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FEM-BEM Couplings without Stabilization

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FEM-BEM COUPLINGS WITHOUT STABILIZATION

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We consider a nonlinear interface problem with the Laplacian, which can equivalently be stated via various FEM-BEM coupling methods. We treat the symmetric coupling [4, 8], the Johnson-Nédélec coupling [9, 16] as well as the one-equation Bielak-MacCamy coupling [2]. Due to constant functions in the kernel of these equations, these formulations are not elliptic and unique solvability cannot be shown directly. Available results concerning solvability of these methods include the following:

- For the symmetric coupling and certain nonlinear problems with additional Dirichlet boundary, Gatica & Hsiao [6] proved unique solvability.
- For the symmetric coupling and certain nonlinear problems, Carstensen & Stephan [3] proved unique solvability for sufficiently fine meshes, but without an additional interior Dirichlet boundary.
- For the Johnson-Nédélec coupling and the linear Laplace and Yukawa transmission problem, Sayas [11] proved unique solvability on polyhedral boundaries, whereas the original work [9] relied on the Fredholm alternative and hence smooth coupling boundaries.
- For the Johnson-Nédélec coupling and a general class of linear problems, Steinbach [13] introduced a stabilization to prove ellipticity of the stabilized coupling equations, cf. also [10]. His approach, however, requires pre- and postprocessing steps which involve the numerical solution of an additional integral equation with the simple-layer potential.

We present a framework based on *implicit stabilization* to prove well-posedness of nonlinear FEM-BEM coupling formulations [1]. We build on the works of Sayas [11] and Steinbach [13] and introduce stabilized coupling equations which are uniquely solvable and have the same solution as the (original) continuous resp. discrete coupling equations. With this *theoretic auxiliary problem*, we obtain unique solvability of the original (non-stabilized) coupling equations. In particular, we have to implement and solve the original coupling equations only, and we avoid the solution of any

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additional equation and corresponding pre- and postprocessing steps as well as any assumption on the mesh-size. Our approach also applies to nonlinear elasticity [5].

1. Model problem

Let $\Omega \subseteq \mathbb{R}^d$ for $d = 2, 3$ be a connected, bounded Lipschitz domain with polyhedral boundary $\Gamma := \partial\Omega$. We refer to Ω as interior domain and to the unbounded domain $\Omega^{\rm ext} := \mathbb{R}^d \backslash \overline{\Omega}$ as exterior domain. For given data $f \in H^1(\Omega), u_0 \in H^{1/2}(\Gamma), \phi_0 \in H^{-1/2}(\Gamma)$, our model problem then reads:

$$-\operatorname{div}(\mathcal{A}\nabla u) = f \quad \text{in } \Omega, \tag{1a}$$

$$-\Delta u^{\rm ext} = 0 \quad \text{in } \Omega^{\rm ext}, \tag{1b}$$

$$u - u^{\rm ext} = u_0 \quad \text{on } \Gamma, \tag{1c}$$

$$(\mathcal{A}\nabla u - \nabla u^{\rm ext}) \cdot \mathbf{n} = \phi_0 \quad \text{on } \Gamma, \tag{1d}$$

$$u^{\rm ext}(x) = \mathcal{O}(1/|x|) \quad \text{for } |x| \to \infty. \tag{1e}$$

Here, $\mathbf{n}$ is the outer normal vector on Γ. Let $\langle \cdot\,,\,\cdot \rangle_\Omega$ denote the $L^2(\Omega)$ scalar product and let $\langle \cdot\,,\,\cdot \rangle_\Gamma$ denote the $L^2(\Gamma)$ scalar product which is continuously extended to the duality bracket between $H^{-1/2}(\Gamma)$ and $H^{1/2}(\Gamma)$. We assume $\mathcal{A} : \mathbb{R}^d \to \mathbb{R}^d$ to be strongly monotone (2a) as well as Lipschitz continuous (2b) with constants $c_{\rm mon}, c_{\rm lip} > 0$, i.e.

$$\langle \mathcal{A}\nabla u - \mathcal{A}\nabla v\,,\, \nabla u - \nabla v \rangle_\Omega \geq c_{\rm mon} \|\nabla u - \nabla v\|^2_{L^2(\Omega)} \tag{2a}$$

as well as

$$\|\mathcal{A}\nabla u - \mathcal{A}\nabla v\|^2_{L^2(\Omega)} \leq c_{\rm lip} \|\nabla u - \nabla v\|^2_{L^2(\Omega)} \quad \text{for all } u, v \in H^1(\Omega). \tag{2b}$$

In 2D, the compatibility condition

$$\langle f\,,\,1 \rangle_\Omega + \langle \phi_0\,,\,1 \rangle_\Gamma = 0 \tag{3}$$

has to be imposed on the data to ensure the radiation condition (1e).

Throughout, K denotes the double-layer potential with adjoint K', V denotes the simple-layer potential, and W the hypersingular integral operator. Moreover, we assume $\operatorname{diam}(\Omega) < 1$ to guarantee ellipticity $\langle \phi\,,\, V\phi \rangle_\Gamma \geq \|\phi\|_{H^{-1/2}(\Gamma)}$ of the simple-layer potential.

Under these assumptions, problem (1) allows for a unique solution $(u, u^{\rm ext}) \in H^1(\Omega) \times H^1_{\rm loc}(\Omega^{\rm ext})$ with finite energy $\|\nabla u^{\rm ext}\|_{L^2(\Omega^{\rm ext})} < \infty$, see e.g. [3].

2. Symmetric coupling

As shown in e.g. [3, 6], the nonlinear transmission problem (1) can equivalently be stated via the symmetric coupling [4, 8], which reads in variational formulation as follows: Find $\mathbf{u} = (u, \phi) \in \mathcal{H} := H^1(\Omega) \times H^{-1/2}(\Gamma)$ such that

$$b(\mathbf{u}, \mathbf{v}) = F(\mathbf{v}) \quad \text{holds for all } \mathbf{v} \in \mathcal{H}, \tag{4}$$

where the mapping $b : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$ and the linear functional $F : \mathcal{H} \to \mathbb{R}$ are defined for all $\mathbf{u} = (u, \phi), \mathbf{v} = (v, \psi) \in \mathcal{H}$ via

$$\begin{aligned} b(\mathbf{u}, \mathbf{v}) := \langle \mathcal{A}\nabla u\,,\, \nabla v\rangle_\Omega + \langle (K' - \tfrac{1}{2})\phi\,,\, v\rangle_\Gamma + \langle Wu\,,\, v\rangle_\Gamma \\ + \langle \psi\,,\, (\tfrac{1}{2} - K)u\rangle_\Gamma + \langle \psi\,,\, V\phi\rangle_\Gamma, \end{aligned} \tag{5a}$$

$$F(\mathbf{v}) := \langle f\,,\, v\rangle_\Omega + \langle \phi_0 + Wu_0\,,\, v\rangle_\Gamma + \langle \psi\,,\, (\tfrac{1}{2} - K)u_0\rangle_\Gamma \tag{5b}$$

By taking $(u, \phi) = (1, 0) = (v, \psi)$ in (5a), we see that $b((1, 0), (1, 0)) = 0$ and thus constant functions are in the kernel of the mapping $\mathbf{u} \mapsto b(\mathbf{u}, \mathbf{u})$. Therefore, $b(\cdot, \cdot)$ is not elliptic and solvability of (4) cannot be shown directly by applying well-known PDE theory, such as the Lax-Milgram lemma for linear problems.

We stress that clearly (4) is uniquely solvable up to a constant in the interior domain Ω. Early works, including [4, 8] as well as [6], use interior Dirichlet boundaries to tackle this constant in Ω. The very first work which circumvented this technical restriction was [3]. However, the latter work requires the mesh-size to be sufficiently fine.

In the following, we introduce the concept of *implicit stabilization* and show that the second equation of (4), i.e. $b((u, \phi), (0, \psi)) = F((0, \psi))$, already fixes the constant in the interior domain. Unfortunately, this information is lost when trying to prove ellipticity of $b(\cdot, \cdot)$, but can be reconstructed by adding appropriate stabilization terms to the mapping $b(\cdot, \cdot)$. This leads to a modified (or stabilized) formulation that admits a unique solution. Moreover, Lemma 2.3 states equivalence of this modified problem to (4) in the sense that both problems have the same solution even in the discrete formulation.

For the remainder, let $\mathcal{H}_h = \mathcal{X}_h \times \mathcal{Y}_h \subseteq \mathcal{H}$ be a closed subspace of $\mathcal{H}$. In particular, $\mathcal{H}_h = \mathcal{H}$ is a valid choice. We stress that $\mathcal{H}$, associated with the natural product norm

$$\|\mathbf{u}\|_{\mathcal{H}}^2 = \|u\|_{H^1(\Omega)}^2 + \|\phi\|_{H^{-1/2}(\Gamma)}^2 \quad \text{for all } \mathbf{u} = (u, \phi) \in \mathcal{H},$$

becomes a Hilbert space. The following theorem from [1] shows solvability of the continuous formulation (4) as well as of its Galerkin discretization.

Theorem 2.1. *The symmetric coupling* (4) *admits a unique solution* $\mathbf{u} \in \mathcal{H}$. *Moreover, assume that*

$$\exists \xi \in \mathcal{Y}_h \quad \langle \xi\,,\, 1\rangle_\Gamma \neq 0. \tag{6}$$

Then, the discretized equations

$$b(\mathbf{u}_h, \mathbf{v}_h) = F(\mathbf{v}_h) \quad \textit{for all } \mathbf{v}_h \in \mathcal{H}_h \tag{7}$$

also admit a unique solution $\mathbf{u}_h \in \mathcal{H}_h$. *There holds the Céa-type estimate*

$$\|\mathbf{u} - \mathbf{u}_h\|_{\mathcal{H}} \le C \min_{\mathbf{v}_h \in \mathcal{H}_h} \|\mathbf{u} - \mathbf{v}_h\|_{\mathcal{H}}, \tag{8}$$

where the constant $C > 0$ *depends only on* $\mathcal{A}, \Omega$, *and* $\xi \in \mathcal{Y}_h$.

Remark 2.2. *(i)* For a sequence $(\mathcal{Y}_h)_{h>0}$ of closed subspaces of $H^{-1/2}(\Gamma)$ with

$$\exists \xi \in \bigcap_{h>0} \mathcal{Y}_h \quad \langle \xi \,,\, 1 \rangle_\Gamma \neq 0, \tag{9}$$

the constant in (8) is independent of $h > 0$.
(ii) For an arbitrary sequence $(\mathcal{E}_h)_{h>0}$ of regular triangulations and $\mathcal{Y}_h = \mathcal{P}^p(\mathcal{E}_h)$ being the space of $\mathcal{E}_h$-piecewise polynomials of degree $p \geq 0$, $\xi = 1 \in \mathcal{Y}_h$ satisfies (9).

Note that the following lemma from [1] holds for arbitrary $\xi \in \mathcal{Y}_h$. But to prove solvability in Theorem 2.1 we have to impose the assumption (6), which also appears in [11].

Lemma 2.3. *For fixed $\xi \in \mathcal{Y}_h$, define*

$$\widetilde{b}(\mathbf{u}_h, \mathbf{v}_h) := b(\mathbf{u}_h, \mathbf{v}_h) + \langle \xi \,,\, (\tfrac{1}{2} - K)u_h + V\phi_h \rangle_\Gamma \langle \xi \,,\, (\tfrac{1}{2} - K)v_h + V\psi_h \rangle_\Gamma \tag{10a}$$

as well as

$$\widetilde{F}(\mathbf{v}_h) := F(\mathbf{v}_h) + \langle \xi \,,\, (\tfrac{1}{2} - K)u_0 \rangle \langle \xi \,,\, (\tfrac{1}{2} - K)v_h + V\psi_h \rangle_\Gamma \tag{10b}$$

for all $\mathbf{u}_h = (u_h, \phi_h), \mathbf{v}_h = (v_h, \psi_h) \in \mathcal{H}_h$. Then, a function $\mathbf{u}_h \in \mathcal{H}_h$ solves (7) *if and only if $\mathbf{u}_h$ solves*

$$\widetilde{b}(\mathbf{u}_h, \mathbf{v}_h) = \widetilde{F}(\mathbf{v}_h) \quad \textit{for all } \mathbf{v} \in \mathcal{H}_h. \tag{11}$$

□

The following lemma is used to prove Theorem 2.1 and essentially states that the stabilization terms added to $b(\cdot,\cdot)$ to obtain $\widetilde{b}(\cdot,\cdot)$ can be used to define an equivalent norm on $\mathcal{H}$.

Lemma 2.4. *Let the linear functional $L : \mathcal{H} \to \mathbb{R}$ fulfill*

$$L((1, 0)) \neq 0. \tag{12}$$

Then, $|||\mathbf{u}|||^2 := \|\nabla u\|^2_{L^2(\Omega)} + \langle \phi \,,\, V\phi \rangle_\Gamma + |L(\mathbf{u})|^2$ for $\mathbf{u} = (u, \phi) \in \mathcal{H}$ defines an equivalent norm $|||\cdot||| \simeq \|\cdot\|_\mathcal{H}$ on $\mathcal{H}$. Moreover, $L(\mathbf{v}) := \langle \xi \,,\, (\tfrac{1}{2} - K)v + V\psi \rangle_\Gamma$ for $\mathbf{v} = (v, \psi) \in \mathcal{H}$ satisfies (12) *if $\xi \in \mathcal{Y}_h$ fulfills* (6). □

Sketch of proof of Theorem 2.1. We consider the operator $\widetilde{B} : \mathcal{H} \to \mathcal{H}^*$ associated to the mapping $\widetilde{b}(\cdot,\cdot)$. With Lemma 2.4, one can show that $\widetilde{B}$ is strongly monotone and Lipschitz continuous. Therefore, standard arguments [15] prove the assertions of Theorem 2.1 for (7) replaced by (11). The equivalence of Lemma 2.3 then concludes the proof. □

3. Johnson-Nédélec coupling

The nonlinear transmission problem (1) can also be reformulated by means of the Johnson-Nédélec coupling [9, 16]. It reads as (4), where the mapping $b : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$ and the linear functional $F : \mathcal{H} \to \mathbb{R}$ are now defined for all $\mathbf{u} = (u, \phi), \mathbf{v} = (v, \psi) \in \mathcal{H}$ via

$$b(\mathbf{u}, \mathbf{v}) := \langle \mathcal{A}\nabla u \,,\, \nabla v\rangle_\Omega - \langle \phi \,,\, v\rangle_\Gamma + \langle \psi \,,\, (\tfrac{1}{2} - K)u\rangle_\Gamma + \langle \psi \,,\, V\phi\rangle_\Gamma, \tag{13a}$$

$$F(\mathbf{v}) := \langle f \,,\, v\rangle_\Omega + \langle \phi_0 \,,\, v\rangle_\Gamma + \langle \psi \,,\, (\tfrac{1}{2} - K)u_0\rangle_\Gamma \tag{13b}$$

As in Section 2, we infer that constant functions lie in the kernel of the mapping $\mathbf{u} \mapsto b(\mathbf{u}, \mathbf{u})$. Thus, solvability cannot be shown directly. Still, Theorem 2.3 holds essentially true for the Johnson-Nédélec coupling [1].

Theorem 3.1. *Assume that* $c_{\rm mon} > c_K/4$, *where* $c_{\rm mon} > 0$ *denotes the monotonicity constant* (2a) *of* $\mathcal{A}$ *and* $0 < c_K < 1$ *denotes the contraction constant of the double-layer potential* [12]. *Then, the assertions of Theorem 2.1 hold true for the Johnson-Nédélec coupling.* □

Remark 3.2. *(i)* The very same results as for the Johnson-Nédélec coupling also hold for the non-symmetric Bielak-MacCamy one-equation coupling [1]. *(ii)* In [10], it is proven that the assumption $c_{\rm mon} > c_K/4$ is not only sufficient but also necessary to prove ellipticity of the bilinear form associated to the stabilized formulation of Steinbach [13]. Nevertheless, numerical experiments in [1] indicate that the assumption $c_{\rm mon} > c_K/4$ is not necessary for existence and uniqueness of discrete solutions of the Johnson-Nédélec coupling.

4. Extensions

The recent work [5] presents how to transfer the ideas of *implicit stabilization* developed for nonlinear Laplace transmission problems [1] to nonlinear elasticity transmission problems. As in [1], we treat the symmetric coupling as well as the non-symmetric one-equation couplings of Johnson-Nédélec and Bielak-MacCamy type. In contrast to Laplace problems, one faces the problem that the kernel of the Lamé operator contains the space of rigid body motions $\mathcal{R}$, with $\dim(\mathcal{R}) = 3$ in 2D and $\dim(\mathcal{R}) = 6$ in 3D. The stabilization process then becomes more complicated, since not only constant functions have to be fixed in the interior domain. However,under the assumption

$$\forall r \in \mathcal{R}\backslash\{0\} \exists \xi \in \mathcal{Y}_h \cap L^2(\Gamma) \quad \langle \xi \,,\, r\rangle_\Gamma \neq 0 \tag{14}$$

on the discrete space $\mathcal{Y}_h$, which is the extension of (6) to elasticity problems, similar results as in Sections 2–3 hold true for nonlinear elasticity problems. It is shown in [5] that assumption (14) is satisfied for $\mathcal{P}^0(\mathcal{E}_h) \subseteq \mathcal{Y}_h$ and regular triangulations $\mathcal{E}_h$ of the boundary into plane surface triangles

Unlike [7], we prove that interior Dirichlet boundaries can be avoided to fix the rigid body motions in the interior domain. Moreover, the explicit stabilization as recently proposed in [14] is avoided by our analysis [5], i.e. we have to implement and solve the original coupling equations only.

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