ are uniformly bounded. Since $\mathbf{P}_{\mathcal{V}}^{-1} \mathbf{A}_{\mathcal{V}}$ is symmetric with respect to $\langle \cdot \, , \, \cdot \rangle_{\mathbf{A}_{\mathcal{V}}}$ and $\langle \cdot \, , \, \cdot \rangle_{\mathbf{P}_{\mathcal{V}}}$, the condition number can be estimated by

$$\mathrm{cond}_{\mathbf{A}_{\mathcal{V}}}(\mathbf{P}_{\mathcal{V}}^{-1} \mathbf{A}_{\mathcal{V}}) = \mathrm{cond}_{\mathbf{P}_{\mathcal{V}}}(\mathbf{P}_{\mathcal{V}}^{-1} \mathbf{A}_{\mathcal{V}}) = \frac{\lambda_{\max}(\mathbf{P}_{\mathcal{V}}^{-1} \mathbf{A}_{\mathcal{V}})}{\lambda_{\min}(\mathbf{P}_{\mathcal{V}}^{-1} \mathbf{A}_{\mathcal{V}})} \lesssim 1 \tag{90}$$

with $\lambda_{\min}(\cdot)$ and $\lambda_{\max}(\cdot)$ being the minimal resp. maximal eigenvalue. On the other hand, it has been proved in [AMT99] that

$$\mathrm{cond}_{\mathbf{A}_{\mathcal{V}}}(\mathbf{P}_{\mathrm{diag}}^{-1} \mathbf{A}_{\mathcal{V}}) \lesssim M_L (1 + |\log(M_L h_{\min})|) \frac{1 + |\log h_{\min}|}{1 + |\log h_{\max}|} =: \beta_L. \tag{91}$$

In Figure 4, we compare the condition numbers of the Galerkin matrix $\mathbf{A}_{\mathcal{V}}$ and the preconditioned matrices $\mathbf{P}_{\mathcal{V}}^{-1} \mathbf{A}_{\mathcal{V}}$ and $\mathbf{P}_{\mathrm{diag}}^{-1} \mathbf{A}_{\mathcal{V}}$. We observe that the condition numbers behave as predicted by the estimates (88) and (90)–(91).

6.2. Transmission problem with adaptive mesh-refinement. Let Ω denote the L-shaped domain from Figure 3. We consider the (stabilized) Johnson-Nédélec FEM-BEM

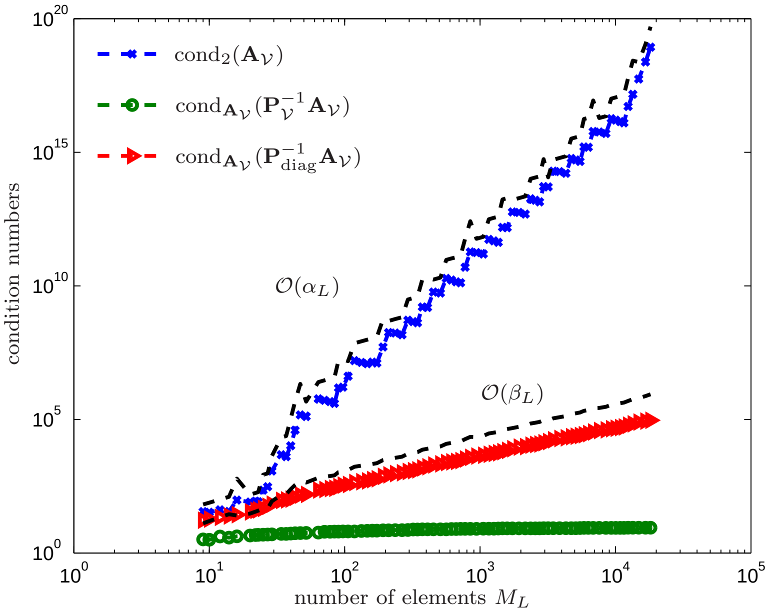

FIGURE 4. Condition numbers for the unpreconditioned matrix $\mathbf{A}_\mathcal{V}$ and the preconditioned matrices $\mathbf{P}_\mathcal{V}^{-1}\mathbf{A}_\mathcal{V}$ and $\mathbf{P}_{\mathrm{diag}}^{-1}\mathbf{A}_\mathcal{V}$ for the weakly-singular integral equation from Section 6.1. Here, α_L resp. β_L are the upper bounds in the estimates (88) resp. (91).

coupling (15) for the transmission problem (4) with $A(x)$ being the 2×2 identity matrix, i.e. $-\mathrm{div}(A\nabla u) = -\Delta u$ in Ω. We prescribe the exact solutions

$$u(x,y) = r^{2/3}\cos(2\varphi/3) \quad \text{for } (x,y) \in \Omega, \tag{92}$$

$$u^{\mathrm{ext}}(x,y) = \frac{1}{10}\frac{x+y-0.125}{(x-0.125)^2+y^2} \quad \text{for } (x,y) \in \Omega^{\mathrm{ext}}, \tag{93}$$

where (r,φ) denote the 2D polar coordinates. The data $f \in L^2(\Omega)$, $u_0 \in H^{1/2}(\Gamma)$, and $\phi_0 \in H^{-1/2}(\Gamma)$ are computed thereof. We stress that u, hence also $u_0 = (u - u^{\mathrm{ext}})|_\Gamma$, exhibits a generic singularity at the reentrant corner $(0,0) \in \mathbb{R}^2$. To steer the mesh-adaptivity, we use the residual-based error estimator from [AFF$^+$13a, AFKP12] which dates back to [CS95] for the symmetric coupling.

For the Johnson-Nédélec coupling, we compare the proposed optimal preconditioner $\mathbf{P}_\mathcal{L}$ with the block-diagonal preconditioner

$$\mathbf{P}_\mathcal{L}^{\mathrm{HB}} = \begin{pmatrix} \mathbf{P}_\mathcal{A}^{\mathrm{HB}} & \mathbf{0} \\ \mathbf{0} & \mathbf{P}_\mathcal{V}^{\mathrm{HB}} \end{pmatrix}, \tag{94}$$

which was proposed and analyzed in [MS98] for the symmetric coupling. Here, $\mathbf{P}_\mathcal{A}^{\mathrm{HB}}$ denotes the hierarchical basis preconditioner corresponding to the operator $\mathcal{A}$, and $\mathbf{P}_\mathcal{V}^{\mathrm{HB}}$ denotes the hierarchical basis preconditioner corresponding to the simple-layer operator $\mathcal{V}$. Basically, the difference between local multilevel preconditioners and hierarchical preconditioners is that the set $\widetilde{\mathcal{N}}_\ell^\Omega$ resp. $\widetilde{\mathcal{N}}_\ell^\Gamma$ is replaced by the set of new nodes $\mathcal{N}_{\ell+1}^\Omega \backslash \mathcal{N}_\ell^\Omega$ resp. $\mathcal{N}_{\ell+1}^\Gamma \backslash \mathcal{N}_\ell^\Gamma$.

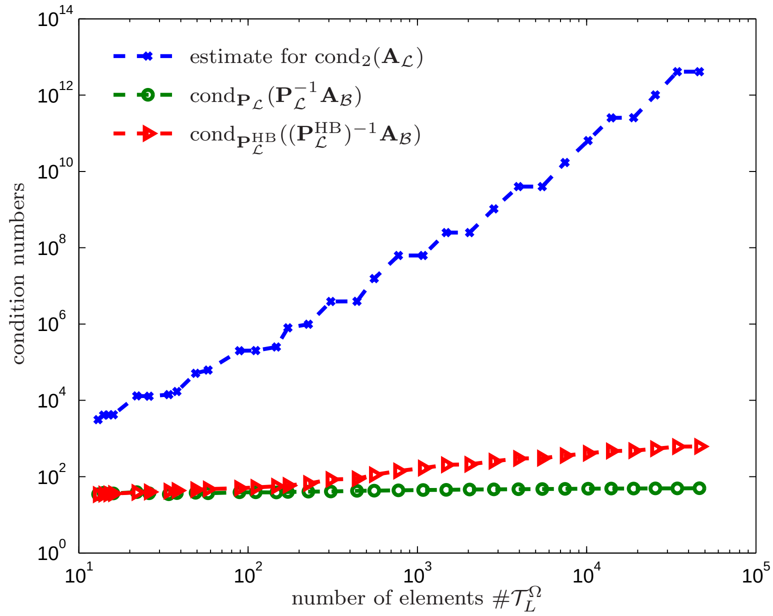

FIGURE 5. Estimate (95) for the condition number of $\mathbf{A}_\mathcal{L}$ and the condition number of $\mathbf{P}_\mathcal{L}^{-1}\mathbf{A}_\mathcal{B}$ resp. $(\mathbf{P}_\mathcal{L}^{\mathrm{HB}})^{-1}\mathbf{A}_\mathcal{B}$ for the stabilized Johnson-Nédélec coupling of Section 6.2.

This means that scaling is only done on the newly created nodes, but not on their neigbours. It is well-known that hierarchical basis preconditioners lead to sub-optimal condition number, which depend on the number of levels L. A more detailed discussion can be found in [Yse86] for FEM problems and in [TSM97] for BEM model problems. See also [XCH10, Section 6] resp. [FFPS13, Section 3] for a numerical comparison between hierarchical basis and local multilevel additive Schwarz preconditioners for some FEM resp. BEM problems on adaptively refined meshes. Sub-optimality of $\mathbf{P}_\mathcal{A}^{\mathrm{HB}}$ and $\mathbf{P}_\mathcal{V}^{\mathrm{HB}}$ lead to sub-optimality of the FEM-BEM preconditioner $\mathbf{P}_\mathcal{L}^{\mathrm{HB}}$, i.e. a dependency on the level L. Thus, also the number of iterations depend on L, which is also seen in our numerical examples.

In Figure 5, we plot $\mathrm{cond}_{\mathbf{P}_\mathcal{L}}(\mathbf{P}_\mathcal{L}^{-1}\mathbf{A}_\mathcal{B})$ which is an upper bound for $\mathrm{cond}_{\mathbf{P}_\mathcal{L}}(\mathbf{P}_\mathcal{L}^{-1}\mathbf{A}_\mathcal{L})$, see Remark 12, and compare it with the condition number $\mathrm{cond}_{\mathbf{P}_\mathcal{L}^{\mathrm{HB}}}((\mathbf{P}_\mathcal{L}^{\mathrm{HB}})^{-1}\mathbf{A}_\mathcal{B})$. We observe that the condition number of the preconditioned matrix $(\mathbf{P}_\mathcal{L}^{\mathrm{HB}})^{-1}\mathbf{A}_\mathcal{L}$ depends on the level L, whereas the condition number of $\mathbf{P}_\mathcal{L}^{-1}\mathbf{A}_\mathcal{L}$ is independent of the level L. This underlines the optimality of the preconditioner $\mathbf{P}_\mathcal{L}$ as stated in Theorem 4. Additionally, we plot the estimate

$$\mathrm{cond}_2(\mathbf{A}_\mathcal{L}) \le \sqrt{\mathrm{cond}_1(\mathbf{A}_\mathcal{L})\mathrm{cond}_1(\mathbf{A}_\mathcal{L}^T)}. \tag{95}$$

for the ℓ_2-condition number of $\mathbf{A}_\mathcal{L}$ in Figure 5. An estimate for the condition number $\mathrm{cond}_1(\mathbf{A}_\mathcal{L}) = \|\mathbf{A}_\mathcal{L}\|_1\|\mathbf{A}_\mathcal{L}^{-1}\|_1$ is computed with the MATLAB function `condest`. Estimate (95) is obtained from

$$\|\mathbf{A}\|_2 \le \sqrt{\|\mathbf{A}\|_1\|\mathbf{A}\|_\infty} \quad \text{for all matrices } \mathbf{A}.$$

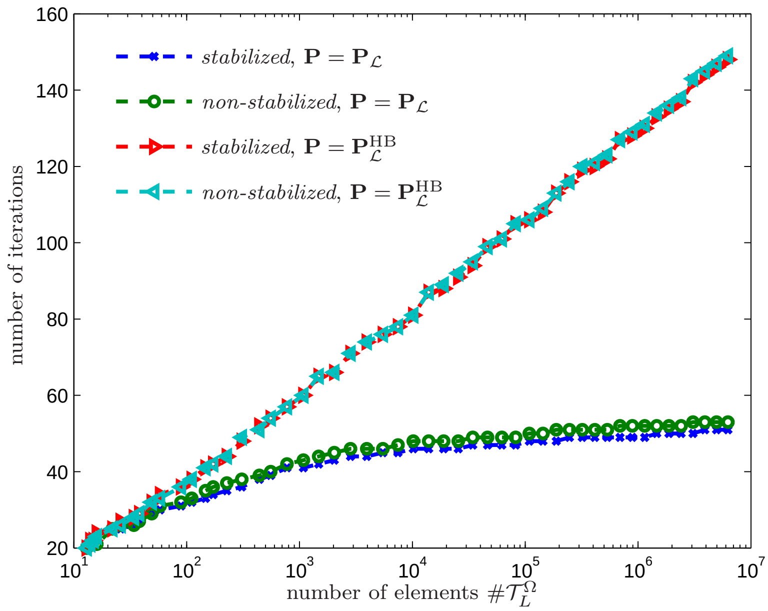

Figure 6. Number of iterations for solving the non-stabilized (96) resp. stabilized Johnson-Nédélec coupling (97) using the preconditioned GMRES Algorithm 3 with tolerance $\tau = 10^{-6}$, inner product $\mathbf{P} = \mathbf{P}_\mathcal{L}$ resp. $\mathbf{P} = \mathbf{P}_\mathcal{L}^{\mathrm{HB}}$, and initial guess $\mathbf{U}_0 = \mathbf{0}$.

In Figure 6, we furthermore consider the *non-stabilized* system

$$\mathbf{P}^{-1}\widehat{\mathbf{A}}_\mathcal{L}\mathbf{U} = \mathbf{P}^{-1}\widehat{\mathbf{F}}, \tag{96}$$

where $\widehat{\mathbf{A}}_\mathcal{L}$ corresponds to the Galerkin matrix of the *non-stabilized* problem (14) and $\widehat{\mathbf{F}}$ corresponds to the right-hand side of (14). The matrix $\mathbf{P}$ is either the preconditioner matrix $\mathbf{P}_\mathcal{L}$ or $\mathbf{P}_\mathcal{L}^{\mathrm{HB}}$. Note that by Lemma 1, the solution $\mathbf{U}$ of (96) is unique and also a solution of

$$\mathbf{P}^{-1}\mathbf{A}_\mathcal{L}\mathbf{U} = \mathbf{P}^{-1}\mathbf{F}. \tag{97}$$

In Figure 6, we plot the number of iterations used in the preconditioned GMRES Algorithm 3 with tolerance $\tau = 10^{-6}$, inner product $\mathbf{P} = \mathbf{P}_\mathcal{L}$ resp. $\mathbf{P} = \mathbf{P}_\mathcal{L}^{\mathrm{HB}}$, and initial guess $\mathbf{U}_0 = \mathbf{0}$ for solving the problem (97) and problem (96). We observe that, both for $\mathbf{P}_\mathcal{L}$ and $\mathbf{P}_\mathcal{L}^{\mathrm{HB}}$, the number of iterations for solving the *non-stabilized* problem (96) is slightly higher than the number of iterations used for solving problem (97) with the *stabilized* system matrix $\mathbf{A}_\mathcal{L}$.

6.3. Symmetric coupling vs. Johnson-Nédélec coupling. In a further experiment, we compare the (stabilized) Johnson-Nédélec coupling (97) and the (stabilized) symmetric coupling (81) with respect to the number of iterations used in the preconditioned GMRES Algorithm 3 with $\tau = 10^{-3}$ and $\mathbf{P} = \mathbf{P}_\mathcal{L}$ resp. $\mathbf{P} = \mathbf{P}_{\widetilde{\mathcal{L}}}$. For the initial guess $\mathbf{U}_0$ we prolongate the solution of (97) resp. (81) at level $L-1$ to level L. Mesh-adaptivity is steered with the solution of (81) and the residual-based error estimator from [CS95]. In Figure 7, we plot the number of iterations used for adaptive refinement. We observe that for both the uniform and adaptive case, the symmetric coupling needs less iterations. However, the symmetric

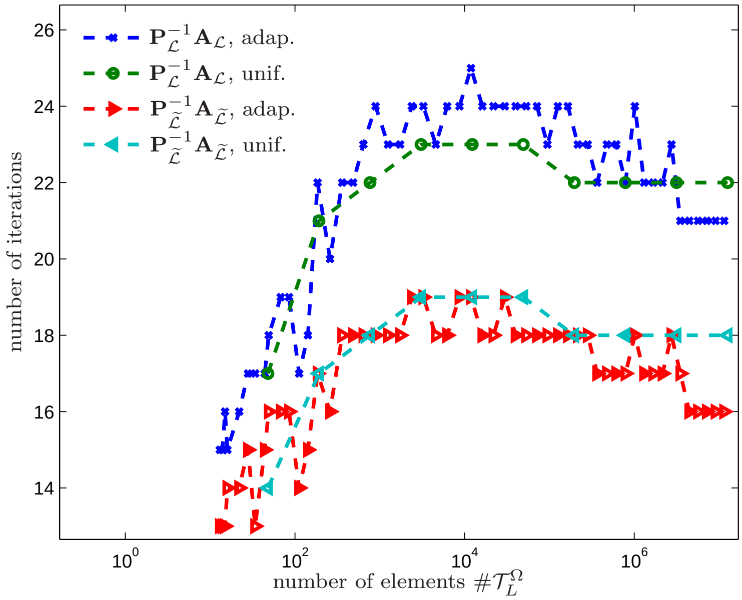

FIGURE 7. Number of iterations for solving the stabilized Johnson-Nédélec coupling (97) resp. symmetric coupling (81) using the preconditioned GMRES Algorithm 3 with tolerance $\tau = 10^{-3}$ and inner product $\mathbf{P} = \mathbf{P}_{\mathcal{L}}$ resp. $\mathbf{P} = \mathbf{P}_{\widetilde{\mathcal{L}}}$ on adaptively and uniformly refined meshes. For the initial guess $\mathbf{U}_0$ we prolongate the solution of (97) resp. (81) at level $L-1$ to level L.

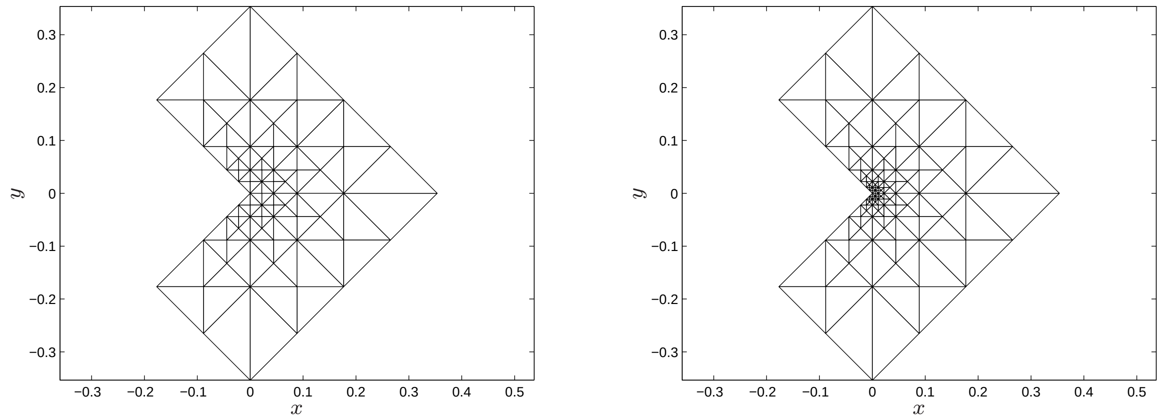

FIGURE 8. Artificially refined meshes at level $L = 3$ (left) with $\#\mathcal{T}_3^\Omega = 114$ resp. $M_3 = \#\mathcal{T}_3^\Gamma = 20$ and level $L = 5$ (right) with $\#\mathcal{T}_5^\Omega = 186$ resp. $M_5 = \#\mathcal{T}_5^\Gamma = 24$ for the stabilized Johnson-Nédélec coupling of Section 6.4.

coupling requires the computation of additional matrix-vector multiplications with discrete BEM operators in each iteration step.

L	$\#\mathcal{T}_L^\Omega$	M_L	$\mathrm{cond}_2(\mathbf{A}_\mathcal{L})$	$\mathrm{cond}_{\mathbf{P}_\mathcal{L}}(\mathbf{P}_\mathcal{L}^{-1}\mathbf{A}_\mathcal{B})$	$\mathrm{cond}_{\mathbf{P}_\mathcal{L}^{\mathrm{HB}}}((\mathbf{P}_\mathcal{L}^{\mathrm{HB}})^{-1}\mathbf{A}_\mathcal{B})$	$h_{\max}$	$h_{\min}$
1	42	16	4.57e+03	36.65	43.72	0.18	8.84e-02
2	78	18	1.60e+04	41.03	46.69	0.18	4.42e-02
3	114	20	6.11e+04	44.47	49.49	0.18	2.21e-02
4	150	22	2.40e+05	47.26	54.62	0.18	1.10e-02
5	186	24	9.54e+05	49.39	67.47	0.18	5.52e-03
6	222	26	3.80e+06	51.03	86.40	0.18	2.76e-03
7	258	28	1.52e+07	52.30	108.98	0.18	1.38e-03
8	294	30	6.07e+07	53.30	134.68	0.18	6.91e-04
9	330	32	2.43e+08	54.10	163.37	0.18	3.45e-04
10	366	34	9.70e+08	54.75	195.00	0.18	1.73e-04
11	402	36	3.88e+09	55.28	229.54	0.18	8.63e-05
12	438	38	1.55e+10	55.72	267.00	0.18	4.32e-05
13	474	40	6.21e+10	56.09	307.35	0.18	2.16e-05
14	510	42	2.48e+11	56.40	350.59	0.18	1.08e-05
15	546	44	9.93e+11	56.67	396.72	0.18	5.39e-06
16	582	46	3.97e+12	56.89	445.73	0.18	2.70e-06
17	618	48	1.59e+13	57.09	497.62	0.18	1.35e-06
18	654	50	6.35e+13	57.26	552.40	0.18	6.74e-07
19	690	52	2.54e+14	57.41	610.05	0.18	3.37e-07
20	726	54	1.02e+15	57.54	670.57	0.18	1.69e-07
21	762	56	4.07e+15	57.66	733.97	0.18	8.43e-08
22	798	58	1.63e+16	57.76	800.24	0.18	4.21e-08
23	834	60	6.51e+16	57.85	869.38	0.18	2.11e-08

Table 1. Condition number estimates for the example with artificial mesh-refinement from Section 6.4. Note that $\mathrm{cond}_{\mathbf{P}_\mathcal{L}}(\mathbf{P}_\mathcal{L}^{-1}\mathbf{A}_\mathcal{B})$ is an upper bound for the condition number $\mathrm{cond}_{\mathbf{P}_\mathcal{L}}(\mathbf{P}_\mathcal{L}^{-1}\mathbf{A}_\mathcal{L})$, see Remark 12, and $\mathrm{cond}_{\mathbf{P}_\mathcal{L}^{\mathrm{HB}}}((\mathbf{P}_\mathcal{L}^{\mathrm{HB}})^{-1}\mathbf{A}_\mathcal{B})$ is an upper bound for $\mathrm{cond}_{\mathbf{P}_\mathcal{L}^{\mathrm{HB}}}((\mathbf{P}_\mathcal{L}^{\mathrm{HB}})^{-1}\mathbf{A}_\mathcal{L})$.

6.4. Transmission problem with artificial refinement. Let Ω denote the L-shaped domain with boundary $\Gamma = \partial\Omega$ and initial triangulations $\mathcal{T}_0^\Omega$, $\mathcal{T}_0^\Gamma$ from Figure 3. We consider an artificial mesh-refinement, where we only mark the elements $T \in \mathcal{T}_\ell^\Omega$, $\mathcal{T}_\ell^\Gamma$ with $(0,0) \in T$ for refinement. Clearly, this leads to strongly adapted meshes towards the origin $(0,0) \in \mathbb{R}^2$, see Figure 8. As for the example from Section 6.2, we compare $\mathrm{cond}_{\mathbf{P}_\mathcal{L}}(\mathbf{P}_\mathcal{L}^{-1}\mathbf{A}_\mathcal{B})$ and $\mathrm{cond}_{\mathbf{P}_\mathcal{L}^{\mathrm{HB}}}((\mathbf{P}_\mathcal{L}^{\mathrm{HB}})^{-1}\mathbf{A}_\mathcal{B})$ as well as the estimate (95) for $\mathrm{cond}_2(\mathbf{A}_\mathcal{L})$. The results are summarized in Table 1. We observe optimality of the proposed preconditioner $\mathbf{P}_\mathcal{L}$, whereas the condition numbers for the hierarchical preconditioner depend on the number of levels L.

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