ASC Report No. 10/2013

Entries of indefinite Nevanlinna matrices

H. Woracek

Institute for Analysis and Scientific Computing
Vienna University of Technology — TU Wien
www.asc.tuwien.ac.at ISBN 978-3-902627-05-6

Most recent ASC Reports

09/2013 *L. Banas, M. Page, and D. Praetorius*
A convergent linear finite element scheme for the Maxwell-Landau-Lifshitz-Gilbert equation

08/2013 *L. Banas, M. Page, D. Praetorius, and J. Rochat*
On the Landau-Lifshitz-Gilbert equations with magnetostriction

07/2013 *P. Amodio, T. Levitina, G. Settanni, and E. B. Weinmüller*
Calculations of the Morphology Dependent Resonances

06/2013 *R. Hammer, W. Pötz, A. Arnold*
A dispersion and norm preserving finite difference scheme with transparent boundary
conditions for the Dirac equation in $(1+1)$D

05/2013 *M. Langer, H. Woracek*
Stabilty of N-extremal measures

04/2013 *J.M. Melenk*
xxxxxxxxxxxxxxxxxxxx xxxxxxxx

03/2013 *A. Jüngel, C.-K. Lin, and K.-C. Wu*
An asymptotic limit of a Navier-Stokes system with capillary effects

02/2013 *M. Tung and E. Weinmüller*
Gravitational frequency shifts in transformation acoustics

01/2013 *W. Auzinger*
A note on similarity to contraction for stable 2×2 companion matrices

52/2012 *M. Feischl, T. Führer, M. Karkulik, and D. Praetorius*
Stability of symmetric and nonsymmetric FEM-BEM couplings for nonlinear elasticity problems

Institute for Analysis and Scientific Computing
Vienna University of Technology
Wiedner Hauptstraße 8–10
1040 Wien, Austria

E-Mail: admin@asc.tuwien.ac.at
WWW: http://www.asc.tuwien.ac.at
FAX: +43-1-58801-10196

ISBN 978-3-902627-05-6

Entries of indefinite Nevanlinna matrices

HARALD WORACEK

Abstract: In the early 1950's M.G.Krein characterised those entire functions which are an entry of some Nevanlinna matrix, and those pairs of entire functions which are a row of some such matrix. In connection with Pontryagin space versions of Krein's theory of entire operators and de Branges' theory of Hilbert spaces of entire functions, an indefinite analogue of Nevanlinna matrices plays a role. In this paper we extend the mentioned characterisations to the indefinite situation and investigate the geometry of associated reproducing kernel Pontryagin spaces.

AMS MSC 2010: 46C20, 34B20, 30D10, 30D15
Keywords: Nevanlinna matrix, Pontryagin space, entire function, Krein class

1 Introduction

We call an entire 2×2-matrix valued function $W(z) = (w_{ij}(z))_{i,j=1}^2$ a Nevanlinna matrix, if it is normalized by $W(0) = I$, its entries take real values on the real line, $\det W(z) = 1$, $z \in \mathbb{C}$, and ($\mathbb{C}^+$ denotes the open upper half-plane)

$$\operatorname{Im} \frac{w_{11}(z)t + w_{12}(z)}{w_{21}(z)t + w_{22}(z)} \geq 0, \quad z \in \mathbb{C}^+,\ t \in \mathbb{R} \cup \{\infty\}\,. \tag{1.1}$$

It is well-known that, equivalently, one could require that $W(0) = I$, $w_{ij}(\overline{z}) = \overline{w_{ij}(z)}$, $\det W(z) = 1$, and that the reproducing kernel (J denotes the signature matrix $J := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$)

$$H_W(w,z) := \frac{W(z)JW(w)^* - J}{z - \overline{w}}, \quad z, w \in \mathbb{C}\,, \tag{1.2}$$

is positive definite[1,2]. This says that the class of Nevanlinna matrices coincides with the class of (normalized) entire (iJ)-inner 2×2-matrix functions, see, e.g., [AD08]. Often in the literature the entries of a Nevanlinna matrix are required to be transcendental entire functions, e.g. in [Akh61, Definition 2.4.3]. We do not include this condition.

Nevanlinna matrices play a fundamental role in many places of functional and complex analysis. Let us mention the following, of course interrelated, instances.

❀ M.G.Krein's theory of entire operators: there they appear as resolvent matrices in the description of all spectral functions, see, e.g., [GG97, Theorem 7.2].

❀ L.de Branges' theory of Hilbert spaces of entire functions: there they are used to characterise whether a de Branges space is invariant under forming difference quotients, or to characterise isometric inclusions of spaces, see, e.g., [dB68, Theorem 27, Theorems 33/34].

❀ Direct and inverse spectral problems for two-dimensional canonical systems:

[1]For $z = \overline{w}$ the expression $H_W(w,z)$ has to be interpreted appropriately as a derivative.
[2]The kernel $H_W(w,z)$ is called positive definite, if for each $L \in \mathbb{N}$, $z_1, \dots, z_L \in \mathbb{C}$, and $a_1, \dots, a_L \in \mathbb{C}^2$, the quadratic form $\sum_{i,j=1}^L a_j^* H_W(z_j, z_i) a_i \cdot \xi_i \overline{\xi_j}$ is positive semidefinite.

there they appear as fundamental solution matrix, and are represented as multiplicative integrals in the sense of V.P.Potapov, see, e.g., [GK67, Chapter VI.1], [AD08, Introduction, Theorem 4.13].

The theorems which hold for Nevanlinna matrices in the above contexts have implications for a number of concrete problems. For example they can be used to describe the solution set of a continuation problem of a positive definite function from an interval. Or, they can be used to describe spectral functions of certain differential operators. For these and other applications, see, e.g., [GG97, Theorems 3.1.6 and 3.2.6], [AD98], [KK68, §3.Fundamental Theorem], [KL]. The -probably- most intensively studied subject where Nevanlinna matrices play a role is the power moment problem. Literature on this topic starts with a historical paper of T.J.Stieltjes from 1894, and ranges from the classical work of H.Hamburger, R.Nevanlinna and M.Riesz from the 1920's to very recent contributions, e.g., [BP94, Ber95, Ped09].

A question which appears naturally is which entire functions may occur as an entry of *some* Nevanlinna matrix, or (especially when having in mind de Branges' theory) which pairs of entire functions may serve as a row of *some* Nevanlinna matrix. These questions were completely answered by M.G.Krein in the early 1950's, cf. [Kre52, §3.Theorems A,B,C][3].

In the theory of indefinite inner product spaces a generalisation of the notion of a Nevanlinna matrix occurs. Namely a certain class of entire matrix functions, we speak of $\mathcal{M}_{<\infty}$, where the positivity requirement for the kernel H_W is weakened; for details see Definition 2.4 below.

Matrices of the class $\mathcal{M}_{<\infty}$ are of similar significance in Pontryagin space theory, as Nevanlinna matrices are in Hilbert space theory. In connection with entire operators see [KL78, Satz 6.9], in connection with de Branges spaces see [KW99a, Proposition 10.3, Theorem 12.2], and in connection with canonical systems see [KW10, 1.3]. Also various applications are found, e.g., to indefinite power moment problems, cf. [KL79, KL80], the continuation problem of a hermitian indefinite function from an interval, cf. [GL74, KL85, KW98b], or to the spectral theory of certain differential operators with singular potentials, cf. [Wor12], [LW].

Our aim in the present paper is to establish the analogues of Krein's Theorems A,B,C for the class $\mathcal{M}_{<\infty}$, and to describe in detail the geometry of reproducing kernel Pontryagin spaces generated by matrices $W \in \mathcal{M}_{<\infty}$ with one prescribed row or entry. Our main results are Theorems 3.1, 3.4, and 3.5, being the full indefinite analogues of Krein's Theorems A,B,C, and Theorems 4.2 and 4.3 where we investigate geometric structure.

The proof of Theorems 3.1, 3.4, and 3.5 is neither too difficult nor too labourious. They can be shown with some Pontryagin space arguments and some standard complex analysis using our previous (involved, but readily available) work [Wor11]. Nevertheless, we regard these results themselves as valuable; the beautifully round up the picture of the indefinite theory. Our method of proof uses the interplay of Nevanlinna function theory and the theory of de Branges-Pontryagin spaces. This approach seems to be new also in the positive definite

[3]Up to our knowledge the relevant paper of Krein has not been translated to english. For the convenience of the reader, we will formulate his results in detail, cf. Theorems 2.1, 2.2, and 2.3 below. A proof of Theorem B can be found in [Akh61, Ch.3,p.133,№12], see also the more detailed and slightly more general exposition [BP95, Theorems 3.6,5.1].

case. In particular, when specialised to the positive definite case, our results provide (probably) new proofs of Krein's Theorems A,B,C.

Matters are getting much more involved in Theorems 4.2 and 4.3. In order to analyse the geometric structure of corresponding reproducing kernel spaces, we show one structure result about indefinite canonical systems and employ several facts from this theory.

Concerning the presentation of the article, one comment is in order. The core content is arranged in two sections. These are Sections 3 and 4. In the first, we prove the indefinite analogues of Krein's theorems. In the latter, we carry out the mentioned analysis of geometric structure. In view of the amount of notions and results required from previous work, it turned out impossible to provide a fully self-contained exposition within an acceptable page range. Hence, we decided for a clear division: All auxiliary notation and knowledge needed for the proof of Theorems 3.1, 3.4, and 3.5 in Section 3 is provided in the preliminary Section 2.2. All what is required in Section 4 from the theory of indefinite canonical systems is clearly and extensively referenced, but otherwise used without further notice. We will comment on this in more detail in the notice on page 19.

2 Preliminaries

First, in Subsection 2.1, we state the classical theorems in detail. Then, in Subsection 2.2, we provide a selection of auxiliary notation and results.

2.1 Krein's Theorems A,B,C

In the below formulations we already include our normalisation $W(0) = I$. Let us moreover remark that Krein used the term "special matrix", whereas nowadays it is more common to speak of Nevanlinna matrices[4].

First, a description of all pairs of entire functions which appear as the second row of some Nevanlinna matrix. Thereby, we call an entire function real, if it takes real values along the real line.

2.1 Theorem ([Kre52, §3.Theorem A])**.** *Let F and G be two real entire functions with $(F(0), G(0)) = (0, 1)$. Then there exists a Nevanlinna matrix W such that $(F, G) = (0, 1)W$, if and only if the following conditions (i)–(iv) hold.*

(i) *F and G have no common zeros, and all zeros of F and G are real and simple.*

(ii) $\operatorname{Im} \frac{G(z)}{F(z)} \geq 0$ *for all $z \in \mathbb{C}^+$.*

(iii) *The nonzero zeros α_n of F and β_n of G satisfy*

$$\sum_n \frac{1}{|F'(\alpha_n)G(\alpha_n)|\alpha_n^2} < \infty, \quad \sum_n \frac{1}{|F(\beta_n)G'(\beta_n)|\beta_n^2} < \infty\,.$$

[4] The term "Nevanlinna matrix" was probably first used in [Akh61], and is motivated since such matrices appear in Nevanlinna's parameterisation of the solutions of a power moment problem.

(iv) *The function* $\frac{1}{FG}$ *has an expansion*

$$\frac{1}{F(z)G(z)} = \frac{c_{-1}}{z} + c_0 + \sum_n \frac{1}{F'(\alpha_n)G(\alpha_n)} \Big[\frac{1}{z-\alpha_n} + \frac{1}{\alpha_n}\Big] + \\ + \sum_n \frac{1}{F(\beta_n)G'(\beta_n)} \Big[\frac{1}{z-\beta_n} + \frac{1}{\beta_n}\Big],$$

with some $c_{-1}, c_0 \in \mathbb{R}$.

Second, a description of all entire functions which appear as an entry of some Nevanlinna matrix.

2.2 Theorem ([Kre52, §3.Theorem B]). *Let F be a real entire function with $F(0) = 0$ or $F(0) = 1$. Then there exists a Nevanlinna matrix W such that F is an entry of W, if and only if all zeros of F are real and simple, the nonzero zeros α_n of F satisfy*

$$\sum_n \frac{1}{|F'(\alpha_n)|\alpha_n^2} < \infty,$$

and the function $\frac{1}{F}$ *has an expansion*

$$\frac{1}{F(z)} = \frac{c_{-1}}{z} + c_0 + \sum_n \frac{1}{F'(\alpha_n)} \Big[\frac{1}{z-\alpha_n} + \frac{1}{\alpha_n}\Big],$$

with some $c_{-1}, c_0 \in \mathbb{R}$.

Third, thinking of the left lower entry of some Nevanlinna matrix being fixed, a description of all entire functions which form together with this function the second row of some (other) Nevanlinna matrix.

2.3 Theorem ([Kre52, §3.Theorem C]). *Let F be a real entire functions with $F(0) = 0$, and assume that F is subject to the conditions of Theorem 2.2. In order that a real entire function G, $G(0) = 1$, forms together with F the second row of some Nevanlinna matrix, it is necessary and sufficient that the functions F and G satisfy the following conditions (i)–(iii).*

(i) *F and G have no common zeros, and all zeros of G are real and simple.*

(ii) $\operatorname{Im} \frac{G(z)}{F(z)} \geq 0$ *for all* $z \in \mathbb{C}^+$.

(iii) $\displaystyle\sum_n \frac{1}{|F'(\alpha_n)G(\alpha_n)|\alpha_n^2} < \infty.$

The analogous statement holds when we regard G as fixed and F as varying.

In [Kre52, §3.Theorem C] actually a different condition is stated, and it is remarked afterwards that this condition is in turn equivalent to the one we state here. However, this equivalence is immediate. Hence, we repeat only one of the conditions (and chose the one which is more suitable in the present context).

2.2 Some classes of analytic and meromorphic functions

We recall some terminology and present some (mainly) well-known facts from complex analysis and Pontryagin space theory.

To start with, let us give the definition of the main players in the present paper: matrices of class $\mathcal{M}_{<\infty}$.

2.4 Definition. Let $W = (w_{ij})_{i,j=1}^2$ be a 2×2-matrix valued function and let $\kappa \in \mathbb{N}_0$. We write $W \in \mathcal{M}_\kappa$ if the following conditions hold.

(M1) The entries w_{ij} of W are real entire functions.

(M2) $\det W(z) = 1$ for $z \in \mathbb{C}$, and $W(0) = I$.

(M3) The reproducing kernel H_W defined in (1.2) has κ negative squares[5].

Moreover, we set

$$\mathcal{M}_{<\infty} := \bigcup_{\kappa\in\mathbb{N}_0} \mathcal{M}_\kappa \,,$$

and write $\operatorname{ind}_- W = \kappa$ to express that $W \in \mathcal{M}_\kappa$. ◊

As we already said in the introduction, the class $\mathcal{M}_0$ is nothing but the class of all Nevanlinna matrices.

2.5 Remark. For the present considerations it is more practical to use the indefinite version of the definition of Nevanlinna matrices via the reproducing kernel H_W, rather than the indefinite version of the initially stated classical definition of Nevanlinna matrices the fractional linear transformations. As in the positive definite case, these two approaches are equivalent, see, e.g., [Kal02, Proposition 2.3, Theorem 6.1]. ◊

It is an important fact that the class $\mathcal{M}_{<\infty}$ is closed with respect to taking products. In fact,

$$\operatorname{ind}_-(W_1W_2) \leq \operatorname{ind}_- W_1 + \operatorname{ind}_- W_2, \quad W_1, W_2 \in \mathcal{M}_{<\infty} \,, \tag{2.1}$$

cf. [KW11, Lemma 2.10].

2.6. Polynomial matrices: Examples for matrices of class $\mathcal{M}_{<\infty}$ are obtained from polynomials. Let p be a real polynomial without constant term. Then we have

$$W_p := \begin{pmatrix} 1 & p \\ 0 & 1 \end{pmatrix} \in \mathcal{M}_{<\infty} \,, \tag{2.2}$$

and ($n := \deg p$, a_n leading coefficient of p, and $\lfloor x \rfloor$ denotes the largest integer not exceeding x)

$$\operatorname{ind}_- \Big\lfloor \frac{n}{2} \Big\rfloor + \begin{cases} 1\,, & n \text{ odd, } a_n < 0 \\ 0\,, & \text{otherwise} \end{cases} \,.$$

This fact is well-known; an explicit reference is, e.g., [KW11, Proposition 2.8].

If W is a real 2×2-matrix polynomial with $W(0) = I$ and $\det W = 1$, then W belongs to $\mathcal{M}_{<\infty}$ and can be factorized into a product of “rotations” of elementary factors of the form (2.2). This is shown in [ADL04, Theorem 3.1], for a purely algebraic approach see [KW06, Theorem 3.1]. ◊

[5] By this we mean that κ is the maximal number of negative squares of quadratic forms $\sum_{i,j=1}^L a_j^* H_W(z_j, z_i) a_i \cdot \xi_i \overline{\xi_j}$ with $L \in \mathbb{N}$, $z_1, \ldots, z_L \in \mathbb{C}$, and $a_1, \ldots, a_L \in \mathbb{C}^2$.

2.7. Changing roles of rows and columns: Each of the classes $\mathcal{M}_\kappa$, $\kappa \in \mathbb{N}_0$, is invariant under the transformations

$$\begin{pmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{pmatrix} \mapsto \begin{pmatrix} w_{22} & -w_{21} \\ -w_{12} & w_{11} \end{pmatrix}, \quad \begin{pmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{pmatrix} \mapsto \begin{pmatrix} w_{11} & -w_{21} \\ -w_{12} & w_{22} \end{pmatrix},$$

$$\begin{pmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{pmatrix} \mapsto \begin{pmatrix} w_{22} & w_{12} \\ w_{21} & w_{11} \end{pmatrix},$$

see, e.g., [KWW06, Lemma 2.3]. Hence, a pair (F, G) is the second row of some matrix $W \in \mathcal{M}_{<\infty}$, if and only if $(G, -F)$ is the first row or if and only $(-F, G)^T$ is the second column or if and only if $(G, F)^T$ is the first column of some matrix $W \in \mathcal{M}_{<\infty}$. $\Diamond$

Next, we discuss some facts from classical complex analysis.

2.8. Entire functions of Cartwright class: An entire function is said to be of Cartwright class if it is of finite exponential type, i.e., satisfies

$$\limsup_{|z|\to\infty} \frac{1}{|z|} \log |F(z)| < \infty \,,$$

and if

$$\int_{-\infty}^{\infty} \frac{\log^+ |F(x)|}{1+x^2}\, dx < \infty \,.$$

Functions of Cartwright class are well-behaved concerning their growth and distribution of zeros, see, e.g., [Lev80, V.4.Theorem 11]. In the present context the following is important: Assume that F is of Cartwright class, and that all but finitely many zeros of F are real and simple. Denote by (α_n) the sequence of all nonzero real and simple zeros of F, let (γ_j) be the remaining nonzero zeros (with multiplicities $d_j \in \mathbb{N}$), and let $d_0 \in \mathbb{N}_0$ be the multiplicity of 0 as a zero of F. Moreover, denote by α_n^+ and α_n^- the sequences of positive or negative, respectively, elements of (α_n) arranged according to increasing modulus. Then

(*i*) the limits[6] $\lim_n \frac{\alpha_n^+}{n}$ and $\lim_n \frac{\alpha_n^-}{n}$ exist in $[0, \infty)$ and are equal;

(*ii*) the limit $\lim_{R\to\infty} \sum_{|\alpha_n|\le R} \frac{1}{\alpha_n}$ exists in $\mathbb{R}$;

(*iii*) the function F has the representation

$$F(z) = \frac{F^{(d_0)}(0)}{d_0!} z^{d_0} \cdot \prod_j \Big(1 - \frac{z}{\gamma_j}\Big)^{d_j} \cdot \lim_{R\to\infty} \prod_{|\alpha_n|\le R} \Big(1 - \frac{z}{\alpha_n}\Big). \tag{2.3}$$

However, in general these properties do not imply Cartwright class. $\Diamond$

2.9. Meromorphic functions of bounded type: Let f be a function which is meromorphic in the half-plane $\mathbb{C}^+$. Then we say that f is of bounded type in this half plane, if it can be written as the quotient of two functions which are analytic and bounded in this half-plane.

The set of all functions of bounded type in $\mathbb{C}^+$ is a field. Two simple but important examples are the following.

[6] We tacitly understand the limit of a finite sequence to be equal to zero.

(i) Each rational function is of bounded type in $\mathbb{C}^+$. This follows since the constant function 1 and the function $(z-i)^{-1}$ (obviously) have this property.

(ii) Each function which is analytic in $\mathbb{C}^+$ and has nonnegative imaginary part throughout this half-plane is of bounded type. To see this consider the fractional linear transformation $L(z) := (z-i)(z+i)^{-1}$. It is analytic in $\mathbb{C} \setminus \{-i\}$ and maps the closed half-plane $\overline{\mathbb{C}^+}$ onto the closed unit disk. If q is analytic with $\operatorname{Im} q \geq 0$ throughout $\mathbb{C}^+$, then $L \circ q$ is analytic and bounded in $\mathbb{C}^+$. Now we obtain the representation

$$q = L^{-1} \circ (L \circ q) = i\frac{1 + (L \circ q)}{1 - (L \circ q)}$$

of q as the quotient of two bounded analytic functions[7].

Functions of bounded type in the lower half-plane $\mathbb{C}^-$ are defined in the same way, and enjoy analogous properties. This follows by conformally invariance. $\Diamond$

2.10. Analytic functions of bounded type: If F is analytic in $\mathbb{C}^+$, a more intrinsic characterisation of the property to be of bounded type reads as follows, cf. [RR94, Theorem 3.20]: An analytic function F in $\mathbb{C}^+$ is of bounded type if and only if the subharmonic function $\log^+|F(z)|$ has a harmonic majorant in $\mathbb{C}^+$.

Let us finally turn to entire functions. It is an important result due to M.G.Krein that an entire function is of Cartwright class, if and only if its restrictions to $\mathbb{C}^+$ and $\mathbb{C}^-$ are both of bounded type in the respective half-plane. Moreover, if F is of bounded type in both half-planes, then the exponential type of F can be computed as

$$\max\Big\{ \lim_{y\to+\infty} \frac{1}{y}\log^+|F(iy)|, \lim_{y\to-\infty} \frac{1}{|y|}\log^+|F(iy)| \Big\}. \tag{2.4}$$

For these facts see [Kre47, Theorems 3 and 2] or, e.g., [RR94, Theorems 6.17 and 6.18]. $\Diamond$

Finally, we recall some notions and results from the indefinite world; among them generalized Nevanlinna functions and de Branges-Pontryagin spaces. These include the most important tools for our present work, and are probably the least commonly known of our required prerequisits.

2.11. Generalized Nevanlinna functions: A function q which is meromorphic in $\mathbb{C} \setminus \mathbb{R}$ is called a generalized Nevanlinna function, if $q(\overline{z}) = \overline{q(z)}$ and the reproducing kernel ($\rho(q)$ denotes the domain of holomorphy of q)

$$N_q(w,z) := \frac{q(z) - q(\overline{w})}{z - \overline{w}}, \quad z, w \in \rho(q),$$

has a finite number of negative squares. The set of all generalized Nevanlinna functions is denoted as $\mathcal{N}_{<\infty}$. If $q \in \mathcal{N}_{<\infty}$, then we write $\operatorname{ind}_- q$ for the actual number of negative squares of the kernel N_q, and set

$$\mathcal{N}_\kappa := \big\{ q \in \mathcal{N}_{<\infty} : \operatorname{ind}_- q = \kappa \big\}, \quad \kappa \in \mathbb{N}_0.$$

[7] Functions with nonnegative imaginary part throughout the half-plane are even outer, cf. [RR94, V.Examples and Addenda 1.].

It is a historical result[8] that a function q belongs to $\mathcal{N}_0$ if and only if it is analytic throughout $\mathbb{C}\setminus\mathbb{R}$, satisfies $q(\overline{z})=\overline{q(z)}$ and $\operatorname{Im} q(z)\geq 0$ for $z\in\mathbb{C}^+$.

Simple examples of generalized Nevanlinna functions are rational functions. Each rational function q with real coefficients belongs to the class $\mathcal{N}_{<\infty}$, and $\operatorname{ind}_- q$ cannot exceed the maximal degree of numerator and denominator of q (this is a well-known fact; a short and explicit proof can be found, e.g., in [Wor97, Theorem 1]). If q is a polynomial, it is easy to compute $\operatorname{ind}_- q$ explicitly: we have ($n:=\deg q$, a_n leading coefficient of q)

$$\operatorname{ind}_- \left\lfloor \frac{n}{2} \right\rfloor + \begin{cases} 1\,, & n \text{ odd},\ a_n<0 \\ 0\,, & \text{otherwise} \end{cases}$$

see, e.g., [KL77, Lemma 3.3] (applied with $\sigma=0$).

Let us mention that the class $\mathcal{N}_{<\infty}$ is closed with respect to sums, in fact, $\operatorname{ind}_-(q_1+q_2)\leq \operatorname{ind}_- q_1+\operatorname{ind}_- q_2$.

It is a deep result shown in [KL77, Satz 3.1] that functions of class $\mathcal{N}_{<\infty}$ have an integral representation analogous to the Herglotz integral representation of functions with nonnegative imaginary part.

Finally, recall that all but finitely poles of a generalized Nevanlinna function are real, simple, and have negative residuum. This follows from the mentioned integral representation or, alternatively, from the multiplicative representation of a generalized Nevanlinna function stated in 2.13 below. $\Diamond$

2.12. Computing negative index: If $q\in\mathcal{N}_{<\infty}$, one can give a formula for $\operatorname{ind}_- q$ based on the structure of the poles of q and on the asymptotic growth of the measure in its integral representation towards certain critical points, cf. [KL77, Satz 3.4]. For our present needs the following estimate is sufficient: Let $q\in\mathcal{N}_{<\infty}$ and assume that q is meromorphic in the whole plane. Denote by (α_n) the sequence of all real and simple poles of q with negative residuum, let (γ_j) be the remaining poles (with multiplicities $d_j\in\mathbb{N}$), and set

$$\delta_j := \begin{cases} 1\,, & d_j \text{ odd},\ \lim_{z\to\gamma_j}(z-\gamma_j)^{d_j}q(z)>0 \\ 0\,, & \text{otherwise} \end{cases}\,.$$

Then

$$\operatorname{ind}_- q \geq \sum_{\gamma_j\in\mathbb{R}} \Big(\Big\lfloor \frac{d_j}{2} \Big\rfloor + \delta_j \Big) + \sum_{\operatorname{Im}\gamma_j>0} d_j + \\ + \min\Big\{ m\in\mathbb{N}_0 : \sum_n \frac{|\operatorname{Res}(q;\alpha_n)|}{1+\alpha_n^{2(m+1)}} <\infty \Big\}\,. \quad (2.5)$$

Sometimes, especially when dealing with functions which are meromorphic in the whole plane, it is practical not to refer to [KL77, Satz 3.4] directly. A useful method for "counting negative squares" is to employ [KL77, Satz 1.13] to split into a sum with one summand well-behaved at ∞ and one well-behaved off ∞. The negative index of the first summand then can be computed using [KL81, Theorem 3.5] and [Lan86], and for the second summand by using [KL77, Lemma 3.3].

[8] It can be traced back as far as to some work of G.Herglotz from 1911.

Let us also explicitly mention the following corollary: Let $q \in \mathcal{N}_\kappa$ be meromorphic in $\mathbb{C}$. Then all but at most 2κ poles of q are real and simple and have negative residuum. Moreover, if (α_n) denotes the sequence of all real and simple poles of q with negative residuum, then

$$\sum_n \frac{|\operatorname{Res}(q;\alpha_n)|}{1+\alpha_n^{2(\kappa+1)}} < \infty\,. \tag{2.6}$$

Using to the sources mentioned above, one can also show the following more refined statement: Let $q \in \mathcal{N}_{<\infty}$ be meromorphic in $\mathbb{C}$, and denote the number on the right side of (2.5) by $\kappa_0(q)$. Then (we denote by $\mathbb{R}[z]$ the set of polynomials with real coefficients)

$$\big\{ \operatorname{ind}_-(q+p) : \, p \in \mathbb{R}[z], p(0)=0 \big\} = \big[\kappa_0(q), \infty\big) \cap \mathbb{N}_0\,. \tag{2.7}$$

$\Diamond$

2.13. Multiplicative representations of $q \in \mathcal{N}_{<\infty}$*:* The following important multiplicative representation of a generalized Nevanlinna function has been established independently in [DLLS00, Corollary] and [DHdS99, Theorem 3.3]: Let a function $q \in \mathcal{N}_{<\infty}$ be given. Then there exist relatively prime polynomials $p, \tilde{p}$ whose zeros are all located in the closed upper half-plane, and a function $q \in \mathcal{N}_0$, such that (we denote $f^\#(z) := \overline{f(\overline{z})}$)

$$q(z) = \frac{\tilde{p}(z)\tilde{p}^\#(z)}{p(z)p^\#(z)} q_0(z)\,. \tag{2.8}$$

Also a converse holds, cf. [DHdS99, Proposition 3.2]: Let $p, \tilde{p}$ and q_0 be as above, then the function q in (2.8) belongs to $\mathcal{N}_\kappa$ with $\kappa := \max\{\deg p, \deg \tilde{p}\}$.

As a consequence of this representation we see that each generalized Nevanlinna function is of bounded type (in both half-planes $\mathbb{C}^+$ and $\mathbb{C}^-$). $\Diamond$

2.14. Indefinite Hermite-Biehler functions: We say that an entire function E belongs to the indefinite Hermite-Biehler class $\mathcal{HB}_{<\infty}$, if it is normalized by $E(0)=1$, the functions E and $E^\#$ have no common zeros, and the reproducing kernel

$$K_E(w,z) := i\frac{E(z)\overline{E(w)} - E^\#(z)E(\overline{w})}{2\pi(z-\overline{w})}, \quad z,w \in \mathbb{C}\,,$$

has a finite number of negative squares. Again we denote the actual number of negative squares of this kernel by $\operatorname{ind}_- E$, and write $\mathcal{HB}_\kappa$ for all functions with $\operatorname{ind}_- E = \kappa$.

The class $\mathcal{HB}_0$ is a classical object: It consists of all entire functions with $E(0)=1$, which are such that E and $E^\#$ have no common zeros, and satisfy

$$|E(\overline{z})| \leq |E(z)|, \quad z \in \mathbb{C}^+\,.$$

Thus is nothing but the class of all Hermite-Biehler functions (somtimes also called “de Branges functions”) as studied, e.g., in [dB68], [Lev80, Chapter VII].

Let $E \in \mathcal{HB}_{<\infty}$. Then the reproducing kernel K_E generates a Pontryagin space of entire functions, cf. [ADRdS97, Theorem 1.1.3]. We denote this space as $\mathcal{P}(E)$, and refer to it as the de Branges–Pontryagin space associated with E. For the theory of such spaces, see [KW99a]. If $\operatorname{ind}_- E = 0$, this notion coincides with the classical concept of de Branges’ Hilbert spaces of entire functions, cf. [dB68]. $\Diamond$

2.15. Functions associated to a de Branges–Pontryagin spaces: A central concept for our present purposes is the notion of functions N-associated to a de Branges space: For $E \in \mathcal{HB}_{<\infty}$ and $N \in \mathbb{N}$, we set

$$\operatorname{Assoc}_N \mathcal{P}(E) := \mathcal{P}(E) + z\mathcal{P}(E) + \ldots + z^N \mathcal{P}(E) .$$

The following statements are crucial tools for our present considerations.

First, the fact whether or not $\operatorname{Assoc}_N \mathcal{P}(E)$ contains a real and zerofree element is related to existence of matrices $W \in \mathcal{M}_{<\infty}$ whose first row equals (A, B) where

$$A := \frac{1}{2}(E + E^{\#}), \quad B := \frac{i}{2}(E - E^{\#}) .$$

Namely, we have

$$1 \in \bigcup_{N\in\mathbb{N}} \operatorname{Assoc}_N \mathcal{P}(E) \quad \Leftrightarrow \quad \exists\, W \in \mathcal{M}_{<\infty} :\ (1,0)W = (A,B) \quad .$$

This is shown in [KW99a, Proposition 10.3] and [Wor11, Proposition 6.1], where the first reference covers the case $N = 1$ and the second the case $N > 1$.

The second major result is [Wor11, Theorem 3.2]. To recall this, we need to introduce some notation. For $E \in \mathcal{HB}_{<\infty}$ and $\varphi \in \mathbb{R}$, set

$$S_\varphi(z) := \frac{1}{2i}\Big(e^{i\varphi}E(z) - e^{-i\varphi}E^{\#}(z)\Big) .$$

Denote by $(\alpha_{\varphi,n})$ the sequence of all nonzero real and simple zeros of S_φ such that the residuum of $S_\varphi^{-1}S_{\varphi+\frac{\pi}{2}}$ at this point is negative, set $\sigma_{\varphi,n} := -\operatorname{Res}\big(S_\varphi^{-1}S_{\varphi+\frac{\pi}{2}}; \alpha_{\varphi,n}\big)$, let $(\gamma_{\varphi,j})$ be the remaining nonzero zeros (multiplicities denoted as $d_{\varphi,j} \in \mathbb{N}$), and let $d_{\varphi,0} \in \mathbb{N}_0$ be the multiplicity of 0 as a zero of S_φ. Moreover, denote by $\alpha^+_{\varphi,n}$ and $\alpha^-_{\varphi,n}$ the sequences of positive or negative, respectively, elements of $(\alpha_{\varphi,n})$ arranged according to increasing modulus. Finally, set

$$F_\varphi(z) = z^{d_{\varphi,0}} \cdot \prod_j \Big(1 - \frac{z}{\gamma_{\varphi,j}}\Big)^{d_{\varphi,j}} \cdot \lim_{R\to\infty} \prod_{|\alpha_{\varphi,n}|\leq R} \Big(1 - \frac{z}{\alpha_{\varphi,n}}\Big) ,$$

provided the product converges.

Now [Wor11, Theorem 3.2] says the following. Assume that $\dim \mathcal{P}(E) = \infty$ and let $N \in \mathbb{N}$. Then $\operatorname{Assoc}_N \mathcal{P}(E)$ contains a real and zerofree functions, if and only if for some $\varphi \in \mathbb{R}$ the above data satisfies the conditions (i) and (ii) of 2.8 and

$$\sum_n |\alpha_{\varphi,n}|^{-2N} \frac{1}{|F'_\varphi(\alpha_{\varphi,n})|^2 \sigma_{\varphi,n}} < \infty .$$

If $\operatorname{Assoc}_N \mathcal{P}(E)$ contains a real and zerofree function, then these conditions hold for all $\varphi \in \mathbb{R}$, and the function $F_\varphi^{-1}S_\varphi$ is the (up to scalar multiples) unique real zerofree element of $\bigcup_{M\in\mathbb{N}} \operatorname{Assoc}_M \mathcal{P}(E)$.

If $\dim \mathcal{P}(E) < \infty$, then always $\operatorname{Assoc}_1 \mathcal{P}(E)$ contains a real and zerofree function. In fact, $\mathcal{P}(E)$ is of the form

$$\mathcal{P}(E) = \big\{U(z)p(z) :\ p \in \mathbb{C}[z], \deg p < \dim \mathcal{P}(E)\big\} ,$$

with U being real and zerofree, see, e.g., [Wor11, Remark 3.3]. $\lozenge$

2.16. Relations between $\mathcal{M}_{<\infty}$, $\mathcal{N}_{<\infty}$, $\mathcal{HB}_{<\infty}$: There is a variety of (analytic and geometric) relations between the classes $\mathcal{M}_{<\infty}$, $\mathcal{N}_{<\infty}$, and $\mathcal{HB}_{<\infty}$, as well as between the respective reproducing kernel space. In the present paper the following facts are used.

(*i*) Let $W = (w_{ij})_{i,j=1}^2 \in \mathcal{M}_\kappa$. Then

$$\frac{w_{11}}{w_{21}}, \frac{w_{12}}{w_{22}}, \frac{w_{12}}{w_{11}}, \frac{w_{22}}{w_{21}} \in \bigcup_{\kappa' \leq \kappa} \mathcal{N}_{\kappa'} ,$$

see, e.g., [KWW06, Corollary 2.10] or [KL78].

(*ii*) Let F, G be real and entire functions, $F(0) = 1$, $G(0) = 0$, which have no common zeros, and set $E := F - iG$. Then ($\kappa \in \mathbb{N}_0$)

$$E \in \mathcal{HB}_\kappa \quad \Longleftrightarrow \quad \frac{G}{F} \in \mathcal{N}_\kappa ,$$

see, e.g., [KW99a, Remark 5.2].

(*iii*) Let us explicitly point out the following fact, which follows by combining items (*i*) and (*ii*). If $W \in \mathcal{M}_{<\infty}$, then $E_+ := w_{11} - iw_{12}, E_- := w_{22} + iw_{21} \in \mathcal{HB}_{<\infty}$, and $\operatorname{ind}_- E_+, \operatorname{ind}_- E_- \leq \operatorname{ind}_- W$.

$\Diamond$

3 The indefinite analogues of Krein's theorems

First, we give the indefinite analogue of Theorem 2.2. Thereby, we include an additional item (III), because it provides an easier accessible condition on F and because it makes the proof more transparent.

3.1 Theorem. *Let F be a real entire function with $F(0) = 0$ or $F(0) = 1$. Then the following are equivalent.*

(I) *There exists a matrix $W \in \mathcal{M}_{<\infty}$ such that F is an entry of W.*

(II) *Denote by (α_n) the (finite or infinite) sequence of all nonzero real and simple zeros of F. Then*

$$\exists\, N \in \mathbb{N} : \quad \sum_n \frac{1}{|F'(\alpha_n)| \cdot |\alpha_n|^{N+1}} < \infty , \tag{3.1}$$

and the function $\frac{1}{F}$ has the expansion

$$\frac{1}{F(z)} = R(z) + \sum_n \frac{1}{F'(\alpha_n)} \Big[\frac{1}{z - \alpha_n} + \frac{1}{\alpha_n} + \frac{z}{\alpha_n^2} + \ldots + \frac{z^{N-1}}{\alpha_n^N} \Big] , \tag{3.2}$$

with some rational function R.

(III) (a) *All but finitely many zeros of F are real and simple.*

Denote by (α_n) the sequence of all nonzero real and simple zeros of F, let (γ_j) be the remaining nonzero zeros (with multiplicities $d_j \in \mathbb{N}$), and let $d_0 \in \mathbb{N}_0$ be the multiplicity of 0 as a zero of F. Moreover, denote by α_n^+ and α_n^- the sequences of positive or negative, respectively, elements of (α_n) arranged according to increasing modulus. Then

(b) *The limits*[9] $\lim_n \frac{\alpha_n^+}{n}$ *and* $\lim_n \frac{\alpha_n^-}{n}$ *exist in* $[0,\infty)$ *and are equal.*

(c) *The limit* $\lim_{R\to\infty} \sum_{|\alpha_n|\le R} \frac{1}{\alpha_n}$ *exists in* $\mathbb{R}$.

(d) *The function* F *has the representation*

$$F(z)=\frac{F^{(d_0)}(0)}{d_0!}z^{d_0}\cdot\prod_j\Big(1-\frac{z}{\gamma_j}\Big)^{d_j}\cdot\lim_{R\to\infty}\prod_{|\alpha_n|\le R}\Big(1-\frac{z}{\alpha_n}\Big).$$

(e) *We have*

$$\begin{aligned}&\exists\ N\in\mathbb{N},\,\sigma_n>0:\\ &\qquad\sum_n\frac{\sigma_n}{|\alpha_n|^{N+1}}<\infty,\ \sum_n\frac{1}{\sigma_n|F'(\alpha_n)|^2\cdot|\alpha_n|^{N+1}}<\infty\,.\end{aligned}\tag{3.3}$$

The proof of this result requires some preparation. First, we provide a variant of [Kre47, Theorem 4], see also [Lev80, V.6.Theorem 13].

3.2 Lemma. *Let* (α_n) *be a (finite or infinite) sequence of pairwise distinct and nonzero real numbers, let* (τ_n) *be a corresponding sequence of real numbers, let* $N\in\mathbb{N}$, *and let* R *be a rational function. Assume that*

$$\sum_n\Big|\frac{\tau_n}{\alpha_n^{N+1}}\Big|<\infty\,,\tag{3.4}$$

and consider the function[10]

$$\begin{aligned}f(z):=R(z)+\sum_n\tau_n\Big[\frac{1}{z-\alpha_n}+\frac{1}{\alpha_n}+\frac{z}{\alpha_n^2}+\ldots+\frac{z^{N-1}}{\alpha_n^N}\Big]=\\ =R(z)+z^N\sum_n\frac{\tau_n}{\alpha_n^N(z-\alpha_n)}\,.\end{aligned}\tag{3.5}$$

Then the meromorphic functions $f|_{\mathbb{C}^+}$ *and* $f|_{\mathbb{C}^-}$ *are of bounded type in the respective half-planes.*

Krein's theorem [Kre47, Theorem 4] is much stronger in the sense that also nonreal points α_n satisfying the Blaschke-condition are permitted. On the other hand, it is assumed a priori that f is the inverse of an entire function, i.e., has no zeros. In the above variant this is not required.

Proof (of Lemma 3.2*).* We rewrite

$$\begin{aligned}f(z)=R(z)+z^{N-1}\Big(\sum_{\tau_n\alpha_n^{N-1}<0}\frac{\tau_n}{\alpha_n^{N-1}}\Big[\frac{1}{z-\alpha_n}+\frac{1}{\alpha_n}\Big]-\\ -\sum_{\tau_n\alpha_n^{N-1}>0}\frac{-\tau_n}{\alpha_n^{N-1}}\Big[\frac{1}{z-\alpha_n}+\frac{1}{\alpha_n}\Big]\Big)\,.\end{aligned}$$

Each of the two sums is analytic in $\mathbb{C}^+$, and has nonnegative imaginary part throughout this half-plane. Thus it is of bounded type, cf. 2.9, (*ii*). ❑

[9] Again we tacitly understand the limit of a finite sequence to be equal to zero.

[10] Due to (3.4) the sum converges locally uniformly off the points α_n. It represents a meromorphic function in $\mathbb{C}$, whose poles are all simple and located at the points α_n.

Second, a variant of an argument which is around (at least) since the basic paper [Kre47], and repeatedly appears more or less explicitly in the literature (e.g., [dB59, Lemma 2], [BP95, Lemma 6.3], or [Bak98, Theorem 3.1]). The formulation taylored to our present needs reads as follows.

3.3 Lemma. *Let F be a real entire function of Cartwright class, such that all but finitely many zeros of F are real and simple, and such that* (3.1) *holds. Then* $\frac{1}{F}$ *has an expansion* (3.2).

Proof. The function F is of bounded type in $\mathbb{C}^+$ and $\mathbb{C}^-$. Due to its representation (2.3), we have $\lim_{|y|\to\infty} |F(iy)| = \infty$.

Choose $N \in \mathbb{N}$ according to (3.1), and consider the function

$$H(z) := \sum_n \frac{1}{F'(\alpha_n)} \Big[\frac{1}{z-\alpha_n} + \frac{1}{\alpha_n} + \frac{z}{\alpha_n^2} + \ldots + \frac{z^{N-1}}{\alpha_n^N} \Big] .$$

By Lemma 3.2, H is of bounded type in $\mathbb{C}^+$ and $\mathbb{C}^-$. By bounded convergence (remember (3.5)), we have $\lim_{|y|\to\infty} |y|^{-(N+1)}|H(iy)| = 0$.

Let R_0 be the sum of all principal parts of the Laurent expansions of $\frac{1}{F}$ at poles different from the points α_n. Then R_0 is rational, hence of bounded type, and $\lim_{|y|\to\infty} R_0(iy) = 0$.

The function $L := \frac{1}{F} - R_0 - H$ is entire, of bounded type in $\mathbb{C}^+$ and $\mathbb{C}^-$, and satisfies $\lim_{|y|\to\infty} |y|^{-(N+1)}|L(iy)| = 0$. Referring to Krein's theorem [RR94, Theorem 6.18], cf. (2.4), we obtain that L is of minimal exponential type. The Phragmen-Lindelöf principle (e.g., in the form [Lev80, I.14.Corollary] adapted with an appropriate power) implies that L is a polynomial of degree at most N. We see that $\frac{1}{F}$ admits an expansion (3.2). ❑

The equivalence of (II) and (III) in Theorem 3.1 is now nearly obvious. The implication "(I) ⇒ (III)" is deduced with mainly standard methods, and "(III) ⇒ (I)" is established with help of [Wor11].

Proof (of Theorem 3.1*).* If F is a polynomial, each of (I), (II), and (III), holds. Thereby, (II) and (III) are trivial, for (I) remember 2.6. Hence, throughout the proof we may assume that F is transcendental.

Step 1; "(II)⇒(III)*":* Only the poles of R may give rise to nonreal poles, or to poles with multiplicity greater than 1 of $\frac{1}{F}$. Thus (a) holds. Lemma 3.2 together with Krein's theorem recalled in 2.10 gives that F is of Cartwright class. Hence, (b), (c), and (d) hold, cf. 2.8. Choose $N \in \mathbb{N}$ according to (3.1) and set

$$\sigma_n := \frac{1}{|F'(\alpha_n)|} .$$

Then (3.3) is immediate from (3.1).

Step 2; "(III)⇒(II)*":* Choose N, σ_n according to (3.3). Since $x + \frac{1}{x} \geq 1$ for all $x > 0$, writing the relations (3.3) in the form

$$\sum_n \sigma_n |F'(\alpha_n)| \cdot \tfrac{1}{|F'(\alpha_n)||\alpha_n|^{N+1}} < \infty, \quad \sum_n \tfrac{1}{\sigma_n |F'(\alpha_n)|} \cdot \tfrac{1}{|F'(\alpha_n)||\alpha_n|^{N+1}} < \infty ,$$

yields (3.1). It has been shown in [LW02, Lemma 5.5][11], that (a)–(d) in conjunction with (3.1) imply that F is of Cartwright class. Now Lemma 3.3 guarantees (3.2).

Step 3; "(I)⇒(III)": We restrict explicit proof to the case that $F(0) = 1$; the case "$F(0) = 0$" is treated in the same way. Moreover, by 2.7, we may restrict to considering the left upper entry of a matrix of class $\mathcal{M}_{<\infty}$.

Let $W \in \mathcal{M}_{<\infty}$ be given, set $\kappa := \operatorname{ind}_- W$, and consider the function $F := w_{11}$. By [LW13, Proposition 2.7][12], this function is of Cartwright class and hence satisfies (b)–(d), cf. 2.8.

The functions $\frac{w_{12}}{F}$ and $-\frac{w_{21}}{F}$ both belong to $\bigcup_{\kappa' \leq \kappa} \mathcal{N}_{\kappa'}$, cf. 2.16. Since F and w_{12} (and F and w_{21}, respectively) have no common zeros, (a) follows, cf. 2.11. Moreover, by (2.6),

$$\sum_n \Big| \frac{w_{12}(\alpha_n)}{F'(\alpha_n)} \Big| \frac{1}{|\alpha_n|^{2(\kappa+1)}} < \infty, \quad \sum_n \Big| \frac{w_{21}(\alpha_n)}{F'(\alpha_n)} \Big| \frac{1}{|\alpha_n|^{2(\kappa+1)}} < \infty .$$

Set $N := 2\kappa + 1$ and $\sigma_n := \big| \frac{w_{12}(\alpha_n)}{F'(\alpha_n)} \big|$. Then the first relation in (3.3) holds. We have

$$1 = \det W(\alpha_n) = -w_{12}(\alpha_n) w_{21}(\alpha_n) \tag{3.6}$$

and hence

$$\Big| \frac{w_{21}(\alpha_n)}{F'(\alpha_n)} \Big| = \frac{1}{\sigma_n |F'(\alpha_n)|^2} .$$

Thus also the second relation in (3.3) holds.

Step 4; "(III)⇒(I)": Again we restrict explicit proof to the case that $F(0) = 1$. Choose $N \in \mathbb{N}$ and $\sigma_n > 0$ such that (3.3) holds. Set

$$\begin{aligned}
q_0(z) &:= \sum_n \frac{\sigma_n}{|\alpha_n|^{N-1}} \Big[\frac{1}{\alpha_n - z} - \frac{1}{\alpha_n} \Big] + \sum_{\substack{j:\gamma_j \in \mathbb{R} \\ d_j \text{ odd}}} \Big[\frac{1}{\gamma_j - z} - \frac{1}{\gamma_j} \Big] , \\
p(z) &:= \prod_{\gamma_j \in \mathbb{R}} (z - \gamma_j)^{\lfloor \frac{d_j}{2} \rfloor} \cdot \prod_{\operatorname{Im} \gamma_j > 0} (z - \gamma_j)^{d_j} , \\
q(z) &:= \frac{1}{p(z) p^{\#}(z)} \cdot q_0(z), \qquad G(z) := F(z) q(z) , \\
M &:= N + \sum_{\gamma_j \in \mathbb{R}} \Big\lfloor \frac{d_j}{2} \Big\rfloor + \sum_{\operatorname{Im} \gamma_j > 0} d_j .
\end{aligned}$$

The function q_0 is well-defined by convergence of the first sum in (3.3), and meromorphic in the whole plane. Moreover, it belongs to $\mathcal{N}_0$. It follows that $q \in \mathcal{N}_{<\infty}$, cf. 2.13. We have

$$\operatorname{Res}(q; \alpha_n) = \frac{-\sigma_n}{|\alpha_n|^{N-1}} \cdot \frac{1}{|p(\alpha_n)|^2} ,$$

and therefore

$$\lim_{n\to\infty} \frac{|\operatorname{Res}(q;\alpha_n)|}{\sigma_n} \cdot \frac{|\alpha_n|^{2M}}{|\alpha_n|^{N+1}} = 1 . \tag{3.7}$$

[11] Probably this fact has a longer history, but we are not aware of another explicit reference.
[12] For asymptotically well-behaved matrices W this also follows by combining [KL78, Satz 4.2] with [KL78, §6.2].

The functions F and G have no common zero and $G(0) = 0$. By what we recalled in 2.16,

$$E := F - iG \in \mathcal{H}B_{<\infty} .$$

Using (3.7), convergence of the second sum in (3.3) implies that

$$\sum_n \alpha_n^{-2M} \frac{1}{|F'(\alpha_n)|^2 |\operatorname{Res}(q; \alpha_n)|} < \infty .$$

Remembering our present hypothesis (b) and (c), we see that [Wor11, Theorem 3.2] is applicable with the function E and the angle "$\varphi = \frac{\pi}{2}$", cf. 2.15. Using (d), it follows that $1 \in \operatorname{Assoc}_M \mathcal{P}(E)$. Now [Wor11, Proposition 6.1], cf. 2.15, provides a matrix $W \in \mathcal{M}_{<\infty}$ with $(1,0)W = (F,G)$. ❑

Let us now proceed to the indefinite versions of Theorems 2.1 and 2.3.

3.4 Theorem. *Let F and G be two real entire functions with $(F(0), G(0)) = (0,1)$. Then there exists a matrix $W \in \mathcal{M}_{<\infty}$ such that $(F,G) = (0,1)W$, if and only if the following conditions (α)–(δ) hold.*

(α) *F and G have no common zeros, and all but finitely many zeros of F and G are real and simple.*

(β) *The reproducing kernel*

$$N_{\frac{G}{F}}(w,z) := \frac{1}{z-\overline{w}} \left(\frac{G(z)}{F(z)} - \frac{G(\overline{w})}{F(\overline{w})} \right)$$

has a finite number of negative squares.

(γ) *The sequences (α_n) and (β_n) of all nonzero real and simple zeros of F and G, respectively, satisfy*

$$\exists\, N \in \mathbb{N} :$$
$$\sum_n \frac{1}{|F'(\alpha_n)G(\alpha_n)||\alpha_n|^{N+1}} < \infty, \ \sum_n \frac{1}{|F(\beta_n)G'(\beta_n)||\beta_n|^{N+1}} < \infty .$$

(δ) *The function $\frac{1}{FG}$ has the expansion*

$$\begin{aligned} \frac{1}{F(z)G(z)} = R(z) + & \\ & + \sum_n \frac{1}{F'(\alpha_n)G(\alpha_n)} \Big[\frac{1}{z-\alpha_n} + \frac{1}{\alpha_n} + \frac{z}{\alpha_n^2} + \ldots + \frac{z^{N-1}}{\alpha_n^N} \Big] + \\ & + \sum_n \frac{1}{F(\beta_n)G'(\beta_n)} \Big[\frac{1}{z-\beta_n} + \frac{1}{\beta_n} + \frac{z}{\beta_n^2} + \ldots + \frac{z^{N-1}}{\beta_n^N} \Big] , \end{aligned}$$

with some rational function R.

3.5 Theorem. *Let F be an entire function with $F(0) = 0$, and assume that F is of Cartwright class. In order that a real entire function G, $G(0) = 1$, forms together with F the second row of some matrix $W \in \mathcal{M}_{<\infty}$, it is necessary and sufficient that the conditions (α) and (β) of Theorem 3.4 and the following condition (γ') holds.*

(γ') *The sequence* (α_n) *of all nonzero real and simple zeros of* F *satisfies*

$$\exists\, N \in \mathbb{N}: \quad \sum_n \frac{1}{|F'(\alpha_n)G(\alpha_n)||\alpha_n|^{N+1}} < \infty\,.$$

The analogous statement holds when we regard G *as fixed and* F *as varying.*

In the positive definite case this result contains a slight improvement of Theorem 2.3; the a priori hypothesis on F is slightly weakened.

3.6 Remark. Following the presentation in [Kre52], we have formulated Theorems 3.4 and 3.5 for (F, G) being the second row (or the left lower element, respectively) of some matrix $W \in \mathcal{M}_{<\infty}$. From 2.7 it is obvious that corresponding statements hold for the first row, or the first or the second column, instead of the second row. $\Diamond$

The proof of Theorems 3.4 and 3.5 is carried out using essentially the same arguments as in the proof of Theorem 3.1.

Proof (of Theorems 3.4 *and* 3.5*).* For necessity assume that $W \in \mathcal{M}_{<\infty}$ is given, and consider $(F, G) := (0, 1)W$. The condition (α) follows since $\det W = 1$ and $\frac{G}{F} \in \mathcal{N}_{<\infty}$, cf. 2.11. Condition (β) holds by 2.16. By [Wor11, Proposition 6.1], cf. 2.15, there exists $M \in \mathbb{N}$ with $1 \in \operatorname{Assoc}_M \mathcal{P}(E)$ where $E := G + iF$ (apply 2.7 to switch lower and upper row). Applying [Wor11, Theorem 3.2], cf. 2.15, with the function E and the angle "$\varphi = \frac{\pi}{2}$" gives

$$\sum_n \frac{1}{|F'(\alpha_n)G(\alpha_n)||\alpha_n|^{2M}} = \sum_n \alpha_n^{-2M} \frac{1}{|F'(\alpha_n)|^2 |\operatorname{Res}\big(\frac{G}{F}; \alpha_n\big)|} < \infty\,.$$

This is (γ') with $N := 2M - 1$. Using the angle "$\varphi = 0$", gives

$$\sum_n \frac{1}{|F(\beta_n)G'(\beta_n)||\beta_n|^{2M}} = \sum_n |\beta_n|^{-2M} \frac{1}{|G'(\beta_n)|^2 |\operatorname{Res}\big(-\frac{F}{G}; \beta_n\big)|} < \infty\,,$$

and we see that even (γ) holds. By [LW13, Proposition 2.7], the functions F and G are of Cartwright class. Hence, also FG is of Cartwright class. Due to (γ), the hypothesis required to apply Lemma 3.3 is satisfied, and (δ) follows.

For sufficiency assume that F and G are given and satisfy the hypothesis of either Theorem 3.4 or of Theorem 3.5. In the latter case, F is of Cartwright class directly from the hypothesis. In the first case, we use Krein's theorem (or Lemma 3.2) to conclude that the function FG is of bounded type in $\mathbb{C}^+$ and $\mathbb{C}^-$. The function $\frac{G}{F}$ belongs to $\mathcal{N}_{<\infty}$, and thus has the same property, cf. 2.13. It follows that F^2 is of Cartwright class, and hence also F is. Again using (β) we see that also G is of Cartwright class. From (α) and (β),

$$E := G + iF \in \mathcal{H}B_{<\infty}\,,$$

cf. 2.16, (ii). By (γ') (hence also by the stronger condition (γ)) the hypothesis necessary to apply [Wor11, Theorem 3.2] with E and "$\varphi = \frac{\pi}{2}$" is fullfilled, cf. 2.15. Now [Wor11, Proposition 6.1], cf. 2.15, provides us with a matrix $W \in \mathcal{M}_{<\infty}$ such that $(G, -F) = (1, 0)W$. It remains to apply 2.7 in order to pass to the lower row. ❑

4 On the geometry of reproducing kernel spaces

Consider a matrix $W \in \mathcal{M}_{<\infty}$. Then the reproducing kernel $H_W(w,z)$ defined in (1.2) generates a reproducing kernel Pontryagin space $\mathcal{K}(W)$ whose elements are 2-vector valued entire functions. This space is obtained as the Pontryagin space completion of the linear space

$$\operatorname{span}\Big\{H_W(w,.)\binom{\alpha}{\beta}:\ w\in\mathbb{C},\ \alpha,\beta\in\mathbb{C}\Big\}$$

which is endowed with an inner product $[.,.]$ defined by linearity and

$$\Big[H_W(w,.)\binom{\alpha}{\beta},H_W(w',.)\binom{\alpha'}{\beta'}\Big]:=\binom{\alpha'}{\beta'}^*H_W(w,.)\binom{\alpha}{\beta},$$
$$w,w'\in\mathbb{C},\ \alpha,\beta,\alpha',\beta'\in\mathbb{C}\,,$$

see, e.g., [ADRdS97, Theorem 1.1.3]. Basic theorems on spaces $\mathcal{K}(W)$, their relation with de Branges–Pontryagin spaces on the one hand, and their relation with the spectral theory of entire operators on the other, can be found in [KW99a], [KW99b] for the first, and in [KW98a], [KL78] for the latter. Standard literature dealing with the positive definite case, is, e.g., [dB68] or [GG97], [KL].

4.1. The subspaces $\mathcal{K}_\pm(W)$: Let $W\in\mathcal{M}_{<\infty}$. In the structure theory of the space $\mathcal{K}(W)$ certain subspaces play a role. Namely (here "cls" stands for "closed linear span")

$$\mathcal{K}_+(W):=\operatorname{cls}\Big\{H_W(w,.)\binom{1}{0}:\ w\in\mathbb{C}\Big\},$$
$$\mathcal{K}_-(W):=\operatorname{cls}\Big\{H_W(w,.)\binom{0}{1}:\ w\in\mathbb{C}\Big\}.$$

As we have observed in 2.16, the functions $E_+:=w_{11}-iw_{12}$ and $E_-:=w_{22}+iw_{21}$ belong to $\mathcal{HB}_{<\infty}$. Hence, they generate de Branges–Pontryagin spaces $\mathcal{P}(E_+)$ and $\mathcal{P}(E_-)$. Denote by π_+ the projection of a vector function onto its first component, i.e., $\pi_+:\binom{F}{G}\mapsto F$. It is shown in [KW99a, Lemma 8.6] that $\pi_+|_{\mathcal{K}_+(W)}$ maps $\mathcal{K}_+(W)$ isometrically and surjectively onto $\mathcal{P}(E_+)$. Moreover, we have $\ker(\pi_+|_{\mathcal{K}(W)})=\mathcal{K}_+(W)^\perp$. In particular,

$$\ker(\pi_+|_{\mathcal{K}_+(W)})=\mathcal{K}_+(W)^\circ\,.$$

The analogous statements hold for the second row of W, i.e., with "+" everywhere replaced by "−", and π_- being the projection onto the second component. $\Diamond$

The geometry of $\mathcal{K}_+(W)$ and $\mathcal{K}_-(W)$ as closed subspaces of the Pontryagin space $\mathcal{K}(W)$ has consequences for the structure theory of W. For instance, two crucial result in this context are [KW99b, Theorem 5.7] and [Wor11, Lemma 6.3].

In the spirit of the present paper the following question appears naturally:

Which values may the quantities

$$\operatorname{ind}_-\mathcal{K}(W),\quad \operatorname{ind}_-\mathcal{K}_-(W),\ \dim\mathcal{K}_-(W)^\circ$$

take when W varies through all matrices having a prescribed second row (or one prescribed entry in this row)?

In the below two theorems we give the answer. First, we regard the second row as prescribed.

4.2 Theorem. *Let F and G be entire functions which are subject to the conditions of either Theorem 3.4 or Theorem 3.5, so that there exist matrices of class $\mathcal{M}_{<\infty}$ which have (F, G) as their second row. Denote by (α_n) the sequence of all nonzero real and simple zeros of F. Then the following statements hold true.*

(*i*) *We have*

$$\big\{ \operatorname{ind}_- \mathcal{K}(W) : \ W \in \mathcal{M}_{<\infty}, \, (0,1)W = (F,G) \big\} \\ = \Big[\operatorname{ind}_- \frac{G}{F} + \dim \mathcal{K}_-(W)^\circ \ , \ \infty \Big) \cap \mathbb{N}_0 \, .$$

(*ii*) *For each $W \in \mathcal{M}_{<\infty}$ with $(0,1)W = (F,G)$ we have*

$$\operatorname{ind}_- \mathcal{K}_-(W) = \operatorname{ind}_- \frac{G}{F} \, .$$

(*iii*) *For each $W \in \mathcal{M}_{<\infty}$ with $(0,1)W = (F,G)$ we have*

$$\dim \mathcal{K}_-(W)^\circ = \min \Big\{ M \in \mathbb{N}_0 : \ \sum_n \frac{1}{|F'(\alpha_n) G(\alpha_n)| |\alpha_n|^{2(M+1)}} < \infty \Big\} \, .$$

Second, we regard the left lower entry as prescribed. A corresponding result holds, if we fix the right lower entry; we do not give details.

4.3 Theorem. *Let F, $F(0) = 0$, be an entire function which satisfies one (and hence each) condition of Theorem 3.1, so that there exist matrices of class $\mathcal{M}_{<\infty}$ which have F as their left lower entry. Let notation "α_n, γ_j, d_j, etc." be as in Theorem 3.1, and set*

$$\nu := \min \Big\{ N \in \mathbb{N}_0 : \sum_n \frac{1}{|F'(\alpha_n)| \cdot |\alpha_n|^{N+1}} < \infty \Big\} \, ,$$

$$\delta := \Big(\Big\lfloor \frac{d_0}{2} \Big\rfloor + \begin{cases} 1, & d_0 \ odd, F^{(d_0)}(0) > 0 \\ 0, & otherwise \end{cases} \Big) + \sum_{\gamma_j \in \mathbb{R}} \Big\lfloor \frac{d_j}{2} \Big\rfloor + \sum_{\operatorname{Im} \gamma_j > 0} d_j \, .$$

Then the following statements hold.

(*i*) *If $W \in \mathcal{M}_{<\infty}$ with $(0,1)W\binom{1}{0} = F$, then*

$$\operatorname{ind}_- \mathcal{K}(W) \geq \operatorname{ind}_- \mathcal{K}_-(W) + \dim \mathcal{K}_-(W)^\circ \geq \delta + \nu - 1 \, , \tag{4.1}$$

$$\operatorname{ind}_- \mathcal{K}_-(W) \geq \delta \, . \tag{4.2}$$

(*ii*) *If $\alpha, \beta, \gamma \in \mathbb{N}_0$ satisfy*

$$\alpha \geq \beta + \gamma \geq \delta + \nu - 1, \qquad \beta \geq \delta \, , \tag{4.3}$$

then there exists $W \in \mathcal{M}_{<\infty}$ such that

$$(0,1)W \begin{pmatrix} 1 \\ 0 \end{pmatrix} = F \, ,$$

$$\operatorname{ind}_- \mathcal{K}(W) = \alpha, \quad \operatorname{ind}_- \mathcal{K}_-(W) = \beta, \ \dim \mathcal{K}_-(W)^\circ = \gamma \, .$$

Let us immediately point out one interesting consequence of this theorem: Since, in the first relation in (4.3) only the sum $\beta+\gamma$ appears, one can trade negative index against dimension of degeneracy.

4.4 Corollary. *Let F, $F(0)=0$, be an entire function which satisfies one (and hence each) condition of Theorem* 3.1, *so that there exist matrices of class $\mathcal{M}_{<\infty}$ which have F as their left lower entry. Then there exist matrices $W_1, W_2 \in \mathcal{M}_{<\infty}$ such that*

$$(0,1)W_1\begin{pmatrix}1\\0\end{pmatrix}=F,\quad \operatorname{ind}_-\mathcal{K}_-(W_1)=\delta\,.$$

$$(0,1)W_2\begin{pmatrix}1\\0\end{pmatrix}=F,\quad \dim\mathcal{K}_-(W_2)^\circ=0\,.$$

The choice of W_2 can be made such that $\mathcal{K}_-(W_2)=\mathcal{K}(W_2)$.

The rest of this section is devoted to the proof of these results.

Notice: In the following discussion we extensively use terminology and results from the theory of indefinite canonical systems and maximal chains of matrices. We refer the reader who wishes to dive into the details to [Wor11, §4], [LWar, §2], or (the most exhaustive reference) [KW11, §2, §3]. There all definitions and a review of most relevant theorems can be found. Detailed references will be provided throughout the subsequent text.

To the reader who is not interested in details we suggest to skip Proposition 4.5, believe in Theorem 4.2, (*i*), and proceed directly to the proofs of Theorem 4.2, (*ii*) and (*iii*), and Theorem 4.3 and its corollary, which start on page 24. To make the dependencies precise:

Theorem 4.2, (*ii*) and (*iii*), Theorem 4.3, Steps 1,2, Corollary 4.4	Proofs can be read without further prerequisits
Proposition 4.5, Theorem 4.2, (*i*)	Requires familiarity with indefinite canonical systems
Theorem 4.3, Step 3	No further prerequisits required, but uses Theorem 4.2, (*i*)

In this context we must say it very clearly that the theory of indefinite canonical systems has entered through the backdoor from the very beginning. The *proofs* in our previous work [Wor11] depend highly on this theory. Only, the theorems required from [Wor11] in the present paper can be *formulated* without using such notions (and this we did in 2.15). $\Diamond$

Throughout the following we agree on a generic notation: if p is a polynomial with real coefficients and without constant term, we denote by P the matrix

$$P(z):=\begin{pmatrix}1 & p\\0 & 1\end{pmatrix}.$$

Let $W\in\mathcal{M}_{<\infty}$. Then we know from [KW99a, Corollary 9.8] that the totality of all matrices of class $\mathcal{M}_{<\infty}$ which have the same second row as W is given as

$$\Big\{\tilde W\in\mathcal{M}_{<\infty}:\ (0,1)\tilde W=(0,1)W\Big\}=\Big\{PW:\ p\in\mathbb{R}[z],p(0)=0\Big\}\,. \tag{4.4}$$

If $\mathcal{K}_-(W)^\circ = \{0\}$, then the spaces $\mathcal{K}(PW)$ are accessible to explicit computation. On the other hand, if $\mathcal{K}_-(W)$ degenerates, this is not anymore the case and things are getting more involved. In the following we describe what it means to pass from W to PW in terms of the associated general Hamiltonians. Thereby, we focus on the more difficult case that $\mathcal{K}_-(W)^\circ \neq \{0\}$. We also include the limit point situation (which means to work with Weyl coefficients instead of matrices of class $\mathcal{M}_{<\infty}$).

4.5 Proposition. *Let $\mathfrak{h}$ be a general Hamiltonian ([KW11, Definition 3.35]) given by data*[13]

$$\sigma_0, \ldots, \sigma_{n+1}, \quad H_0, \ldots, H_n, \quad \ddot{o}_i, b_{i,j}, d_{i,j},\ i = 1, \ldots, n, \quad E\,.$$

Assume that $n \geq 2$, that (σ_0, σ_1) is indivisible of type 0 ([KW11, p.259]), and that σ_1 is not left endpoint of an indivisible interval. Let $\mathfrak{h}'$ be another general Hamiltonian, and assume that $\mathfrak{h}$ and $\mathfrak{h}'$ are either both regular or both singular ([KW11, Definition 3.35,(3.4)]). Then, with (i), (ii_r), (ii_s) as written out below: if $\mathfrak{h}$ is regular we have "$(i) \Leftrightarrow (ii_r)$", and if $\mathfrak{h}$ is singular we have "$(i) \Leftrightarrow (ii_s)$".

(i) *The general Hamiltonians $\mathfrak{h}$ and $\mathfrak{h}'$ differ only in their data part at their first singularity. Precisely formulated, by this we mean that there exists a reparameterization ([KW11, Remark 3.38]) of $\mathfrak{h}'$ which is given by the data*

$$\sigma_0, \ldots, \sigma_{n+1}, \quad H_0, \ldots, H_n, \quad \ddot{o}_i', b_{i,j}', d_{i,j}',\ i = 1, \ldots, n, \quad E\,, \tag{4.5}$$

where

$$\ddot{o}_i' = \ddot{o}_i, b_{i,j}' = b_{i,j},\ d_{i,j}' = d_{i,j} \quad \text{for} \quad i = 2, \ldots, n\,. \tag{4.6}$$

(ii_r) *There exists $p \in \mathbb{R}[z]$, $p(0) = 0$, such that the monodromy matrices (this are the matrices "$\omega(\mathfrak{B}(\mathfrak{h}))$" in [KW11, Theorem 5.1,Proposition 4.29], cf. [KW11, top of p.226]) W and W' of $\mathfrak{h}$ and $\mathfrak{h}'$ are related as $W' = PW$.*

(ii_s) *There exists $p \in \mathbb{R}[z]$, $p(0) = 0$, such that the Weyl coefficients ([KW11, Definition 5.2]) $q_{\mathfrak{h}}$ and $q_{\mathfrak{h}'}$ of $\mathfrak{h}$ and $\mathfrak{h}'$ are related as $q_{\mathfrak{h}'} = q_{\mathfrak{h}} + p$.*

In the proof we use the following lemma.

4.6 Lemma. *Let $W \in \mathcal{M}_{<\infty}$ with $\mathcal{K}_-(W)^\circ \neq \{0\}$, and let $p \in \mathbb{R}[z]$, $p(0) = 0$. Let ω be a finite maximal chain going down from W ([KW11, Definition 3.7,3.1]), and denote its domain by $[\sigma_0, \sigma_{n+1}] \setminus \{\sigma_1, \ldots, \sigma_n\}$. Moreover, set*

$$\omega_p(x) := \begin{cases} \omega(x) \quad , & x \in [\sigma_0, \sigma_1) \\ P\omega(x)\,, & x \in (\sigma_1, \sigma_{n+1}] \setminus \{\sigma_2, \ldots, \sigma_n\} \end{cases}.$$

Then ω_p is a finite maximal chain going down from PW.

Proof. By [Wor11, Lemma 6.3] (applied with $-JWJ$) the interval $[\sigma_0, \sigma_1)$ in ω is indivisible of type 0 and σ_1 is not left endpoint of an indivisible interval.

Since $PW \in \mathcal{M}_{<\infty}$, cf. 2.6 and (2.1), there exists a finite maximal chain going down from PW ([KW11, 3.9,p.253]). Let ω' be one such. By [LW13,

[13] We agree that $b_{\ddot{o}_i+1} = 0$ unless indivisible intervals ([KW11, p.259]) adjoin to both sides of σ_i.

5.16 (p.310)], this chain starts with an indivisible interval of infinite length and type 0, and its first singularity is not left endpoint of an indivisible interval. By [LW13, Lemma 5.7] there exists an endsection of ω' which is a reparameterisation ([KW11, Definition 3.4]) of $P\omega|_{(\sigma_1,\sigma_{n+1}]\setminus\{\sigma_2,\dots,\sigma_n\}}$. The left endpoint of this endsection is a singularity of ω' since (notation "$\mathfrak{t}$" as in [KW11, Definition 2.1,(2.0)])

$$\lim_{x\searrow\sigma_1} \mathfrak{t}[P\omega(x)] = \mathfrak{t}[P] + \lim_{x\searrow\sigma_1} \mathfrak{t}[\omega(x)] = -\infty\,.$$

The intermediate Weyl coefficient ([KW03, p.284,Proposition 5.1]) of ω at its first singularity σ_1 is equal to the constant ∞, and ω' has the same property. We have (notation "$\star$" as in [KW11, p.246])

$$\lim_{x\searrow\sigma_1} \big[(P\omega(x))\star\tau\big] = P\star\big(\lim_{x\searrow\sigma_1}[\omega(x))\star\tau]\big) = P\star\infty = \infty, \quad \tau\in\mathbb{R}\cup\{\infty\}\,.$$

Hence the left endpoint of the mentioned endsection of ω' must be the first singularity of ω' (argue, e.g., as in [KW11, Proposition 3.10]). It follows that ω' and ω_p are related by a reparameterisation, in particular, ω_p is a finite maximal chain. ❑

4.7 Corollary. *Let ω be a maximal chain ([KW11, Definition 3.1]) defined on $[\sigma_0,\sigma_{n+1})\setminus\{\sigma_1,\dots,\sigma_n\}$, and assume that $[\sigma_0,\sigma_1)$ is indivisible of type 0 and that σ_1 is not left endpoint of an indivisible interval. Moreover, let $p\in\mathbb{R}[z]$, $p(0)=0$, and set*

$$\omega_p(x) := \begin{cases} \omega(x) \quad , & x\in[\sigma_0,\sigma_1) \\ P\omega(x)\,, & x\in(\sigma_1,\sigma_{n+1})\setminus\{\sigma_2,\dots,\sigma_n\}\end{cases}.$$

Then ω_p is a maximal chain. The Weyl coefficients ([KW11, Definition 3.5]) q_ω and q_{ω_p} of ω and ω_p, respectively, are related as

$$q_{\omega_p} = q_\omega + p\,.$$

Proof. Apply Lemma 4.6 to each beginning section of ω (and argue using [KW11, Remark 3.15]). ❑

Proof (of Proposition 4.5). First assume (i), i.e. that $\mathfrak{h}$ and $\mathfrak{h}'$ differ only in their data part at their first singularity. Without loss of generality, we may assume that the parameterisation of $\mathfrak{h}'$ is chosen such that it is given by data (4.5) with (4.6). Let ω and ω' be the (finite) maximal chains constructed from $\mathfrak{h}$ and $\mathfrak{h}'$ (as in [KW11, Definition 5.3]). Since the Hamiltonian functions of $\mathfrak{h}$ and $\mathfrak{h}'$ coincide, [KW11, Corollary 5.6] implies

$$\omega|_{[\sigma_0,\sigma_1)} = \omega'|_{[\sigma_0,\sigma_1)}\,, \tag{4.7}$$

$$\omega(x)^{-1}\omega(y) = \omega'(x)^{-1}\omega'(y), \quad \sigma_1<x\le y,\ x,y\in\operatorname{dom}\omega\,. \tag{4.8}$$

We are going to define two more general Hamiltonians $\tilde{\mathfrak{h}}$ and $\tilde{\mathfrak{h}}'$. Set $s_1 := \min(E\cap(\sigma_1,\sigma_2))$, choose $\psi\in\mathbb{R}$ such that s_1 is not right endpoint of an indivisible interval of type ψ, and set

$$\tilde{H}_1(x) := \begin{cases} H_1(x) & , \quad x\in(\sigma_1,s_1] \\ (\cos\psi,\sin\psi)^T(\cos\psi,\sin\psi)\,, & x\in(s_1,\infty)\end{cases}.$$

Now we define $\tilde{\mathfrak{h}}$ and $\tilde{\mathfrak{h}}'$ as the sets of data

$$\tilde{\mathfrak{h}}: \quad \sigma_0, \sigma_1, \infty, \quad H_0, \tilde{H}_1, \quad \ddot{o}_1, b_{1,j}, d_{1,j}, \quad \{\sigma_0, s_1, \infty\}$$

$$\tilde{\mathfrak{h}}': \quad \sigma_0, \sigma_1, \infty, \quad H_0, \tilde{H}_1, \quad \ddot{o}_1', b_{1,j}', d_{1,j}', \quad \{\sigma_0, s_1, \infty\}$$

Since $\tilde{\mathfrak{h}}$ and $\mathfrak{h}$, and $\tilde{\mathfrak{h}}'$ and $\mathfrak{h}'$, respectively, coincide to the left of s_1, we have

$$\tilde{\omega}|_{[\sigma_0,s_1]\setminus\{\sigma_1\}} = \omega|_{[\sigma_0,s_1]\setminus\{\sigma_1\}}, \quad \tilde{\omega}'|_{[\sigma_0,s_1]\setminus\{\sigma_1\}} = \omega'|_{[\sigma_0,s_1]\setminus\{\sigma_1\}}\,. \tag{4.9}$$

We apply [LWar, Corollary 5.9] with $\tilde{\mathfrak{h}}$ and $\tilde{\mathfrak{h}}'$. This provides us with a polynomial $p \in \mathbb{R}[z]$, $p(0) = 0$, such that the Weyl coefficients $q_{\tilde{\mathfrak{h}}}$ and $q_{\tilde{\mathfrak{h}}'}$ of $\tilde{\mathfrak{h}}$ and $\tilde{\mathfrak{h}}'$, respectively, are related as

$$q_{\tilde{\mathfrak{h}}'} = q_{\tilde{\mathfrak{h}}} + p\,.$$

Consider the maximal chain $\tilde{\omega}_p$ defined as in Corollary 4.7 starting from $\tilde{\omega}$. Then the Weyl coefficient $q_{\tilde{\omega}_p}$ is equal to $q_{\tilde{\omega}} + p$, and it follows that

$$q_{\tilde{\omega}_p} = q_{\tilde{\omega}} + p = q_{\tilde{\mathfrak{h}}} + p = q_{\tilde{\mathfrak{h}}'} = q_{\tilde{\omega}'}\,.$$

Hence, $\tilde{\omega}_p$ and $\tilde{\omega}'$ are reparameterisations of each other ([KW11, 3.6,p.251]).

Let α be an increasing bijection of $[\sigma_0, \infty) \setminus \{\sigma_1\}$ onto itself, such that $\tilde{\omega}_p = \tilde{\omega}' \circ \alpha$. The interval $[s_1, \infty)$ is maximal indivisible in the chain $\tilde{\omega}'$, and in the chain $\tilde{\omega}$, hence also in $\tilde{\omega}_p$. It follows that $\alpha(s_1) = s_1$, and hence (remember (4.9))

$$\omega'(s_1) = \tilde{\omega}'(s_1) = \tilde{\omega}_p(s_1) = P\tilde{\omega}(s_1) = P\omega(s_1)\,.$$

Using (4.8), it follows that

$$\omega'(y) = P\omega(y), \quad y \geq s_1, y \in \operatorname{dom}\omega\,.$$

If $\mathfrak{h}$ is regular, we may evaluate at $y = \sigma_{n+1}$ and obtain that monodromy matrices are related as $W' = PW$. If $\mathfrak{h}$ is singular, we pass to the limit $y \nearrow \sigma_{n+1}$ and obtain that Weyl coefficients are related as $q_{\mathfrak{h}'} = q_{\mathfrak{h}} + p$.

Second, assume that $\mathfrak{h}$ (and hence also $\mathfrak{h}'$) is regular, and that the corresponding monodromy matrices are related as $W' = PW$ with some $p \in \mathbb{R}[z]$, $p(0) = 0$. Let ω and ω' be the finite maximal chains associated with $\mathfrak{h}$ and $\mathfrak{h}'$ (as in [KW11, Definition 5.3]). Moreover, let ω_p be the finite maximal chain defined as in Lemma 4.6. Then (σ_{n+1} and $\sigma'_{n'+1}$ denote the maximum of the domains of ω and ω', respectively)

$$\omega_p(\sigma_{n+1}) = P\omega(\sigma_{n+1}) = PW = W' = \omega'(\sigma'_{n+1})\,.$$

It follows that ω_p and ω' are reparameterisations of each other ([KW11, 3.9,p.253]). Let $\mathfrak{h}_p$ denote the regular general Hamiltonian constructed from ω_p (as in [KW10, §2,§3] using the same splitting points E as in $\mathfrak{h}$). Then $\mathfrak{h}_p$ and $\mathfrak{h}'$ are reparameterisations of each other ([KW10, 1.3,p.514]).

From the definition of ω_p we have

$$\omega_p(x) = \omega(x), \quad x \in [\sigma_0, \sigma_1)\,,$$

$$\omega_p(x)^{-1}\omega_p(y) = \omega(x)^{-1}\omega(y), \quad \sigma_1 < x \leq y,\ x, y \in \operatorname{dom}\omega\,.$$

Hence, the Hamiltonian functions of $\mathfrak{h}_p$ and $\mathfrak{h}$ all coincide and the data parts $\ddot{o}_i, b_{i,j}, d_{i,j}$ for $i > 1$ also coincide (apply [KW11, 3.6,3.9] and notice that no proper reparameterisation is possible).

Finally, consider the case that $\mathfrak{h}$ (and hence also $\mathfrak{h}'$) is singular, and that the corresponding Weyl coefficients are related as $q_{\mathfrak{h}'} = q_{\mathfrak{h}} + p$ with some $p \in \mathbb{R}[z]$, $p(0) = 0$. In this case the required form of $\mathfrak{h}'$ follows with word-by-word the same argument as above, only using to Corollary 4.7 instead of Lemma 4.6 and the references for existence, uniqueness, etc., for maximal chains instead of finite maximal chains. We skip the details. ❑

Having available this result on general Hamiltonians we are ready for the proof of Theorem 4.2. We start with the proof of item (i), which actually is the hard part where Proposition 4.5 is needed.

Proof (of Theorem 4.2, (i)*).* By our a priori hypothesis on F and G, we may choose a matrix $W \in \mathcal{M}_{<\infty}$ with $(0,1)W = (F,G)$. As we have readily noticed, cf. (4.4), the task is to consider all matrices of the form PW with $p \in \mathbb{R}[z]$, $p(0) = 0$.

Assume first that $\mathcal{K}_-(W)^\circ = \{0\}$. Then, by [KW99a, Proposition 10.3, Corollary 10.4], there exists a matrix W_1 with $(0,1)W_1 = (0,1)W$ and $\mathcal{K}_-(W_1) = \mathcal{K}(W_1)$. Since W_1 is of the form P_1W with some $p_1 \in \mathbb{R}[z]$, $p_1(0) = 0$, we have

$$\big\{PW_1 : p \in \mathbb{R}[z], p(0) = 0\big\} = \big\{PW : p \in \mathbb{R}[z], p(0) = 0\big\}\,.$$

The general Hamiltonian whose monodromy matrix equals W_1 does not start with an indivisible interval of type 0 ([KW99b, Lemma 7.5], [Wor11, Lemma 6.3]). Hence, we have ([KW11, Proposition 3.17])

$$\operatorname{ind}_-(PW_1) = \operatorname{ind}_- P + \operatorname{ind}_- W_1\,.$$

Remembering what we said in 2.6 and (2.1), thus

$$\Big\{\operatorname{ind}_- PW_1 : p \in \mathbb{R}[z], p(0) = 0\Big\} = \Big[\operatorname{ind}_- W_1\ ,\ \infty\Big) \cap \mathbb{N}_0\,.$$

However, $\operatorname{ind}_- W_1 = \operatorname{ind}_- \mathcal{K}_-(W_1) = \operatorname{ind}_- \frac{G}{F}$.

Assume now that $\mathcal{K}_-(W)^\circ \neq \{0\}$. Consider a general Hamiltonian $\mathfrak{h}$ whose monodromy matrix equals W, and denote the data $\mathfrak{h}$ is composed of as in (4.12). Then, by [Wor11, Lemma 6.3], (σ_0, σ_1) is indivisible of type 0 and σ_1 is not left endpoint of an indivisible interval. By Proposition 4.5, the totality of general Hamiltonians with monodromy matrices PW, $p \in \mathbb{R}[z]$, $p(0) = 0$, is given (up to reparameterisation) by all sets of data (4.5) subject to (4.6). We have ([KW11, Proposition 4.29])

$$\begin{aligned}
\operatorname{ind}_- PW =& \sum_{i=1}^{n}\bigg(\Delta_i + \bigg\lfloor\frac{\ddot{o}'_i}{2}\bigg\rfloor\bigg) + \big|\big\{1 \le i \le n : \ddot{o}'_i \text{ odd}, b'_{i,1} > 0\big\}\big| = \\
=& \bigg(\Delta_1 + \bigg\lfloor\frac{\ddot{o}'_1}{2}\bigg\rfloor\bigg) + \begin{cases} 1, & \ddot{o}'_1 \text{ odd}, b'_{1,1} > 0 \\ 0, & \text{otherwise} \end{cases} + \\
&+ \sum_{i=2}^{n}\bigg(\Delta_i + \bigg\lfloor\frac{\ddot{o}_i}{2}\bigg\rfloor\bigg) + \big|\big\{2 \le i \le n : \ddot{o}_i \text{ odd}, b_{i,1} > 0\big\}\big|\,.
\end{aligned}$$

Since $\ddot{o}'_1, b'_{1,j}, d'_{1,j}$ may be chosen arbitrarily, it follows that

$$\begin{aligned}&\big\{ \operatorname{ind}_- PW : p \in \mathbb{R}[z], p(0) = 0 \big\} = \\ &\quad = \Big[\Delta_1 + \sum_{i=2}^{n} \Big(\Delta_i + \Big\lfloor \frac{\ddot{o}_i}{2} \Big\rfloor \Big) + \big| \{ 2 \leq i \leq n : \ddot{o}_i \text{ odd}, b_{i,1} > 0 \} \big| \ , \ \infty \Big) \cap \mathbb{N}_0 \,.\end{aligned}$$

By [Wor11, Lemma 6.3], we have

$$\Delta_1 = \dim \mathcal{K}_-(W)^\circ \,.$$

Consider the matrix $\operatorname{rev} W$ ([KW11, Definition 2.6]). The general Hamiltonian $\operatorname{rev} \mathfrak{h}$ ([KW11, Definition 3.40]) ends with an indivisible interval of infinite length and type 0, and hence $(\operatorname{rev} W) \star \infty$ is the intermediate Weyl coefficient of $\operatorname{rev} \mathfrak{h}$ at the singularity $-\sigma_1$. It follows that ([KW11, Theorem 5.1])

$$\operatorname{ind}_- \big[(\operatorname{rev} W) \star \infty \big] = \sum_{i=2}^{n} \Big(\Delta_i + \Big\lfloor \frac{\ddot{o}_i}{2} \Big\rfloor \Big) + \big| \{ 2 \leq i \leq n : \ddot{o}_i \text{ odd}, b_{i,1} > 0 \} \big| \,.$$

However, as a short computation shows, $(\operatorname{rev} W_1) \star \infty = -\frac{G}{F}$, and we obtain

$$\sum_{i=2}^{n} \Big(\Delta_i + \Big\lfloor \frac{\ddot{o}_i}{2} \Big\rfloor \Big) + \big| \{ 2 \leq i \leq n : \ddot{o}_i \text{ odd}, b_{i,1} > 0 \} \big| = \operatorname{ind}_- \frac{G}{F} \,.$$

❑

The proof of items *(ii)* and *(iii)* in Theorem 4.2 is again more elementary.

Proof (of Theorem 4.2, *(ii) and (iii)).* Item *(ii)* follows since the projection π_- maps $\mathcal{K}_-(W)$ surjectively and isometrically onto $\mathcal{P}(G+iF)$. In fact, using 2.16, *(ii)*, we obtain

$$\operatorname{ind}_- \mathcal{K}_-(W) = \operatorname{ind}_- \mathcal{P}(G + iF) = \operatorname{ind}_-(G + iF) = \operatorname{ind}_- \frac{G}{F} \,.$$

We come to item *(iii)*. By our a priori hypothesis on F and G there exist matrices $W \in \mathcal{M}_{<\infty}$ such that $(0,1)W = (F,G)$. Hence, $1 \in \bigcup_{N\in\mathbb{N}} \operatorname{Assoc}_N \mathcal{P}(G + iF)$, cf. 2.15. If $1 \in \operatorname{Assoc}_1 \mathcal{P}(G + iF)$, set $N_0 := 0$. Otherwise, let N_0 be the unique positive integer with

$$1 \in \operatorname{Assoc}_{N_0+1} \mathcal{P}(G + iF) \setminus \operatorname{Assoc}_{N_0} \mathcal{P}(G + iF) \,.$$

Then, by [Wor11, Proposition 6.1] and [KW99a, Proposition 10.3], we have

$$\dim \mathcal{K}_-(W)^\circ = N_0, \quad W \in \mathcal{M}_{<\infty}, (0,1)W = (F,G) \,.$$

Applying [Wor11, Theorem 3.2], cf. 2.15, with $E := G + iF$ and the angle "$\varphi = 0$", gives that

$$\sum_n \frac{1}{|F'(\alpha_n)G(\alpha_n)||\alpha_n|^{2(N_0+1)}} < \infty \quad \text{but} \quad \sum_n \frac{1}{|F'(\alpha_n)G(\alpha_n)||\alpha_n|^{2N_0}} = \infty \,.$$

❑

For the proof of Theorem 4.3, we need a preparatory lemma which contains a refinement of the argument used in the proof of Theorem 3.1, Steps 3,4 [14]. Thereby, we denote by $\mathfrak{d}_f$ the divisor of a meromorphic function f, i.e., $\mathfrak{d}_f(w)$ is the minimal integer n such that the n-th coefficient in the Laurent expansion of q at w is nonzero (see, e.g., [Rem98, Ch.3,§1.1]).

4.8 Lemma. *Let F be as in Theorem 4.3. Then the assignement*

$$q \mapsto Fq$$

establishes a bijection between the sets

$$\Big\{q \in \mathcal{N}_{<\infty} : \quad q \text{ meromorphic in } \mathbb{C}, \\ -\min\{\mathfrak{d}_q, 0\} = \mathfrak{d}_F,\ \lim_{z\to 0}[F(z)q(z)] = 1, \\ \exists M \in \mathbb{N}_0 : \sum_n \frac{1}{|F'(\alpha_n)|^2 \cdot |\operatorname{Res}(q;\alpha_n)| \cdot |\alpha_n|^{2(M+1)}} < \infty\Big\} \tag{4.10}$$

and

$$\Big\{G \text{ entire} : \ \exists W \in \mathcal{M}_{<\infty} \text{ with } (0,1)W = (F,G)\Big\}. \tag{4.11}$$

Thereby, for each q in the set (4.10) *and each $W \in \mathcal{M}_{<\infty}$ with $(0,1)W = (F, Fq)$,*

$$\operatorname{ind}_- \mathcal{K}_-(W) = \operatorname{ind}_- q\,, \tag{4.12}$$

$$\dim \mathcal{K}_-(W)^\circ = \min\Big\{M \in \mathbb{N}_0 : \sum_n \frac{1}{|F'(\alpha_n)|^2 \cdot |\operatorname{Res}(q;\alpha_n)| \cdot |\alpha_n|^{2(M+1)}} < \infty\Big\}. \tag{4.13}$$

Proof. Since F satisfies the conditions in Theorem 3.1, F is of Cartwright class, all but finitely many zeros of F are real and simple, and (3.1) holds.

Let q be an element of the set (4.10), and set $G := Fq$. Our aim is to apply Theorem 3.5. The function G is entire, has no common zeros with F, and satisfies $G(0) = 1$. The zeros of G coincide with the zeros of q including multiplicities. Since $q \in \mathcal{N}_{<\infty}$, all but finitely many zeros of G are real and simple, cf. 2.11. Clearly, $\frac{G}{F} = q \in \mathcal{N}_{<\infty}$. Moreover, we have

$$\operatorname{Res}(q;\alpha_n) = \frac{G(\alpha_n)}{F'(\alpha_n)}\,,$$

and hence (with some appropriate $N' \in \mathbb{N}$)

$$\sum_n \frac{1}{|F'(\alpha_n)G(\alpha_n)||\alpha_n|^{N'}} < \infty\,.$$

[14] At this point our presentation contains a slight redundancy. Accurately tracing back the logic of the proofs, one sees that one could skip a part of Steps 3,4 in the proof of Theorem 3.1 and substitute it by the corresponding argument from Lemma 4.8. For the following reason we decided to arrange matters in this way, and accept a slight repetition. We find it interesting that a proof of the pure existence statement Theorem 3.1, (I), is not only technically simpler but can be carried out by using much coarser methods compared to what is required to get hands on negative indices and dimensions of degeneracy. The argument in the proof of Lemma 4.8 is not "just the same, only more complicated" as the argument carried out in the proof of Theorem 3.1, Step 3,4; for instance there we construct the function q in a multiplicative way, we use the trick (3.6), and we refer only to the rough estimate (2.6).

Alltogether, Theorem 3.5 applies and provides us with a matrix $W \in \mathcal{M}_{<\infty}$ such that $(0,1)W = (F,G)$. Hence, the assignment $q \mapsto Fq$ indeed maps (4.10) into (4.11). Clearly, it is injective.

Assume that G is entire and $(F,G) = (0,1)W$ for some $W \in \mathcal{M}_{<\infty}$. The function $q := \frac{G}{F}$ is meromorphic in $\mathbb{C}$ and satisfies $-\min\{\mathfrak{d}_q, 0\} = \mathfrak{d}_F$ since F and G have no common zeros. Moreover, $\lim_{z\to 0}[F(z)q(z)] = G(0) = 1$. Due to Theorem 3.4, (β), we have $q \in \mathcal{N}_{<\infty}$. Finally, Theorem 3.4, (γ), shows that q belongs to the set (4.10). Thus $q \mapsto Fq$ mapst (4.10) surjectively onto (4.11).

The formulas (4.12) and (4.13) are immediate from Theorem 4.2, (ii) and (iii). ❑

Proof (of Theorem 4.3*).*
Step 1: Assume that $W \in \mathcal{M}_{<\infty}$ with $(0,1)W\binom{1}{0} = F$ is given. Set

$$G := (0,1)W\begin{pmatrix}0\\1\end{pmatrix}, \quad q := \frac{G}{F},$$

$$m_0 := \min\Big\{m \in \mathbb{N}_0 :\ \sum_n \frac{|\operatorname{Res}(q;\alpha_n)|}{|\alpha_n|^{2(m+1)}} < \infty\Big\},$$

$$M_0 := \min\Big\{M \in \mathbb{N}_0 :\ \sum_n \frac{1}{|F'(\alpha_n)|^2|\operatorname{Res}(q;\alpha_n)||\alpha_n|^{2(M+1)}} < \infty\Big\}.$$

Then $M_0 = \dim \mathcal{K}_-(W)^\circ$, cf. (4.13). We use (2.5) to estimate $\operatorname{ind}_- q$. To this end we have to match notation: the "set of points γ_j" in the present notation is a subset of the "set of points γ_j" in 2.12, and the "set of points α_n" in 2.12 is a subset of the present "set of points α_n". However, they differ only by finitely many points, namely the real and simple poles of q with positive residuum and the pole at 0. We conclude that the minimum in (2.5) equals m_0. Moreover, the numer δ_0 in (2.5) is nonzero if and only if d_0 is odd and $F^{(d_0)}(0)$ is positive. Thus we obtain that the sum in the first line of (2.5) is not less than δ, and it follows that

$$\operatorname{ind}_- \mathcal{K}_-(W) + \dim \mathcal{K}_-(W)^\circ = \operatorname{ind}_- q + \dim \mathcal{K}_-(W)^\circ \geq \delta + m_0 + M_0\,.$$

By the Schwarz inequality in ℓ^2, we have

$$\sum_n \frac{1}{|F'(\alpha_n)||\alpha_n|^{m_0+M_0+2}} \leq$$
$$\leq \Big(\sum_n \frac{|\operatorname{Res}(q;\alpha_n)|}{|\alpha_n|^{2(m_0+1)}}\Big)^{\frac12}\Big(\sum_n \frac{1}{|F'(\alpha_n)|^2|\operatorname{Res}(q;\alpha_n)||\alpha_n|^{2(M_0+1)}}\Big)^{\frac12} < \infty\,.$$

This shows that $m_0 + M_0 + 1 \geq \nu$, and the second inequality in (4.1) follows. The first inequality in (4.1) is obvious, and (4.2) follows from (2.5) by dropping the summands δ_j (not δ_0) and the minimum.

Step 2: Let $\beta, \gamma \in \mathbb{N}_0$ be given according to (4.3), and set

$$\tau_n := \frac{|\alpha_n|^{\beta-\delta-\gamma}}{|F'(\alpha_n)|}\,.$$

Then

$$\sum_n \frac{\tau_n}{|\alpha_n|^{2([\beta-\delta]+1)}} = \sum_n \frac{1}{|F'(\alpha_n)||\alpha_n|^{\beta-\delta+\gamma+2}} = \sum_n \frac{1}{|F'(\alpha_n)|^2\tau_n|\alpha_n|^{2(\gamma+1)}}\,.$$

Due to (4.3), we have $\beta-\delta+\gamma+2\geq\nu+1$, and hence the middle series converges.

Set

$$\sigma_n := \begin{cases} \tau_n & , \quad \tau_n \leq \frac{1}{|F'(\alpha_n)|^2|\alpha_n|^{2\gamma}} \\ \frac{1}{|F'(\alpha_n)|^2|\alpha_n|^{2\gamma}} , & \text{otherwise} \end{cases} .$$

Then $\sigma_n \leq \tau_n$, and hence

$$\sum_n \frac{\sigma_n}{|\alpha_n|^{2([\beta-\delta]+1)}} < \infty .$$

Remembering that $\beta-\delta\geq 0$ by (4.3), thus

$$m_0 := \min\Big\{ m\in\mathbb{N}_0 :\ \sum_n \frac{\sigma_n}{|\alpha_n|^{2(m+1)}} < \infty \Big\} \leq \beta-\delta .$$

Since $\sigma_n|F'(\alpha_n)|^2|\alpha_n|^{2\gamma}\leq 1$, we have

$$\sum_n \frac{1}{|F'(\alpha_n)|^2\sigma_n|\alpha_n|^{2\gamma}} = \infty .$$

On the other hand,

$$\frac{1}{|F'(\alpha_n)|^2\sigma_n|\alpha_n|^{2(\gamma+1)}} = \begin{cases} \frac{1}{|F'(\alpha_n)|^2\tau_n|\alpha_n|^{2(\gamma+1)}} , & \tau_n \leq \frac{1}{|F'(\alpha_n)|^2|\alpha_n|^{2\gamma}} \\ \frac{1}{|\alpha_n|^2} & , \quad \text{otherwise} \end{cases} .$$

By Theorem 3.1, (III.b), the convergence exponent of the sequence $(\alpha_n)_n$ cannot exceed 1. Hence

$$\sum_n \frac{1}{|F'(\alpha_n)|^2\sigma_n|\alpha_n|^{2(\gamma+1)}} \leq \sum_n \frac{1}{|F'(\alpha_n)|^2\tau_n|\alpha_n|^{2(\gamma+1)}} + \sum_n \frac{1}{|\alpha_n|^2} < \infty ,$$

and we conclude that

$$\min\Big\{ M\in\mathbb{N}_0 :\ \sum_n \frac{1}{|F'(\alpha_n)|^2\sigma_n|\alpha_n|^{2(M+1)}} < \infty \Big\} = \gamma .$$

Set

$$\begin{aligned} q_0(z) := & \Big[\frac{F^{(d_0)}(0)}{d_0!}\Big]^{-1}\frac{1}{z^{d_0}} + \\ & + \sum_{\gamma_j\in\mathbb{R}} \frac{1}{(\gamma_j-z)^{d_j}} + \sum_{\operatorname{Im}\gamma_j>0} \Big(\frac{1}{(\gamma_j-z)^{d_j}} + \frac{1}{(\overline{\gamma_j}-z)^{d_j}} \Big) + \\ & + \sum_n \sigma_n \Big(\frac{1}{\alpha_n-z} - \frac{1}{\alpha_n} - \ldots - \frac{z^{2m_0}}{\alpha_n^{2m_0+1}} \Big) . \end{aligned}$$

Since $\beta\geq\delta+m_0$, there exists a polynomial $p\in\mathbb{R}[z]$, $p(0)=0$, such that

$$\operatorname{ind}_-(q_0+p)=\beta ,$$

cf. (2.7). Now Lemma 4.8, applied with q_0+p, provides us with a matrix $W\in\mathcal{M}_{<\infty}$ such that

$$(0,1)W\binom{1}{0} = F, \quad \operatorname{ind}_-\mathcal{K}_-(W)=\beta,\ \dim\mathcal{K}_-(W)^\circ=\gamma . \tag{4.14}$$

Step 3: Let, in addition to β and γ as in Step 2, also a number $\alpha \in \mathbb{N}_0$ with $\alpha \geq \beta + \gamma$ be given. Choose $W \in \mathcal{M}_{<\infty}$ with (4.14). Then Theorem 4.2, (i), applied with the functions F and $G := (0,1)W\binom{0}{1}$, provides us with a (possibly different) matrix $\tilde{W} \in \mathcal{M}_{<\infty}$ such that

$$(0,1)\tilde{W} = (0,1)W, \quad \operatorname{ind}_- \mathcal{K}(\tilde{W}) = \alpha .$$

Since the quantities "$\operatorname{ind}_- \mathcal{K}_-(W)$" and "$\dim \mathcal{K}_-(W)^\circ$" depend only on the second row of a matrix, we have

$$\operatorname{ind}_- \mathcal{K}_-(\tilde{W}) = \beta, \ \dim \mathcal{K}_-(\tilde{W})^\circ = \gamma .$$

Clearly, $(0,1)\tilde{W}\binom{1}{0} = F$. ❑

Finally, the proof of Corollary 4.4.

Proof (of Corollary 4.4). We apply Theorem 4.3, (ii). Choosing $\alpha := \delta + \nu - 1$, $\beta := \delta$, $\gamma := \nu - 1$, we obtain $W_1 \in \mathcal{M}_{<\infty}$ with

$$(0,1)W_1\begin{pmatrix}1\\0\end{pmatrix} = F, \quad \operatorname{ind}_- \mathcal{K}_-(W) = \delta .$$

Choosing $\alpha := \delta + \nu - 1$, $\beta := \delta + \nu - 1$, $\gamma := 0$, we obtain $W_2 \in \mathcal{M}_{<\infty}$ with

$$(0,1)W_2\begin{pmatrix}1\\0\end{pmatrix} = F, \quad \dim \mathcal{K}_-(W_2)^\circ = 0 .$$

It is easy to alter W_2 so to achieve that "$\mathcal{K}_-(W_2) = \mathcal{K}(W_2)$". We apply [KW99b, Theorem 5.7, (iii)] with some parameter τ, say $\tau := 0$. This provides us with a real polynomial p having the properties stated there. An application of [KW99b, Theorem 5.7, (ii)] with the matrix and parameter

$$\tilde{W}_2 := PW_2, \qquad \tau := 0 ,$$

yields that $\mathcal{K}_-(\tilde{W}_2) = \mathcal{K}(\tilde{W}_2)$. Clearly, the left lower entry of $\tilde{W}_2$ equals F. ❑

References

[AD98] D.Z. Arov and H. Dym, *On three Krein extension problems and some generalizations*, Integral Equations Operator Theory **31** (1998), no. 1, 1–91. MR 1618628 (99f:47019)

[AD08] ———, *J-contractive matrix valued functions and related topics*, Encyclopedia of Mathematics and its Applications, vol. 116, Cambridge University Press, Cambridge, 2008. MR 2474532 (2009m:47031)

[ADL04] D. Alpay, A. Dijksma, and H. Langer, *Factorization of J-unitary matrix polynomials on the line and a Schur algorithm for generalized Nevanlinna functions*, Linear Algebra Appl. **387** (2004), 313–342. MR 2069282 (2005b:47029)

[ADRdS97] D. Alpay, A. Dijksma, J. Rovnyak, and H. de Snoo, *Schur functions, operator colligations, and reproducing kernel Pontryagin spaces*, Operator Theory: Advances and Applications, vol. 96, Birkhäuser Verlag, Basel, 1997. MR 1465432 (2000a:47024)

[Akh61] N.I. Akhiezer, Классическая проблема моментов и некоторые вопросы анализа, связанные с нею, Gosudarstv. Izdat. Fiz.-Mat. Lit., Moscow, 1961 (Russian), English translation: *The classical moment problem and some related topics in analysis*, Oliver & Boyd, Edinburgh, 1965.

[Bak98] A.G. Bakan, *Polynomial approximation in $L_p(\mathbf{R}, d\mu)$. I. General results and representation theorem*, Nats. Akad. Nauk Ukraïn. Īnst. Mat. Preprint (1998), no. 7, 45. MR 1734341 (2000m:41002)

[Ber95] C. Berg, *Indeterminate moment problems and the theory of entire functions*, Proceedings of the International Conference on Orthogonality, Moment Problems and Continued Fractions (Delft, 1994), vol. 65, 1995, pp. 27–55. MR 1379118 (97d:44002)

[BP94] C. Berg and H.L. Pedersen, *On the order and type of the entire functions associated with an indeterminate Hamburger moment problem*, Ark. Mat. **32** (1994), no. 1, 1–11. MR 1277917 (95i:30027)

[BP95] ______, *Nevanlinna matrices of entire functions*, Math. Nachr. **171** (1995), 29–52. MR 1316350 (96a:47028)

[dB59] L. de Branges, *The Bernstein problem*, Proc. Amer. Math. Soc. **10** (1959), 825–832. MR 0114080 (22 #4907)

[dB68] Louis de Branges, *Hilbert spaces of entire functions*, Prentice-Hall Inc., Englewood Cliffs, N.J., 1968. MR 0229011 (37 #4590)

[DHdS99] Vladimir Derkach, Seppo Hassi, and Henk de Snoo, *Operator models associated with Kac subclasses of generalized Nevanlinna functions*, Methods Funct. Anal. Topology **5** (1999), no. 1, 65–87. MR 1771251 (2001f:47063)

[DLLS00] A. Dijksma, H. Langer, A. Luger, and Yu. Shondin, *A factorization result for generalized Nevanlinna functions of the class $\mathcal{N}_\kappa$*, Integral Equations Operator Theory **36** (2000), no. 1, 121–125. MR 1736921 (2000i:47027)

[GG97] M.L. Gorbachuk and V.I. Gorbachuk, *M. G. Krein's lectures on entire operators*, Operator Theory: Advances and Applications, vol. 97, Birkhäuser Verlag, Basel, 1997. MR 1466698 (99f:47001)

[GK67] I.C. Gohberg and M.G. Krein, Теория волтерровых операторов в гилбертовом пространстве и ее приложения, Izdat. 'Nauka', Moscow, 1967 (Russian), English translation: *Theory and applications of Volterra operators in Hilbert space*, Translations of Mathematical Monographs, Vol.24, American Mathematical Society, Providence, R.I., 1970.

[GL74] M. Großmann and H. Langer, *Über indexerhaltende Erweiterungen eines hermiteschen Operators im Pontrjaginraum*, Math. Nachr. **64** (1974), 289–317 (German). MR 0370257 (51 #6484)

[Kal02] M. Kaltenbäck, *Indefinite analogues of j_{pq}-contractive matrix functions and linear fractional transformations*, Acta Sci. Math. (Szeged) **68** (2002), no. 1-2, 349–371. MR 1916585 (2003f:47025)

[KK68] I.S. Kac and M.G. Krein, *On spectral functions of a string*, pp. 648–737, Izdat. "Mir", Moscow, 1968 (Russian), Addition II in F.V.Atkinson, Дискретные и непрерывные граничные задачи (Russian translation). English translation: Amer. Math. Soc. Transl. (2) 103 (1974), 19–102.

[KL] M.G. Krein and H. Langer, *Continuation of Hermitian positive definite functions and related questions*, unpublished manuscript.

[KL77] M. G. Kreĭn and H. Langer, *Über einige Fortsetzungsprobleme, die eng mit der Theorie hermitescher Operatoren im Raume Π_κ zusammenhängen. I. Einige Funktionenklassen und ihre Darstellungen*, Math. Nachr. **77** (1977), 187–236. MR 0461188 (57 #1173)

[KL78] ______, *Über einige Fortsetzungsprobleme, die eng mit der Theorie hermitescher Operatoren im Raume Π_κ zusammenhängen. II. Verallgemeinerte Resolventen, u-Resolventen und ganze Operatoren*, J. Funct. Anal. **30** (1978), no. 3, 390–447. MR 518342 (80h:47045)

[KL79] M. G. Kreĭn and Heinz Langer, *On some extension problems which are closely connected with the theory of Hermitian operators in a space Π_κ. III. Indefinite analogues of the Hamburger and Stieltjes moment problems. Part I*, Beiträge Anal. (1979), no. 14, 25–40 (loose errata). MR 563344 (83b:47047a)

[KL80] ______, *On some extension problems which are closely connected with the theory of Hermitian operators in a space Π_κ. III. Indefinite analogues of the Hamburger and Stieltjes moment problems. Part II*, Beiträge Anal. (1980), no. 15, 27–45 (1981). MR 614775 (83b:47047b)

[KL81] M. G. Kreĭn and H. Langer, *Some propositions on analytic matrix functions related to the theory of operators in the space Π_κ*, Acta Sci. Math. (Szeged) **43** (1981), no. 1-2, 181–205. MR 621369 (82i:47053)

[KL85] Mark G. Kreĭn and Heinz Langer, *On some continuation problems which are closely related to the theory of operators in spaces Π_κ. IV. Continuous analogues of orthogonal polynomials on the unit circle with respect to an indefinite weight and related continuation problems for some classes of functions*, J. Operator Theory **13** (1985), no. 2, 299–417. MR 776000 (87h:47084)

[Kre47] M. Krein, *A contribution to the theory of entire functions of exponential type*, Izvestiya Akad. Nauk SSSR **11** (1947), 309–326. MR 0022252 (9,179e)

[Kre52] M. G. Krein, *On the indeterminate case of the Sturm-Liouville boundary problem in the interval* $(0,\infty)$, Izvestiya Akad. Nauk SSSR. Ser. Mat. **16** (1952), 293–324. MR 0052004 (14,558g)

[KW98a] M. Kaltenbäck and H. Woracek, *Generalized resolvent matrices and spaces of analytic functions*, Integral Equations Operator Theory **32** (1998), no. 3, 282–318. MR 1652697 (2000a:47076)

[KW98b] ______, *On extensions of Hermitian functions with a finite number of negative squares*, J. Operator Theory **40** (1998), no. 1, 147–183. MR 1642542 (2000f:47026)

[KW99a] ______, *Pontryagin spaces of entire functions. I*, Integral Equations Operator Theory **33** (1999), no. 1, 34–97. MR 1664343 (2000a:46039)

[KW99b] ______, *Pontryagin spaces of entire functions. II*, Integral Equations Operator Theory **33** (1999), no. 3, 305–380. MR 1671482 (2000a:46040)

[KW03] ______, *Pontryagin spaces of entire functions. III*, Acta Sci. Math. (Szeged) **69** (2003), no. 1-2, 241–310. MR 1991668 (2004h:46022)

[KW06] ______, *Unique prime factorization in a partial semigroup of matrix-polynomials*, Discuss. Math. Gen. Algebra Appl. **26** (2006), no. 1, 21–43. MR 2227628 (2007b:20123)

[KW10] ______, *Pontryagin spaces of entire functions. VI*, Acta Sci. Math. (Szeged) **76** (2010), no. 3-4, 511–560. MR 2789685 (2012e:46046)

[KW11] ______, *Pontryagin spaces of entire functions. V*, Acta Sci. Math. (Szeged) **77** (2011), no. 1-2, 223–336. MR 2841150 (2012e:46047)

[KWW06] M. Kaltenbäck, H. Winkler, and H. Woracek, *Singularities of generalized strings*, Operator theory and indefinite inner product spaces, Oper. Theory Adv. Appl., vol. 163, Birkhäuser, Basel, 2006, pp. 191–248. MR 2215864 (2007f:47040)

[Lan86] Heinz Langer, *A characterization of generalized zeros of negative type of functions of the class N_κ*, Advances in invariant subspaces and other results of operator theory (Timişoara and Herculane, 1984), Oper. Theory Adv. Appl., vol. 17, Birkhäuser, Basel, 1986, pp. 201–212. MR 901070 (88j:47051)

[Lev80] B. Ja. Levin, *Distribution of zeros of entire functions*, revised ed., Translations of Mathematical Monographs, vol. 5, American Mathematical Society, Providence, R.I., 1980, Translated from the Russian by R. P. Boas, J. M. Danskin, F. M. Goodspeed, J. Korevaar, A. L. Shields and H. P. Thielman. MR 589888 (81k:30011)

[LW] M. Langer and H. Woracek, *Direct and inverse spectral theorems for a class of canonical systems with two singular endpoints*, manuscript in preparation.

[LW02] ______, *A characterization of intermediate Weyl coefficients*, Monatsh. Math. **135** (2002), no. 2, 137–155. MR 1894093 (2003g:46028)

[LW13] ______, *The Exponential Type of the Fundamental Solution of an Indefinite Hamiltonian System*, Complex Anal. Oper. Theory **7** (2013), no. 1, 285–312. MR 3010801

[LWar] ______, *Indefinite Hamiltonian systems whose Titchmarsh–Weyl coefficients have no finite generalized poles of non-negative type*, Oper. Matrices (to appear), 75 pp.,
ASC Report: http://www.asc.tuwien.ac.at/preprint/2012/asc04x2012.pdf.

[Ped09] Henrik L. Pedersen, *Logarithmic order and type of indeterminate moment problems. II*, J. Comput. Appl. Math. **233** (2009), no. 3, 808–814. MR 2583020 (2011b:44007)

[Rem98] R. Remmert, *Classical topics in complex function theory*, Graduate Texts in Mathematics, vol. 172, Springer-Verlag, New York, 1998, Translated from the German by Leslie Kay. MR 1483074 (98g:30002)

[RR94] Marvin Rosenblum and James Rovnyak, *Topics in Hardy classes and univalent functions*, Birkhäuser Advanced Texts: Basler Lehrbücher. [Birkhäuser Advanced Texts: Basel Textbooks], Birkhäuser Verlag, Basel, 1994. MR 1307384 (97a:30047)

[Wor97] H. Woracek, *Nevanlinna-Pick interpolation: the degenerated case*, Linear Algebra Appl. **252** (1997), 141–158. MR 1428633 (98e:30041)

[Wor11] ______, *Existence of zerofree functions N-associated to a de Branges Pontryagin space*, Monatsh. Math. **162** (2011), no. 4, 453–506. MR 2784432 (2012i:46033)

[Wor12] ______, *An inverse spectral theorem for Kreĭn strings with a negative eigenvalue*, Monatsh. Math. **167** (2012), no. 1, 105–149. MR 2942966

H. Woracek
Institut for Analysis and Scientific Computing
Vienna University of Technology
Wiedner Hauptstraße. 8–10/101
1040 Wien
AUSTRIA
email: harald.woracek@tuwien.ac.at