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# QUASI-OPTIMAL ADAPTIVE BEM

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There are two main reasons to use a posteriori error estimation in finite and boundary element methods: First, to check whether a computed solution is accurate enough. Second, if it is not, to find out where to refine the mesh to obtain the optimal improvement in accuracy. The latter leads immediately to the idea of the following adaptive feedback loop

$$\boxed{\text{solve}} \longrightarrow \boxed{\text{estimate}} \longrightarrow \boxed{\text{mark}} \longrightarrow \boxed{\text{refine}} \tag{1}$$

where the computed solution is checked for accuracy and then improved by marking elements with big error contributions and by refining at least these elements to obtain an improved mesh. Hereafter, the index $\ell$ indicates the iteration index in the adaptive loop of (1).

The usual observation is that the improvement compared to uniform mesh-refinement is vast in terms of error versus degrees of freedom, see also Figure 1 below for some typical model problem of 2D BEM. This is due to singularities of the given data and/or the (unknown) solution which lead to suboptimal convergence rates if the meshes are refined uniformly. However, besides these empirical observations, two important questions arise from a mathematical point of view:

(i) Convergence? Since the mesh-size does not necessarily tend to zero everywhere in $\Omega$, it is far from obvious that the approximate solutions $\Phi_\ell$ converge towards the exact solution $\phi$ as $\ell \to \infty$.
(ii) Speed of convergence? Empirically, adaptive schemes recover the optimal convergence rate even in the presence of singularities. For adaptive FEM, mathematical explanations for this observation are available, see e.g. [4]. Unfortunately, the contemporary FEM proofs cannot be directly transferred to adaptive BEM, as the non-locality of the involved operators and norms leads to severe technicalities.

The following sections state the model problem, clarify the adaptive algorithm (1), and give positive answers to both (i) and (ii). Moreover, we shall give a short overview on the current state of research.

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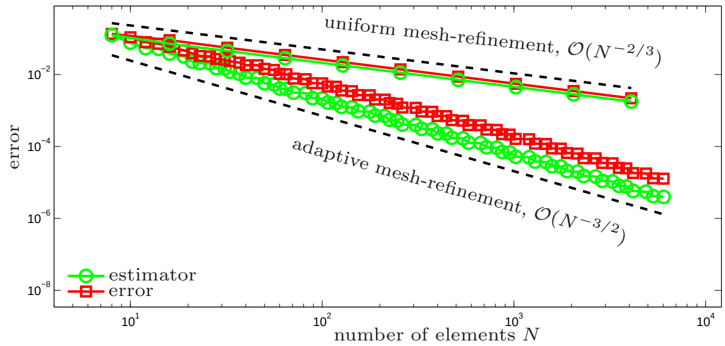


FIGURE 1. Adaptive mesh-refinement yields an improved convergence rate compared to uniform mesh-refinement.

## 1. Model problem & state of the art

With $V$ and $K$ being the simple-layer and double-layer potential of the 2D or 3D Laplacian, we consider the weakly singular integral equation

$$V\phi = (1/2 + K)g \quad \text{on } \Gamma \tag{2}$$

for some given Dirichlet data $g \in H^{1/2}(\Gamma)$. Here, $\Gamma = \partial\Omega$ is the boundary of a polygonal Lipschitz domain $\Omega \subset \mathbb{R}^d$ for $d = 2, 3$. For $d = 2$, we ensure $\mathrm{diam}(\Omega) < 1$ and hence ellipticity of $V$ by scaling. The equation (2) is thus the prototype of an elliptic integral equation of the first-kind.

Convergence of the adaptive algorithm (1) with quasi-optimal algebraic rates has independently first been proved in [8, 9]. Therein, the discretization is restricted to lowest-order elements, and the right-hand side is assumed to be computed exactly. While the wavelet based analysis of [9] covers general integral kernels, but requires the boundary to be smooth, the work [8] restricts to the Laplace kernel, but also covers piecewise polygonal resp. polyhedral boundaries. The work [7] extends the analysis to arbitrary but fixed-order polynomials and includes the adaptive resolution of the right-hand side. For an overview, we refer to the recent PhD thesis [10], where also the hypersingular integral equation is analyzed. Finally, we also refer to [1], where the present analysis is used to show convergence of an adaptive FEM-BEM coupling method by means of the estimator reduction principle.

For the ease of presentation, we restrict to the weakly singular integral equation (2) with lowest-order elements as in [8], but also include the approximation of the given Dirichlet data [7].

## 2. The adaptive algorithm

We now specify the four modules of algorithm (1). For $\boxed{\text{solve}}$, we use a Galerkin BEM to approximate the solution $\phi$ of (2) by a piecewise constant function $\Phi_\ell \in \mathcal{P}^0(\mathcal{T}_\ell)$ which is given as the solution of the linear system

$$\langle V\Phi_\ell\,,\,\Psi_\ell\rangle = \langle (1/2+K)P_\ell g\,,\,\Psi_\ell\rangle \quad \text{for all } \Psi_\ell \in \mathcal{P}^0(\mathcal{T}_\ell). \tag{3}$$

Here, we use an arbitrary $H^{1/2}$-stable projection $P_\ell$ onto piecewise affine, globally continuous functions on a mesh $\mathcal{T}_\ell$, e.g. the $L^2$-projection [11] or the Scott-Zhang projection [12], and approximate $g \approx G_\ell := P_\ell g$. For 2D and continuous $g$, one may also use nodal interpolation. Then, the linear system (3) relies only on operator matrices of $V$ and $K$ which can be assembled fast and, in certain cases, even exactly.

For $\boxed{\text{estimate}}$, we build on the weighted residual estimator from [5, 6]

$$\eta_\ell^2 = \sum_{T\in\mathcal{T}_\ell} \eta_\ell(T)^2 \text{ with } \eta_\ell(T)^2 := \operatorname{diam}(T)\|\nabla(V\Phi_\ell - (1/2+K)G_\ell)\|_{L^2(T)}^2.$$

Here, $\nabla$ denotes the surface gradient on the $(d-1)$-dimensional manifold $\Gamma$. In 2D, $\nabla$ is simply the arclength derivative. To take account of the additional data approximation error, we assume $g \in H^1(\Gamma)$ and define

$$\operatorname{osc}_\ell^2 = \sum_{T\in\mathcal{T}_\ell} \operatorname{osc}_\ell(T)^2 \quad \text{with} \quad \operatorname{osc}_\ell(T)^2 := \operatorname{diam}(T)\|(1-\Pi_\ell^0)\nabla g\|_{L^2(T)}^2,$$

where $\Pi_\ell^0 : L^2(\Gamma) \to \mathcal{P}^0(\mathcal{T}_\ell)$ denotes the $\mathcal{T}_\ell$-elementwise integral mean. The advantage of this oscillation term is that it incorporates only a fairly easy-to-compute projection $\Pi_\ell^0$. Moreover, it is independent of the exact choice of $P_\ell$ for data approximation. The combined error estimator

$$\rho_\ell^2 = \sum_{T\in\mathcal{T}_\ell} \rho_\ell(T)^2 \quad \text{with} \quad \rho_\ell(T)^2 = \eta_\ell(T)^2 + \operatorname{osc}_\ell(T)^2$$

will then be provided by the module $\boxed{\text{estimate}}$. We stress the upper bound

$$|||\phi - \Phi_\ell||| \le C_{\rm rel}\rho_\ell \tag{4}$$

where $|||\cdot||| := \langle V\cdot\,,\,\cdot\rangle^{1/2} \simeq \|\cdot\|_{H^{-1/2}(\Gamma)}$ denotes the energy norm induced by $V$. Moreover, in 2D and under certain regularity assumptions on the Dirichlet data $g$, we also obtain a lower bound [2]

$$C_{\rm eff}^{-1}\rho_\ell \le |||\phi - \Phi_\ell||| + \operatorname{osc}_\ell + \operatorname{hot}_\ell, \tag{5}$$

where $\operatorname{hot}_\ell$ is a term of higher order, compared to the generic rate of convergence $\mathcal{O}(h^{3/2})$ of lowest-order BEM for uniform meshes and smooth solutions. Exactly speaking, there holds for $g \in H^{2+\varepsilon}(\Gamma)$

$$\operatorname{hot}_\ell = \mathcal{O}(h^{3/2+\varepsilon}) = o(|||\phi - \Phi_\ell|||)$$

for uniform meshes $\mathcal{T}_\ell$ with mesh-size $h > 0$.

In the module $\boxed{\text{mark}}$, we use a separate Dörfler marking strategy also employed in [3]: For given parameters $0 < \theta, \vartheta < 1$, we determine the set of marked elements $\mathcal{M}_\ell \subseteq \mathcal{T}_\ell$ as a set of minimal cardinality which satisfies

$$\theta\,\eta_\ell^2 \le \sum_{T\in\mathcal{M}_\ell} \eta_\ell(T)^2 \quad \text{for } \operatorname{osc}_\ell^2 \le \vartheta\,\eta_\ell^2 \tag{6a}$$

resp.

$$\theta\,\mathrm{osc}_\ell^2 \le \sum_{T\in\mathcal{M}_\ell} \mathrm{osc}_\ell(T)^2 \quad \text{for } \mathrm{osc}_\ell^2 > \vartheta\,\eta_\ell^2. \tag{6b}$$

For the special case when $P_\ell$ is the Scott-Zhang projection (or for $d = 2$ the nodal interpolant), we may also use the standard Dörfler marking

$$\theta\rho_\ell^2 \le \sum_{T\in\mathcal{M}_\ell} \rho_\ell(T)^2, \tag{7}$$

and all results below hold accordingly.

Finally, the module refine consists of a fixed refinement rule. For 3D, we use the so-called *newest vertex bisection*, see e.g. [4, 11], whereas we use the bisection algorithm of [2] for 2D.

## 3. Convergence

By proving certain (local) inverse-type estimates [1], we are able to conclude that $\rho_\ell$ satisfies the estimator reduction estimate

$$\rho_{\ell+1}^2 \le \widetilde{\kappa}\,\rho_\ell^2 + C_{\rm inv}\Big(|\!|\!|\Phi_{\ell+1}-\Phi_\ell|\!|\!|^2 + \|G_{\ell+1}-G_\ell\|_{H^{1/2}(\Gamma)}^2\Big) \tag{8}$$

with $\ell$-independent constants $0 < \widetilde{\kappa} < 1$ and $C_{\rm inv} > 0$. Abstract functional analysis provides that the discrete solutions $\Phi_\ell$ and discretized data $G_\ell$ converge in $H^{-1/2}(\Gamma)$ resp. $H^{1/2}(\Gamma)$ to certain (yet unknown) limits. In particular, $\rho_\ell$ is thus contractive up to a perturbation which tends to zero as $\ell \to \infty$. In particular, we thus obtain the following convergence result.

**Theorem 1.** *For arbitrary marking parameters $0 < \theta, \vartheta < 1$, there holds*

$$C_{\rm rel}^{-1}\,|\!|\!|\phi-\Phi_\ell|\!|\!| \le \rho_\ell \to 0 \quad \text{as } \ell\to\infty, \tag{9}$$

*i.e. algorithm* (1) *always yields convergence.* □

The observation in Figure 1 is that the adaptive algorithm converges even with the best possible rate for lowest-order BEM for this problem type. A first step towards a mathematical justification for this observation is to prove linear convergence of the estimator $\rho_\ell$ in the sense of

$$\rho_\ell \le C_{\rm conv} q_{\rm conv}^\ell \tag{10}$$

where $C_{\rm conv} > 0$ and $0 < q_{\rm conv} < 1$ are independent of $\ell \in \mathbb{N}$. However, due to the changing right-hand side $(1/2 + K)G_\ell$ in each step of the algorithm, we cannot rely on the usual Galerkin orthogonality as e.g. in [4], i.e.

$$|\!|\!|\phi-\Phi_\ell|\!|\!|^2 \ne |\!|\!|\phi-\Phi_{\ell+1}|\!|\!|^2 + |\!|\!|\Phi_{\ell+1}-\Phi_\ell|\!|\!|^2$$

in general. Nevertheless, one may prove the following contraction result [7].

**Theorem 2.** *For sufficiently small $0 < \vartheta < 1$, but arbitrary $0 < \theta < 1$, there exist constants $0 < \kappa < 1$ and $\alpha, \beta > 1$ such that the contraction quantity*

$$\Delta_\ell := |\!|\!|\phi_\ell-\Phi_\ell|\!|\!|^2 + \alpha\,\eta_\ell^2 + \beta\,\mathrm{osc}_\ell^2 \tag{11}$$

*with* $\phi_\ell := V^{-1}(1/2 + K)G_\ell$ *satisfies the estimate*

$$\Delta_{\ell+1} \le \kappa \Delta_\ell \quad \text{for all } \ell \in \mathbb{N}. \tag{12}$$

*Since* $\min\{\alpha, \beta\}\, \rho_\ell^2 \le \Delta_\ell \le (C_{\rm rel}^2 + \max\{\alpha, \beta\})\, \rho_\ell^2$, *this implies linear convergence* (10). □

## 4. Optimal convergence rates

Given the initial mesh $\mathcal{T}_0$, let $\mathbb{T}$ be the set of all triangulations which can be generated by means of the fixed mesh-refinement strategy. For arbitrary $s > 0$, we say that $\phi$ belongs to the *approximation class* $\mathbb{A}_s$ provided that there exists a sequence $(\widetilde{\mathcal{T}}_\ell)_{\ell \in \mathbb{N}_0}$ of meshes in $\mathbb{T}$ such that

$$\widetilde{\rho}_\ell \le C\,(\#\widetilde{\mathcal{T}}_\ell - \#\mathcal{T}_0)^{-s} \quad \text{for all } \ell \in \mathbb{N}. \tag{13}$$

Here, $\widetilde{\rho}_\ell$ is the error estimator corresponding to the Galerkin approximations $\widetilde{\Phi}_\ell$ on $\widetilde{\mathcal{T}}_\ell$, and (13) states the decay $\widetilde{\rho}_\ell = \mathcal{O}(N^{-s})$ for the estimator sequence.

We emphasize that in the definition of $\mathbb{A}_s$, the sequence of meshes can be chosen in an *optimal* way, i.e., to ensure (13) with $C > 0$ as small as possible. However, this sequence is chosen *a priori* and is hence not computable. The following theorem from [7] states that the approximation classes to which $\phi$ belongs, i.e. each possible decay $\rho_\ell = \mathcal{O}(N^{-s})$ of the error estimator, is in fact characterized and will be achieved by the adaptive algorithm (1).

**Theorem 3.** *Assume that the adaptivity parameters* $0 < \theta, \vartheta < 1$ *are sufficiently small. Then, for all* $s > 0$, *there holds equivalence*

$$\phi \in \mathbb{A}_s \quad \Longleftrightarrow \quad \forall \ell \in \mathbb{N}:\ \rho_\ell \le C_{\rm opt}(\#\mathcal{T}_\ell - \#\mathcal{T}_0)^{-s}, \tag{14}$$

*i.e. algorithm* (1) *will lead to optimal algebraic convergence rates for the error estimator.* □

Finally, we restrict ourselves to the 2D case on a 1D manifold $\Gamma = \partial\Omega$. We use the efficiency result (5) from [2] to characterize the class of problems $\phi$ for which (13) is possible, by means of the best approximation error only. Recall that the optimal rate of convergence for lowest-order BEM is $\mathcal{O}(h^{3/2})$ which corresponds to $s = 3/2$ for $d = 2$.

**Theorem 4.** *Let* $d = 2$ *and* $g \in H^{2+\varepsilon}(\Gamma)$. *Assume that* $0 < \theta, \vartheta < 1$ *are sufficiently small. Then, for all* $0 < s \le 5/2$, *there holds equivalence*

$$\phi \in \mathbb{A}_s \quad \Longleftrightarrow \quad \forall \ell \in \mathbb{N}:\ \min_{\Psi_\ell \in \mathcal{P}^0(\mathcal{T}_\ell)} |\!|\!|\phi - \Psi_\ell|\!|\!| \le C_{\rm opt}(\#\mathcal{T}_\ell - \#\mathcal{T}_0)^{-s}, \tag{15}$$

*i.e. algorithm* (1) *will also lead to the optimal algebraic convergence rate for the Galerkin error.* □

In particular, we see that the weighted-residual error estimator $\eta_\ell$ from [5, 6] performs (in terms of convergence rate) at least as good as any other error estimator and especially better than uniform mesh refinement.

## 5. Summary and extensions

We have empirically observed that the use of adaptivity for the model problem (2) is advantageous in terms of error versus degrees of freedom. This empirical observation can now be explained and guaranteed mathematically. In experiments, we even observe that adaptivity is superior in terms of computational time and memory consumption. The used techniques are also applicable to the Neumann problem for the Laplacian. Moreover, other elliptic equations as the Lamé system might be analysed in the same fashion.

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