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Congruences of Convex Algebras

Ana Sokolova[1,*] and Harald Woracek[2]

Abstract

We provide a full description of congruence relations of convex, positive convex, and totally convex algebras. As a consequence of this result we obtain that finitely generated convex (positive convex, totally convex) algebras are finitely presentable. Convex algebras, in particular positive convex algebras, are important in the area of probabilistic systems. They are the Eilenberg-Moore algebras of the subdistribution monad.

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1 Introduction

In this paper we present a study of the equational classes CA, PCA, and TCA, of convex, positive convex, and totally convex algebras. We describe all congruence relations of such algebras. Knowing the congruences, we obtain that finitely generated convex (positive convex, totally convex) algebras are finitely presentable.

A convex algebra is an algebra (with nonempty carrier set) with an infinite set of operations of arbitrary positive arities providing convex combinations of the arguments, which satisfy two axioms (axiom schemes): (1) the projection axiom stating that a convex combination with a single coefficient equal to 1 equals the identity map, and (2) the baricenter axiom stating that a convex combination of convex combinations equals the convex combination with suitably multiplied and summed coefficients. Positive convex and totally convex algebras are defined in a similar way from larger convex structures (sub-convex combinations for positive convex algebras and linear combinations with coefficients whose absolute values are sub-convex for totally convex algebras). The precise definitions and details follow in Section 3.

Examples of convex algebras are provided by convex subsets of a vector space over the scalar field $\mathbb{R}$ (a subset of a vector space is convex, if it contains with each two points the whole line segment connecting them): If K is such, then K is a convex algebra with the operations inherited from the vector space. However, these examples do not exhaust the class CA; the major obstacle being possible failure of cancellation laws in general convex algebras. Examples of positive convex algebras are provided by convex subsets of a vector space over $\mathbb{R}$ which contain the zero vector. Examples of totally convex algebras are provided by convex subsets of a vector space $\mathbb{R}$ which are symmetric around the zero vector.

Convex algebras appear in a categorical context. To explain this, e.g. for the PCA-situation, consider the category Vec_1^+ whose objects are regularly ordered

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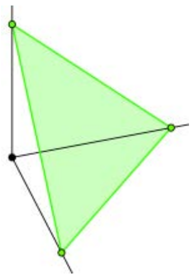
normed vector spaces over the scalar field $\mathbb{R}$ and morphisms are positive and linear contractions between such spaces. The functor $\Delta : \mathbf{Vec}_1^+ \to \mathbf{Sets}$ which acts on objects as

$$\Delta(V) := \big\{x \in V \mid \|x\| \le 1, x \ge 0\big\},$$

and on morphisms as restriction to $\Delta(V)$, has a left adjoint. It turns out that the algebraic category PCA is the category of Eilenberg-Moore algebras of the monad induced by this adjunction, cf. [Pu84]. Moreover, the monad in question is actually the discrete subprobability distribution monad, hence PCA is the category of Eilenberg-Moore algebras of the subprobability distribution monad [Do06, Do08].

Our aim in this paper is to achieve full understanding of the structure of finitely generated algebras in CA (PCA and TCA). We manage this with Theorem 4.3 and 4.4 below, where we describe the congruences on any polytope in the euclidean space $\mathbb{R}^n$ considered as a convex algebra.

It is simple to check that, for each $n \in \mathbb{N}^+$, the free algebra F_n in CA with n generators is given by the standard $(n–1)$-simplex in $\mathbb{R}^n$ (a particular polytope). For $n = 3$, we can picture this algebra as[1]



(1.1)

Clearly, knowing all congruences of the free algebras F_n, $n \in \mathbb{N}^+$, is enough to understand all finitely generated algebras in CA. These results on CA can be transferred to PCA and TCA. Therefore, we also achieve full understanding of the structure of finitely generated positive convex or totally convex algebras.

Besides its obvious intrinsic interest, our motivation to investigate finitely generated algebras in CA (PCA or TCA) originates in a categorical problem. The probability subdistribution monad arising from the above mentioned adjunction, including the functor Δ, and its Eilenberg-Moore algebras play a crucial role in connection with the axiomatization of trace semantics for probabilistic systems given in [SS11]. There the question arose whether or not each finitely generated algebra in PCA is also finitely presentable. Using the newly established knowledge about congruence relations, we can answer this question affirmatively, cf. Corollary 5.3.

Historically, work on convex algebras can be traced back (at least) to [Se67]. The theory of totally convex algebras (and their analogues allowing infinitary operations) started in [PR84], where they were realized to be the Eilenberg-Moore algebras associated with the adjunction induced by the unit ball functor from the category of Banach spaces (with linear contractions) to **Sets**. A similar treatment of positive convex algebras was given shortly after in [Pu84]. Later on these notions were extensively studied, mainly focussing on the categorical viewpoint

[1]Picture source: http://en.wikipedia.org/wiki/Simplex

and topological questions, see, e.g., [PR85, BK93, Ke98, Pu01a, Pu01b, Pu03] and the references therein. A far reaching generalization, namely the concept of convexity theories, has been developed in a series of papers involving several authors which started with [Rö94], and went on (at least) till [Rö01].

Previous work which is the closest to our approach is [PR90, Ke99, Ke00], where congruence relations in (infinitary or p-) totally convex modules (algebras) are studied. Some parts of our results read similarly and several geometric ideas employed there can also be used in the present setting. In order to prevent confusion concerning terminology, let us note explicitly that in the literature algebras in CA (PCA or TCA) are also called "finitely (positively/totally) convex modules". The term "finitely" thereby refers to the fact that they carry only finitary operations. However, in the present paper we stick to the purely algebraic setting and do not touch upon the possibility of allowing infinitary operations. Hence, we omit the prefix "finitely" from the notation. Moreover, we also choose the term "algebra" over "module" since it has been used in recent work regarding positive convex algebras [Do06, Do08, SS11].

The structure of the paper is as follows. After the introduction, we recall some notions and facts from convex geometry in Section 2. In Section 3, we present the equational classes of convex, positive convex and totally convex algebras. After collecting some basic facts, we investigate the relationship between CA, PCA, and TCA. Interestingly, it turns out that CA and PCA are in essence just the same, whereas TCA carries a significantly stronger structure. Section 4 is the core of the paper. There we formulate and prove Theorems 4.3 and 4.4 that describe the congruence relations on a polytope K in euclidean space. It turns out that a congruence on K is fully determined by two ingredients: (1) a family of linear subspaces, describing the congruence classes in the interior of K and in the interior of each of its lower dimensional facets; (2) a graph, describing how the interiors of K and each of its facets are related to the lower dimensional facets forming the respective boundary. Finally, in Section 5, we give the already mentioned application, and show that in each of the algebraic categories corresponding to CA, PCA, and TCA, the notions "finitely generated" and "finitely presentable" coincide.

Our basic reference concerning terminology and results of universal algebra is the (old but still excellent) book [Grä68], or the more recent treatment [BS00]. For the (few) notions from category theory which are used in this paper, we refer the reader to [La98]. Our standard reference concerning convex geometry is [Grü03].

2 Preliminaries from convex geometry

Basic universal algebra and euclidean topology notions and results will be recalled when they are needed. In this section we explicitly recall some definitions and results from convex geometry. We give even proofs of simple properties, to set the mood for what follows in the paper.

We start with the definition of a convex set.

Definition 2.1. A subset C of a vector space V over the field $\mathbb{R}$ is convex if for all $x, y \in C$ and any scalar $\lambda \in [0, 1]$ it holds that $\lambda x + (1 - \lambda)y \in C$. Geometrically, this means that C contains, together with each two points, the whole line segment connecting them.

Convex sets are mapped by linear functions to convex sets. The following simple property shows that convexity is the same as being closed under arbitrary convex linear combinations.

Lemma 2.2. *Let C be a subset of a vector space V over $\mathbb{R}$. Then C is convex if and only if*

$$\sum_{i=1}^{n} p_i x_i \in C$$

for all $n \in \mathbb{N}^+$, all $x_i \in C$, and all $p_i \in [0,1]$ with $i = 1, \ldots, n$ such that $\sum_{i=1}^{n} p_i = 1$.

Proof. It is clear that being closed under convex linear combinations implies convexity. The other direction is proved by induction. Let C be a convex subset and let $x_i \in C$, $p_i \in [0,1]$, for $i = 1, \ldots, n$ be given such that $\sum_{i=1}^{n} p_i = 1$. If $n = 1$ or $p_i = 1$ for some i, the statement is trivial. If $n = 2$, the statement holds by convexity. Assume that $n > 2$, $p_1 \neq 1$, and the statement holds for $n - 1$. Then

$$\sum_{i=1}^{n} p_i x_i = p_1 x_1 + (1 - p_1) \cdot \sum_{i=2}^{n} \frac{p_i}{1 - p_1} x_i$$

and $\sum_{i=2}^{n} \frac{p_i}{1-p_1} = 1$. Hence, from the inductive hypothesis $\sum_{i=2}^{n} \frac{p_i}{1-p_1} x_i \in C$ and by convexity also $\sum_{i=1}^{n} p_i x_i \in C$. □

The following subsets, associated with a finite subset Y of V, play an important role:

$$\begin{aligned}
\operatorname{span} Y &:= \Big\{ \sum_{y \in Y} \lambda_y y \mid \lambda_y \in \mathbb{R} \Big\}, \\
\operatorname{dir} Y &:= \Big\{ \sum_{y \in Y} \lambda_y y \mid \lambda_y \in \mathbb{R}, \sum_{y \in Y} \lambda_y = 0 \Big\}, \\
\operatorname{aff} Y &:= \Big\{ \sum_{y \in Y} \lambda_y y \mid \lambda_y \in \mathbb{R}, \sum_{y \in Y} \lambda_y = 1 \Big\}, \\
\operatorname{co} Y &:= \Big\{ \sum_{y \in Y} \lambda_y y \mid \lambda_y \in [0,1], \sum_{y \in Y} \lambda_y = 1 \Big\}, \\
\check{\operatorname{co}} Y &:= \Big\{ \sum_{y \in Y} \lambda_y y \mid \lambda_y \in (0,1], \sum_{y \in Y} \lambda_y = 1 \Big\}.
\end{aligned}$$

We refer to $\operatorname{co} Y$ as the closed convex hull of Y and $\check{\operatorname{co}} Y$ as the open convex hull or the interior of $\operatorname{co} Y$. We will see later that this choice of terminology is indeed justified, cf. Lemma 2.5. The linear span $\operatorname{span} Y$ is the smallest vector subspace that contains Y. Moreover, we refer to $\operatorname{aff} Y$ as the affine space generated by Y and $\operatorname{dir} Y$ as the directions of $\operatorname{aff} Y$. Note that $\operatorname{dir} Y$ is a vector subspace. Clearly, for each finite set Y,

$$\check{\operatorname{co}} Y \subseteq \operatorname{co} Y \subseteq \operatorname{aff} Y \subseteq \operatorname{span} Y \quad \text{and} \quad \operatorname{dir} Y \subseteq \operatorname{span} Y.$$

If Y contains only one element, then $\check{\operatorname{co}} Y = \operatorname{co} Y = \operatorname{aff} Y = Y$ and $\operatorname{dir} Y = \{0\}$. If $|Y| \geq 2$, then $\check{\operatorname{co}} Y \subset \operatorname{co} Y \subset \operatorname{aff} Y$ and $\operatorname{dir} Y \neq \{0\}$.

First, some simple geometric properties of these sets.

Lemma 2.3. *Let Y be a finite subset of a vector space V over $\mathbb{R}$. Then the following hold:*

(*i*) *For each $z \in \operatorname{aff} Y$ we have*

$$\operatorname{aff} Y = z + \operatorname{dir} Y\,.$$

(*ii*) *For each $z \in \operatorname{aff} Y$, we have*

$$\operatorname{dir} Y = \big\{ w - z \mid w \in \operatorname{aff} Y \big\}\,.$$

(*iii*) *We have*

$$\operatorname{dir} Y = \big\{ w - z \mid z, w \in \operatorname{aff} Y \big\}\,.$$

(*iv*) *And, for every $y_0 \in Y$,*

$$\operatorname{dir} Y = \operatorname{span}\{ y - y_0 \mid y \in Y \big\}\,.$$

Proof. Let $z \in \operatorname{aff} Y$ be given, and write $z = \sum_{y\in Y} \mu_y y$ with $\sum_{y\in Y} \mu_y = 1$. Given $x \in \operatorname{aff} Y$, write $x = \sum_{y\in Y} \lambda_y y$ with $\sum_{y\in Y} \lambda_y = 1$. Then

$$x - z = \sum_{y\in Y} (\lambda_y - \mu_y) y\,,$$

and $\sum_{y\in Y}(\lambda_y - \mu_y) = 0$. This means that $x - z \in \operatorname{dir} Y$, and we see that the inclusion "$\subseteq$" in item (*i*) holds. All other inclusions in item (*i*) - item (*iii*) follow in the same way. Item (*iv*) is straightforward by the definition of $\operatorname{dir} Y$. □

In the situation that $V = \mathbb{R}^n$ some important topological properties hold. These are expressed in the following two lemmas.

Lemma 2.4. *Let Y be a finite subset of $\mathbb{R}^n$. Then $\operatorname{co} Y$ is compact and convex. Moreover, $\operatorname{co} Y$ is the closure of $\check{\operatorname{co}} Y$.*

Proof. It is easy to check by the definitions that $\operatorname{co} Y$ is convex. Recall that a set is compact if it is closed and bounded. Moreover, continuous functions map a compact set to a compact set. The set

$$\Lambda = \{(\lambda_y)_{y\in Y} \in \mathbb{R}^{|Y|} \mid \lambda_y \in [0,1], \sum_{y\in Y} \lambda_y = 1\}$$

is compact, the function $(\lambda_y)_{y\in Y} \mapsto \sum_{y\in Y} \lambda_y y$ is continuous, and it maps Λ to $\operatorname{co} Y$. Hence, $\operatorname{co} Y$ is compact.

Since $\check{\operatorname{co}} Y \subseteq \operatorname{co} Y$ and $\operatorname{co} Y$ is closed, we get $\operatorname{Clos}(\check{\operatorname{co}} Y) \subseteq \operatorname{co} Y$. For the opposite inclusion, take an element $y \in \operatorname{co} Y$. Write $Y = \{y_1, \ldots, y_n\}$, and $y = \sum_{i=1}^n \lambda_i y_i$ with $\lambda_i \in [0,1]$ and $\sum_{i=1}^n \lambda_i = 1$. We will construct a sequence of elements in $\check{\operatorname{co}} Y$ whose limit is y, showing that $y \in \operatorname{Clos}(\check{\operatorname{co}} Y)$. Without loss of generality, we may assume that $\lambda_n > 0$ and $\lambda_1, \ldots, \lambda_k$ are all coefficients that are equal to 0. Then for $y_\epsilon = \sum_{i=1}^n \hat{\lambda}_i y_i$ where $\varepsilon \in (0, \frac{\lambda_n}{k})$ and

$$\hat{\lambda}_i := \begin{cases} \varepsilon & i = 1, \ldots, k \\ \lambda_i & i = k+1, \ldots, n-1 \\ \lambda_n - k\varepsilon & i = n \end{cases}$$

we have $y_\varepsilon \in \check{\operatorname{co}} Y$. The elements y_ε tend to y when ε tends to 0. □

Lemma 2.5. *Let Y be a finite subset of $\mathbb{R}^n$ with $|Y| \geq 2$. Then $\breve{\mathrm{co}}\, Y$ is open considered as a subset of* $\operatorname{aff} Y$.

Proof. Let $Y = \{y_0, \dots, y_m\} \subseteq \mathbb{R}^n$ with $m \geq 1$. From Lemma 2.3, for each $z \in \operatorname{aff} Y$, $\operatorname{aff} Y = z + \operatorname{span}\{y_k - y_0 \mid k = 1, \dots, m\}$. Let $\{b_1, \dots, b_l\}$ be a maximal linearly independent subset of $\{y_k - y_0 \mid k = 1, \dots, m\}$. Then $\operatorname{aff} Y = z + \operatorname{span}\{b_1, \dots, b_l\}$.

Choose (if necessary) $b_{l+1}, \dots, b_n$ such that $\{b_1, \dots, b_n\}$ is a basis of $\mathbb{R}^n$ and consider the norm

$$||x|| := \max_{i=1,\dots,n} |\beta_i|, \quad x \in \mathbb{R}^n\,,$$

where β_i are the unique coefficients in $x = \sum_{i=1}^n \beta_i b_i$.

For each $\varepsilon > 0$, the set $U_\varepsilon = \{x \in \mathbb{R}^n \mid ||x|| < \varepsilon\}$ is open with respect to the euclidean topology in $\mathbb{R}^n$ (by equivalence of norms[2]). Thus, for each $z \in \operatorname{aff} Y$, the set $(z + U_\varepsilon) \cap \operatorname{aff} Y$ is an open subset of $\operatorname{aff} Y$. We have

$$(z + U_\varepsilon) \cap \operatorname{aff} Y = z + \{\sum_{i=1}^{l} \beta_i b_i \mid |\beta_i| < \varepsilon\}\,,$$

where "$\supseteq$" is trivial and "$\subseteq$" follows since $\{b_1, \dots, b_n\}$ is linearly independent.

Assume now that $z \in \breve{\mathrm{co}}\, Y$, and write $z = \sum_{i=0}^m \lambda_i y_i$ with $\lambda_i \in (0, 1)$. This is always possible since $|Y| \geq 2$ and hence no λ_i equals 1. Set

$$\varepsilon := \frac{1}{m} \min\left(\{\lambda_i \mid i = 0, \dots, m\} \cup \{1 - \lambda_i \mid i = 0, \dots, m\}\right).$$

Then $\varepsilon > 0$ and it is not difficult to check that the choice of ε guarantees that

$$(z + U_\varepsilon) \cap \operatorname{aff} Y \subseteq \breve{\mathrm{co}}\, Y.$$

Hence we have found an open set in $\operatorname{aff} Y$ containing z and being contained in $\breve{\mathrm{co}}\, Y$. Since $z \in \breve{\mathrm{co}}\, Y$ was arbitrary, this shows that $\breve{\mathrm{co}}\, Y$ is open as a subset of $\operatorname{aff} Y$. □

Lemma 2.5 implies an alternative characterization of $\operatorname{dir} Y$.

Lemma 2.6. *Let Y be a finite subset of $\mathbb{R}^n$. Then we have*

$$\operatorname{dir} Y = \operatorname{span}\{y - z \mid y \in \breve{\mathrm{co}}\, Y\}, \quad z \in \operatorname{aff} Y\,,$$

and

$$\operatorname{dir} Y = \operatorname{span}\{y_2 - y_1 \mid y_1, y_2 \in \breve{\mathrm{co}}\, Y\}\,.$$

Proof. If $|Y| = 1$, we have $\operatorname{dir} Y = \{0\}$ and $\breve{\mathrm{co}}\, Y = \operatorname{aff} Y = Y$. Hence, the stated equalities hold in this case. Moreover, the second asserted equality will follow immediately once the first is shown, and the inclusion "$\supseteq$" in the first one is trivial.

Assume that $|Y| \geq 2$, and let $z \in \operatorname{aff} Y$ be given. Since $\operatorname{span}\{y - z \mid y \in \breve{\mathrm{co}}\, Y\} \subseteq \operatorname{dir} Y$ and both are vector subspaces, we get that $\operatorname{span}\{y - z \mid y \in \breve{\mathrm{co}}\, Y\}$ is a subspace of $\operatorname{dir} Y$. Note that no proper subspace of $\operatorname{dir} Y$ contains a non-empty open subset of $\operatorname{dir} Y$ (think as an illustration of a line in a plane and

[2] $\max_{i=1,\dots,n} |\beta_i| \leq \left(\sum_{i=1}^n \beta_i^2\right)^{\frac{1}{2}} \leq \sqrt{n} \max_{i=1,\dots,n} |\beta_i|$.

a full open disc in this plane). In order to show the needed equality, we will show that $\operatorname{span}\{y - z \mid y \in \breve{\operatorname{co}} Y\}$ does contain a non-empty open set in $\operatorname{dir} Y$. Consider the translation map $T_{-z} : x \mapsto x - z$. It is a homeomorphism, i.e., a continuous bijective map whose inverse is also continuous. Homeomorphisms map open subsets onto open subsets. Note that T_{-z} maps $\operatorname{aff} Y$ onto $\operatorname{dir} Y$. As a consequence, also by Lemma 2.5, $T_{-z}(\breve{\operatorname{co}} Y)$ is a non-empty open subset of $\operatorname{dir} Y$. Clearly, $T_{-z}(\breve{\operatorname{co}} Y) = \{y - z \mid y \in \breve{\operatorname{co}} Y\} \subseteq \operatorname{span}\{y - z \mid y \in \breve{\operatorname{co}} Y\}$. □

Let $C \subseteq \mathbb{R}^n$ be convex. A point $e \in C$ is called an extremal point if

$$e = tx + (1-t)y \quad \text{with } x, y \in C, t \in (0,1) \quad \Rightarrow \quad x = y = e\,.$$

Geometrically, this means that e does not lie in the interior of any line segment with endpoints in C. We denote the set of all extremal points of C by $\operatorname{ext} C$.

It is an important fact that compact convex sets can be recovered from their extremal points. The Kreĭn-Milman theorem states in a very general context that each compact convex set is the closed convex hull of its extremal points, see, e.g., [Ru91, 3.23]. The version of this theorem for subsets C of $\mathbb{R}^n$, and this is what we use here, can be found in [Grü03, 2.4.5].

We mainly deal with a certain kind of geometric objects called polytopes.

Definition 2.7. Let K be a subset of the euclidean space $\mathbb{R}^n$. The set K is a polytope if it is of the form $K = \operatorname{co} Y$ for some finite set $Y \subseteq \mathbb{R}^n$.

A fundamental example of a polytope is a simplex.

Example 2.8 (A d-dimensional simplex)**.** Let $a \in \mathbb{R}^n$, and let $\{u_1, \ldots, u_d\}$ be a linearly independent subset of $\mathbb{R}^n$. Then the polytope

$$K := \operatorname{co}\big(\{a\} \cup \{a + u_i \mid i = 1, \ldots, d\}\big)$$

is called a d-dimensional simplex.

For instance, for $d = n = 3$ and

$$a := \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad u_1 := \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \; u_2 := \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \; u_3 := \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \tag{2.1}$$

we obtain the pyramid having the triangle with corner points $(0,0,0)$, $(1,0,0)$, $(0,1,0)$ as its base and the point $(0,0,1)$ as its apex.

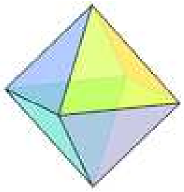
Another concrete example of a polytope is an octahedron.

Example 2.9 (A d-dimensional octahedron)**.** Let $a \in \mathbb{R}^n$, and let $\{u_1, \ldots, u_d\}$ be a linearly independent subset of $\mathbb{R}^n$. Then the polytope

$$K := \operatorname{co}\big(\{a + u_i \mid i = 1, \ldots, d\} \cup \{a - u_i \mid i = 1, \ldots, d\}\big)$$

is called a d-dimensional octahedron.

For instance, if $d = n = 3$ and a, u_1, u_2, u_3 again as in (2.1), we obtain a regular octahedron with center at the origin.[3]



(2.2)

Polytopes can be defined in several equivalent ways. The definition used in [Grü03] is presented in the next lemma. The fact that this definition is equivalent to the one above, i.e., the proof of the lemma is, in essence, a consequence of the Kreĭn-Milman theorem; we skip the details.

Lemma 2.10. *A subset $K \subseteq \mathbb{R}^n$ is a polytope if and only if K is compact, convex, and the set $\operatorname{ext} K$ of its extremal points is finite.* □

Note that if K is a polytope and $K = \operatorname{co} Y$ for some finite set Y, then $\operatorname{ext} K \subseteq Y$ and $K = \operatorname{co}(\operatorname{ext} K)$.

Remark 2.11. The mentioned concrete examples, the simplex (2.1) and the octahedron (2.2), are of particular interest in the present context. They are the free algebras with 3 generators in the equational classes PCA and TCA, respectively (similarly as the standard simplex (1.1) is in CA). This fact (of course for dimension n instead of 3), together with the results of Section 3 below, shows that understanding polytopes as convex algebras suffices to understand all finitely generated algebras in CA, PCA, and TCA.

3 The equational classes CA, PCA, and TCA

In this section we investigate the three convexity theories of convex, positive convex, and totally convex algebras and their induced equational classes. To start with, let us recall the definitions.

Definition 3.1. A convex algebra is an algebra (with nonempty carrier set) of type

$$\mathcal{T}_{\text{ca}} := \Big\{ (p_i)_{i=1}^n \in \mathbb{R}^n \mid \ n \in \mathbb{N}^+, \, p_1, \dots, p_n \geq 0, \, \sum_{i=1}^n p_i = 1 \Big\},$$

which satisfies (we denote by $f_{(p_i)_{i=1}^n}$, $(p_i)_{i=1}^n \in \mathcal{T}_{\text{ca}}$, the operations of the algebra)

(1) The projection axiom:

$$f_{(\delta_{ij})_{i=1}^n}(x_1, \dots, x_n) = x_j, \quad n \in \mathbb{N}^+, \ j = 1, \dots, n,$$

where δ_{ij} denotes the Kronecker-delta

$$\delta_{ij} := \begin{cases} 1\,, & i = j \\ 0\,, & i \neq j \end{cases}$$

[3]Picture source: http://en.wikipedia.org/wiki/Octahedron

(2) The baricenter axiom:

$$f_{(p_i)_{i=1}^n}\big(f_{(p_{1j})_{j=1}^m}(x_1,\dots,x_m),\dots,f_{(p_{nj})_{j=1}^m}(x_1,\dots,x_m)\big) = \\ = f_{(\sum_{i=1}^n p_i p_{ij})_{j=1}^m}(x_1,\dots,x_m),$$

whenever $n,m \in \mathbb{N}^+$, $(p_i)_{i=1}^n \in \mathcal{T}_{\text{ca}}$, and $(p_{ij})_{j=1}^m \in \mathcal{T}_{\text{ca}}$, $i=1,\dots,n$.

The operation appearing on the right hand side of the baricenter axiom is well-defined since

$$\sum_{j=1}^m \Big(\sum_{i=1}^n p_i p_{ij}\Big) = \sum_{i=1}^n p_i \Big(\sum_{j=1}^m p_{ij}\Big), \tag{3.1}$$

and hence $(\sum_{i=1}^n p_i p_{ij})_{j=1}^m \in \mathcal{T}_{\text{ca}}$.

By CA we denote the equational class of convex algebras.

Definition 3.2. A positive convex algebra is an algebra (with nonempty carrier set) of type

$$\mathcal{T}_{\text{pca}} := \Big\{(p_i)_{i=1}^n \in \mathbb{R}^n \mid n \in \mathbb{N}^+,\, p_1,\dots,p_n \geq 0,\, \sum_{i=1}^n p_i \leq 1\Big\},$$

which satisfies the projection axiom and the baricenter axiom, where in the latter $(p_i)_{i=1}^n$ and $(p_{ij})_{j=1}^m$ vary through $\mathcal{T}_{\text{pca}}$.

We denote the equational class of all positive convex algebras as PCA.

Definition 3.3. A totally convex algebra is an algebra (with nonempty carrier set) of type

$$\mathcal{T}_{\text{tca}} := \Big\{(p_i)_{i=1}^n \in \mathbb{R}^n \mid n \in \mathbb{N}^+,\, \sum_{i=1}^n |p_i| \leq 1\Big\},$$

which satisfies the projection axiom and the baricenter axiom, where in the latter $(p_i)_{i=1}^n$ and $(p_{ij})_{j=1}^m$ vary through $\mathcal{T}_{\text{tca}}$.

We denote the equational class of all positive convex algebras as TCA.

Note that, again because of (3.1), the operation appearing on the right hand side of the baricenter axiom is always well-defined, i.e., is of type $\mathcal{T}_{\text{pca}}$ or $\mathcal{T}_{\text{tca}}$, respectively (to see this for $\mathcal{T}_{\text{tca}}$, use the triangle inequality).

It is obvious that each positive convex algebra can be considered as a convex algebra. To be precise, if $\langle P, (f_\gamma)_{\gamma\in\mathcal{T}_{\text{pca}}}\rangle$ is a positive convex algebra, then $\langle P, (f_\gamma)_{\gamma\in\mathcal{T}_{\text{ca}}}\rangle$ is a convex algebra. Similarly: If $\langle P, (f_\gamma)_{\gamma\in\mathcal{T}_{\text{tca}}}\rangle$ is a totally convex algebra, then $\langle P, (f_\gamma)_{\gamma\in\mathcal{T}_{\text{pca}}}\rangle$ is a positive convex algebra, and in turn $\langle P, (f_\gamma)_{\gamma\in\mathcal{T}_{\text{ca}}}\rangle$ is a convex algebra. Due to this fact, many results can immediately be transferred from CA to PCA and TCA.

Another interesting fact, which we shall explain in the sequel, is that CA and PCA are essentially the same from an algebraic viewpoint, whereas TCA carries more structure than CA (compare Proposition 3.6 with Proposition 3.8 below).

When working with algebras of one of the types $\mathcal{T}_{\text{ca}}$, $\mathcal{T}_{\text{pca}}$, or $\mathcal{T}_{\text{tca}}$, it is practical (and customary) to write operations as formal sums and/or to use vector notation:

$$f_{(p_i)_{i=1}^n}(x_1,\dots,x_n) = \sum_{i=1}^n p_i x_i = (p_1,\dots,p_n)\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}.$$

With this notation the projection axiom writes as

$$(0,\dots,\underset{\substack{\uparrow\\ j\text{-th place}}}{1},\dots,0)\begin{pmatrix}x_1\\ \vdots\\ x_n\end{pmatrix}=x_j,$$

and the baricenter axiom as

$$(p_1,\dots,p_n)\left[\begin{pmatrix}p_{11} & \dots & p_{1m}\\ \vdots & & \vdots\\ p_{n1} & \dots & p_{nm}\end{pmatrix}\begin{pmatrix}x_1\\ \vdots\\ x_m\end{pmatrix}\right]=$$

$$=\left[(p_1,\dots,p_n)\begin{pmatrix}p_{11} & \dots & p_{1m}\\ \vdots & & \vdots\\ p_{n1} & \dots & p_{nm}\end{pmatrix}\right]\begin{pmatrix}x_1\\ \vdots\\ x_m\end{pmatrix}.$$

In the next lemma we provide some simple but useful identities which follow from the projection and baricenter axioms. For the convenience of the reader, we provide an explicit proof. In the setting of TCA (with infinitary operations) these identities were shown in [PR84, Theorem 2.4] using a different proof.

Lemma 3.4. *For items (i)–(iii), let P be an algebra in any of the classes* CA, PCA, *or* TCA. *For items (iv) and (v), assume that P belongs to* PCA *or* TCA. *Let c stand for ca, pca, tca, if P is in* CA, PCA, TCA, *respectively.*

(i) *The above introduced sum notation already suggests that operations are in a sense commutative. They indeed are, we have*

$$f_{(p_i)_{i=1}^n}(x_1,\dots,x_n)=f_{(p_{\sigma(i)})_{i=1}^n}(x_{\sigma(1)},\dots,x_{\sigma(n)}),$$

whenever $n\in\mathbb{N}^+$, σ is a permutation of $\{1,\dots,n\}$, and $(p_i)_{i\in I}\in\mathcal{T}_c$.

(ii) *The extended projection law:*

$$f_{(p_i)_{i=1}^n}(x_1,\dots,x_n)=f_{(p_{i_k})_{k=1}^m}(x_{i_1},\dots,x_{i_m}),$$

holds whenever $(p_i)_{i=1}^n\in\mathcal{T}_c$ and $i_1,\dots,i_m$ satisfy

$$i_1<\dots<i_m,\quad \{i_1,\dots,i_m\}\supseteq\{i\in\{1,\dots,n\}\mid p_i\neq 0\}.$$

(iii) *Whenever $(p_i)_{i=1}^n\in\mathcal{T}_c$ and $x\in P$, we have*

$$f_{(p_i)_{i=1}^n}(x,\dots,x)=f_{(\sum_{i=1}^n p_i)}(x).$$

(iv) *The elements*

$$f_{(0)_{i=1}^n}(x_1,\dots,x_n),\quad n\in\mathbb{N}^+,\ x_1,\dots,x_n\in P,$$

all coincide. We denote this element as 0_P.

(*v*) *The element* 0_P *plays the role of a zero element: Let* $(p_i)_{i=1}^n \in \mathcal{T}_c$ *and let* $i_1, \dots, i_m$ *satisfy*

$$i_1 < \cdots < i_m, \quad \{i_1, \dots, i_m\} \supseteq \{i \in \{1, \dots, n\} \mid x_i \neq 0_P\}.$$

Then

$$f_{(p_i)_{i=1}^n}(x_1, \dots, x_n) = \begin{cases} f_{(p_{i_k})_{k=1}^m}(x_{i_1}, \dots, x_{i_m}), & m \geq 1 \\ 0_P &, \quad m = 0 \end{cases}.$$

Proof. We denote, here and in the sequel, by I_n the $n \times n$ identity matrix and by $0_{n,m}$ the $n \times m$ zero matrix.

(*i*) Set $p_{ij} := \delta_{i,\sigma^{-1}(j)}$ and compute

$$\begin{aligned} & f_{(p_{\sigma(i)})_{i=1}^n}\big(x_{\sigma(1)}, \dots, x_{\sigma(n)}\big) = \\ & \quad = f_{(p_{\sigma(i)})_{i=1}^n}\big(f_{(p_{1j})_{j=1}^n}(x_1, \dots, x_n), \dots, f_{(p_{nj})_{j=1}^n}(x_1, \dots, x_n)\big) = \\ & \quad = f_{(\sum_{i=1}^n p_{\sigma(i)} p_{ij})_{j=1}^n}(x_1, \dots, x_n) = f_{(p_j)_{j=1}^n}(x_1, \dots, x_n). \end{aligned}$$

(*ii*) By commutativity it is enough to consider the case that $i_k = k$, $k = 1, \dots, m$. Then we compute, using the projection and baricenter axioms,

$$\begin{aligned} (p_1, \dots, p_m) \begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix} & = (p_1, \dots, p_m) \left[\begin{pmatrix} I_m & 0_{m,n-m} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_m \\ x_{m+1} \\ \vdots \\ x_n \end{pmatrix} \right] = \\ & = \Big[(p_1, \dots, p_m) \begin{pmatrix} I_m & 0_{m,n-m} \end{pmatrix} \Big] \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \\ & = \underbrace{(p_1, \dots, p_m, 0, \dots, 0)}_{(p_1, \dots, p_n)} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}. \end{aligned}$$

(*iii*) $f_{(p_i)_{i=1}^n}(x, \dots, x) = f_{(p_i)_{i=1}^n}(f_{(1)}(x), \dots, f_{(1)}(x)) = f_{(\sum_{i=1}^n p_i)}(x)$.

(*iv*) Apply the extended projection law twice to obtain

$$f_{(0)_{i=1}^n}(x_1, \dots, x_n) = f_{(0)_{i=1}^{n+n'}}(x_1, \dots, x_n, x'_1, \dots, x'_{n'}) = f_{(0)_{i=1}^{n'}}(x'_1, \dots, x'_{n'}).$$

(*v*) Consider first the case that $m = 0$. We have, using (*iv*),

$$\begin{aligned} (p_1, \dots, p_n) \begin{pmatrix} 0_P \\ \vdots \\ 0_P \end{pmatrix} & = (p_1, \dots, p_n) \left[0_{n,n} \begin{pmatrix} 0_P \\ \vdots \\ 0_P \end{pmatrix} \right] = \\ & = \Big[(p_1, \dots, p_n) 0_{n,n} \Big] \begin{pmatrix} 0_P \\ \vdots \\ 0_P \end{pmatrix} = (0, \dots, 0) \begin{pmatrix} 0_P \\ \vdots \\ 0_P \end{pmatrix} = 0_P. \end{aligned}$$

Assume now that $m \geq 1$. Again, it is enough to consider the case that $i_k = k$, $k = 1, \ldots, m$. Then we can compute

$$
\begin{aligned}
(p_1, \ldots, p_n) \begin{pmatrix} x_1 \\ \vdots \\ x_m \\ 0_P \\ \vdots \\ 0_P \end{pmatrix} &= (p_1, \ldots, p_n) \left[\begin{pmatrix} I_m & 0_{m,n-m} \\ 0_{n-m,m} & 0_{n-m,n-m} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_m \\ 0_P \\ \vdots \\ 0_P \end{pmatrix} \right] = \\
&= \left[(p_1, \ldots, p_n) \begin{pmatrix} I_m & 0_{m,n-m} \\ 0_{n-m,m} & 0_{n-m,n-m} \end{pmatrix} \right] \begin{pmatrix} x_1 \\ \vdots \\ x_m \\ 0_P \\ \vdots \\ 0_P \end{pmatrix} = \\
&= (p_1, \ldots, p_m, 0, \ldots, 0) \begin{pmatrix} x_1 \\ \vdots \\ x_m \\ 0_P \\ \vdots \\ 0_P \end{pmatrix} \overset{(ii)}{=} (p_1, \ldots, p_m) \begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix}.
\end{aligned}
$$

□

Note that items (iv) and (v) would not at all make sense within CA, since the operations appearing in them do not belong to $\mathcal{T}_{\text{ca}}$.

Remark 3.5. Let $P \in \mathsf{CA}$. A consequence of the extended projection law is that the baricenter axiom remains valid when the vectors $(p_{ij})_{j=1}^m$ appearing therein are no more of the same length m and no more bound to the same variables. More precisely, let

$$
\begin{aligned}
& n, m \in \mathbb{N}^+, \quad K_i \subseteq \mathbb{N}^+, \ i = 1, \ldots, n \text{ with } \bigcup_{i=1}^n K_i = \{1, \ldots, m\}, \\
& (p_i)_{i=1}^n \in \mathcal{T}_{\text{ca}}, \qquad (3.2) \\
& (p_{ij})_{j \in K_i} \text{ with } p_{ij} \geq 0, \sum_{j \in K_i} p_{ij} = 1, \quad i = 1, \ldots, n. \qquad (3.3)
\end{aligned}
$$

Set $m_i := |K_i|$, and write $K_i = \{\kappa_k^i \mid k = 1, \ldots, m_i\}$ with $\kappa_k^i < \kappa_{k+1}^i$. Then

$$
\sum_{i=1}^n p_i \Big(\sum_{k=1}^{m_i} p_{i\kappa_k^i} x_{\kappa_k^i} \Big) = \sum_{j=1}^m \Big(\sum_{\substack{i=1 \\ j \in K_i}}^n p_i p_{ij} \Big) x_j,
$$

whenever $x_1, \ldots, x_m \in P$.

The same holds when $P \in \mathsf{PCA}$ and the conditions (3.2) and (3.3) are replaced by

$$
\begin{aligned}
& (p_i)_{i=1}^n \in \mathcal{T}_{\text{pca}}, \\
& (p_{ij})_{j \in K_i} \text{ with } p_{ij} \geq 0, \sum_{j \in K_i} p_{ij} \leq 1, \quad i = 1, \ldots, n.
\end{aligned}
$$

Correspondingly, the same holds when $P \in \mathsf{TCA}$ and

$$
\begin{aligned}
& (p_i)_{i=1}^n \in \mathcal{T}_{\text{tca}}, \\
& (p_{ij})_{j \in K_i} \text{ with } p_{ij} \in \mathbb{R}, \sum_{j \in K_i} |p_{ij}| \leq 1, \quad i = 1, \ldots, n.
\end{aligned}
$$

This remark also clarifies the note made in [Pu03, p.110] immediately after the definition of positively convex algebra (in which the above stronger form of the baricenter axiom is required).

3.1 Positive convex algebras vs. convex algebras

In this subsection we make precise what we meant above by claiming that CA and PCA are "just the same".

Let $\langle P,(f_\gamma)_{\gamma\in\mathcal{T}_{\rm ca}}\rangle\in\mathsf{CA}$. We call a collection $\{\bar f_\gamma|\gamma\in\mathcal{T}_{\rm pca}\}$ of operations on P an operational extension of $\langle P,(f_\gamma)_{\gamma\in\mathcal{T}_{\rm ca}}\rangle$ to PCA, if

$$\bar f_\gamma=f_\gamma,\ \gamma\in\mathcal{T}_{\rm ca},\quad\text{and}\quad\langle P,(\bar f_\gamma)_{\gamma\in\mathcal{T}_{\rm pca}}\rangle\in\mathsf{PCA}\,. \tag{3.4}$$

Proposition 3.6. *Let $\langle P,(f_\gamma)_{\gamma\in\mathcal{T}_{ca}}\rangle\in\mathsf{CA}$. Then the set of all operational extensions of $\langle P,(f_\gamma)_{\gamma\in\mathcal{T}_{ca}}\rangle$ to PCA corresponds bijectively to P. In particular, since $P\neq\emptyset$, P can be operationally extended to a positive convex algebra.*

Proof. In the first step we show existence of operational extensions. Choose an element $z\in P$, and define for $(p_i)_{i=1}^n\in\mathcal{T}_{\rm pca}$

$$\bar f_{(p_i)_{i=1}^n}(x_1,\dots,x_n):=f_{(p_1,\dots,p_n,\bar p)}(x_1,\dots,x_n,z),$$

where

$$\bar p:=1-\sum_{i=1}^n p_i.$$

The extended projection law in $\langle P,(f_\gamma)_{\gamma\in\mathcal{T}_{\rm ca}}\rangle$ gives $\bar f_\gamma=f_\gamma$, $\gamma\in\mathcal{T}_{\rm ca}$.

Knowing that $\bar f_\gamma=f_\gamma$, $\gamma\in\mathcal{T}_{\rm ca}$, it is clear that the projection axiom holds in $\langle P,(\bar f_\gamma)_{\gamma\in\mathcal{T}_{\rm pca}}\rangle$; the operations involved all belong to $\mathcal{T}_{\rm ca}$. To show the PCA-baricenter axiom, let $(p_i)_{i=1}^n\in\mathcal{T}_{\rm pca}$ and $(p_{ij})_{i=1}^m\in\mathcal{T}_{\rm pca}$, $i=1,\dots,n$, be given. Denote

$$\bar p:=1-\sum_{i=1}^n p_i,\quad \bar p_i:=1-\sum_{j=1}^m p_{ij},\ i=1,\dots,n.$$

Then (δ_{ij} again denotes the Kronecker-delta)

$$(p_1,\dots,p_n,\bar p),\quad (p_{i1},\dots,p_{im},\bar p_i),\ i=1,\dots,n,\quad (\delta_{i,m+1})_{i=1}^{m+1}$$

all belong to $\mathcal{T}_{\rm ca}$, and hence we may apply the CA-baricenter axiom. This gives

$$\bar f_{(p_i)_{i=1}^n}\big(\bar f_{(p_{1j})_{j=1}^m}(x_1,\dots,x_m),\dots,\bar f_{(p_{nj})_{j=1}^m}(x_1,\dots,x_m)\big)=$$

$$=(p_1,\dots,p_n,\bar p)\begin{pmatrix}\bar f_{(p_{1j})_{j=1}^m}(x_1,\dots,x_m)\\ \vdots\\ \bar f_{(p_{nj})_{j=1}^m}(x_1,\dots,x_m)\\ z\end{pmatrix}=$$

$$=(p_1,\dots,p_n,\bar p)\left[\begin{pmatrix}p_{11}&\dots&p_{1m}&\bar p_1\\ \vdots&&\vdots&\vdots\\ p_{n1}&\dots&p_{nm}&\bar p_n\\ 0&\dots&0&1\end{pmatrix}\begin{pmatrix}x_1\\ \vdots\\ x_m\\ z\end{pmatrix}\right]=$$

$$= \left[(p_1,\dots,p_n,\bar p) \begin{pmatrix} p_{11} & \dots & p_{1m} & \bar p_1 \\ \vdots & & \vdots & \vdots \\ p_{n1} & \dots & p_{nm} & \bar p_n \\ 0 & \dots & 0 & 1 \end{pmatrix} \right] \begin{pmatrix} x_1 \\ \vdots \\ x_m \\ z \end{pmatrix} =$$

$$= \Big(\sum_{i=1}^n p_i p_{i1}, \dots, \sum_{i=1}^n p_i p_{im}, \sum_{i=1}^n p_i \bar p_i + \bar p \Big) \begin{pmatrix} x_1 \\ \vdots \\ x_m \\ z \end{pmatrix} =$$

$$= \bar f_{(\sum_{i=1}^n p_i p_{ij})_{j=1}^m}(x_1,\dots,x_m).$$

The last equality holds since, due to (3.1),

$$\sum_{i=1}^n p_i \bar p_i + \bar p = 1 - \Big(\sum_{i=1}^n p_i p_{i1} + \dots + \sum_{i=1}^n p_i p_{im} \Big).$$

Moreover, we have for arbitrary $x \in P$

$$0_P = \bar f_{(0)}(x) = f_{(0,1)}(x,z) = z\,.$$

Second, assume conversely that operations $\bar f_\gamma$, $\gamma \in \mathcal{T}_{\mathrm{pca}}$, with (3.4) are given. Then, by Lemma 3.4(v),

$$\bar f_{(p_i)_{i=1}^n}(x_1,\dots,x_n) = \bar f_{(p_1,\dots,p_n,\bar p)}(x_1,\dots,x_n,0_P) = \\ = f_{(p_1,\dots,p_n,\bar p)}(x_1,\dots,x_n,0_P), \quad (p_i)_{i=1}^n \in \mathcal{T}_{\mathrm{pca}}\,,$$

where $\bar p := 1 - \sum_{i=1}^n p_i$. Hence, the operations $\bar f_\gamma$ coincide with the operaions constructed in the first step with the choice $z = 0_P$.

Finally, the above computation also shows that an operational extension according to (3.4) is uniquely determined by the corresponding element 0_P, showing that P indeed corresponds bijectively to the set of all operational extensions. □

Next, we show that congruence relations within CA or PCA are just the same. Let P be an algebra in CA or PCA and Θ an equivalence relation on P. Recall that Θ is a congruence if whenever $(x_i, x_i') \in \Theta$, for $i \in \{1,\dots,n\}$ then also $(\sum_{i=1}^n p_i x_i, \sum_{i=1}^n p_i x_i') \in \Theta$ for $(p_i)_{i=1}^n \in \mathcal{T}_{\mathrm{ca}}$ or $(p_i)_{i=1}^n \in \mathcal{T}_{\mathrm{pca}}$, respectively. By $\mathrm{Con}_{\mathsf{CA}}\,P$ or $\mathrm{Con}_{\mathsf{PCA}}\,P$ we denote the sets of all CA- or PCA-congruence relations on P, respectively.

Lemma 3.7. *Let $\langle P, (f_\gamma)_{\gamma\in\mathcal{T}_{ca}}\rangle \in$ CA and let $\{\bar f_\gamma | \gamma \in \mathcal{T}_{pca}\}$ be an operational extension of $\langle P, (f_\gamma)_{\gamma\in\mathcal{T}_{ca}}\rangle$ to PCA. Moreover, let Θ be an equivalence relation on P. Then $\Theta \in \mathrm{Con}_{\mathsf{CA}}\langle P, (f_\gamma)_{\gamma\in\mathcal{T}_{ca}}\rangle$ if and only if $\Theta \in \mathrm{Con}_{\mathsf{PCA}}\langle P, (\bar f_\gamma)_{\gamma\in\mathcal{T}_{pca}}\rangle$.*

Proof. Clearly, each PCA-congruence on P is also a CA-congruence. Conversely, assume that $\Theta \in \mathrm{Con}_{\mathsf{CA}}\,P$. Let $(p_i)_{i=1}^n \in \mathcal{T}_{\mathrm{pca}}$ and $(x_i, x_i') \in \Theta$, $i = 1,\dots,n$, be given. Again set $\bar p := 1 - \sum_{i=1}^n p_i$, so that $(p_1,\dots,p_n,\bar p) \in \mathcal{T}_{\mathrm{ca}}$. Since $\Theta \in \mathrm{Con}_{\mathsf{CA}}\,P$, it follows that

$$\Big(\sum_{i=1}^n p_i x_i, \sum_{i=1}^n p_i x_i' \Big) = \\ = \big(p_1 x_1 + \dots + p_n x_n + \bar p \cdot 0_P\,,\ p_1 x_1' + \dots + p_n x_n' + \bar p \cdot 0_P \big) \in \Theta$$

□

3.2 Totally convex algebras vs. convex algebras

In this subsection we characterize those convex algebras which can be operationally extended to totally convex algebras. In view of Proposition 3.6 this problem is equivalent to characterizing those positive convex algebras that can be operationally extended[4] to totally convex algebras. It turns out that not every convex algebra has this property, contrasting the situation for passing from CA to PCA.

The decisive additional property which allows to operationally extend to TCA is existence of an involutory endomorphism satisfying a particular computation rule. Recall that an endomorphism ω of an algebra is called involutory if it is inverse to itself, i.e., if $\omega^2 = id$.

Proposition 3.8. *Let* $\langle P, (f_\gamma)_{\gamma\in\mathcal{T}_{pca}}\rangle \in$ PCA. *Then there exists an operational extension* $\{\bar f_\gamma | \gamma \in \mathcal{T}_{tca}\}$ *of* P *to* TCA *if and only if there exists an involutory* PCA*-endomorphism* ω *of* P *which has the property that*

$$\begin{aligned} f_{(p_1,\dots,p_n,q_1,\dots,q_n)}(x_1,\dots,x_n,\omega x_1,\dots,\omega x_n) = \\ = f_{(p'_1,\dots,p'_n,q'_1,\dots,q'_n)}(x_1,\dots,x_n,\omega x_1,\dots,\omega x_n), \end{aligned} \tag{3.5}$$

whenever $n \in \mathbb{N}^+$,

$$(p_1,\dots,p_n,q_1,\dots,q_n),(p'_1,\dots,p'_n,q'_1,\dots,q'_n) \in \mathcal{T}_{pca},$$

$$p_k - q_k = p'_k - q'_k, \quad k = 1,\dots,n.$$

Proof. Consider first the case that $|P| = 1$. Then $P = \{0_P\}$ and trivially P can be operationally extended to a totally convex algebra. Equally trivially the identity map ω is an involutory endomorphism and satisfies (3.5). Hence, in this case, the asserted equivalence holds. For the rest of the proof we may thus assume that P contains more than one element.

Necessity. Assume that an operational extension $\{\bar f_\gamma | \gamma \in \mathcal{T}_{\rm tca}\}$ is given, and consider the map $\omega\colon P \to P$ defined as

$$\omega(x) = \bar f_{(-1)}(x).$$

Let $(p_i)_{i=1}^n \in \mathcal{T}_{\rm tca}$, then

$$\begin{aligned} \bar f_{(-1)}\big(\bar f_{(p_i)_{i=1}^n}(x_1,\dots,x_n)\big) &= \bar f_{(-p_i)_{i=1}^n}(x_1,\dots,x_n) \\ &= \bar f_{(p_i)_{i=1}^n}\big(\bar f_{(-1)}(x_1),\dots,\bar f_{(-1)}(x_n)\big). \end{aligned}$$

This shows that ω is an endomorphism (even a TCA-endomorphism). Moreover, we have

$$\bar f_{(-1)}(\bar f_{(-1)}(x)) = \bar f_{(1)}(x) = x,$$

i.e. ω is involutory.

In order to show (3.5), it is certainly sufficient to show that

$$\begin{aligned} \bar f_{(p_1,\dots,p_n,q_1,\dots,q_n)}(x_1,\dots,x_n,\omega x_1,\dots,\omega x_n) = \\ = \bar f_{(p_1-q_1,\dots,p_n-q_n)}(x_1,\dots,x_n), \end{aligned} \tag{3.6}$$

[4] Operational extensions of an algebra in CA or PCA to an algebra in TCA are defined analogously as operational extensions of an algebra in CA to one in PCA.

whenever $(p_1,\dots,p_n,q_1,\dots,q_n)\in\mathcal{T}_{\rm tca}$. Note here that

$$\sum_{i=1}^n |p_i-q_i| \le \sum_{i=1}^n (|p_i|+|q_i|) = \sum_{i=1}^n |p_i| + \sum_{i=1}^n |q_i| \le 1,$$

and hence the operation written on the right side is legitimate. To see (3.6), we compute

$$(p_1,\dots,p_n,q_1,\dots,q_n)\begin{pmatrix} x_1\\ \vdots\\ x_n\\ \omega x_1\\ \vdots\\ \omega x_n\end{pmatrix} = (p_1,\dots,p_n,q_1,\dots,q_n)\left[\begin{pmatrix} I_n\\ -I_n\end{pmatrix}\begin{pmatrix} x_1\\ \vdots\\ x_n\end{pmatrix}\right] =$$

$$= \left[(p_1,\dots,p_n,q_1,\dots,q_n)\begin{pmatrix} I_n\\ -I_n\end{pmatrix}\right]\begin{pmatrix} x_1\\ \vdots\\ x_n\end{pmatrix} = (p_1-q_1,\dots,p_n-q_n)\begin{pmatrix} x_1\\ \vdots\\ x_n\end{pmatrix}.$$

Sufficiency. Assume that an involutory PCA-endomorphism ω on P with (3.5) is given. For each $(p_i)_{i=1}^n\in\mathcal{T}_{\rm tca}$ we define an operation $\bar f_{(p_i)_{i=1}^n}$ as

$$\bar f_{(p_i)_{i=1}^n}(x_1,\dots,x_n) := f_{(p_1^+,\dots,p_n^+,p_1^-,\dots,p_n^-)}(x_1,\dots,x_n,\omega x_1,\dots,\omega x_n), \tag{3.7}$$

where

$$p^+ := \max\{p,0\},\ \ p^- := -\min\{p,0\}, \quad p\in\mathbb{R}. \tag{3.8}$$

Note that $p=p^+-p^-$. Moreover, we have

$$\sum_{i=1}^n p_i^+ + \sum_{i=1}^n p_i^- = \sum_{i=1}^n |p_i|,$$

and hence the operation in $\mathcal{T}_{\rm pca}$ on the right side of (3.7) is well-defined. If $(p_i)_{i=1}^n\in\mathcal{T}_{\rm pca}$, then $p_i^+=p_i$ and $p_i^-=0$ for all $i\in\{1,\dots,n\}$, and the extended projection law for $\langle P,(f_\gamma)_{\gamma\in\mathcal{T}_{\rm pca}}\rangle$ gives

$$\begin{aligned}\bar f_{(p_i)_{i=1}^n}(x_1,\dots,x_n) &= f_{(p_1,\dots,p_n,0,\dots,0)}(x_1,\dots,x_n,\omega x_1,\dots,\omega x_n)\\ &= f_{(p_i)_{i=1}^n}(x_1,\dots,x_n).\end{aligned}$$

It is not difficult to see that the projection axiom holds for $\langle P,(\bar f_\gamma)_{\gamma\in\mathcal{T}_{\rm tca}}\rangle$; we skip the details. To check the TCA-baricenter axiom, let $(p_i)_{i=1}^n\in\mathcal{T}_{\rm tca}$ and $(p_{ij})_{j=1}^m\in\mathcal{T}_{\rm tca}$, $i=1,\dots,n$, be given. First, compute

$$\begin{aligned}\omega\bar f_{(p_{ij})_{j=1}^m}(x_1,\dots,x_m) &= \omega f_{(p_{i1}^+,\dots,p_{im}^+,p_{i1}^-,\dots,p_{im}^-)}(x_1,\dots,x_m,\omega x_1,\dots,\omega x_m)\\ &= f_{(p_{i1}^+,\dots,p_{im}^+,p_{i1}^-,\dots,p_{im}^-)}(\omega x_1,\dots,\omega x_m,x_1,\dots,x_m)\\ &= f_{(p_{i1}^-,\dots,p_{im}^-,p_{i1}^+,\dots,p_{im}^+)}(x_1,\dots,x_m,\omega x_1,\dots,\omega x_m).\end{aligned}$$

Next, set

$$P^+ := \left(p_{ij}^+\right)_{\substack{i=1,\dots,n\\ j=1,\dots,m}}, \quad P^- := \left(p_{ij}^-\right)_{\substack{i=1,\dots,n\\ j=1,\dots,m}},$$

and use the baricenter axiom for $\langle P, (f_\gamma)_{\gamma\in\mathcal{T}_{\text{pca}}}\rangle$ to compute

$$\bar{f}_{(p_i)_{i=1}^n}\big(\bar{f}_{(p_{1j})_{j=1}^m}(x_1,\dots,x_m),\dots,\bar{f}_{(p_{nj})_{j=1}^n}(x_1,\dots,x_m)\big) =$$

$$=(p_1^+,\dots,p_n^+,p_1^-,\dots,p_n^-)\left[\begin{pmatrix}P^+ & P^-\\ P^- & P^+\end{pmatrix}\begin{pmatrix}x_1\\ \vdots\\ x_n\\ \omega x_1\\ \vdots\\ \omega x_n\end{pmatrix}\right] =$$

$$=\left[(p_1^+,\dots,p_n^+,p_1^-,\dots,p_n^-)\begin{pmatrix}P^+ & P^-\\ P^- & P^+\end{pmatrix}\right]\begin{pmatrix}x_1\\ \vdots\\ x_n\\ \omega x_1\\ \vdots\\ \omega x_n\end{pmatrix} =$$

$$=\Big(\sum_{i=1}^n p_i^+p_{i1}^+ + \sum_{i=1}^n p_i^-p_{i1}^-,\dots\ ,\sum_{i=1}^n p_i^+p_{i1}^- + \sum_{i=1}^n p_i^-p_{i1}^+,\dots\Big)\begin{pmatrix}x_1\\ \vdots\\ x_n\\ \omega x_1\\ \vdots\\ \omega x_n\end{pmatrix} =$$

$$=\Big(\Big(\sum_{i=1}^n p_ip_{i1}\Big)^+,\dots\ ,\Big(\sum_{i=1}^n p_ip_{i1}\Big)^-,\dots\Big)\begin{pmatrix}x_1\\ \vdots\\ x_n\\ \omega x_1\\ \vdots\\ \omega x_n\end{pmatrix}$$

where the last equality follows from (3.5) since

$$\Big(\sum_{i=1}^n p_i^+p_{ij}^+ + \sum_{i=1}^n p_i^-p_{ij}^-\Big) - \Big(\sum_{i=1}^n p_i^+p_{ij}^- + \sum_{i=1}^n p_i^-p_{ij}^+\Big) =$$

$$= \sum_{\substack{i=1\\ p_i\geq 0, p_{ij}\geq 0}}^n p_ip_{ij} + \sum_{\substack{i=1\\ p_i<0, p_{ij}<0}}^n p_ip_{ij} - \sum_{\substack{i=1\\ p_i\geq 0, p_{ij}<0}}^n p_i(-p_{ij}) - \sum_{\substack{i=1\\ p_i<0, p_{ij}\geq 0}}^n (-p_i)p_{ij} =$$

$$=\sum_{i=1}^n p_ip_{ij} = \Big(\sum_{i=1}^n p_ip_{ij}\Big)^+ - \Big(\sum_{i=1}^n p_ip_{ij}\Big)^-.$$

□

In order to show that TCA is (in the sense of operational extendability) indeed strictly smaller than CA, we need to provide an example of a positive convex algebra which does not admit an involutory endomorphism satisfying (3.5). Such examples can already be found in euclidean space. First, an auxilliary observation.

Lemma 3.9. *Let $P\in\mathsf{PCA}$, and let ω be an involutory PCA-endomorphism on P which satisfies* (3.5)*. Then ω has exactly one fixed point, namely 0_P.*

Proof. Let $x\in P$ with $\omega x = x$. Then, using (3.5),

$$x = f_{(\frac12,\frac12)}(x,x) = f_{(\frac12,\frac12)}(x,\omega x) = f_{(0,0)}(x,\omega x) = 0_P.$$

To see that 0_P indeed is a fixed point, compute

$$\omega 0_P = \omega f_{(0)}(0_P) = f_{(0)}(\omega 0_P) = 0_P.$$

□

Proposition 3.10. *Let $K \subseteq \mathbb{R}^n$ be a polytope with $|K| > 1$, $0 \in K$, and let K be endowed with the operations f_γ, $\gamma \in \mathcal{T}_{pca}$, given as the usual sum of vectors in $\mathbb{R}^n$*

$$f_{(p_i)_{i=1}^n}(x_1, \ldots, x_n) := \sum_{i=1}^n p_i x_i, \quad (p_i)_{i=1}^n \in \mathcal{T}_{pca}.$$

If K can be operationally extended to a totally convex algebra, then K has an even number of extremal points.

Proof. Assume that K can be operationally extended to a totally convex algebra and let, by Proposition 3.8, ω be an involutory PCA-endomorphism of K which satisfies (3.5). Note that ω is bijective.

The first step is to show that $\omega(\operatorname{ext} K) = \operatorname{ext} K$. Let $x \in K \setminus \operatorname{ext} K$, and choose $x_1, x_2 \in K$, $x_1, x_2 \neq x$, and $t \in (0,1)$ such that $x = tx_1 + (1-t)x_2$. Then also

$$\omega x = t\omega x_1 + (1-t)\omega x_2.$$

Since ω is injective, we conclude that ωx cannot be an extremal point of K. Thus we have $\omega(K \setminus \operatorname{ext} K) \subseteq K \setminus \operatorname{ext} K$. Furthermore,

$$K \setminus \operatorname{ext} K = \omega^2(K \setminus \operatorname{ext} K) \subseteq \omega(K \setminus \operatorname{ext} K),$$

showing that $\omega(K \setminus \operatorname{ext} K) = K \setminus \operatorname{ext} K$. Since ω is bijective, this implies $\omega(\operatorname{ext} K) = \operatorname{ext} K$.

Next, consider the restriction $\omega|_{\operatorname{ext} K}$. It is an involutory permutation on $\operatorname{ext} K$. Consider the orbits of $\omega|_{\operatorname{ext} K}$, i.e., the sets $\{(\omega|_{\operatorname{ext} K})^k x \mid k \in \mathbb{Z}\}$ for $x \in \operatorname{ext} K$. Each orbit can contain at most two points. Choose $x \in K \setminus \{0\}$. Then also $\omega x \neq 0$. From (3.5),

$$\frac{1}{2}x + \frac{1}{2}\omega x = f_{(\frac{1}{2}, \frac{1}{2})}(x, \omega x) = 0,$$

where the sum on the left denotes the usual vector sum, and we see that $0 \notin \operatorname{ext} K$. However, we know by Lemma 3.9 that 0 is the only fixed point of ω. Hence, each orbit of $\omega|_{\operatorname{ext} K}$ must contain exactly two points. Since $\operatorname{ext} K$ is the disjoint union of all orbits, it contains an even number of points. □

We close the discussion of TCA by making explicit the relationship between TCA-congruences and CA-congruences. Since we already know that CA-congruences are the same as PCA-congruences, we may equally well compare TCA-congruences with PCA-congruences. Denote by $\operatorname{Con}_{\mathsf{TCA}} P$ the set of all congruences of P in TCA.

Lemma 3.11. *Let $\langle P, (f_\gamma)_{\gamma \in \mathcal{T}_{pca}} \rangle \in \mathsf{PCA}$, let $\{\bar{f}_\gamma | \gamma \in \mathcal{T}_{tca}\}$ be an operational extension to TCA, and set $\omega(x) := \bar{f}_{(-1)}(x)$, $x \in P$. Moreover, let Θ be an equivalence relation on P. Then $\Theta \in \operatorname{Con}_{\mathsf{TCA}}\langle P, (\bar{f}_\gamma)_{\gamma \in \mathcal{T}_{tca}} \rangle$ if and only if $\Theta \in \operatorname{Con}_{\mathsf{PCA}}\langle P, (f_\gamma)_{\gamma \in \mathcal{T}_{pca}} \rangle$ and $(\omega \times \omega)\Theta \subseteq \Theta$.*

Proof. Clearly, each TCA-congruence on P is also a PCA-congruence and satisfies $(\omega \times \omega)\Theta \subseteq \Theta$. We have to show the converse. Let $(p_i)_{i=1}^n \in \mathcal{T}_{\text{tca}}$ and

$(x_i, x_i') \in \Theta$, $i = 1, \dots, n$, be given. Using the notation (3.8), we obtain from (3.6) that

$$\bar{f}_{(p_i)_{i=1}^n}(x_1, \dots, x_n) = f_{(p_1^+, \dots, p_n^+, p_1^-, \dots, p_n^-)}(x_1, \dots, x_n, \omega x_1, \dots, \omega x_n).$$

The same equation holds for x_i' instead of x_i. Since Θ is a PCA-congruence, $x_i \Theta x_i'$, and $\omega x_i \Theta \omega x_i'$ by the assumptions, we see that indeed

$$\bar{f}_{(p_i)_{i=1}^n}(x_1, \dots, x_n) \, \Theta \, \bar{f}_{(p_i)_{i=1}^n}(x_1', \dots, x_n') \,.$$

□

4 Convex equivalences on polytopes

Let $K \subseteq \mathbb{R}^n$ be a polytope. We consider K as a convex algebra endowed with the operations inherited from $\mathbb{R}^n$ (as in Proposition 3.10). The following property is a direct consequence of the definitions and Lemma 2.2 but it is an important observation for what follows.

Lemma 4.1. *An equivalence relation Θ on a polytope K is in $\operatorname{Con}_{\mathsf{CA}} K$ if and only if it is convex as a subset of $K \times K \subseteq \mathbb{R}^n \times \mathbb{R}^n$, with operations defined component-wise.* □

Throughout the paper, we denote

$$V_K := \mathcal{P}(\operatorname{ext} K) \setminus \{\emptyset\},$$

where $\mathcal{P}(M)$ denotes the power set of a set M, and consider V_K as a join-subsemilattice of the lattice $\mathcal{P}(\operatorname{ext} K)$. Moreover, we denote by $\operatorname{Sub} \mathbb{R}^n$ the set of all linear subspaces of $\mathbb{R}^n$, and consider $\operatorname{Sub} \mathbb{R}^n$ as being ordered by inclusion.

In the following definition we introduce the crucial concepts for describing $\operatorname{Con}_{\mathsf{CA}} K$.

Definition 4.2. Let $K \subseteq \mathbb{R}^n$ be a polytope and let $\Theta \in \operatorname{Con}_{\mathsf{CA}} K$. We define a map $\varphi_\Theta \colon V_K \to \operatorname{Sub} \mathbb{R}^n$ by

$$\varphi_\Theta(Y) = \operatorname{span}\big\{x_2 - x_1 \mid x_1, x_2 \in \check{\operatorname{co}}\, Y,\, x_1 \Theta x_2\big\}, \quad Y \in V_K.$$

We define a graph $\mathcal{G}_\Theta$ with vertices V_K and (undirected) edges E_Θ given by

$$\{Y_1, Y_2\} \in E_\Theta \iff \Theta \cap (\check{\operatorname{co}}\, Y_1 \times \check{\operatorname{co}}\, Y_2) \neq \emptyset, \quad Y_1, Y_2 \in V_K.$$

We denote by $\approx_\Theta$ the equivalence relation on V_K defined as

$$Y_1 \approx_\Theta Y_2 \iff Y_1 \text{ and } Y_2 \text{ are connected by a path in } \mathcal{G}_\Theta.$$

The equivalence classes of $\approx_\Theta$ are the connected components of $\mathcal{G}_\Theta$, see, e.g., [Di00]. Note that there is always an edge in $\mathcal{G}_\Theta$ connecting a vertex Y with itself, even if $|Y| = 1$, due to the definition of $\check{\operatorname{co}}\, Y$.

In the following three statements we give a complete description of the congruence lattice $\operatorname{Con}_{\mathsf{CA}} K$.

Theorem 4.3. *Let K be a polytope in $\mathbb{R}^n$ and let $\Theta \in \operatorname{Con}_{\mathsf{CA}} K$. Then* *(i)* *The map* φ_Θ *is monotone.*

(ii) *Let* $\mathcal{C}$ *be a component of the graph* $\mathcal{G}_\Theta$*. Then* $\mathcal{C}$ *contains a largest element with respect to inclusion. Denoting this largest element by* $Y(\mathcal{C})$*, we have* $\{Y, Y(\mathcal{C})\} \in E_\Theta$*,* $Y \in \mathcal{C}$*.*

(iii) *The relation* $\approx_\Theta$ *is a congruence of the join-semilattice* V_K*.*

(iv) *Let* $\mathcal{C}$ *be a component of* $\mathcal{G}_\Theta$ *and* $Y(\mathcal{C})$ *its largest element. Then*

$$\varphi_\Theta(Y) = \varphi_\Theta(Y(\mathcal{C})) \cap \operatorname{dir} Y \quad \text{for } Y \in \mathcal{C}, \tag{4.1}$$

$$\big[\check{\text{co}}\, Y + \varphi_\Theta(Y(\mathcal{C}))\big] \cap \check{\text{co}}\, Y(\mathcal{C}) \neq \emptyset \quad \text{for } Y \in \mathcal{C}. \tag{4.2}$$

Set

$$Z(\mathcal{C}) := \bigcup_{Y \in \mathcal{C}} \check{\text{co}}\, Y, \tag{4.3}$$

then the congruence Θ *can be recovered from* φ_Θ *and* $\mathcal{G}_\Theta$ *as*

$$\Theta = \bigcup_{\substack{\mathcal{C} \text{ component} \\ \text{of } \mathcal{G}_\Theta}} \big\{(x_1, x_2) \in Z(\mathcal{C}) \times Z(\mathcal{C}) : x_2 - x_1 \in \varphi_\Theta(Y(\mathcal{C}))\big\}. \tag{4.4}$$

Let us point out that reconstructing Θ by means of the formula (4.4) only requires knowledge of the classes of $\approx_\Theta$ and the values of φ_Θ on the respective largest elements of these classes.

Theorem 4.4. *Let* K *be a polytope in* $\mathbb{R}^n$*. Let* $\sim$ *be a congruence relation of the join-semilattice* V_K *with the property that each congruence class* $\mathcal{C}$ *of* $\sim$ *contains a largest element, say* $Y(\mathcal{C})$*. Moreover, let*

$$\varphi\colon \{Y(\mathcal{C}) \mid \mathcal{C} \text{ class of } \sim\} \longrightarrow \operatorname{Sub} \mathbb{R}^n$$

be a monotone map such that, for each class $\mathcal{C}$ *of* $\sim$*,*

$$\varphi(Y(\mathcal{C})) \subseteq \operatorname{dir} Y(\mathcal{C}), \quad \big[\check{\text{co}}\, Y + \varphi(Y(\mathcal{C}))\big] \cap \check{\text{co}}\, Y(\mathcal{C}) \neq \emptyset \ \text{ for } Y \in \mathcal{C}.$$

Then there exists a unique congruence $\Theta \in \operatorname{Con}_{\mathsf{CA}} K$ *such that*

$$\approx_\Theta \, = \, \sim, \quad \varphi_\Theta(Y(\mathcal{C})) = \varphi(Y(\mathcal{C})) \ \text{ for } \mathcal{C} \text{ a class of } \sim.$$

This congruence Θ *can be computed from* $\sim$ *and* φ *by means of the formula*

$$\Theta = \bigcup_{\substack{\mathcal{C} \text{ class} \\ \text{of } \sim}} \big\{(x_1, x_2) \in Z(\mathcal{C}) \times Z(\mathcal{C}) : x_2 - x_1 \in \varphi(Y(\mathcal{C}))\big\}, \tag{4.5}$$

where again $Z(\mathcal{C}) := \bigcup_{Y \in \mathcal{C}} \check{\text{co}}\, Y$*. Its associated function* φ_Θ *is given as*

$$\varphi_\Theta(Y) = \varphi\big(Y(\mathcal{C})\big) \cap \operatorname{dir} Y \quad \text{for } Y \in \mathcal{C}, \tag{4.6}$$

and the set of edges E_Θ *of its associated graph* $\mathcal{G}_\Theta$ *is given as*

$$\{Y_1, Y_2\} \in E_\Theta \iff \Big(Y_1 \sim Y_2 \wedge \big[\check{\text{co}}\, Y_1 + \varphi\big(Y([Y_1]_\sim)\big)\big] \cap \check{\text{co}}\, Y_2 \neq \emptyset\Big) \tag{4.7}$$

where $[Y_1]_\sim$ *denotes the equivalence class of* Y_1*.*

Remark 4.5. Note that for any semi-lattice congruence $\sim$ with largest element in every class (as in Theorem 4.4) there is at least one possible choice for the assignment φ in Theorem 4.4, namely $\varphi(Y(\mathcal{C})) = \operatorname{dir} Y(\mathcal{C})$ satisfies all conditions.

We do not have a simple description of all semi-lattices congruences which are admissible in the sense of Theorem 4.4. However, some examples are easily obtained. Let $K \subseteq \mathbb{R}^n$, $|K| > 1$, be a polytope, and let $Y_0 \subseteq \operatorname{ext} K$. Define an equivalence relation $\sim$ on V_K by specifying its equivalence classes to be

$$\mathcal{C}_y := \big\{\{y\}\big\},\ y \in Y_0, \qquad \mathcal{C}_0 := V_K \setminus \bigcup_{y \in Y_0} \mathcal{C}_y\,.$$

Clearly, each of these classes contains a largest element $Y(\mathcal{C})$ (for $\mathcal{C}_0$ it is $Y(\mathcal{C}_0) = \operatorname{ext} K$), and it is straightforward to verify that $\sim$ is a congruence of the join-semilattice V_K.

In order to illustrate these results, let us consider a (toy) example.

Example 4.6. Let $Y := \{0, 1\}$ and $K := \operatorname{co} Y$ in $\mathbb{R}$. Then $K = [0, 1]$ and K is in CA. We will show, using Theorem 4.3 and Theorem 4.4, that there are exactly five CA-congruences on K. These are:

$$\begin{aligned} \Theta_1 &= \Delta \\ \Theta_2 &= \{(0,0),(1,1)\} \cup (0,1) \times (0,1) \\ \Theta_3 &= \{(0,0)\} \cup (0,1] \times (0,1] \\ \Theta_4 &= [0,1) \times [0,1) \cup \{(1,1)\}, \text{ and} \\ \Theta_5 &= [0,1] \times [0,1]. \end{aligned}$$

We have $V_K = \{\{0\}, \{1\}, \{0, 1\}\}$. To ease the notation we write

$$\mathbf{0} = \{0\},\ \ \mathbf{1} = \{1\},\ \ \mathbf{01} = \{0, 1\},$$

hence $V_K = \{\mathbf{0}, \mathbf{1}, \mathbf{01}\}$. Next we list all join-semilattice congruences of V_K with the property that each class has a largest element. There are four such, given by their partitions:

$$\begin{aligned} V_K/\sim_1 &= \{\{\mathbf{0}\}, \{\mathbf{1}\}, \{\mathbf{01}\}\} \\ V_K/\sim_2 &= \{\{\mathbf{0}\}, \{\mathbf{1}, \mathbf{01}\}\} \\ V_K/\sim_3 &= \{\{\mathbf{1}\}, \{\mathbf{0}, \mathbf{01}\}\} \\ V_K/\sim_4 &= \{\{\mathbf{0}, \mathbf{1}, \mathbf{01}\}\}. \end{aligned}$$

Note that $\mathbb{R}$ has only two vector subspaces, the trivial ones, $\operatorname{Sub} \mathbb{R} = \{\mathbf{0}, \mathbb{R}\}$. Furthermore, $\operatorname{dir} \mathbf{0} = \mathbf{0}, \operatorname{dir} \mathbf{1} = \mathbf{0}$, and $\operatorname{dir} \mathbf{01} = \mathbb{R}$.

For each of the join-semilattice congruences we need to consider all monotone maps φ mapping the largest elements of each class to $\operatorname{Sub} \mathbb{R}$ and satisfying the conditions $\varphi(Y(\mathcal{C})) \subseteq \operatorname{dir} Y(\mathcal{C})$ and $\big[\,\breve{\operatorname{co}}\, Y + \varphi(Y(\mathcal{C}))\big] \cap \breve{\operatorname{co}}\, Y(\mathcal{C}) \neq \emptyset$ for $Y \in \mathcal{C}$.

Consider $\sim_1 = \Delta$. The second condition here is always satisfied, and the first implies that $\varphi(\mathbf{0}) = \mathbf{0}$ and $\varphi(\mathbf{1}) = \mathbf{0}$. Hence, there are two possibilities for defining φ, namely

$$\varphi_1 = (\mathbf{0} \mapsto \mathbf{0}, \mathbf{1} \mapsto \mathbf{0}, \mathbf{01} \mapsto \mathbf{0}) \quad \text{and} \quad \varphi_2 = (\mathbf{0} \mapsto \mathbf{0}, \mathbf{1} \mapsto \mathbf{0}, \mathbf{01} \mapsto \mathbb{R})\,.$$

From (4.5) we then get two convex congruences on K and these are exactly Θ_1 and Θ_2. Consider next $\sim_2$. Due to the conditions on φ, there is a unique possibility to define this map, namely

$$\varphi_3 = (\mathbf{0} \mapsto \mathbf{0}, \mathbf{01} \mapsto \mathbb{R})$$

and (4.5) then gives Θ_3. The case $\sim_3$ is symmetric. Again there is a unique possibility to define the map φ, leading to

$$\varphi_4 = (\mathbf{1} \mapsto \mathbf{0}, \mathbf{01} \mapsto \mathbb{R})$$

and Θ_4. Finally, consider $\sim_4$. There is a unique map

$$\varphi_4 = (\mathbf{01} \mapsto \mathbb{R})$$

which satisfies the conditions imposed on φ leading to Θ_5. By Theorem 4.3, there are no other convex congruences on K.

The next theorem shows that also the order relation on the congruence lattice $\operatorname{Con}_{\mathsf{CA}} K$ can be characterised in terms of φ_Θ and $\mathcal{G}_\Theta$.

Theorem 4.7. *Let K be a polytope in $\mathbb{R}^n$. If $\Theta_1, \Theta_2 \in \operatorname{Con}_{\mathsf{CA}} K$, then*

$$\Theta_1 \subseteq \Theta_2 \quad \Longleftrightarrow \quad \Big(E_{\Theta_1} \subseteq E_{\Theta_2} \wedge \varphi_{\Theta_1} \leq \varphi_{\Theta_2} \Big)$$

where $\leq$ denotes the pointwise order by inclusion.

The remainder of this section is entirely devoted to the proof of Theorem 4.3, Theorem 4.4, and Theorem 4.7. Thereby, the hard part is to establish Theorem 4.3. Once this is achieved, Theorem 4.4 and Theorem 4.7 are fairly straightforward. We are going to proceed according to the following schedule:

- In Subsection 4.1 we provide some simple geometric consequences of convexity which are used throughout. Moreover, we show Theorem 4.3, (*i*).
- In Subsection 4.2 we investigate the structure of $\mathcal{G}_\Theta$, and show Theorem 4.3, (*ii*).
- In Subsection 4.3 we investigate the behaviour of a congruence relation in the interior of K (or one of its lower dimensional facets).
- In Subsection 4.4 we establish the representation (4.4).
- In Subsection 4.5 we use (4.4) to deduce items (*iii*) and (*iv*) of Theorem 4.3.
- In Subsection 4.6 we carry out the converse construction and provide evidence to Theorem 4.4.
- Finally, in Subsection 4.7, we show that the order structure of $\operatorname{Con}_{\mathsf{CA}} K$ can be characterised as stated in Theorem 4.7.

For the rest of this section let a polytope K be fixed, and throughout Sections 4.1– 4.5 let a congruence $\Theta \in \operatorname{Con}_{\mathsf{CA}} K$ be fixed.

4.1 Geometric consequences of convexity

We start with a definition of a useful concept of perspective.

Definition 4.8. For each $z \in K$ we denote by $\Phi_z\colon [0,1] \times \mathbb{R}^n \to \mathbb{R}^n$ the map defined as

$$\Phi_z(s,x) := sz + (1-s)x, \quad s \in [0,1], x \in \mathbb{R}^n.$$

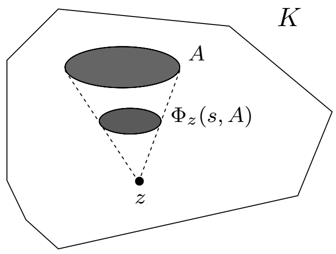
Based on geometric intuition, we refer to Φ_z as *the perspective with center z.*

Since K is convex and $z \in K$, we have $\Phi_z(s,K) \subseteq K$ for any $s \in [0,1]$. Moreover, $\Phi_z(0,.)$ is the identity map, and $\Phi_z(1,.)$ is the constant map with value z.

The following observation is simple but important, and we further refer to it as perspective invariance. Intuitively speaking, perspective invariance means that a congruence class cannot split and distribute over several different classes when moved with a perspective.

Lemma 4.9. *Let $A \subseteq K$ be an equivalence class of Θ. Then, for each $z \in K$ and $s \in [0,1]$, there exists an equivalence class $A_{z,s} \subseteq K$ of Θ with*

$$\Phi_z(s,A) \subseteq A_{z,s}.$$



Moreover, perspective invariance implies that each equivalence class of Θ is convex.

Proof. Let $x_1, x_2 \in K$ with $x_1 \Theta x_2$, $z \in K$, and $s \in [0,1]$ be given. Since $z \Theta z$ and $(x,y) \mapsto sx + (1-s)y$ is an operation of $K \in \mathsf{CA}$, we have

$$\big(\Phi_z(s,x_1), \Phi_z(s,x_2)\big) = (sz + (1-s)x_1, sz + (1-s)x_2) \in \Theta\,.$$

To deduce that equivalence classes are convex, let $x_1, x_2 \in K$ with $x_1 \Theta x_2$, and $s \in [0,1]$ be given. Perspective invariance with $z := x_1$ gives

$$(x_1, sx_1 + (1-s)x_2) = \big(\Phi_{x_1}(s,x_1), \Phi_{x_1}(s,x_2)\big) \in \Theta.$$

□

Also the next fact is simple but useful.

Lemma 4.10.

(*i*) *Let $Y_1, Y_2 \in V_K$ be given. Then*

$$\forall z \in \check{\text{co}}\, Y_1, x \in \check{\text{co}}\, Y_2 : \ \Phi_z(s,x) \in \check{\text{co}}(Y_1 \cup Y_2), \quad s \in (0,1).$$

(*ii*) *Let $Y \in V_K$ be given. Then*

$$\forall z \in \operatorname{co} Y, x \in \check{\operatorname{co}}\, Y: \ \Phi_z(s,x) \in \check{\operatorname{co}}\, Y, \quad s \in [0,1).$$

$$\forall z \in \check{\operatorname{co}}\, Y, x \in \operatorname{co} Y: \ \Phi_z(s,x) \in \check{\operatorname{co}}\, Y, \quad s \in (0,1].$$

Proof. For the proof of (*i*) write

$$z = \sum_{y \in Y_1} \lambda_y^1 y, \quad \lambda_y^1 \in (0,1], \sum_{y \in Y_1} \lambda_y^1 = 1,$$
$$x = \sum_{y \in Y_2} \lambda_y^2 y, \quad \lambda_y^2 \in (0,1], \sum_{y \in Y_2} \lambda_y^2 = 1.$$

Then

$$\Phi_z(s,x) = s\Big(\sum_{y \in Y_1} \lambda_y^1 y\Big) + (1-s)\Big(\sum_{y \in Y_2} \lambda_y^2 y\Big) =$$
$$= \sum_{y \in Y_1 \setminus Y_2} s\lambda_y^1 y + \sum_{y \in Y_1 \cap Y_2} \big(s\lambda_y^1 + (1-s)\lambda_y^2\big) y + \sum_{y \in Y_2 \setminus Y_1} (1-s)\lambda_y^2 y.$$

All coefficients in these sums are positive and they sum up to 1. Hence, $\Phi_z(s,x) \in \check{\operatorname{co}}(Y_1 \cup Y_2)$. Item (*ii*) follows in the same manner. □

Let us immediately exploit perspective invariance to show that φ_Θ is monotone.

Proof (of Theorem 4.3, (i)). Let $Y_1, Y_2 \in V_K$ with $Y_1 \subseteq Y_2$ be given. If $Y_1 = Y_2$, there is nothing to prove. Hence, assume that $Y_1 \subset Y_2$.

Let $w \in \varphi_\Theta(Y_1)$, and write according to the definition of φ_Θ

$$w = \sum_{k=1}^{m} \lambda_k (x_2^k - x_1^k),$$

with some

$$\lambda_k \in \mathbb{R}, \quad x_1^k, x_2^k \in \check{\operatorname{co}}\, Y_1, \quad x_1^k \Theta x_2^k, \ k = 1, \ldots, m.$$

Set $z := \frac{1}{|Y_2 \setminus Y_1|} \sum_{y \in Y_2 \setminus Y_1} y$, and fix $s \in (0,1)$. Note that the assumption $Y_1 \subset Y_2$ ensures that z is well defined. Perspective invariance gives

$$\Phi_z(s, x_1^k) \,\Theta\, \Phi_z(s, x_2^k), \quad k = 1, \ldots, m.$$

By Lemma 4.10(i), since $z \in \check{\operatorname{co}}(Y_2 \setminus Y_1)$ and $Y_2 = (Y_2 \setminus Y_1) \cup Y_1$ by the assumption, we have $\Phi_z(s, x_1^k), \Phi_z(s, x_2^k) \in \check{\operatorname{co}}\, Y_2$, and hence

$$w' := \sum_{k=1}^{m} \lambda_k \big(\Phi_z(s, x_2^k) - \Phi_z(s, x_1^k)\big) \in \varphi_\Theta(Y_2).$$

However,

$$\Phi_z(s, x_2^k) - \Phi_z(s, x_1^k) = (1-s)(x_2^k - x_1^k),$$

and hence $w' = (1-s)w$. Since $\varphi_\Theta(Y_2)$ is a vector subspace, it follows that also $w \in \varphi_\Theta(Y_2)$. □

The property shown in the next lemma will be used in several instances.

Lemma 4.11. *Let $Y \in V_K$, $|Y| \geq 2$, and let $x_1, x_2 \in \check{\operatorname{co}}\, Y$, $x_1 \neq x_2$. Consider the map $\Gamma\colon \mathbb{R} \to \operatorname{aff} Y$ given by*

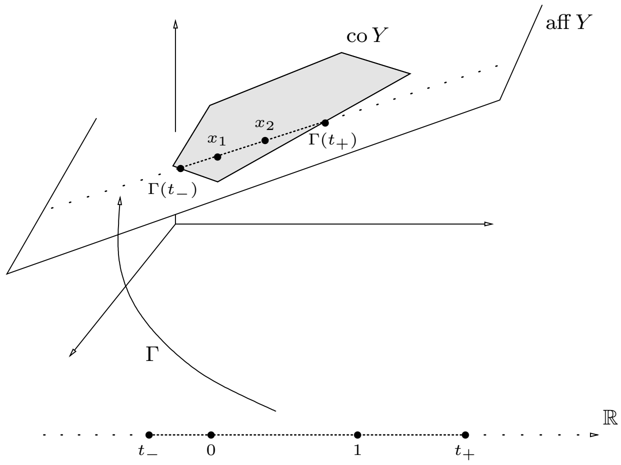
$$\Gamma(t) := tx_2 + (1-t)x_1, \quad t \in \mathbb{R}.$$

Clearly, $\Gamma(\mathbb{R}) = \operatorname{aff}\{x_1, x_2\}$ is the line containing x_1 and x_2.[5] *Then*

$$\Phi_{\Gamma(r)}(s, \Gamma(t)) = \Gamma\big(sr + (1-s)t\big), \quad r, t \in \mathbb{R}, s \in [0,1]. \tag{4.8}$$

There exist numbers $t_- < 0$ and $t_+ > 1$, such that

$$\Gamma^{-1}(\check{\operatorname{co}}\, Y) = (t_-, t_+), \quad \Gamma^{-1}(\operatorname{co} Y) = [t_-, t_+]$$



Proof. To show (4.8), we compute

$$\begin{aligned}\Phi_{\Gamma(r)}(s, \Gamma(t)) = s\Gamma(r) + (1-s)\Gamma(t) =& \\ =&s\big(rx_2 + (1-r)x_1\big) + (1-s)\big(tx_2 + (1-t)x_1\big) = \\ =&\big(sr + (1-s)t\big)x_2 + \big(s(1-r) + (1-s)(1-t)\big)x_1 = \\ =&\big(sr + (1-s)t\big)x_2 + \big(1 - (sr + (1-s)t)\big)x_1 = \Gamma(sr + (1-s)t).\end{aligned}$$

Note that $\mathbb{R}$ and $\operatorname{aff} Y$ inherit the euclidean topology from $\mathbb{R}^n$. Moreover they also inherit the euclidean metric from $\mathbb{R}^n$. Also, a line is continuous, i.e., Γ is a continuous function. Consider the set $\Gamma^{-1}(\check{\operatorname{co}}\, Y)$. Since Γ is continuous and $\check{\operatorname{co}}\, Y$ is an open subset of $\operatorname{aff} Y$ by Lemma 2.5, this set is an open subset of $\mathbb{R}$. Since Γ is linear and $\check{\operatorname{co}}\, Y$ is convex, cf. Section 2, it is convex. We will now show that $\Gamma^{-1}(\check{\operatorname{co}}\, Y)$ is bounded. Note that $\Gamma(t) = x_1 + t(x_2 - x_1)$. Therefore, for $\|\cdot\|$ denoting the euclidean norm, using the downward triangle inequality, we get

$$\|\Gamma(t)\| \geq |\|t(x_2 - x_1)\| - \|x_1\|| \geq |t| \cdot \|x_2 - x_1\| - \|x_1\|.$$

Hence, for each positive real number R, if $|t| > \frac{R + \|x_1\|}{\|x_2 - x_1\|}$, then $\|\Gamma(t)\| > R$. This shows that the inverse image by Γ of a bounded set in $\operatorname{aff} Y$ is a bounded set in $\mathbb{R}$. Now, since $\check{\operatorname{co}}\, Y \subseteq \operatorname{co} Y$ and the latter is bounded by Lemma 2.4, we get

[5] Note that Γ actually depends on the points x_1 and x_2 but we prefer a light, overloaded notation.

that $\Gamma^{-1}(\breve{\mathrm{co}}\, Y)$ is bounded. Note that in $\mathbb{R}$ a set is open, convex, and bounded if and only if it is a finite open interval. Hence

$$\Gamma^{-1}(\breve{\mathrm{co}}\, Y) = (t_-, t_+)$$

with some numbers $t_-, t_+ \in \mathbb{R}$. Since $\Gamma(0) = x_1 \in \breve{\mathrm{co}}\, Y$ and $\Gamma(1) = x_2 \in \breve{\mathrm{co}}\, Y$, we must have $t_- < 0$ and $t_+ > 1$.

Again by Lemma 2.4, $\mathrm{co}\, Y$ is the closure of $\breve{\mathrm{co}}\, Y$ and hence continuity of Γ implies that $\Gamma^{-1}(\mathrm{co}\, Y)$ is closed. Since $\Gamma^{-1}(\breve{\mathrm{co}}\, Y) \subseteq \Gamma^{-1}(\mathrm{co}\, Y)$ and $[t_-, t_+]$ is the closure of $\Gamma^{-1}(\breve{\mathrm{co}}\, Y) = (t_-, t_+)$, we also have $[t_-, t_+] \subseteq \Gamma^{-1}(\mathrm{co}\, Y)$.

To show the opposite inclusion, let $t \in \mathbb{R}$ with $\Gamma(t) \in \mathrm{co}\, Y$ be given. If $t \in [0, 1]$, we have $\Gamma(t) \in \mathrm{co}\{x_1, x_2\} \subseteq \breve{\mathrm{co}}\, Y$, and hence $t \in (t_-, t_+)$. Next, consider the case that $t > 1$. Choose $t' \in (1, t)$ and set $s := \frac{t'}{t}$. Then $s \in (0, 1)$ and

$$\begin{aligned}\Phi_{\Gamma(t)}(s, x_1) = s\Gamma(t) + (1-s)x_1 = s(tx_2 + (1-t)x_1) + (1-s)x_1 = \\ = (st)x_2 + (1-st)x_1 = t'x_2 + (1-t')x_1 = \Gamma(t').\end{aligned}$$

By Lemma 4.10, we have $\Gamma(t') \in \breve{\mathrm{co}}\, Y$. Thus $t' \in (t_-, t_+)$. Since t' was arbitrary in $(1, t)$, we have $(1, t) \subseteq (t_-, t_+)$ and hence $[1, t] \subseteq [t_-, t_+]$ showing $t \in [t_-, t_+]$. The case that $t < 0$ is analogous. □

4.2 The structure of $\mathcal{G}_\Theta$

In order to understand the structure of $\mathcal{G}_\Theta$, we need one preparatory result.

Lemma 4.12. *Let $Y_1, Y_2 \in V_K$, and assume that $Y_1 \subseteq Y_2$. If $\{Y_1, Y_2\} \in E_\Theta$, i.e.,*

$$\exists x_1 \in \breve{\mathrm{co}}\, Y_1, x_2 \in \breve{\mathrm{co}}\, Y_2 :\ x_1 \Theta x_2,$$

then actually

$$\forall x_1' \in \breve{\mathrm{co}}\, Y_1,\ \exists x_2' \in \breve{\mathrm{co}}\, Y_2 :\ x_1' \Theta x_2'.$$

Proof. If $Y_1 = Y_2$, we can always choose $x_2' := x_1'$. If $|Y_1| = 1$, also $|\breve{\mathrm{co}}\, Y_1| = 1$, and hence the assertion is trivial. Note that the assertion is not trivial in general if $Y_1 \subset Y_2$ as it often happens that then $\breve{\mathrm{co}}\, Y_1 \cap \breve{\mathrm{co}}\, Y_2 = \emptyset$, think for example of $|Y_1| = 2$ and $|Y_2| = 3$ when $\breve{\mathrm{co}}\, Y_2$ is the interior of a triangle and $\breve{\mathrm{co}}\, Y_1$ is the interior of one of its sides.

Assume that $Y_1 \subset Y_2$, $|Y_1| > 1$, and choose $x_1 \in \breve{\mathrm{co}}\, Y_1$ and $x_2 \in \breve{\mathrm{co}}\, Y_2$ with $x_1 \Theta x_2$.

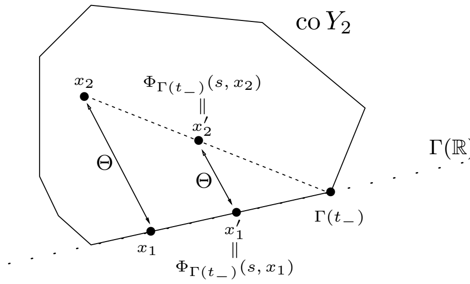
Let $x_1' \in \breve{\mathrm{co}}\, Y_1$ be given. If $x_1' = x_1$, we can take $x_2' := x_2$. Hence, assume that $x_1' \neq x_1$. Consider the line Γ containing the points x_1', x_1, i.e.,

$$\Gamma(t) := tx_1 + (1-t)x_1', \quad t \in \mathbb{R}.$$

By Lemma 4.11, we have $\Gamma^{-1}(\breve{\mathrm{co}}\, Y_1) = (t_-, t_+)$ and $\Gamma^{-1}(\mathrm{co}\, Y_1) = [t_-, t_+]$ with some $t_- < 0$ and $t_+ > 1$. Set $s := \frac{1}{1-t_-}$, then $s \in (0, 1)$ and $st_- + (1-s) \cdot 1 = 0$. By (4.8) since $\Gamma(1) = x_1$ we have

$$\Phi_{\Gamma(t_-)}(s, x_1) = \Gamma(st_- + (1-s) \cdot 1) = \Gamma(0) = x_1'.$$

Take $x_2' := \Phi_{\Gamma(t_-)}(s, x_2)$. By perspective invariance and the above equality, we get $x_1' \Theta x_2'$. Since $\Gamma(t_-) \in \mathrm{co}\, Y_1 \subseteq \mathrm{co}\, Y_2$ and $x_2 \in \breve{\mathrm{co}}\, Y_2$, Lemma 4.10(ii) implies $x_2' = \Phi_{\Gamma(t_-)}(s, x_2) \in \breve{\mathrm{co}}\, Y_2$, which completes the proof.



□

Corollary 4.13. *Let $Y_1, Y_2, Y_3 \in V_K$, and assume that $\{Y_1, Y_2\}, \{Y_2, Y_3\} \in E_\Theta$. If $Y_2 \subseteq Y_3$, then also $\{Y_1, Y_3\} \in E_\Theta$.*

Proof. Choose $x_1 \in \check{\mathrm{co}}\, Y_1$ and $x_2 \in \check{\mathrm{co}}\, Y_2$ with $x_1 \Theta x_2$. According to Lemma 4.12 there exists $x_3 \in \check{\mathrm{co}}\, Y_3$ with $x_2 \Theta x_3$. Transitivity gives $x_1 \Theta x_3$, and we see that $\{Y_1, Y_3\} \in E_\Theta$. □

We can now establish the result on the structure of components of $\mathcal{G}_\Theta$.

Proof (of Theorem 4.3, (ii)). We proceed in three steps.

The first step is to show that

$$\{Y_1, Y_2\} \in E_\Theta \quad \Rightarrow \quad \{Y_1, Y_1 \cup Y_2\}, \{Y_2, Y_1 \cup Y_2\} \in E_\Theta$$

To this end, choose $x_1 \in \check{\mathrm{co}}\, Y_1$ and $x_2 \in \check{\mathrm{co}}\, Y_2$ with $x_1 \Theta x_2$. By convexity of equivalence classes, we have

$$\frac{1}{2}(x_1 + x_2)\, \Theta\, x_j, \quad j = 1, 2,$$

and since $\Phi_{x_1}\big(\frac{1}{2}, x_2\big) = \frac{1}{2}(x_1 + x_2)$, Lemma 4.10 gives $\frac{1}{2}(x_1 + x_2) \in \check{\mathrm{co}}(Y_1 \cup Y_2)$ which shows the above property.

Let $\mathcal{C}$ be a component of $\mathcal{G}_\Theta$. The second step is to show that

$$Y_1, Y_2 \in \mathcal{C} \quad \Rightarrow \quad \exists Y_3 \in \mathcal{C} : \; Y_1 \cup Y_2 \subseteq Y_3, \{Y_1, Y_3\}, \{Y_2, Y_3\} \in E_\Theta$$

To established existence of Y_3 with the required properties, we use induction on the length of a path in $\mathcal{G}_\Theta$ connecting Y_1 and Y_2.

If Y_1 and Y_2 can be connected by a path of length 1, i.e., if $\{Y_1, Y_2\} \in E_\Theta$, then set $Y_3 := Y_1 \cup Y_2$. By the property shown in the first step, we have $\{Y_1, Y_3\}, \{Y_2, Y_3\} \in E_\Theta$. This also implies that $Y_3 \in \mathcal{C}$.

For the inductive step, let $m \in \mathbb{N}^+$ be given, and assume that Y_1 and Y_2 can be connected by a path of length $m + 1$. This means that there exist

$$Y_0', \ldots, Y_{m+1}' \in V_K, \; Y_0' = Y_1, Y_{m+1}' = Y_2 \text{ with } \{Y_k', Y_{k+1}'\} \in E_\Theta$$

for $k \in \{0, \ldots, m\}$. Clearly, all Y_k' belong to $\mathcal{C}$. The inductive hypothesis provides a vertex $Y' \in \mathcal{C}$ with

$$Y_1 \cup Y_m' \subseteq Y', \quad \{Y_1, Y'\}, \{Y_m', Y'\} \in E_\Theta.$$

Now, since $Y'_m \subseteq Y'$ and $\{Y_2, Y'_m\}, \{Y'_m, Y'\} \in E_\Theta$, from Corollary 4.13, we get $\{Y_2, Y'\} \in E_\Theta$. From the property shown in the first step, we get $\{Y_2, Y_2 \cup Y'\}, \{Y', Y_2 \cup Y'\} \in E_\Theta$. Another application of Corollary 4.13, this time since $\{Y_1, Y'\}, \{Y', Y_2 \cup Y'\} \in E_\Theta$ and obviously $Y' \subseteq Y_2 \cup Y'$, gives that also $\{Y_1, Y_2 \cup Y'\} \in E_\Theta$. Thus the element $Y_3 := Y_2 \cup Y'$ has all required properties. This finishes the proof of the second step.

Let $Y \in \mathcal{C}$. If there exists $Y' \in \mathcal{C}$ with $Y' \nsubseteq Y$, the property shown in the second step gives an element $Y'' \in \mathcal{C}$ with $Y \cup Y' \subseteq Y''$ and $Y \subset Y''$ (since $Y \subset Y \cup Y'$). Hence, Y is not maximal in $\mathcal{C}$. We conclude that if an element is maximal in $\mathcal{C}$, then it must also be the largest in $\mathcal{C}$. Since $\mathcal{C}$ is finite (the whole graph is finite), there certainly exists a maximal element. Let Y_0 be such. Then, since it is the largest, we have

$$Y_0 \supseteq \bigcup_{Y \in \mathcal{C}} Y,$$

and hence $Y_0 = \bigcup_{Y \in \mathcal{C}} Y$.

Let $Y \in \mathcal{C}$ be given. By the property shown in the second step, applied to Y and Y_0, we obtain $Y_3 \in \mathcal{C}$ with

$$Y \cup Y_0 \subseteq Y_3, \quad \{Y, Y_3\}, \{Y_0, Y_3\} \in E_\Theta.$$

However, since Y_0 is the largest element of $\mathcal{C}$, it follows that $Y_3 = Y_0$. Hence, we have shown that $\{Y, Y_0\} \in E_\Theta$. □

4.3 Behaviour of Θ inside of $\check{\mathrm{co}}\, Y$

Let $Y \in V_K$. Our aim is to describe $\Theta \cap (\check{\mathrm{co}}\, Y \times \check{\mathrm{co}}\, Y)$. If Y contains only one element, then $\check{\mathrm{co}}\, Y = Y$ and clearly $\Theta \cap (\check{\mathrm{co}}\, Y \times \check{\mathrm{co}}\, Y) = Y \times Y$. Hence, assume that $|Y| \geq 2$.

The main observation is that an equivalence class of Θ which contains two different points of $\check{\mathrm{co}}\, Y$ must already stretch all the way to the boundary of $\check{\mathrm{co}}\, Y$. This is the analogue of [PR90, Theorem 1.3], where a similar result was established for congruences of totally convex algebras. However, the proof given there does not immediately carry over to the presently considered situation, since we are bound to operations in $\mathcal{T}_{\mathrm{ca}}$, i.e., true convex combinations.

Lemma 4.14. *Let $x_1, x_2 \in \check{\mathrm{co}}\, Y$, $x_1 \neq x_2$, with $x_1 \Theta x_2$. Then*

$$\mathrm{aff}\{x_1, x_2\} \cap \check{\mathrm{co}}\, Y \subseteq [x_1]_\Theta,$$

where $[x_1]_\Theta$ denotes the Θ-equivalence class of x_1.

Proof. Consider the line containing the points x_1 and x_2, i.e., the map

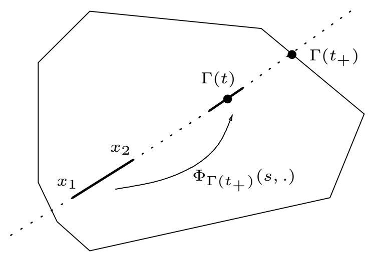
$$\Gamma(t) := t x_2 + (1-t) x_1, \quad t \in \mathbb{R},$$

and let t_-, t_+ be as given by Lemma 4.11. As a first step we show that, for each $t \in (t_-, t_+)$, the set $\Gamma^{-1}([\Gamma(t)]_\Theta \cap \check{\mathrm{co}}\, Y)$ contains an open neighbourhood of t.

Consider first the case that $t \geq \frac{1}{2}$. Choose $s \in [0,1)$ such that $st_+ + (1-s)\frac{1}{2} = t$. This is doable since under the assumption for t we have $t_+ > \frac{1}{2}$. Then by

(4.8)

$$\begin{aligned}
&\Phi_{\Gamma(t_+)}(s, x_1) = \Phi_{\Gamma(t_+)}(s, \Gamma(0)) = \Gamma(st_+),\\
&\Phi_{\Gamma(t_+)}(s, x_2) = \Phi_{\Gamma(t_+)}(s, \Gamma(1)) = \Gamma(st_+ + (1-s)),\\
&\Phi_{\Gamma(t_+)}\big(s, \tfrac{1}{2}(x_1 + x_2)\big) = \Phi_{\Gamma(t_+)}\big(s, \Gamma(\tfrac{1}{2})\big) = \Gamma(t).
\end{aligned}$$



Projective invariance implies

$$\Gamma(st_+)\,\Theta\,\Gamma(st_+ + (1-s)).$$

Note that $\mathrm{co}(Y+z) = \mathrm{co}(Y)+z$ and for any linear map f it holds that $f(\mathrm{co}\,Y) = \mathrm{co}(f(Y))$. Since the map $t \mapsto \Gamma(t) - x_1$ is linear, we have

$$\Gamma\big([st_+, st_+ + (1-s)]\big) = \mathrm{co}\,\big\{\Gamma(st_+), \Gamma(st_+ + (1-s))\big\} \tag{4.9}$$

and we conclude from convexity of equivalence classes that

$$\Gamma\big([st_+, st_+ + (1-s)]\big) \subseteq [\Gamma(st_+)]_\Theta.$$

Since $st_+ < t < st_+ + (1-s)$, in particular $\Gamma(t) \in [\Gamma(st_+)]_\Theta$ and so $[\Gamma(st_+)]_\Theta = [\Gamma(t)]_\Theta$. It follows that

$$t \in (st_+, st_+ + (1-s)) \subseteq \Gamma^{-1}\big([\Gamma(t)]_\Theta\big).$$

We have $t_- < 0 \le st_+$ and (since $t_+ > 1$) $st_+ + (1-s) < st_+ + (1-s)t_+ = t_+$. Therefore,

$$\Gamma\big((st_+, st_+ + (1-s))\big) \subseteq \Gamma\big((t_-, t_+)\big) \subseteq \breve{\mathrm{co}}\,Y\,.$$

Hence, we have found an open neighbourhood of t which is contained in $\Gamma^{-1}([\Gamma(t)]_\Theta \cap \breve{\mathrm{co}}\,Y)$. The case that $t \le \frac{1}{2}$ is treated in the same way, using the perspective $\Phi_{\Gamma(t_-)}$.

From this it is easy to deduce that actually for each $t \in (t_-, t_+)$ the set $\Gamma^{-1}([\Gamma(t)]_\Theta \cap \breve{\mathrm{co}}\,Y)$ is open: Let $t' \in \Gamma^{-1}([\Gamma(t)]_\Theta \cap \breve{\mathrm{co}}\,Y)$. Then $\Gamma(t') \in [\Gamma(t)]_\Theta \cap \breve{\mathrm{co}}\,Y$ and hence, by Lemma 4.11, $t' \in (t_-, t_+)$ and $\Gamma(t')\Theta\Gamma(t)$. Then, by what we showed above, there exists an open neighbourhood U of t' with

$$U \subseteq \Gamma^{-1}([\Gamma(t')]_\Theta \cap \breve{\mathrm{co}}\,Y)\,.$$

However, $[\Gamma(t')]_\Theta = [\Gamma(t)]_\Theta$.

Let $I \subseteq (t_-, t_+)$ be such that $\{\Gamma(t) \mid t \in I\}$ is a set of class representatives of the equivalence relation $\Theta \cap [\Gamma((t_-, t_+)) \times \Gamma((t_-, t_+))]$. Recall that $(t_-, t_+) = \Gamma^{-1}(\check{\text{co}}\, Y)$ by Lemma 4.11. We have,

$$(t_-, t_+) = \bigcup_{t \in I} \left[\Gamma^{-1}([\Gamma(t)]_\Theta) \cap (t_-, t_+)\right],$$

where the "$\supseteq$" inclusion is obvious, and the other one is a consequence of $\bigcup_{t\in I}([\Gamma(t)]_\Theta) \supseteq \Gamma(t_-, t_+)$ and properties of the inverse image Γ^{-1}.

Now, the sets $\Gamma^{-1}([\Gamma(t)]_\Theta) \cap (t_-, t_+)$, for $t \in I$, are all nonempty (each contains t), disjoint (since the corresponding equivalence classes are disjoint), and open (by the above shown). Since (t_-, t_+) is connected, it must be that $|I| = 1$. This just says that all of $\Gamma((t_-, t_+))$ is contained in a single class of Θ. On the other hand,

$$(t_-, t_+) = \Gamma^{-1}(\check{\text{co}}\, Y) = \Gamma^{-1}(\Gamma(\mathbb{R}) \cap \check{\text{co}}\, Y) = \Gamma^{-1}(\text{aff}\{x_1, x_2\} \cap \check{\text{co}}\, Y)$$

and hence

$$\Gamma((t_-, t_+)) = \Gamma(\Gamma^{-1}(\text{aff}\{x_1, x_2\} \cap \check{\text{co}}\, Y)) = \text{aff}\{x_1, x_2\} \cap \check{\text{co}}\, Y$$

where the last equality holds since $\text{aff}\{x_1, x_2\} \cap \check{\text{co}}\, Y$ is contained in the image of Γ. The statement of the lemma is now a direct consequence of this as $x_1 \in \text{aff}\{x_1, x_2\} \cap \check{\text{co}}\, Y$. □

Next we present a description of Θ inside of $\check{\text{co}}\, Y$.

Lemma 4.15. *We have*

$$\Theta \cap (\check{\text{co}}\, Y \times \check{\text{co}}\, Y) = \big\{(x_1, x_2) \in \check{\text{co}}\, Y \times \check{\text{co}}\, Y : x_2 - x_1 \in \varphi_\Theta(Y)\big\}.$$

Proof. The inclusion "$\subseteq$" is immediate from the definition of φ_Θ, Definition 4.2. We have to show the reverse inclusion. Let $x_1, x_2 \in \check{\text{co}}\, Y$ be given, and assume that $x_2 - x_1 \in \varphi_\Theta(Y)$. If $x_1 = x_2$, there is nothing to prove. Hence, assume in addition that $x_1 \neq x_2$. Note that this implies that $|Y| > 1$.

Since $x_2 - x_1 \in \varphi_\Theta(Y)$, we can write

$$x_2 - x_1 = \sum_{i=1}^{m} \lambda_i (x_2^i - x_1^i),$$

with some $\lambda_i \in \mathbb{R}$ and $x_1^i, x_2^i \in \check{\text{co}}\, Y$, $x_1^i \Theta x_2^i$. Clearly, we may assume that always $x_1^i \neq x_2^i$.

Choose $\varepsilon > 0$, and define elements z_k for $k = 0, \ldots, m$

$$z_k := x_1 + \varepsilon \sum_{i=1}^{k} \lambda_i (x_2^i - x_1^i).$$

Note that for $k = 1, \ldots, m$ we have

$$z_k = z_{k-1} + \varepsilon \lambda_k (x_2^k - x_1^k).$$

Then it is easy to see that

$$z_0 = x_1, \quad z_k \in \text{aff}\, Y,\ k = 0, \ldots, m.$$

By Lemma 2.5, $\check{\text{co}}\, Y$ is an open subset of $\operatorname{aff} Y$. Now from $x_1 \in \check{\text{co}}\, Y$, we can make the choice of $\varepsilon > 0$ such that

$$z_k \in \check{\text{co}}\, Y, \quad k = 0, \ldots, m.$$

We will next show that

$$z_{k-1} \Theta z_k, \quad k = 1, \ldots, m.$$

Let $k \in \{1, \ldots, m\}$ be given. First consider the case that $x_1^k = z_{k-1}$. Then we have

$$z_k = x_1^k + \varepsilon \lambda_k (x_2^k - x_1^k) \in \operatorname{aff}\{x_1^k, x_2^k\} \cap \check{\text{co}}\, Y.$$

By Lemma 4.14, the set on the right side is contained in $[x_1^k]_\Theta$, and thus $z_k \Theta z_{k-1}$.

Assume now that $x_1^k \neq z_{k-1}$. Let Γ be the line containing the points z_{k-1} and x_1^k, that is

$$\Gamma(t) := t x_1^k + (1-t) z_{k-1}, \quad t \in \mathbb{R}.$$

Let t_-, t_+ be as in Lemma 4.11. Choose $s \in (0,1)$ such that $s t_- + (1-s) = 0$ (this is always possible, namely $s = \frac{1}{1-t_-}$ is such). Then, using Lemma 4.11,

$$\Phi_{\Gamma(t_-)}(s, x_1^k) = \Phi_{\Gamma(t_-)}(s, \Gamma(1)) = \Gamma(s t_- + (1-s) \cdot 1) = \Gamma(0) = z_{k-1}.$$

By perspective invariance

$$\Phi_{\Gamma(t_-)}(s, x_2^k) \,\Theta\, z_{k-1}.$$

Also

$$\Phi_{\Gamma(t_-)}(s, x_2^k) - z_{k-1} = \Phi_{\Gamma(t_-)}(s, x_2^k) - \Phi_{\Gamma(t_-)}(s, x_1^k) = (1-s)(x_2^k - x_1^k).$$

It follows that

$$z_k = z_{k-1} + \frac{\varepsilon \lambda_k}{1-s} \Big(\Phi_{\Gamma(t_-)}(s, x_2^k) - z_{k-1} \Big) \in \operatorname{aff}\left\{ \Phi_{\Gamma(t_-)}(s, x_2^k), z_{k-1} \right\} \cap \check{\text{co}}\, Y \subseteq [z_{k-1}]_\Theta,$$

where the last inclusion holds by Lemma 4.14 since $\Phi_{\Gamma(t_-)}(s, x_2^k) \in \check{\text{co}}\, Y$ by Lemma 4.10 using also Lemma 4.11, $\Phi_{\Gamma(t_-)}(s, x_2^k) \,\Theta\, z_{k-1}$ as shown above, and $\Phi_{\Gamma(t_-)}(s, x_2^k) \neq z_{k-1}$ since $x_2^k \neq x_1^k$ as assumed above, and we again have $z_k \Theta z_{k-1}$.

By transitivity we have $z_0 \Theta z_m$, i.e. $x_1 \Theta z_m$, and another application of Lemma 4.14 gives

$$x_2 = x_1 + \sum_{i=1}^{m} \lambda_i (x_2^i - x_1^i) = x_1 + \frac{1}{\varepsilon}(z_m - x_1) \in \operatorname{aff}\{x_1, z_m\} \cap \check{\text{co}}\, Y \subseteq [x_1]_\Theta,$$

where again $x_1 \neq z_m$ by the made assumption $x_1 \neq x_2$. $\square$

4.4 Recovering Θ from φ_Θ and $\mathcal{G}_\Theta$

We are now in position to establish the representation (4.4) of Θ.

Lemma 4.16. *Let $\mathcal{C}$ be a component of $\mathcal{G}_\Theta$ and $Y(\mathcal{C})$ its largest element. Then*

$$\Theta \cap (Z(\mathcal{C}) \times Z(\mathcal{C})) = \big\{(x_1, x_2) \in Z(\mathcal{C}) \times Z(\mathcal{C}) : x_2 - x_1 \in \varphi_\Theta(Y(\mathcal{C}))\big\} \quad (4.10)$$

where $Z(\mathcal{C})$ is as in (4.3).

Proof. As a first step, we show the following statement: If $x \in Z(\mathcal{C})$, $x' \in \check{\operatorname{co}}\, Y(\mathcal{C})$, and $x\Theta x'$, then $x - x' \in \varphi_\Theta(Y(\mathcal{C}))$.

If $x = x'$, this is trivial. Hence, assume that $x \neq x'$. Let $z := \frac{1}{2}(x + x')$. Since equivalence classes of Θ are convex, we have $z\Theta x'$.

Recall that $Y(\mathcal{C}) = \bigcup_{Y\in\mathcal{C}} Y$ and $\check{\operatorname{co}}$ is monotone, implying that $Z(\mathcal{C}) \subseteq \check{\operatorname{co}}\, Y(\mathcal{C}) \subseteq \operatorname{co} Y(\mathcal{C})$. Hence $x \in \operatorname{co} Y(\mathcal{C})$ and $x' \in \check{\operatorname{co}}\, Y(\mathcal{C})$, which by Lemma 4.10 gives $z \in \check{\operatorname{co}}\, Y(\mathcal{C})$ as $z = \Phi_x(\frac{1}{2}, x')$. Lemma 4.15 now implies that $z - x' \in \varphi_\Theta(Y(\mathcal{C}))$. It follows that

$$x - x' = 2(z - x') \in \varphi_\Theta(Y(\mathcal{C}))$$

which completes the first step.

We now prove the inclusion "$\subseteq$" in (4.10). Let $x_1, x_2 \in Z(\mathcal{C})$ with $x_1\Theta x_2$ be given. Since, by the already proved Theorem 4.3(ii), for any $Y \in \mathcal{C}$ we have $\{Y, Y(\mathcal{C})\} \in E_\Theta$, we can choose $x_1', x_2' \in \check{\operatorname{co}}\, Y(\mathcal{C})$ with $x_1\Theta x_1'$ and $x_2\Theta x_2'$. Transitivity implies $x_1'\Theta x_2'$, and Lemma 4.15 thus gives $x_1' - x_2' \in \varphi_\Theta(Y(\mathcal{C}))$. By the property shown in the first step, also $x_1 - x_1', x_2 - x_2' \in \varphi_\Theta(Y(\mathcal{C}))$, and together

$$x_2 - x_1 = (x_2 - x_2') + (x_2' - x_1') + (x_1' - x_1) \in \varphi_\Theta(Y(\mathcal{C})).$$

Finally, we show the inclusion "$\supseteq$" in (4.10). This is done by reversing the argument in the previous paragraph. Let $x_1, x_2 \in Z(\mathcal{C})$ with $x_2 - x_1 \in \varphi_\Theta(Y(\mathcal{C}))$ be given. Again choose $x_1', x_2' \in \check{\operatorname{co}}\, Y(\mathcal{C})$ with $x_1\Theta x_1'$ and $x_2\Theta x_2'$. Again from the property shown in the first step, we have $x_1 - x_1', x_2 - x_2' \in \varphi_\Theta(Y(\mathcal{C}))$. Hence, also

$$x_2' - x_1' = (x_2' - x_2) + (x_2 - x_1) + (x_1 - x_1') \in \varphi_\Theta(Y(\mathcal{C})),$$

and Lemma 4.15 implies that $x_1'\Theta x_2'$. Transitivity gives $x_1\Theta x_2$. □

Proof (of Theorem 4.3, relation (4.4)*).* The main observation to make is that

$$K = \bigcup_{Y\in V_K} \check{\operatorname{co}}\, Y = \bigcup_{\mathcal{C}} \bigcup_{Y\in\mathcal{C}} \check{\operatorname{co}}\, Y.$$

Hence, we can write K as the disjoint union

$$K = \bigcup_{\substack{\mathcal{C} \text{ component} \\ \text{of } \mathcal{G}_\Theta}} Z(\mathcal{C})$$

where disjointness is easy to check: Assume $x \in Z(\mathcal{C}) \cap Z(\mathcal{C}')$. Then $x \in \check{\operatorname{co}}\, Y$ for some $Y \in \mathcal{C}$ and $x \in \check{\operatorname{co}}\, Y'$ for some $Y' \in \mathcal{C}'$. From the reflexivity of Θ we have $(x,x) \in \Theta$ and from the definition of $\mathcal{G}_\Theta$ this yields $\{Y, Y'\} \in E_\Theta$, which implies that $\mathcal{C} = \mathcal{C}'$.

As a consequence, Θ is the disjoint union

$$\Theta = \bigcup_{\substack{\mathcal{C},\mathcal{C}' \text{ components} \\ \text{of } \mathcal{G}_\Theta}} \big[\Theta \cap (Z(\mathcal{C}) \times Z(\mathcal{C}'))\big].$$

With a similar argument as above, the definition of $\mathcal{G}_\Theta$ ensures that

$$\Theta \cap (Z(\mathcal{C}) \times Z(\mathcal{C}')) = \emptyset, \quad \mathcal{C} \neq \mathcal{C}',$$

and hence

$$\Theta = \bigcup_{\substack{\mathcal{C} \text{ component} \\ \text{of } \mathcal{G}_\Theta}} \big[\Theta \cap (Z(\mathcal{C}) \times Z(\mathcal{C}))\big].$$

The desired representation (4.4) follows now directly from Lemma 4.16. □

4.5 Further properties of φ_Θ and $\mathcal{G}_\Theta$

The properties stated as *(iii)* and *(iv)* of Theorem 4.3 can now be deduced using the already established relation (4.4).

Proof (of Theorem 4.3, (iii)). Let $Y_1, Y_2 \in V_K$, and let $\mathcal{C}_1$ and $\mathcal{C}_2$ be the components which contain Y_1 and Y_2, respectively. Moreover, let $\mathcal{C}$ be the component which contains $Y(\mathcal{C}_1) \cup Y(\mathcal{C}_2)$. We show that then $Y_1 \cup Y_2 \in \mathcal{C}$. This suffices for the proof that $\approx_\Theta$ is a congruence on the join-semilattice V_K since if $Y_1 \approx_\Theta Y_1'$ and $Y_2 \approx_\Theta Y_2'$, i.e., $Y_1' \in \mathcal{C}_1$ and $Y_2' \in \mathcal{C}_2$, then both $Y_1 \cup Y_2$ and $Y_1' \cup Y_2'$ are in $\mathcal{C}$ showing that $Y_1 \cup Y_2 \approx_\Theta Y_1' \cup Y_2'$.

For $j = 1, 2$ choose

$$x_j \in \breve{\text{co}}\, Y_j,\ x_j' \in \breve{\text{co}}\, Y(\mathcal{C}_j) \text{ with } x_j \Theta x_j'.$$

Since Θ is convex, we have

$$\big(\frac{1}{2}(x_1 + x_1'), \frac{1}{2}(x_2 + x_2')\big) = \frac{1}{2}(x_1, x_1') + \frac{1}{2}(x_2, x_2') \in \Theta.$$

By Lemma 4.10 we have

$$\begin{aligned} \Phi_{x_1}(\frac{1}{2}, x_2) &= \frac{1}{2}(x_1 + x_2) \in \breve{\text{co}}(Y_1 \cup Y_2) \\ \Phi_{x_1'}(\frac{1}{2}, x_2') &= \frac{1}{2}(x_1' + x_2') \in \breve{\text{co}}\,\big(Y(\mathcal{C}_1) \cup Y(\mathcal{C}_2)\big) \end{aligned}$$

and hence $Y_1 \cup Y_2 \approx_\Theta Y(\mathcal{C}_1) \cup Y(\mathcal{C}_2)$. This just means that $Y_1 \cup Y_2 \in \mathcal{C}$. □

Proof (of Theorem 4.3, (iv)). Let $\mathcal{C}$ be a component of $\mathcal{G}_\Theta$, $Y \in \mathcal{C}$, and $Y(\mathcal{C})$ the largest element of $\mathcal{C}$.

If $|Y| = 1$, we have $\varphi_\Theta(Y) = \{0\}$ and $\operatorname{dir} Y = \{0\}$. Hence, in this case, equality (4.1) holds trivially. Assume that $|Y| \geq 2$. The inclusion '$\subseteq$' in (4.1) follows since φ_Θ is monotone, $Y \subseteq Y(\mathcal{C})$, and

$$\varphi_\Theta(Y) \subseteq \operatorname{span}\big(\breve{\text{co}}\, Y - \breve{\text{co}}\, Y\big) = \operatorname{dir} Y,$$

where the last equality follows by Lemma 2.6. To show the reverse inclusion, let $u \in \varphi_\Theta(Y(\mathcal{C})) \cap \operatorname{dir} Y$ be given. Choose $x_1 \in \breve{\text{co}}\, Y$, let $\varepsilon > 0$, and set $x_2 := x_1 + \varepsilon u$. Since $u \in \operatorname{dir} Y$ and $x_1 \in \operatorname{aff} Y$, we have $x_1 + \operatorname{span}\{u\} \subseteq \operatorname{aff} Y$. Since $\breve{\text{co}}\, Y$ is an open subset of $\operatorname{aff} Y$, we can choose $\varepsilon > 0$ so small that $x_2 \in \breve{\text{co}}\, Y$. We have $x_2 - x_1 = \varepsilon u \in \varphi_\Theta(Y(\mathcal{C}))$, and hence (4.4) implies that $x_1 \Theta x_2$. This is enough to conclude that $u = \frac{1}{\varepsilon} \cdot \varepsilon u = \frac{1}{\varepsilon}(x_2 - x_1) \in \varphi_\Theta(Y)$.

It remains to prove (4.2). To this end, remember that $\{Y, Y(\mathcal{C})\} \in E_\Theta$, i.e., there exist $x_1 \in \breve{\text{co}}\, Y$, $x_2 \in \breve{\text{co}}\, Y(\mathcal{C})$, with $x_1 \Theta x_2$. By (4.4), we have $x_2 - x_1 \in \varphi_\Theta(Y(\mathcal{C}))$, and hence

$$x_2 \in \big[\breve{\text{co}}\, Y + \varphi_\Theta(Y(\mathcal{C}))\big] \cap \breve{\text{co}}\, Y(\mathcal{C}).$$

□

By now all assertions of Theorem 4.3 are proved.

4.6 The converse construction

In this subsection let a relation $\sim$ and a map φ as in the statement of Theorem 4.4 be given. For each equivalence class $\mathcal{C}$ of $\sim$, we denote its largest element by $Y(\mathcal{C})$ and set

$$Z(\mathcal{C}) := \bigcup_{Y \in \mathcal{C}} \breve{\text{co}}\, Y.$$

Moreover, we define a relation on K as

$$\Theta = \bigcup_{\substack{\mathcal{C} \text{ class} \\ \text{of } \sim}} \big\{(x_1, x_2) \in Z(\mathcal{C}) \times Z(\mathcal{C}) \mid x_2 - x_1 \in \varphi(Y(\mathcal{C}))\big\}.$$

The main step is to show:

Lemma 4.17. *The relation Θ is a convex congruence on K, i.e., $\Theta \in \operatorname{Con}_{\mathsf{CA}} K$, and*

$$\approx_\Theta = \sim, \qquad \varphi_\Theta(Y(\mathcal{C})) = \varphi(Y(\mathcal{C})) \text{ for } \mathcal{C} \text{ class of } \sim .$$

Proof. First of all, Θ is an equivalence relation as a direct consequence of $\varphi(Y(\mathcal{C}))$ being a linear subspace.

Next, notice the following: If $x \in \breve{\text{co}}\, Y$, $x' \in \breve{\text{co}}\, Y'$, and $x \Theta x'$, then x and x' must both belong to the same of the sets $Z(\mathcal{C})$, and hence $Y \sim Y'$.

To show that Θ is a convex congruence, let $(x_1, x_1'), (x_2, x_2') \in \Theta$ and $s \in (0,1)$ be given. Choose Y_j, Y_j', $j = 1, 2$, such that $x_j \in \breve{\text{co}}\, Y_j, x_j' \in \breve{\text{co}}\, Y_j'$. Then $Y_j \sim Y_j'$, and hence also

$$Y_1 \cup Y_2 \sim Y_1' \cup Y_2'.$$

Let $\mathcal{C}$ be the class of $\sim$ which contains $Y_1 \cup Y_2$. Lemma 4.10 gives

$$x := sx_1 + (1-s)x_2 \in \breve{\text{co}}(Y_1 \cup Y_2), \quad x' := sx_1' + (1-s)x_2' \in \breve{\text{co}}(Y_1' \cup Y_2'),$$

and hence $x, x' \in Z(\mathcal{C})$.

Let $\mathcal{C}_j$ be the class which contains Y_j. Since $Y_j \sim Y(\mathcal{C}_j)$, it follows that $Y_1 \cup Y_2 \sim Y(\mathcal{C}_1) \cup Y(\mathcal{C}_2)$, and hence $Y(\mathcal{C}_1) \cup Y(\mathcal{C}_2) \subseteq Y(\mathcal{C})$. Since φ is monotone, we conclude that

$$\varphi(Y(\mathcal{C}_j)) \subseteq \varphi(Y(\mathcal{C})), \quad j = 1, 2.$$

We compute

$$x' - x = s(x_1' - x_1) + (1 - s)(x_2' - x_2) \in \varphi(Y(\mathcal{C}_1)) + \varphi(Y(\mathcal{C}_2)) \subseteq \varphi(Y(\mathcal{C})),$$

and this shows that $(x, x') \in \Theta$ and therefore Θ is a convex congruence.

Next, we show that $\varphi_\Theta(Y(\mathcal{C})) = \varphi(Y(\mathcal{C}))$ whenever $\mathcal{C}$ is a class of $\sim$. By the definition of Θ, we have

$$\big\{x_2 - x_1 \mid x_1, x_2 \in \breve{\text{co}}\, Y(\mathcal{C}), x_1 \Theta x_2\big\} \subseteq \varphi(Y(\mathcal{C})),$$

and hence $\varphi_\Theta(Y(\mathcal{C})) \subseteq \varphi(Y(\mathcal{C}))$. Conversely, let $u \in \varphi(Y(\mathcal{C}))$ be given. Choose $x_1 \in \breve{\text{co}}\, Y(\mathcal{C}) \subseteq \text{aff}\, Y(\mathcal{C})$, let $\varepsilon > 0$, and set $x_2 := x_1 + \varepsilon u$. Since $\varphi(Y(\mathcal{C})) \subseteq \text{dir}\, Y(\mathcal{C})$, we have $x_2 \in \text{aff}\, Y(\mathcal{C})$. Since $\breve{\text{co}}\, Y(\mathcal{C})$ is an open subset of $\text{aff}\, Y(\mathcal{C})$, we may choose $\varepsilon > 0$ so small that $x_2 \in \breve{\text{co}}\, Y(\mathcal{C})$. Then, by the definition of Θ, we have $x_1 \Theta x_2$. It follows that $u = \frac{1}{\varepsilon}(x_2 - x_1) \in \varphi_\Theta(Y(\mathcal{C}))$.

In order to establish the inclusion "$\approx_\Theta \subseteq \sim$", it is enough to show that $\{Y_1, Y_2\} \in E_\Theta$ implies $Y_1 \sim Y_2$. This, however, is clear from the note made in the beginning of this proof. We next prove the reverse inclusion. First, let one element $Y \in V_K$ be given, and denote by $\mathcal{C}$ the class of $\sim$ which contains Y, i.e., $\mathcal{C} = [Y]_\sim$. By the hypothesis that $[\breve{\text{co}}\, Y + \varphi(Y(\mathcal{C}))] \cap \breve{\text{co}}\, Y(\mathcal{C}) \neq \emptyset$, we can choose $x_1 \in \breve{\text{co}}\, Y$, $u \in \varphi(Y(\mathcal{C}))$, and $x_2 \in \breve{\text{co}}\, Y(\mathcal{C})$, such that $x_2 = x_1 + u$. By the definition of Θ, we have $x_1 \Theta x_2$. This shows that $\{Y, Y(\mathcal{C})\} \in E_\Theta$. Let now $Y_1, Y_2 \in V_K$ with $Y_1 \sim Y_2$, and denote by $\mathcal{C}$ the class of $\sim$ which contains Y_1 (and hence also Y_2). By what we just showed, $\{Y_1, Y(\mathcal{C})\}, \{Y_2, Y(\mathcal{C})\} \in E_\Theta$. This implies that $Y_1 \approx_\Theta Y_2$. □

The fact that Θ can be computed by means of (4.5) is just the above definition of Θ. The fact that Θ is unique, is clear from (4.4). The remaining assertions of Theorem 4.4 now follow easily, as shown below.

Proof (of Theorem 4.4, (4.6), (4.7)*).* Let $Y \in V_K$, and let $\mathcal{C}$ be the class of $\sim$ which contains Y. Using (4.1) and Lemma 4.17, we obtain

$$\varphi_\Theta(Y) = \varphi_\Theta(Y(\mathcal{C})) \cap \text{dir}\, Y = \varphi(Y(\mathcal{C})) \cap \text{dir}\, Y,$$

showing (4.6). For (4.7), it is enough to note that our definition of Θ ensures

$$\forall\, Y_1, Y_2 \in V_K. \quad \{Y_1, Y_2\} \in E_\Theta \;\Rightarrow\; Y_1 \sim Y_2$$

as already noted in the proof of Lemma 4.17 and

$$\forall\, Y_1, Y_2 \in \mathcal{C}. \quad \{Y_1, Y_2\} \in E_\Theta \;\Leftrightarrow\; \big[\breve{\text{co}}\, Y_1 + \varphi(Y(\mathcal{C}))\big] \cap \breve{\text{co}}\, Y_2 \neq \emptyset$$

which is easy to show unfolding the definitions. □

4.7 Characterising inclusions

In this subsection we prove Theorem 4.7. Having Theorem 4.3, this is no longer difficult.

Proof (of Theorem 4.7). The implication "⇒" is immediate from the definition of E_{Θ_j} and φ_{Θ_j}. Conversely, assume that $E_{\Theta_1} \subseteq E_{\Theta_2}$ and $\varphi_{\Theta_1} \leq \varphi_{\Theta_2}$. The former implies that each component $\mathcal{C}_2$ of $\mathcal{G}_{\Theta_2}$ is the union of components of $\mathcal{G}_{\Theta_1}$. This implies that

$$Z(\mathcal{C}_2) = \bigcup_{\substack{\mathcal{C}_1 \text{ comp. of } \mathcal{G}_{\Theta_1} \\ \mathcal{C}_1 \subseteq \mathcal{C}_2}} Z(\mathcal{C}_1).$$

Using this fact, the representation (4.4) for Θ_1 and for Θ_2, monotonicity of φ_{Θ_1} and φ_{Θ_2}, and the assumption $\varphi_{\Theta_1} \leq \varphi_{\Theta_2}$, we compute

$$\begin{aligned}
\Theta_1 = & \bigcup_{\substack{\mathcal{C}_1 \text{ comp.} \\ \text{of } \mathcal{G}_{\Theta_1}}} \big\{(x_1, x_2) \in Z(\mathcal{C}_1) \times Z(\mathcal{C}_1) : \, x_2 - x_1 \in \varphi_{\Theta_1}(Y(\mathcal{C}_1))\big\} \\
\subseteq & \bigcup_{\substack{\mathcal{C}_2 \text{ comp.} \\ \text{of } \mathcal{G}_{\Theta_2}}} \left[\bigcup_{\substack{\mathcal{C}_1 \text{ comp.} \\ \text{of } \mathcal{G}_{\Theta_1} \\ \mathcal{C}_1 \subseteq \mathcal{C}_2}} \big\{(x_1, x_2) \in Z(\mathcal{C}_1) \times Z(\mathcal{C}_1) : \, x_2 - x_1 \in \varphi_{\Theta_2}(Y(\mathcal{C}_2))\big\} \right] \\
\subseteq & \bigcup_{\substack{\mathcal{C}_2 \text{ comp.} \\ \text{of } \mathcal{G}_{\Theta_2}}} \big\{(x_1, x_2) \in Z(\mathcal{C}_2) \times Z(\mathcal{C}_2) : \, x_2 - x_1 \in \varphi_{\Theta_2}(Y(\mathcal{C}_2))\big\} \\
= & \ \Theta_2.
\end{aligned}$$

□

5 Finitely presentable (positive, totally) convex algebras

We have already mentioned in the introduction that the free algebra $F_n(\mathsf{CA})$ with n generators is the standard $(n-1)$-simplex in $\mathbb{R}^n$. Moreover, $F_n(\mathsf{PCA})$ is the n-dimensional simplex in $\mathbb{R}^n$ generated by the point 0 and the unit vectors $e_1, \dots, e_n$, cf. Example 2.8, and $F_n(\mathsf{TCA})$ is the octahedron in $\mathbb{R}^n$ centered at 0 and having the $2n$ corners $\{\pm e_i :, \ i = 1, \dots, n\}$, cf. Example 2.9. Hence, all these free algebras are polytopes whose congruences we have fully described. Using this description, we can prove the theorem below. Before that, we recall some more notions: An algebra A is finitely generated if it is a quotient (under a congruence) of a free algebra $F_X(\mathcal{V})$ for X a finite set. A congruence Θ on an algebra A is finitely generated if there exists a finite subset $R \subseteq A \times A$ such that Θ is the smallest congruence which contains R.

Theorem 5.1. *Let $\mathcal{V}$ be one of the equational classes* CA*,* PCA*, or* TCA*, and let $A \in \mathcal{V}$ be finitely generated. Then every congruence relation on A is finitely generated.*

Proof. The proof is carried out in three steps. First, we give a straightforward equivalent formulation of the theorem. Then we verify the assertion of the given reformulation for polytopes. After that we deduce the theorem, using that finitely generated free algebras are polytopes.

Reformulation. We show that the assertion of the theorem is equivalent to the following statement: *Let $n \in \mathbb{N}^+$. Let Θ_1, Θ_2 be congruence relations on the free* *algebra $F_n(\mathcal{V})$ in $\mathcal{V}$ with n generators, and assume that $\Theta_1 \subseteq \Theta_2$. Then there exists a finite subset $R \subseteq F_n(\mathcal{V}) \times F_n(\mathcal{V})$, such that Θ_2 is generated by $\Theta_1 \cup R$.*

For the proof of equivalence, assume first that the assertion of the theorem holds, and let $F_n(\mathcal{V})$, Θ_1, Θ_2 be given as in the above statement. Consider the canonical projection $\pi : F_n(\mathcal{V}) \to F_n(\mathcal{V})/_{\Theta_1}$ with $\ker \pi = \Theta_1$. It is a surjective homomorphism, and hence $\pi \times \pi$ induces an order isomorphism

$$\pi \times \pi : \{\Theta \in \mathrm{Con}_{\mathcal{V}} F_n(\mathcal{V}) \mid \Theta \supseteq \Theta_1\} \to \mathrm{Con}_{\mathcal{V}} \big(F_n(\mathcal{V})/_{\Theta_1}\big).$$

According to the theorem, as $F_n(\mathcal{V})/_{\Theta_1}$ is finitely generated, we can choose a finite subset R of $F_n(\mathcal{V})/_{\Theta_1} \times F_n(\mathcal{V})/_{\Theta_1}$, such that $(\pi\times\pi)\Theta_2$ is generated by R. Choose a subset R' of Θ_2 with $|R'| = |R|$ and $(\pi\times\pi)R' = R$. If $\Theta \in \mathrm{Con}_{\mathcal{V}} F_n(\mathcal{V})$ contains $\Theta_1 \cup R'$, then $(\pi \times \pi)\Theta \supseteq R$, and hence $(\pi \times \pi)\Theta \supseteq (\pi \times \pi)\Theta_2$. We are going to show still that then $\Theta \supseteq \Theta_2$ and hence the reformulated assertion holds. Let $(x_1, x_2) \in \Theta_2$, then $(\pi \times \pi)(x_1, x_2) \in (\pi \times \pi)\Theta_2 \subseteq (\pi \times \pi)\Theta$ and hence there exists a pair $(y_1, y_2) \in \Theta$ with $(\pi \times \pi)(x_1, x_2) = (\pi \times \pi)(y_1, y_2)$. But this means that $\pi(x_1) = \pi(y_1)$ and $\pi(x_2) = \pi(y_2)$ which due to $\ker \pi = \Theta_1$ implies $(x_1, y_1) \in \Theta_1$ and $(x_2, y_2) \in \Theta_1$. But then $(x_1, x_2) \in \Theta_1 \circ \Theta \circ \Theta_1$ and since $\Theta_1 \subseteq \Theta$ we get $(x_1, x_2) \in \Theta \circ \Theta \circ \Theta \subseteq \Theta$.

Assume now that the above reformulated statement holds and let A and $\Theta \in \mathrm{Con}_{\mathcal{V}} A$ be given according to the theorem. Choose $n \in \mathbb{N}^+$ and $\Theta_1 \in \mathrm{Con}_{\mathcal{V}} F_n(\mathcal{V})$, such that $A \cong F_n(\mathcal{V})/_{\Theta_1}$. Let π be the corresponding surjective homomorphism of $F_n(\mathcal{V})$ onto A, and set $\Theta_2 := (\pi \times \pi)^{-1}\Theta$. Note that Θ_2 is a convex congruence and $\Theta_2 \supseteq \ker \pi = \Theta_1$. Choose a finite subset $R \subseteq \Theta_2$ such that Θ_2 is generated by $\Theta_1 \cup R$ and set $R' := (\pi \times \pi)R$. Note that R' is finite and $R' \subseteq \Theta$. If $\Theta' \in \mathrm{Con}_{\mathcal{V}} A$ and $\Theta' \supseteq R'$, then $(\pi \times \pi)^{-1}\Theta' \supseteq \Theta_1 \cup R$, and hence $(\pi \times \pi)^{-1}\Theta' \supseteq \Theta_2$. This implies that $\Theta' \supseteq \Theta$. Hence we have shown that Θ is finitely generated, namely it is generated by R'.

Generators for convex congruences on polytopes. Let K be a polytope in $\mathbb{R}^n$, and consider K as an algebra in CA. Let $n \in \mathbb{N}^+$, and $\Theta_1, \Theta_2 \in \mathrm{Con}_{\mathsf{CA}} K$ with $\Theta_1 \subseteq \Theta_2$ be given. We will now construct a finite subset $R \subseteq \Theta_2$, such that Θ_2 is generated by $\Theta_1 \cup R$.

Define $R \subseteq \Theta_2$ as the collection of pairs of the following two kinds:

(*i*) For each edge $\{Y_1, Y_2\} \in E_{\Theta_2} \setminus E_{\Theta_1}$ choose a pair $(a, b) \in \Theta_2 \cap (\check{\mathrm{co}}\, Y_1 \times \check{\mathrm{co}}\, Y_2)$. Take this pair into R.

(*ii*) For each $Y \in V_K$, choose pairs $(a_i, b_i) \in \Theta_2 \cap (\check{\mathrm{co}}\, Y \times \check{\mathrm{co}}\, Y)$, $i = 1, \ldots, \dim \varphi_{\Theta_2}(Y) - \dim \varphi_{\Theta_1}(Y)$, such that

$$\varphi_{\Theta_1}(Y) + \mathrm{span}\{b_i - a_i \,|\, i = 1, \ldots, \dim \varphi_{\Theta_2}(Y) - \dim \varphi_{\Theta_1}(Y)\} = \varphi_{\Theta_2}(Y). \tag{5.1}$$

Take these pairs into R.

Let $\Theta \in \mathrm{Con}_{\mathsf{CA}} K$, and assume that $\Theta \supseteq \Theta_1 \cup R$. Then $E_\Theta \supseteq E_{\Theta_1}$ since $\Theta \supseteq \Theta_1$, and $E_\Theta \supseteq E_{\Theta_2} \setminus E_{\Theta_1}$, since $\Theta \supseteq R$. Hence $E_\Theta \supseteq E_{\Theta_2}$. Similarly, since $\Theta \supseteq \Theta_1$ we have $\varphi_\Theta(Y) \supseteq \varphi_{\Theta_1}(Y)$, for all $Y \in V_K$, and since R contains the pairs with (5.1), also $\varphi_\Theta(Y) \supseteq \varphi_{\Theta_2}(Y)$, for all $Y \in V_K$. By Theorem 4.7, $\Theta \supseteq \Theta_2$. Hence Θ_2 is indeed generated by $\Theta_1 \cup R$.

Proof of the theorem. Let $\mathcal{V}$ be one of the classes CA, PCA, or TCA. Then as discussed above $F_n(\mathcal{V})$ is a polytope in euclidean space (endowed with the usual vector operations). Let $\Theta_1, \Theta_2 \in \operatorname{Con}_{\mathcal{V}} F_n(\mathcal{V})$, and assume that $\Theta_1 \subseteq \Theta_2$. We can consider $F_n(\mathcal{V})$ as an element of CA, and then $\Theta_j \in \operatorname{Con}_{\mathsf{CA}} F_n(\mathcal{V})$, cf. Lemma 3.7 and Lemma 3.11, respectively. By the second step, there exists a finite subset $R \subseteq \Theta_2$, such that Θ_2 is generated by $\Theta_1 \cup R$ as an element of $\operatorname{Con}_{\mathsf{CA}} F_n(\mathcal{V})$. This immediately implies that Θ_2 is generated by $\Theta_1 \cup R$ as an element of $\operatorname{Con}_{\mathcal{V}} F_n(\mathcal{V})$. We have thus verified the statement shown to be equivalent to the theorem in the first step. □

Let us now recall the (algebraic) definition of a finitely presentable algebra, see, e.g., [AR94, 3.10]. The connection to the categorical concept of presentability is made in [AR94, Theorem 3.12], see also [ARV11, Corollary 11.33].

Definition 5.2. Let $\mathcal{V}$ be an equational class, and let $A \in \mathcal{V}$. Let $F_X(\mathcal{V})$ be the free algebra in $\mathcal{V}$ with the set X as free generators. A presentation of A is a pair (X, R_X) where X is a set and R_X is a subset of $F_X(\mathcal{V}) \times F_X(\mathcal{V})$ such that A is isomorphic to the quotient $F_X(\mathcal{V})/_{\Theta}$ where Θ is the smallest congruence containing R_X.

An algebra A is finitely presentable if there exists a presentation (X, R_X) with both X and R_X finite.

Obviously, an algebra is finitely generated if and only if there exists a presentation (X, R_X) with X finite. Hence, trivially, each finitely presented algebra is finitely generated. Having shown the previous theorem, we obtain as an immediate consequence that for CA, PCA, and TCA also the converse holds.

Corollary 5.3. *Let $\mathcal{V}$ be one of the equational classes* CA*,* PCA*, or* TCA*, and let $A \in \mathcal{V}$. If A is finitely generated, then A is also finitely presentable. Hence a convex, positive convex, or totally convex algebra is finitely presentable if and only if it is finitely generated.* □

Remark 5.4. A sufficient condition in order that each finitely generated algebra of an equational class is finitely presentable, appeared recently in a categorical context. This condition is that the subclass of all finitely generated algebras of the equational class is closed under kernel pairs, cf. [BMS11, Lemma 3.21]. Formulated algebraically, it means that each congruence of any finitely generated algebra P is finitely generated as a subalgebra of $P \times P$.

We see that the condition of being closed under kernel pairs is quite strong. In order to have "finitely generated ⇒ finitely presented", it is certainly sufficient that each congruence of a free algebra F with finitely many generators is finitely generated as a congruence of F[6].

Remark 5.5. The question whether or not every finitely generated algebra of an equational class if finitely presented is classical. Some previously known examples where the answer is positive are

- *commutative groups:* Due to the classification of finitely generated commutative groups.
- *semimodules over a Noetherian semiring*: Due to a "kernel pair argument", cf. [BMS11, Proposition 2.3].

[6]Being finitely generated as a subalgebra is in general a significantly stronger property than being finitely generated as a congruence, see the below Proposition 5.6.

- *commutative semigroups:* This is essentially a particular case of the previous item (units can be adjoined easily), but has a longer history. It was first shown by L.Redei, cf. [Re63, Satz 72] or [CP67, §9.3]. A short proof based on Hilberts basis theorem (i.e., a "Noetherian" argument) is given in [Fr68].

There are many equational classes where the answer is negative. For example, the equational class of all groups: Not every finitely generated group is finitely presented. In fact, there exist only countable many non-isomorphic finitely presentable groups, but already $2^{\aleph_0}$ non-isomorphic 2-generator groups. The (probably) simplest example of a finitely generated but not finitely presented group is the standard wreath product $\mathbb{Z} \,\mathrm{wr}\, \mathbb{Z}$, cf. [Ro93, §14.1]. The question whether a specific finitely generated group is finitely presented may already be involved, see for example [RS76], [KC97], or [GS97].

In view of Remark 5.4, it is interesting to observe that for $\mathcal{V}$ being any of CA, PCA, TCA, the free algebras $F_n(\mathcal{V})$ for $n \in \mathbb{N}^+$ (except for $F_1(\mathsf{CA})$ which contains only one element) always contain congruence relations which are not finitely generated as a subalgebra of $F_n(\mathcal{V}) \times F_n(\mathcal{V})$, cf. Example 5.8 below. This follows immediately from the next proposition, where we characterise the congruences on a polytope that are finitely generated as subalgebras of $K \times K$, showing that a "kernel pair argument" cannot be applied in these equational classes.

Proposition 5.6. *Let $\mathcal{T}$ be one of $\mathcal{T}_{ca}$, $\mathcal{T}_{pca}$, or $\mathcal{T}_{tca}$. Let K be a polytope in $\mathbb{R}^n$. Then $\langle K, (f_\gamma)_{\gamma \in \mathcal{T}} \rangle \in \mathsf{CA}$ (PCA or TCA, respectively) with operations given by*

$$f_{(p_i)_{i=1}^n}(x_1, \ldots, x_n) = \sum_{i=1}^n p_i x_i, \quad (p_i)_{i=1}^n \in \mathcal{T},$$

where the linear combination on the right side denotes the usual vector operation on $\mathbb{R}^n$.

Let $\Theta \in \mathrm{Con}_{\mathsf{CA}} K$ ($\mathrm{Con}_{\mathsf{PCA}} K$ or $\mathrm{Con}_{\mathsf{TCA}} K$, respectively). Then Θ is finitely generated as a CA- (PCA- or TCA-, respectively) subalgebra of $K \times K$ if and only if Θ is closed as a subset of $\mathbb{R}^n \times \mathbb{R}^n$.

Before we can prove this fact, we describe the closure of Θ in terms of φ_Θ.

Lemma 5.7. *Let K be a polytope in $\mathbb{R}^n$, and let $\Theta \in \mathrm{Con}_{\mathsf{CA}} K$. Then*

$$\mathrm{Clos}\, \Theta = \big\{ (x_1, x_2) \in K \times K : \, x_2 - x_1 \in \varphi_\Theta(\mathrm{ext}\, K) \big\}.$$

Proof. Denote the set on the right side of the desired equality as Θ_0. By the representation (4.4) and monotonicity of φ_Θ, we have

$$\Theta \subseteq \Theta_0.$$

It is easy to show that Θ_0 is closed (the limit of any converging sequence of elements of Θ_0 is in Θ_0) since K is closed and $\varphi_\Theta(\mathrm{ext}\, K)$ is closed as a linear subspace.

Let $(x_1, x_2) \in \Theta_0$ be given. Choose a point $z \in \check{\mathrm{co}}(\mathrm{ext}\, K)$. Then, since $x_1, x_2 \in \mathrm{co}(\mathrm{ext}\, K)$, we obtain from Lemma 4.10 that $\Phi_z(s, x_1), \Phi_z(s, x_2) \in \check{\mathrm{co}}(\mathrm{ext}\, K)$, $s \in (0, 1)$, and we have

$$\Phi_z(s, x_2) - \Phi_z(s, x_1) = (1 - s)(x_2 - x_1) \in \varphi_\Theta(\mathrm{ext}\, K).$$

Now using (4.4), since $\operatorname{ext} K$ is a vertex of $\mathcal{G}_\Theta$ and hence in some connected component $\mathcal{C}$, and φ_Θ is monotone, we get

$$\big(\Phi_z(s,x_1),\Phi_z(s,x_2)\big)\in\Theta,\quad s\in(0,1).$$

Letting s tend to 0, $\Phi_z(s,x_1)$ tends to x_1 and $\Phi_z(s,x_2)$ to x_2 and it follows that $(x_1,x_2)\in\operatorname{Clos}\Theta$. Hence

$$\Theta_0\subseteq\operatorname{Clos}\Theta$$

and therefore $\Theta_0=\operatorname{Clos}\Theta$. □

This lemma tells us, in particular, that Θ is closed if and only if $\Theta=\Theta_0$.

Proof (of Proposition 5.6).
Necessity. Assume that R is a finite subset of $K\times K$, such that Θ is generated by R as a subalgebra of $K\times K\in\mathsf{CA}$ (PCA or TCA, respectively). Then we have

$$\Theta=\begin{cases}\operatorname{co}R & (\,\mathsf{CA}-\text{situation}\,)\\ \operatorname{co}\big(\{0_K\}\cup R\big) & (\,\mathsf{PCA}-\text{situation}\,)\\ \operatorname{co}\big(R\cup(\omega R)\big) & (\,\mathsf{TCA}-\text{situation}\,)\end{cases}$$

where ω is as in Proposition 3.8. For the PCA or TCA situation, note here that each PCA– or TCA– linear combination can be written as a convex combination by making use of the zero element 0_K and the ω-images of the involved elements, and that for convex combinations the operations coincide with the usual vector operations.

In any case, Θ is a polytope in $\mathbb{R}^{2n}$ and thus in particular closed.

Sufficiency. Assume that $\Theta\in\operatorname{Con}_{\mathsf{CA}}K$, and that Θ is closed as a subset of $\mathbb{R}^n\times\mathbb{R}^n$. We will now show that Θ is finitely generated as a CA-subalgebra of $K\times K$. Set $d:=n-\dim\varphi_\Theta(\operatorname{ext}K)$, and choose a linear and surjective map $\pi:\mathbb{R}^n\to\mathbb{R}^d$ whose kernel equals $\varphi_\Theta(\operatorname{ext}K)$. Note that this means

$$(x_1,x_2)\in\ker\pi\Leftrightarrow\pi(x_1)=\pi(x_2)\Leftrightarrow x_2-x_1\in\{x\mid\pi(x)=0\}$$

$$\Leftrightarrow x_2-x_1\in\operatorname{Ker}\pi\Leftrightarrow x_2-x_1\in\varphi_\Theta(\operatorname{ext}K).$$

We denote by Δ the diagonal in $\mathbb{R}^n\times\mathbb{R}^n$. Since Θ is closed, by Lemma 5.7 and the above, we have

$$\Theta=\operatorname{Clos}\Theta=(K\times K)\cap\ker\pi=(K\times K)\cap(\pi\times\pi)^{-1}(\Delta).$$

The set $K\times K$ is a compact (since K is compact) and convex subset of $\mathbb{R}^{2n}$, and one can easily show unfolding the definition of an extremal point that

$$\operatorname{ext}(K\times K)=\operatorname{ext}K\times\operatorname{ext}K,$$

so $\operatorname{ext}(K\times K)$ is a finite set. Hence, $K\times K$ is a polytope in $\mathbb{R}^{2n}$.

Now we will employ some non-trivial geometric arguments from [Grü03]. The diagonal Δ is a linear subspace, and hence can be written as a finite intersection of halfspaces, i.e., it is a polyhedral set in the sense of [Grü03, §2.6]. Since $(\pi\times\pi)$ is linear and surjective, the inverse image of a halfspace is again a halfspace. Hence, $(\pi\times\pi)^{-1}(\Delta)$ is again a polyhedral set.

As an intersection of a polytope with a polyhedral set, Θ is a polytope, cf. [Grü03, §3.1]. Hence, Θ has only finitely many extremal points and $\Theta = \mathrm{co}(\mathrm{ext}\,\Theta)$. This means that the finite set $\mathrm{ext}\,\Theta$ generates Θ as a CA-subalgebra of $K \times K$.

It remains to consider PCA and TCA congruences. Assume that $\Theta \in \mathrm{Con}_{\mathsf{PCA}}\, K$ (or $\mathrm{Con}_{\mathsf{TCA}}\, K$, respectively) and is closed. Then also $\Theta \in \mathrm{Con}_{\mathsf{CA}}\, K$, and hence by what we have proven so far Θ is finitely generated as a CA-subalgebra of $K \times K$. In particular, it is therefore also finitely generated as a PCA- (TCA-, respectively) subalgebra of $K \times K$. □

Example 5.8. Let $K \subseteq \mathbb{R}^n$ be a polytope with $|\,\mathrm{ext}\,K| \geq 2$, and let Y_0 be a nonempty and proper subset of $\mathrm{ext}\,K$. Consider the join-semilattice congruence $\sim$ on V_K defined by specifying its equivalence classes to be

$$\mathcal{C}_y := \big\{\{y\}\big\},\ y \in Y_0, \qquad \mathcal{C}_0 := V_K \setminus \bigcup_{y \in Y_0} \mathcal{C}_y\,,$$

and the map $\varphi : \{Y(\mathcal{C}) | \mathcal{C} \text{ class of } \sim\} \to \mathrm{Sub}\,\mathbb{R}^n$ defined as

$$\varphi\big(Y(\mathcal{C}_y)\big) := \{0\},\ y \in Y_0, \qquad \varphi\big(Y(\mathcal{C}_0)\big) := \mathrm{dir}(\mathrm{ext}\,K)\,,$$

cf. Remark 4.5. Applying Theorem 4.4, we obtain that the relation

$$\Theta := \big\{(x,x) \mid x \in Y_0\big\} \cup \big\{(x_1,x_2) \in Z(\mathcal{C}_0) \times Z(\mathcal{C}_0) \mid x_2 - x_1 \in \mathrm{dir}(\mathrm{ext}\,K)\big\}$$

is a congruence of K when K is considered as a convex algebra with the usual vector space operations of $\mathbb{R}^n$.

Let $y_0 \in Y_0$. Since y_0 is an extremal point, it cannot be an element of $Z(\mathcal{C}_0)$. However, it can be approximated by elements from $Z(\mathcal{C}_0)$: Choose $y \in (\mathrm{ext}\,K) \setminus Y_0$, and set $x_\varepsilon := \varepsilon y + (1-\varepsilon) y_0$, $\varepsilon \in (0,1]$. Then one can check from the definitions that $(x_\varepsilon, y) \in \Theta$ and $\lim_{\varepsilon \downarrow 0}(x_\varepsilon, y) = (y_0, y) \notin \Theta$, and this shows that Θ is not closed.

This example applies immediately to the free algebras $F_n(\mathsf{CA})$, $n \geq 2$, and $F_n(\mathsf{PCA})$, $n \in \mathbb{N}^+$, and shows that they contain congruences which are not finitely generated as subalgebras. For $F_n(\mathsf{TCA})$, choose Y_0 symmetric around the zero vector.

6 Conclusion

We fully describe the congruence lattice of a polytope in finite-dimensional euclidean space when considered as a convex algebra. This description applies in particular to the free algebras with a finite number of generators in this equational class, and hence clarifies the structure of finitely generated algebras. In particular, we see that each finitely generated convex algebra is finitely presented. The proofs are algebraic in their nature and use the geometry of euclidean space.

We show that the equational classes of positive- or totally convex algebras (and their respective congruence lattices) are closely related with convex algebras. Using this relation, similar structure results for these equational classes follow.

The presented results have an interpretation in a categorical context, since the considered equational classes are the categories of Eilenberg-Moore algebras associated with certain monads.

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