ASC Report No. 16/2013

ZZ-type a posteriori error estimators for adaptive boundary element methods on a curve

M. Feischl, T. Führer, M. Karkulik, D. Praetorius

Institute for Analysis and Scientific Computing
Vienna University of Technology — TU Wien
www.asc.tuwien.ac.at ISBN 978-3-902627-05-6

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Institute for Analysis and Scientific Computing
Vienna University of Technology
Wiedner Hauptstraße 8–10
1040 Wien, Austria

E-Mail: admin@asc.tuwien.ac.at
WWW: http://www.asc.tuwien.ac.at
FAX: +43-1-58801-10196

ISBN 978-3-902627-05-6

ZZ-Type A Posteriori Error Estimators for Adaptive Boundary Element Methods on a Curve

Michael Feischl, Thomas Führer, Michael Karkulik, Dirk Praetorius*

Abstract

In the context of the adaptive finite element method (FEM), ZZ-error estimators named after Zienkiewicz and Zhu [ZZ87] are mathematically well-established and widely used in practice. In this work, we propose and analyze ZZ-type error estimators for the adaptive boundary element method (BEM). We consider weakly-singular and hyper-singular integral equations and prove, in particular, convergence of the related adaptive mesh-refining algorithms. Throughout, the theoretical findings are underlined by numerical experiments.

Keywords: boundary element method, local mesh-refinement, adaptive algorithm, ZZ-type error estimator
2010 MSC: 65N30, 65N38, 65N50

1. Introduction

Since the seminal work of Zienkiewicz and Zhu [ZZ87], averaging techniques became popular in engineering and applied sciences for the a posteriori error control of the finite element solution of partial differential equations. To sketch the idea, we consider the most simple context of the 2D Poisson equation

$$\begin{aligned} -\Delta \mathfrak{u} &= \mathfrak{f} \quad \text{in } \Omega, \\ \mathfrak{u} &= 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{1}$$

Here and throughout the work, $\Omega \subset \mathbb{R}^2$ is a bounded Lipschitz domain with polygonal boundary $\partial\Omega$.

Let $\mathcal{T}_h$ denote a regular triangulation of Ω into compact, nondegenerate triangles. Let $\mathcal{P}^0(\mathcal{T}_h)$ be the space of all $\mathcal{T}_h$-piecewise constant functions and $\mathcal{S}^1(\mathcal{T}_h)$ be the space of all $\mathcal{T}_h$-piecewise affine and globally continuous splines. The lowest-order finite element solution $\mathfrak{u}_h \in \mathcal{S}^1_0(\mathcal{T}_h) := \big\{\mathfrak{v}_h \in \mathcal{S}^1(\mathcal{T}_h) \,:\, \mathfrak{v}_h = 0 \text{ on } \partial\Omega\big\}$ is the unique solution of the Galerkin formulation

$$\int_\Omega \nabla \mathfrak{u}_h \cdot \nabla \mathfrak{v}_h \, dx = \int_\Omega \mathfrak{f}\mathfrak{v}_h \, dx \tag{2}$$

for all test functions $\mathfrak{v}_h \in \mathcal{S}^1_0(\mathcal{T}_h)$. In this context, the ZZ error estimator reads

$$\eta_h = \|(1-\mathcal{A}_h)\nabla \mathfrak{u}_h\|_{L^2(\Omega)}, \tag{3}$$

where $\mathcal{A}_h : \mathcal{P}^0(\mathcal{T}_h)^2 \to \mathcal{S}^1(\mathcal{T}_h)^2$ is some averaging operator which maps the $\mathcal{T}_h$-piecewise constant gradient $\nabla\mathfrak{u}_h \in \mathcal{P}^0(\mathcal{T}_h)^2$ onto some continuous and piecewise affine function $\mathcal{A}_h\nabla\mathfrak{u}_h \in \mathcal{S}^1(\mathcal{T}_h)^2$. Possible choices for $\mathcal{A}_h$ are the usual Clément-type operators like

$$(\mathcal{A}_h\mathfrak{v})(z) = \frac{1}{\text{area}(\omega_z)} \int_{\omega_z} \mathfrak{v} \, dx \tag{4}$$

for all nodes $z \in \mathcal{K}_h$ of $\mathcal{T}_h$, where

$$\omega_z := \bigcup \big\{T \in \mathcal{T}_h \,:\, z \in T\big\} \tag{5}$$

denotes the patch of z, i.e., the union of all elements $T \in \mathcal{T}_h$ which have z as a node. Although ZZ error estimators are strikingly simple and mathematically well-developed for the finite element method, see e.g. [BC02a, BC02b, Car04, Rod94], they have not been considered for boundary element methods, yet. Available error estimators from the literature include residual-based error estimators for weakly-singular [CS95, CS96, Car97, CF01, CMS01, Fae00, Fae02] and hyper-singular integral equations [Car97, CMPS04], hierarchical error estimators for weakly-singular [EH06, HMS01, MSW98] and hyper-singular integral equations [Heu02, HMS01], $(h-h/2)$-based error estimators [EFLFP09, EFGP12, FLP08], averaging on large patches [CP06, CP07b, CP07a], and estimators based on the use of the full Calderón system [MPM99, SS00, Ste00]. The reader is also referred to the overviews given in [CF01, EFGP12] and the references therein.

This note proposes ZZ-type error estimators in the context of the boundary element method. As model problems serve the hyper-singular and the weakly-singular integral equation associated with the 2D Laplacian. Difficulties arise from the fact that neither the involved integral operators nor the energy norms are local.

*Corresponding Author
Email addresses: Michael.Feischl@tuwien.ac.at (Michael Feischl), Thomas.Fuehrer@tuwien.ac.at (Thomas Führer), mkarkulik@mat.puc.cl (Michael Karkulik), Dirk.Praetorius@tuwien.ac.at (Dirk Praetorius)
URL: http://www.asc.tuwien.ac.at/abem (Dirk Praetorius)

The outline of this paper reads as follows: In Section 2, we consider the hyper-singular integral equation, introduce a ZZ-type error estimator, and provide numerical evidence for its successful use on a slit model problem as well as for the first-kind integral formulation of some Neumann problem. In Section 3, we apply this approach in the context of the weakly-singular integral equation. While Section 2 and Section 3 are written for a general audience, Section 4 collects the preliminaries for the numerical analysis of the proposed a posteriori error estimators. A rigorous a posteriori error analysis is postponed to Section 5. The final Section 6 even proves convergence of the standard adaptive mesh-refining algorithm steered by the ZZ-type error estimators proposed.

2. Hyper-singular integral equation

We suppose that $\Omega \subset \mathbb{R}^2$ is simply connected, i.e., Ω has no holes and $\partial\Omega$ thus is connected. We denote the fundamental solution of the 2D Laplacian by

$$G(z) := -\frac{1}{2\pi} \log|z| \quad \text{for } z \in \mathbb{R}^2 \backslash \{0\}. \tag{6}$$

Let Γ be some relatively open and connected subset of the boundary $\partial\Omega$. Then, the hyper-singular integral operator is formally defined by

$$(Wu)(x) = -\partial_{n(x)} \int_\Gamma \partial_{n(y)} G(x-y) u(y)\, d\Gamma(y) \tag{7}$$

for $x \in \Gamma$. Here, $\int_\Gamma d\Gamma$ denotes integration along the curve and $\partial_{n(x)}$ is the normal derivative at some point $x \in \Gamma$. The hyper-singular integral equation reads

$$Wu = f \quad \text{on } \Gamma. \tag{8}$$

For the following facts on the functional analytic setting as well as for proofs and further details, the reader is referred to e.g. the monographs [HW08, McL00, SS11].

2.1. Slit model problem

Assume that $\Gamma \subsetneqq \partial\Omega$ is not closed. Let $\widetilde{H}^{1/2}(\Gamma)$ denote the space of all $H^{1/2}(\Gamma)$-functions which vanish at the tips of Γ. Then, W is a linear, bounded and elliptic operator between the fractional-order Sobolev space $\widetilde{H}^{1/2}(\Gamma)$ and its dual space $H^{-1/2}(\Gamma)$, where duality is understood with respect to the extended $L^2(\Gamma)$-scalar product $\langle \cdot\,,\, \cdot \rangle_{L^2(\Gamma)}$. Let $f \in H^{-1/2}(\Gamma)$. The variational form of (8) reads

$$\langle Wu\,,\, v\rangle_{L^2(\Gamma)} = \langle f\,,\, v\rangle_{L^2(\Gamma)} \quad \text{for all } v \in \widetilde{H}^{1/2}(\Gamma). \tag{9}$$

Since the left-hand side defines a scalar product on $\widetilde{H}^{1/2}(\Gamma)$, the Lax-Milgram lemma provides existence and uniqueness of the solution u.

2.2. Model problem on closed boundaries

Assume that $\Gamma = \partial\Omega$ is closed. Then, W is a linear and bounded operator from $H^{1/2}(\partial\Omega)$ to $H^{-1/2}_\star(\partial\Omega) := \big\{\psi \in H^{-1/2}(\partial\Omega) \,:\, \langle \psi\,,\, 1\rangle_{L^2(\partial\Omega)} = 0\big\}$. Moreover, W is elliptic on the subspace $H^{1/2}(\partial\Omega)/\mathbb{R} \equiv H^{1/2}_\star(\partial\Omega) := \big\{v \in H^{1/2}(\partial\Omega) \,:\, \int_{\partial\Omega} v\, d\Gamma = 0\big\}$, where connectedness of $\partial\Omega$ is required. Let $f \in H^{-1/2}_\star(\partial\Omega)$. The variational form of (8) now reads

$$\langle Wu\,,\, v\rangle_{L^2(\partial\Omega)} = \langle f\,,\, v\rangle_{L^2(\partial\Omega)} \text{ for all } v \in H^{1/2}_\star(\partial\Omega). \tag{10}$$

As before, the left-hand side defines a scalar product on $H^{1/2}_\star(\partial\Omega)$, and the Lax-Milgram lemma thus provides existence and uniqueness of the solution u.

We note that, for certain right-hand sides f and $\Gamma = \partial\Omega$, (8) is an equivalent formulation of the Neumann problem

$$\begin{aligned} -\Delta \mathfrak{u} &= \mathfrak{f} \quad \text{in } \Omega, \\ \partial_n \mathfrak{u} &= \mathfrak{g} \quad \text{on } \partial\Omega. \end{aligned} \tag{11}$$

In this case, the solution u of (8) is, up to some additive constant, the trace $\mathfrak{u}|_{\partial\Omega}$ of the solution $\mathfrak{u}$ of (11).

2.3. Galerkin boundary element discretization

Let $\mathcal{T}_h$ be a partition of Γ into affine line segments. Let $\mathcal{S}^1(\mathcal{T}_h)$ denote the space of all functions v_h which are continuous and $\mathcal{T}_h$-piecewise affine with respect to the arclength. For $\Gamma \subsetneqq \partial\Omega$, $\mathcal{S}^1_0(\mathcal{T}_h) := \mathcal{S}^1(\mathcal{T}_h) \cap \widetilde{H}^{1/2}(\Gamma)$ denotes the subspace of all functions $v_h \in \mathcal{S}^1(\mathcal{T}_h)$ which additionally vanish at the tips of Γ. For $\Gamma = \partial\Omega$, $\mathcal{S}^1_0(\mathcal{T}_h) := \mathcal{S}^1(\mathcal{T}_h) \cap H^{1/2}_\star(\Gamma)$ denotes the subspace of all functions $v_h \in \mathcal{S}^1(\mathcal{T}_h)$ which satisfy $\int_\Gamma v_h\, d\Gamma = 0$. In either case, $\mathcal{S}^1_0(\mathcal{T}_h)$ is a conforming subspace of $\widetilde{H}^{1/2}(\Gamma)$ resp. $H^{1/2}_\star(\partial\Omega)$. In particular, the Galerkin formulation of (9) resp. (10) reads

$$\langle Wu_h\,,\, v_h\rangle_{L^2(\Gamma)} = \langle f\,,\, v_h\rangle_{L^2(\Gamma)} \text{ for all } v_h \in \mathcal{S}^1_0(\mathcal{T}_h) \tag{12}$$

and admits a unique Galerkin solution $u_h \in \mathcal{S}^1_0(\mathcal{T}_h)$.

2.4. ZZ-type error estimator

Let $h \in L^\infty(\Gamma)$ be the local mesh-size function defined by

$$h|_T := \text{length}(T) \quad \text{for } T \in \mathcal{T}_h \tag{13}$$

with the arclength $\text{length}(\cdot)$. With $(\cdot)'$ denoting the arclength derivative, we propose the following ZZ-type error estimator

$$\eta_h = \|h^{1/2}(1-\mathcal{A}_h)u_h'\|_{L^2(\Gamma)}, \tag{14}$$

where $\mathcal{A}_h : L^2(\Gamma) \to \mathcal{S}^1(\mathcal{T}_h)$ denotes the Clément operator defined by

$$(\mathcal{A}_h v)(z) := \frac{1}{\text{length}(\omega_z)} \int_{\omega_z} v\, d\Gamma \tag{15}$$

for all nodes $z \in \mathcal{K}_h$ of $\mathcal{T}_h$ with $\omega_z = \bigcup\big\{T \in \mathcal{T}_h \,:\, z \in T\big\}$ the node patch.

2.5. Adaptive mesh-refining algorithm

Given a right-hand side $f \in H^{-1/2}(\Gamma)$, an initial partition $\mathcal{T}_h$ of Γ, and some adaptivity parameter $0 < \theta < 1$, the proposed adaptive algorithm reads as follows:

(i) Compute discrete solution $u_h \in \mathcal{S}_0^1(\mathcal{T}_h)$.

(ii) For all $T \in \mathcal{T}_h$, compute the refinement indicators

$$\eta_h(T)^2 := \text{length}(T)\, \|(1-\mathcal{A}_h)u_h'\|_{L^2(T)}^2. \qquad (16)$$

(iii) Determine a set $\mathcal{M}_h \subseteq \mathcal{T}_h$ such that

$$\theta\, \eta_h^2 \le \sum_{T\in\mathcal{M}_h} \eta_h(T)^2. \qquad (17)$$

(iv) Generate a new mesh $\mathcal{T}_h$ by bisection of at least all elements in $\mathcal{M}_h$.

(v) goto (i) and iterate.

For the proof of quasi-optimal convergence rates in the frame of adaptive FEM, e.g. [Ste07, CKNS08], and adaptive BEM [FKMP13, Tso13], the set $\mathcal{M}_h$ in step (iii) is usually chosen with minimal cardinality. A greedy algorithms sorts the indicators in descending order and then iteratively splits $\mathcal{T}_h$ into $\mathcal{M}_h$ and $\mathcal{T}_h\backslash\mathcal{M}_h$ by moving the largest indicator from $\mathcal{T}_h\backslash\mathcal{M}_h$ to $\mathcal{M}_h$ until the Dörfler criterion (17) is satisfied.

For our implementation, we use the MATLAB BEM library `HILBERT` [AEF+11]. The local mesh-refinement in step (iv) of the algorithm is done by some bisection-based algorithm from [AFF+13] which guarantees that the local mesh-ratio

$$\kappa(\mathcal{T}_h) := \max\Big\{\frac{\text{length}(T)}{\text{length}(T')} \,:\, T, T' \in \mathcal{T}_h \text{ neighbors}\Big\} \quad (18)$$

stays uniformly bounded $\kappa(\mathcal{T}_h) \le \gamma$ for some $\gamma \ge 2$ which depends only on the initial partition. We stress that such a property is required for the numerical analysis of η_h in Section 5 and Section 6 below.

We recall from the literature [SS11] that the optimal rate of convergence with lowest-order BEM is $\mathcal{O}(h^{3/2})$ if the exact solution is smooth. This corresponds to $\mathcal{O}(N^{-3/2})$ with respect to the number N of elements on adaptively generated meshes.

2.6. Numerical experiment for slit problem

We consider the hyper-singular integral equation

$$Wu = 1 \quad \text{on } \Gamma = (-1,1)\times\{0\}. \qquad (19)$$

The exact solution is known and reads $u(x,0) = 2\sqrt{1-x^2}$. Note that $u \in \widetilde{H}^{1/2}(\Gamma)\cap H^{1-\varepsilon}(\Gamma)$ for all $\varepsilon > 0$. In particular, we expect an empirical convergence order $\mathcal{O}(h^{1/2})$ for uniform mesh-refinement.

The initial mesh $\mathcal{T}_h$ for the computation is shown in Figure 1. We compare adaptive mesh-refinement with parameter $\theta = 1/2$ with uniform mesh-refinement. The corresponding convergence graphs are visualized in Figure 2. While uniform mesh-refinement leads to the predicted suboptimal order $\mathcal{O}(h^{1/2}) = \mathcal{O}(N^{-1/2})$, the proposed adaptive strategy regains the optimal rate $\mathcal{O}(N^{3/2})$.

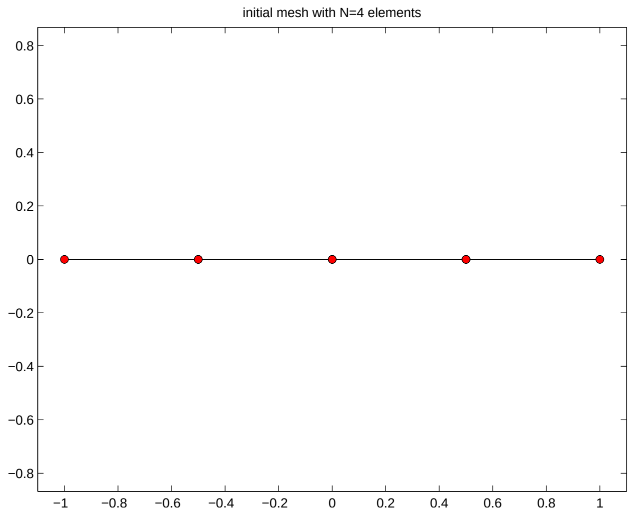

Figure 1: Slit $\Gamma = (-1,1)\times\{0\}$ and initial mesh $\mathcal{T}_h$ with $N = 4$ elements of the numerical experiment for the hyper-singular integral equation from Section 2.6.

2.7. Numerical experiment on closed boundary

We consider the Z-shaped domain with reentrant corner at the origin $(0,0)$, see Figure 3 for a sketch. The right-hand side $f = (1/2-K')(\partial_n\mathfrak{u}) \in H^{-1/2}(\Gamma)$ with $\Gamma = \partial\Omega$ and K' the adjoint double layer-potential is chosen such that the hyper-singular integral equation (8) is equivalent to some Neumann problem (11) with $\mathfrak{f} = 0$. The exact solution reads

$$\mathfrak{u}(x) = r^{4/7}\cos(4\varphi/7) \qquad (20)$$

in 2D polar coordinates $x = r\,(\cos\varphi, \sin\varphi)$. The exact solution u of (8) is, up to some additive constant, the trace $\mathfrak{u}|_\Gamma$. Moreover, u admits a generic singularity at the reentrant corner. Note that $u \in H_0^{1/2}(\partial\Omega)\cap H^{4/7+1/2-\varepsilon}(\partial\Omega)$ for all $\varepsilon > 0$. Theoretically, this predicts an expected convergence order $\mathcal{O}(h^{4/7})$ for uniform mesh-refinement.

The Z-shaped domain as well as the initial mesh $\mathcal{T}_h$ for the computation are shown in Figure 3. We compare adaptive mesh-refinement with parameter $\theta = 1/2$ with uniform mesh-refinement. The corresponding convergence graphs are visualized in Figure 4. While uniform mesh-refinement leads to the expected rate $\mathcal{O}(h^{4/7}) = \mathcal{O}(N^{-4/7})$, the proposed adaptive strategy regains the optimal rate $\mathcal{O}(N^{-3/2})$.

3. Weakly-singular integral equation

In this section, we consider the simple-layer potential

$$(V\phi)(x) = \int_\Gamma G(x-y)\phi(y)\,d\Gamma(y) \quad \text{for } x \in \Gamma, \qquad (21)$$

where $G(\cdot)$ denotes the fundamental solution of the 2D Laplacian from (6). We assume that $\Gamma \subseteq \partial\Omega$ is a relatively

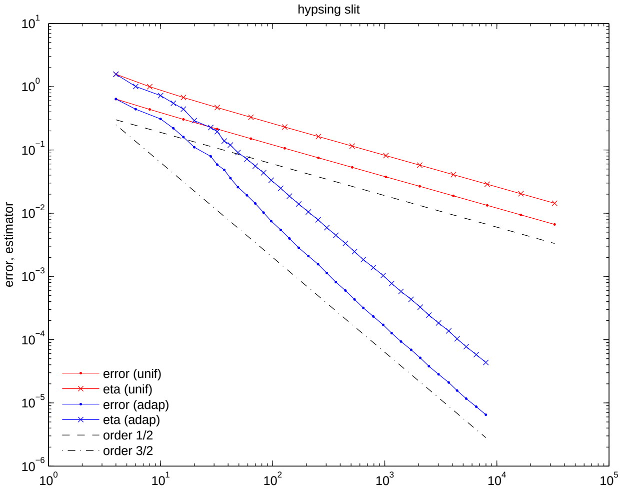

Figure 2: Numerical outcome of the experiment for the hyper-singular integral equation from Section 2.6.

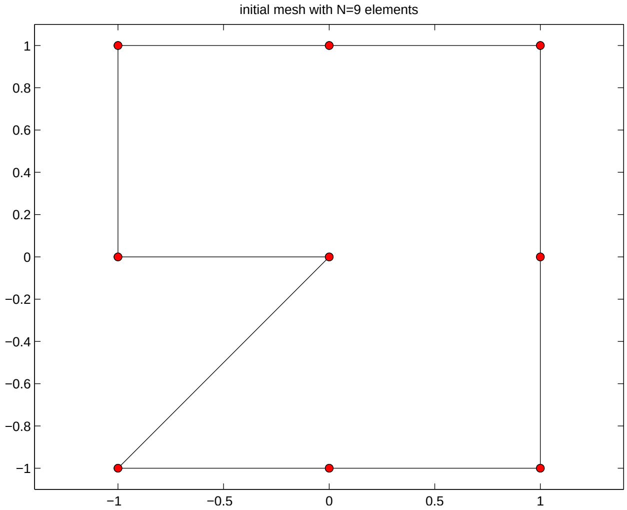

Figure 3: Boundary $\Gamma = \partial\Omega$ and initial mesh $\mathcal{T}_h$ with $N = 9$ elements of the numerical experiment for the hyper-singular integral equation from Section 2.7.

open but possibly non-connected subset of the boundary $\partial\Omega$ and that $\operatorname{diam}(\Omega) < 1$. For the following facts on the functional analytic setting as well as for proofs and further details, we again refer to e.g. the monographs [HW08, McL00, SS11].

3.1. Model problem

It is well-known that V is a linear, bounded, and elliptic operator from $\widetilde{H}^{-1/2}(\Gamma)$ to its dual $H^{1/2}(\Gamma)$, where ellipticity follows from $\operatorname{diam}(\Omega) < 1$. Given some $f \in H^{1/2}(\Gamma)$, we aim at the numerical solution of the weakly-singular integral equation

$$V\phi = f. \tag{22}$$

We use the variational form

$$\langle V\phi\,,\,\psi\rangle_{L^2(\Gamma)} = \langle f\,,\,\psi\rangle_{L^2(\Gamma)} \quad \text{for all } \psi \in \widetilde{H}^{-1/2}(\Gamma). \tag{23}$$

The left-hand side defines an equivalent scalar product on $\widetilde{H}^{-1/2}(\Gamma)$, and the Lax-Milgram lemma thus provides existence and uniqueness of the solution $\phi \in \widetilde{H}^{-1/2}(\Gamma)$ of (23).

We stress that, for certain right-hand sides f and $\Gamma = \partial\Omega$, (22) is an equivalent formulation of the Dirichlet problem

$$\begin{aligned} -\Delta \mathfrak{u} &= \mathfrak{f} \quad \text{in } \Omega,\\ \mathfrak{u} &= \mathfrak{g} \quad \text{on } \Gamma. \end{aligned} \tag{24}$$

In this case, it holds $\phi = \partial_n \mathfrak{u}$. In particular, one cannot expect that ϕ is locally smooth, where the outer normal vector n is not.

3.2. Galerkin boundary element discretization

Let $\mathcal{T}_h$ be a partition of Γ into affine line segments. Let $\mathcal{P}^0(\mathcal{T}_h)$ denote the space of all $\mathcal{T}_h$-piecewise constant functions ψ_h. For the Galerkin discretization, we replace $\phi, \psi \in \widetilde{H}^{-1/2}(\Gamma)$ by discrete functions $\phi_h, \psi_h \in \mathcal{P}^0(\mathcal{T}_h)$. Then, $\mathcal{P}^0(\mathcal{T}_h) \subset \widetilde{H}^{-1/2}(\Gamma)$ is a conforming subspace, and the Galerkin formulation

$$\langle V\phi_h\,,\,\psi_h\rangle_{L^2(\Gamma)} = \langle f\,,\,\psi_h\rangle_{L^2(\Gamma)} \text{ for all } \psi_h \in \mathcal{P}^0(\mathcal{T}_h) \tag{25}$$

admits a unique Galerkin solution $\phi_h \in \mathcal{P}^0(\mathcal{T}_h)$.

3.3. ZZ-type error estimator

With $h \in L^\infty(\Gamma)$ the local mesh-size function from (13), we propose the following ZZ-type error estimator

$$\eta_h = \|h^{1/2}(1 - \mathcal{A}_h)\phi_h\|_{L^2(\Gamma)}. \tag{26}$$

As noted before, we may expect that ϕ is non-smooth at points $x \in \Gamma$, where the normal mapping $x \mapsto n(x)$ is non-smooth. Therefore, we slightly modify the Clément operator $\mathcal{A}_h : L^2(\Gamma) \to \mathcal{P}^1(\mathcal{T}_h)$ from (15) as follows:

- First, if $\{z\} = T_j \cap T_k$ is the node between the elements $T_j, T_k \in \mathcal{T}_h$ and if the normal vector of T_j and T_k does not jump at z, we define

$$(\mathcal{A}_h v)(z) := \frac{1}{\operatorname{length}(\omega_z)} \int_{\omega_z} v\, d\Gamma \tag{27}$$

  with $\omega_z = \bigcup \{T \in \mathcal{T}_h \,:\, z \in T\} = T_j \cup T_k$ the node patch.

- Second, if the normal vectors of T_j and T_k differ at z, we allow $\mathcal{A}_h v$ to jump at z as well, namely

$$\begin{aligned} (\mathcal{A}_h v)|_{T_j}(z) &= \frac{1}{\operatorname{length}(T_j)} \int_{T_j} v\, d\Gamma,\\ (\mathcal{A}_h v)|_{T_k}(z) &= \frac{1}{\operatorname{length}(T_k)} \int_{T_k} v\, d\Gamma. \end{aligned} \tag{28}$$

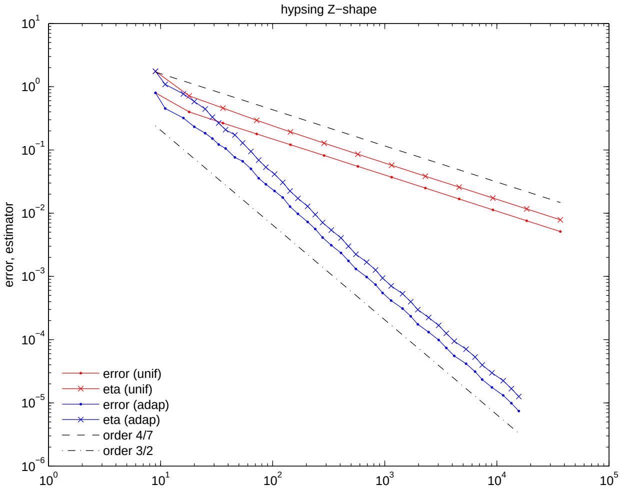

Figure 4: Numerical outcome of the experiment for the hypersingular integral equation from Section 2.7.

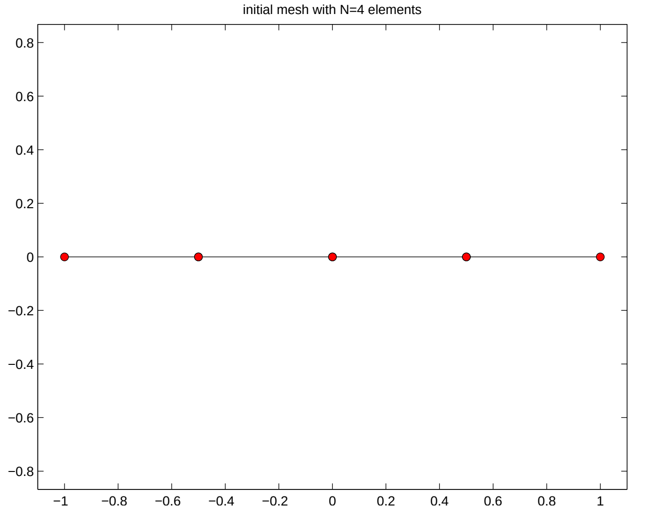

Figure 5: Slit $\Gamma = (-1,1) \times \{0\}$ and initial mesh $\mathcal{T}_h$ with $N = 4$ elements of the numerical experiment from Section 3.5.

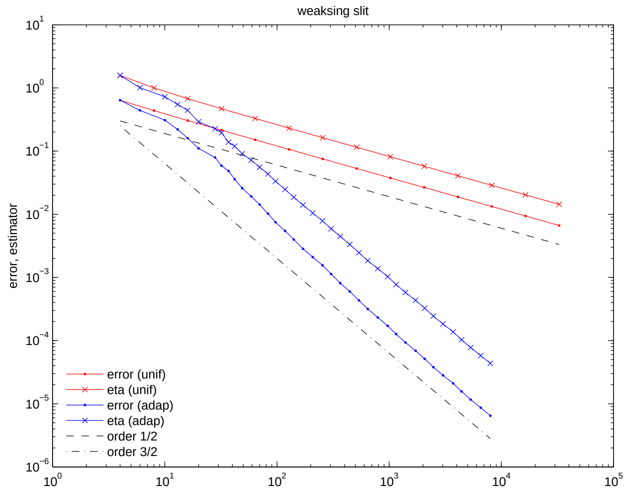

Figure 6: Numerical outcome of experiment from Section 3.5.

Note that this definition can only be meaningful if each connected component $\gamma \subseteq \Gamma$ on which the normal mapping $x \mapsto n(x)$ is smooth, consists of at least two elements. Otherwise, $\gamma = T_j$ would lead to $\phi_h|_\gamma = (\mathcal{A}_h\phi_h)|_\gamma$ so that η_h vanishes on γ, i.e. T_j would never be marked for refinement by an adaptive algorithm.

3.4. Adaptive algorithm

We consider the adaptive algorithm from Section 2.5 with the obvious modifications., i.e. we compute $\phi_h \in \mathcal{P}^0(\mathcal{T}_h)$ in step (i) as well as the local contributions

$$\eta_h(T)^2 := \mathrm{length}(T)\,\|(1-\mathcal{A}_h)\phi_h\|^2_{L^2(T)} \tag{29}$$

in step (ii). We refer to the literature, e.g. [SS11], that the optimal rate of lowest-order BEM is $\mathcal{O}(h^{3/2})$ for a smooth solution ϕ, and the adaptive algorithm thus aims to regain a convergence order $\mathcal{O}(N^{-3/2})$ with respect to the number of elements.

3.5. Numerical experiment for slit problem

We consider the weakly-singular integral equation

$$V\phi = 1 \quad \text{on } \Gamma = (-1,1)\times\{0\}. \tag{30}$$

The unique exact solution of this equation is known and reads $\phi(x,0) = -2x/\sqrt{1-x^2}$. Note that $\phi \in \widetilde{H}^{-1/2}(\Gamma) \cap H^{-\varepsilon}(\Gamma)$ for all $\varepsilon > 0$. In particular, we expect an empirical convergence order $\mathcal{O}(h^{1/2})$ for uniform mesh-refinement.

The initial mesh $\mathcal{T}_h$ for the computation is shown in Figure 5. We compare adaptive mesh-refinement with parameter $\theta = 1/2$ with uniform mesh-refinement. The corresponding convergence graphs are visualized in Figure 6. While uniform mesh-refinement leads to the expected rate $\mathcal{O}(h^{1/2}) = \mathcal{O}(N^{-1/2})$, the adaptive algorithm regains the optimal rate $\mathcal{O}(N^{-3/2})$.

3.6. Numerical experiment on closed boundary

We consider the rotated L-shaped domain from Figure 7 with reentrant corner at the origin $(0,0)$. We consider $\Gamma = \partial\Omega$ and choose the right-hand side $f = (K+1/2)(\mathfrak{u}|_\Gamma) \in H^{1/2}(\Gamma)$ with K the double-layer potential, so that the weakly-singular integral equation (22) is equivalent to some Dirichlet problem (24) with $\mathfrak{f} = 0$. The exact solution of (24) is prescribed as

$$\mathfrak{u}(x) = r^{2/3}\cos(2\varphi/3) \tag{31}$$

in 2D polar coordinates $x = r\,(\cos\varphi, \sin\varphi)$ and admits a generic singularity at the reentrant corner. The exact solution ϕ of (22) is the normal derivative $\phi = \partial_n\mathfrak{u}$. We note that $\phi \in H^{2/3-1/2-\varepsilon}(\Gamma)$ for all $\varepsilon > 0$, and we may hence expect convergence of order $\mathcal{O}(h^{2/3})$ for uniform mesh-refinement.

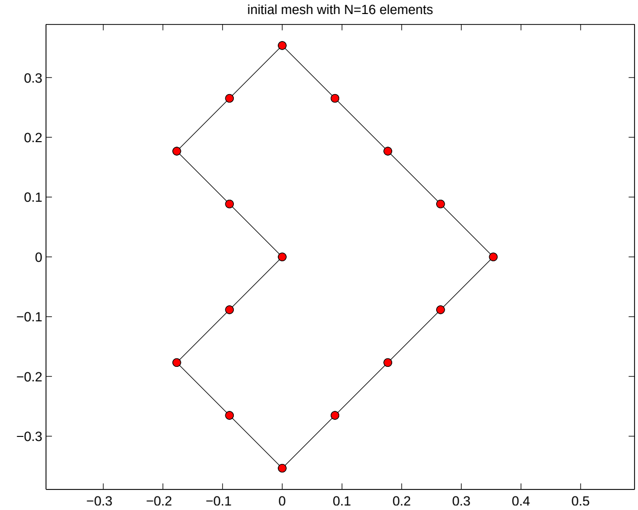

Figure 7: Boundary $\Gamma = \partial\Omega$ and initial mesh $\mathcal{T}_h$ with $N = 16$ elements of the numerical experiment from Section 3.6.

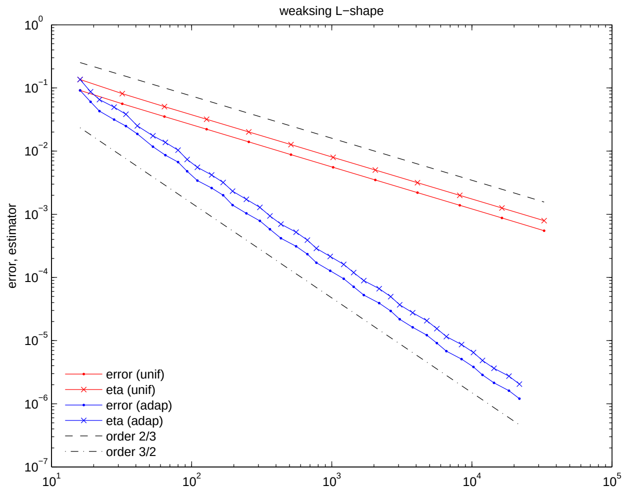

Figure 8: Numerical outcome of experiment from Section 3.6.

The L-shaped domain as well as the initial mesh $\mathcal{T}_h$ for the computation are shown in Figure 7. We compare adaptive mesh-refinement with parameter $\theta = 1/2$ with uniform mesh-refinement. The corresponding convergence graphs are visualized in Figure 8. The proposed adaptive algorithm recovers the optimal order of convergence.

4. Preliminaries

The purpose of this short section is to fix the notation of the spaces involved and to recall standard results used in the following.

4.1. Interpolation spaces

Let X_0 and X_1 be Hilbert spaces with $X_0 \supseteq X_1$ and continuous inclusion, i.e., there exists some constant $C > 0$ such that

$$\|x\|_{X_0} \leq C\,\|x\|_{X_1} \quad \text{for all } x \in X_1. \tag{32}$$

Interpolation theory, e.g. [BL76], provides a means to define intermediate spaces

$$X_1 \subseteq X_s := [X_0; X_1]_s \subseteq X_0 \quad \text{for all } 0 < s < 1, \tag{33}$$

where $[\cdot;\cdot]_s$ denotes the interpolation operator of, e.g., the K-method. The norm related to the intermediate interpolation space X_s satisfies

$$\|x\|_{X_s} \leq \|x\|_{X_0}^{1-s}\|x\|_{X_1}^{s} \quad \text{for all } x \in X_1. \tag{34}$$

The most important consequence, however, is the so-called *interpolation estimate*: Let $X_0 \supseteq X_1$ and $Y_0 \supseteq Y_1$ be Hilbert spaces with continuous inclusions. Let $T : X_0 \to Y_0$ be a linear operator with $T(X_1) \subseteq Y_1$. Assume that $T : X_0 \to Y_0$ as well as $T : X_1 \to Y_1$ are continuous, i.e.,

$$\begin{aligned} \|Tx\|_{Y_0} &\leq C_1\,\|x\|_{X_0} && \text{for all } x \in X_0, \\ \|Tx\|_{Y_1} &\leq C_2\,\|x\|_{X_1} && \text{for all } x \in X_1, \end{aligned} \tag{35}$$

with the respective operator norms $C_1, C_2 > 0$. Let $0 < s < 1$ and $X_s = [X_0; X_1]_s$ and $Y_s = [Y_0; Y_1]_s$. Then, $T : X_s \to Y_s$ is a well-defined linear and continuous operator with

$$\|Tx\|_{Y_s} \leq C_1^{1-s} C_2^{s}\,\|x\|_{X_s} \quad \text{for all } x \in X_s. \tag{36}$$

Note that for other interpolation methods than the real K-method, the previous estimates (34) and (36) hold only up to some additional constant which depends only on Γ, see e.g. [BL76].

4.2. Function spaces

Let $L^2(\Gamma)$ denote the space of square integrable functions on Γ, associated with the Hilbert norm

$$\|v\|_{L^2(\Gamma)}^2 := \int_\Gamma v^2\,d\Gamma. \tag{37}$$

Note that $\|\cdot\|_{L^2(\Gamma)}$ stems from the scalar product

$$\langle v\,,\,w\rangle_{L^2(\Gamma)} := \int_\Gamma vw\,d\Gamma. \tag{38}$$

Let $H^1(\Gamma)$ denote the closure of all Lipschitz continuous functions on Γ with respect to the Hilbert norm

$$\|v\|_{H^1(\Gamma)}^2 := \|v\|_{L^2(\Gamma)}^2 + \|v'\|_{L^2(\Gamma)}^2. \tag{39}$$

Let $\widetilde{H}^1(\Gamma)$ denote the closure of all Lipschitz continuous functions on Γ with respect to the $H^1(\Gamma)$-norm which vanish at the tips of Γ. We stress that both $H^1(\Gamma)$ and $\widetilde{H}^1(\Gamma)$ are dense subspaces of $L^2(\Gamma)$ with respect to the $L^2(\Gamma)$-norm. Moreover, it holds $H^1(\Gamma) = \widetilde{H}^1(\Gamma)$ in case of a closed boundary $\Gamma = \partial\Omega$.

Sobolev spaces of fractional order $0 < s < 1$ are defined by interpolation

$$\begin{aligned} H^s(\Gamma) &:= [L^2(\Gamma); H^1(\Gamma)]_s, \\ \widetilde{H}^s(\Gamma) &:= [L^2(\Gamma); \widetilde{H}^1(\Gamma)]_s. \end{aligned} \tag{40}$$

To abbreviate notation, we shall also write $L^2(\Gamma) = H^0(\Gamma) = \widetilde{H}^0(\Gamma)$. It follows that all $H^s(\Gamma)$ and $\widetilde{H}^s(\Gamma)$ are dense subspaces of $L^2(\Gamma)$ with respect to the $L^2(\Gamma)$-norm. Therefore, the dual spaces can be understood with respect to the extended $L^2(\Gamma)$-scalar product. For $-1 \le s < 0$, we define

$$\begin{aligned} H^{-s}(\Gamma) &:= \widetilde{H}^s(\Gamma)^*, \\ \widetilde{H}^{-s}(\Gamma) &:= H^s(\Gamma)^*. \end{aligned} \tag{41}$$

It follows that $L^2(\Gamma)$ is dense in $H^{-s}(\Gamma)$ and $\widetilde{H}^{-s}(\Gamma)$ with respect to the associated norms. For $s = 0$, we let $\widetilde{H}^0(\Gamma) := L^2(\Gamma) =: H^0(\Gamma)$.

We stress that interpolation theory also states the equalities

$$\begin{aligned} H^{-s}(\Gamma) &= [H^{-1}(\Gamma); L^2(\Gamma)]_s, \\ \widetilde{H}^{-s}(\Gamma) &= [\widetilde{H}^{-1}(\Gamma); L^2(\Gamma)]_s \end{aligned} \tag{42}$$

in the sense of sets and equivalent norms [McL00]. Moreover, interpolation reveals the continuous inclusions $\widetilde{H}^{\pm s}(\Gamma) \subseteq H^{\pm s}(\Gamma)$ as well as $\widetilde{H}^{\pm s}(\partial\Omega) = H^{\pm s}(\partial\Omega)$.

The analysis of the hyper-singular integral equation further requires

$$H^{\pm s}_\star(\partial\Omega) := \big\{ v \in H^{\pm s}(\partial\Omega) \,:\, \langle v\,,\,1\rangle_{L^2(\partial\Omega)} = 0 \big\} \tag{43}$$

for $0 \le s \le 1$. We define $L^2_\star(\Gamma) := H^0_\star(\Gamma)$. We again note that interpolation yields the equality

$$H^{\pm s}_\star(\partial\Omega) = [L^2_\star(\partial\Omega); H^1_\star(\partial\Omega)]_s. \tag{44}$$

Finally, $H^s_0(\Gamma)$ denotes either $\widetilde{H}^s(\Gamma)$ for $\Gamma \subsetneqq \partial\Omega$ resp. $H^s_\star(\partial\Omega)$ for $\Gamma = \partial\Omega$. In either case, $H^s_0(\Gamma)$ contains no constant function different from zero provided that Γ is connected.

4.3. Discrete spaces

We assume that $\mathcal{T}_h = \{T_1, \dots, T_N\}$ is a partition of Γ into finitely many compact and affine line segments $T \in \mathcal{T}_h$. With each element $T \in \mathcal{T}_h$, we associate an affine bijection $\gamma_T : [0,1] \to T$.

For $q \in \mathbb{N}_0$, let $\mathcal{P}^q$ denote the space of polynomials of degree $\le q$ on $\mathbb{R}$. With this, we define the space of $\mathcal{T}_h$-piecewise polynomials by

$$\mathcal{P}^q(\mathcal{T}_h) := \big\{ v_h : \Gamma \to \mathbb{R} \,:\, \forall T \in \mathcal{T} \quad v_h \circ \gamma_T \in \mathcal{P}^q \big\}. \tag{45}$$

Note that functions $v_h \in \mathcal{P}^q(\mathcal{T}_h)$ are discontinuous in general. Special attention is paid to the piecewise constants $\mathcal{P}^0(\mathcal{T}_h)$.

If continuity is required, we use the space

$$\mathcal{S}^q(\mathcal{T}_h) := \mathcal{P}^q(\mathcal{T}_h) \cap C(\Gamma) \tag{46}$$

of continuous splines of piecewise degree $q \ge 1$. Special attention is paid to the Courant space $\mathcal{S}^1(\mathcal{T}_h)$ of lowest order.

For the treatment of the hyper-singular integral equation, we additionally define

$$\widetilde{\mathcal{S}}^q(\mathcal{T}_h) := \mathcal{S}^q(\mathcal{T}_h) \cap \widetilde{H}^1(\Gamma), \tag{47}$$

$$\mathcal{S}^q_\star(\mathcal{T}_h) := \mathcal{S}^q(\mathcal{T}_h) \cap H^1_\star(\Gamma). \tag{48}$$

Finally, $\mathcal{S}^q_0(\mathcal{T}_h)$ denotes either $\widetilde{\mathcal{S}}^q(\mathcal{T}_h)$ for $\Gamma \subsetneqq \partial\Omega$ resp. $\mathcal{S}^q_\star(\mathcal{T}_h)$ for $\Gamma = \partial\Omega$.

4.4. Projections

Let X_h be a finite dimensional subspace of a Hilbert space X. The X-orthogonal projection onto X_h is the unique linear operator $\mathbb{P}_h : X \to X_h$ such that, for all $x \in X$ and $x_h \in X_h$, it holds

$$\begin{aligned} \mathbb{P}_h x_h &= x_h, \\ \langle \mathbb{P}_h x\,,\, x_h\rangle_X &= \langle x\,,\, x_h\rangle_X. \end{aligned} \tag{49}$$

This implies the Pythagoras theorem

$$\|x\|_X^2 = \|\mathbb{P}_h x\|_X^2 + \|(1-\mathbb{P}_h)x\|_X^2 \tag{50}$$

and consequently

$$\|(1-\mathbb{P}_h)x\|_X = \min_{x_h \in X_h} \|x - x_h\|_X. \tag{51}$$

In [SZ90], a quasi-interpolation operator $\mathcal{J}_h^\Omega : H^1(\Omega) \to \mathcal{S}^1(\mathcal{T}_h^\Omega)$ is introduced. Here, $\Omega \subset \mathbb{R}^d$ for $d \ge 2$ is a Lipschitz domain, $\mathcal{T}_h^\Omega$ is a conforming triangulation of Ω into simplices, and $\mathcal{S}^1(\mathcal{T}_h^\Omega)$ is the lowest-order Courant finite element space. It is shown that $\mathcal{J}_h^\Omega$ has a local first-order approximation property and is a linear and continuous projection onto $\mathcal{S}^1(\mathcal{T}_h^\Omega)$. Moreover, $\mathcal{J}_h^\Omega$ preserves discrete boundary data, since the boundery values $(\mathcal{J}_h v)|_\Gamma$ depend only on the trace $v|_\Gamma$ with $\Gamma = \partial\Omega$.

Let $\mathcal{T}_h$ denote the partition of Γ induced by $\mathcal{T}_h^\Omega$. Then, the mentioned properties of $\mathcal{J}_h$ yield that the restriction $\mathcal{J}_h := \mathcal{J}_h^\Omega(\cdot)|_\Gamma : H^{1/2}(\Gamma) \to \mathcal{S}^1(\mathcal{T}_h)$ to the trace space $H^{1/2}(\Gamma)$ yields a well-defined, linear, and continuous projection onto $\mathcal{S}^1(\mathcal{T}_h)$ with respect to the $H^{1/2}(\Gamma)$-norm. However, arguing along the lines of the domain-based proof from [SZ90], we see that $\mathcal{J}_h$ has the following properties. For an element $T \in \mathcal{T}_h$, we denote by

$$\omega_T := \bigcup \big\{ T' \in \mathcal{T}_h \,:\, T \cap T' \neq \emptyset \big\} \tag{52}$$

its patch, i.e., the union of T and its (at most two) neighbours. We shall use the following properties of $\mathcal{J}_h$:

(i) $\mathcal{J}_h v$ is well-defined for all $v \in L^2(\Gamma)$.

(ii) $(\mathcal{J}_h v)|_T$ depends only on the function values $v|_{\omega_T}$ on the patch of $T \in \mathcal{T}_h$.

(iii) $\mathcal{J}_h$ is locally L^2-stable, for all $v \in L^2(\Gamma)$,

$$\|(1-\mathcal{J}_h)v\|_{L^2(T)} \le C_3 \|v\|_{L^2(\omega_T)}. \tag{53}$$

(iv) $\mathcal{J}_h$ is locally H^1-stable, for all $v \in H^1(\Gamma)$,

$$\|((1-\mathcal{J}_h)v)'\|_{L^2(T)} \le C_3 \|v'\|_{L^2(\omega_T)}. \tag{54}$$

(v) $\mathcal{J}_h$ has a first-order approximation property, for all $v \in H^1(\Gamma)$,

$$\|(1-\mathcal{J}_h)v\|_{L^2(T)} \le C_3 \|hv'\|_{L^2(\omega_T)}. \tag{55}$$

(vi) The constant $C_3 > 0$ depends only on the local mesh-ratio $\kappa(\mathcal{T}_h)$.

Since ω_T consists of at most three elements, the ℓ_2-sums of the estimates (53)–(55) also provide global estimates with T and ω_T replaced by Γ. From (iii), we thus see that $\mathcal{J}_h \in L(L^2(\Gamma); L^2(\Gamma))$. The combination of (iii)–(iv) yields $\mathcal{J}_h \in L(H^1(\Gamma); H^1(\Gamma))$. In particular, the interpolation estimate (36) provides $\mathcal{J}_h \in L(H^s(\Gamma); H^s(\Gamma))$, for all $0 \le s \le 1$.

5. A posteriori error analysis

In this section, we show that under appropriate assumptions, the ZZ-type error estimators proposed provide an upper bound for the error (*reliability*) and, up to some higher-order terms, also a lower bound for the error (*efficiency*). Our analysis builds on equivalence of seminorms on finite dimensional spaces and scaling arguments. The elementary, but abstract result employed reads as follows: If X is a finite dimensional space with seminorms $|\cdot|_1$ and $|\cdot|_2$, an estimate of the type

$$|x|_1 \le C\,|x|_2 \quad \text{for all } x \in X \tag{56}$$

and some independent constant $C > 0$ is equivalent to the inclusion

$$\big\{x \in X \,:\, |x|_2 = 0\big\} \subseteq \big\{x \in X \,:\, |x|_1 = 0\big\} \tag{57}$$

of the respective null spaces. This result is used for polynomial spaces on *element patches*. To this end, the restricted partition of the patch ω_T from (52) is denoted by

$$\mathcal{T}_h|_{\omega_T} := \big\{T' \in \mathcal{T}_h \,:\, T \cap T' \neq \emptyset\big\} \tag{58}$$

for all $T \in \mathcal{T}$.

5.1. Hyper-singular integral equation

Recall the abbreviate notation $H_0^{1/2}(\Gamma)$ from Section 4.2 and note that

$$|\!|\!| v |\!|\!|^2 := \langle Wv\,,\,v\rangle_{L^2(\Gamma)} \tag{59}$$

defines an equivalent Hilbert norm on $H_0^{1/2}(\Gamma)$. Because of $H^{1/2}(\partial\Omega) = \widetilde{H}^{1/2}(\partial\Omega)$ even with equal norms, we can simply use the norm $\|\cdot\|_{\widetilde{H}^{1/2}(\Gamma)} \simeq |\!|\!|\cdot|\!|\!|$ throughout the section.

We start with the derivation of an upper bound. The proof relies on the assumption that $\mathcal{T}_h$ is the uniform refinement of some coarser mesh $\mathcal{T}_{2h}$ and on some saturation assumption (61). While the first assumption can easily be achieved implementationally, the latter is essentially equivalent to the assumption that the numerical scheme has reached an asymptotic regime, see [FLP08, Section 5.2] for discussion and numerical evidence.

Theorem 1. *Let $\mathcal{T}_h$ be the uniform refinement of some mesh $\mathcal{T}_{2h}$, i.e. all elements $T \in \mathcal{T}_{2h}$ are bisected into two sons $T_1, T_2 \in \mathcal{T}_h$ of half length. Let $u_h \in \mathcal{S}_0^1(\mathcal{T}_h)$ and $u_{2h} \in \mathcal{S}_0^1(\mathcal{T}_{2h})$ be the respective Galerkin solutions. Then, it holds*

$$|\!|\!| u_h - u_{2h} |\!|\!| \le C_4\,\eta_h \tag{60}$$

with some constant $C_4 > 0$ which depends only on Γ and all possible shapes of element patches (52)*. Under the saturation assumption*

$$|\!|\!| u - u_h |\!|\!| \le C_{\rm sat}\,|\!|\!| u - u_{2h} |\!|\!| \tag{61}$$

with some uniform constant $0 < C_{\rm sat} < 1$, there holds

$$C_{\rm sat}^{-1}\,|\!|\!| u - u_h |\!|\!| \le |\!|\!| u - u_{2h} |\!|\!| \le \frac{C_4}{(1 - C_{\rm sat}^2)^{1/2}}\,\eta_h. \tag{62}$$

Proof. Let $\Pi_{2h} : L^2(\Gamma) \to \mathcal{P}^0(\mathcal{T}_{2h})$ denote the L^2-orthogonal projection onto the $\mathcal{T}_{2h}$-piecewise constants, i.e. the piecewise integral mean operator

$$(\Pi_{2h}\psi)|_{\widehat{T}} = \frac{1}{\mathrm{length}(\widehat{T})}\int_{\widehat{T}} \psi\,d\Gamma \quad \text{for all } \widehat{T} \in \mathcal{T}_{2h}. \tag{63}$$

According to [EFGP12], it holds that

$$|\!|\!| u_h - u_{2h} |\!|\!| \simeq \|h^{1/2}(1-\Pi_{2h})u_h'\|_{L^2(\Gamma)},$$

where the hidden constants depend only on Γ and the local mesh-ratio $\kappa(\mathcal{T}_h)$ from (18). To prove (60), we will verify

$$\|h^{1/2}(1-\Pi_{2h})u_h'\|_{L^2(T)} \lesssim \|h^{1/2}(1-\mathcal{A}_h)u_h'\|_{L^2(\omega_T)} \tag{64}$$

for all $T \in \mathcal{T}_h$ in the following. Both sides of (64) define seminorms on $\mathcal{P}^0(\mathcal{T}_h|_{\omega_T})$, where u_h' is replaced by an arbitrary $\psi_h \in \mathcal{P}^0(\mathcal{T}_h|_{\omega_T})$. It thus suffices to show that $\|h^{1/2}(1-\mathcal{A}_h)\psi_h\|_{L^2(\omega_T)} = 0$ implies $\|h^{1/2}(1-\Pi_{2h})\psi_h\|_{L^2(T)} = 0$. From $\|h^{1/2}(1-\mathcal{A}_h)\psi_h\|_{L^2(\omega_T)} = 0$ and hence $\psi_h = \mathcal{A}_h\psi_h$ on ω_T, we see that ψ_h is constant on ω_T, since ψ_h is both, $\mathcal{T}_h$-piecewise constant and continuous on ω_T. By assumption, T has a brother $T' \in \mathcal{T}_h$ such that $\widehat{T} = T \cup T' \in \mathcal{T}_{2h}$. Moreover, the definition of the patch and $T \cap T' \neq \emptyset$ yield $\widehat{T} \subseteq \omega_T$. Therefore, ψ_h is constant on $\widehat{T}$ so that

$\psi_h = \Pi_{2h}\psi_h$ on $\widehat{T}$. This proves $\|h^{1/2}(1-\Pi_{2h})\psi_h\|_{L^2(T)} = 0$ and thus verifies

$$\|h^{1/2}(1-\Pi_{2h})\psi_h\|_{L^2(T)} \lesssim \|h^{1/2}(1-\mathcal{A}_h)\psi_h\|_{L^2(\omega_T)}$$

for all $T \in \mathcal{T}_h$ and $\psi_h \in \mathcal{P}^0(\mathcal{T}_h)$. Finally, a scaling argument proves that the hidden constant depends only on the shape of the patch ω_T. We note that each element $T' \in \mathcal{T}_h$ is contained in at most three patches. Taking the ℓ_2-sum in (64) over all elements $T \in \mathcal{T}_h$, we arrive at

$$\|h^{1/2}(1-\Pi_{2h})\psi_h\|_{L^2(\Gamma)} \lesssim \|h^{1/2}(1-\mathcal{A}_h)\psi_h\|_{L^2(\Gamma)} \quad (65)$$

for all $\psi_h \in \mathcal{P}^0(\mathcal{T}_h)$. Plugging in $\psi_h = u_h'$, we conclude the proof of (60).

The proof of (62) follows from abstract principles. According to the Galerkin orthogonality

$$\langle W(u-u_h)\,,\,v_h\rangle_{L^2(\Gamma)} = 0 \quad \text{for all } v_h \in \mathcal{S}^1_0(\mathcal{T}_h),$$

we obtain a Pythagoras theorem for the induced Hilbert norm

$$|||u-u_h|||^2 + |||u_h-u_{2h}|||^2 = |||u-u_{2h}|||^2,$$

where we use $v_h = u_h - u_{2h}$. Together with the saturation assumption (61), this results in

$$C_{\rm sat}^{-1}\,|||u-u_h||| \le |||u-u_{2h}||| \le \frac{1}{(1-C_{\rm sat}^2)^{1/2}}\,|||u_h-u_{2h}|||,$$

and (62) follows. ■

Remark 2. *With the same techniques as in the proof of Theorem 1, one can prove that the ZZ-type error estimator η_h is an upper bound for the estimator μ_h from [CP07b] which is based on averaging on large patches. The analysis then requires that $\mathcal{T}_h$ is a refinement of a coarser mesh $\mathcal{T}_{kh}$ for some $k \ge 2$ which depends only on Γ. Then, the saturation assumption (61) is formally avoided. However, the parameter k is still unknown, although $k = 2$ empirically appears to be sufficient, see e.g. the numerical experiments in [CP07b]. Moreover, the upper bound (62) holds only up to some additional best approximation error*

$$|||u-u_h||| \lesssim \eta_h + \min_{U_h \in \mathcal{S}^2_0(\mathcal{T}_{kh})} |||u-U_h|||$$

with higher-order elements $\mathcal{S}^2_0(\mathcal{T}_{kh}) := \mathcal{P}^2(\mathcal{T}_{kh}) \cap \widetilde{H}^{1/2}(\Gamma) \subseteq H^1(\Omega)$ which are piecewise quadratic and globally continuous. If the exact solution u is smooth or if the mesh is appropriately graded to the singularities of u, this additional term is of higher-order. The reader is referred to [CP07a] for further discussions. ■

We next prove the lower bound. Unlike the reliability estimate (62), the following efficiency estimate (66) does not rely on the saturation assumption (61), but holds only up to some further best approximation error with higher-order elements. If the exact solution solution u of (8) is smooth or if the mesh is properly adapted to the singularities of u, this term becomes a higher-order term.

Let $\mathcal{S}^{2,1}(\mathcal{T}_h) := \mathcal{P}^2(\mathcal{T}_h) \cap C^1(\Gamma)$ denote the set of all $\mathcal{T}_h$-piecewise quadratic polynomials p such that p as well as its derivative p' are continuous. With $\mathcal{S}^{2,1}_0(\mathcal{T}_h) := \mathcal{S}^{2,1}(\mathcal{T}_h) \cap H^{1/2}_0(\Gamma)$, our efficiency result then reads as follows:

Theorem 3. *It holds*

$$C_5^{-1}\,\eta_h \le |||u-u_h||| + \min_{U_h \in \mathcal{S}^{2,1}_0(\mathcal{T}_h)} |||u-U_h|||. \quad (66)$$

The constant $C_5 > 0$ depends only on Γ and all possible shapes of element patches (52).

The proof requires the following probably well-known lemma. For the convenience of the reader, we include the proof also here.

Lemma 4. *For $0 \le s \le 1$, the arc-length derivative induces linear and continuous operators $(\cdot)' : H^s(\Gamma) \to H^{s-1}(\Gamma)$ and $(\cdot)' : \widetilde{H}^s(\Gamma) \to \widetilde{H}^{s-1}(\Gamma)$.*

Proof. For $s = 1$, it holds

$$\|v'\|_{L^2(\Gamma)} \le \|v\|_{H^1(\Gamma)} \quad \text{for all } v \in H^1(\Gamma)$$

and, by integration by parts,

$$\langle v'\,,\,w\rangle_{L^2(\Gamma)} = -\langle v\,,\,w'\rangle_{L^2(\Gamma)} \le \|v\|_{L^2(\Gamma)}\|w\|_{H^1(\Gamma)}.$$

for all $w \in \widetilde{H}^1(\Gamma)$. Note that here we require either that $\Gamma = \partial\Omega$ or that w (or v) vanishes at the tips of Γ. By definition of the duality $H^{-1}(\Gamma) = \widetilde{H}^1(\Gamma)^*$, this yields

$$\|v'\|_{H^{-1}(\Gamma)} \le \|v\|_{L^2(\Gamma)} \quad \text{for all } v \in H^1(\Gamma).$$

Since $H^1(\Gamma)$ is dense in $L^2(\Gamma)$, we obtain continuity of $(\cdot)' : L^2(\Gamma) \to H^{-1}(\Gamma)$, i.e. the last estimate holds even for all $v \in L^2(\Gamma)$. Finally, the interpolation estimate (36) reveals

$$\|v'\|_{H^{s-1}(\Gamma)} \le \|v\|_{H^s(\Gamma)} \quad \text{for all } v \in H^s(\Gamma),$$

i.e. $(\cdot)' : H^s(\Gamma) \to H^{s-1}(\Gamma)$ is a linear and continuous operator, even with operator norm 1.

To prove the same statement for $(\cdot)' : \widetilde{H}^s(\Gamma) \to \widetilde{H}^{s-1}(\Gamma)$, recall the duality $\widetilde{H}^{-1}(\Gamma) = H^1(\Gamma)^*$. With $v \in \widetilde{H}^1(\Gamma)$ and $w \in H^1(\Gamma)$ all foregoing steps remain valid with nothing but the obvious modifications. ■

Proof of Theorem 3. Let $J_h : L^2(\Gamma) \to \mathcal{S}^1(\Gamma)$ denote the Scott-Zhang projection from Section 4.4. We first show that

$$\|h^{1/2}(1-\mathcal{A}_h)\psi_h\|_{L^2(\Gamma)} \lesssim \|h^{1/2}(1-J_h)\psi_h\|_{L^2(\Gamma)} \quad (67)$$

for all $\psi_h \in \mathcal{P}^0(\mathcal{T}_h)$. To that end, we use a seminorm argument on $\mathcal{P}^0(\mathcal{T}_h|_{\omega_T})$: From $\|h^{1/2}(1-J_h)\psi_h\|_{L^2(\omega_T)} = 0$, it follows that ψ_h is constant on ω_T. By definition (15) of $\mathcal{A}_h$ this yields $\mathcal{A}_h\psi_h = \psi_h$ on T. Therefore, we see $\|h^{1/2}(1-\mathcal{A}_h)\psi_h\|_{L^2(T)} = 0$, and

$$\|h^{1/2}(1-\mathcal{A}_h)\psi_h\|_{L^2(T)} \lesssim \|h^{1/2}(1-J_h)\psi_h\|_{L^2(\omega_T)}$$

follows. A scaling argument proves that the hidden constant depends only on the shape of the patch ω_T. Taking the ℓ_2-sum of the last estimate over all elements $T \in \mathcal{T}_h$, we obtain (67).

- Second, we show that

$$\|h^{1/2}(1-J_h)\psi_h\|_{L^2(\Gamma)} \lesssim \|\psi_h - \Psi_h\|_{\widetilde{H}^{-1/2}(\Gamma)} \tag{68}$$

for all $\psi_h \in \mathcal{P}^0(\mathcal{T}_h)$ and $\Psi_h \in \mathcal{S}^1(\mathcal{T}_h)$. Since the Scott-Zhang projection is stable with respect to the $h^{1/2}$-weighted L^2-norm, see Section 4.4, the projection property of J_h gives

$$\begin{aligned}\|h^{1/2}(1-J_h)\psi_h\|_{L^2(\Gamma)} &= \|h^{1/2}(1-J_h)(\psi_h-\Psi_h)\|_{L^2(\Gamma)}\\ &\lesssim \|h^{1/2}(\psi_h-\Psi_h)\|_{L^2(\Gamma)}.\end{aligned}$$

The inverse estimate of [GHS05, Thm 3.6] then concludes the proof of (68).

- Finally, let $\mathbb{P}_h : H_0^{1/2}(\Gamma) \to \mathcal{S}_0^{2,1}(\mathcal{T}_h)$ denote the $H_0^{1/2}(\Gamma)$-orthogonal projection onto $\mathcal{S}_0^{2,1}(\mathcal{T}_h)$ with respect to the energy norm $|\!|\!|\cdot|\!|\!|$. Combining norm equivalence $|\!|\!|\cdot|\!|\!| \simeq \|\cdot\|_{\widetilde{H}^{1/2}(\Gamma)}$ with the estimates (67) and (68) for $\psi_h = u_h'$ and $\Psi_h = (\mathbb{P}_h u_h)'$, we obtain

$$\begin{aligned}\|h^{1/2}(1-\mathcal{A}_h)u_h'\|_{L^2(\Gamma)} &\lesssim \|(u_h-\mathbb{P}_h u_h)'\|_{\widetilde{H}^{-1/2}(\Gamma)}\\ &\lesssim \|(1-\mathbb{P}_h)u_h\|_{\widetilde{H}^{1/2}(\Gamma)} \simeq |\!|\!|(1-\mathbb{P}_h)u_h|\!|\!|.\end{aligned}$$

The triangle inequality and stability of $\mathbb{P}_h$ yield

$$|\!|\!|(1-\mathbb{P}_h)u_h|\!|\!| \le |\!|\!|(1-\mathbb{P}_h)u|\!|\!| + |\!|\!|u-u_h|\!|\!|.$$

Since $\mathbb{P}_h u$ is the best approximation (51) of u in $\mathcal{S}_0^{2,1}(\mathcal{T}_h)$ with respect to $|\!|\!|\cdot|\!|\!|$, this proves (66). ∎

5.2. Weakly-singular integral equation

We stress that the same results hold as for the hyper-singular integral equation. By

$$|\!|\!|w|\!|\!|^2 := \langle Vw\,,\,w\rangle_{L^2(\Gamma)}, \tag{69}$$

we now denote the Hilbert norm which is induced by the weakly-singular integral operator, and note that $|\!|\!|\cdot|\!|\!| \simeq \|\cdot\|_{\widetilde{H}^{-1/2}(\Gamma)}$ is an equivalent norm on $\widetilde{H}^{-1/2}(\Gamma)$. The reliability result reads as follows:

Theorem 5. *Let $\mathcal{T}_h$ be the uniform refinement of some mesh $\mathcal{T}_{2h}$, i.e. all elements $T \in \mathcal{T}_{2h}$ are bisected into two sons $T_1, T_2 \in \mathcal{T}_h$ of half length. Let $\phi_h \in \mathcal{P}^0(\mathcal{T}_h)$ and $\phi_{2h} \in \mathcal{P}^0(\mathcal{T}_{2h})$ be the respective Galerkin solutions. Then, it holds*

$$|\!|\!|\phi_h - \phi_{2h}|\!|\!| \le C_6\,\eta_h \tag{70}$$

with some constant $C_6 > 0$ which depends only on Γ and all possible shapes of element patches (52). Under the saturation assumption

$$|\!|\!|\phi - \phi_h|\!|\!| \le C_{\rm sat}\,|\!|\!|\phi-\phi_{2h}|\!|\!| \tag{71}$$

with some uniform constant $0 < C_{\rm sat} < 1$, there holds

$$C_{\rm sat}^{-1}\,|\!|\!|\phi-\phi_h|\!|\!| \le |\!|\!|\phi-\phi_{2h}|\!|\!| \le \frac{C_6}{(1-C_{\rm sat}^2)^{1/2}}\,\eta_h. \tag{72}$$

Remark 6. *We refer to [AFF+13], where the saturation assumption (71) is proved in the frame of the weakly-singular integral equation for the Dirichlet problem (24) and $\mathcal{T}_{2h}$ replaced by some coarser mesh $\mathcal{T}_{kh}$ with $k \ge 2$ depending only on Γ.* ∎

Proof of Theorem 5. We adopt the notation from the proof of Theorem 1. According to [EFLFP09], it holds that

$$|\!|\!|\phi_h - \phi_{2h}|\!|\!| \simeq \|h^{1/2}(1-\Pi_{2h})\phi_h\|_{L^2(\Gamma)},$$

where the hidden constants depend only on Γ and the local mesh-ratio $\kappa(\mathcal{T}_h)$ from (18). Recall that the operator $\mathcal{A}_h$ is now slightly different to the case of the hyper-singular integral equation. However, the same arguments as in the proof of Theorem 1 show that (65) remains valid. As before the hidden constant involved depends on all possible shapes of element patches in $\mathcal{T}_h$. This yields (70), and (72) follows as before. ∎

We next prove the lower bound. As before, the following efficiency estimate (73) does not rely on the saturation assumption (71), but holds only up to some further best approximation error with higher-order elements.

Theorem 7. *It holds*

$$C_7^{-1}\,\eta_h \le |\!|\!|\phi-\phi_h|\!|\!| + \min_{\Phi_h \in \mathcal{S}^1(\mathcal{T}_h)} |\!|\!|\phi - \Phi_h|\!|\!|. \tag{73}$$

The constant $C_7 > 0$ depends only on Γ and all possible shapes of element patches (52).

Proof. Arguing along the lines of the proof of Theorem 3, we see that

$$\|h^{1/2}(1-\mathcal{A}_h)\phi_h\|_{L^2(\Gamma)} \lesssim \|\phi_h - \Psi_h\|_{\widetilde{H}^{-1/2}(\Gamma)}$$

for all $\Psi_h \in \mathcal{S}^1(\mathcal{T}_h)$. Let $\mathbb{P}_h : \widetilde{H}^{-1/2}(\Gamma) \to \mathcal{S}^1(\mathcal{T}_h)$ be the orthogonal projection onto $\mathcal{S}^1(\mathcal{T}_h)$ with respect to the energy norm $|\!|\!|\cdot|\!|\!|$. With norm equivalence $|\!|\!|\cdot|\!|\!| \simeq \|\cdot\|_{\widetilde{H}^{-1/2}(\Gamma)}$ and the triangle inequality, we see for $\Psi_h = \mathbb{P}_h\phi_h$

$$\begin{aligned}|\!|\!|\phi_h - \Psi_h|\!|\!| &= |\!|\!|(1-\mathbb{P}_h)\phi_h|\!|\!|\\ &\le |\!|\!|(1-\mathbb{P}_h)\phi|\!|\!| + |\!|\!|(1-\mathbb{P}_h)(\phi-\phi_h)|\!|\!|\\ &\le |\!|\!|(1-\mathbb{P}_h)\phi|\!|\!| + |\!|\!|\phi-\phi_h|\!|\!|.\end{aligned}$$

Since $\mathbb{P}_h\phi$ is the best approximation of ϕ in $\mathcal{S}^1(\mathcal{T}_h)$ with respect to $|\!|\!|\cdot|\!|\!|$, we conclude the proof. ∎

6. Adaptive mesh-refinement

In this section, we prove that the constants in the a posteriori estimates of Section 5 are uniformly bounded and that the adaptive algorithms of Section 2.5 and Section 3.4 are convergent.

6.1. Notation

For the following analysis, we slightly change the notation for the discrete quantities. Let $\mathcal{T}_0$ be the given initial partition of Γ, the adaptive algorithm is started with. Let $\ell = 0, 1, 2, \dots$ denote the counter for the adaptive loop, i.e. we start with $\ell = 0$, and $\ell \mapsto \ell + 1$ is increased in step (v) of the adaptive algorithm.

The mesh in the ℓ-th step of the adaptive loop is denoted by $\mathcal{T}_\ell$. With $\mathcal{T}_\ell$, we associate the local mesh-size $h_\ell \in L^\infty(\Gamma)$ defined in (13). Moreover, $u_\ell \in \mathcal{S}^1_0(\mathcal{T}_\ell)$ resp. $\phi_\ell \in \mathcal{P}^0(\mathcal{T}_\ell)$ are the corresponding discrete solutions with respective ZZ-type error estimators η_ℓ.

Throughout, we assume that mesh-refinement is based on bisection only, i.e. refined elements are bisected into two sons of half length. In step (iv) of the adaptive algorithm, we ensure

$$\kappa(\mathcal{T}_\ell) \le 2\,\kappa(\mathcal{T}_0) \tag{74}$$

Algorithmically, this mesh-refinement is stated and analyzed in [AFF$^+$13]. In addition to (74), the properties of the mesh-refinement necessary in current proofs of quasi-optimal convergence rates for adaptive boundary element methods [FKMP13, Tso13] and adaptive finite element methods [CKNS08, Ste07, Ste08] are satisfied, i.e. the so-called *overlay estimate* and *mesh-closure estimate* are valid. Moreover, bisection and boundedness (74) of the local mesh-ratio guarantee that only a finite number of shapes of element patches (52) can occur. Therefore, the constants in the a posteriori analysis of Section 5 are uniformly bounded.

6.2. Hyper-singular integral equation

The proof of the following theorem follows the concept of *estimator reduction* proposed in [AFLP12] for $(h - h/2)$-type error estimators. We show that the ZZ-type error estimator is contractive up to some vanishing perturbation

$$\eta_{\ell+1} \le q\,\eta_\ell + \alpha_\ell \text{ with } 0 \le \alpha_\ell \xrightarrow{\ell\to\infty} 0 \tag{75}$$

for some ℓ-independent constant $0 < q < 1$. In the current frame, however, the proof that the perturbation α_ℓ tends to zero, is much more involved than in [AFLP12], since it does not only rely on the a priori convergence of Lemma 9, but also on a pointwise convergence property of the averaging operator $\mathcal{A}_h$.

Theorem 8. *Let $(u_\ell)_{\ell\in\mathbb{N}}$ and $(\eta_\ell)_{\ell\in\mathbb{N}}$ be the sequences of discrete solutions and error estimators generated by the adaptive algorithm. Then, it holds estimator convergence*

$$\lim_{\ell\to\infty} \eta_\ell = 0. \tag{76}$$

Provided that $|||u - u_\ell||| \lesssim \eta_\ell$, cf. Theorem 1, we may thus conclude $\lim\limits_{\ell\to\infty} u_\ell = u$.

The proof requires the following lemmas. The first is already found in the early work [BV84] and will be applied for $H = H^{1/2}_0(\Gamma)$ and $X_\ell = \mathcal{S}^1_0(\mathcal{T}_\ell)$ for the hyper-singular integral equation as well as for $H = \widetilde{H}^{-1/2}(\Gamma)$ and $X_\ell = \mathcal{P}^0(\mathcal{T}_\ell)$ for the weakly-singular integral equation.

Lemma 9 (A priori convergence of Galerkin solutions)**.** *Suppose that H is a Hilbert space and $(X_\ell)_{\ell\in\mathbb{N}}$ is a sequence of discrete subspaces with $X_\ell \subseteq X_{\ell+1}$. For $u \in H$ and $\ell \in \mathbb{N}$, let $u_\ell \in X_\ell$ be the best approximation of u. Then, there exists a limit $u_\infty \in H$ such that $\lim\limits_{\ell\to\infty} \|u_\infty - u_\ell\|_X = 0$.* ∎

The following lemma recalls local L^2-stability and first-order approximation property of the averaging operator $\mathcal{A}_\ell$ used.

Lemma 10. *Let $T \in \mathcal{T}_\ell$. Then, the operators $\mathcal{A}_\ell : L^2(\Gamma) \to L^2(\Gamma)$ defined in (15) resp. (28) are locally L^2-stable*

$$\|\mathcal{A}_\ell v\|_{L^2(T)} \le C_8\,\|v\|_{L^2(\omega_T)}, \tag{77}$$

for all $v \in L^2(\Gamma)$, are local H^1-stable

$$\|(\mathcal{A}_\ell v)'\|_{L^2(T)} \le C_8\,\|v\|_{H^1(\omega_T)}, \tag{78}$$

for all $v \in H^1(\Gamma)$, and have a local first-order approximation property

$$\|(1 - \mathcal{A}_\ell)v\|_{L^2(T)} \le C_8\,\|h_\ell v'\|_{L^2(\omega_T)}, \tag{79}$$

for all $v \in H^1(\Gamma)$. Here, ω_T denotes the element patch (52) of $T \in \mathcal{T}_\ell$, and $C_8 > 0$ depends only on Γ and the mesh-refinement chosen.

Proof. The proof follows as for usual Clément-type operators in finite element analysis, cf e.g. [BS08, SZ90]. Scaling arguments prove that the constants involved depend only on the shape of the element patch ω_T. The mesh-refinement chosen guarantees that only finitely many patches occur so that these constants depend, in fact, only on the boundary Γ and the mesh-refinement strategy. ∎

The following proposition is more general than required for the proof of Theorem 8. However, it might be of general interest and might have further applications, since it also applies to FEM and higher dimensions even with the same proof.

Proposition 11 (A priori convergence of averaging operators)**.** *Given the sequence $(\mathcal{T}_\ell)_{\ell\in\mathbb{N}}$ of adaptively generated meshes, let $\mathcal{A}_\ell : L^2(\Gamma) \to H^1(\Gamma)$ be a linear operator which satisfies (77)–(79). Assume that, for all elements $T \in \mathcal{T}_\ell$ and all functions $v \in L^2(\Gamma)$, $(\mathcal{A}_\ell v)|_T$ depends only on the function values $v|_{\omega_T}$ on the element patch (52). Then, there a exists a limit operator $\mathcal{A}_\infty : L^2(\Gamma) \to L^2(\Gamma)$ which satisfies the following:*

(i) *For all* $0 \le s \le 1$, $\mathcal{A}_\infty : H^s(\Gamma) \to H^s(\Gamma)$ *is a well-defined linear and continuous operator.*

(ii) *For all* $0 \le s < 1$, $\mathcal{A}_\infty$ *is the pointwise limit of* $\mathcal{A}_\ell$, *i.e., for all* $v \in H^s(\Gamma)$ *it holds*

$$\lim_{\ell\to\infty} \|(\mathcal{A}_\infty - \mathcal{A}_\ell)v\|_{H^s(\Gamma)} = 0. \tag{80}$$

(iii) *For all* $v \in H^1(\Gamma)$, $\mathcal{A}_\ell v$ *converges weakly in* $H^1(\Gamma)$ *towards* $\mathcal{A}_\infty v$ *as* $\ell \to \infty$.

Proof. For the proof, let $\omega_\ell(\gamma) := \bigcup\{T \in \mathcal{T}_\ell : T \cap \overline{\gamma} \neq \emptyset\}$ denote the patch of subsets $\gamma \subseteq \Gamma$ with respect to $\mathcal{T}_\ell$. We follow the ideas from [MSV08] and define the following subsets of Γ:

$$\begin{aligned}
\Gamma_\ell^0 &:= \bigcup\Big\{T \in \mathcal{T}_\ell : \omega_\ell(T) \subseteq \bigcup\big(\bigcap_{j=\ell}^{\infty} \mathcal{T}_j\big)\Big\},\\
\Gamma_\ell &:= \bigcup\Big\{T \in \mathcal{T}_\ell : \text{Exists } k \ge 0 \text{ s.t. } \omega_T \text{ is at least}\\
&\qquad\qquad \text{uniformly refined in } \mathcal{T}_{\ell+k}\Big\},\\
\Gamma_\ell^* &:= \Gamma\backslash(\Gamma_\ell \cup \Gamma_\ell^0).
\end{aligned}$$

According to [MSV08, Corollary 4.1], it holds that

$$\|h_\ell\|_{L^\infty(\omega_\ell(\Gamma_\ell))} \simeq \|h_\ell\|_{L^\infty(\Gamma_\ell)} \xrightarrow{\ell\to\infty} 0. \tag{81}$$

Let $v \in L^2(\Gamma)$ and $\varepsilon > 0$ be arbitrary. Since $H^1(\Omega)$ is dense in $L^2(\Gamma)$, we find $v_\varepsilon \in H^1(\Gamma)$ such that $\|v - v_\varepsilon\|_{L^2(\Gamma)} \le \varepsilon$. Due to the local L^2-stability (77) and the approximation property (79) of $\mathcal{A}_\ell$, we obtain

$$\begin{aligned}
\|(1-\mathcal{A}_\ell)v\|_{L^2(\Gamma_\ell)} &\lesssim \|(1-\mathcal{A}_\ell)v_\varepsilon\|_{L^2(\Gamma_\ell)} + \varepsilon\\
&\lesssim \|h_\ell \nabla v_\varepsilon\|_{L^2(\omega_\ell(\Gamma_\ell))} + \varepsilon.
\end{aligned}$$

According to (81), we find $\ell_0 \in \mathbb{N}$ such that

$$\|h_\ell \nabla v_\varepsilon\|_{L^2(\omega_\ell(\Gamma_\ell))} \le \|h_\ell\|_{L^\infty(\omega_\ell(\Gamma_\ell))} \|\nabla v_\varepsilon\|_{L^2(\Gamma)} \le \varepsilon$$

for all $\ell \ge \ell_0$. This proves

$$\|(1-\mathcal{A}_\ell)v\|_{L^2(\Gamma_\ell)} \lesssim \varepsilon \quad \text{for } \ell \ge \ell_0. \tag{82}$$

[MSV08, Proposition 4.2] states $|\Gamma_\ell^*| \to 0$ as $\ell \to \infty$. Due to the non-concentration of Lebesgue functions, this yields

$$\|v\|_{L^2(\omega_\ell(\Gamma_\ell^*))} \le \varepsilon \quad \text{for some } \ell_1 \in \mathbb{N} \text{ and all } \ell \ge \ell_1. \tag{83}$$

Let $\ell \ge \max\{\ell_0, \ell_1\}$ and $k \ge 0$. For $T \in \mathcal{T}_\ell$, the definition of $(\mathcal{A}_\ell v)|_T$ depends only on $v|_{\omega_\ell(T)}$. By definition of Γ_ℓ^0, we obtain

$$\|(\mathcal{A}_\ell - \mathcal{A}_{\ell+k})v\|_{L^2(\Gamma_\ell^0)} = 0.$$

With local L^2-stability (77) and (83), we see

$$\begin{aligned}
\|(\mathcal{A}_\ell - \mathcal{A}_{\ell+k})v\|_{L^2(\Gamma_\ell^*)} &\lesssim \|v\|_{L^2(\omega_\ell(\Gamma_\ell^*))} + \|v\|_{L^2(\omega_{\ell+k}(\Gamma_\ell^*))}\\
&\le 2\,\|v\|_{L^2(\omega_\ell(\Gamma_\ell^*))} \lesssim \varepsilon.
\end{aligned}$$

Moreover, (82) and a triangle inequality prove

$$\|(\mathcal{A}_\ell - \mathcal{A}_{\ell+k})v\|_{L^2(\Gamma_\ell)} \lesssim \varepsilon.$$

The combination of the last three estimates yields

$$\|(\mathcal{A}_\ell - \mathcal{A}_{\ell+k})v\|_{L^2(\Gamma)} \lesssim \varepsilon.$$

Altogether, $(\mathcal{A}_\ell v)_\ell$ is thus a Cauchy sequence in $L^2(\Gamma)$ and hence convergent to some limit $\mathcal{A}_\infty v := \lim_\ell \mathcal{A}_\ell v \in L^2(\Gamma)$. Elementary calculus predicts that this provides a well-defined linear operator $\mathcal{A}_\infty : L^2(\Gamma) \to L^2(\Gamma)$, and the Banach-Steinhaus theorem even predicts continuity $\mathcal{A}_\infty \in L(L^2(\Gamma); L^2(\Gamma))$.

Second, the H^1-stability (78) yields that $\mathcal{A}_\ell \in L^2(H^1(\Gamma); H^1(\Gamma))$ are uniformly continuous operators. For $v \in H^1(\Gamma)$, the sequence $(\mathcal{A}_\ell v)_\ell$ is hence bounded in $H^1(\Gamma)$ and thus admits a weakly convergent subsequence $\mathcal{A}_{\ell_k} v \rightharpoonup w$ weakly in $H^1(\Gamma)$ as $k \to \infty$. The Rellich compactness theorem yields $\mathcal{A}_{\ell_k} v \to w$ strongly in $L^2(\Omega)$. Uniqueness of limits therefore reveals $\mathcal{A}_\infty v = w \in H^1(\Gamma)$. Iterating this argument, we see that each subsequence of $\mathcal{A}_\ell v$ admits a further subsequence such that $\mathcal{A}_{\ell_{k_j}} v$ converges to $\mathcal{A}_\infty v \in H^1(\Gamma)$ weakly in $H^1(\Gamma)$. By elementary calculus, this implies weak convergence $\mathcal{A}_\ell v \rightharpoonup \mathcal{A}_\infty v$ in $H^1(\Gamma)$ for the entire sequence. Again, the Banach-Steinhaus theorem applies and proves that $\mathcal{A}_\infty \in L(H^1(\Gamma); H^1(\Gamma))$.

Third, the remaining claims follow from interpolation. The interpolation estimate (36) implies that the operator $\mathcal{A}_\infty \in L(H^s(\Gamma); H^s(\Gamma))$ is well-defined, linear, and continuous. Moreover, the estimate (34) of the interpolation norm and boundedness of weakly convergent sequences yields

$$\begin{aligned}
&\|(\mathcal{A}_\infty - \mathcal{A}_\ell)v\|_{H^s(\Gamma)}\\
&\quad \le \|(\mathcal{A}_\infty - \mathcal{A}_\ell)v\|_{L^2(\Gamma)}^{1-s} \|(\mathcal{A}_\infty - \mathcal{A}_\ell)v\|_{H^1(\Gamma)}^{s} \xrightarrow{\ell\to 0} 0
\end{aligned}$$

for all $0 < s < 1$ and $v \in H^1(\Gamma)$. By density of $H^1(\Gamma)$ in $H^s(\Gamma)$ and stability of $\mathcal{A}_\ell$, this results in pointwise convergence $\|(\mathcal{A}_\infty - \mathcal{A}_\ell)v\|_{H^s(\Gamma)} \to 0$ for all $v \in H^s(\Gamma)$. ∎

Proof of Theorem 8. The triangle inequality shows

$$\begin{aligned}
\eta_{\ell+1} \le& \|h_{\ell+1}^{1/2}(1-\mathcal{A}_\ell)u_\ell'\|_{L^2(\Gamma)}\\
&+ \|h_{\ell+1}^{1/2}(1-\mathcal{A}_{\ell+1})(u_{\ell+1}-u_\ell)'\|_{L^2(\Gamma)}\\
&+ \|h_{\ell+1}^{1/2}(\mathcal{A}_{\ell+1}-\mathcal{A}_\ell)u_\ell'\|_{L^2(\Gamma)}.
\end{aligned} \tag{84}$$

● For the first term, we argue analogously to [AFLP12]: According to bisection, we have $h_{\ell+1}|_T = \frac{1}{2}\, h_\ell|_T$ for refined elements $T \in \mathcal{T}_\ell\backslash\mathcal{T}_{\ell+1}$. This gives

$$\begin{aligned}
&\|h_{\ell+1}^{1/2}(1-\mathcal{A}_\ell)u_\ell'\|_{L^2(\Gamma)}^2\\
&\quad \le \sum_{T\in\mathcal{T}_\ell\cap\mathcal{T}_{\ell+1}} \eta_\ell(T)^2 + \frac{1}{2}\sum_{T\in\mathcal{T}_\ell\backslash\mathcal{T}_{\ell+1}} \eta_\ell(T)^2\\
&\quad = \eta_\ell^2 - \frac{1}{2}\sum_{T\in\mathcal{T}_\ell\backslash\mathcal{T}_{\ell+1}} \eta_\ell(T)^2.
\end{aligned}$$

Since at least all marked elements are refined, the Dörfler marking strategy (17) in step (iii) of the adaptive algorithm yields

$$\sum_{T\in\mathcal{T}_\ell\setminus\mathcal{T}_{\ell+1}} \eta_\ell(T)^2 \geq \sum_{T\in\mathcal{M}_\ell} \eta_\ell(T)^2 \geq \theta\,\eta_\ell^2.$$

Combining the last two estimates, we see

$$\|h_{\ell+1}^{1/2}(1-\mathcal{A}_\ell)u_\ell'\|_{L^2(\Gamma)} \leq (1-\theta/2)^{1/2}\,\eta_\ell. \qquad (85)$$

● Next, we consider the second term in (84). The local H^1-stability (77) yields

$$\|h_{\ell+1}^{1/2}(1-\mathcal{A}_{\ell+1})(u_{\ell+1}-u_\ell)'\|_{L^2(\Gamma)} \lesssim \|h_{\ell+1}^{1/2}(u_{\ell+1}-u_\ell)'\|_{L^2(\Gamma)}.$$

The inverse estimate of [GHS05, Thm. 3.6] gives

$$\begin{aligned}\|h_{\ell+1}^{1/2}(u_{\ell+1}-u_\ell)'\|_{L^2(\Gamma)} &\lesssim \|(u_{\ell+1}-u_\ell)'\|_{\widetilde{H}^{-1/2}(\Gamma)}\\ &\lesssim \|u_{\ell+1}-u_\ell\|_{\widetilde{H}^{1/2}(\Gamma)} \simeq |\!|\!|u_{\ell+1}-u_\ell|\!|\!|.\end{aligned}$$

Together with the a priori convergence of Lemma 9, we thus see

$$\|(1-\mathcal{A}_{\ell+1})(u_{\ell+1}-u_\ell)'\|_{L^2(\Gamma)} \xrightarrow{\ell\to\infty} 0. \qquad (86)$$

● Third, we consider the last term in (84): Let $\varepsilon > 0$. According to the a priori convergence of Lemma 9, there exists an index $k_0 \in \mathbb{N}$ such that

$$\|u_\ell - u_k\|_{\widetilde{H}^{1/2}(\Gamma)} \leq \varepsilon \quad \text{for all } k,\ell \geq k_0.$$

According to the pointwise a priori convergence of $\mathcal{A}_\ell$ from Lemma 11, there exists an index $\ell_0 \in \mathbb{N}$ such that

$$\|(\mathcal{A}_{\ell+1}-\mathcal{A}_\ell)u_{k_0}'\|_{L^2(\Gamma)} \leq \varepsilon \quad \text{for all } \ell \geq \ell_0.$$

Moreover, the local L^2-stability (77) of the operators yields

$$\|h_{\ell+1}^{1/2}(\mathcal{A}_{\ell+1}-\mathcal{A}_\ell)\psi\|_{L^2(\Gamma)} \lesssim \|h_{\ell+1}^{1/2}\psi\|_{L^2(\Gamma)}.$$

Plugging in $\psi = (u_\ell - u_{k_0})'$, the usual inverse estimate from [GHS05, Thm. 3.6] shows

$$\begin{aligned}\|h_{\ell+1}^{1/2}(\mathcal{A}_{\ell+1}-\mathcal{A}_\ell)(u_\ell-u_{k_0})'\|_{L^2(\Gamma)} &\lesssim \|h_{\ell+1}^{1/2}(u_\ell-u_{k_0})'\|_{L^2(\Gamma)}\\ &\lesssim \|(u_\ell-u_{k_0})'\|_{\widetilde{H}^{-1/2}(\Gamma)} \lesssim \|u_\ell-u_{k_0}\|_{\widetilde{H}^{1/2}(\Gamma)},\end{aligned}$$

where the hidden constants depend only on Γ and uniform boundedness of the local mesh-ratio $\kappa(\mathcal{T}_\ell)$. For $\ell \geq \max\{k_0,\ell_0\}$, we thus obtain

$$\begin{aligned}&\|h_{\ell+1}^{1/2}(\mathcal{A}_{\ell+1}-\mathcal{A}_\ell)u_\ell'\|_{L^2(\Gamma)}\\ &\quad\lesssim \|(\mathcal{A}_{\ell+1}-\mathcal{A}_\ell)u_{k_0}'\|_{L^2(\Gamma)} + \|u_\ell-u_{k_0}\|_{\widetilde{H}^{1/2}(\Gamma)}\\ &\quad\leq 2\varepsilon.\end{aligned}$$

This proves

$$\|h_{\ell+1}^{1/2}(\mathcal{A}_{\ell+1}-\mathcal{A}_\ell)u_\ell'\|_{L^2(\Gamma)} \xrightarrow{\ell\to\infty} 0. \qquad (87)$$

● Altogether, (85)–(87) prove

$$\eta_{\ell+1} \leq (1-\theta/2)^{1/2}\,\eta_\ell + \alpha_\ell \text{ with } 0 \leq \alpha_\ell \xrightarrow{\ell\to\infty} 0.$$

Since $0 < \theta \leq 1$, the error estimator is thus contractive up to a zero sequence. Therefore, elementary calculus concludes (76). ∎

6.3. Weakly-singular integral equation

As for the hyper-singular integral equation, we have the following convergence result for the adaptive algorithm of Section 3.4.

Theorem 12. *Let $(\phi_\ell)_{\ell\in\mathbb{N}}$ and $(\eta_\ell)_{\ell\in\mathbb{N}}$ be the sequences of discrete solutions and error estimators generated by the adaptive algorithm. Then, it holds*

$$\lim_{\ell\to\infty} \eta_\ell = 0. \qquad (88)$$

Provided that $|\!|\!|\phi-\phi_\ell|\!|\!| \lesssim \eta_\ell$, cf. Theorem 5, we may thus conclude $\lim_{\ell\to\infty}\phi_\ell = \phi$.

Proof. The proof follows analogously to that of Theorem 8. ∎

Acknowledgement. The authors MF, TF, and DP acknowledge support through the Austrian Science Fund (FWF) under grant P21732 *Adaptive Boundary Element Method*. MK acknowledges support by CONICYT project Anillo ACT1118 (ANANUM).

References

[AEF+11] Markus Aurada, Michael Ebner, Michael Feischl, Samuel Ferraz-Leite, Thomas Führer, Petra Goldenits, Michael Karkulik, Markus Mayr, and Dirk Praetorius. HILBERT – a MATLAB implementation of adaptive 2D-BEM. *ASC Report*, 24/2011, Vienna University of Technology, 2011.

[AFF+13] Markus Aurada, Michael Feischl, Thomas Führer, Michael Karkulik, and Dirk Praetorius. Efficiency and optimality of some weighted-residual error estimator for adaptive 2D boundary element methods. *Comput. Methods Appl. Math.*, in print, 2013.

[AFLP12] Markus Aurada, Samuel Ferraz-Leite, and Dirk Praetorius. Estimator reduction and convergence of adaptive BEM. *Appl. Numer. Math.*, 62(6):787–801, 2012.

[BC02a] Sören Bartels and Carsten Carstensen. Each averaging technique yields reliable a posteriori error control in FEM on unstructured grids. I. Low order conforming, nonconforming, and mixed FEM. *Math. Comp.*, 71(239):945–969, 2002.

[BC02b] Sören Bartels and Carsten Carstensen. Each averaging technique yields reliable a posteriori error control in FEM on unstructured grids. II. Higher order FEM. *Math. Comp.*, 71(239):971–994, 2002.

[BL76] Jöran Bergh and Jörgen Löfström. *Interpolation spaces. An introduction.* Springer-Verlag, Berlin, 1976. Grundlehren der Mathematischen Wissenschaften, No. 223.

[BS08] Susanne C. Brenner and L. Ridgway Scott. *The mathematical theory of finite element methods*, volume 15 of *Texts in Applied Mathematics*. Springer, New York, third edition, 2008.

[BV84] I. Babuška and M. Vogelius. Feedback and adaptive finite element solution of one-dimensional boundary value problems. *Numer. Math.*, 44(1):75–102, 1984.

[Car97] C. Carstensen. An a posteriori error estimate for a first-kind integral equation. *Math. Comp.*, 66(217):139–155, 1997.

[Car04] C. Carstensen. All first-order averaging techniques for a posteriori finite element error control on unstructured grids are efficient and reliable. *Math. Comp.*, 73(247):1153–1165, 2004.

[CF01] Carsten Carstensen and Birgit Faermann. Mathematical foundation of a posteriori error estimates and adaptive mesh-rening algorithms for boundary integral equations of the rst kind. *Eng. Anal. Bound. Elem.*, 25:497–509, 2001.

[CKNS08] J. Manuel Cascon, Christian Kreuzer, Ricardo H. Nochetto, and Kunibert G. Siebert. Quasi-optimal convergence rate for an adaptive finite element method. *SIAM J. Numer. Anal.*, 46(5):2524–2550, 2008.

[CMPS04] Carsten Carstensen, M. Maischak, D. Praetorius, and E. P. Stephan. Residual-based a posteriori error estimate for hypersingular equation on surfaces. *Numer. Math.*, 97(3):397–425, 2004.

[CMS01] C. Carstensen, M. Maischak, and E. P. Stephan. A posteriori error estimate and h-adaptive algorithm on surfaces for Symm's integral equation. *Numer. Math.*, 90(2):197–213, 2001.

[CP06] Carsten Carstensen and Dirk Praetorius. Averaging techniques for the effective numerical solution of Symm's integral equation of the first kind. *SIAM J. Sci. Comput.*, 27(4):1226–1260, 2006.

[CP07a] Carsten Carstensen and Dirk Praetorius. Averaging techniques for a posteriori error control in finite element and boundary element analysis. *Lect. Notes Appl. Comput. Mech.*, 29:29–59, 2007.

[CP07b] Carsten Carstensen and Dirk Praetorius. Averaging techniques for the a posteriori BEM error control for a hypersingular integral equation in two dimensions. *SIAM J. Sci. Comput.*, 29(2):782–810, 2007.

[CS95] Carsten Carstensen and Ernst P. Stephan. A posteriori error estimates for boundary element methods. *Math. Comp.*, 64(210):483–500, 1995.

[CS96] Carsten Carstensen and Ernst P. Stephan. Adaptive boundary element methods for some first kind integral equations. *SIAM J. Numer. Anal.*, 33(6):2166–2183, 1996.

[EFGP12] Christoph Erath, Stefan Funken, Petra Goldenits, and Dirk Praetorius. Simple error estimators for the Galerkin BEM for some hypersingular integral equation in 2D. *Appl. Anal.*, published online first, 2012.

[EFLFP09] Christoph Erath, Samuel Ferraz-Leite, Stefan Funken, and Dirk Praetorius. Energy norm based a posteriori error estimation for boundary element methods in two dimensions. *Appl. Numer. Math.*, 59(11):2713–2734, 2009.

[EH06] Vincent J. Ervin and Norbert Heuer. An adaptive boundary element method for the exterior Stokes problem in three dimensions. *IMA J. Numer. Anal.*, 26(2):297–325, 2006.

[Fae00] Birgit Faermann. Localization of the Aronszajn-Slobodeckij norm and application to adaptive boundary element methods. I. The two-dimensional case. *IMA J. Numer. Anal.*, 20(2):203–234, 2000.

[Fae02] Birgit Faermann. Localization of the Aronszajn-Slobodeckij norm and application to adaptive boundary element methods. II. The three-dimensional case. *Numer. Math.*, 92(3):467–499, 2002.

[FKMP13] M. Feischl, M. Karkulik, J. M. Melenk, and D. Praetorius. Quasi-optimal Convergence Rate for an Adaptive Boundary Element Method. *SIAM J. Numer. Anal.*, 51(2):1327–1348, 2013.

[FLP08] Samuel Ferraz-Leite and Dirk Praetorius. Simple a posteriori error estimators for the h-version of the boundary element method. *Computing*, 83(4):135–162, 2008.

[GHS05] I. G. Graham, W. Hackbusch, and S. A. Sauter. Finite elements on degenerate meshes: inverse-type inequalities and applications. *IMA J. Numer. Anal.*, 25(2):379–407, 2005.

[Heu02] Norbert Heuer. An hp-adaptive refinement strategy for hypersingular operators on surfaces. *Numer. Methods Partial Differential Equations*, 18(3):396–419, 2002.

[HMS01] N. Heuer, M. E. Mellado, and E. P. Stephan. hp-adaptive two-level methods for boundary integral equations on curves. *Computing*, 67(4):305–334, 2001.

[HW08] George C. Hsiao and Wolfgang L. Wendland. *Boundary integral equations*, volume 164 of *Applied Mathematical Sciences*. Springer-Verlag, Berlin, 2008.

[McL00] William McLean. *Strongly elliptic systems and boundary integral equations*. Cambridge University Press, Cambridge, 2000.

[MPM99] Govind Menon, Glaucio H. Paulino, and Subrata Mukherjee. Analysis of hypersingular residual error estimates in boundary element methods for potential problems. *Comput. Methods Appl. Mech. Engrg.*, 173(3-4):449–473, 1999.

[MSV08] Pedro Morin, Kunibert G. Siebert, and Andreas Veeser. A basic convergence result for conforming adaptive finite elements. *Math. Models Methods Appl. Sci.*, 18(5):707–737, 2008.

[MSW98] P. Mund, E. P. Stephan, and J. Weiße. Two-level methods for the single layer potential in $\mathbf{R}^3$. *Computing*, 60(3):243–266, 1998.

[Rod94] Rodolfo Rodríguez. Some remarks on Zienkiewicz-Zhu estimator. *Numer. Methods Partial Differential Equations*, 10(5):625–635, 1994.

[SS00] H. Schulz and O. Steinbach. A new a posteriori error estimator in adaptive direct boundary element methods: the Dirichlet problem. *Calcolo*, 37(2):79–96, 2000.

[SS11] Stefan A. Sauter and Christoph Schwab. *Boundary element methods*, volume 39 of *Springer Series in Computational Mathematics*. Springer-Verlag, Berlin, 2011.

[Ste00] O. Steinbach. Adaptive boundary element methods based on computational schemes for Sobolev norms. *SIAM J. Sci. Comput.*, 22(2):604–616, 2000.

[Ste07] Rob Stevenson. Optimality of a standard adaptive finite element method. *Found. Comput. Math.*, 7(2):245–269, 2007.

[Ste08] Rob Stevenson. The completion of locally refined simplicial partitions created by bisection. *Math. Comp.*, 77(261):227–241, 2008.

[SZ90] L. Ridgway Scott and Shangyou Zhang. Finite element interpolation of nonsmooth functions satisfying boundary conditions. *Math. Comp.*, 54(190):483–493, 1990.

[Tso13] Gantumur Tsogtorel. Adaptive boundary element methods with convergence rates. *Numer. Math.*, in print, 2013.

[ZZ87] O. C. Zienkiewicz and J. Z. Zhu. A simple error estimator and adaptive procedure for practical engineering analysis. *Internat. J. Numer. Methods Engrg.*, 24(2):337–357, 1987.