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A NOTE ON SIMILARITY TO CONTRACTION FOR STABLE 2×2 COMPANION MATRICES

Winfried Auzinger [1]

We consider companion matrices of dimension 2 with general complex spectra satisfying a root condition with respect to the closed complex unit circle or the closed left complex half plane. For both cases, smooth and naturally conditioned basis transformations are constructed such that the resulting, transformed matrix is contractive or dissipative, respectively.

1 Introduction

For discrete or continuous evolution processes of the form

$$\boldsymbol{y}_{\nu+1} = A\boldsymbol{y}_\nu, \quad \nu \geq 0, \quad \text{or} \quad \boldsymbol{y}'(t) = A\boldsymbol{y}(t), \quad t \geq 0, \quad \text{with} \quad A \in \mathbb{C}^{n\times n},$$

the asymptotic behavior for $\nu \to \infty$ or $t \to \infty$ is determined by the location of the spectrum of A, while the initial, transient behavior is governed by $\|A\|$ or $\mu(A)$, respectively, where $\mu(A)$ denotes the logarithmic matrix norm. It is well known that, for non-normal A significant transient growth can occur even if the system has a stable spectrum. The questions of describing the transient behavior, or of bounding the evolution operator $\|A^\nu\|$ or $\|e^{tA}\|$ uniformly in ν or t, respectively, has been studied in many papers on linear stability theory.

One of the classical results on this topic is the Kreiss Matrix Theorem (KMT) (see, e.g., [4, 6] and references therein), which involves several equivalent conditions on A which in turn are equivalent to uniform boundedness of families of evolution operators in the ℓ_2-norm $\|\cdot\|_2$. All these equivalent conditions are not constructive and usually difficult to verify. Therefore it is a relevant question in what cases auch bounds can be derived in a more or less explicit manner, depending on the spectrum and making use of certain additional information about the matrix A.

A closer inspection of the literature indeed reveals that results of this type are naturally restricted to cases for which additional structural properties are known (or assumed). A very special case is the family of 2×2 companion matrices $C \in \mathbb{C}^{2\times 2}$ describing the evolution of the BDF2 approximation to scalar ODEs $y'(t) = \lambda y(t)$. This method is A-stable, and a uniform, well-conditioned transformation is known such that the transformed matrix is contractive for arbitrary $\mathrm{Re}(\lambda) \leq 0$ and $h > 0$. This is a direct consequence of the G-stability of the scheme, which is equivalent to A-stability; see [4] for details. Higher order $A(\alpha)$-stable BDF schemes have been considered in [2]: Here, the distribution of the spectrum is analyzed, and combination with the resolvent condition in the KMT leads to growth bounds uniformly valid with respect to the stability domain of the scheme. Some further results of related type can, e.g., be found in [3] and [5].

In this short study we consider families of 2×2 companion matrices with arbitrary complex stable spectra (with respect to the unit circle or the left half, plane, respectively). We construct a natural basis transformation such that the transformed matrix behaves contractive or dissipative, respectively, with respect to $\|\cdot\|_2$. This basis transformation should feature a natural conditioning behavior and depend smoothly on the spectrum.

At first sight, this problem seems to be a rather simple one, but from our results it can be seen that already the case $n = 2$ is technically rather intricate. We provide a solution for both cases (contractivity and dissipativity). Our results include a quantitative measure for the 'distance to instability', defined in terms of the location of the spectrum. Our results and their proofs are specified in Sections 3 and 4 below. $\|\cdot\|_2$, respectively. Examples are also given.

The construction used in the proofs of Propositions 3.1 and 4.1 cannot be extended to dimension $n \geq 3$ due to the significantly more complicated algebra involved. (Some numerical solutions for numerical data are, however, given in [1].) For more general classes of matrices, explicit, quantitative results of this type seem to be very hard to obtain.

In the sequel, for a square matrix S, $S > 0$ means that S is positive definite (analogously for $<, \geq, \leq$).

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2 Problem setting

Consider

$$C = \begin{pmatrix} 0 & 1 \\ -c_0 & -c_1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\zeta_1\zeta_2 & \zeta_1+\zeta_2 \end{pmatrix} \in \mathbb{C}^{2\times 2} \tag{2.1}$$

with characteristic polynomial

$$\pi(\zeta) = \zeta^2 + c_1\zeta + c_0 = (\zeta-\zeta_1)(\zeta-\zeta_2) \tag{2.2}$$

In the following sections we study the problem of finding a basis transformation, preferably well-conditioned, converting a given companion matrix C with a [weakly] stable spectrum into a ℓ_2-contractive, or ℓ_2-dissipative matrix, respectively. The assumption is that the spectrum of C satisfies a [weak] stability condition w.r.t. the closed complex unit circle or the closed complex left half plane, respectively.

For this purpose we consider a similarity transformation of C of the form

$$C = L\,T\,L^{-1}, \tag{2.3}$$

with

$$L = \begin{pmatrix} 1 & 0 \\ \mu & \delta \end{pmatrix}, \quad \text{and} \quad T = \begin{pmatrix} \mu & \delta \\ \frac{\sigma}{\delta} & \mu \end{pmatrix}. \tag{2.4}$$

Here we define

$$\mu = \frac{\zeta_1+\zeta_2}{2}, \qquad \sigma = \Big(\frac{\zeta_1-\zeta_2}{2}\Big)^2 = \mu^2 - \zeta_1\zeta_2\,, \tag{2.5}$$

and $\delta > 0$ is a scaling parameter which will be chosen in a suitable way. Relation (2.3) holds true for arbitrary $\delta \neq 0$ because

$$C\,L = \begin{pmatrix} 0 & 1 \\ -\zeta_1\zeta_2 & 2\,\mu \end{pmatrix}\begin{pmatrix} 1 & 0 \\ \mu & \delta \end{pmatrix} = \begin{pmatrix} \mu & \delta \\ 2\,\mu^2 - \zeta_1\zeta_2 & 2\,\mu\,\delta \end{pmatrix},$$

$$L\,T = \begin{pmatrix} 1 & 0 \\ \mu & \delta \end{pmatrix}\begin{pmatrix} \mu & \delta \\ \frac{\sigma}{\delta} & \mu \end{pmatrix} = \begin{pmatrix} \mu & \delta \\ \mu^2 + \sigma & 2\,\mu\,\delta \end{pmatrix} = C\,L$$

due to (2.5).

3 Contractivity for stable spectra in the closed unit circle

Assume that C from (2.1) satisfies a stability condition (root condition) with respect to the closed complex unit circle, i.e.,

$$|\zeta_1| \le 1, \quad |\zeta_2| \le 1, \quad \text{and} \quad |\zeta_1| < 1 \ \text{ if } \ \zeta_1 = \zeta_2. \tag{3.1}$$

Proposition 3.1 (similarity to contraction)**.** *Consider a companion matrix of the form* (2.1), $C \in \mathbb{C}^{2\times 2}$ *with spectrum* $\{\zeta_1, \zeta_2\}$*, satisfying the stability condition* (3.1)*. Let*

$$\delta = \sqrt{\tfrac{1}{2}(1-|\zeta_1|^2)(1-|\zeta_2|^2) + \tfrac{1}{4}|\zeta_1-\zeta_2|^2} > 0. \tag{3.2}$$

Then the transformed matrix T from (2.3),(2.4) *satisfies*

$$\|T\|_2 \le 1. \tag{3.3}$$

The parameter δ from (3.2) is a measure for 'the distance to instability' of the spectrum $\{\zeta_1, \zeta_2\}$. It vanishes exactly in the limiting, unstable case $\zeta_1 = \zeta_2$ with $|\zeta_1| = |\zeta_2| = 1$. For further details concerning this similarity transformation, see Remark 3.1 below.

Proof. We consider (2.3),(2.4) with δ unspecified for the moment. The norm $\|T\|_2$ cannot be expressed in a simple way as a function of δ. Alternatively, we aim for finding $\delta > 0$ such that the requirement

$$S := \Delta^2 - (T\Delta)^*(T\Delta) \geq [>]\ 0, \quad \text{with} \quad \Delta = \begin{pmatrix} 1 & 0 \\ 0 & \delta \end{pmatrix}, \tag{3.4}$$

is satisfied, which is equivalent to the requirement $\|T\|_2 \leq [<]\ 1$.

The matrix S evaluates to

$$S = \begin{pmatrix} 1 - |\mu|^2 & -\mu\bar{\sigma} \\ -\sigma\bar{\mu} & -|\sigma|^2 \end{pmatrix} + \delta^2 \begin{pmatrix} -1 & -\bar{\mu} \\ -\mu & 1 - |\mu|^2 \end{pmatrix},$$

and its determinant is given by

$$\det S = -\delta^4 + \big(1 - 2\,|\mu|^2 + |\mu^2 - \sigma|^2\big)\delta^2 - |\sigma|^2. \tag{3.5}$$

This assumes its maximal value for

$$\begin{aligned} \delta^2 &= \tfrac{1}{2}\big(1 - 2|\mu|^2 + |\mu^2 - \sigma|^2\big) \\ &= \tfrac{1}{2}(1 - |\zeta_1|^2)(1 - |\zeta_2|^2) + (1 - |\mu|^2) \\ &= \tfrac{1}{2}(1 - |\zeta_1|^2)(1 - |\zeta_2|^2) + \tfrac{1}{4}|\zeta_1 - \zeta_2|^2 \geq 0. \end{aligned} \tag{3.6}$$

With this choice for $\delta > 0$, i.e., δ according to (3.2), $\det S$ evaluates to

$$\begin{aligned} \det S = \delta^4 - |\sigma|^2 &= \big(\delta^2 - \tfrac{1}{4}|\zeta_1 - \zeta_2|^2\big)\big(\delta^2 + \tfrac{1}{4}|\zeta_1 - \zeta_2|^2\big) \\ &= \tfrac{1}{2}(1 - |\zeta_1|^2)(1 - |\zeta_2|^2)\big(\delta^2 + \tfrac{1}{4}|\zeta_1 - \zeta_2|^2\big) \\ &= \tfrac{1}{4}(1 - |\zeta_1|^2)(1 - |\zeta_2|^2)\,|1 - \zeta_1\bar{\zeta}_2|^2 \geq 0. \end{aligned}$$

Now we check requirement (3.4) for S with δ^2 from (3.6). To this end, we note that

$$\operatorname{trace} S = \big(1 - |\mu|^2 - \delta^2\big) + \big(-|\sigma|^2 + \delta^2(1 - |\mu|^2)\big),$$

and

$$|\mu|^2 + |\sigma| = \tfrac{1}{4}\,|\zeta_1 + \zeta_2|^2 + \tfrac{1}{4}\,|\zeta_1 - \zeta_2|^2 = \tfrac{1}{2}\big(|\zeta_1|^2 + |\zeta_2|^2\big). \tag{3.7}$$

We consider three different cases of a stable spectrum (in all cases, $|\mu| < 1$ and $\delta > 0$):

(i) $|\zeta_1| < 1,\ |\zeta_2| < 1$: Here,

$$\delta^2 < 1 - |\mu|^2, \quad S_{11} > 0, \quad \det S > 0.$$

This implies $S > 0$.

(ii) $|\zeta_1| = 1$, $|\zeta_2| < 1$: Here,

$$\delta^2 = 1 - |\mu|^2, \quad \det S = 0, \quad \operatorname{trace} S > 0,$$

where the estimate for the trace easily follows from (3.7). This implies that the eigenvalues of S must be $\lambda_1 = 0$ and $\lambda_2 > 0$, hence $S \geq 0$ with $\operatorname{rank}(S) = 1$.

(iii) $|\zeta_1| = |\zeta_2| = 1$, with $\zeta_1 \neq \zeta_2$: Here,

$$\delta^2 = 1 - |\mu|^2, \quad \det S = 0, \quad \operatorname{trace} S = 0,$$

where $\operatorname{trace} S = 0$ again follows from (3.7). This implies $S = 0$.

In all these cases, $\operatorname{rank}(S)$ equals the number of eigenvalues ζ_k with $|\zeta_k| < 1$. Summarizing (i)–(iii) concludes the proof. □

Remark 3.1. For $\rho(C)=1$ with $S=0$ (case (iii) above), C is diagonalizable, and $S=0$ implies that T is unitary.

For $\rho(C)=1$ with $0\neq S\geq 0$ (case (ii) above), C is diagonalizable. In this case it follows from $\delta^2=|\sigma|$ that T is normal, with $\|T\|_2=1$. Thus, up to unitary transformation the outcome amounts to diagonalization of C. We may call T a normalization of C.

In cases (ii) and (iii), $\delta=\sqrt{1-|\mu|^2}$ is approximately proportional to the distance between 1 and the modulus of the arithmetic mean μ of the eigenvalues of the matrix C.

The more interesting, general case is $\rho(C)<1$, with $S>0$ (case (i) above):

For $\zeta_1\neq\zeta_2$, T is not related to a diagonalization, or normalization, of C, which gets undefined in the limit $\zeta_1\to\zeta_2$. The transformation matrix L is well-conditioned also for $\zeta_1\to\zeta_2$ unless the spectrum is close to the unit circle. Here we have $S>0$ and $\|T\|_2<1$. The value of $\|T\|_2$ depends on the location of the spectrum of C in a rather complicated way.

In the confluent case $\zeta_1=\zeta_2=\mu$ we obtain $\delta=\frac{\sqrt{2}}{2}(1-|\mu|^2)$, thus

$$T=\begin{pmatrix}\mu & \frac{\sqrt{2}}{2}(1-|\mu|^2)\\ 0 & \mu\end{pmatrix},$$

i.e., T is a rescaled Jordan form.

In all cases, the condition number of the transformation matrix L is $\mathcal{O}(\delta^{-1})$ for $\delta\to 0$, which is quite natural and related to the transient behavior of the powers $\|C^\nu\|_2$ (the so-called hump phenomenon).

Summarizing, we see that Proposition 3.1 describes a similarity transformation leading to a contraction which is based on a *smooth transition* between normalization and Jordan decomposition. The construction appears quite natural, but it is not unique, because one may think of different ways to 'balance' between '$\|T\|_2$ small' and '$\kappa(L)$ not too large'.

Example 3.1 (Second order difference equations)**.** Consider the homogeneous difference equation

$$y_{\nu+2}+c_1\,y_{\nu+1}+c_0\,y_\nu=0,\quad \nu\geq 0,$$

for given y_0,y_1. For the characteristic polynomial $\pi(\zeta)=\zeta^2+c_1\zeta+c_0=(\zeta-\zeta_1)(\zeta-\zeta_2)$ we assume that $\{\zeta_1,\zeta_2\}$ satisfies the stability condition (3.1). With $\boldsymbol{y}_\nu=(y_\nu,y_{\nu+1})^T$ this is equivalent to $\boldsymbol{y}_{\nu+1}=C\,\boldsymbol{y}_\nu$ with C from (2.1), or equivalently, $L^{-1}\boldsymbol{y}_{\nu+1}=T\,L^{-1}\boldsymbol{y}_\nu$ with L,T from (2.4). Here,

$$L^{-1}\boldsymbol{y}_\nu=\begin{pmatrix}y_\nu\\ \frac{1}{\delta}(y_{\nu+1}-\mu\,y_\nu)\end{pmatrix},$$

and Proposition 3.1 asserts that

$$\delta^2\,\|L^{-1}\boldsymbol{y}_\nu\|_2^2=|\delta\,y_\nu|^2+|y_{\nu+1}-\mu\,y_\nu|^2$$

is always monotonously decreasing with ν.

4 Dissipativity for stable spectra in the closed left half plane

Assume that C from (2.1) satisfies a stability condition (root condition) with respect to the closed complex left half plane, i.e.,

$$\mathrm{Re}\,\zeta_1\leq 0,\quad \mathrm{Re}\,\zeta_2\leq 0,\quad \text{and}\quad \mathrm{Re}\,\zeta_1<0\ \text{ if }\ \zeta_1=\zeta_2. \tag{4.1}$$

Proposition 4.1 (similarity to dissipation)**.** *Consider a companion matrix of the form* (2.1)*,* $C\in\mathbb{C}^{2\times 2}$ *with spectrum* $\{\zeta_1,\zeta_2\}$*, satisfying the stability condition* (4.1)*. Let*

$$\delta=\sqrt{2\,\mathrm{Re}\,\zeta_1\,\mathrm{Re}\,\zeta_2+\tfrac{1}{4}|\zeta_1-\zeta_2|^2}>0. \tag{4.2}$$

Then the transformed matrix T from (2.3)*,*(2.4) *satisfies*

$$\mathrm{Re}\,T=\tfrac{1}{2}(T+T^*)\leq 0. \tag{4.3}$$

The parameter δ from (4.2) is a measure for 'the distance to instability' of the spectrum $\{\zeta_1, \zeta_2\}$. It vanishes exactly in the limiting, unstable case $\zeta_1 = \zeta_2$ with $\mathrm{Re}\,\zeta_1 = \mathrm{Re}\,\zeta_2 = 0$.

Proof. We aim for finding $\delta > 0$ such that the requirement

$$S := \mathrm{Re}\,(2\,\delta T) \le [<]\; 0 \tag{4.4}$$

is satisfied, which is equivalent to the requirement $\mathrm{Re}\,T \le [<]\; 0$.

The matrix S evaluates to

$$S = \begin{pmatrix} 2\,\delta\,\mathrm{Re}\,\mu & \delta^2 + \bar\sigma \\ \delta^2 + \sigma & 2\,\delta\,\mathrm{Re}\,\mu \end{pmatrix},$$

and its determinant is given by

$$\det S = -\delta^4 + 2\big(2\,(\mathrm{Re}\,\mu)^2 - \mathrm{Re}\,\sigma\big)\delta^2 - |\sigma|^2. \tag{4.5}$$

This assumes its maximal value for

$$\begin{aligned} \delta^2 &= 2\,(\mathrm{Re}\,\mu)^2 - \mathrm{Re}\,\sigma \\ &= 2\;\mathrm{Re}\,\zeta_1\,\mathrm{Re}\,\zeta_2 + \tfrac{1}{4}|\zeta_1 - \zeta_2|^2 \ge 0. \end{aligned} \tag{4.6}$$

With this choice for $\delta > 0$, i.e., δ according to (4.2), $\det S$ evaluates to

$$\begin{aligned} \det S = \delta^4 - |\sigma|^2 &= \big(\delta^2 - \tfrac{1}{4}|\zeta_1 - \zeta_2|^2\big)\big(\delta^2 + \tfrac{1}{4}|\zeta_1 - \zeta_2|^2\big) \\ &= 2\;\mathrm{Re}\,\zeta_1\,\mathrm{Re}\,\zeta_2\,\big(2\;\mathrm{Re}\,\zeta_1\,\mathrm{Re}\,\zeta_2 + \tfrac{1}{2}|\zeta_1 - \zeta_2|^2\big) \\ &= \mathrm{Re}\,\zeta_1\,\mathrm{Re}\,\zeta_2\;|\zeta_1 + \zeta_2|^2. \end{aligned}$$

Now we check requirement (4.4) for S with δ^2 from (4.6).

We consider three different cases of a stable spectrum (in all cases, $\mathrm{Re}\,\mu < 0$ and $\delta > 0$):

(i) $\mathrm{Re}\,\zeta_1 < 0,\ \mathrm{Re}\,\zeta_2 < 0$: Here,

$$\delta^2 > |\sigma|, \quad \mathrm{Re}\,\mu < 0, \quad S_{11} < 0, \quad \det S > 0.$$

This implies $S < 0$.

(ii) $\mathrm{Re}\,\zeta_1 = 0,\ \mathrm{Re}\,\zeta_2 < 0$: Here,

$$\delta^2 = |\sigma|, \quad \mathrm{Re}\,\mu < 0, \quad \det S = 0, \quad \mathrm{trace}\,S < 0.$$

This implies that the eigenvalues of S must be $\lambda_1 = 0$ and $\lambda_2 < 0$, hence $S \le 0$ with $\mathrm{rank}(S) = 1$.

(iii) $\mathrm{Re}\,\zeta_1 = \mathrm{Re}\,\zeta_2 = 0$, with $\zeta_1 \ne \zeta_2$: Here,

$$\delta^2 = |\sigma|, \quad \mathrm{Re}\,\mu = 0, \quad \det S = 0, \quad \mathrm{trace}\,S = 0.$$

This implies $S = 0$.

In all these cases, $\mathrm{rank}(S)$ equals the number of eigenvalues ζ_k with $\mathrm{Re}\,\zeta_k < 0$. Summarizing (i)–(iii) concludes the proof. □

Remark 4.1. Similar remarks as those following Proposition 3.1 apply. For case (iii), in particular, $S = 0$ implies $\mathrm{Re}\,T = 0$, i.e., T is skew-Hermitian. For case (ii), T is normal.

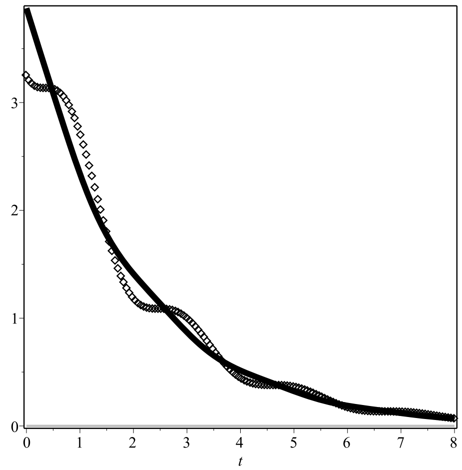

Figure 1: Damped harmonic oscillator. Energy $E(y(t),\dot y(t))$ [$\diamond\diamond\diamond$] and mean energy $\tilde E(y(t),\dot y(t))$ [—] for $\omega = 1.5$, $\rho = 0.25$, and initial values $y(0) = \dot y(0) = 1$.

Example 4.1 (Damped harmonic oscillator)**.** In this example we show that, in the context of a simple ODE problem, Proposition 4.1 provides a physically meaningful dissipation functional.

Consider the second order linear ODE for the free damped harmonic oscillator in the dimensionless variable y,

$$\ddot y(t) + 2\,\rho\,\dot y(t) + \omega^2\,y(t) = 0,$$

with damping parameter $\rho \ge 0$ and angular frequency $\omega > 0$. For $\boldsymbol{y}(t) = (y(t),\dot y(t))^T$ we have

$$\dot{\boldsymbol{y}}(t) = C\,\boldsymbol{y}(t), \quad C = \begin{pmatrix} 0 & 1 \\ -\omega^2 & -2\,\rho \end{pmatrix},$$

with eigenvalues $\zeta_{1,2} = -\rho \pm \sqrt{\rho^2-\omega^2}$ and $\mu = \frac{1}{2}(\zeta_1+\zeta_2) = -\rho$. Consider the assertion from Proposition 4.1. In all cases (over- or underdamping, critical damping) for δ from (4.2) we obtain $\delta = \sqrt{\rho^2+\omega^2}$, and

$$T = \begin{pmatrix} -\rho & \sqrt{\rho^2+\omega^2} \\ \frac{\rho^2-\omega^2}{\sqrt{\rho^2+\omega^2}} & -\rho \end{pmatrix} \quad \text{with} \quad \operatorname{Re} T \le \Big(\tfrac{\rho}{\sqrt{\rho^2+\omega^2}} - 1\Big)\rho I \le 0.$$

Together with $\frac{d}{dt}(L^{-1}\boldsymbol{y}(t)) = T\,(L^{-1}\boldsymbol{y}(t))$ this implies

$$\|L^{-1}\boldsymbol{y}(t)\|_2 \le e^{-\tilde\rho t}\,\|L^{-1}\boldsymbol{y}(0)\|_2, \quad \text{with} \quad \tilde\rho := -\mu_2(T) = \Big(1 - \tfrac{\rho}{\sqrt{\rho^2+\omega^2}}\Big)\rho \ge 0.$$

(Here, $\mu_2(T)$ denotes the logarithmic norm of T, i.e. the rightmost eigenvalue of $\operatorname{Re} T$.) Equivalently, this means that

$$\tilde E(y,\dot y) := (\rho^2+\omega^2)\|L^{-1}\boldsymbol{y}\|_2^2 = (\rho^2+\omega^2)\,y^2 + (\dot y + \rho\,y)^2$$

is always a Lyapunov functional for the oscillator, i.e., $d\tilde E \le 0$ along solution trajectories. In the undamped case, $\tilde E$ is identical with the total energy functional $E(y,\dot y) = \omega^2\,y^2 + \dot y^2$ which is conserved, $d\tilde E \equiv 0$ for $\rho = 0$. For $\rho > 0$ we have $dE < 0$, and $d\tilde E < 0$ due to $\tilde\rho > 0$, where $\tilde E \ne E$. A straightforward calculation shows $d\tilde E = -2\rho E$, i.e., $\tilde E(t)$ represents a form of mean energy; see Fig. 1.

A remarkable special case occurs, e.g., at confluence, $\omega = \rho \gg 0$, critical damping at high stiffness. Here,

$$\mu_2(C) = \tfrac{1}{2}(\omega-1)^2 = \mathcal{O}(\omega^2) \gg 0, \quad \text{in contrast to} \quad \mu_2(T) = \big(\tfrac{\sqrt{2}}{2} - 1\big)\omega \ll 0.$$

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