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ADAPTIVE FEM, BEM, AND FEM-BEM COUPLING WITH OPTIMAL RATES FOR STRONGLY NON-SYMMETRIC PROBLEMS

M. FEISCHL, T. FÜHRER, AND D. PRAETORIUS

Abstract. We prove an abstract summability result which is motivated by the usual *Pythagoras* theorem for symmetric problems. This allows to prove convergence with optimal algebraic rates for adaptive algorithms for a wide range of non-symmetric problems as long as they fit into the abstract setting of the Lax-Milgram lemma. Possible applications include the coupling of finite elements and boundary elements as well as boundary element formulations for mixed boundary conditions. The operators of these model problems cannot be decomposed into a symmetric and elliptic principle part plus a compact perturbation, but the non-symmetric part is non-compact as well. While this prevents the use of available techniques from the literature, these problems fit into the framework of the Lax-Milgram lemma and are hence covered by our analysis.

1. Introduction

Suppose a continuous and elliptic bilinear form $b(\cdot,\cdot)$ on a real Hilbert space $\mathcal{X}$. Given a functional $f \in \mathcal{X}^*$, the Lax-Milgram lemma guarantees existence and uniqueness of $u \in \mathcal{X}$ with

$$b(u, v) = f(v) \quad \text{for all } v \in \mathcal{X}. \tag{1}$$

Given an initial finite dimensional subspace $\mathcal{X}_0 \subseteq \mathcal{X}$ based on a triangulation $\mathcal{T}_0$, an adaptive algorithm of the form

$$\boxed{\texttt{Solve}} \longrightarrow \boxed{\texttt{Estimate}} \longrightarrow \boxed{\texttt{Mark}} \longrightarrow \boxed{\texttt{Refine}} \tag{2}$$

generates a sequence of nested triangulations $\mathcal{T}_\ell$ with corresponding discrete spaces $\mathcal{X}_\ell \subseteq \mathcal{X}$ and approximates the exact solution by computing Galerkin approximations $U_\ell \in \mathcal{X}_\ell$ for all $\ell \in \mathbb{N}$. The module `Solve` in (2) assumes an exact solver which computes the unique solution $U_\ell \in \mathcal{X}_\ell$ of

$$b(U_\ell, V) = f(V) \quad \text{for all } V \in \mathcal{X}_\ell. \tag{3}$$

The ellipticity of $b(\cdot,\cdot)$ provides a constant $C_{\rm Cea} > 0$ which depends only on $b(\cdot,\cdot)$, such that U_ℓ satisfies the Céa-type estimate

$$\|u - U_\ell\|_{\mathcal{X}} \le C_{\rm Cea} \min_{V_\ell \in \mathcal{X}_\ell} \|u - V_\ell\|_{\mathcal{X}} \quad \text{for all } \ell \in \mathbb{N}. \tag{4}$$

The module `Estimate` in (2) assumes a computable error estimator

$$\rho_\ell^2 := \sum_{T \in \mathcal{T}_\ell} \rho_\ell(T)^2 \quad \text{for all } \ell \in \mathbb{N}_0. \tag{5}$$

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The module `Mark` in (2) uses the Dörfler marking criterion to determine a set of marked element domains $\mathcal{M}_\ell \subseteq \mathcal{T}_\ell$ as a set of minimal cardinality to satisfy

$$\theta \rho_\ell^2 \le \sum_{T \in \mathcal{M}_\ell} \rho_\ell(T)^2 \tag{6}$$

for some fixed parameter $0 < \theta < 1$.

The module `Refine` in (2) enriches the space $\mathcal{X}_\ell$ by refining the underlying triangulation $\mathcal{T}_\ell$ to generate $\mathcal{T}_{\ell+1}$ and $\mathcal{X}_{\ell+1} \supseteq \mathcal{X}_\ell$ (see Section 3.1 for details and discussion).

The goal of this work is to prove optimal convergence rates for the estimator ρ_ℓ in the following sense: If theoretically there exist meshes $\widetilde{\mathcal{T}}_\ell$ which are refinements of $\mathcal{T}_0$ such that a certain rate of convergence $s > 0$ is possible for the corresponding error estimator $\widetilde{\rho}_\ell$, i.e.

$$\widetilde{\rho}_\ell \lesssim (\#\widetilde{\mathcal{T}}_\ell - \#\mathcal{T}_0)^{-s} \quad \text{for all } \ell \in \mathbb{N}, \tag{7}$$

then, the adaptively generated meshes recover at least this rate, i.e.,

$$\rho_\ell \lesssim (\#\mathcal{T}_\ell - \#\mathcal{T}_0)^{-s} \quad \text{for all } \ell \in \mathbb{N}, \tag{8}$$

see Theorem 14 below for a precise statement of the result.

A common tool in the proofs of such optimality statements (7)–(8) in e.g. [Ste07, CKNS08, FKMP13, Gan13] is a Pythagoras identity of the form

$$\|u - U_{\ell+1}\|_{\mathcal{X}}^2 + \|U_{\ell+1} - U_\ell\|_{\mathcal{X}}^2 = \|u - U_\ell\|_{\mathcal{X}}^2 \quad \text{for all } \ell \in \mathbb{N}. \tag{9}$$

This is essential to relate successive solutions U_ℓ, $U_{\ell+1}$ with each other. In case of a symmetric bilinear form with induced norm $\|\cdot\|_{\mathcal{X}} := b(\cdot,\cdot)^{1/2}$, such an identity follows immediately from the Galerkin orthogonality

$$b(u - U_{\ell+1}, U_{\ell+1} - U_\ell) = 0 = b(U_{\ell+1} - U_\ell, u - U_{\ell+1}) \quad \text{for all } \ell \in \mathbb{N}, \tag{10}$$

but fails to hold in many other cases as, e.g., non-symmetric problems or FEM-BEM coupling formulations. In the frame of the Lax-Milgram lemma and in many applications, convergence

$$\lim_{\ell \to \infty} \|u - U_\ell\|_{\mathcal{X}} = 0 \tag{11}$$

is a priori available by means of the estimator reduction principle [AFLP12] or follows from more general concepts [MSV08, Sie11]. The Pythagoras identity (9) therefore implies

$$\sum_{k=\ell}^{\infty} \|U_{k+1} - U_k\|_{\mathcal{X}}^2 = \|u - U_\ell\|_{\mathcal{X}}^2 \quad \text{for all } \ell \in \mathbb{N}. \tag{12}$$

The generalization of (12) to *general quasi-orthogonality*

$$\sum_{k=\ell}^{\infty} \|U_{k+1} - U_k\|_{\mathcal{X}}^2 \le C_1 \|u - U_\ell\|_{\mathcal{X}}^2 \quad \text{for all } \ell \in \mathbb{N} \tag{13}$$

for some ℓ-independent constant $C_1 > 0$ still enables the analysis to prove optimal convergence rates in the spirit of [Ste07] (see Section 3 and Section 8), but allows to include a much wider variety of problem classes.

This improves on the existing literature on rate optimality. The seminal work [Ste07] proves optimal convergence rates for the Laplacian, whereas [CKNS08] applies to linear symmetric and elliptic second-order PDEs, and [AFK$^+$13, FPP14] include non-homogeneous boundary conditions. For boundary element methods, the works [FKMP13, Gan13] were the first to prove optimal convergence rates, where [FKMP13] is concerned with the weakly-singular integral equation for the Laplacian on polygonal domains, while [Gan13] considers weakly-singular and hyper-singular integral equation on smooth domains. The work [Gan08] proves optimal rates for certain non-symmetric problems for a wavelet method. The work [CN12] considers mildly non-symmetric, linear elliptic second-order PDEs and proves optimal convergence rates for standard conforming FEM under the assumption that the initial mesh is sufficiently fine. The recent work [FFP12] improves on that by dropping the assumption on the initial mesh-width and including general linear elliptic second-order PDEs into the analysis. There also exist optimality results [FFP12, GMZ12, BDK12, HTZ10] for certain classes of non-linear second-order PDEs.

The present work is the first to prove optimal convergence rates for linear, but non-symmetric problems beyond conforming FEM for second-order PDEs, as for example the coupling of FEM and BEM (see Sections 5–6) or the boundary element formulation for mixed boundary conditions (see Section 7). The proposed *general quasi-orthogonality* (13) is a true generalization of the existing quasi-orthogonality concepts found in e.g. [CN12, AFK+13, FFP12] and allows to tackle even problems where the non-symmetry is not compact (as opposed to [CN12, FFP12]).

The remainder of the work is organized as follows: The first half, comprising of Section 2, provides the proof of (13) for general linear and elliptic problems (1). In the second half, Section 3 states the adaptive algorithm as well as the main result on quasi-optimal convergence (Theorem 14). The short Section 4 gives a new proof of the result for linear second-order PDEs in [FFP12]. Sections 5–6 apply the results of the previous sections to two formulations of the FEM-BEM coupling. Section 7 considers the so-called symmetric boundary integral formulation of some mixed boundary value problem. Finally, Section 8 contains the postponed proof of Theorem 14.

Throughout the paper, $\lesssim$ denotes $\leq$ up to a multiplicative constant and $\simeq$ denotes that both $\lesssim$ and $\gtrsim$ hold.

2. General quasi-orthogonality

The following theorem is the main result of this section and it will serve as the main tool to prove optimal convergence rates in the following sections.

Theorem 1. *Suppose a constant $C_2 > 0$ such that the bilinear form $b(\cdot,\cdot)$ is continuous and elliptic in the sense of*

$$b(v,w) \leq C_2 \|v\|_{\mathcal{X}} \|w\|_{\mathcal{X}} \quad \textit{and} \quad b(v,v) \geq C_2^{-1} \|v\|_{\mathcal{X}}^2 \quad \textit{for all } v,w \in \mathcal{X}. \tag{14}$$

Suppose that $\mathcal{X}_\ell$ are nested subspaces of $\mathcal{X}$, i.e. $\mathcal{X}_\ell \subseteq \mathcal{X}_{\ell+1} \subseteq \mathcal{X}$ for all $\ell \in \mathbb{N}_0$. Let $U_\ell \in \mathcal{X}_\ell$ and $u \in \mathcal{X}$ denote the unique solutions of (1)–(3)*. Then, convergence* (11) *implies general quasi-orthogonality* (13)*.*

Remark 2. *Without loss of generality, we may assume $U_\ell \neq U_{\ell+1}$ for all $\ell \in \mathbb{N}$, since otherwise the respective terms vanish in the general quasi-orthogonality* (13)*. The unique solvability therefore implies $U_{\ell+1} \notin \mathcal{X}_\ell$ for all $\ell \in \mathbb{N}$.*

For the proof, and for convenience of the presentation, an equivalent operator formulation replaces the variational setting above. To that end, define the operator

$$B : \mathcal{X} \to \mathcal{X}^*, \; v \mapsto Bv := b(v,\cdot), \tag{15}$$

as well as for all $\mathcal{X}_\star \subseteq \mathcal{X}$ the restriction $P_{\mathcal{X}_\star} : \mathcal{X}^* \to \mathcal{X}_\star^*$ by

$$P_{\mathcal{X}_\star} f = f|_{\mathcal{X}_\star} \quad \text{for all } f \in \mathcal{X}^*. \tag{16}$$

The problem (1) as well as its discretization (3) equivalently read as

$$Bu = f \quad \text{and} \quad P_{\mathcal{X}_\ell} B|_{\mathcal{X}_\ell} U_\ell = P_{\mathcal{X}_\ell} f \quad \text{for all } \ell \in \mathbb{N}_0. \tag{17}$$

The proof of Theorem 1 is split into two parts which mark the following two subsections. In Section 2.1, the result is proved for the simpler case of symmetric but indefinite bilinear forms $b(\cdot,\cdot)$ with the additional restriction that $\dim(\mathcal{X}_\ell)+1=\dim(\mathcal{X}_{\ell+1})$. In Section 2.2, it is shown that the general case can be reduced to the case of Section 2.1

2.1. **Step 1: The symmetric but possibly indefinite case.** Since this section poses additional assumptions on the spaces $\mathcal{X}_\ell$ as well as on the operator B, it seems useful to change the notation slightly. All the previous definitions (1)–(3) and (13) transfer likewise in the sense that $\boldsymbol{u}\in\boldsymbol{X}$ is the unique solution of

$$\boldsymbol{b}(\boldsymbol{u},\boldsymbol{v})=\boldsymbol{f}(\boldsymbol{v})\quad\text{for all } \boldsymbol{v}\in\boldsymbol{X} \tag{18}$$

with Galerkin discretization

$$\boldsymbol{b}(\boldsymbol{U}_\ell,\boldsymbol{V})=\boldsymbol{f}(\boldsymbol{V})\quad\text{for all } \boldsymbol{V}\in\boldsymbol{X}_\ell. \tag{19}$$

Proposition 3. *Let $\boldsymbol{X}$ denote a real Hilbert space and let $\boldsymbol{X}_\ell$ be finite dimensional subspaces of $\boldsymbol{X}$ with*

$$\boldsymbol{X}=\overline{\bigcup_{\ell\in\mathbb{N}_0}\boldsymbol{X}_\ell}\quad\textit{and}\quad \dim(\boldsymbol{X}_\ell)+1=\dim(\boldsymbol{X}_{\ell+1})\quad\textit{for all } \ell\in\mathbb{N}_0.$$

Let $\boldsymbol{B}:\boldsymbol{X}\to\boldsymbol{X}^*$ denote a symmetric operator in the sense $\langle \boldsymbol{B}v\,,\,w\rangle=\langle \boldsymbol{B}w\,,\,v\rangle$ for all $v,w\in\mathcal{X}$ which satisfies

$$\max\{\|\boldsymbol{B}\|_{\boldsymbol{X}\to\boldsymbol{X}^*},\|\boldsymbol{B}^{-1}\|_{\boldsymbol{X}^*\to\boldsymbol{X}}\}\le C_3 \tag{20a}$$

as well as for all $\ell\in\mathbb{N}_0$

$$\|(P_{\boldsymbol{X}_\ell}\boldsymbol{B}|_{\boldsymbol{X}_\ell})^{-1}\|_{\boldsymbol{X}_\ell^*\to\boldsymbol{X}_\ell}\le C_3 \tag{20b}$$

for some constant $C_3>0$. Then, the problems (18) and (19) allow for unique solutions, and there holds general quasi-orthogonality

$$\sum_{k=\ell}^{\infty}\|\boldsymbol{U}_{k+1}-\boldsymbol{U}_k\|_{\boldsymbol{X}}^2\le C_1\|\boldsymbol{u}-\boldsymbol{U}_\ell\|_{\boldsymbol{X}}^2\quad\textit{for all } \ell\in\mathbb{N}_0. \tag{21}$$

The proof requires some preparations. To that end, let $(\boldsymbol{v}_\ell)_{\ell\in\mathbb{N}_0}\subset\boldsymbol{X}$ be an orthonormal basis of $\boldsymbol{X}$ such that

$$\boldsymbol{X}_\ell=\operatorname{span}\{\boldsymbol{v}_k\,:\,k=0,\dots,\ell\}\quad\text{for all } \ell\in\mathbb{N}_0.$$

Note that such a basis can always be constructed e.g. via Gram-Schmidt orthogonalization.

Lemma 4. *There exists a sequence $(\boldsymbol{w}_\ell)_{\ell\in\mathbb{N}}\subset\boldsymbol{X}$ and a constant $C_4>0$ such that there holds*

(i) *Nestedness:*

$$\textit{span}\{\boldsymbol{w}_k\,:\,k=0,\dots,\ell\}=\boldsymbol{X}_\ell\quad\textit{for all } \ell\in\mathbb{N}_0. \tag{22}$$

(ii) *Boundedness:*

$$C_4^{-1}\le\|\boldsymbol{w}_\ell\|_{\boldsymbol{X}}\le C_4\quad\textit{for all } \ell\in\mathbb{N}_0. \tag{23}$$

(iii) *$\boldsymbol{B}$-orthogonality:*

$$\langle \boldsymbol{B}\boldsymbol{w}_\ell\,,\,\boldsymbol{w}_k\rangle=0\quad\textit{for all } \ell\neq k\quad\textit{and}\quad\langle \boldsymbol{B}\boldsymbol{w}_\ell\,,\,\boldsymbol{w}_\ell\rangle\in\{-1,1\}\quad\textit{for all } \ell\in\mathbb{N}_0. \tag{24}$$

Proof. Define $\boldsymbol{w}_0 = \boldsymbol{v}_0$ and for all $\ell \geq 1$

$$\boldsymbol{w}_\ell := \boldsymbol{v}_\ell - (P_{\boldsymbol{X}_{\ell-1}} \boldsymbol{B}|_{\boldsymbol{X}_{\ell-1}})^{-1} P_{\boldsymbol{X}_{\ell-1}} \boldsymbol{B} \boldsymbol{v}_\ell.$$

From $\boldsymbol{w}_\ell \in \boldsymbol{v}_\ell + \boldsymbol{X}_{\ell-1}$ and the choice of $\boldsymbol{v}_\ell$, we derive (22). Obviously, there holds $\|\boldsymbol{w}_0\|_{\boldsymbol{X}} = 1$ and since $(\boldsymbol{v}_\ell)_{\ell \in \mathbb{N}_0}$ is an orthonormal basis

$$\|\boldsymbol{w}_\ell\|_{\boldsymbol{X}}^2 = 1 + \|(P_{\boldsymbol{X}_{\ell-1}} \boldsymbol{B}|_{\boldsymbol{X}_{\ell-1}})^{-1} P_{\boldsymbol{X}_{\ell-1}} \boldsymbol{B} \boldsymbol{v}_\ell\|_{\boldsymbol{X}}^2$$

for all $\ell \in \mathbb{N}$. Since continuity of $\boldsymbol{B}$ implies continuity of $P_{\boldsymbol{X}_\ell} \boldsymbol{B}|_{\boldsymbol{X}_\ell}$ even with the same stability constant $\|P_{\mathcal{X}_\ell} \boldsymbol{B}|_{\mathcal{X}_\ell}\|_{\mathcal{X}_\ell \to \mathcal{X}_\ell^*} \leq \|\boldsymbol{B}\|_{\mathcal{X} \to \mathcal{X}^*} \leq C_3$, the last identity and (20) prove

$$1 \leq \|\boldsymbol{w}_\ell\|_{\boldsymbol{X}}^2 \leq 1 + C_3^4 \quad \text{for all } \ell \in \mathbb{N}.$$

Moreover, there holds for $0 \leq k < \ell$

$$\langle B\boldsymbol{w}_\ell \,,\, \boldsymbol{v}_k \rangle = \langle P_{\boldsymbol{X}_{\ell-1}} \boldsymbol{B} \boldsymbol{v}_\ell - P_{\boldsymbol{X}_{\ell-1}} \boldsymbol{B}|_{\boldsymbol{X}_{\ell-1}} (P_{\boldsymbol{X}_{\ell-1}} \boldsymbol{B}|_{\boldsymbol{X}_{\ell-1}})^{-1} P_{\boldsymbol{X}_{\ell-1}} \boldsymbol{B} \boldsymbol{v}_\ell \,,\, \boldsymbol{v}_k \rangle = 0.$$

This shows $\langle \boldsymbol{B}\boldsymbol{w}_\ell \,,\, \boldsymbol{x} \rangle = 0$ for all $\boldsymbol{x} \in \boldsymbol{X}_{\ell-1}$. Symmetry of $\boldsymbol{B}$ proves the first statement of (24). For the second part, observe that each $\boldsymbol{x} \in \boldsymbol{X}_\ell$ has a representation $\boldsymbol{x} = \sum_{i=0}^{\ell} x_i \boldsymbol{v}_i$. The orthogonality from above together with continuity of $\boldsymbol{B}$ and (20b) imply

$$1 \simeq \|P_{\boldsymbol{X}_\ell} \boldsymbol{B}|_{\boldsymbol{X}_\ell} \boldsymbol{w}_\ell\|_{\boldsymbol{X}_\ell^\star} = \sup_{\boldsymbol{x} \in \boldsymbol{X}_\ell,\, \|\boldsymbol{x}\|_{\boldsymbol{X}} \leq 1} \langle \boldsymbol{B}\boldsymbol{w}_\ell \,,\, \boldsymbol{x} \rangle = \sup_{x_\ell \in \mathbb{R},\, |x_\ell| \leq 1} x_\ell \langle \boldsymbol{B}\boldsymbol{w}_\ell \,,\, \boldsymbol{v}_\ell \rangle = |\langle \boldsymbol{B}\boldsymbol{w}_\ell \,,\, \boldsymbol{v}_\ell \rangle|.$$

By definition of $\boldsymbol{w}_\ell$, we get $|\langle \boldsymbol{B}\boldsymbol{w}_\ell \,,\, \boldsymbol{w}_\ell \rangle| = |\langle \boldsymbol{B}\boldsymbol{w}_\ell \,,\, \boldsymbol{v}_\ell \rangle| \simeq 1$. This, together with (23), allows to scale the basis $(\boldsymbol{w}_\ell)_{\ell \in \mathbb{N}}$, such that there holds both, the orthogonality (24) as well as (23). This concludes the proof. $\square$

Lemma 5. *Let $(\boldsymbol{w}_\ell)_{\ell \in \mathbb{N}}$ denote the basis from Lemma 4. Define the spaces*

$$\begin{aligned} \boldsymbol{X}_+ &:= \overline{\operatorname{span}\big\{\boldsymbol{w}_\ell \,:\, \langle \boldsymbol{B}\boldsymbol{w}_\ell \,,\, \boldsymbol{w}_\ell \rangle = 1,\, \ell \in \mathbb{N}\big\}}, \\ \boldsymbol{X}_- &:= \overline{\operatorname{span}\big\{\boldsymbol{w}_\ell \,:\, \langle \boldsymbol{B}\boldsymbol{w}_\ell \,,\, \boldsymbol{w}_\ell \rangle = -1,\, \ell \in \mathbb{N}\big\}}, \end{aligned} \tag{25}$$

where the closure is understood in $\boldsymbol{X}$. Then, there holds the direct decomposition $\boldsymbol{X}_+ \oplus \boldsymbol{X}_- = \boldsymbol{X}$. Hence, there exist continuous projections

$$\begin{aligned} P_+ &: \boldsymbol{X} \to \boldsymbol{X}_+ \quad \textit{with } \ker P_+ = \boldsymbol{X}_-, \\ P_- &: \boldsymbol{X} \to \boldsymbol{X}_- \quad \textit{with } \ker P_+ = \boldsymbol{X}_+ \end{aligned} \tag{26}$$

with $\|P_+\|_{\boldsymbol{X} \to \boldsymbol{X}}, \|P_-\|_{\boldsymbol{X} \to \boldsymbol{X}} \leq C_5 < \infty$.

Proof. Due to (22), it is clear that $\boldsymbol{X}_+ + \boldsymbol{X}_- = \boldsymbol{X}$. To see $\boldsymbol{X}_+ \cap \boldsymbol{X}_- = \{0\}$, let $\boldsymbol{x} \in \boldsymbol{X}_+ \cap \boldsymbol{X}_-$. Then, there exists a sequence $\boldsymbol{x}_j \in \boldsymbol{X}_-^\circ := \operatorname{span}\big\{\boldsymbol{w}_\ell \,:\, \langle \boldsymbol{B}\boldsymbol{w}_\ell \,,\, \boldsymbol{w}_\ell \rangle = -1,\, \ell \in \mathbb{N}\big\}$ such that $\boldsymbol{x}_j \to \boldsymbol{x}$ in $\boldsymbol{X}$ as $j \to \infty$. Then, the orthogonality (24) proves

$$\langle \boldsymbol{B}\boldsymbol{x} \,,\, \boldsymbol{w}_\ell \rangle = \lim_{j \to \infty} \langle \boldsymbol{B}\boldsymbol{x}_j \,,\, \boldsymbol{w}_\ell \rangle = 0 \quad \text{for all } \ell \in \mathbb{N}_0 \text{ with } \boldsymbol{w}_\ell \in \boldsymbol{X}_+.$$

For $\ell \in \mathbb{N}$ with $\boldsymbol{w}_\ell \in \boldsymbol{X}_-$, we obtain the same result. Therefore, it holds $\langle \boldsymbol{B}\boldsymbol{x} \,,\, \boldsymbol{w}_\ell \rangle = 0$ for all $\ell \in \mathbb{N}$ and hence $\boldsymbol{x} = 0$. Therefore, each $\boldsymbol{x} \in \boldsymbol{X}$ has a unique decomposition into $\boldsymbol{x} = \boldsymbol{x}_+ + \boldsymbol{x}_-$ with $\boldsymbol{x}_+ \in \boldsymbol{X}_+$ and $\boldsymbol{x}_- \in \boldsymbol{X}_-$. According to linear algebra, this gives rise to $P_+\boldsymbol{x} = \boldsymbol{x}_+$ and $P_-\boldsymbol{x} = \boldsymbol{x}_-$. To see the continuity of the projections, we employ Banach's closed-graph theorem: Suppose $\boldsymbol{x}_j \to \boldsymbol{x}$ and $P_+\boldsymbol{x}_j \to \boldsymbol{y}$ in $\boldsymbol{X}$. It remains to show that $P_+\boldsymbol{x} = \boldsymbol{y}$. Since $\boldsymbol{X}_+$ is closed, we see that $\boldsymbol{y} \in \boldsymbol{X}_+$, i.e. $P_+\boldsymbol{y} = \boldsymbol{y}$. Moreover, $P_+(\boldsymbol{x}_j - P_+\boldsymbol{x}_j) = 0$ and hence $\boldsymbol{x}_j - P_+\boldsymbol{x}_j \in \ker P_+ = \boldsymbol{X}_-$. Therefore, it holds $\boldsymbol{x} - \boldsymbol{y} = \lim_{j \to \infty}(\boldsymbol{x}_j - P_+\boldsymbol{x}_j) \in \boldsymbol{X}_- = \ker P_+$. Altogether, this yields

$$P_+\boldsymbol{x} - \boldsymbol{y} = P_+(\boldsymbol{x} - \boldsymbol{y}) = 0.$$

The boundedness of P_- follows analogously. $\square$

Lemma 6. *Define the operator*

$$\iota : \bigcup_{\ell\in\mathbb{N}} \boldsymbol{X}_\ell \to \ell^2(\mathbb{N}), \quad \iota\big(\sum_{\ell=1}^{N} \lambda_j \boldsymbol{w}_\ell\big) = \boldsymbol{\lambda} := (\lambda_1, \dots, \lambda_N, 0, \dots) \tag{27}$$

for all $N \in \mathbb{N}$ and $\lambda_j \in \mathbb{R}$. The operator ι may be continuously extended to an operator $\iota : \boldsymbol{X} \to \ell^2(\mathbb{N})$ such that

$$\|\iota(\boldsymbol{x})\|_{\ell^2(\mathbb{N})} := \big(\sum_{j=1}^{\infty} (\iota(\boldsymbol{x})_j)^2\big)^{1/2} \le C_4 \|\boldsymbol{x}\|_{\boldsymbol{X}} \quad \textit{for all } \boldsymbol{x} \in \boldsymbol{X}. \tag{28}$$

Moreover, it holds

$$\iota(\boldsymbol{x})_\ell = \langle \boldsymbol{B}\boldsymbol{w}_\ell \,,\, \boldsymbol{w}_\ell \rangle \langle \boldsymbol{B}\boldsymbol{x} \,,\, \boldsymbol{w}_\ell \rangle \quad \textit{for all } \boldsymbol{x} \in \boldsymbol{X} \textit{ and } \ell \in \mathbb{N}. \tag{29}$$

Proof. Define $N_+ := \big\{\ell \in \mathbb{N} \,:\, \boldsymbol{w}_\ell \in \boldsymbol{X}_+\big\}$ and $N_- := \big\{\ell \in \mathbb{N} \,:\, \boldsymbol{w}_\ell \in \boldsymbol{X}_-\big\} = \mathbb{N} \setminus N_+$ and note that for $\boldsymbol{x} = \sum_{j=0}^{N} \lambda_j \boldsymbol{w}_j \in \boldsymbol{X}_N$, it holds

$$\langle \boldsymbol{B}\boldsymbol{x} \,,\, \boldsymbol{w}_\ell \rangle = \langle \boldsymbol{B}\boldsymbol{w}_\ell \,,\, \boldsymbol{w}_\ell \rangle \lambda_\ell \quad \text{for all } 0 \le \ell \le N.$$

Since $\langle \boldsymbol{B}\boldsymbol{w}_\ell \,,\, \boldsymbol{w}_\ell \rangle \in \{1, -1\}$, this proves

$$\lambda_\ell = \langle \boldsymbol{B}\boldsymbol{x} \,,\, \boldsymbol{w}_\ell \rangle \langle \boldsymbol{B}\boldsymbol{w}_\ell \,,\, \boldsymbol{w}_\ell \rangle \quad \text{for all } 0 \le \ell \le N, \tag{30}$$

as well as

$$\lambda_\ell := \langle \boldsymbol{B}\boldsymbol{x} \,,\, \boldsymbol{w}_\ell \rangle \langle \boldsymbol{B}\boldsymbol{w}_\ell \,,\, \boldsymbol{w}_\ell \rangle = 0 \quad \text{for all } N < \ell < \infty.$$

Moreover, we see immediately $P_+\boldsymbol{x} = \sum_{j\in N_+} \lambda_j \boldsymbol{w}_j$ as well as $P_-\boldsymbol{x} = \sum_{j\in N_-} \lambda_j \boldsymbol{w}_j$. Together with the continuity of P_+ and P_-, this implies

$$\begin{aligned}
\|\iota(\boldsymbol{x})\|_{\ell_2(\mathbb{N})}^2 &= \sum_{\ell=1}^{N} \lambda_\ell^2 = \sum_{\ell=1}^{\infty} \langle \boldsymbol{B}\boldsymbol{x} \,,\, \boldsymbol{w}_\ell \rangle^2 \\
&= \sum_{\ell\in N_+} \langle \boldsymbol{B}\boldsymbol{x} \,,\, \boldsymbol{w}_\ell \rangle^2 + \sum_{\ell\in N_-} \langle \boldsymbol{B}\boldsymbol{x} \,,\, \boldsymbol{w}_\ell \rangle^2 \\
&= \sum_{\ell\in N_+} \langle \boldsymbol{B}\boldsymbol{x} \,,\, \langle \boldsymbol{B}\boldsymbol{x} \,,\, \boldsymbol{w}_\ell \rangle \boldsymbol{w}_\ell \rangle - \sum_{\ell\in N_-} \langle \boldsymbol{B}\boldsymbol{x} \,,\, -\langle \boldsymbol{B}\boldsymbol{x} \,,\, \boldsymbol{w}_\ell \rangle \boldsymbol{w}_\ell \rangle \\
&= \langle \boldsymbol{B}\boldsymbol{x} \,,\, P_+\boldsymbol{x} \rangle - \langle \boldsymbol{B}\boldsymbol{x} \,,\, P_-\boldsymbol{x} \rangle \lesssim \|\boldsymbol{x}\|_{\boldsymbol{X}}^2.
\end{aligned}$$

The constants in the estimate above do not depend on $N \in \mathbb{N}$. As $\bigcup_{N\in\mathbb{N}_0} \boldsymbol{X}_N$ is dense in $\boldsymbol{X}$, ι can be extended continuously to $\iota : \boldsymbol{X} \to \ell^2(\mathbb{N})$. Since evaluation of one component is a continuous operation in $\ell^2(\mathbb{N})$, there holds for $\lim_{j\to\infty} \boldsymbol{x}_j = \boldsymbol{x}$ with $\boldsymbol{x}_j \in \bigcup_{\ell\in\mathbb{N}} \boldsymbol{X}_\ell$ together with (30)

$$\iota(\boldsymbol{x})_\ell = \lim_{j\to\infty} \iota(\boldsymbol{x}_j)_\ell = \lim_{j\to\infty} \langle \boldsymbol{B}\boldsymbol{x}_j \,,\, \boldsymbol{w}_\ell \rangle \langle \boldsymbol{B}\boldsymbol{w}_\ell \,,\, \boldsymbol{w}_\ell \rangle = \langle \boldsymbol{B}\boldsymbol{x} \,,\, \boldsymbol{w}_\ell \rangle \langle \boldsymbol{B}\boldsymbol{w}_\ell \,,\, \boldsymbol{w}_\ell \rangle.$$

This proves (29) and concludes the proof. □

Proof of Proposition 3. Obviously, there holds $\boldsymbol{U}_{k+1} - \boldsymbol{U}_k \in \boldsymbol{X}_{k+1}$ as well as Galerkin orthogonality

$$\langle \boldsymbol{B}(\boldsymbol{U}_{k+1} - \boldsymbol{U}_k) \,,\, \boldsymbol{w}_j \rangle = 0 \quad \text{for all } j \le k.$$

Recall the basis $(\boldsymbol{w}_\ell)_{\ell\in\mathbb{N}_0}$ from Lemma 4. The above orthogonality implies $\boldsymbol{U}_{k+1}-\boldsymbol{U}_k = \alpha_{k+1}\boldsymbol{w}_{k+1}$ for some $\alpha_{k+1}\in\mathbb{R}$ and hence $\iota(\boldsymbol{U}_{k+1}-\boldsymbol{U}_k) = \alpha_{k+1}e_{k+1}$ for some $\alpha_{k+1}\in\mathbb{R}$ by definition of ι. Due to (23), it holds

$$\|\boldsymbol{U}_{k+1}-\boldsymbol{U}_k\|_{\boldsymbol{X}} = |\alpha_{k+1}|\|\boldsymbol{w}_{k+1}\|_{\boldsymbol{X}} \simeq |\alpha_{k+1}| = \|\iota(\boldsymbol{U}_{k+1}-\boldsymbol{U}_k)\|_{\ell^2(\mathbb{N})} \quad \text{for all } k\in\mathbb{N}_0. \tag{31}$$

We have

$$(\iota(\boldsymbol{U}_{k+1})-\iota(\boldsymbol{U}_k))_j = 0 \quad \text{for all } j\neq k+1.$$

Furthermore, the representation (29) together with (1)–(3) imply for $j\leq k$

$$\iota(\boldsymbol{U}_k)_j = \langle \boldsymbol{B}\boldsymbol{w}_j\,,\,\boldsymbol{w}_j\rangle\langle \boldsymbol{B}\boldsymbol{U}_k\,,\,\boldsymbol{w}_j\rangle = \langle \boldsymbol{B}\boldsymbol{w}_j\,,\,\boldsymbol{w}_j\rangle\langle \boldsymbol{B}\boldsymbol{u}\,,\,\boldsymbol{w}_j\rangle = \iota(u)_j.$$

This yields

$$(\iota(\boldsymbol{u})-\iota(\boldsymbol{U}_{k+1}))_j = 0 \quad \text{for all } 0\leq j\leq k+1.$$

Consequently, we see

$$\langle \iota(\boldsymbol{u})-\iota(\boldsymbol{U}_{k+1})\,,\,\iota(\boldsymbol{U}_{k+1})-\iota(\boldsymbol{U}_k)\rangle_{\ell^2(\mathbb{N})} = 0 \quad \text{for all } k\in\mathbb{N}.$$

This orthogonality proves the Pythagoras theorem

$$\|\iota(\boldsymbol{u}-\boldsymbol{U}_{k+1})\|^2_{\ell^2(\mathbb{N})} + \|\iota(\boldsymbol{U}_{k+1}-\boldsymbol{U}_k)\|^2_{\ell^2(\mathbb{N})} = \|\iota(\boldsymbol{u}-\boldsymbol{U}_k)\|^2_{\ell^2(\mathbb{N})}. \tag{32}$$

With (31)–(32) and with the stability of $\iota:\boldsymbol{X}\to\ell^2(\mathbb{N})$, it follows

$$\begin{aligned}\sum_{k=\ell}^\infty \|\boldsymbol{U}_{k+1}-\boldsymbol{U}_k\|^2_{\boldsymbol{X}} &\simeq \sum_{k=\ell}^\infty \|\iota(\boldsymbol{U}_{k+1}-\boldsymbol{U}_k)\|^2_{\ell^2(\mathbb{N})}\\ &= \sum_{k=\ell}^\infty \big(\|\iota(\boldsymbol{u}-\boldsymbol{U}_k)\|^2_{\ell^2(\mathbb{N})} - \|\iota(\boldsymbol{u}-\boldsymbol{U}_{k+1})\|^2_{\ell^2(\mathbb{N})}\big)\\ &\leq \|\iota(\boldsymbol{u}-\boldsymbol{U}_\ell)\|^2_{\ell^2(\mathbb{N})} \lesssim \|\boldsymbol{u}-\boldsymbol{U}_\ell\|^2_{\boldsymbol{X}}.\end{aligned}$$

This concludes the proof. □

2.2. **Step 2: Reduction to the symmetric case.** The reduction step depends on the construction of an equivalent operator, which satisfies the claims of Proposition 3.

For the following proposition, recall the right-hand side $f\in\mathcal{X}^*$ and the operator B from (15) as well as the solutions u and U_ℓ from (1)–(3), resp. (17).

Proposition 7. *Suppose the assumptions of Theorem 1. Then, there exists a Hilbert space $\boldsymbol{X}$ which is a closed subspace of $\mathcal{X}\times\mathcal{X}$ and a sequence of nested subspaces $\boldsymbol{X}_\ell$ of $\boldsymbol{X}$ such that $\boldsymbol{X} = \overline{\bigcup_{\ell\in\mathbb{N}}\boldsymbol{X}_\ell}$ and $\dim(\boldsymbol{X}_\ell)+1 = \dim(\boldsymbol{X}_{\ell+1})$ for all $\ell\in\mathbb{N}_0$. There exists a symmetric operator $\boldsymbol{B}:\boldsymbol{X}\to\boldsymbol{X}^\star$ which satisfies* (20). *Given $\boldsymbol{f} := (f,-f)$ the unique solutions of* (18)–(19) *satisfy $\boldsymbol{u} := (u,u)\in\boldsymbol{X}$ and $\boldsymbol{U}_{2\ell+1} := (U_\ell,U_\ell)\in\boldsymbol{X}_{2\ell+1}$ for all $\ell\in\mathbb{N}$.*

Proof. With the notation (15)–(17), consider the transposed operator $B^T:\mathcal{X}\to\mathcal{X}^\star$, $B^T\boldsymbol{v} := b(\cdot,\boldsymbol{v})$ and define the symmetric part $S := (B+B^T)/2$ as well as the antisymmetric part $A := (B-B^T)/2$. Obviously, there holds $S^T = S$, $A^T = -A$, and $B = S+A$. With this, define the symmetric operator $\boldsymbol{B}:\mathcal{X}\times\mathcal{X}\to\mathcal{X}^\star\times\mathcal{X}^\star$ as

$$\boldsymbol{B} := \begin{pmatrix} S & A\\ -A & -S\end{pmatrix}.$$

By definition of $\boldsymbol{f}$ as well as of $\boldsymbol{u}$, there holds

$$\boldsymbol{B}\boldsymbol{u} = \boldsymbol{f}. \tag{33}$$

Define the subspaces $\boldsymbol{X}_\ell$ inductively for $\ell \in \mathbb{N}$ by

$$\begin{aligned}
\boldsymbol{X}_0 &:= \operatorname{span}\{(U_0, 0)\},\\
\boldsymbol{X}_{2\ell-1} &:= \operatorname{span}(\boldsymbol{X}_{2\ell-2} \cup \{(0, U_{\ell-1})\},\\
\boldsymbol{X}_{2\ell} &:= \operatorname{span}(\boldsymbol{X}_{2\ell-1} \cup \{(U_\ell, 0)\}.
\end{aligned}$$

Note that there holds $\dim(\boldsymbol{X}_\ell) + 1 = \dim(\boldsymbol{X}_{\ell+1})$ due to Remark 2. Consequently, define

$$\boldsymbol{X} := \overline{\bigcup_{\ell \in \mathbb{N}} \boldsymbol{X}_\ell}.$$

Since $u = \lim_{\ell\to\infty} U_\ell$, there also holds $\boldsymbol{u} \in \boldsymbol{X}$.

To show that $\boldsymbol{B}$ satisfies (20), consider for $\boldsymbol{V} = (V_1, V_2)$ and $\boldsymbol{W} = (W_1, W_2)$

$$\begin{aligned}
&\inf_{\boldsymbol{V}\in\boldsymbol{X}_\ell} \sup_{\boldsymbol{V}\in\boldsymbol{X}_\ell} \frac{\langle \boldsymbol{B}\boldsymbol{V}\,,\,\boldsymbol{W}\rangle}{\|\boldsymbol{V}\|_{\boldsymbol{X}}\|\boldsymbol{W}\|_{\boldsymbol{X}}}\\
&\quad= \inf_{\boldsymbol{V}\in\boldsymbol{X}_\ell} \sup_{\boldsymbol{W}\in\boldsymbol{X}_\ell} \frac{\langle SV_1\,,\,W_1\rangle + \langle AV_2\,,\,W_1\rangle - \langle AV_1\,,\,W_2\rangle - \langle SV_2\,,\,W_2\rangle}{\|\boldsymbol{V}\|_{\boldsymbol{X}}\|\boldsymbol{W}\|_{\boldsymbol{X}}}\\
&\quad\geq \inf_{\boldsymbol{V}\in\boldsymbol{X}_\ell} \frac{\langle SV_1\,,\,V_1\rangle + \langle AV_2\,,\,V_1\rangle + \langle AV_1\,,\,V_2\rangle + \langle SV_2\,,\,V_2\rangle}{\|\boldsymbol{V}\|_{\boldsymbol{X}}\|(V_1, -V_2)\|_{\boldsymbol{X}}}\\
&\quad= \inf_{\boldsymbol{V}\in\boldsymbol{X}_\ell} \frac{\langle SV_1\,,\,V_1\rangle + \langle SV_2\,,\,V_2\rangle}{\|\boldsymbol{V}\|_{\boldsymbol{X}}^2}.
\end{aligned}$$

Since $\langle Sv\,,\,v\rangle = \langle Bv\,,\,v\rangle \geq C_2^{-1}\|v\|_{\mathcal{X}}^2$, this implies

$$\inf_{\boldsymbol{V}\in\boldsymbol{X}_\ell} \sup_{\boldsymbol{V}\in\boldsymbol{X}_\ell} \frac{\langle \boldsymbol{B}\boldsymbol{V}\,,\,\boldsymbol{W}\rangle}{\|\boldsymbol{V}\|_{\boldsymbol{X}}\|\boldsymbol{W}\|_{\boldsymbol{X}}} \geq C_2^{-1} \inf_{\boldsymbol{V}\in\boldsymbol{X}_\ell} \frac{\|V_1\|_{\mathcal{X}}^2 + \|V_2\|_{\mathcal{X}}^2}{\|\boldsymbol{V}\|_{\boldsymbol{X}}^2} = C_2^{-1} > 0.$$

This implies (20b). The same arguments on $\boldsymbol{X}$ together with the continuity of B imply (20a). Hence, the problems (18) and (19) allow for a unique solution. Since $(u, u) \in \boldsymbol{X}$ solves (18) and $(U_\ell, U_\ell) \in \boldsymbol{X}_{2\ell+1}$ solves (19), this concludes the proof. □

Proof of Theorem 1. The operator $\boldsymbol{B}$ and the constructed spaces $\boldsymbol{X}$ and $(\boldsymbol{X}_\ell)_{\ell\in\mathbb{N}_0}$ from Proposition 7 satisfy all claims of Proposition 3. Recall that Proposition 7 states $\boldsymbol{u} = (u, u)$ and $\boldsymbol{U}_{2\ell+1} = (U_\ell, U_\ell)$. Hence, Proposition 3 implies

$$\begin{aligned}
\sum_{k=\ell}^{\infty} \|U_{k+1} - U_k\|_{\mathcal{X}}^2 &\leq \sum_{k=\ell}^{\infty} \|\boldsymbol{U}_{2k+3} - \boldsymbol{U}_{2k+1}\|_{\boldsymbol{X}}^2\\
&\leq 2\sum_{k=\ell}^{\infty} \Big(\|\boldsymbol{U}_{2k+3} - \boldsymbol{U}_{2k+2}\|_{\boldsymbol{X}}^2 + \|\boldsymbol{U}_{2k+2} - \boldsymbol{U}_{2k+1}\|_{\boldsymbol{X}}^2\Big)\\
&= \sum_{k=2\ell+1}^{\infty} \|\boldsymbol{U}_{k+1} - \boldsymbol{U}_k\|_{\boldsymbol{X}}^2\\
&\lesssim \|\boldsymbol{u} - \boldsymbol{U}_{2\ell+1}\|_{\boldsymbol{X}}^2 \leq 2\|u - U_\ell\|_{\mathcal{X}}^2.
\end{aligned}$$

This concludes the proof of Theorem 1. □

3. Abstract optimality analysis

This section builds the framework to apply the quasi-orthogonality result from the previous sections and to thus analyze the convergence and quasi-optimality of adaptive mesh-refining algorithms for the problem class stated in the introduction of Section 1. The outline of this section is as follows: Algorithm 8 is the commonly used formulation of the adaptive loop (2). Under certain assumptions (Assumption 10–11), which are later verified for particular model problems, convergence with optimal algebraic rates is shown (Theorem 14). The abstract rate optimality analysis is first found in [CFPP13] and is recalled here only for the convenience of the reader and to underline the important contribution of Theorem 1.

The following formulation of the adaptive loop (2) iteratively generates triangulations $\mathcal{T}_\ell$, $\ell \in \mathbb{N}$, by local refinement of an initial conforming triangulation $\mathcal{T}_0$ of a d-dimensional manifold with $d \geq 1$ and hence corresponding discrete spaces $\mathcal{X}_\ell$ which are supposed to be nested and conforming, i.e. $\mathcal{X}_\ell \subseteq \mathcal{X}_{\ell+1} \subseteq \mathcal{X}$ for all $\ell \in \mathbb{N}_0$ (see Section 3.1 for details).

Algorithm 8. ***Input:*** *initial mesh $\mathcal{T}_0$, adaptivity parameter $0 < \theta < 1$, and $\ell = 0$*

(i) *Compute solution $U_\ell \in \mathcal{X}_\ell$ of* (3).
(ii) *Compute error estimator $\rho_\ell(T)$ from* (5) *for all $T \in \mathcal{T}_\ell$.*
(iii) *Determine a set of marked elements $\mathcal{M}_\ell \subseteq \mathcal{T}_\ell$ with minimal cardinality which satisfies the Dörfler marking* (6).
(iv) *Refine the marked elements $\mathcal{T}_{\ell+1} = \texttt{refine}(\mathcal{T}_\ell, \mathcal{M}_\ell)$ to obtain an enriched space $\mathcal{X}_{\ell+1} \supseteq \mathcal{X}_\ell$ (see Section 3.1 for details).*
(v) *Increment $\ell \leftarrow \ell + 1$ and goto* (i).

Output: *sequence of error estimators $(\rho_\ell)_{\ell\in\mathbb{N}}$ and sequence of Galerkin solutions $(U_\ell)_{\ell\in\mathbb{N}}$.*

Remark 9. *Note that the restriction to an exact solver in* (i) *is only to allow for a convenient presentation. As shown in [CFPP13, Section 7], an approximate solver can easily be integrated in the analysis.*

3.1. Mesh refinement.

For $d = 1$, the bisection algorithm from [AFF+13b] is used for mesh refinement. For $d = 2$ and $d = 3$, we use newest vertex bisection, see e.g. [KPP13] (for $d = 2$) and [Ste08] (for $d \geq 3$) as well as the references therein. Constrained by these refinement rules, we suppose that $\mathcal{T}_{\ell+1} = \texttt{refine}(\mathcal{T}_\ell, \mathcal{M}_\ell)$ is the coarsest conforming refinement of $\mathcal{T}_\ell$ such that all marked elements $T \in \mathcal{M}_\ell$ have been bisected.

The notation $\mathcal{T}_\star \in \texttt{refine}(\mathcal{T}_\ell)$ denotes that there exists a sequence of marked element sets $\widetilde{\mathcal{M}}_j$, $j = 1, \ldots, N-1$ and a sequence of intermediate meshes $\widetilde{\mathcal{T}}_j$, $j = 1, \ldots, N$ such that $\mathcal{T}_\ell = \widetilde{\mathcal{T}}_0$, $\mathcal{T}_\star = \widetilde{\mathcal{T}}_N$, and $\widetilde{\mathcal{T}}_{j+1} = \texttt{refine}(\widetilde{\mathcal{T}}_j; \widetilde{\mathcal{M}}_j)$ for all $j = 1, \ldots, N-1$. We suppose that each refinement $\mathcal{T}_\star \in \texttt{refine}(\mathcal{T}_0)$ induces a discrete space $\mathcal{X}_\star := \mathcal{X}(\mathcal{T}_\star) \subseteq \mathcal{X}$ such that $\mathcal{X}_\star \supseteq \mathcal{X}_\ell$ whenever $\mathcal{T}_\star \in \texttt{refine}(\mathcal{T}_\ell)$.

First, the choice of these mesh-refinement strategies guarantees that the meshes $\mathcal{T}_\ell$ generated by Algorithm 8 are uniformly γ-shape regular, where $\gamma > 0$ depends only on the initial mesh $\mathcal{T}_0$. For $d = 1$, this means

$$\frac{|T|}{|T'|} \leq \gamma \quad \text{for all neighbouring } T, T' \in \mathcal{T}_0, \tag{34a}$$

whereas for $d \geq 2$, γ-shape regularity is understood as

$$\frac{\mathrm{diam}(T)}{|T|^{1/d}} \leq \gamma \quad \text{for all } T \in \mathcal{T}_0. \tag{34b}$$

Here, $|\cdot|$ denotes the d-dimensional (surface) measure.

Second, it has first been observed in [BDD04] for 2D newest vertex bisection that the number $\#\mathcal{T}_\ell$ of elements in $\mathcal{T}_\ell$ can be controlled by the number of marked elements, i.e.

$$\#\mathcal{T}_\ell - \#\mathcal{T}_0 \leq C_{\mathrm{mesh}} \sum_{j=0}^{\ell-1} \#\mathcal{M}_j, \tag{35}$$

where $C_{\mathrm{mesh}} > 0$ depends only on $\mathcal{T}_0$. While [BDD04] requires an additional assumption on $\mathcal{T}_0$, this assumption has been removed in [KPP13], so that $\mathcal{T}_0$ is in fact an arbitrary conforming triangulation. For $d = 1$, the estimate (35) is proved in [AFF$^+$13b] for a bisection-based refinement, where additional bisections of non-marked elements are required to ensure uniform γ-shape regularity (34a). For $d \geq 3$, the result is proved in [Ste08], but requires an admissibility condition for the initial mesh $\mathcal{T}_0$.

Finally, for two triangulations $\mathcal{T}_\ell, \mathcal{T}_\star \in \mathtt{refine}(\mathcal{T}_0)$, let $\mathcal{T}_\ell \oplus \mathcal{T}_\star \in \mathtt{refine}(\mathcal{T}_\ell) \cap \mathtt{refine}(\mathcal{T}_\star)$ be the coarsest common refinement of both $\mathcal{T}_\ell$ and $\mathcal{T}_\star$. Then, $\mathcal{T}_\ell \oplus \mathcal{T}_\star$ is in fact the overlay, and it holds

$$\#(\mathcal{T}_\ell \oplus \mathcal{T}_\star) \leq \#\mathcal{T}_\ell + \#\mathcal{T}_\star - \#\mathcal{T}_0, \tag{36}$$

see [Ste07] for $d = 2$ and [CKNS08] for $d \geq 3$ resp. [AFF$^+$13b] for $d = 1$.

For any subset $\mathcal{S} \subseteq \mathcal{T}_\ell$ of a mesh $\mathcal{T}_\ell$, define the patch

$$\omega_\ell(\mathcal{S}) := \big\{ T \in \mathcal{T}_\ell \,:\, \text{exists } T' \in \mathcal{S} \text{ with } T \cap T' \neq \emptyset \big\}. \tag{37}$$

For simplicity, we write $\omega_\ell(T)$ instead of $\omega_\ell(\{T\})$ for single elements $T \in \mathcal{T}_\ell$.

Define the local mesh-size function $h_\ell \in L^\infty(\bigcup \mathcal{T}_\ell)$ by $h_\ell|_T = |T|^{1/d}$ for all $T \in \mathcal{T}_\ell$. By definition of the refinement rules, there exists $0 < q_{\mathrm{ref}} < 1$, which depends only on $d = 1, 2, 3$, such that

$$h_\star|_{T'} \leq q_{\mathrm{ref}} h_\ell|_T \quad \text{for all sons } T' \subsetneqq T, \quad T' \in \mathcal{T}_\star, T \in \mathcal{T}_\ell. \tag{38}$$

3.2. **Optimal convergence rates.** To quantify the quality of the convergence rate of Algorithm 8, we introduce for all $s > 0$ the approximability quasi-norm

$$\|\rho\|_{\mathbb{A}_s} := \sup_{N \in \mathbb{N}_0} \inf_{\substack{\mathcal{T}_\star \in \mathrm{refine}(\mathcal{T}_0) \\ \#\mathcal{T}_\star - \#\mathcal{T}_0 \leq N}} (N+1)^s \rho_\star.$$

The fact $\|\rho\|_{\mathbb{A}_s} < \infty$ for a particular $s > 0$ implies that the theoretically achievable convergence rate of Algorithm 8 is at least

$$\widetilde{\rho}_\ell \lesssim (\#\widetilde{\mathcal{T}}_\ell - \#\mathcal{T}_0)^{-s} \quad \text{for all } \ell \in \mathbb{N},$$

if the optimal meshes $\widetilde{\mathcal{T}}_\ell \in \mathtt{refine}(\mathcal{T}_0)$ are chosen. Theorem 14 below states that this is, in fact, asymptotically the case for Algorithm 8 if the following Assumptions 10–11 hold. Moreover, Theorem 14 states that the empirical convergence rate of Algorithm 8, in fact, characterises those $s > 0$ for which $\|\rho\|_{\mathbb{A}_s} < \infty$.

Assumption 10. *There holds* (i)–(iii)*:*

(i) *Any refinement $\mathcal{T}_\star \in \mathtt{refine}(\mathcal{T}_\ell)$ of $\mathcal{T}_\ell \in \mathtt{refine}(\mathcal{T}_0)$ satisfies*

$$\Big| \Big(\sum_{T \in \mathcal{T}_\star \cap \mathcal{T}_\ell} \rho_\star(T)^2 \Big)^{1/2} - \Big(\sum_{T \in \mathcal{T}_\ell \cap \mathcal{T}_\star} \rho_\ell(T)^2 \Big)^{1/2} \Big| \leq C_{\mathrm{stab}} \, \|U_\star - U_\ell\|_{\mathcal{X}}. \tag{39}$$

(ii) *Any refinement $\mathcal{T}_\star \in \mathtt{refine}(\mathcal{T}_\ell)$ of $\mathcal{T}_\ell \in \mathtt{refine}(\mathcal{T}_0)$ satisfies*

$$\sum_{T \in \mathcal{T}_\star \setminus \mathcal{T}_\ell} \rho_\star(T)^2 \leq q_{\mathrm{red}} \sum_{T \in \mathcal{T}_\ell \setminus \mathcal{T}_\star} \rho_\ell(T)^2 + C_{\mathrm{red}} \|U_\star - U_\ell\|_{\mathcal{X}}^2. \tag{40}$$

(iii) *Any refinement $\mathcal{T}_\star \in \texttt{refine}(\mathcal{T}_\ell)$ of $\mathcal{T}_\ell \in \texttt{refine}(\mathcal{T}_0)$ satisfies discrete reliability*

$$\|U_\star - U_\ell\|_{\mathcal{X}}^2 \le C_{\mathrm{Rel}}^2 \sum_{T \in \mathcal{R}(\mathcal{T}_\ell, \mathcal{T}_\star)} \rho_\ell(T)^2 \tag{41}$$

with the augmented set of refined elements

$$\mathcal{R}(\mathcal{T}_\ell, \mathcal{T}_\star) := \omega_\ell(\mathcal{T}_\ell \setminus \mathcal{T}_\star).$$

While the previous assumptions may not be necessary for the optimality proof (but turn out to be sufficient), the following assumption is even necessary for plain convergence of Algorithm 8.

Assumption 11. *Suppose that the given model problem converges under uniform refinement, i.e. for all $\varepsilon > 0$ exists a maximal mesh-width $h_\varepsilon > 0$ such that all meshes $\mathcal{T}_\star \in \texttt{refine}(\mathcal{T}_0)$ with $h_\star \le h_\varepsilon$ pointwise almost everywhere satisfy*

$$\|u - U_\star\|_{\mathcal{X}} \le \varepsilon$$

for the related Galerkin solution $U_\star$.

Remark 12. *Assumption 11 is usually verified via density arguments in combination with the Céa-type estimate* (4) *and a priori estimates of the type*

$$\|u - U_\ell\|_{\mathcal{X}} \lesssim \|u - v\|_{\mathcal{X}} + \min_{V_\ell \in \mathcal{X}_\ell} \|v - V_\ell\|_{\mathcal{X}} \lesssim \|u - v\|_{\mathcal{X}} + \max_{T \in \mathcal{T}_\ell} (h_\ell|_T)^\gamma$$

for some $\gamma > 0$ which depends on the regularity of the smooth approximation v.

Lemma 13. *Suppose that Assumptions 10 and Assumption 11 hold. Then, the error estimator ρ_ℓ is reliable*

$$\|u - U_\ell\|_{\mathcal{X}} \le C_{\mathrm{Rel}} \rho_\ell \quad \text{for all } \mathcal{T}_\ell \in \texttt{refine}(\mathcal{T}_0), \tag{42}$$

satisfies the estimator reduction

$$\rho_{\ell+1}^2 \le q_{\mathrm{est}} \rho_\ell^2 + C_{\mathrm{est}} \|U_{\ell+1} - U_\ell\|_{\mathcal{X}}^2 \quad \text{for all } \ell \in \mathbb{N}_0, \tag{43}$$

and there holds convergence

$$\|u - U_\ell\|_{\mathcal{X}} \le C_{\mathrm{Rel}} \rho_\ell \to 0 \quad \text{as } \ell \to \infty \tag{44}$$

as well as the general quasi-orthogonality (13)*. The constant $0 < q_{\mathrm{est}} < 1$ depends only on θ and q_{red}. The constant $C_{\mathrm{est}} > 0$ depends on θ, q_{red}, C_{red}, and C_{stab}.*

Proof. As shown in [CFPP13, Lemma 3.3], discrete reliability (41) together with Assumption 11 implies reliability (42). Stability (39), reduction (40), and Young's inequality $(a + b)^2 \le (1 + \delta)a^2 + (1 + \delta^{-1})b^2$ for all $a, b \in \mathbb{R}$ and $\delta > 0$ show with $C_{\mathrm{est}} := (C_{\mathrm{red}} + (1 + \delta^{-1})C_{\mathrm{stab}}^2)$

$$\begin{aligned}\rho_{\ell+1}^2 &\le q_{\mathrm{red}} \sum_{T \in \mathcal{T}_\ell \setminus \mathcal{T}_{\ell+1}} \rho_\ell^2 + (1 + \delta) \sum_{T \in \mathcal{T}_\ell \cap \mathcal{T}_{\ell+1}} \rho_\ell^2 + (C_{\mathrm{red}} + (1 + \delta^{-1})C_{\mathrm{stab}}^2) \|U_{\ell+1} - U_\ell\|_{\mathcal{X}}^2 \\ &\le (1 + \delta)\rho_\ell^2 + (q_{\mathrm{red}} - 1 - \delta) \sum_{T \in \mathcal{T}_\ell \setminus \mathcal{T}_{\ell+1}} \rho_\ell^2 + C_{\mathrm{est}} \|U_{\ell+1} - U_\ell\|_{\mathcal{X}}^2.\end{aligned}$$

Dörfler marking (6) and $\mathcal{M}_\ell \subseteq \mathcal{T}_\ell \setminus \mathcal{T}_{\ell+1}$ imply

$$\rho_{\ell+1}^2 \le (1 + \delta - \theta(1 + \delta - q_{\mathrm{red}}))\rho_\ell^2 + C_{\mathrm{est}} \|U_{\ell+1} - U_\ell\|_{\mathcal{X}}^2.$$

Sufficiently small $\delta > 0$ shows $0 < q_{\mathrm{est}} := 1 + \delta - \theta(1 + \delta - q_{\mathrm{red}}) < 1$. This proves (43). The Céa lemma (4) together with nestedness $\mathcal{X}_\ell \subseteq \mathcal{X}_{\ell+1}$ and elementary calculus (as laid out, e.g., in [CFPP13, Section 3.6]) shows $\lim_{\ell\to\infty} \rho_\ell = 0$, and therefore reliability (42) implies (44).

With convergence (44), the claims of Theorem 1 are satisfied. This implies the general quasi-orthogonality (13) and concludes the proof. $\square$

By means of the general quasi-orthogonality (13), the convergence result (44) can further be improved to obtain R-linear convergence of the estimator (45). However, as the validity of (13) depends on the much weaker plain convergence of the estimator (44), it seems that (44) is a necessary intermediate result.

Theorem 14. *Suppose that Assumptions 10–11 hold. Then, for all $0 < \theta \leq 1$, there exist constants $C_{\mathrm{R}} > 0$ and $0 < q_{\mathrm{R}} < 1$ such that there holds R-linear convergence of Algorithm 8 in the sense*

$$\rho_{\ell+k}^2 \leq C_{\mathrm{R}}\, q_{\mathrm{R}}^k \rho_\ell^2 \quad \textit{for all } \ell, k \in \mathbb{N}. \tag{45}$$

For $\theta < \theta_{\mathrm{opt}} := (1 + C_{\mathrm{stab}}^2 C_{\mathrm{Rel}}^2)^{-1}$, Algorithm 8 converges with the optimal rate in the sense of

$$c_{\mathrm{opt}} \|\rho\|_{\mathbb{A}_s} \leq \sup_{\ell\in\mathbb{N}_0} \rho_\ell (|\mathcal{T}_\ell| - |\mathcal{T}_0| + 1)^s \leq C_{\mathrm{opt}} \|\rho\|_{\mathbb{A}_s} \tag{46}$$

for all $s > 0$. The constants $C_{\mathrm{opt}}, C_{\mathrm{R}}, q_{\mathrm{R}} > 0$ depend only on θ and on the constants from Assumption 10, whereas C_{opt} depends additionally on s. The constant $c_{\mathrm{opt}} > 0$ depends only on d.

The proof of the above main result is postponed to Section 8 and utilizes only the quantities and estimates of Assumptions 10–11 and the general quasi-orthogonality (13). Therefore, it is sufficient to prove each of the assumptions separately for the concrete applications in Section 5–7.

4. Application 1: Fem for linear second-order PDEs

4.1. **Model Problem and definitions.** We consider the model problem from [FFP12]. Given a Lipschitz domain $\Omega \subseteq \mathbb{R}^d$ for $d = 2, 3$, this section considers the second-order elliptic PDE

$$\begin{aligned} \mathcal{L}u &= f_0 \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega \end{aligned} \tag{47}$$

with the non-symmetric linear operator

$$\mathcal{L}u := -\mathrm{div}(\boldsymbol{A}\nabla u) + \boldsymbol{b}\cdot\nabla u + cu. \tag{48}$$

The given right-hand side satisfies $f_0 \in L^2(\Omega)$. Moreover, $\boldsymbol{A} = \boldsymbol{A}(x) \in \mathbb{R}^{d\times d}$ with $\boldsymbol{A} \in \big(W_1^\infty(\Omega)\big)^{d\times d}$ is a symmetric matrix, $\boldsymbol{b} = \boldsymbol{b}(x) \in \mathbb{R}^d$ with $\boldsymbol{b} \in \big(L^\infty(\Omega)\big)^d$ is a vector, and $c = c(x) \in \mathbb{R}$ with $c \in L^\infty(\Omega)$ is a scalar. Here, $W_1^\infty(\Omega)$ denotes the space of Lipschitz continuous functions. This allows to write down the weak formulation of (47) as follows: Find $u \in \mathcal{X} := H_0^1(\Omega) := \big\{v \in H^1(\Omega) \,:\, v|_\Gamma = 0 \text{ in the sense of traces}\big\}$ such that

$$b(u, v) := \int_\Omega \boldsymbol{A}\nabla u\cdot\nabla v + \boldsymbol{b}\cdot\nabla u\, v + cuv\, dx = f(v) \quad \text{for all } v \in H_0^1(\Omega) \tag{49}$$

with $f(v) := \int_\Omega f_0 v\, dx$. The assumptions on the given data guarantee the boundedness of $b(\cdot,\cdot)$ in the natural norm $\|\cdot\|_{\mathcal{X}} := \|\nabla(\cdot)\|_{L^2(\Omega)}$. Moreover, we suppose the coefficients to be chosen such that $b(\cdot,\cdot)$ is also elliptic in the sense

$$b(v, v) \geq C_{\mathrm{ell}} \|v\|_{\mathcal{X}}^2 \quad \text{for all } v \in \mathcal{X}.$$

Given an admissible triangulation $\mathcal{T}_\star \in \texttt{refine}(\mathcal{T}_0)$ and a polynomial degree $p \geq 1$, define

$$\mathcal{S}^p(\mathcal{T}_\star) := \big\{V_\star \in C(\Omega) \,:\, V_\star|_T \text{ is polynomial of degree } \leq p \text{ for all } T \in \mathcal{T}_\star\big\}.$$

The discrete form of (49) is given by (3), where uniqueness and solvability are guaranteed by the Lax-Milgram lemma. The corresponding weighted-residual error estimator reads

$$\rho_\star^2 := \sum_{T\in\mathcal{T}_\star} \rho_\star(T)^2 \text{ with } \rho_\star(T)^2 := |T|^{2/d}\|\mathcal{L}|_T U_\star - f\|_{L^2(T)}^2 + |T|^{1/d}\|[\boldsymbol{A}\nabla U_\star\cdot n]\|_{L^2(\partial T\cap\Omega)}^2.$$

Here, $[\cdot]$ denotes the jump over ∂T, and n denotes the outer unit normal vector of each element $T \in \mathcal{T}_\star$.

4.2. **Proof of Assumptions 10–11.**

Proof of Assumptions 10. The proof of (i)–(ii) is part of the proof of [FFP12, Lemma 2] or in different notation also in [CKNS08, Corollary 3.4]. The discrete reliability (iii) follows as for the symmetric case $\boldsymbol{b} = 0$ with $\mathcal{R}(\mathcal{T}_\ell, \mathcal{T}_\star) = \mathcal{T}_\ell \setminus \mathcal{T}_\star$, see e.g. [CKNS08, Lemma 3.6]. The constants $C_{\rm stab}, C_{\rm red}, q_{\rm red}, C_{\rm Rel}$ depend only on the domain Ω, the coefficients of $\mathcal{L}$ the γ-shape regularity of $\mathcal{T}_\ell$ and $\mathcal{T}_\star$ as well as on the polynomial degree p. □

Proof of Assumptions 11. Let $\varepsilon > 0$. By density, find a function $v \in C^\infty(\overline{\Omega}) \cap H^1_0(\Omega)$ such that

$$\|u - v\|_{\mathcal{X}} \leq \varepsilon.$$

The Céa lemma (4) together with the standard nodal approximation result proves

$$\|u - U_\star\|_{\mathcal{X}} \lesssim \|u - v\|_{\mathcal{X}} + \min_{V_\star\in\mathcal{X}_\star} \|v - V_\star\|_{\mathcal{X}} \lesssim \varepsilon + \|h_\star\|_{L^\infty(\Omega)}\|D^2 v\|_{L^2(\Omega)} \leq 2\varepsilon$$

for sufficiently fine meshes $\mathcal{T}_\star \in \texttt{refine}(\mathcal{T}_0)$. □

Consequence 15. *The adaptive finite element discretization of* (47) *in Algorithm 8 converges with optimal rates in the sense of Theorem 14.*

5. Application 2: Non-symmetric Johnson-Nédélec FEM/BEM-coupling

5.1. **Model problem and definitions.** Given a Lipschitz domain $\Omega \subseteq \mathbb{R}^d$ for $d \in \{2, 3\}$, this section considers a FEM-BEM reformulation of the linear Laplace-type transmission problem

$$\begin{aligned} -\mathrm{div}(\boldsymbol{A}\nabla u^{\rm int}) &= f_0 && \text{in } \Omega,\\ -\Delta u^{\rm ext} &= 0 && \text{in } \mathbb{R}^d \setminus \overline{\Omega},\\ u^{\rm int} - u^{\rm ext} &= u_0 && \text{on } \Gamma := \partial\Omega,\\ (\boldsymbol{A}\nabla u^{\rm int})\cdot n - \partial_n u^{\rm ext} &= \phi_0 && \text{on } \Gamma, \end{aligned} \tag{50a}$$

where $f_0 \in L^2(\Omega)$, $u_0 \in H^1(\Gamma)$, $\phi_0 \in L^2(\Gamma)$, $\boldsymbol{A} \in W^{1,\infty}(\Omega)$ with $\boldsymbol{A}(x) \in \mathbb{R}^{d\times d}$ being a symmetric matrix, and the normal derivative $\partial_n(\cdot)$ is understood with respect to the outer unit normal vector n on Γ. Let $0 < c_{\mathfrak{K}} < 1$ denote the contraction constant of the double layer potential $\mathfrak{K}$ defined below, see e.g. [Ste11]. We suppose that there exist $c_{\mathfrak{K}}/4 < \lambda_{\min} \leq \lambda_{\max} < \infty$ such that there holds $\lambda_{\min} \leq \boldsymbol{A}(x) \leq \lambda_{\max}$ for all $x \in \Omega$ in the sense that the eigenvalues of $\boldsymbol{A}(x)$ are bounded from below and from above. The uniqueness of the solution is guaranteed via the following radiation condition

$$u^{\rm ext} = \mathcal{O}(|x|^{-1}) \quad \text{as } |x| \to \infty, \tag{50b}$$

which requires the compatibility condition $\int_\Omega f\,dx + \int_\Gamma \phi_0\,ds = 0$ for $d = 2$. Moreover, we assume $\mathrm{diam}(\Omega) < 1$ for $d = 2$.

The presence of the unbounded domain $\mathbb{R}^d \setminus \overline{\Omega}$ motivates the use of boundary elements for the exterior problem. One possible formulation is known as the one-equation coupling of Johnson and Nédélec [JN80] and employs the simple-layer integral operator $\mathfrak{V} : H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma)$ as well as the double layer integral operator $\mathfrak{K} : H^{1/2}(\Gamma) \to H^{1/2}(\Gamma)$ which are formally defined via the Newton kernel

$$G(z) := \begin{cases} -\frac{1}{2\pi} \log |z|, & d = 2, \\ \frac{1}{4\pi}|z|^{-1}, & d = 3, \end{cases} \tag{51a}$$

as

$$(\mathfrak{V}\phi)(x) := \int_\Gamma G(x-y)\phi(y)\, dy \quad \text{and} \quad (\mathfrak{K}g)(x) := \int_\Gamma \partial_{n(y)} G(x-y) g(y)\, dy \tag{51b}$$

for all $x \in \Gamma$. With these operators, (50) is equivalently reformulated as follows: Find $u := (u^{\rm int}, \phi) \in \mathcal{X} := H^1(\Omega) \times H^{-1/2}(\Gamma)$ such that

$$b(u, v) = \widetilde{f}(v) \quad \text{for all } v := (v^{\rm int}, \psi) \in \mathcal{X}, \tag{52}$$

where the bilinear form reads

$$b(u, v) := \langle \boldsymbol{A}\nabla u^{\rm int} \,,\, \nabla v^{\rm int} \rangle_{L^2(\Omega)} + \langle \phi \,,\, v^{\rm int} \rangle_{L^2(\Gamma)} + \langle \psi \,,\, (\tfrac{1}{2} - \mathfrak{K}) u^{\rm int} \rangle_{L^2(\Gamma)} + \langle \psi \,,\, \mathfrak{V}\phi \rangle_{L^2(\Gamma)}$$

and the right-hand side is defined by

$$f(v) := \langle f_0 \,,\, v^{\rm int} \rangle_{L^2(\Omega)} + \langle \phi_0 \,,\, v^{\rm int} \rangle_{L^2(\Gamma)} + \langle \psi \,,\, (\tfrac{1}{2} - \mathfrak{K}) u_0 \rangle_{L^2(\Gamma)}.$$

The two formulations (50) and (52) are linked as follows: Given $(u^{\rm int}, u^{\rm ext})$, there holds $\partial_n u^{\rm ext} = \phi$. Given $u = (u^{\rm int}, \phi)$, the exterior solution is available via the representation formula $u^{\rm ext} := \widetilde{\mathfrak{K}} u^{\rm int} - \widetilde{\mathfrak{V}}\phi$. Here, $\widetilde{\mathfrak{V}}$ and $\widetilde{\mathfrak{K}}$ denote the integral operators from (51b), but are now evaluated in $\mathbb{R}^d \setminus \overline{\Omega}$.

We consider the natural norm on the product space $\mathcal{X} = H^1(\Omega) \times H^{-1/2}(\Gamma)$

$$\|v\|_{\mathcal{X}}^2 := \|v^{\rm int}\|_{H^1(\Omega)}^2 + \|\psi\|_{H^{-1/2}(\Gamma)}^2 \quad \text{for all } v = (v^{\rm int}, \psi) \in \mathcal{X}.$$

Given an admissible triangulation $\mathcal{T}_\star \in \texttt{refine}(\mathcal{T}_0)$ of Ω, define

$$\mathcal{P}^{p-1}(\mathcal{T}_\star|_\Gamma) := \big\{ \Psi_\star \in L^2(\Gamma) \,:\, \Psi_\star|_{T\cap\Gamma} \text{ polynomial of degree } \le p-1 \text{ for all } T \in \mathcal{T}_\star \big\}.$$

With this, the discrete analogue of (52) reads: Find $U_\star := (U_\star^{\rm int}, \Phi_\star) \in \mathcal{X}_\star := \mathcal{S}^p(\mathcal{T}_\star) \times \mathcal{P}^{p-1}(\mathcal{T}_\star|_\Gamma)$ such that

$$b(U_\star, V_\star) = f(V_\star) \quad \text{for all } V_\star := (V_\star^{\rm int}, \Psi_\star) \in \mathcal{X}_\star. \tag{53}$$

Note that $b((1,0),(1,0)) = 0$, so that $b(\cdot,\cdot)$ is not elliptic and unique solvability is thus not obvious. The solvability of (52) and (53) were firstly proved for smooth boundaries Γ in [JN80]. For polyhedral boundaries, the case was open until [Say09]. Inspired by [Ste11, Say09], the recent work [AFF$^+$13a] uses a novel technique to generalize the available results and prove unique solvability even for strongly monotone operators. The residual-based error estimator (see e.g. [AFP12, AFF$^+$13a] for the derivation) reads elementwise for all $T \in \mathcal{T}_\star$

$$\begin{aligned} \rho_\star(T)^2 := {}& \operatorname{diam}(T)^2 \|f_0 + \operatorname{div}(\boldsymbol{A}\nabla U_\star^{\rm int})\|_{L^2(T)}^2 + \operatorname{diam}(T) \|[(\boldsymbol{A}\nabla U_\star^{\rm int}) \cdot n]\|_{L^2(\partial T\cap\Omega)}^2 \\ & + \operatorname{diam}(T) \|\phi_0 + \Phi_\star - (\boldsymbol{A}\nabla U_\star^{\rm int}) \cdot n\|_{L^2(\partial T\cap\Gamma)}^2 \\ & + \operatorname{diam}(T) \|\nabla_\Gamma((\tfrac{1}{2} - \mathfrak{K})(u_0 - U_\star^{\rm int}) - \mathfrak{V}\Phi_\star)\|_{L^2(\partial T\cap\Gamma)}^2, \end{aligned}$$

where $[\cdot]$ denotes the jump over interior edges of $\mathcal{T}_\star$ and n is the outer unit normal vector of each element $T \in \mathcal{T}_\star$. Note that the exterior problem affects the estimator only on elements $T \in \mathcal{T}_\star$ with $T \cap \Gamma \neq \emptyset$. The overall estimator reads

$$\rho_\star := \Big(\sum_{T \in \mathcal{T}_\star} \rho_\star(T)^2 \Big)^{1/2} \quad \text{for all } \mathcal{T}_\star \in \texttt{refine}(\mathcal{T}_0).$$

5.2. **Proof of Assumptions 10–11.** The method of *implicit stabilization* from [AFF$^+$13a] introduces an equivalent problem

$$\begin{aligned} \widetilde{b}(u,v) &:= b(u,v) + \langle 1\,,\, (\tfrac{1}{2} - \mathfrak{K})u^{\rm int} + \mathfrak{V}\phi \rangle_{L^2(\Gamma)} \langle 1\,,\, (\tfrac{1}{2} - \mathfrak{K})v^{\rm int} + \mathfrak{V}\psi \rangle_{L^2(\Gamma)}, \\ \widetilde{f}(v) &:= f(v) + \langle 1\,,\, (\tfrac{1}{2} - \mathfrak{K})u_0 \rangle_{L^2(\Gamma)}. \end{aligned} \tag{54}$$

The bilinear form $\widetilde{b}(\cdot,\cdot)$ is elliptic (see [AFF$^+$13a, Theorem 14] for a proof) in the sense of

$$\widetilde{b}(v,v) \geq C_2^{-1} \|v\|_{\mathcal{X}}^2 \quad \text{for all } v = (v^{\rm int}, \psi) \in \mathcal{X}, \tag{55}$$

and the solutions $u, U_\star$ of (52) and (53) satisfy for all $\mathcal{T}_\star \in \texttt{refine}(\mathcal{T}_0)$

$$\widetilde{b}(u,v) = \widetilde{f}(v) \quad \text{for all } v \in \mathcal{X} \qquad \text{and} \qquad \widetilde{b}(U_\star, V_\star) = \widetilde{f}(V_\star) \quad \text{for all } V_\star \in \mathcal{X}_\star.$$

Thus, (54) serves as an equivalent reformulation of (52) with the additional property that $\widetilde{b}(\cdot,\cdot)$ is strongly elliptic and hence fits in the frame of the Lax-Milgram lemma outlined in the introduction of Section 1. Hence, we prove Assumption 10 for the equivalent formulation (54).

Proof of Assumption 10 (i)–(ii). The statements (i) and (ii) are part of the proof of [AFF$^+$13a, Theorem 25] and follow from the triangle inequality and local inverse estimates for the non-local operators $\mathfrak{V}$ and $\mathfrak{K}$ from [AFF$^+$12]. The constants $C_{\rm stab}, C_{\rm red}, q_{\rm red}$ depend only on Γ, the polynomial degree p, the γ-shape regularity of $\mathcal{T}_\ell$ and $\mathcal{T}_\star$, and on $\boldsymbol{A}$. □

Proof of Assumption 10 (iii). The proof is essentially the combination of the corresponding proofs for FEM in [Ste07, CKNS08] and BEM in [FKMP13]. There holds with ellipticity [AFF$^+$13a, Theorem 14] and $V_\star = (V_\star^{\rm int}, \Psi_\star) := U_\star - U_\ell \in \mathcal{X}_\star$

$$\begin{aligned} \|U_\star - U_\ell\|_{\mathcal{X}}^2 &\lesssim \widetilde{b}(U_\star - U_\ell, V_\star) = b(U_\star - U_\ell, V_\star) \\ &= f(V_\star - V_\ell) - b(U_\ell, V_\star - V_\ell) \quad \text{for all } V_\ell \in \mathcal{X}_\ell. \end{aligned}$$

Recall the Scott-Zhang operator $J_\ell : H^1(\Omega) \to \mathcal{S}^p(\mathcal{T}_\ell)$ from [SZ90] as well as the $L^2(\Gamma)$-orthogonal projection $\Pi_\ell : L^2(\Gamma) \to \mathcal{P}^{p-1}(\mathcal{T}_\ell|_\Gamma)$. With this, define

$$V_\ell := (J_\ell V_\star^{\rm int}, \Pi_\ell \Psi_\star) \in \mathcal{X}_\ell.$$

This implies

$$\begin{aligned} \|U_\star - U_\ell\|_{\mathcal{X}}^2 \lesssim\ & \langle f_0\,,\, (1 - J_\ell) V_\star^{\rm int} \rangle_{L^2(\Omega)} - \langle \boldsymbol{A} \nabla U_\ell^{\rm int}\,,\, \nabla (1 - J_\ell) V_\star^{\rm int} \rangle_{L^2(\Omega)} \\ &+ \langle \phi_0 - \Phi_\ell\,,\, (1 - J_\ell) V_\star^{\rm int} \rangle_{L^2(\Gamma)} \\ &+ \langle (1 - \Pi_\ell)\Psi_\star\,,\, (\tfrac{1}{2} - \mathfrak{K})(u_0 - U_\ell^{\rm int}) - \mathfrak{V}\Phi_\ell \rangle_{L^2(\Gamma)}. \end{aligned} \tag{56}$$

$\mathcal{T}_\ell$-piecewise integration by parts shows

$$
\begin{aligned}
&\langle f_0\,,\,(1-J_\ell)V_\star^{\rm int}\rangle_{L^2(\Omega)} - \langle \boldsymbol{A}\nabla U_\ell^{\rm int}\,,\,\nabla(1-J_\ell)V_\star^{\rm int}\rangle_{L^2(\Omega)} + \langle \phi_0-\Phi_\ell\,,\,(1-J_\ell)V_\star^{\rm int}\rangle_{L^2(\Gamma)}\\
&\quad\lesssim \sum_{T\in\mathcal{T}_\ell}\|f_0+\operatorname{div}(\boldsymbol{A}\nabla U_\ell^{\rm int})\|_{L^2(T)}\|(1-J_\ell)V_\star^{\rm int}\|_{L^2(T)}\\
&\qquad+\sum_{T\in\mathcal{T}_\ell}\Big(\|[(\boldsymbol{A}\nabla U_\ell^{\rm int})\cdot n]\|_{L^2(\partial T\cap\Omega)}\\
&\qquad\quad+\|\phi_0-\Phi_\ell-(\boldsymbol{A}\nabla U_\ell^{\rm int})\cdot n\|_{L^2(\partial T\cap\Gamma)}\Big)\|(1-J_\ell)V_\star^{\rm int}\|_{H^{1/2}(T)}.
\end{aligned}
$$

Since all $T\in\mathcal{T}_\ell$ with $T\notin\omega_\ell(\mathcal{T}_\ell\setminus\mathcal{T}_\star)$ satisfy $((1-J_\ell)V_\star^{\rm int})|_T=0$ (This is can be improved to $\mathcal{T}_\ell\cap\mathcal{T}_\star$ as shown in [CKNS08, FFK+13b]. However, the improvement is not necessary here and therefore omitted.) and by use of the first-order approximation properties of J_ℓ, the above estimate implies

$$
\begin{aligned}
&\langle f_0\,,\,(1-J_\ell)V_\star^{\rm int}\rangle_{L^2(\Omega)} - \langle \boldsymbol{A}\nabla U_\ell^{\rm int}\,,\,\nabla(1-J_\ell)V_\star^{\rm int}\rangle_{L^2(\Omega)} + \langle \phi_0-\Phi_\ell\,,\,(1-J_\ell)V_\star^{\rm int}\rangle_{L^2(\Gamma)}\\
&\quad\lesssim \sum_{T\in\omega_\ell(\mathcal{T}_\ell\setminus\mathcal{T}_\star)}\Big(\operatorname{diam}(T)\|f_0+\operatorname{div}(\boldsymbol{A}\nabla U_\ell^{\rm int})\|_{L^2(T)}+\operatorname{diam}(T)^{1/2}\|[(\boldsymbol{A}\nabla U_\ell^{\rm int})\cdot n]\|_{L^2(\partial T\cap\Omega)}\\
&\qquad+\operatorname{diam}(T)^{1/2}\|\phi_0-\Phi_\ell-(\boldsymbol{A}\nabla U_\ell^{\rm int})\cdot n\|_{L^2(\partial T\cap\Gamma)}\Big)\|\nabla V_\star^{\rm int}\|_{L^2(T)},
\end{aligned}\tag{57}
$$

where the hidden constant depends only on γ-shape regularity of $\mathcal{T}_\ell$ and Ω. Consider a partition of unity of Γ in the sense

$$\sum_{\substack{z\in\Gamma\\ z\text{ node of }\mathcal{T}_\ell}}\xi_z=1\quad\text{on }\Gamma$$

with the nodal hat functions $\xi_z\in\mathcal{P}^1(\mathcal{T}_\ell|_\Gamma)\cap C(\Gamma)$ which satisfy $\xi_z(z')=\delta_{z,z'}$ for all boundary nodes $z'\in\Gamma$ of $\mathcal{T}_\ell$ with Kronecker's $\delta_{z,z'}$. Since $(1-\Pi_\ell)\Psi_\star=0$ on $\mathcal{T}_\ell\cap\mathcal{T}_\star$, the last term on the right-hand side of (56) satisfies

$$
\begin{aligned}
&\langle(1-\Pi_\ell)\Psi_\star\,,(\tfrac12-\mathfrak{K})(u_0-U_\ell^{\rm int})-\mathfrak{V}\Phi_\ell\rangle_{L^2(\Gamma)}\\
&\qquad=\langle(1-\Pi_\ell)\Psi_\star\,,\sum_{\substack{z\in\bigcup(\mathcal{T}_\ell\setminus\mathcal{T}_\star)\cap\Gamma\\ z\text{ node of }\mathcal{T}_\ell}}\xi_z\big((\tfrac12-\mathfrak{K})(u_0-U_\ell^{\rm int})-\mathfrak{V}\Phi_\ell\big)\rangle_{L^2(\Gamma)}.
\end{aligned}
$$

The fact $\langle 1\,,\,(\frac12-\mathfrak{K})(u_0-U_\ell^{\rm int})-\mathfrak{V}\Phi_\ell\rangle_{L^2(T\cap\Gamma)}=0$ for all $T\in\mathcal{T}_\ell$ allows to follow the arguments of the proof of [FKMP13, Proposition 5.3] resp. [FFK+13a, Proposition 4]. This shows

$$
\begin{aligned}
&\langle(1-\Pi_\ell)\Psi_\star\,,\,(\tfrac12-\mathfrak{K})(u_0-U_\ell^{\rm int})-\mathfrak{V}\Phi_\ell\rangle_{L^2(\Gamma)}\\
&\quad\lesssim\Big(\sum_{T\in\omega_\ell(\mathcal{T}_\ell\setminus\mathcal{T}_\star)}\operatorname{diam}(T)^{1/2}\|\nabla_\Gamma\big((\tfrac12-\mathfrak{K})(u_0-U_\ell^{\rm int})-\mathfrak{V}\Phi_\ell\big)\|_{L^2(T\cap\Gamma)}\Big)\|\Psi_\star\|_{H^{-1/2}(\Gamma)}.
\end{aligned}\tag{58}
$$

The combination of (57)–(58) with (56) concludes the proof of the discrete reliability (41). The constant $C_{\rm Rel}$ depends only on Γ, the coefficient matrix $\boldsymbol{A}$, the γ-shape regularity of $\mathcal{T}_\ell$ and $\mathcal{T}_\star$, and the polynomial degree p. □

Proof of Assumption 11. The proof follows with the same arguments as in Section 4. □

Consequence 16. *Algorithm 8 for the Johnson-Nédélec formulation of FEM-BEM coupling converges with optimal rates in the sense of Theorem 14.*

6. Application 3. Symmetric FEM/BEM-coupling

6.1. **Model problem and definitions.** As in the previous section, we consider the transmission problem (50). To state the so-called *symmetric* coupling, define the hypersingular integral operator $\mathfrak{W} : H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$ formally as

$$\mathfrak{W}\phi(x) := -\partial_{n(x)} \int_\Gamma \partial_{n(y)} G(x-y)\phi(y)\, dy.$$

With this, (50) reformulates as: Find $u := (u^{\rm int}, \phi) \in \mathcal{X} := H^1(\Omega) \times H^{-1/2}(\Gamma)$ such that

$$b(u, v) = f(v) \quad \text{for all } v := (v^{\rm int}, \psi) \in \mathcal{X}, \tag{59}$$

where the bilinear form reads

$$\begin{aligned} b(u, v) := {}& \langle \boldsymbol{A}\nabla u^{\rm int} \, , \, \nabla v^{\rm int} \rangle_{L^2(\Omega)} + \langle (\mathfrak{K}' - \tfrac{1}{2})\phi \, , \, v^{\rm int} \rangle_{L^2(\Gamma)} + \langle \mathfrak{W} u^{\rm int} \, , \, v^{\rm int} \rangle_{L^2(\Gamma)} \\ & + \langle \psi \, , \, (\tfrac{1}{2} - \mathfrak{K}) u^{\rm int} \rangle_{L^2(\Gamma)} + \langle \psi \, , \, \mathfrak{V}\phi \rangle_{L^2(\Gamma)} \end{aligned}$$

and the right-hand side is defined by

$$f(v) := \langle f_0 \, , \, v^{\rm int} \rangle_{L^2(\Omega)} + \langle \phi_0 + \mathfrak{W} u_0 \, , \, v^{\rm int} \rangle_{L^2(\Gamma)} + \langle \phi_0 \, , \, v^{\rm int} \rangle_{L^2(\Gamma)} + \langle \psi \, , \, (\tfrac{1}{2} - \mathfrak{K}) u_0 \rangle_{L^2(\Gamma)}.$$

The two formulations (50) and (59) are linked as for the Johnson-Nédélec coupling from Section 5, and the space $\mathcal{X}$ is equipped with the same norm.

The discretization of (59) is straightforward. Given a triangulation $\mathcal{T}_\star \in \texttt{refine}(\mathcal{T}_0)$ and a polynomial degree $p \geq 1$: Find $U_\star := (U_\star^{\rm int}, \Phi_\star) \in \mathcal{X}_\star := \mathcal{S}^p(\mathcal{T}_\star) \times \mathcal{P}^{p-1}(\mathcal{T}_\star|_\Gamma)$

$$b(U_\star, V_\star) = f(V_\star) \quad \text{for all } V_\star := (V_\star^{\rm int}, \Psi_\star) \in \mathcal{X}_\star. \tag{60}$$

The solvability of (59) and (60) were firstly proved in [Cos88]. The recent work [AFF+13a] uses the *implicit stabilization* technique to give a much simplified version of the proof. The residual-based error estimator (see e.g. [CS95, AFF+13a] for a reliability proof) reads elementwise for all $T \in \mathcal{T}_\star$

$$\begin{aligned} \rho_\star(T)^2 := {}& \operatorname{diam}(T)^2 \| f + \operatorname{div}(\boldsymbol{A}\nabla U_\star^{\rm int}) \|_{L^2(T)}^2 + \operatorname{diam}(T) \| [(\boldsymbol{A}\nabla U_\star^{\rm int}) \cdot n] \|_{L^2(\partial T \cap \Omega)}^2 \\ & + \operatorname{diam}(T) \| \phi_0 - (\mathfrak{K}' - \tfrac{1}{2})\Phi_\star - \mathfrak{W}(U_\star^{\rm int} - u_0) - (\boldsymbol{A}\nabla U_\star^{\rm int}) \cdot n \|_{L^2(\partial T \cap \Gamma)}^2 \\ & + \operatorname{diam}(T) \| \nabla_\Gamma ((\tfrac{1}{2} - \mathfrak{K})(u_0 - U_\star^{\rm int}) - \mathfrak{V}\Phi_\star) \|_{L^2(\partial T \cap \Gamma)}^2. \end{aligned}$$

and hence

$$\rho_\star := \Big(\sum_{T \in \mathcal{T}_\star} \rho_\star(T)^2 \Big)^{1/2} \quad \text{for all } \mathcal{T}_\star \in \texttt{refine}(\mathcal{T}_0).$$

As above $[\cdot]$ denotes the jump over interior edges of $\mathcal{T}_\star$, and n is the outer unit normal vector of each element $T \in \mathcal{T}_\star$.

6.2. **Proof of Assumption 10–11.** The proof of Assumption 10–11 is very similar to that of the previous section and therefore omitted.

Consequence 17. *Algorithm 8 for the symmetric formulation of FEM-BEM coupling converges with optimal rates in the sense of Theorem 14.*

7. Application 4: BEM for mixed boundary value problems

7.1. **Model problem and definitions.** Given a Lipschitz domain $\Omega \subset \mathbb{R}^2$ and relatively open, disjoint boundary parts $\overline{\Gamma_D} \cup \overline{\Gamma_N} = \partial\Omega$, consider the mixed boundary value problem

$$\begin{aligned} -\Delta w &= 0 && \text{in } \Omega, \\ w &= u_D && \text{on } \Gamma_D, \\ \partial_n w &= \phi_N && \text{on } \Gamma_N, \end{aligned} \tag{61}$$

where $(u_D, \phi_N) \in H^1(\Gamma_D) \times L^2(\Gamma_N)$ are given boundary data and the sought solution satisfies $u \in H^1(\Omega)$. The Dirichlet boundary is non-trivial $|\Gamma_D| > 0$ and for $d = 2$, the domain satisfies $\operatorname{diam}(\Omega) < 1$. For the boundary integral formulation of (61), let $u_D \in H^{1/2}(\Gamma)$ and $\phi_N \in H^{-1/2}(\Gamma)$ be arbitrary extensions of the given data from Γ_D resp. Γ_N to the whole boundary $\partial\Omega$. Define the spaces $\widetilde{H}^{1/2}(\Gamma_N) := \big\{ v \in H^{1/2}(\partial\Omega) \,:\, \operatorname{supp}(v) \subseteq \Gamma_N \big\}$ and $\widetilde{H}^{-1/2}(\Gamma_D) := H^{1/2}(\Gamma_D)^*$. With this, the so-called *symmetric* formulation reads: Find $u := (u_N, \phi_D) \in \mathcal{X} := \widetilde{H}^{1/2}(\Gamma_N) \times \widetilde{H}^{-1/2}(\Gamma_D)$ such that

$$\begin{pmatrix} \mathfrak{W}_{NN} & \mathfrak{K}'_{DN} \\ -\mathfrak{K}_{ND} & \mathfrak{V}_{DD} \end{pmatrix} \begin{pmatrix} u_N \\ \phi_D \end{pmatrix} = \begin{pmatrix} -\mathfrak{W} & 1/2 - \mathfrak{K}' \\ 1/2 + \mathfrak{K} & -\mathfrak{V} \end{pmatrix} \begin{pmatrix} u_D \\ \phi_N \end{pmatrix}, \tag{62}$$

with $\mathfrak{V}_{DD} : \widetilde{H}^{-1/2}(\Gamma_D) \to H^{1/2}(\Gamma_D)$, $\mathfrak{K}_{ND} : \widetilde{H}^{1/2}(\Gamma_N) \to H^{1/2}(\Gamma_D)$, $\mathfrak{K}'_{DN} : \widetilde{H}^{-1/2}(\Gamma_D) \to H^{-1/2}(\Gamma_N)$, and $\mathfrak{W}_{NN} : \widetilde{H}^{1/2}(\Gamma_N) \to H^{-1/2}(\Gamma_N)$ denoting the boundary integral operators $\mathfrak{V}, \mathfrak{K}, \mathfrak{W}$ and the adjoint $\mathfrak{K}' \in L(H^{-1/2}(\partial\Omega), H^{-1/2}(\partial\Omega))$ restricted to the respective boundary parts $\Gamma_D, \Gamma_N \subseteq \Gamma$. The formulations (61) and (62) are linked as follows: Given the solution $w \in H^1(\Omega)$ from (61), $u_N := w - u_D \in \widetilde{H}^{1/2}(\Gamma_N)$ and $\phi_D := \partial_n w - \phi_N \in \widetilde{H}^{-1/2}(\Gamma_D)$ solve (62). Given the solution $u = (u_N, \phi_D) \in \mathcal{X}$ from (62), the representation formula provides a solution of (61), i.e.

$$w = \widetilde{\mathfrak{V}}(\phi_D + \phi_N) - \widetilde{\mathfrak{K}}(u_D + u_N).$$

Here, $\widetilde{\mathfrak{V}}$ and $\widetilde{\mathfrak{K}}$ formally denote the integral operators from (51), but now they are evaluated in Ω.

This motivates the definition of the following bilinear form for all $u = (u_N, \phi_D), v := (v_N, \psi_D) \in \mathcal{X}$

$$b(u, v) := \langle \mathfrak{W}_{NN} u_N + \mathfrak{K}'_{DN} \phi_D \,,\, v_N \rangle_{L^2(\Gamma_N)} + \langle \mathfrak{V}_{DD} \phi_D - \mathfrak{K}_{ND} u_N \,,\, \psi_D \rangle_{L^2(\Gamma_D)}. \tag{63}$$

Obviously, the bilinear form is elliptic with

$$\begin{aligned} b(v, v) &= \langle \mathfrak{W}_{NN} v_N \,,\, v_N \rangle_{L^2(\Gamma_N)} + \langle \mathfrak{V}_{DD} \psi_D \,,\, \psi_D \rangle_{L^2(\Gamma_D)} \\ &\simeq \|v_N\|^2_{\widetilde{H}^{1/2}(\Gamma_N)} + \|\psi_D\|^2_{\widetilde{H}^{-1/2}(\Gamma_D)} =: \|v\|^2_{\mathcal{X}} \quad \text{for all } v \in \mathcal{X} \end{aligned} \tag{64}$$

and fits in the frame of the Lax-Milgram lemma. This guarantees uniqueness and solvability of

$$b(u, v) = f(v) \quad \text{for all } v \in \mathcal{X} \tag{65}$$

with

$$f(v) := \langle -\mathfrak{W} u_D + (1/2 - \mathfrak{K}') \phi_N \,,\, v_N \rangle_{L^2(\Gamma_N)} + \langle (1/2 + \mathfrak{K}) u_D - \mathfrak{V} \phi_N \,,\, \psi_D \rangle_{L^2(\Gamma_D)}.$$

The discretization of (65) is straightforward. Suppose a triangulation $\mathcal{T}_\star \in \texttt{refine}(\mathcal{T}_0)$. With the spline space $\mathcal{S}_0^{p+1}(\mathcal{T}_\star|_{\Gamma_N}) := \mathcal{S}^{p+1}(\mathcal{T}_\star|_{\Gamma_N}) \cap \widetilde{H}^{1/2}(\Gamma_N)$, let $\mathcal{X}_\star := \mathcal{S}_0^{p+1}(\mathcal{T}_\star|_{\Gamma_N}) \times \mathcal{P}^p(\mathcal{T}_\star|_{\Gamma_D})$, and find $U_\star := (U_{\star,N}, \Phi_{\star,D}) \in \mathcal{X}_\star$ such that all $V_\star := (V_{\star,N}, \Psi_{\star,D}) \in \mathcal{X}_\star$ satisfy

$$b(U_\star, V_\star) = f(V_\star). \tag{66}$$

The weighted-residual error estimator reads elementwise for all $T \in \mathcal{T}_\star$

$$\rho_\star(T)^2 := \mathrm{diam}(T)\|\mathfrak{W}_{NN}U_{\star,N} + \mathfrak{K}'_{DN}\Phi_{\star,D} + \mathfrak{W}u_D - (1/2 - \mathfrak{K}')\phi_N\|^2_{L^2(T\cap\Gamma_N)}$$
$$+ \mathrm{diam}(T)\|\nabla(\mathfrak{V}_{DD}\Phi_{\star,D} - \mathfrak{K}_{ND}U_{\star,N} - (1/2 + \mathfrak{K})u_D + \mathfrak{V}\phi_N)\|^2_{L^2(T\cap\Gamma_D)}$$

and

$$\rho_\star := \Big(\sum_{T\in\mathcal{T}_\star} \rho_\star(T)^2\Big)^{1/2} \quad \text{for all } \mathcal{T}_\star \in \texttt{refine}(\mathcal{T}_0).$$

This follows from the combination of the respective proofs for the weakly-singular integral equation [CMS01] and the hyper-singular integral equation [CMPS04]. Moreover, we refer to the proof of Assumption 10 (iii) below, where the main arguments of the derivation are reused.

7.2. **Proof of Assumption 10–11.**

Proof of Assumption 10 (i)–(ii). Let $\mathcal{T}_\ell$ denote a triangulation and let $\mathcal{T}_\star$ denote an arbitrary refinement. With the triangle inequality and $h_\star = h_\ell$ on $\omega := \bigcup(\mathcal{T}_\star \cap \mathcal{T}_\ell) \subseteq \partial\Omega$, it holds

$$\Big|\Big(\sum_{T\in\mathcal{T}_\star\cap\mathcal{T}_\ell} \rho_\star(T)^2\Big)^{1/2} - \Big(\sum_{T\in\mathcal{T}_\ell\cap\mathcal{T}_\star} \rho_\ell(T)^2\Big)^{1/2}\Big|$$
$$\lesssim \|h_\star^{1/2}\big(\mathfrak{W}_{NN}(U_{\star,N} - U_{\ell,N}) + \mathfrak{K}'_{DN}(\Phi_{\star,D} - \Phi_{\ell,D})\big)\|_{L^2(\omega)}$$
$$+ \|h_\star^{1/2}\nabla\big(\mathfrak{V}_{DD}(\Phi_{\star,D} - \Phi_{\ell,D}) - \mathfrak{K}_{ND}(U_{\star,N} - U_{\ell,N})\big)\|_{L^2(\omega)}$$
$$\lesssim \mathrm{RHS} := \|h_\star^{1/2}\big(\mathfrak{W}_{NN}(U_{\star,N} - U_{\ell,N})\big)\|_{L^2(\Gamma_N)} + \|h_\ell^{1/2}\big(\mathfrak{K}'_{DN}(\Phi_{\star,D} - \Phi_{\ell,D})\big)\|_{L^2(\Gamma_D)}$$
$$+ \|h_\star^{1/2}\nabla\big(\mathfrak{V}_{DD}(\Phi_{\star,D} - \Phi_{\ell,D})\|_{L^2(\Gamma_D)} + \|h_\ell^{1/2}\nabla\big(\mathfrak{K}_{ND}(U_{\star,N} - U_{\ell,N})\big)\|_{L^2(\Gamma_N)}.$$

The inverse estimates from [AFF$^+$12] show

$$\mathrm{RHS} \lesssim \|U_{\star,N} - U_{\ell,N}\|_{\widetilde{H}^{1/2}(\Gamma_N)} + \|\Phi_{\star,D} - \Phi_{\ell,D}\|_{\widetilde{H}^{-1/2}(\Gamma_D)} \simeq \|U_\star - U_\ell\|_{\mathcal{X}}. \tag{67}$$

This concludes the proof of (i). The constant C_{stab} depends only on Γ_N and Γ_D, the γ-shape regularity of $\mathcal{T}_\ell$ and $\mathcal{T}_\star$, and the polynomial degree p.

For the proof of (ii), the mesh size reduction (38) is exploited. It holds $h_\star \le q_{\mathrm{ref}} h_\ell$ on $\omega := \bigcup(\mathcal{T}_\ell\setminus\mathcal{T}_\star) = \bigcup(\mathcal{T}_\star\setminus\mathcal{T}_\ell)$. Hence, the same arguments as above and Young's inequality $(a+b)^2 \le (1+\delta)a^2 + (1+\delta^{-1})b^2$ for all $a, b \in \mathbb{R}$ and $\delta > 0$ show

$$\sum_{T\in\mathcal{T}_\star\setminus\mathcal{T}_\ell} \rho_\star(T)^2 \le (1+\delta)\Big(\|h_\star^{1/2}\big(\mathfrak{W}_{NN}U_{\ell,N} + \mathfrak{K}'_{DN}\Phi_{\ell,D} + \mathfrak{W}u_D - (1/2 - \mathfrak{K}')\phi_N\big)\|^2_{L^2(\omega)}$$
$$+ \|h_\star^{1/2}\nabla(\mathfrak{V}_{DD}\Phi_{\ell,D} - \mathfrak{K}_{ND}U_{\ell,N} - (1/2 + \mathfrak{K})u_D + \mathfrak{V}\phi_N)\|^2_{L^2(\omega)}\Big)$$
$$+ (1+\delta^{-1})\,\mathrm{RHS}^2$$
$$\le (1+\delta)q_{\mathrm{ref}} \sum_{T\in\mathcal{T}_\ell\setminus\mathcal{T}_\star} \rho_\ell(T)^2 + (1+\delta^{-1})\,\mathrm{RHS}^2.$$

Sufficiently small $\delta > 0$ and (67) conclude the proof of (ii). The constant C_{red} depends only on Γ_N and Γ_D, the γ-shape regularity of $\mathcal{T}_\ell$ and $\mathcal{T}_\star$, and the polynomial degree p. $\square$

Proof of Assumption 10 (iii). Ellipticity (64) and Galerkin orthogonality show for all $V_\ell \in \mathcal{X}_\ell$

$$\|U_\star - U_\ell\|^2_{\mathcal{X}} \simeq b(U_\star - U_\ell, U_\star - U_\ell - V_\ell) = b(u - U_\ell, U_\star - U_\ell - V_\ell).$$

With the Scott-Zhang projection $J_\ell : \widetilde{H}^{1/2}(\Gamma_N) \to \mathcal{S}^{p+1}(\mathcal{T}_\ell|_{\Gamma_N})$ (see [AFF+13c, Section 3.2] for the definition and discussion on $\widetilde{H}^{1/2}(\Gamma_N)$, since the original construction [SZ90] requires $H^{1/2+\varepsilon}$ regularity for some $\varepsilon > 0$) and the L^2-orthogonal projection $\Pi_\ell : L^2(\Gamma_D) \to \mathcal{P}^p(\mathcal{T}_\ell|_{\Gamma_D})$, define

$$V_\ell := (J_\ell(U_{\star,N} - U_{\ell,N}), \Pi_\ell(\Phi_{\star,D} - \Phi_{\ell,D})) \in \mathcal{X}_\ell.$$

This implies

$$\begin{aligned}\|U_\star - U_\ell\|_{\mathcal{X}}^2 \simeq \langle \mathfrak{W}_{NN}(u_N - U_{\ell,N}) + \mathfrak{K}'_{DN}(\phi_D - \Phi_{\ell,D})\,,\,(1-J_\ell)(U_{\star,N} - U_{\ell,N})\rangle_{L^2(\Gamma_N)} \\ + \langle \mathfrak{V}_{DD}(\phi_D - \Phi_{\ell,D}) - \mathfrak{K}_{ND}(u_N - U_{\ell,N})\,,\,(1-\Pi_\ell)(\Phi_{\star,D} - \Phi_{\ell,D})\rangle_{L^2(\Gamma_D)}.\end{aligned} \tag{68}$$

There holds $(1-J_\ell)(U_{\star,N} - U_{\ell,N}) = 0$ in $\mathcal{T}_\ell \setminus \omega_\ell(\mathcal{T}_\star \setminus \mathcal{T}_\ell)$ and $(1-\Pi_\ell)(\Phi_{\star,D} - \Phi_{\ell,D}) = 0$ in $\mathcal{T}_\ell \cap \mathcal{T}_\star$. The first term on the right-hand side of (68) is estimated as in [FFK+13b]. Since there also holds

$$\langle \mathfrak{V}_{DD}(\phi_D - \Phi_{\ell,D}) - \mathfrak{K}_{ND}(u_N - U_{\ell,N})\,,\,1\rangle_{L^2(T)} = 0 \quad \text{for all } T \in \mathcal{T}_\ell|_{\Gamma_D},$$

the estimate for the second term on the right-hand side of (68) follows with the arguments from [FKMP13] for $p = 0$ resp. [FFK+13a] for general $p \geq 0$. This concludes the proof. The constant $C_{\rm Rel}$ depends only on Γ_N and Γ_D, the γ-shape regularity of $\mathcal{T}_\ell$ and $\mathcal{T}_\star$ and the polynomial degree p. □

Proof of Assumption 11. The proof follows with the same arguments as in Section 4. □

Consequence 18. *Algorithm 8 for the symmetric boundary element formulation of some mixed boundary value problem converges with optimal rates in the sense of Theorem 14.*

8. Proof of Theorem 14

For this section, assume that Assumptions 10–11 hold for a given model problem with error estimator $\rho(\cdot)$.

Proposition 19. *For any $\theta < \theta_{\rm opt} := (1 + C_{\rm stab}^2 C_{\rm Rel}^2)^{-1}$, there exists $0 < \kappa_{\rm opt} < 1$ such that all $\mathcal{T}_\star \in \texttt{refine}(\mathcal{T}_\ell)$ satisfy the implication*

$$\rho_\star^2 \leq \kappa_{\rm opt}\rho_\ell^2 \qquad \Longrightarrow \qquad \theta\rho_\ell^2 \leq \sum_{T\in\mathcal{R}(\mathcal{T}_\ell,\mathcal{T}_\star)} \rho_\ell(T)^2. \tag{69}$$

Proof. Recall Young's inequality $(a+b)^2 \leq (1+\delta)a^2 + (1+\delta^{-1})b^2$ for all $a, b \in \mathbb{R}$ and $\delta > 0$. The stability (39) shows

$$\begin{aligned}\rho_\ell^2 &= \sum_{T\in\mathcal{T}_\ell\cap\mathcal{T}_\star} \rho_\ell(T)^2 + \sum_{T\in\mathcal{T}_\ell\setminus\mathcal{T}_\star} \rho_\ell(T)^2 \\ &\leq (1+\delta) \sum_{T\in\mathcal{T}_\star\cap\mathcal{T}_\ell} \rho_\star(T)^2 + \sum_{T\in\mathcal{T}_\ell\setminus\mathcal{T}_\star} \rho_\ell(T)^2 + (1+\delta^{-1})C_{\rm stab}^2\|U_\star - U_\ell\|_{\mathcal{X}}^2.\end{aligned}$$

The assumption $\rho_\star^2 \leq \kappa\rho_\ell^2$ together with discrete reliability (41) implies

$$\rho_\ell^2 \leq (1+\delta)\kappa\rho_\ell^2 + (1 + (1+\delta^{-1})C_{\rm stab}^2 C_{\rm Rel}) \sum_{T\in\mathcal{R}(\mathcal{T}_\ell,\mathcal{T}_\star)} \rho_\ell(T)^2$$

and rearrangement of the terms proves $\theta\rho_\ell^2 \leq \sum_{T\in\mathcal{R}(\mathcal{T}_\ell,\mathcal{T}_\star)} \rho_\ell(T)^2$ for all

$$0 \leq \theta < \theta(\kappa) := \sup_{\delta>0} \frac{1-(1+\delta)\kappa}{1+(1+\delta^{-1})C_{\rm stab}^2 C_{\rm Rel}^2}.$$

For each $\theta < \theta_{\rm opt}$, there exist $\delta, \kappa > 0$ such that

$$\theta < \frac{1-(1+\delta)\kappa}{1+(1+\delta^{-1})C_{\rm stab}^2 C_{\rm Rel}^2} < \frac{1}{1+C_{\rm stab}^2 C_{\rm Rel}^2} = \theta_{\rm opt}$$

and hence $\theta < \theta(\kappa)$. This concludes the proof. $\square$

The definition of the approximability quasi-norm $\|\rho\|_{\mathbb{A}_s}$ allows to find *optimal* meshes, which compare with the adaptively generated meshes. This is stated in the following lemma.

Lemma 20. *Let $0 < \kappa_{\rm opt} < 1$ and let $s > 0$ such that $\|\rho\|_{\mathbb{A}_s} < \infty$. For all meshes $\mathcal{T}_\ell$, there exists a refinement $\mathcal{T}_\star \in$ `refine`$(\mathcal{T}_\ell)$ with*

$$\rho_\star^2 \le \kappa_{\rm opt}\rho_\ell^2 \quad \text{and} \quad \#\mathcal{T}_\star - \#\mathcal{T}_\ell + 1 \le C_6 \|\rho\|_{\mathbb{A}_s}^{1/s} \rho_\ell^{-1/s}. \tag{70}$$

The constant $C_6 > 0$ depends only on $C_{\rm stab}$, $C_{\rm red}$, $C_{\rm Rel}$, $\kappa_{\rm opt}$, and $s > 0$.

Proof. Stability (39), reduction (40), and discrete reliability (41) prove for all refinements $\mathcal{T}_+ \in$ `refine`$(\mathcal{T}_\ell)$

$$\rho_+^2 \le 2\rho_\ell + 2(C_{\rm stab}^2 + C_{\rm red})\|U_+ - U_\ell\|_{\mathcal{X}}^2 \le (2 + 2(C_{\rm stab}^2 + C_{\rm red})C_{\rm Rel}^2)\rho_\ell^2. \tag{71}$$

Define the constant $C_{\rm mon} := (2 + 2(C_{\rm stab}^2 + C_{\rm red})C_{\rm Rel}^2)^{1/2}$. Arguing as e.g. in [CFPP13, Ste07, CKNS08], find a mesh $\mathcal{T}_+ \in$ `refine`$(\mathcal{T}_0)$ which satisfies

$$\#\mathcal{T}_+ - \#\mathcal{T}_0 + 1 \lesssim \|\rho\|_{\mathbb{A}_s}^{1/s} \rho_\ell^{-1/s} \quad \text{and} \quad \rho_+^2 \le C_{\rm mon}^{-2}\kappa_{\rm opt}\rho_\ell^2.$$

Define $\mathcal{T}_\star := \mathcal{T}_\ell \oplus \mathcal{T}_+$ and use the overlay estimate (36) to verify

$$\#\mathcal{T}_\star - \#\mathcal{T}_\ell + 1 \le \#\mathcal{T}_+ - \#\mathcal{T}_0 + 1 \lesssim \|\rho\|_{\mathbb{A}_s}^{1/s} \rho_\ell^{-1/s}.$$

Since $\mathcal{T}_\star \in$ `refine`$(\mathcal{T}_+)$, the quasi-monotonicity (71) shows

$$\rho_\star^2 \le C_{\rm mon}^2 \rho_+^2 \le \kappa_{\rm opt}\rho_\ell^2.$$

This concludes the proof. $\square$

Proof of Theorem 14. To see linear convergence (45), apply the estimator reduction (43) for $\ell, N \in \mathbb{N}$

$$\sum_{j=\ell+1}^{\ell+N} \rho_j^2 \le q_{\rm est} \sum_{j=\ell+1}^{\ell+N} \rho_{j-1}^2 + C_{\rm est} \sum_{j=\ell+1}^{\ell+N} \|U_j - U_{j-1}\|_{\mathcal{X}}^2.$$

The general quasi-orthogonality (13) and reliability (42) show

$$(1 - q_{\rm est}) \sum_{j=\ell+1}^{\ell+N} \rho_j^2 \le q_{\rm est}\rho_\ell^2 + C_{\rm est}C_1\|u - U_\ell\|_{\mathcal{X}}^2 \le (q_{\rm est} + C_{\rm est}C_1C_{\rm rel}^2)\rho_\ell^2.$$

Since the involved constants do not depend on $N \in \mathbb{N}$, there holds with $C_{\rm R} := 1 + (q_{\rm est} + C_{\rm est}C_1C_{\rm rel}^2)/(1 - q_{\rm est}) > 1$

$$\sum_{j=\ell}^{\infty} \rho_j^2 \le C_{\rm R}\rho_\ell^2.$$

This and mathematical induction on $k \in \mathbb{N}$ show

$$\sum_{j=\ell+k}^{\infty} \rho_j^2 = \Big(\sum_{j=\ell+k-1}^{\infty} \rho_j^2 \Big) - \rho_{\ell+k-1}^2 \le (1 - C_{\rm R}^{-1}) \sum_{j=\ell+k-1}^{\infty} \rho_j^2 \le \ldots \le (1 - C_{\rm R}^{-1})^k \sum_{j=\ell}^{\infty} \rho_j^2,$$

which implies immediately

$$\rho_{\ell+k}^2 \le C_{\rm R}(1-C_{\rm R}^{-1})^k\rho_\ell^2.$$

This concludes the proof of (45) with $0 < q_{\rm R} := (1-C_{\rm R}^{-1}) < 1$.

The optimality statement (46) follows as a consequence. Choose $\kappa_{\rm opt} > 0$ sufficiently small such that the implication (69) holds true. Given $\mathcal{T}_\ell$, Lemma 20 provides a mesh $\mathcal{T}_\star \in \texttt{refine}(\mathcal{T}_\ell)$ with (70). Therefore, Proposition 19 implies that $\mathcal{T}_\ell \setminus \mathcal{T}_\star$ satisfies the Dörfler marking (6). Since $\mathcal{M}_\ell$ is a set of minimal cardinality which satisfies the Dörfler marking (6), there holds

$$\#\mathcal{M}_\ell + 1 \le \#(\mathcal{T}_\ell\setminus\mathcal{T}_\star) + 1 \le \#\mathcal{T}_\star - \#\mathcal{T}_\ell + 1 \le C_6\|\rho\|_{\mathbb{A}_s}^{1/s}\rho_\ell^{-1/s}.$$

This and the mesh closure estimate (35) imply

$$\#\mathcal{T}_\ell - \#\mathcal{T}_0 + 1 \lesssim \sum_{j=0}^{\ell-1}(\#\mathcal{M}_j+1) \lesssim \|\rho\|_{\mathbb{A}_s}^{1/s}\sum_{j=0}^{\ell-1}\rho_j^{-1/s}.$$

The R-linear convergence (45) together with the convergence of the geometric series show

$$\#\mathcal{T}_\ell - \#\mathcal{T}_0 + 1 \lesssim \|\rho\|_{\mathbb{A}_s}^{1/s}\rho_\ell^{-1/s}C_{\rm R}^{-1/s}\sum_{j=0}^{\ell-1}q_{\rm R}^{(\ell-j)/s} \lesssim \|\rho\|_{\mathbb{A}_s}^{1/s}\rho_\ell^{-1/s}.$$

This implies the upper bound in (46). The lower bound in (46) follows form elementary arguments and the fact that each refined element is split into at most two sons for $d = 1$, four sons for $d = 2$, and into a uniformly bounded number of sons for $d = 3$ (the proof follows with arguments from [Ste08]) as pointed out by R. Stevenson in a private communication (we refer to [CFPP13, Section 2.5] for details). This concludes the proof. □

Remark 21. *Our proof of* (45) *shows that the* modified general quasi-orthogonality

$$\sum_{k=\ell}^{\ell+N}\|U_{k+1}-U_k\|_{\mathcal{X}}^2 \le C_1\rho_\ell^2 \quad \textit{for all } \ell, N\in\mathbb{N} \tag{72}$$

is sufficient. In our frame, (72) *follows from Theorem 1 and reliability* (42)*. We refer to [CFPP13, Proposition 4.10–4.11] for the proof that, under Assumption 10 (i)–(ii) and reliability* (42)*,* (72) *is, in fact, equivalent to R-linear convergence* (45)*.*

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