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A general integrator for the Landau-Lifshitz-Gilbert equation

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***Abstract*—In our contribution, we extend a P1 finite element scheme for the discretization of the Landau-Lifshitz-Gilbert equation (LLG), which has originally been proposed by Alouges [1] for a simpler model problem. Unlike prior works [2], [5], we allow arbitrary contributions to the effective field and elaborate the circumstances under which weak subconvergence towards a weak solution can mathematically be guaranteed. Our analysis particularly includes nonlinear, non-local, and/or time-dependent operators. In addition, we investigate coupling of LLG to the full Maxwell's equations and to the conservation of momentum equation in order to include magnetostrictive effects.**

***Index Terms*—Landau-Lifshitz-Gilbert equation, multiscale problems, magnetostriction, Maxwell, finite elements, convergence analysis.**

I. Introduction

THE UNDERSTANDING of magnetization dynamics, especially on a microscale, is of utter relevance, for example in the development of magnetic sensors, recording heads, and magneto-resistive storage devices. In the literature, it is well-accepted that dynamic micromagnetic phenomena are modeled best by the Landau-Lifshitz-Gilbert equation (LLG) which describes the behaviour of the magnetization $\mathbf{m}$ under the influence of some effective field $\mathbf{h}_{\text{eff}}(\mathbf{m})$ that consists of several contributions such as e.g. the microcrystalline anisotropy or the demagnetization field.

In our contribution, we analyze a weak solver for LLG which is based on the approach of [1]. By exploiting an abstract framework, we can cover general field contributions which might be nonlinear, non-local, and/or time-dependent and are only supposed to fulfil a couple of prerequisites. Applications include multiscale modeling [4], coupling of LLG to the full Maxwell's equations [3], or even to the conservation of momentum equation to include magnetostrictive effects [6]. For both, the basic LLG equation and the coupled systems, we derive algorithms that are shown to be unconditionally convergent as temporal and spatial mesh-size tend to zero. In addition, we stress that our analysis is constructive, in the sense that it even shows the existence of weak solutions of the various investigated problems. Finally, although LLG is nonlinear and even in this general setting, we only need to solve one (resp. two for coupled problems) *linear* systems per timestep.

II. Problem Formulation

We consider a ferromagnetic body $\Omega \subset \mathbb{R}^3$ with polyhedral boundary $\partial\Omega$ and a fixed time-interval $(0, \tau_{\text{end}})$. For a given damping parameter $\alpha > 0$ and an initial magnetization $\mathbf{m}^0 : \Omega \to \mathbb{S}^2 := \{x \in \mathbb{R}^3 : |x| = 1\}$, the dimensionless formulation of LLG on $\Omega_\tau := (0, \tau_{\text{end}}) \times \Omega$ then reads

$$\mathbf{m}_\tau - \alpha\mathbf{m} \times \mathbf{m}_\tau = \mathbf{h}_{\text{eff}}(\mathbf{m}) \times \mathbf{m}. \tag{1a}$$

supplemented by initial and boundary conditions

$$\mathbf{m}(0) = \mathbf{m}^0 \quad \text{in } \Omega, \tag{1b}$$

$$\partial_{\boldsymbol{\nu}}\mathbf{m} = 0 \qquad \text{in } (0, \tau_{\text{end}}) \times \partial\Omega, \tag{1c}$$

where $\boldsymbol{\nu}$ denotes the outer unit normal vector on $\partial\Omega$ and $\mathbf{m}$ the sought magnetization.The effective field, in terms of a general energy contribution $\boldsymbol{\pi}(\cdot)$ that depends on the magnetization $\mathbf{m}$, is given by

$$\mathbf{h}_{\text{eff}}(\mathbf{m}) = C_e\Delta\mathbf{m} - \boldsymbol{\pi}(\mathbf{m}) + \mathbf{f}, \tag{2}$$

where C_e denotes the exchange constant. In this context, the general contribution is only required to fulfil a few assumptions specified below. The generality of the above ansatz and the corresponding convergence analysis now allows to cover many interesting applications and field contributions, such as:

- Linear and nonlinear pointwise operators like the ones arising from unaxial- resp. cubic anisotropy.
- Linear and continuous operators that require a further numerical discretization like the strayfield operator that is computed via e.g. the Fredkin-Köhler approach.
- Uniformly monotone operators like the multiscale operator from [4]. Here, LLG is solved only on a small domain where micromagnetic interactions are relevant. The monotone operator then models the coupling to another, larger domain which is homogenized by means of some nonlinear material law.
- The analysis can also be extended to cover time-dependent operators that do not act pointwise in time, i.e. $\boldsymbol{\pi}_h(\mathbf{m}_{hk}^-)(\tau_j) \neq \boldsymbol{\pi}_h(\mathbf{m}_h^j)$ with the notation from Section III. In this case, the analysis becomes quite involved and additionally depends on the convergence of certain Riemann sums. The interested reader is referred to [6].

We like to emphasize that our analysis, in particular, includes the case

$$\mathbf{h}_{\text{eff}}(\mathbf{m}) = C_e\Delta\mathbf{m} - C_a D\mathbf{\Phi}(\mathbf{m}) - \mathcal{P}(\mathbf{m}) + \mathbf{f} \tag{3}$$

from [2], [5], where $\mathbf{\Phi}$ denotes the microcrystalline anisotropy density with corresponding constant C_a, and $\mathcal{P}(\mathbf{m})$ is the demagnetization field. As mentioned above, also a numerical discretization of those contributions is covered. In view of the subsequent algorithm and our notion of a weak solution, we recall the following two equivalent formulations of LLG

$$\mathbf{m}_\tau = -\frac{1}{1+\alpha^2}\,\mathbf{m}\times\mathbf{h}_{\text{eff}}(\mathbf{m}) - \frac{\alpha}{1+\alpha^2}\,\mathbf{m}\times\big(\mathbf{m}\times\mathbf{h}_{\text{eff}}(\mathbf{m})\big) \tag{4}$$

$$\text{and} \quad \alpha\mathbf{m}_\tau + \mathbf{m}\times\mathbf{m}_\tau = \mathbf{h}_{\text{eff}}(\mathbf{m}) - \big(\mathbf{m}\cdot\mathbf{h}_{\text{eff}}(\mathbf{m})\big)\mathbf{m}. \tag{5}$$

Note that (1a) as well as (4) imply $0 = \mathbf{m}\cdot\mathbf{m}_\tau = \partial_\tau|\mathbf{m}|^2/2$ and thus the modulus constraint $|\mathbf{m}| = 1$ almost everywhere in Ω_τ. In this work, we exploit (1a) for the definition of a weak solution and (5) for construction of our numerical integrator.

III. Basic Numerical Scheme

The key idea of our numerical scheme is to treat the time derivative $\mathbf{m}_\tau$ of $\mathbf{m}$ as an independent variable $\mathbf{v}$ and thus work with a linear equation in $\mathbf{v}$. Recall that $0 = \mathbf{m}\cdot\mathbf{m}_\tau = \mathbf{m}\cdot\mathbf{v}$

3.1. Numerical Integrator. Let $\mathcal{T}_h$ be a conforming triangulation of Ω into compact and non-degenerate tetrahedra $T \in \mathcal{T}_h$ with spatial mesh-size $h > 0$. For the discretization of the magnetization $\mathbf{m}$ and its time derivative $\mathbf{v} = \mathbf{m}_\tau$, we consider the standard P1-FEM space of globally continuous and piecewise affine functions $\mathcal{S}^1(\mathcal{T}_h)^3 =: \mathcal{V}_h$. Given the set of nodes $\mathcal{N}_h$ of $\mathcal{T}_h$ and fixed time τ_j, the discrete magnetization $\mathbf{m}_h^j \approx \mathbf{m}(\tau_j)$ is sought in the set

$$\mathcal{M}_h := \big\{\mathbf{n}_h \in \mathcal{V}_h \,:\, |\mathbf{n}_h(z)| = 1 \text{ for all } z\in\mathcal{N}_h\big\},$$

whereas the discrete time derivative $\mathbf{v}_h^j \approx \mathbf{v}(\tau_j)$ is sought in the discrete tangent space

$$\mathcal{K}_{\mathbf{m}_h^j} := \big\{\mathbf{n}_h \in \mathcal{V}_h \,:\, \mathbf{n}_h(z)\cdot\mathbf{m}_h^j(z) = 0 \text{ for all } z\in\mathcal{N}_h\big\}.$$

This space is used to ensure a discrete version of the non-convex modulus constraint on $\mathbf{m}$. For time discretization, we impose uniform timesteps $0 = \tau_0 < \tau_1 < \ldots < \tau_N = \tau_{\text{end}}$, where the constant timestep size is denoted by $k = \tau_{j+1} - \tau_j$.

As in [5], our analysis also allows to incorporate discrete versions of the field contributions. This is, in particular, interesting for complicated contributions like the demagnetization field which is usually computed numerically, e.g. by means of some fast FEM-BEM methods. In this work, we assume that $\boldsymbol{\pi}$ is a spatial operator which maps the magnetization $\mathbf{m}(\tau) \in \mathbf{L}^2(\Omega)$ to some field $\big(\boldsymbol{\pi}(\mathbf{m})\big)(\tau) = \boldsymbol{\pi}\big(\mathbf{m}(\tau)\big) \in \mathbf{L}^2(\Omega)$. For given $h > 0$, let $\boldsymbol{\pi}_h$ be a numerical realization which maps $\mathbf{m}(\tau_j) \approx \mathbf{m}_h^j \in \mathcal{M}_h$ to some $\boldsymbol{\pi}_h(\mathbf{m}_h^j) \in \mathbf{L}^2(\Omega)$. For any given initial data $\boldsymbol{\varphi}^0$, we denote its spatial approximation in the corresponding discrete space by $\boldsymbol{\varphi}_h^0$. For the upcoming time derivatives, we introduce the difference quotients

$$d_\tau z_i = \frac{z_i - z_{i-1}}{k}, \quad d_\tau^2 z_i = \frac{d_\tau z_i - d_\tau z_{i-1}}{k}.$$

By $\mathbf{f}_h^j$, we denote an approximation of $\mathbf{f}(\tau_j)$ specified below. Finally, let $(\boldsymbol{\varphi},\boldsymbol{\phi}) = \int_\Omega \boldsymbol{\varphi}\cdot\boldsymbol{\phi}$ be the $L^2(\Omega)$-scalar product and $\|\boldsymbol{\varphi}\| = (\boldsymbol{\varphi},\boldsymbol{\varphi})^{1/2}$ the corresponding norm. Then, our numerical time integrator reads as follows:

Algorithm 1: INPUT: Initial approximation $\mathbf{m}_h^0 \in \mathcal{M}_h$ and Gilbert damping parameter $\alpha > 0$, parameter $\theta \in [0,1]$. Then, for $j = 0, 1, 2, \ldots, N-1$ do:

(i) Compute $\mathbf{v}_h^j \in \mathcal{K}_{\mathbf{m}_h^j}$ such that for all $\boldsymbol{\phi}_h \in \mathcal{K}_{\mathbf{m}_h^j}$ holds

$$\begin{aligned}\alpha(\mathbf{v}_h^j,\boldsymbol{\phi}_h) + C_e\,k\theta(\nabla\mathbf{v}_h^j,\nabla\boldsymbol{\phi}_h) + (\mathbf{m}_h^j\times\mathbf{v}_h^j,\boldsymbol{\phi}_h)\\ = -C_e(\nabla\mathbf{m}_h^j,\nabla\boldsymbol{\phi}_h) - \big(\boldsymbol{\pi}_h(\mathbf{m}_h^j),\boldsymbol{\phi}_h\big) + (\mathbf{f}_h^j,\boldsymbol{\phi}_h).\end{aligned} \tag{6}$$

(ii) Define $\mathbf{m}_h^{j+1} \in \mathcal{M}_h$ by $\mathbf{m}_h^{j+1}(\mathbf{z}) = \frac{\mathbf{m}_h^j(\mathbf{z}) + k\mathbf{v}_h^j(\mathbf{z})}{|\mathbf{m}_h^j(\mathbf{z}) + k\mathbf{v}_h^j(\mathbf{z})|}$ for all nodes $\mathbf{z}\in\mathcal{N}_h$ ■

Standard arguments easily show that the above algorithm is well-defined, i.e. it admits a unique solution in each step of the loop. The normalization step (ii) is feasible due to $\mathbf{v}_h^j \in \mathcal{K}_{\mathbf{m}_h^j}$. Note that only the exchange part is treated implicitely, whereas the remaining field contributions are treated explicitely. This is especially relevant for computationally demanding terms like the strayfield contribution. Overall, the above algorithm only requires the solution of one linear system per timestep and is thus computationally very attractive. We refer to [5] for a short discussion on the implementation. The next section shows that the proposed algorithm is also interesting from an analytical point of view.

3.2. Weak Solution and Convergence Result. We start with the definition of a weak solution which is based on (1a).

Definition 2: A function $\mathbf{m}$ is called a *weak solution* of LLG in Ω_τ, if

(i) $\mathbf{m} \in \mathbf{H}^1(\Omega_\tau)$ with $|\mathbf{m}| = 1$ almost everywhere in Ω_τ and $\mathbf{m}(0) = \mathbf{m}^0$ in the sense of traces;

(ii) For all $\boldsymbol{\phi} \in C^\infty(\Omega_\tau)$, we have

$$\begin{aligned}\int_{\Omega_\tau}\mathbf{m}_\tau\cdot\boldsymbol{\phi} - \alpha\int_{\Omega_\tau}(\mathbf{m}\times\mathbf{m}_\tau)\cdot\boldsymbol{\phi} =\\ -C_e\int_{\Omega_\tau}(\nabla\mathbf{m}\times\mathbf{m})\cdot\nabla\boldsymbol{\phi}\\ -\int_{\Omega_\tau}\big(\boldsymbol{\pi}(\mathbf{m})\times\mathbf{m}\big)\cdot\boldsymbol{\phi} + \int_{\Omega_\tau}(\mathbf{f}\times\mathbf{m})\cdot\boldsymbol{\phi}\end{aligned} \tag{7}$$

(iii) for almost all $t \in (0,\tau_{\text{end}})$, we have

$$\|\nabla\mathbf{m}(t)\|_\Omega^2 + \|\mathbf{m}_\tau\|_{\Omega_\tau}^2 \le C, \tag{8}$$

for some $C > 0$ which depends only on $\mathbf{m}^0$ and $\mathbf{f}$. ■

Remark 3: For vanishing external field $\mathbf{f}$ and under certain assumptions on the general field contribution $\boldsymbol{\pi}$, namely boundedness in $\mathbf{L}^4(\Omega_\tau)$ and self-adjointness, the energy estimate (8) can be improved to

$$\mathcal{E}\big(\mathbf{m}(t)\big) + 2\alpha\|\mathbf{m}_\tau\|_{\Omega_\tau}^2 \leq \mathcal{E}\big(\mathbf{m}(0)\big), \tag{9}$$

with the energy

$$\mathcal{E}\big(\mathbf{m}(t)\big) := C_e\|\nabla\mathbf{m}(t)\|_\Omega^2 + \big(\boldsymbol{\pi}(\mathbf{m}(t)), \mathbf{m}(t)\big), \tag{10}$$

see [4]. Similar estimates remain valid for the coupled systems of Section IV below. ■

Now, we interpret the output of Algorithm 1 as discrete functions $\mathbf{m}_{hk}, \mathbf{m}_{hk}^-$ in space-time which are for $\tau \in [\tau_j, \tau_{j+1})$ defined by

$$\mathbf{m}_{hk}(\tau, x) := \frac{\tau - jk}{k}\,\mathbf{m}_h^{j+1}(x) + \frac{(j+1)k - \tau}{k}\,\mathbf{m}_h^j(x), \text{ and}$$
$$\mathbf{m}_{hk}^-(\tau, x) := \mathbf{m}_h^j(x),$$

i.e. piecewise affine resp. constant in time. Then, the following theorem from [4] guarantees weak convergence of a proper subsequence of $\mathbf{m}_{hk}$ towards a weak solution of LLG.

Theorem 4: **(a)** Let $\theta \in (1/2, 1]$ and suppose that the spatial meshes $\mathcal{T}_h$ are uniformly shape regular and satisfy the angle condition

$$(\nabla\eta_i, \nabla\eta_j) \leq 0 \tag{11}$$

for all nodal basis functions $\eta_i, \eta_j \in \mathcal{S}^1(\mathcal{T}_h)^3$ with $i \neq j$. Define the function $\mathbf{f}_{hk}^- : \Omega_\tau \to \mathbb{R}^3$ by $\mathbf{f}_{hk}^-(\tau) := \mathbf{f}_h^j$ for $\tau_j \leq \tau < \tau_{j+1}$. We suppose that

$$\mathbf{f}_{hk}^- \rightharpoonup \mathbf{f} \quad \text{weakly convergent in } \mathbf{L}^2(\Omega_\tau) \tag{12}$$

Moreover, we suppose that $\boldsymbol{\pi}_h(\cdot)$ satisfies

$$\|\boldsymbol{\pi}_h(\mathbf{n})\|_\Omega \leq C_1 \tag{13}$$

for all $h, k > 0$ and all $\mathbf{n} \in \mathbf{L}^2(\Omega)$ with $|\mathbf{n}| \leq 1$ almost everywhere in Ω. Here, $C_1 > 0$ denotes a constant that is independent of h, k, and $\mathbf{n}$, but may depend on Ω. Under these assumptions, a subsequence of $\mathbf{m}_{hk}$ converges strongly in $\mathbf{L}^2(\Omega_\tau)$ towards some function $\mathbf{m}$.

(b) In addition to the above, we assume $\mathbf{m}_h^0 \rightharpoonup \mathbf{m}^0$ weakly in $\mathbf{L}^2(\Omega)$ and

$$\boldsymbol{\pi}_h(\mathbf{m}_{hk}^-) \rightharpoonup \boldsymbol{\pi}(\mathbf{m}) \quad \text{weakly subconv. in } \mathbf{L}^2(\Omega_\tau). \tag{14}$$

Then, a subsequence of the computed FE solutions $\mathbf{m}_{hk}$ weakly converges in $\mathbf{H}^1(\Omega_\tau)$ to a weak solution $\mathbf{m} \in \mathbf{H}^1(\Omega_\tau)$ of LLG. In particular, this even shows existence of weak solutions. ■

The proof basically relies on the following three steps and the reader is referred to [4] for a detailed analysis.

(i) Boundedness of the discrete quantities.
(ii) Existence of weakly convergent subsequences.
(iii) Identification of the limits with weak solutions.

We stress that all the field contributions mentioned in Section II indeed satisfy the assumptions (13)–(14) and are thus covered by the theorem, see [4].

IV. Coupling to other PDEs

In this section, we comment on the coupling of LLG to other equations like the full Maxwell equations or the conservation of momentum equation. In both cases, we will see that the equations can be decoupled in each timestep of Algorithm 1, so that only two linear systems have to be solved. Nevertheless, the analysis shows convergence towards weak solutions of the coupled systems and thus, as before, even existence.

4.1. The Maxwell-Landau-Lifshitz-Gilbert system. We consider a ferromagnetic domain $\Omega \Subset \widehat{\Omega} \subseteq \mathbb{R}^3$. For a given damping parameter $\alpha > 0$, electric and magnetic permeability ε_0, μ_0, and $\Omega_\tau := (0, \tau_{\text{end}}) \times \Omega, \widehat{\Omega}_\tau := (0, \tau_{\text{end}}) \times \widehat{\Omega}$, the Maxwell-Landau-Lifshitz-Gilbert system (MLLG) reads

$$\mathbf{m}_\tau - \alpha\mathbf{m} \times \mathbf{m}_\tau = \mathbf{h}_{\text{eff}}(\mathbf{m}) \times \mathbf{m} \quad \text{in } \Omega_\tau, \tag{15a}$$
$$\varepsilon_0\mathbf{E}_\tau - \nabla \times \mathbf{H} + \sigma\chi_\Omega\mathbf{E} = -\mathbf{J} \quad \text{in } \widehat{\Omega}_\tau, \tag{15b}$$
$$\mu_0\mathbf{H}_\tau + \nabla \times \mathbf{E} = -\mu_0\mathbf{m}_\tau \quad \text{in } \widehat{\Omega}_\tau. \tag{15c}$$

This system is supplemented by the initial and boundary conditions $\mathbf{m}(0, \cdot) = \mathbf{m}^0$ in $\Omega, \mathbf{E}(0, \cdot) = \mathbf{E}^0$ in $\widehat{\Omega}, \mathbf{H}(0, \cdot) = \mathbf{H}^0$ in $\widehat{\Omega}$ and $\partial_{\boldsymbol{\nu}}\mathbf{m} = 0$ on $\partial\Omega_\tau, \mathbf{E} \times \boldsymbol{\nu} = 0$ on $\partial\widehat{\Omega}_\tau$. Here, $\mathbf{h}_{\text{eff}}(\mathbf{m})$ is given by $\mathbf{h}_{\text{eff}}(\mathbf{m}) = C_e\Delta\mathbf{m} + \mathbf{H} - \boldsymbol{\pi}(\mathbf{m})$ and thus the equations are coupled via $\mathbf{H}$. In analogy to Algorithm 1, we propose the following fully decoupled scheme for the solution of (15):

Algorithm 5: INPUT: Initital data $\mathbf{m}^0$, $\mathbf{E}^0$, and $\mathbf{H}^0$, parameter $\theta \in [0, 1]$, $\alpha > 0$. For all $j = 0, \ldots, N-1$ iterate:

(i) Compute unique solution $\mathbf{v}_h^j \in \mathcal{K}_{\mathbf{m}_h^j}$ such that for all $\boldsymbol{\phi}_h \in \mathcal{K}_{\mathbf{m}_h^j}$ holds

$$\begin{aligned}\alpha(\mathbf{v}_h^j, \boldsymbol{\phi}_h) &+ \big((\mathbf{m}_h^j \times \mathbf{v}_h^j), \boldsymbol{\phi}_h\big) \\ &= -C_e\big(\nabla(\mathbf{m}_h^j + \theta k\mathbf{v}_h^j), \nabla\boldsymbol{\phi}_h\big) \\ &\quad + (\mathbf{H}_h^j, \boldsymbol{\phi}_h) - \big(\pi(\mathbf{m}_h^j), \boldsymbol{\phi}_h\big)\end{aligned} \tag{16a}$$

(ii) Compute unique solution $(\mathbf{E}_h^{j+1}, \mathbf{H}_h^{j+1}) \in (\mathcal{X}_h, \mathcal{Y}_h)$ such that for all $(\boldsymbol{\varphi}_h, \boldsymbol{\zeta}_h) \in \mathcal{X}_h \times \mathcal{Y}_h$ holds

$$\begin{aligned}\varepsilon_0(d_\tau\mathbf{E}_h^{j+1}, \boldsymbol{\varphi}_h) - (\mathbf{H}_h^{j+1}, \nabla \times \boldsymbol{\varphi}_h) + \sigma(\chi_\Omega\mathbf{E}_h^{j+1}, \boldsymbol{\varphi}_h) \\ = -(\mathbf{J}^j, \boldsymbol{\varphi}_h)\end{aligned} \tag{16b}$$

$$\mu_0(d_\tau\mathbf{H}_h^{j+1}, \boldsymbol{\zeta}_h) + (\nabla \times \mathbf{E}_h^{j+1}, \boldsymbol{\zeta}_h) = -\mu_0(\mathbf{v}_h^j, \boldsymbol{\zeta}_h) \tag{16c}$$

(iii) Define $\mathbf{m}_h^{j+1} \in \mathcal{M}_h$ by $\mathbf{m}_h^{j+1}(\mathbf{z}) = \frac{\mathbf{m}_h^j(\mathbf{z}) + k\mathbf{v}_h^j(\mathbf{z})}{|\mathbf{m}_h^j(\mathbf{z}) + k\mathbf{v}_h^j(\mathbf{z})|}$ for all nodes $\mathbf{z} \in \mathcal{N}_h$ ■

For the discretization of $\mathbf{E}$ and $\mathbf{H}$, we choose first order edge elements $\mathcal{X}_h \subset \mathbf{H}^0(\mathbf{curl}; \widehat{\Omega})$ and piecewise constants $\mathcal{Y}_h \subset \mathbf{L}^2(\widehat{\Omega})$ such that there holds $\nabla \times \mathcal{X}_h \subset \mathcal{Y}_h$. Then, the above algorithm is well-defined and admits unique solutions in each step. As usual and in addition to the above notation, we assume $\mathbf{m}^0 \in \mathbf{H}^1(\Omega, \mathbb{S}^2), \mathbf{H}^0, \mathbf{E}^0 \in \mathbf{L}^2(\widehat{\Omega}, \mathbb{R}^3), \mathbf{J} \in \mathbf{L}^2(\widehat{\Omega}_\tau, \mathbb{R}^3)$, $\text{div}\mathbf{H}^0 = 0$ as well as

$$\text{div}(\mathbf{H}^0 + \chi_\Omega\mathbf{m}^0) = 0 \text{ in } \widehat{\Omega}, \text{ and } (\mathbf{H}^0 + \chi_\Omega\mathbf{m}^0, \mathbf{n}) = 0 \text{ on } \partial\widehat{\Omega}.$$

We can now define a weak solution of (15).

Definition 6: Given the above assumptions, the tupel $(\mathbf{m}, \mathbf{E}, \mathbf{H})$ is called a weak solution of MLLG if,

(i) $\mathbf{m} \in H^1(\Omega_\tau)$ with $|\mathbf{m}| = 1$ almost everywhere in Ω_τ and $(\mathbf{E}, \mathbf{H}) \in \mathbf{L}^2(\widehat{\Omega}_\tau)$ and $\mathbf{m}(0) = \mathbf{m}^0$ in the sense of traces;

(ii) for all $\boldsymbol{\phi} \in C^\infty(\Omega_\tau)$ and $\boldsymbol{\zeta} \in C_c^\infty\big([0,T); C^\infty(\widehat{\Omega}) \cap \mathbf{H}^0(\boldsymbol{curl}, \widehat{\Omega})\big)$, we have

$$\int_{\Omega_\tau} \mathbf{m}_\tau \cdot \boldsymbol{\phi} - \alpha \int_{\Omega_\tau} (\mathbf{m} \times \mathbf{m}_\tau) \cdot \boldsymbol{\phi} = -C_e \int_{\Omega_\tau} (\nabla \mathbf{m} \times \mathbf{m}) \cdot \nabla \boldsymbol{\phi} + \int_{\Omega_\tau} (\mathbf{H} \times \mathbf{m}) \cdot \boldsymbol{\phi} - \int_{\Omega_\tau} (\boldsymbol{\pi}(\mathbf{m}) \times \mathbf{m}) \cdot \boldsymbol{\phi} \tag{17a}$$

$$- \varepsilon_0 \int_{\widehat{\Omega}_\tau} \mathbf{E} \cdot \boldsymbol{\zeta}_\tau - \int_{\widehat{\Omega}_\tau} \mathbf{H} \cdot (\nabla \times \boldsymbol{\zeta}) + \sigma \int_{\Omega_\tau} \mathbf{E} \cdot \boldsymbol{\zeta} = - \int_{\widehat{\Omega}_\tau} \mathbf{J} \cdot \boldsymbol{\zeta} + \varepsilon_0 \int_{\widehat{\Omega}} \mathbf{E}^0 \cdot \boldsymbol{\zeta}(0, \cdot) \tag{17b}$$

$$- \mu_0 \int_{\widehat{\Omega}_\tau} \mathbf{H} \cdot \boldsymbol{\zeta}_\tau + \int_{\widehat{\Omega}_\tau} \mathbf{E} \cdot (\nabla \times \boldsymbol{\zeta}) = -\mu_0 \int_{\Omega_\tau} \mathbf{m}_\tau \cdot \boldsymbol{\zeta} + \mu_0 \int_{\widehat{\Omega}} \mathbf{H}^0 \cdot \boldsymbol{\zeta}(0, \cdot) \tag{17c}$$

(iii) for almost all $t \in (0, \tau_{\text{end}})$, we have bounded energy $\|\nabla \mathbf{m}(t)\|^2_\Omega + \|\mathbf{m}_\tau\|^2_{\Omega_\tau} + \|\mathbf{H}(t)\|^2_{\widehat{\Omega}} + \|\mathbf{E}(t)\|^2_{\widehat{\Omega}} \le C$ where $C > 0$ is independent of h and k. ■

Remark 7: Note that the weak solution only requires boundedness of $\mathbf{H}$ and $\mathbf{E}$ in $\mathbf{L}^2(\widehat{\Omega}_\tau)$. This is due to the fact, that the analysis shows boundedness of the discrete quantities in this space only and lead to (17b) and (17c), where the time derivative is moved to the test function. ■

The next theorem states the convergence result for the MLLG system from [3].

Theorem 8: Let $(\mathbf{m}_{hk}, \mathbf{v}_{hk}, \mathbf{H}_{hk}, \mathbf{E}_{hk})$ be the quantities obtained by Algorithm 5 and assume (11), (13), (14), $\theta \in (1/2, 1]$, and convergence of the initial data $\mathbf{m}_h^0, \mathbf{E}_h^0$ and $\mathbf{H}_h^0$ weakly in $\mathbf{L}^2$. Then, as $(h,k) \to (0,0)$ independently of each other, a subsequence of $(\mathbf{m}_{hk}, \mathbf{H}_{hk}, \mathbf{E}_{hk})$ converges weakly in $\mathbf{H}^1(\Omega_\tau) \times \mathbf{L}^2(\widehat{\Omega}_\tau) \times \mathbf{L}^2(\widehat{\Omega}_\tau)$ to a weak solution $(\mathbf{m}, \mathbf{H}, \mathbf{E})$ of MLLG. In particular, this yields existence of a weak solution of MLLG in the sense of Definition 6. ■

4.2. LLG with magnetostriction. Finally, we would like to comment on the coupling of LLG to the conservation of momentum equation in order to include magnetostrictive effects. In this context, we consider the LLG equation (1) on Ω_τ with $\mathbf{h}_{\text{eff}}(\mathbf{m}) = \Delta \mathbf{m} + \mathbf{h}_{\text{m}} - \boldsymbol{\pi}(\mathbf{m})$, where $\mathbf{h}_{\text{m}}$ denotes the magnetostrictive component given by

$$(\mathbf{h}_{\text{m}})_s := \sum_{pqr} \lambda^m_{pqrs} \sigma_{pq}(\mathbf{m})_r \quad s = 1,2,3.$$

Here, we implicitly assume linear dependence of the stress tensor $\boldsymbol{\sigma} = \{\sigma_{pq}\}$, with $\sigma_{pq} = \sum_{rs} \lambda^e_{pqrs} \varepsilon^e_{rs}$, on the elastic part of the total strain $\boldsymbol{\varepsilon}^e = \{\varepsilon^e_{pq}\}$ given by $\boldsymbol{\varepsilon}^e(\mathbf{u}, \mathbf{m}) := \boldsymbol{\varepsilon}(\mathbf{u}) - \boldsymbol{\varepsilon}^m(\mathbf{m})$. The vector field $\mathbf{u}$ denotes the *displacement*, and the *total strain* is defined by the symmetric part of the gradient of $\mathbf{u}$, i.e. $\boldsymbol{\varepsilon}(\mathbf{u}) = \big(\nabla \mathbf{u} + (\nabla \mathbf{u})^T\big)/2$. The magnetic part of the total strain tensor is finally given by $\varepsilon^m_{pq}(\mathbf{m}) = \sum_{rs} \lambda^m_{pqrs}(\mathbf{m})_r(\mathbf{m})_s$. In addition, we assume the material tensors $\boldsymbol{\lambda}^e$ and $\boldsymbol{\lambda}^m$ to be symmetric ($\lambda_{pqrs} = \lambda_{qprs} = \lambda_{pqsr} = \lambda_{rspq}$) and positive definite, i.e. $\sum_{pqrs} \lambda_{pqrs} \xi_{pq} \xi_{rs} \ge \lambda^\star \sum_{pq} \xi^2_{pq}$ with bounded entries for some $\lambda^\star > 0$. The coupling of LLG to the conservation of momentum equation

$$\varrho \mathbf{u}_{\tau\tau} - \nabla \cdot \boldsymbol{\sigma} = 0 \quad \text{in } \Omega_\tau, \tag{18}$$

is now realized via the stress tensor $\boldsymbol{\sigma}$, where the mass density ϱ is assumed to be constant and independent of the deformation. Equation (18) is additionally supplemented by the initial and boundary conditions $\mathbf{u}(0) = \mathbf{u}^0, \mathbf{u}_\tau(0) = \dot{\mathbf{u}}^0$ in Ω, and $\mathbf{u} = 0$ on $\partial\Omega$. Again, we propose a fully decoupled algorithm based on the general scheme (6) to tackle this problem.

Algorithm 9: INPUT: Initital data $\mathbf{m}^0$, $\mathbf{u}^0$, and $\dot{\mathbf{u}}^0$, parameter $\theta \in [0,1]$, $\alpha > 0$. For $j = 0, \ldots, N-1$ iterate:

(i) Compute unique solution $\mathbf{v}_h^j \in \mathcal{K}_{\mathbf{m}_h^j}$ such that for all $\boldsymbol{\phi}_h \in \mathcal{K}_{\mathbf{m}_h^j}$ we have

$$\alpha(\mathbf{v}_h^j, \boldsymbol{\phi}_h) + \big((\mathbf{m}_h^j \times \mathbf{v}_h^j), \boldsymbol{\phi}_h\big) = -C_e\big(\nabla(\mathbf{m}_h^j + \theta k \mathbf{v}_h^j), \nabla \boldsymbol{\phi}_h\big) + \big(\mathbf{h}_{\text{m}}(\mathbf{u}_h^j, \mathbf{m}_h^j), \boldsymbol{\phi}_h\big) - \big(\boldsymbol{\pi}(\mathbf{m}_h^j), \boldsymbol{\phi}_h\big) \tag{19}$$

(ii) Define $\mathbf{m}_h^{j+1} \in \mathcal{M}_h$ by $\mathbf{m}_h^{j+1}(\mathbf{z}) = \frac{\mathbf{m}_h^j(\mathbf{z}) + k\mathbf{v}_h^j(\mathbf{z})}{|\mathbf{m}_h^j(\mathbf{z}) + k\mathbf{v}_h^j(\mathbf{z})|}$ for all nodes $\mathbf{z} \in \mathcal{N}_h$

(iii) Compute unique solution $\mathbf{u}_h^{j+1} \in \mathcal{S}^1_0(\mathcal{T}_h)^3$ such that for all $\boldsymbol{\varphi}_h \in \mathcal{S}^1_0(\mathcal{T}_h)^3$ we have

$$\varrho(d_\tau^2 \mathbf{u}_h^{j+1}, \boldsymbol{\varphi}_h) + \big(\boldsymbol{\lambda}^e \boldsymbol{\varepsilon}(\mathbf{u}_h^{j+1}), \boldsymbol{\varepsilon}(\boldsymbol{\varphi}_h)\big) = \big(\boldsymbol{\lambda}^e \boldsymbol{\varepsilon}^m(\mathbf{m}_h^{j+1}), \boldsymbol{\varepsilon}(\boldsymbol{\varphi}_h)\big) \tag{20}$$

■

The second equation (20) stems from the variational form of (18). As before, the algorithm is well-defined and we derive a convergence theorem that also shows existence of weak solutions. Here, a weak solution of the coupled system is defined analogously to Definition 6 and, as in (17), one time derivative of $\mathbf{u}$ is moved to the test function $\boldsymbol{\varphi}$.

Theorem 10: Let $(\mathbf{v}_h^j, \mathbf{u}_h^{j+1})$ solve Algorithm 9 and assume (11), (13), (14), $\theta \in (1/2, 1]$ as well as weak convergence of the initial data $\mathbf{m}_h^0, \mathbf{u}_h^0$ and $\dot{\mathbf{u}}_h^0$. Then, as $(h,k) \to (0,0)$ independently of each other, a subsequence of $(\mathbf{m}_{hk}, \mathbf{u}_{hk})$ converges weakly in $\mathbf{H}^1(\Omega_\tau) \times \mathbf{H}^1(\Omega_\tau)$ to a weak solution $(\mathbf{m}, \mathbf{u})$ of (1) with (18). In particular, LLG with magnetostriction admits a weak solution. ■

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