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Positive solutions of nonlinear Dirichlet BVPs in ODEs with time and space singularities

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Dedicated to Jean Mawhin on the occasion of his 70th birthday

Abstract

In this paper, we discuss the existence of positive solutions to the singular Dirichlet boundary value problems (BVPs) in ordinary differential equations (ODEs) of the form

$$u''(t) + \frac{a}{t}u'(t) - \frac{a}{t^2}u(t) = f(t, u(t), u'(t)), \quad u(0) = 0, \;\; u(T) = 0,$$

where $a \in (-1, 0)$. The nonlinearity $f(t, x, y)$ may be singular for the space variables $x = 0$ and/or $y = 0$. Moreover, since $a \neq 0$ the differential operator on the left hand side of the differential equation is singular at $t = 0$. Sufficient conditions for the existence of positive solutions of the above BVPs are formulated and asymptotic properties of solutions are specified. The theory is illustrated by numerical experiments computed using the open domain MATLAB code `bvpsuite`, based on polynomial collocation.

Mathematics Subject Classification 2010: 34B18, 34B16, 34A12

Key words: Singular ordinary differential equation of the second order, time singularities, space singularities, positive solutions, existence of solutions, polynomial collocation.

1 Introduction

In the present work, we analyze the existence of positive solutions to the singular Dirichlet BVP,

$$u''(t) + \frac{a}{t}u'(t) - \frac{a}{t^2}u(t) = f(t, u(t), u'(t)), \tag{1a}$$

$$u(0) = 0, \quad u(T) = 0. \tag{1b}$$

Here, we assume that $T > 0$, $a \in (-1, 0)$ and f satisfies the local Carathéodory conditions on $[0, T] \times \mathcal{D}$, where $\mathcal{D} := \mathbb{R}_+ \times \mathbb{R}_0$ and $\mathbb{R}_+ := (0, \infty)$, $\mathbb{R}_0 := \mathbb{R} \setminus \{0\}$. Let us recall that a function $h : [0, T] \times \mathcal{A} \to \mathbb{R}$, $\mathcal{A} \subset \mathbb{R} \times \mathbb{R}$, satisfies the *local Carathéodory conditions* on $[0, T] \times \mathcal{A}$, if

(i) $h(\cdot, x, y) : [0, T] \to \mathbb{R}$ is measurable for all $(x, y) \in \mathcal{A}$,

(ii) $h(t, \cdot, \cdot) : \mathcal{A} \to \mathbb{R}$ is continuous for a.e. $t \in [0, T]$,

(iii) for each compact set $\mathcal{U} \subset \mathcal{A}$ there exists a function $m_{\mathcal{U}} \in L^1[0, T]$ such that

$$|h(t, x, y)| \le m_{\mathcal{U}}(t) \text{ for a.e. } t \in [0, T] \text{ and all } (x, y) \in \mathcal{U}.$$

For such functions we use the notation $h \in Car([0, T] \times \mathcal{A})$. Moreover, $f(t, x, y)$ may become singular when the space variables x and/or y vanish, which means that $f(t, x, y)$ may become unbounded for $x = 0$ and a.e. $t \in [0, T]$ and all $y \in \mathbb{R}_0$, and/or it may be unbounded for $y = 0$ and a.e. $t \in [0, T]$ and all $x \in \mathbb{R}_+$. Finally, since $a \neq 0$, equation (1a) has a singularity of the first kind at the time variable $t = 0$ because

$$\int_0^T \frac{1}{t}\,\mathrm{d}t = \infty, \quad \int_0^T \frac{1}{t^2}\,\mathrm{d}t = \infty.$$

The differential operator on the left hand side of equation (1a) can be equivalently written as $(t^{-a}(t^a u)')'$ and, after the substitution $x(t) = t^a u(t)$, it takes the form $(t^{-a}x')'$, which arises in numerous important applications. Operators of such type were studied in phase transitions of Van der Waals fluids [5, 10, 15, 29], in population genetics, especially in models for the spatial distribution of the genetic composition of a population [8, 9], in the homogeneous nucleation theory [1], in relativistic cosmology for description of particles which can be treated as domains in the universe [18], and in the nonlinear field theory [11], in particular, when describing bubbles generated by scalar fields of the Higgs type in the Minkowski spaces [7].

The aim of this paper is to study the case $a \in (-1, 0)$ which is fundamentally different from the case $a \in (-\infty, -1)$. The latter setting was studied in [25, 26], where structure and properties of the set of all positive solutions to (1) were investigated (the cardinality of this set is a continuum).

In the sequel, we introduce the basic notation and state the preliminary results required in the analysis of problem (1). Here, we focuss our attention on the case $a \in (-1, 0)$ and prove the existence of at least one positive solution of (1). In contrast to [25, 26], we consider the more general situation in which f may be also singular at $y = 0$. This means that we have to deal with the following additional difficulties:
Let u be a positive solution of problem (1), where $f(t, x, y)$ has a singularity at $y = 0$. Then there exists $t_0 \in (0, T)$ such that $u(t_0) > 0$, $u'(t_0) = 0$ and hence f is unbounded in a neighborhood of the point $(t_0, u(t_0), u'(t_0))$. Unfortunately, we do not know the exact position of t_0 and therefore, it is not possible to construct a universal Lebesgue integrable majorant for all functions $f(t, u_n(t), u_n'(t))$, where u_n are positive solutions of a sequence of auxiliary regular problems. Consequently, the Lebesgue dominated convergence theorem is not applicable and we have to use arguments based on the Vitali convergence theorem instead, see Lemma 2. Another tool used in the proofs is a combination of regularization and sequential techniques with the Leray-Schauder nonlinear alternative.

The investigation of singular Dirichlet BVPs has a long history and a lot of methods for their analysis are available. One of the most important, is the topological degree method providing various fixed point theorems and existence alternative theorems, see e.g. Lemma 1. For more information on the topological degree method and its application to numerous BVPs, including Dirichlet problems, we refer the reader to the monographs by J. Mawhin [19, 20, 21].

Throughout this paper, we work with the following conditions on the function f in (1a):

(H_1) $f \in Car([0, T] \times \mathcal{D})$, where $\mathcal{D} = \mathbb{R}_+ \times \mathbb{R}_0$.

(H_2) There exists an $\varepsilon > 0$ such that

$$-f(t, x, y) \geq \varepsilon \quad \text{for a.e. } t \in [0, T] \text{ and all } (x, y) \in \mathcal{D}.$$

(H_3) For a.e. $t \in [0, T]$ and all $(x, y) \in \mathcal{D}$, the estimate

$$-f(t, x, y) \leq \varphi(t) h(x, |y|) + g(x) + r(|y|)$$

holds, where $\varphi \in L^1[0, T]$, $h \in C([0, \infty) \times [0, \infty))$, $g, r \in C(\mathbb{R}_+)$ are positive, h is nondecreasing in both its arguments, g and r are nonincreasing, and

$$\lim_{z \to \infty} \frac{h(z, z)}{z} = 0,$$

$$\int_0^b g(s)\, \mathrm{d}s < \infty, \ \int_0^b r(s)\, \mathrm{d}s < \infty \quad \text{for each } b \in \mathbb{R}_+.$$

By

$$\|x\|_\infty = \max\{|x(t)| : t \in [0,T]\}, \quad \|x\|_1 = \int_0^T |x(t)|\,\mathrm{d}t$$

we denote the norms in $C[0,T]$ and $L^1[0,T]$, respectively. $AC^1[0,T]$ denotes the set of functions whose first derivative is absolutely continuous on $[0,T]$, while $AC^1_{loc}(0,T]$ is the set of functions having absolutely continuous first derivative on each compact subinterval of $(0,T]$. We use the symbol meas$\,(\mathcal{M})$ to denote the Lebesgue measure of $\mathcal{M}$.

Definition 1 We say that a function $u \in AC^1[0,T]$ is *a positive solution of problem* (1) if $u > 0$ on $(0,T)$, u satisfies the boundary conditions (1b) and (1a) holds for a.e. $t \in [0,T]$.

Remark 1 Let a function g have the properties specified in (H_3). Then for each $b, c \in \mathbb{R}_+$, $\int_0^b g(cs)\,\mathrm{d}s < \infty$, and it follows from the inequality

$$t(T-t) \geq \begin{cases} \dfrac{T}{2}\,t \ \text{ for } t \in \left[0, \dfrac{T}{2}\right], \\ \dfrac{T}{2}\,(T-t) \ \text{ for } t \in \left[\dfrac{T}{2}, T\right], \end{cases}$$

that

$$\int_0^T g(ct(T-t))\,\mathrm{d}t < \infty \ \text{ for each } c \in \mathbb{R}_+. \tag{2}$$

In order to prove that the singular problem (1) has a positive solution, we use regularization and sequential techniques. To this end for $n \in \mathbb{N}$ we define functions $f_n^* : [0,T] \times (\mathbb{R} \times \mathbb{R}_0) \to \mathbb{R}$ and $f_n : [0,T] \times \mathbb{R}^2 \to \mathbb{R}$ by

$$f_n^*(t,x,y) = \begin{cases} f(t,x,y) \ \text{ if } x \geq \dfrac{1}{n}, \\ f\left(t, \dfrac{1}{n}, y\right) \ \text{ if } x < \dfrac{1}{n}, \end{cases}$$

and

$$f_n(t,x,y) = \begin{cases} f_n^*(t,x,y) \ \text{ if } |y| \geq \dfrac{1}{n}, \\ \dfrac{n}{2}\left[f_n^*\left(t,x,\dfrac{1}{n}\right)\left(y+\dfrac{1}{n}\right) - f_n^*\left(t,x,-\dfrac{1}{n}\right)\left(y-\dfrac{1}{n}\right)\right] \ \text{ if } |y| < \dfrac{1}{n}, \end{cases}$$

respectively. Then, it follows from (H_1) that $f_n \in Car([0,T]\times\mathbb{R}^2)$ and (H_2) and (H_3) yield

$$-f_n(t,x,y) \geq \varepsilon \ \text{ for a.e. } t \in [0,T] \text{ and all } x, y \in \mathbb{R}, \tag{3}$$

$$\left.\begin{array}{c} -f_n(t,x,y) \le \varphi(t)h(1+|x|,1+|y|)+g(|x|)+r(|y|) \\ \text{for a.e. } t\in[0,T] \text{ and all } x,y\in\mathbb{R}_0. \end{array}\right\} \tag{4}$$

Hence

$$\varepsilon \le -\lambda f_n(t,x,y)+(1-\lambda)\varepsilon \quad \text{for a.e. } t\in[0,T] \text{ and all } x,y\in\mathbb{R},\ \lambda\in[0,1], \tag{5}$$

and

$$\left.\begin{array}{r} -\lambda f_n(t,x,y)+(1-\lambda)\varepsilon \le \varphi(t)h(1+|x|,1+|y|)+g(|x|)+r(|y|) \\ \text{for a.e. } t\in[0,T] \text{ and all } x,y\in\mathbb{R}_0,\ \lambda\in[0,1]. \end{array}\right\} \tag{6}$$

As a first step in the analysis, we investigate auxiliary regular BVPs of the form

$$u''(t)+\frac{a}{t}u'(t)-\frac{a}{t^2}u(t)=f_n(t,u(t),u'(t)), \tag{7a}$$

$$u(0)=0, \quad u(T)=0. \tag{7b}$$

To show the solvability of (7), we use the following alternative of the Leray-Schauder type which follows from [2, Theorem 5.1].

Lemma 1 *Let E be a Banach space, U an open subset of E and $\ell\in U$. Assume that $\mathcal{F}:\overline{U}\to E$ is a compact operator. Then either*

$A1:$ $\mathcal{F}$ has a fixed point in $\overline{U}$, or

$A2:$ there exists an element $u\in\partial U$ and $\lambda\in(0,1)$ with $u=\lambda\mathcal{F}(u)+(1-\lambda)\ell$.

In limit processes we apply the following Vitali convergence theorem, cf. [4, 12, 22].

Lemma 2 *Let $\{\rho_n\}\subset L^1[0,T]$ and let $\lim_{n\to\infty}\rho_n(t)=\rho(t)$ for a.e. $t\in[0,T]$. Then the following statements are equivalent:*

(i) $\rho\in L^1[0,T]$ and $\lim_{n\to\infty}\|\rho_n-\rho\|_1=0$,

(ii) the sequence $\{\rho_n\}$ is uniformly integrable on $[0,T]$.

We recall that a sequence $\{\rho_n\}\subset L^1[0,T]$ is called *uniformly integrable on* $[0,T]$ if for any $\varepsilon>0$ there exists $\delta>0$ such that if $\mathcal{M}\subset[0,T]$ and $\text{meas}\,(\mathcal{M})<\delta$, then

$$\int_{\mathcal{M}}|\rho_n(t)|\,\mathrm{d}t<\varepsilon, \quad n\in\mathbb{N}.$$

The paper is organized as follows. In Section 2, we collect auxiliary results used in the subsequent analysis. Section 3 is devoted to the study of limit properties of solutions to equation (7a). In Section 4, we investigate auxiliary regular problems associated with the singular problem (1). We show their solvability and describe properties of their solutions. An existence result for the singular problem (1) is given in Section 5. Finally, in Section 6, we illustrate the theoretical findings by means of numerical experiments.

Throughout the paper $a\in(-1,0)$.

2 Preliminaries

In this section, auxiliary statements necessary for the subsequent analysis are formulated.

Lemma 3 *Let $\rho \in L^1[0,T]$ and let*

$$r(t) = \frac{1}{t^{a+1}} \int_0^t s^{a+1} \rho(s)\,\mathrm{d}s \text{ for } t \in (0,T],$$

$$H(t) = t \int_t^T \frac{1}{s^{a+2}} \left(\int_0^s \xi^{a+1} \rho(\xi)\,\mathrm{d}\xi \right) \mathrm{d}s \text{ for } t \in (0,T].$$

Then

(i) r can be extended on $[0,T]$ with $r \in C[0,T]$ and $r(0) = 0$,

(ii) H can be extended on $[0,T]$ with $H \in AC^1[0,T]$, and the equality

$$H''(t) + \frac{a}{t} H'(t) - \frac{a}{t^2} H(t) = -\rho(t) \tag{8}$$

holds for a.e. $t \in [0,T]$.

Proof. (i) It is clear that $r \in C(0,T]$. Since

$$\left| \frac{1}{t^{a+1}} \int_0^t s^{a+1} \rho(s)\,\mathrm{d}s \right| \le \int_0^t |\rho(s)|\,\mathrm{d}s \quad \text{for } t \in (0,T], \tag{9}$$

we have $\lim_{t \to 0^+} r(t) = 0$. Setting $r(0) := 0$, $r \in C[0,T]$ follows.

(ii) Let

$$p(t) = \int_t^T \frac{1}{s^{a+2}} \left(\int_0^s \xi^{a+1} \rho(\xi)\,\mathrm{d}\xi \right) \mathrm{d}s \quad \text{for } t \in (0,T].$$

Then $p \in C(0,T]$ and $H(t) = tp(t)$ for $t \in (0,T]$. We now show that p can be extended on $[0,T]$ in such a way that $p \in C[0,T]$. Integrating by parts yields

$$\begin{aligned} \int_t^T \frac{1}{s^{a+2}} \left(\int_0^s \xi^{a+1} \rho(\xi)\,\mathrm{d}\xi \right) \mathrm{d}s &= \frac{1}{(a+1)t^{a+1}} \int_0^t s^{a+1} \rho(s)\,\mathrm{d}s \\ &\quad - \frac{1}{(a+1)T^{a+1}} \int_0^T s^{a+1} \rho(s)\,\mathrm{d}s + \frac{1}{a+1} \int_t^T \rho(s)\,\mathrm{d}s \end{aligned} \tag{10}$$

for $t \in (0,T]$. Hence $\lim_{t \to 0^+} p(t) = A$, where

$$A = -\frac{1}{(a+1)T^{a+1}} \int_0^T s^{a+1} \rho(s)\,\mathrm{d}s + \frac{1}{a+1} \int_0^T \rho(s)\,\mathrm{d}s.$$

Let $p(0) := A$. Then $p \in C[0,T]$. Since $H'(t) = p(t) + tp'(t) = p(t) - r(t)$ for $t \in (0,T]$, we see that H can be extended on $[0,T]$ with $H \in C^1[0,T]$. Moreover,

$$\begin{aligned} H''(t) = p'(t) - r'(t) &= -\frac{1}{t^{a+2}} \int_0^t s^{a+1} \rho(s)\,\mathrm{d}s + \frac{a+1}{t^{a+2}} \int_0^t s^{a+1} \rho(s)\,\mathrm{d}s - \rho(t) \\ &= \frac{a}{t^{a+2}} \int_0^t s^{a+1} \rho(s)\,\mathrm{d}s - \rho(t). \end{aligned}$$

In particular,

$$H''(t) = \frac{a}{t^{a+2}} \int_0^t s^{a+1} \rho(s)\,\mathrm{d}s - \rho(t) \quad \text{for a.e. } t \in [0,T]. \tag{11}$$

Hence, cf. (10),

$$\int_0^T H''(s)\,\mathrm{d}s = a \int_0^T \frac{1}{s^{a+2}} \left(\int_0^s \xi^{a+1} \rho(\xi)\,\mathrm{d}\xi \right) \mathrm{d}s - \int_0^T \rho(t)\,\mathrm{d}t = aA - \int_0^T \rho(t)\,\mathrm{d}t,$$

and therefore, $H'' \in L^1[0,T]$. Consequently, $H \in AC^1[0,T]$. Finally, it follows from $H'(t) = p(t) - r(t)$ and $p(t) = \frac{H(t)}{t}$ that $r(t) = \frac{H(t)}{t} - H'(t)$. Since, by (11), $\frac{a}{t} r(t) = H''(t) + \rho(t)$, we see that equality (8) is satisfied for a.e. $t \in [0,T]$ which completes the proof. □

Lemma 4 *Let $\{\rho_n\} \subset L^1[0,T]$ be a uniformly integrable sequence on $[0,T]$ and let $\lim_{n\to\infty} \rho_n(t) = \rho(t)$ for a.e. $t \in [0,T]$. Then the sequence*

$$\left\{ \frac{1}{t^{a+1}} \int_0^t s^{a+1} \rho_n(s)\,\mathrm{d}s \right\} \quad \textit{is equicontinuous on } [0,T]. \tag{12}$$

Proof. It follows from Lemma 2 that $\|\rho_n\|_1 \le L$ for $n \in \mathbb{N}$, where L is a positive constant. Recall that by Lemma 3(i), $\left\{ \frac{1}{t^{a+1}} \int_0^t s^{a+1} \rho_n(s)\,\mathrm{d}s \right\} \subset C[0,T]$. Let us assume that (12) does not hold. Then there exist $\varepsilon > 0$, $\{k_n\} \subset \mathbb{N}$ and $\{\xi_n\}$, $\{\eta_n\} \subset [0,T]$ such that $\lim_{n\to\infty} k_n = \infty$, $\lim_{n\to\infty}(\xi_n - \eta_n) = 0$ and

$$\left| \frac{1}{\xi_n^{a+1}} \int_0^{\xi_n} s^{a+1} \rho_{k_n}(s)\,\mathrm{d}s - \frac{1}{\eta_n^{a+1}} \int_0^{\eta_n} s^{a+1} \rho_{k_n}(s)\,\mathrm{d}s \right| \ge \varepsilon \quad \text{for } n \in \mathbb{N}. \tag{13}$$

Since $\{\xi_n\}$ and $\{\eta_n\}$ are bounded sequences, we may assume that they are convergent, and $\lim_{n\to\infty} \xi_n = \tau = \lim_{n\to\infty} \eta_n$. If $\tau = 0$, then (cf. (9))

$$\lim_{n\to\infty} \frac{1}{\xi_n^{a+1}} \int_0^{\xi_n} s^{a+1} \rho_{k_n}(s)\,\mathrm{d}s = 0, \quad \lim_{n\to\infty} \frac{1}{\eta_n^{a+1}} \int_0^{\eta_n} s^{a+1} \rho_{k_n}(s)\,\mathrm{d}s = 0,$$

which contradicts (13). Let $\tau \in (0,T]$. Since $\lim_{n\to\infty} \left(\frac{1}{\xi_n^{a+1}} - \frac{1}{\eta_n^{a+1}} \right) = 0$ and since the uniform integrability of the sequence $\{\rho_n\}$ on $[0,T]$ results in

$$\lim_{n\to\infty} \left| \int_{\eta_n}^{\xi_n} |\rho_{k_n}(t)|\,\mathrm{d}t \right| = 0,$$

we conclude from the relation

$$\begin{aligned}
&\left|\frac{1}{\xi_n^{a+1}}\int_0^{\xi_n} s^{a+1}\rho_{k_n}(s)\,\mathrm{d}s-\frac{1}{\eta_n^{a+1}}\int_0^{\eta_n} s^{a+1}\rho_{k_n}(s)\,\mathrm{d}s\right|\\
&\quad\le\left|\frac{1}{\xi_n^{a+1}}-\frac{1}{\eta_n^{a+1}}\right|\int_0^{\xi_n} s^{a+1}|\rho_{k_n}(s)|\,\mathrm{d}s+\frac{1}{\eta_n^{a+1}}\left|\int_{\xi_n}^{\eta_n} s^{a+1}|\rho_{k_n}(s)|\,\mathrm{d}s\right|\\
&\quad\le\left|\frac{1}{\xi_n^{a+1}}-\frac{1}{\eta_n^{a+1}}\right|T^{a+1}L+\frac{T^{a+1}}{\eta_n^{a+1}}\left|\int_{\xi_n}^{\eta_n}|\rho_{k_n}(s)|\,\mathrm{d}s\right|
\end{aligned}$$

that

$$\lim_{n\to\infty}\left|\frac{1}{\xi_n^{a+1}}\int_0^{\xi_n} s^{a+1}\rho_{k_n}(s)\,\mathrm{d}s-\frac{1}{\eta_n^{a+1}}\int_0^{\eta_n} s^{a+1}\rho_{k_n}(s)\,\mathrm{d}s\right|=0$$

The last equality contradicts (13). Consequently, (12) holds and the result follows. □

Lemma 5 *Let $\rho\in L^1[0,T]$. Then*

$$\left|\int_t^T\frac{1}{s^{a+2}}\left(\int_0^s\xi^{a+1}\rho(\xi)\,\mathrm{d}\xi\right)\mathrm{d}s\right|\le\frac{2\|\rho\|_1}{a+1}\quad\text{for } t\in[0,T]. \tag{14}$$

Proof. Since (cf. (10))

$$\begin{aligned}
\left|\int_t^T\frac{1}{s^{a+2}}\left(\int_0^s\xi^{a+1}\rho(\xi)\,\mathrm{d}\xi\right)\mathrm{d}s\right|&\le\frac{1}{a+1}\left(\int_0^t|\rho(s)|\,\mathrm{d}s+\int_t^T|\rho(s)|\,\mathrm{d}s+\|\rho\|_1\right)\\
&=\frac{2\|\rho\|_1}{a+1}
\end{aligned}$$

for $t\in[0,T]$, estimate (14) holds. □

3 Limit properties of solutions to equation (7a)

Here, we investigate asymptotic properties of solutions of (7a). We also provide a related integral equation this solution satisfies.

Lemma 6 *Let (H_1) hold. Let $u\in AC^1_{loc}(0,T]$ satisfy equation (7a) for a.e. $t\in[0,T]$ and $L:=\sup\{|u(t)|+|u'(t)|:t\in(0,T]\}<\infty$. Then u can be extended on $[0,T]$ with $u\in AC^1[0,T]$, and there exists $c\in\mathbb{R}$ such that the integral equation*

$$u(t)=t\left(c-\int_t^T\frac{1}{s^{a+2}}\left(\int_0^s\xi^{a+1}f_n(\xi,u(\xi),u'(\xi))\,\mathrm{d}\xi\right)\mathrm{d}s\right) \tag{15}$$

holds for $t\in[0,T]$.

Proof. Choose $n \in \mathbb{N}$ and denote by $\rho(t) = f_n(t, u(t), u'(t))$ for a.e. $t \in [0,T]$. In order to prove that $\rho \in L^1[0,T]$, define for $m \in \mathbb{N}$, $\frac{1}{m} < T$,

$$v_m(t) := \begin{cases} u(t) & \text{if } t \in (\frac{1}{m}, T], \\ u\left(\frac{1}{m}\right) & \text{if } t \in [0, \frac{1}{m}], \end{cases} \qquad w_m(t) := \begin{cases} u'(t) & \text{if } t \in (\frac{1}{m}, T], \\ u'\left(\frac{1}{m}\right) & \text{if } t \in [0, \frac{1}{m}], \end{cases}$$

and

$$\rho_m(t) := f_n(t, v_m(t), w_m(t)) \text{ for a.e. } t \in [0,T].$$

Then $\rho_m \in L^1[0,T]$ and $\lim_{m\to\infty} \rho_m(t) = \rho(t)$ for a.e. $t \in [0,T]$. Moreover, $|\rho_m(t)| \le \mu(t)$ for a.e. $t \in [0,T]$ and all $m \in \mathbb{N}$, where $\mu(t) = \sup\{|f_n(t,x,y)| : |x| \le L, |y| \le L\} \in L^1[0,T]$. Consequently, by the Lebesgue dominated convergence theorem, $\rho \in L^1[0,T]$.

We now discuss the linear Euler differential equation

$$v''(t) + \frac{a}{t} v'(t) - \frac{a}{t^2} v(t) = \rho(t). \tag{16}$$

Let H be the function given in Lemma 3. By Lemma 3(ii), H can be extended on $[0,T]$ with $H \in AC^1[0,T]$ and $-H$ satisfies (16) for a.e. $t \in [0,T]$. Therefore, each function $v \in AC^1_{loc}(0,T]$ which satisfies equation (16) a.e. on $[0,T]$ has the form $v(t) = c^* t + d^* t^{-a} - H(t)$ for $t \in (0,T]$, with some $c^*, d^* \in \mathbb{R}$. By assumption we know that $u \in AC^1_{loc}(0,T]$ satisfies (16) a.e. on $[0,T]$, and therefore there exist $c, d \in \mathbb{R}$ such that $u(t) = ct + dt^{-a} - H(t)$, $t \in (0,T]$. Since by assumption, $|u'| \le L$ on $(0,T]$, we have $d = 0$. Consequently, the function u can be extended on the interval $[0,T]$ in the class $AC^1[0,T]$ and (15) holds on $[0,T]$. □

Corollary 1 *Let* (H_1) *hold. Let* $u \in AC^1[0,T]$ *be a solution of equation* (7a)*. Then there exists a constant* $c \in \mathbb{R}$ *such that equality* (15) *is satisfied for* $t \in [0,T]$.

Proof. The result holds by Lemma 6 with $\sup\{|u(t)| + |u'(t)| : t \in [0,T]\} < \infty$. □

Remark 2 Corollary 1 says that the set of all solutions $u \in AC^1[0,T]$ of equation (7a) depends on one parameter $c \in \mathbb{R}$ and $u(0) = 0$.

4 Auxiliary regular problems

In order to prove the solvability of problem (7), we first have to investigate the problem

$$u''(t) + \frac{a}{t} u'(t) - \frac{a}{t^2} u(t) = \lambda f_n(t, u(t), u'(t)) - (1-\lambda)\varepsilon, \quad \lambda \in [0,1], \tag{17a}$$

$$u(0) = 0, \quad u(T) = 0, \tag{17b}$$

depending on parameter λ. Here, $\varepsilon > 0$ is from (H_2) and $n \in \mathbb{N}$.

The following result shows that the solvability of problem (17) is equivalent to the solvability of an integral equation in the set $C^1[0,T]$.

Lemma 7 *Let* (H_1) *hold. Then* u *is a solution of problem* (17) *if and only if* u *is a solution of the integral equation*

$$u(t) = -t\int_t^T \frac{1}{s^{a+2}}\left(\int_0^s \xi^{a+1}[\lambda f_n(\xi,u(\xi),u'(\xi)) - (1-\lambda)\varepsilon]\,\mathrm{d}\xi\right)\mathrm{d}s \tag{18}$$

in the set $C^1[0,T]$.

Proof. Let u be a solution of equation (17a). Then $u \in AC^1[0,T]$ and, by Corollary 1 (with f_n replaced by $\lambda f_n - (1-\lambda)\varepsilon$), there exists $c \in \mathbb{R}$ such that the equation

$$u(t) = t\left(c - \int_t^T \frac{1}{s^{a+2}}\left(\int_0^s \xi^{a+1}[\lambda f_n(\xi,u(\xi),u'(\xi)) - (1-\lambda)\varepsilon]\,\mathrm{d}\xi\right)\mathrm{d}s\right)$$

holds for $t \in [0,T]$. Hence, $u(0) = 0$ and $u(T) = 0$ if and only if $c = 0$. Consequently, if u is a solution of problem (17), then u is a solution of equation (18) in $C^1[0,T]$.

Let u be a solution of equation (18) in $C^1[0,T]$. Then $f_n(t,u(t),u'(t)) \in L^1[0,T]$. Hence Lemma 3(ii) (with ρ replaced by $-\lambda f_n + (1-\lambda)\varepsilon$) guarantees that $u \in AC^1[0,T]$ and u is a solution of equation (17a). Moreover, $u(0) = u(T) = 0$. Consequently, u is a solution of problem (17) which completes the proof. □

The following results provide bounds for solutions of problem (17).

Lemma 8 *Let* (H_1)*–*(H_3) *hold. Then, there exists a positive constant* S (*independent on* $n \in \mathbb{N}$ *and* $\lambda \in [0,1]$) *such that for all solutions* u *of problem* (17) *the estimates*

$$u(t) \ge \frac{\varepsilon}{a+2}t(T-t) \text{ for } t \in [0,T], \tag{19}$$

$$\|u\|_\infty < ST, \quad \|u'\|_\infty < S, \tag{20}$$

hold. Moreover, for any solution u *of problem* (17) *there exists* $\xi \in (0,T)$ *such that*

$$|u'(t)| \ge \frac{2\varepsilon}{a+2}|t-\xi| \text{ for } t \in [0,T]. \tag{21}$$

Proof. Let u be a solution of problem (17). Then, by Lemma 7, equation (18) holds for $t \in [0,T]$. Since by (5), $\lambda f_n(t,u(t),u'(t)) - (1-\lambda)\varepsilon \le -\varepsilon$ for a.e. $t \in [0,T]$, relation

$$u(t) \ge \varepsilon t\int_t^T \frac{1}{s^{a+2}}\left(\int_0^s \xi^{a+1}\,\mathrm{d}\xi\right)\mathrm{d}s = \frac{\varepsilon}{a+2}t(T-t) \quad \text{for } t \in [0,T]$$

follows from (18). Hence $g(u(t)) \le g\left(\frac{\varepsilon}{a+2}t(T-t)\right)$ for $t \in (0,T)$ because g is nonincreasing on $\mathbb{R}_+$. Due to Remark 1, $L = \left\|g\left(\frac{\varepsilon}{a+2}t(T-t)\right)\right\|_1 < \infty$, which means that

$$\int_0^T g(u(t))\,\mathrm{d}t \le L. \tag{22}$$

It is clear that L is independent on the choice of solution u to problem (17) and independent on $n \in \mathbb{N}$, $\lambda \in [0,1]$.

We now show that inequality (21) holds for some $\xi \in (0,T)$. Differentiating (18) gives

$$\begin{aligned} u''(t) &= -\frac{a}{t^{a+2}}\int_0^t s^{a+1}[\lambda f_n(s,u(s),u'(s)) - (1-\lambda)\varepsilon]\,\mathrm{d}s \\ &\qquad + \lambda f_n(t,u(t),u'(t)) - (1-\lambda)\varepsilon \ \text{ for a.e. } t \in [0,T]. \end{aligned} \tag{23}$$

Since $a<0$, it follows from (5) and (23) that

$$u''(t) \le \frac{a\varepsilon}{t^{a+2}}\int_0^t s^{a+1}\,\mathrm{d}s - \varepsilon = -\frac{2\varepsilon}{a+2} \ \text{ for a.e. } t \in [0,T].$$

Hence u' is decreasing on $[0,T]$, and therefore u' vanishes at a unique point $\xi \in (0,T)$ due to $u(0) = u(T) = 0$. The inequality (21) now follows from the relations

$$\begin{aligned} u'(t) &= -\int_t^\xi u''(s)\,\mathrm{d}s \ge \frac{2\varepsilon}{a+2}(\xi - t) \text{ for } t \in [0,\xi], \\ u'(t) &= \int_\xi^t u''(s)\,\mathrm{d}s \le -\frac{2\varepsilon}{a+2}(t-\xi) \text{ for } t \in [\xi,T]. \end{aligned}$$

Hence $r(|u'(t)|) \le r\left(\frac{2\varepsilon}{a+2}|t-\xi|\right)$ on $[0,T]\setminus\{\xi\}$, and

$$\begin{aligned} \int_0^T (r(|u'(t)|))\,\mathrm{d}t &\le \int_0^\xi \left(r\left(\frac{2\varepsilon}{a+2}(\xi-t)\right)\right)\mathrm{d}t + \int_\xi^T \left(r\left(\frac{2\varepsilon}{a+2}(t-\xi)\right)\right)\mathrm{d}t \\ &< \frac{a+2}{\varepsilon}\int_0^{(2\varepsilon T)/(a+2)} r(s)\,\mathrm{d}s = V \text{ for } t \in [0,T]. \end{aligned}$$

In particular,

$$\|r(|u'(t)|)\|_1 < V. \tag{24}$$

Let $W = \frac{a+3}{a+1}$. Taking into account (6), (9), (18), (22), (24) and Lemma 5, we obtain

$$\begin{aligned} |u'(t)| &= \left| -\int_t^T \frac{1}{s^{a+2}}\left(\int_0^s \xi^{a+1}[\lambda f_n(\xi,u(\xi),u'(\xi)) - (1-\lambda)\varepsilon]\,\mathrm{d}\xi\right)\mathrm{d}s \right. \\ &\qquad \left. + \frac{1}{t^{a+1}}\int_0^t s^{a+1}[\lambda f_n(s,u(s),u'(s)) - (1-\lambda)\varepsilon]\,\mathrm{d}s \right| \end{aligned}$$

$$\leq W \int_0^T |\lambda f_n(t, u(t), u'(t) - (1-\lambda)\varepsilon| \, \mathrm{d}t$$

$$\leq W \int_0^T \Big(\varphi(t) h(1 + \|u\|_\infty, 1 + \|u'\|_\infty) + g(u(t)) + r(|u'(t)|)\Big) \, \mathrm{d}t$$

$$\leq W\Big(h(1 + \|u\|_\infty, 1 + \|u'\|_\infty)\|\varphi\|_1 + L + V\Big), \quad t \in [0, T].$$

It follows from $u(t) = \int_0^t u'(s)\,\mathrm{d}s$ for $t \in [0, T]$,

$$\|u\|_\infty \leq T\|u'\|_\infty, \tag{25}$$

and therefore, we have

$$\|u'\|_\infty \leq Kh(1 + T\|u'\|_\infty, 1 + \|u'\|_\infty) + M, \tag{26}$$

where $K = W\|\varphi\|_1$ and $M = W(L + V)$. By (H_3),

$$\lim_{z\to\infty} (1/z)(Kh(1 + Tz, 1 + z) + M) = 0.$$

Consequently, there exists $S > 0$ such that $Kh(1+Tz, 1+z)+M < z$ for $z \geq S$. Now, due to (26), $\|u'\|_\infty < S$, and therefore, by (25), $\|u\|_\infty < ST$. □

We are now in the position to prove the existence result for problem (7).

Lemma 9 *Let (H_1)–(H_3) hold. Then, for each $n \in \mathbb{N}$, problem* (7) *has a solution u satisfying inequalities* (19) – (21)*, where S is a positive constant independent on n.*

Proof. Let S be a positive constant in Lemma 8 and let us define

$$\Omega := \{x \in C^1[0, T] : \|x\| < ST, \ \|x'\| < S\}.$$

Then Ω is an open and bounded subset of the Banach space $C^1[0, T]$. Keeping in mind Lemma 3, define an operator $\mathcal{K} : [0, T] \times \overline{\Omega} \to C^1[0, T]$ by the formula

$$\mathcal{K}(\lambda, x)(t) = -t \int_t^T \frac{1}{s^{a+2}} \left(\int_0^s \xi^{a+1} [\lambda f_n(\xi, x(\xi), x'(\xi)) - (1-\lambda)\varepsilon] \, \mathrm{d}\xi \right) \mathrm{d}s. \tag{27}$$

By Lemma 7, any fixed point of the operator $\mathcal{K}(1, \cdot)$ is a solution of problem (7). In order to show the existence of a fixed point of $\mathcal{K}(1, \cdot)$, we apply Lemma 1 with $E = C^1[0, T]$, $U = \Omega$, $\mathcal{F} = \mathcal{K}(1, \cdot)$ and $\ell = \frac{\varepsilon}{a+2} t(T - t)$. Especially, we show that

(i) $\mathcal{K}(1, \cdot) : \overline{\Omega} \to C^1[0, T]$ is a compact operator,
and
(ii) $\mathcal{K}(\lambda, x) \neq x$ for each $\lambda \in (0, 1]$ and $x \in \partial\Omega$.

We first verify that $\mathcal{K}(1,\cdot)$ is a continuous operator. To this end, let $\{x_m\} \subset \overline{\Omega}$ be a convergent sequence, and let $\lim_{m\to\infty} x_m = x$ in $C^1[0,T]$. Let

$$r_m(t) := f_n(t, x_m(t), x_m'(t)) - f_n(t, x(t), x'(t)) \quad \text{for a.e. } t \in [0,T].$$

It follows from Lemma 5 and from (9) that

$$\begin{aligned}
|\mathcal{K}(1,x_m)(t) - \mathcal{K}(1,x)(t)| &= \left| -t \int_t^T \frac{1}{s^{a+2}} \left(\int_0^s \xi^{a+1} r_m(\xi)\,\mathrm{d}\xi \right) \mathrm{d}s \right| \\
&\le \frac{2T\|r_m\|_1}{a+1}, \\
|\mathcal{K}(1,x_m)'(t) - \mathcal{K}(1,x)'(t)| &= \left| - \int_t^T \frac{1}{s^{a+2}} \left(\int_0^s \xi^{a+1} r_m(\xi)\,\mathrm{d}\xi \right) \mathrm{d}s \right. \\
&\qquad \left. + \frac{1}{t^{a+1}} \int_0^t s^{a+1} r_m(\xi)\,\mathrm{d}\xi \right| \\
&\le \frac{2\|r_m\|_1}{a+1} + \|r_m\|_1
\end{aligned}$$

for $t \in [0,T]$. Here $\mathcal{K}(1,x)' = \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{K}(1,x)$. In particular, for $m \in \mathbb{N}$,

$$\begin{aligned}
\|\mathcal{K}(1,x_m) - \mathcal{K}(1,x)\|_\infty &\le \frac{2T\|r_m\|_1}{a+1}, \\
\|\mathcal{K}(1,x_m)' - \mathcal{K}(1,x)'\|_\infty &\le \frac{(a+3)\|r_m\|_1}{a+1}.
\end{aligned} \tag{28}$$

Since $\lim_{m\to\infty} f_n(t, x_m(t), x_m'(t)) = f_n(t, x(t), x'(t))$ for a.e. $t \in [0,T]$ and there exists $\rho \in L^1[0,T]$ such that

$$|f_n(t, x_m(t), x_m'(t))| \le \rho(t) \text{ for a.e. } t \in [0,T] \text{ and all } m \in \mathbb{N},$$

we have $\lim_{m\to\infty} \|r_m\|_1 = 0$ by the Lebesgue dominated convergence theorem. Hence, by (28), $\mathcal{K}(1,\cdot)$ is a continuous operator. We now show that the set $\mathcal{K}(1,\overline{\Omega})$ is relatively compact in $C^1[0,T]$. It follows from $f_n \in Car([0,T]\times\mathbb{R}^2)$ and $\overline{\Omega}$ bounded in $C^1[0,T]$, that there exists $\mu \in L^1[0,T]$ such that

$$|f_n(t, x(t), x'(t))| \le \mu(t) \text{ for a.e. } t \in [0,T] \text{ and all } x \in \overline{\Omega}.$$

Then, by Lemma 5 and (9), the inequalities

$$|\mathcal{K}(1,x)(t)| \le \frac{2T\|\mu\|_1}{a+1}, \quad |\mathcal{K}(1,x)'(t)| \le \frac{(a+3)\|\mu\|_1}{a+1}$$

are satisfied for $t \in [0,T]$ and $x \in \overline{\Omega}$, and therefore, the set $\mathcal{K}(1,\overline{\Omega})$ is bounded in $C^1[0,T]$. Moreover, the relation

$$\begin{aligned} |\mathcal{K}(1,x)''(t)| &= \left| -\frac{a}{t^{a+2}} \int_0^t s^{a+1} f_n(s,x(s),x'(s))\,\mathrm{d}s + f_n(t,x(t),x'(t)) \right| \\ &\le \frac{|a|}{t^{a+2}} \int_0^t s^{a+1}\mu(s)\,\mathrm{d}s + \mu(t) \in L^1[0,T] \end{aligned}$$

holds for a.e. $t \in [0,T]$ and all $x \in \overline{\Omega}$ (cf. (9)). Consequently, the set $\{\mathcal{K}(1,x)' : x \in \overline{\Omega}\}$ is equicontinuous on $[0,T]$. Hence, the set $\mathcal{K}(1,\overline{\Omega})$ is relatively compact in $C^1[0,T]$ by the Arzelà-Ascoli theorem. As a result, $\mathcal{K}(1,\cdot)$ is a compact operator and the condition (i) follows.

Due to the fact that by Lemma 7 any fixed point u of the operator $\mathcal{K}(\lambda,\cdot)$ is a solution of problem (17), Lemma 8 guarantees that u satisfies inequality (20). Therefore, $\mathcal{K}$ has property (ii). Consequently, by Lemmas 1 and 8, for each $n \in \mathbb{N}$ problem (7) has a solution u satisfying estimates (19)-(21). □

Let u_n be a solution of problem (7) for $n \in \mathbb{N}$. The following property of the sequence $\{|f_n(t,u_n(t),u_n'(t))|\}$ is an important prerequisite for solving problem (1).

Lemma 10 *Let $(H_1)-(H_3)$ hold. Let u_n be a solution of problem* (7) *for $n \in \mathbb{N}$. Then the sequence $\{|f_n(t,u_n(t),u_n'(t))|\}$ is uniformly integrable on $[0,T]$.*

Proof. By Lemma 9, the inequalities

$$u_n(t) \ge \frac{\varepsilon}{a+2} t(T-t) \quad \text{for } t \in [0,T],\ n \in \mathbb{N}, \tag{29}$$

$$\|u_n\| < ST, \quad \|u_n'\| < S \quad \text{for } n \in \mathbb{N}, \tag{30}$$

$$|u_n'(t)| \ge \frac{2\varepsilon}{a+2}|t-\xi_n| \quad \text{for } t \in [0,T],\ n \in \mathbb{N}, \tag{31}$$

hold, where S is a positive constant and $\xi_n \in (0,T)$. Hence, by (3) and (4),

$$0 < -f_n(t,u_n(t),u_n'(t)) \le \varphi(t)h(1+ST,1+S) + g_*(t) + r\left(\frac{2\varepsilon}{a+2}|t-\xi_n|\right) \tag{32}$$

for a.e. $t \in [0,T]$, where $g_*(t) = g\left(\frac{\varepsilon}{a+2}t(T-t)\right) \in L^1[0,T]$, see Remark 1. Since the sequence $\left\{r\left(\frac{2\varepsilon}{a+2}|t-\xi_n|\right)\right\}$ is uniformly integrable on $[0,T]$ (cf. [28, criterion A.4], [3, 24]), it follows from (32) that $\{|f_n(t,u_n(t),u_n'(t))|\}$ is uniformly integrable on $[0,T]$ and the result follows. □

5 The existence result for the BVP (1)

This section is devoted to the main result on the existence of positive solutions to the original BVP (1).

Theorem 1 *Let* $(H_1) - (H_3)$ *hold. Then problem* (1) *has at least one positive solution.*

Proof. By Lemma 9, for each $n \in \mathbb{N}$ problem (7) has a solution u_n satisfying inequalities (29)-(31), where S is a positive constant and $\xi_n \in (0, T)$. Moreover, by Lemma 10, the sequence $\{|f_n(t, u_n(t), u_n'(t))|\}$ is uniformly integrable on $[0, T]$. We now prove that $\{u_n'\}$ is equicontinuous on $[0, T]$. Since u_n is a fixed point of the operator $\mathcal{K}(1, \cdot)$ given in (27), the equality

$$u_n''(t) = -\frac{a}{t^{a+2}} \int_0^t s^{a+1} f_n(s, u_n(s), u_n'(s))\,\mathrm{d}s + f_n(t, u_n(t), u_n'(t))$$

holds for a.e. $t \in [0, T]$ and all $n \in \mathbb{N}$. Let $0 \le t_1 < t_2 \le T$. Then

$$\begin{aligned} |u_n'(t_2) - u_n'(t_1)| \; = \; & \left| -a \int_{t_1}^{t_2} \frac{1}{s^{a+2}} \left(\int_0^s \xi^{a+1} f_n(\xi, u_n(\xi), u_n'(\xi))\,\mathrm{d}\xi \right) \mathrm{d}s \right. \\ & \left. + \int_{t_1}^{t_2} f_n(s, u_n(s), u_n'(s))\,\mathrm{d}s \right|. \end{aligned}$$

Let $r_n(t) = \frac{1}{t^{a+1}} \int_0^t s^{a+1} f_n(s, u_n(s), u_n'(s))\,\mathrm{d}s$. By Lemma 3(i), $\{r_n\} \subset C[0, T]$ and $r_n(0) = 0$. Integrating by parts yields

$$\begin{aligned} & \int_{t_1}^{t_2} \frac{1}{s^{a+2}} \left(\int_0^s \xi^{a+1} f_n(\xi, u_n(\xi), u_n'(\xi))\,\mathrm{d}\xi \right) \mathrm{d}s \\ & \quad = \frac{1}{a+1} \left(r_n(t_1) - r_n(t_2) + \int_{t_1}^{t_2} f_n(s, u(s), u'(s))\,\mathrm{d}s \right), \end{aligned}$$

and

$$|u_n'(t_2) - u_n'(t_1)| = \left| \frac{a}{a+1} (r_n(t_2) - r_n(t_1)) + \frac{1}{a+1} \int_{t_1}^{t_2} f_n(s, u(s), u'(s))\,\mathrm{d}s \right| \quad (33)$$

follows. By Lemma 4 (for $\rho_n(t) = f_n(t, u_n(t), u_n'(t))$), the sequence $\{r_n\}$ is equicontinuous on $[0, T]$. Since the sequence $\{|f_n(t, u_n(t), u_n'(t))|\}$ is uniformly integrable on $[0, T]$, the sequence $\left\{ \int_0^t f_n(s, u(s), u'(s))\,\mathrm{d}s \right\}$ is equicontinuous on $[0, T]$. Hence, it follows from (33), that $\{u_n'\}$ is equicontinuous on $[0, T]$. We summarize: $\{u_n\}$ is bounded in $C^1[0, T]$ and $\{u_n'\}$ is equicontinuous on $[0, T]$. Also, $\{\xi_n\} \subset (0, T)$. Using appropriate subsequences, if necessary, we can assume, by the Arzelà-Ascoli theorem and the Bolzano-Weierstrass theorem, that

$\{u_n\}$ is convergent in $C^1[0,T]$ and $\{\xi_n\}$ is convergent in $\mathbb{R}$. Let $\lim_{n\to\infty} u_n =: u$ and $\lim_{n\to\infty}\xi_n =: \xi$. With $n \to \infty$ in (29)-(31), we conclude

$$u(t) \ge \frac{\varepsilon}{a+2} t(T-t), \quad |u'(t)| \ge \frac{2\varepsilon}{a+2}|t-\xi| \text{ for } t \in [0,T],$$

$$\|u\| \le ST, \quad \|u'\| \le S.$$

In addition, $u(0) = 0$, $u(T) = 0$. Since

$$\lim_{n\to\infty} f_n(t, u_n(t), u_n'(t)) = f(t, u(t), u'(t)) \text{ for a.e. } t \in [0,T], \tag{34}$$

it follows from Lemma 2 that

$$\lim_{n\to\infty} \|f_n(t, u_n(t), u_n'(t)) - f(t, u(t), u'(t))\|_1 = 0$$

and $f(t, u(t), u'(t)) \in L^1[0,T]$. We now deduce from the inequality (cf. Lemma 5)

$$\left| t \int_t^T \frac{1}{s^{a+2}} \int_0^s \xi^{a+1} \left((f_n(\xi, u_n(\xi), u_n'(\xi)) - f(\xi, u(\xi), u'(\xi))) \, \mathrm{d}\xi \right) \mathrm{d}s \right|$$
$$\le \frac{2T}{a+1} \|f_n(t, u_n(t), u_n'(t)) - f(t, u(t), u'(t))\|_1 \text{ for } t \in [0,T]$$

that

$$\lim_{n\to\infty} t \int_t^T \frac{1}{s^{a+2}} \left(\int_0^s \xi^{a+1} f_n(\xi, u_n(\xi), u_n'(\xi)) \, \mathrm{d}\xi \right) \mathrm{d}s$$
$$= t \int_t^T \frac{1}{s^{a+2}} \left(\int_0^s \xi^{a+1} f(\xi, u(\xi), u'(\xi)) \, \mathrm{d}\xi \right) \mathrm{d}s \text{ for } t \in [0,T].$$

Taking the limit $n \to \infty$ in

$$u_n(t) = -t \int_t^T \frac{1}{s^{a+2}} \left(\int_0^s \xi^{a+1} f_n(\xi, u_n(\xi), u_n'(\xi)) \, \mathrm{d}\xi \right) \mathrm{d}s,$$

we have

$$u(t) = -t \int_t^T \frac{1}{s^{a+2}} \left(\int_0^s \xi^{a+1} f(\xi, u(\xi), u'(\xi)) \, \mathrm{d}\xi \right) \mathrm{d}s \text{ for } t \in [0,T]. \tag{35}$$

Hence,

$$u''(t) + \frac{a}{t} u'(t) - \frac{a}{t^2} u(t) = f(t, u(t), u'(t)) \text{ for a.e. } t \in [0,T],$$

and $u \in AC^1[0,T]$ by Lemma 3(ii). This means that u is a positive solution of problem (1) and the result follows. □

6 Numerical simulations

For the numerical simulation, we choose $T = 1$ and use an alternative formulation of problem (1),

$$u''(t) + \frac{a}{t}u'(t) - \frac{a}{t^2}u(t) = f(t, u(t), u'(t)), \tag{36a}$$

$$u(1) = 0, \quad u'(1) = -c, \tag{36b}$$

where $c > 0$ is a parameter. We can use the above formulation because problem (1) is solvable for f satisfying the assumptions of Theorem 1 and, therefore, solutions of problem (1) can be computed as solutions of problem (36) using the proper value $c \in (0, c^*)$ depending on f. The values c^* are provided for given f in Examples 1 and 2, below.

The reason for changing the boundary conditions from (1b) to (36b) is that the differential equation (36a) subject to (1b) is not well-posed, see [30]. However, to enable successful numerical treatment, well-posedness of the model is crucial. This property means that equation (36a) subject to proper boundary conditions has at least a *locally unique* solution[1], and this solution depends continuously on the problem data. The well-posedness of the problem is important for two reasons. First of all, it allows to express the errors in the solution of the analytical problem in terms of the modeling errors and the data errors (all measured via appropriate norms). Therefore, when the errors in the data become smaller due to more precise modeling or smaller measurement inaccuracies, the errors in the solution will decrease. The second reason is that the well-posedness decides if the numerical simulation will be at all successful. If the analytical problem is ill-posed, then the inevitable round-off errors can become extremely magnified and fully spoil the accuracy of the approximation.

In what follows, we work with $f(t, x, y) = q(t) + h(x, y)$ for a.e. $t \in [0, 1]$ and all $x \in \mathbb{R}_+$, $y \in \mathbb{R} \setminus \{0\}$ and, according to the next numerical approach (see Section 6.2), we consider equation (36a), where $h \equiv 0$, that is

$$u''(t) + \frac{a}{t}u'(t) - \frac{a}{t^2}u(t) = q(t). \tag{37}$$

By [30], problem (37), (36b) is well-posed and therefore it is suitable for the numerical treatment. To see this, we need to look at the general solution of the homogeneous equation

$$u''(t) + \frac{a}{t}u'(t) - \frac{a}{t^2}u(t) = 0, \quad t \in [0, 1]. \tag{38}$$

If we set $u(t) = t^\lambda$, we arrive at the characteristical polynomial of (38),

$$\lambda(\lambda - 1) + a\lambda - a = 0$$

[1] This BVP can have more than one solution, but they may not lay close together.

whose roots $\lambda_1 = 1$ and $\lambda_2 = -a$ are positive. Therefore conditions for u and u' can be prescribed at $t = 1$ as it is done in (36b).

6.1 Matlab Code `bvpsuite`

To illustrate the analytical results discussed in the previous section, we solved numerically examples of the form (36) using a MATLAB™ software package `bvpsuite` designed to solve BVPs in ODEs and differential algebraic equations. The solver routine is based on a class of collocation methods whose orders may vary from 2 to 8. Collocation has been investigated in context of singular differential equations of first and second order in [14, 31], respectively. This method could be shown to be robust with respect to *singularities in time* and retains its high convergence order in case that the analytical solution is appropriately smooth. The code also provides an asymptotically correct estimate for the global error of the numerical approximation. To enhance the efficiency of the method, a mesh adaptation strategy is implemented, which attempts to choose grids related to the solution behavior, in such a way that the tolerance is satisfied with the least possible effort. Error estimate procedure and the mesh adaptation work dependably provided that the solution of the problem and its global error are appropriately smooth[2]. The code and the manual can be downloaded from `http://www.math.tuwien.ac.at/∼ewa`. For further information see [16]. This software proved useful for the approximation of numerous singular BVPs important for applications, see e.g. [6, 11, 15, 23].

6.2 Preliminaries

Before dealing with two nonlinear models specified in Sections 6.3 and 6.4, we have to compute numerical solutions for a simpler linear[3] model of the form

$$u''(t) + \frac{a}{t}u'(t) - \frac{a}{t^2}u(t) = -\frac{1}{\sqrt{t}}, \quad t \in [0,1], \tag{39a}$$

$$u(1) = 0, \quad u'(1) = -\frac{1}{a+1.5}, \tag{39b}$$

where a was chosen as $a = -0.1, -0.5, -0.9$. Since in this case the exact solution is given, $u(t) = \frac{2t(1-\sqrt{t})}{a+1.5}$, the value $u'(1)$ is available, $u'(1) = -0.72, -1, -1.67$, respectively. In Figure 1, the numerical solutions of BVPs (39) are shown. They will be used as starting values for the numerical solution of Examples 1 and 2, see Sections 6.3 and 6.4, respectively. All numerical results have been obtained using collocation with five Gaussian collocation points on an equidistant grid (justified

[2]The required smoothness of higher derivatives is related to the order of the used collocation method.

[3]The nonlinear term in f has been omitted, see (40a) and (46a).

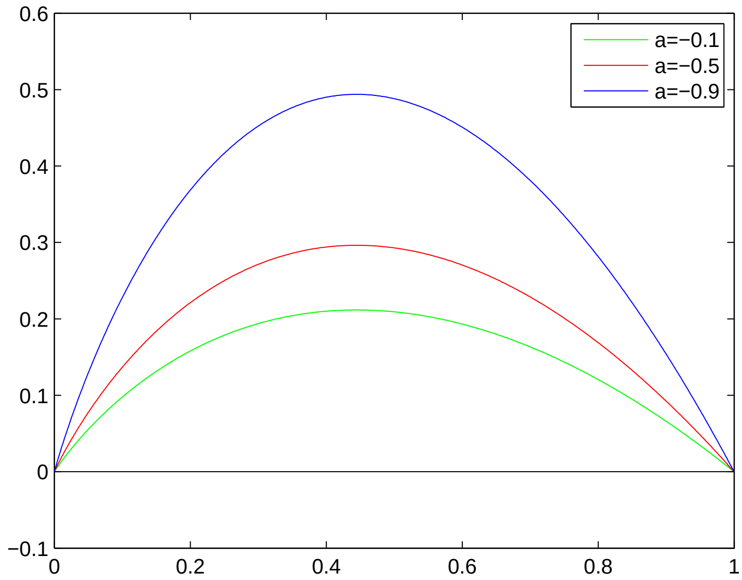

Figure 1: Problem (39): Numerical solutions for different values of a.

by a very simple solution structure) with the step size 0.01.

6.3 Example 1

We first investigate the following problem

$$u''(t) + \frac{a}{t}u'(t) - \frac{a}{t^2}u(t) = -\frac{1}{\sqrt{t}} - u^{\frac{1}{3}}(t), \quad t \in [0,1], \tag{40a}$$

$$u(1) = 0, \quad u'(1) = -c. \tag{40b}$$

The nonlinearity f in (40a) has the form

$$f(t,x) = -\frac{1}{\sqrt{t}} - x^{\frac{1}{3}} \tag{41}$$

and it satisfies $(H_1) - (H_3)$ with $\varepsilon = 1$, $\varphi(t) = \frac{1}{\sqrt{t}}$ for $t \in (0,1]$ and

$$h(x,y) = 1, \quad g(x) = x^{\frac{1}{3}}, \quad r(y) = 0 \quad \text{for } x, y \in \mathbb{R}_+.$$

It follows from Theorem 1 that there exists at least one value of $c > 0$ such that the related solution u of problem (40) with $u'(1) = c$, is positive on $(0,1)$ with $u(0) = 0$. Using formula (35), we now determine an interval $(0, c^*) \subset (0, \infty)$ containing all admissible values of c.

Let u be a solution of problem (1) with f from (41). Then, by (35), we obtain

$$|u(t)| \le t \int_t^1 \frac{1}{s^{a+2}} \left(\int_0^s \xi^{a+1} \left(\frac{1}{\sqrt{\xi}} + \|u\|_\infty^{\frac{1}{3}} \right) \mathrm{d}\xi \right) \mathrm{d}s$$

$$< \frac{4}{2a+3} + \frac{\|u\|_\infty^{\frac{1}{3}}}{4(a+2)}, \; t \in [0,1].$$

Therefore,

$$\|u\|_\infty < \frac{4}{2a+3} + \frac{\|u\|_\infty^{\frac{1}{3}}}{4(a+2)}. \tag{42}$$

Let $K > 0$ satisfy

$$K = \frac{4}{2a+3} + \frac{K^{\frac{1}{3}}}{4(a+2)}. \tag{43}$$

Then (42) implies $\|u\|_\infty < K$ and, due to (35) and (41)

$$u'(1) = \int_0^1 t^{a+1} f(t, u(t), u'(t)) \mathrm{d}t = \int_0^1 t^{a+1} \left(-\frac{1}{\sqrt{t}} - u^{\frac{1}{3}}(t) \right) \mathrm{d}t$$

$$\ge - \int_0^1 \left(t^{a+\frac{1}{2}} + K^{\frac{1}{3}} t^{a+1} \right) \mathrm{d}t = -\frac{2}{2a+3} - K^{\frac{1}{3}} \frac{1}{a+2}.$$

Consequently

$$c^* = \frac{2}{2a+3} + K^{\frac{1}{3}} \frac{1}{a+2}. \tag{44}$$

In order to solve the nonlinear problem (40), we first have to solve a series of auxiliary problems, whose underlying parameter dependent differential equations read:

$$u''(t) + \frac{a}{t} u'(t) - \frac{a}{t^2} u(t) = -\frac{1}{\sqrt{t}} - \delta u^{\frac{1}{3}}(t), \quad t \in (0,1], \quad \delta > 0. \tag{45}$$

We begin the calculations with $\delta = 0$ and increase its value gradually until we arrive at $\delta = 1$, cf. (40a). In each step we use the solution of the previous problem to solve the next one. The aim is to find a good starting value for both, the solution u and the value $u'(1)$, before solving the BVP (40), i.e. find the final value of $c \in (0, c^*)$, such that $u(0) = 0$.

In case of Example 1 and $a = -0.1$, this chain has the following structure:

1. Numerical approximation of BVP (39) is used as an initial guess for ODE (45) with $\delta = 0$, subject to terminal conditions $u(1) = 0$, $u'(1) = -0.5$.

2. Use above approximation as an initial guess for ODE (45) with $\delta = 0.01$, subject to terminal conditions $u(1) = 0$, $u'(1) = -1.0$.

a	K	c^*	c	$u(0)$
-0.1	1.5819	1.327538328	1.00569659944	1.40264071382347 E-14
-0.5	2.2173	1.869327784	1.41953539630	1.18953445347016 E-13
-0.9	3.6844	3.070760848	2.33615892300	4.13495149060736 E-14

Table 1: Problem (40): Complete data of the numerical simulation for different values of a.

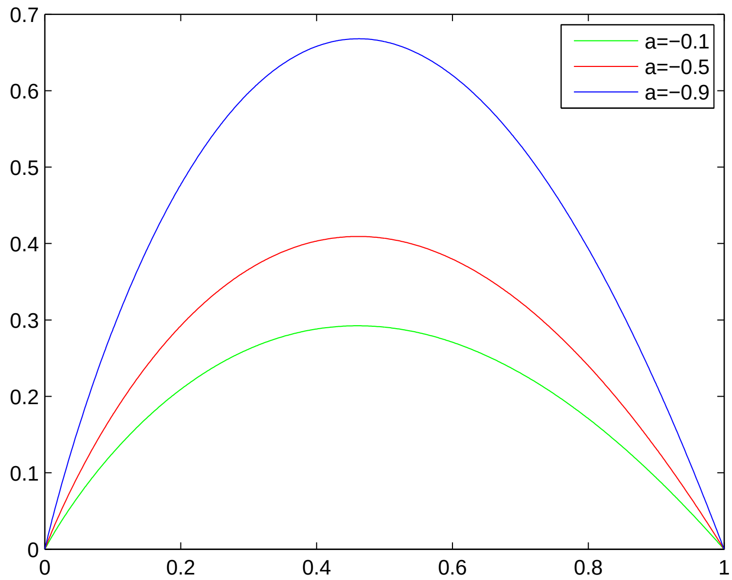

Figure 2: Problem (40): Numerical solutions for different values of a. Values of $u(0) \approx 10^{-14}$.

3. Use above approximation as an initial guess for ODE (45) with $\delta = 0.1$, subject to terminal conditions $u(1) = 0$, $u'(1) = -1.0$.

4. Use above approximation as an initial guess for ODE (45) with $\delta = 1.0$, subject to terminal conditions $u(1) = 0$, $u'(1) = -1.0$.

After the last step, we have solved the problem (40), subject to boundary conditions $u(1) = 0$, $u'(1) = -1.0$. In this case the value of $u(0)$ was not small enough, to consider it a reasonable approximation for $u(0) = 0$. Therefore, we use a shooting idea combined with a bisection strategy to find a better value for $c = -u'(1)$. The complete numerical results for Example 1 can found in Table 1 and Figure 2.

6.4 Example 2

The above approach has been also accordingly applied for Example 2. Here, we consider the problem

$$u''(t) + \frac{a}{t}u'(t) - \frac{a}{t^2}u(t) = -\frac{1}{\sqrt{t}} - u^{-\frac{1}{3}}(t), \quad t \in [0,1], \tag{46a}$$

$$u(1) = 0, \quad u'(1) = -c. \tag{46b}$$

The right hand side f in equation (46a) now reads:

$$f(t,x) = -\frac{1}{\sqrt{t}} - x^{-\frac{1}{3}} \tag{47}$$

and has a singularity at $x = 0$. The function f satisfies conditions (H_1)–(H_3) with $\varepsilon = 1$, $\varphi(t) = \frac{1}{\sqrt{t}}$ for $t \in (0,1]$ and

$$h(x,y) = 1, \quad g(x) = x^{-\frac{1}{3}}, \quad r(y) = 0 \text{ for } x, y \in \mathbb{R}_+.$$

Theorem 1 guarantees the existence of at least one $c > 0$ such that a solution u of problem (46) is positive on $(0,1)$ and $u(0) = 0$ holds. We now again determine an interval $(0, c^*) \subset (0, \infty)$ containing all such values of c. Let u be a solution of problem (1) with f given in (47). Inequality (19) yields

$$u(t) \geq \frac{1}{a+2}t(1-t), \quad t \in [0,1],$$

and hence, by (35),

$$u'(1) = \int_0^1 t^{a+1} f(t, u(t), u'(t))\mathrm{d}t = \int_0^1 t^{a+1}\left(-\frac{1}{\sqrt{t}} - u^{-\frac{1}{3}}(t)\right)\mathrm{d}t$$

$$\geq \int_0^1 t^{a+1}\left(-\frac{1}{\sqrt{t}} - \left(\frac{1}{a+2}t(1-t)\right)^{-\frac{1}{3}}\right)\mathrm{d}t.$$

Consequently

$$c^* = \int_0^1 t^{a+1}\left(\frac{1}{\sqrt{t}} + \frac{(a+2)^{1/3}}{t^{1/3}(1-t)^{1/3}}\right)\mathrm{d}t. \tag{48}$$

For Example 2, the auxiliary ODE is constructed using the ODE (39a),

$$u''(t) + \frac{a}{t}u'(t) - \frac{a}{t^2}u(t) = -\frac{1}{\sqrt{t}} - \delta u^{-\frac{1}{3}}, \quad t \in [0,1], \quad \delta > 0. \tag{49}$$

For all values of a, we choose $u'(1) \in (-c^*, 0)$ and analogously carried out the path-following in δ first. The related chain for $a = -0.1$ is

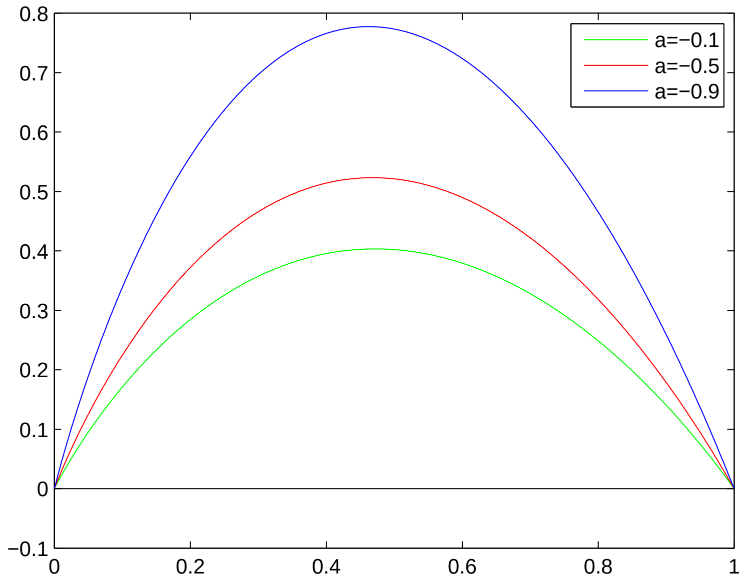

Figure 3: Problem (46): Numerical solutions for different values of a. Values of $u(0) \approx 10^{-12}$.

1. Numerical approximation of BVP (39) is used as an initial guess for ODE (49) with $\delta = 0$, subject to terminal conditions $u(1) = 0$, $u'(1) = -0.8$.

2. Use above approximation as an initial guess for ODE (49) with $\delta = 0.01$, subject to terminal conditions $u(1) = 0$, $u'(1) = -0.8$.

3. Use above approximation as an initial guess for ODE (49) with $\delta = 0.1$, subject to terminal conditions $u(1) = 0$, $u'(1) = -1.2$.

4. Use above approximation as an initial guess for ODE (49) with $\delta = 0.5$, subject to terminal conditions $u(1) = 0$, $u'(1) = -1.3$.

5. Use above approximation as an initial guess for ODE (49) with $\delta = 1.0$, subject to terminal conditions $u(1) = 0$, $u'(1) = -1.7$.

After the last step, we have solved the BVP (46), subject to boundary conditions $u(1) = 0$, $u'(1) = -1.7$, but also in this case the value of $u(0)$ is too large and we have to find a better value for $c = -u'(1)$. The complete numerical results for Example 2 can found in Table 2 and Figure 3.

a	c^*	c	$u(0)$
-0.1	2.044582190	1.63000971355	4.57015034596816 e-12
-0.5	2.528760387	2.04278888650	1.82943058378521 e-13
-0.9	3.566085670	2.91010561000	1.16498324337590 e-12

Table 2: Problem (46): Complete data of the numerical simulation for different values of a.

7 Conclusions

In the present article, we deal with the existence of positive solutions to the singular Dirichlet problem of the form

$$u''(t) + \frac{a}{t}u'(t) - \frac{a}{t^2}u(t) = f(t, u(t), u'(t)), \quad u(0) = 0, \ u(T) = 0,$$

where $a \in (-1, 0)$ and the nonlinearity $f(t, x, y)$ may be singular at the space variables $x = 0$ and/or $y = 0$. Moreover, since $a \neq 0$ the differential operator is singular at $t = 0$. We formulate sufficient conditions for the existence of positive solutions of the above BVP and study their limit properties. The theory is illustrated by numerical experiments carried out using the MATLAB code `bvpsuite`, based on polynomial collocation. For the successful numerical treatment the above problem has to be reformulated to obtain its well-posed form

$$u''(t) + \frac{a}{t}u'(t) - \frac{a}{t^2}u(t) = f(t, u(t), u'(t)), \quad u(T) = 0, \ u'(T) = -c.$$

Here, it is only known that $c \in (0, c^*)$, where c^* can be specified depending on functions f arising in Examples 1 and 2. Now, simple shooting method combined with the bisection idea is used to find c in such a way that $u(0) = 0$.

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