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A review of stability and error theory for collocation methods applied to linear boundary value problems

W. Auzinger and M. Tutz

Institute for Analysis and Scientific Computing
Vienna University of Technology — TU Wien
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Institute for Analysis and Scientific Computing
Vienna University of Technology
Wiedner Hauptstraße 8–10
1040 Wien, Austria

E-Mail: admin@asc.tuwien.ac.at
WWW: http://www.asc.tuwien.ac.at
FAX: +43-1-58801-10196

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ASC TU WIEN

A review of stability and error theory for collocation methods applied to linear boundary value problems

Seminar report by *Winfried Auzinger* and *Martin Tutz* [1]

1 Introduction

The error theory for collocation methods applied to systems of first order boundary value problems is described in [1, Sec. 5.4]. In this report we present a systematic, didactically remastered revision of the convergence proof. The presentation is (only slightly) informal in some details, with emphasis on the structure of the convergence argument.

A word on notation: By $\|\cdot\|$ we denote the sup-norm in $\mathbb{R}^n$, and also the resulting sup-norm on the space $C[a,b]$ of vector-valued functions as well as the sup-norm of vector-valued functions defined over a discrete grid. Matrix norms are defined accordingly.

2 Linear boundary value problems

Consider the n-dimensional linear boundary value problem including a linear boundary condition

$$y'(t) = A(t)\,y(t) + g(t), \tag{1a}$$

$$R_a\,y(a) + R_b\,y(b) = r, \tag{1b}$$

with exact a solution $y(t) \in C^{s+1}[a,b]$, where $y(t) : [a,b] \to \mathbb{R}^n$ $(R_a, R_b \in \mathbb{R}^{n\times n})$. We are assuming that $A(t) \in \mathbb{R}^{n\times n}$ and $g(t) \in \mathbb{R}^n$ are sufficiently smooth. According to the theory of ordinary differential equations (c.f. [4]) the solution of (1) is unique provided $R_a\,Y(a) + R_b\,Y(b)$ is invertible, and it can be represented in the form

$$y(t) = Y(t)\,c + \int_a^b G(t,\tau)\,g(\tau)\,d\tau, \tag{2a}$$

$$\text{with} \quad c = [R_a\,Y(a) + R_b\,Y(b)]^{-1}\,r, \tag{2b}$$

where $Y(t)$ denotes any fundamental matrix of $y'(t) = A(t)\,y(t)$, and $G(t,\tau)$ is Green's matrix (Green's function)

$$G(t,\tau) = \begin{cases} \;\;Y(t)[R_a\,Y(a) + R_b\,Y(b)]^{-1}\,R_a\,Y(a)\,Y^{-1}(\tau), & \tau < t, \\ -Y(t)[R_a\,Y(a) + R_b\,Y(b)]^{-1}\,R_b\,Y(b)\,Y^{-1}(\tau)\,, & \tau > t. \end{cases} \tag{3}$$

Without repeating all the properties of Green's matrix which can, e.g., be found in [4], we only remark that $G(t,\tau)$ is discontinuous at $t = \tau$.

3 Convergence of collocation methods

3.1 Collocation methods

For the numerical solution of (1) via polynomial collocation, we define the following grid (see Fig. 1).

[1] Institute for Analysis and Scientific Computing, Vienna University of Technology.

Definition 3.1. The interval $[a,b] = [t_0, t_N]$ is divided into N subintervals of length $h_\nu = t_\nu - t_{\nu-1}$, with $a = t_0 < t_1 < \cdots < t_{N-1} < t_N = b$. In each subinterval we choose s grid points (collocation nodes) $\tau_{\nu,i}$ ($\nu = 1 \ldots N$, $i = 1 \ldots s$). The relative position of the $\tau_{\nu,i}$ within these subinterval is determined by s pairwise distinct parameters $c_i \in [0,1]$,

$$\tau_{\nu,i} = t_{\nu-1} + c_i\, h_\nu, \quad \nu = 1 \ldots N, \quad i = 1 \ldots s. \tag{4}$$

Let

$$h := \max\{h_1 \ldots h_N\}. \tag{5}$$

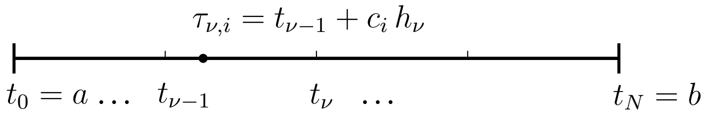

Figure 1: Collocation grid.

The collocation solution is defined as a continuous piecewise polynomial function $p\colon [a,b] \to \mathbb{R}^n$, where $p(t) = p_\nu(t)$ with degree s on the ν-th subinterval. It is fixed by the collocation and boundary conditions

$$p_\nu'(\tau_{\nu,i}) = A(\tau_{\nu,i})\, p_\nu(\tau_{\nu,i}) + g(\tau_{\nu,i}), \quad \nu = 1 \ldots N,\ i = 1 \ldots s, \tag{6a}$$
$$R_a\, p(a) + R_b\, p(b) = r, \tag{6b}$$

together with the continuity equations at the interior endpoints of the subintervals.

Proposition 3.2. *Assume that the boundary value problem (1) has a unique and sufficiently smooth solution. Then, for sufficiently small h, the collocation error $e(t) = p(t) - y(t)$ and its first derivative* [2] *$e'(t) = p'(t) - y'(t)$ satisfy*

$$e(t) = \mathcal{O}(h^s), \tag{7a}$$
$$e'(t) = \mathcal{O}(h^s), \tag{7b}$$

uniformly in t.

In the following sections we give a detailed proof of Proposition 3.2. In particular, the estimates (7a) and (7b) are verified in Sec. 3.3. For sharper results valid at the endpoints t_ν of the subintervals see Sec. 3.4.

In the sequel, $Y_\nu(t)$, $\nu = 1 \ldots N$, denotes the 'local fundamental matrices', defined as the solutions of the matrix initial value problems

$$Y_\nu'(t) = A(t)\, Y_\nu(t), \tag{8a}$$
$$Y_\nu(t_{\nu-1}) = I. \tag{8b}$$

[2]The derivative $e'(t)$ is discontinuous at the endpoints $t = t_\nu$, $\nu = 1 \ldots N-1$, of the subintervals. At these points, (7b) is valid for the left-hand and right-hand limits.

3.2 Stability

Step 1. Local fundamental collocation matrices

For the proof of stability of the collocation method we consider local *local fundamental collocation matrices* $P_\nu(t) \in \mathbb{R}^{n\times n}$, $\nu = 1\dots N$, which are defined as the collocation approximations to the local fundamental matrices $Y_\nu(t)$. I.e., the $P_\nu(t)$ are fixed by the requirements

$$P_\nu'(\tau_{\nu,i}) = A(\tau_{\nu,i})P_\nu(\tau_{\nu,i}), \quad i = 1\dots s, \tag{9a}$$
$$P_\nu(t_{\nu-1}) = I. \tag{9b}$$

An implicit Runge-Kutta formulation of the collocation equations (9) is obtained in the following way. Integration gives

$$\begin{aligned} P_\nu(t) &= I + \int_{t_{\nu-1}}^{t} P_\nu'(\tau)\,d\tau \\ &= I + h_\nu \sum_{j=1}^{s} \omega_{\nu,j}(t)\,P_\nu'(\tau_{\nu,j}) \\ &= I + h_\nu \sum_{j=1}^{s} \omega_{\nu,j}(t) A(\tau_\nu) P_\nu(\tau_{\nu,j}), \end{aligned} \tag{10a}$$

with weights

$$\omega_{\nu,j}(t) = \frac{1}{h_\nu}\int_{t_{\nu-1}}^{t} L_{\nu,j}(\tau)\,d\tau. \tag{10b}$$

Here, $L_{\nu,j}(t)$ denotes the Lagrange polynomial of degree $s-1$ with respect to the grid points $\tau_{\nu,j}$,

$$L_{\nu,j}(t) = \prod_{i=1,i\neq j}^{s} \frac{t-\tau_{\nu,i}}{\tau_{\nu,j}-\tau_{\nu,i}}. \tag{11}$$

Considering (10a) at $t = \tau_{\nu,i}$, $i = 1\dots s$, and defining the the Butcher coefficients[3]

$$a_{i,j} = \frac{1}{h_\nu}\int_{t_{\nu-1}}^{\tau_{\nu,i}} L_{\nu,j}(\tau)\,d\tau, \tag{12a}$$
$$b_j = \frac{1}{h_\nu}\int_{t_{\nu-1}}^{t_\nu} L_{\nu,j}(\tau)\,d\tau, \tag{12b}$$

we obtain the Runge-Kutta formulation of the fundamental collocation equations (9):

$$P_\nu(\tau_{\nu,i}) = I + h_\nu \sum_{j=1}^{s} a_{i,j}\, A(\tau_{\nu,j})P_\nu(\tau_{\nu,j}), \quad i = 1\dots s, \tag{13a}$$
$$P_\nu(t_\nu) = I + h_\nu \sum_{j=1}^{s} b_j\, A(\tau_{\nu,j})P_\nu(\tau_{\nu,j}). \tag{13b}$$

[3]Since the distribution of collocation nodes is assumed to be the same in all subintervals, these quadrature coefficients do not depend on ν. As usual, they can also be defined with respect to the interval $[0,1]$ and nodes $c_1,\dots,c_s$, see (4).

The solution is unique for h_ν sufficiently small. Eq. (13) is a discrete analog of the Volterra integral equation satisfied by $Y_\nu(t)$ (see (8a)),

$$Y_\nu(t) = I + \int_{t_{\nu-1}}^{t} A(\tau)\, Y_\nu(\tau)\, d\tau. \tag{14}$$

Substituting the integral in (14) by the appropriate polynomial quadrature formula of degree $s-1$ with weights from (10b) generates a quadrature error of order[4] $\mathcal{O}(h^{s+1})$, i.e.,

$$Y_\nu(t) = I + h_\nu \sum_{j=1}^{s} \omega_{\nu,j}(t) A(\tau_{\nu,j}) Y_\nu(\tau_{\nu,j}) + \mathcal{O}(h^{s+1}). \tag{15}$$

The error of the local fundamental collocation matrix with respect to the exact local fundamental matrix is denoted by $E_\nu(t) := P_\nu(t) - Y_\nu(t),$ with $E_\nu(t_{\nu-1}) = 0$. Combining (10a) and (15) leads to the error equation

$$E_\nu(t) = h_\nu \sum_{j=1}^{s} \omega_{\nu,j}(t) A(\tau_{\nu,j}) E_\nu(\tau_{\nu,j}) + \mathcal{O}(h^{s+1}). \tag{16}$$

Evaluating (16) at the collocation nodes $t = \tau_{\nu,i}$ and at the endpoints $t = t_\nu$ of the subintervals we obtain the corresponding Runge-Kutta formulation of the error equation,

$$E_\nu(\tau_{\nu,i}) = h_\nu \sum_{j=1}^{s} a_{i,j}\, A(\tau_{\nu,j}) E_\nu(\tau_{\nu,j}) + \mathcal{O}(h^{s+1}), \tag{17a}$$

$$E_\nu(t_\nu) = h_\nu \sum_{j=1}^{s} b_j\, A(\tau_{\nu,j}) E_\nu(\tau_{\nu,j}) + \mathcal{O}(h^{s+1}). \tag{17b}$$

With

$$\mathcal{E}_\nu = \begin{pmatrix} E_\nu(\tau_{\nu,1}) \\ \vdots \\ E_\nu(\tau_{\nu,s}) \end{pmatrix} \tag{18}$$

we can rewrite (17) as a system of linear equations of the form

$$\mathcal{E}_\nu = h_\nu\, X_\nu\, \mathcal{E}_\nu + \mathcal{O}(h^{s+1}), \tag{19}$$

where the matrix $X_\nu \in \mathbb{R}^{ns \times ns}$ consists of the block entries $X_{\nu;i,j} = a_{i,j}\, A(\tau_{\nu,j})$. For sufficiently small h_ν this gives

$$(I - h_\nu\, X_\nu)\, \mathcal{E}_\nu = \mathcal{O}(h^{s+1}), \tag{20a}$$

$$\mathcal{E}_\nu = (I - h_\nu\, X_\nu)^{-1}\, \mathcal{O}(h^{s+1}). \tag{20b}$$

Thus,

$$\|\mathcal{E}_\nu\| = \mathcal{O}(h^{s+1}) \tag{21}$$

[4]Note that the interpolation error is $\mathcal{O}(h^s)$ and thus, the quadrature error over an interval of length $\mathcal{O}(h)$ is $\mathcal{O}(h^{s+1})$.

holds for sufficiently small h_ν. In other words: The local fundamental collocation matrix is a uniformly consistent approximation to the exact local fundamental matrix, i.e.,

$$P_\nu(t) = Y_\nu(t) + \mathcal{O}(h^{s+1}). \tag{22a}$$

This also implies that $P_\nu(t)$ is invertible for sufficiently small h_ν, due to the invertibility of $Y_\nu(t)$. Furthermore, we conclude

$$P_\nu^{-1}(t) = Y_\nu^{-1}(t) + \mathcal{O}(h^{s+1}), \tag{22b}$$

and

$$P_\nu(t)\, P_\nu^{-1}(\tau) = Y_\nu(t)\, Y_\nu^{-1}(\tau) + \mathcal{O}(h^{s+1}), \tag{22c}$$

uniformly for $t, \tau \in [t_{\nu-1}, t_\nu]$.

Step 2. Discrete variation of constants: Expressing $p_\nu(t)$ via $P_\nu(t)$

Consider the solution

$$p(t) = \big(p_1(t), \dots, p_N(t)\big) \tag{23a}$$

of the collocation equations (6), and let

$$\pi_{\nu-1} := p_\nu(t_{\nu-1}). \tag{23b}$$

Now we express the $p_\nu(t)$ by means of the local fundamental collocation matrices $P_\nu(t)$. To this end we use the discrete variation-of-constants ansatz

$$p_\nu(t) = P_\nu(t)\, c_\nu(t). \tag{24a}$$

Here, $c_\nu(t)$ is a polynomial of degree s which is to be determined. The derivative of (24a) is

$$p_\nu'(t) = P_\nu'(t)\, c_\nu(t) + P_\nu(t)\, c_\nu'(t). \tag{24b}$$

The polynomial $p_\nu(t)$ satisfies the collocation equations (6a). With (9a) and (24a) this implies

$$P_\nu'(\tau_{\nu,i})\, c_\nu(\tau_{\nu,i}) + P_\nu(\tau_{\nu,i})\, c_\nu'(\tau_{\nu,i}) = A(\tau_{\nu,i})\, \underbrace{p_\nu(\tau_{\nu,i})}_{=P_\nu(\tau_{\nu,i})\, c_\nu(\tau_{\nu,i})} + g(\tau_{\nu,i}),$$

$$P_\nu'(\tau_{\nu,i})\, c_\nu(\tau_{\nu,i}) + P_\nu(\tau_{\nu,i})\, c_\nu'(\tau_{\nu,i}) = \underbrace{A(\tau_{\nu,i}) P_\nu(\tau_{\nu,i})}_{=P_\nu'(\tau_{\nu,i})}\, c_\nu(\tau_{\nu,i}) + g(\tau_{\nu,i}),$$

hence

$$c_\nu'(\tau_{\nu,i}) = P_\nu^{-1}(\tau_{\nu,i})\, g(\tau_{\nu,i}), \quad i = 1 \dots s,$$

where $c_\nu(t_{\nu-1}) = \pi_{\nu-1}$ is given by (23b). The Lagrange representation of $c_\nu'(t)$ reads

$$c_\nu'(t) = \sum_{j=1}^{s} L_{\nu,j}(t) P_\nu^{-1}(\tau_{\nu,j})\, g(\tau_{\nu,j}),$$

with $L_{\nu,j}(t)$ from (11). Integration leads to

$$c_\nu(t) = \pi_{\nu-1} + h_\nu \sum_{j=1}^{s} \omega_{\nu,j}(t) P_\nu^{-1}(\tau_{\nu,i})\, g(\tau_{\nu,i}), \tag{25}$$

with weights $\omega_{\nu,j}$ from (10b) . Inserting (25) into (24a) yields the discrete variation-of-constants representation for $p_\nu(t)$,

$$p_\nu(t) = P_\nu(t)\, \pi_{\nu-1} + h_\nu \sum_{j=1}^{s} \omega_{\nu,j}(t)\, P_\nu(t) P_\nu^{-1}(\tau_{\nu,j})\, g(\tau_{\nu,j}). \tag{26}$$

Step 3. Stability estimate via 'theoretical multiple shooting'

Now we combine the previously considered local initial value problems with the continuity conditions and the boundary condition. The continuity conditions read $\pi_\nu = p_\nu(t_\nu),\ \nu = 1 \ldots N$. This gives a condensed system of equations for the π_ν :

$$\begin{aligned} \pi_1 &= P_1(t_1)\, \pi_0 + h_1 f_1, \\ &\vdots \\ \pi_N &= P_N(t_N)\, \pi_{N-1} + h_N f_N, \end{aligned} \tag{27a}$$

$$R_a\, \pi_0 + R_b\, \pi_N = r, \tag{27b}$$

where (see (26))

$$\begin{aligned} f_\nu &= \sum_{j=1}^{s} \omega_{\nu,j}(t_\nu) P_\nu(t_\nu) P_\nu^{-1}(\tau_{\nu,j})\, g(\tau_{\nu,j}) \\ &= \sum_{j=1}^{s} b_j\, P_\nu(t_\nu) P_\nu^{-1}(\tau_{\nu,j})\, g(\tau_{\nu,j}). \end{aligned} \tag{28}$$

Thus, the π_ν satisfy a linear system with the 'collocation shooting matrix' $\tilde{S}_h$,

$$\underbrace{\begin{pmatrix} -\frac{1}{h_1}P_1(t_1) & \frac{1}{h_1}I & & & \\ & -\frac{1}{h_2}P_2(t_2) & \frac{1}{h_2}I & & \\ & & \ddots & \ddots & \\ & & & \ddots & \ddots \\ & & & -\frac{1}{h_N}P_N(t_N) & \frac{1}{h_N}I \\ R_a & & & & R_b \end{pmatrix}}_{=:\, \tilde{S}_h} \begin{pmatrix} \pi_0 \\ \pi_1 \\ \vdots \\ \vdots \\ \vdots \\ \pi_N \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ \vdots \\ f_N \\ r \end{pmatrix}. \tag{29}$$

This is an approximation to the system of equations satisfied by the exact solution values $y(t_\nu)$ which is directly obtained via local variation-of-constants representations for $y(t)$ involving the local fundamental matrices $Y_\nu(t)$:

$$\begin{aligned} y_1 &= Y_1(t_1)\, y_0 + h_1 \varphi_1, \\ &\vdots \\ y_N &= Y_N(t_N)\, y_{N-1} + h_N \varphi_N, \end{aligned} \tag{30a}$$

$$R_a\, y_0 + R_b\, y_N = r, \tag{30b}$$

with

$$\varphi_\nu = \frac{1}{h_\nu} \int_{t_{\nu-1}}^{t_\nu} Y_\nu(t_\nu) Y_\nu^{-1}(\tau)\, g(\tau)\, d\tau. \tag{31}$$

In matrix representation, this reads

$$\underbrace{\begin{pmatrix} -\frac{1}{h_1}Y_1(t_1) & \frac{1}{h_1}I & & & & \\ & -\frac{1}{h_2}Y_2(t_2) & \frac{1}{h_2}I & & & \\ & & \ddots & \ddots & & \\ & & & \ddots & \ddots & \\ & & & & -\frac{1}{h_N}Y_N(t_N) & \frac{1}{h_N}I \\ R_a & & & & & R_b \end{pmatrix}}_{=:S} \begin{pmatrix} y_0 \\ y_1 \\ \vdots \\ \vdots \\ \vdots \\ y_N \end{pmatrix} = \begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \vdots \\ \vdots \\ \varphi_N \\ r \end{pmatrix}, \tag{32}$$

with the 'exact shooting matrix' S. Now, (22), shows

$$\tilde{S}_h = S + \mathcal{O}(h^s). \tag{33}$$

Conclusions:

- Solving the system (32) would be equivalent to applying an exact multiple shooting method to the given boundary value problem. This motivates the term 'theoretical multiple shooting' for the present style of stability analysis.
- Via (33) and for h sufficiently small, the unique solution of the condensed collocation equations (29) follows from the unique solution of the exact boundary value problem. In fact, (29) may be considered as the system of an approximate multiple shooting method, with collocation approximations for the local fundamental matrices.
- Due to (26) this also yields the unique solution of the original collocation equations (6).
- Last but not least, (33) implies the stability of the collocation system, i.e., the uniform boundedness of $\tilde{S}_h^{-1}$ for $h \to 0$, since

$$\tilde{S}_h^{-1} = S^{-1} + \mathcal{O}(h^s), \quad \text{with} \quad \|S^{-1}\| \le C. \tag{34}$$

We note that, more specifically, S^{-1} from (32) can be expressed by means of Green's matrix (3):

$$S^{-1} = \begin{pmatrix} h_1\,G(t_0,t_1) & \dots & h_N\,G(t_0,t_N) & Y(t_0)[R_a\,Y(a) + R_b\,Y(b)]^{-1} \\ h_1\,G(t_1,t_1) & \dots & h_N\,G(t_1,t_N) & Y(t_1)[R_a\,Y(a) + R_b\,Y(b)]^{-1} \\ \vdots & & \vdots & \vdots \\ h_1\,G(t_N,t_1) & \dots & h_N\,G(t_N,t_N) & Y(t_N)[R_a\,Y(a) + R_b\,Y(b)]^{-1} \end{pmatrix}, \tag{35}$$

where $G(t,t) := \lim_{\tau\uparrow t} G(t,\tau)$. A proof of this identity for $h \equiv h_\nu = const.$ is given in [3]. This can be directly extended to variable h_ν, leading to (35). We see that the stability of the collocation shooting matrix is directly related to the size of Green's matrix and thus, to the stability (conditioning) of the given boundary value problem.

3.3 Pointwise and uniform error estimates

The solution values $y_\nu = y_\nu(t_\nu)$ of the boundary value problem (1) satisfy equations (30), with φ_ν from (31). These can be interpreted as perturbed collocation equations. Inserting (22) into (31) leads to

$$\varphi_\nu = \frac{1}{h_\nu}\int_{t_{\nu-1}}^{t_\nu} P_\nu(t_\nu)P_\nu^{-1}(\tau)g(\tau)\,d\tau + \mathcal{O}(h^{s+1}). \tag{36}$$

Furthermore, approximating the integral (36) by quadrature according to (28) yields

$$\begin{aligned}\varphi_\nu &= \sum_{j=1}^{s} b_j\, P_\nu(t_\nu)P_\nu^{-1}(\tau_{\nu,j})\, g(\tau_{\nu,j}) + \mathcal{O}(h^{s+1}) + \underbrace{\mathcal{O}(h^s)}_{=\,\text{quadrature error}} \\ &= f_\nu + \mathcal{O}(h^s). \end{aligned} \tag{37}$$

With

$$y = \begin{pmatrix} y_0 \\ y_1 \\ \vdots \\ y_N \end{pmatrix}, \qquad f = \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_N \\ r \end{pmatrix},$$

this leads to

$$\big(\tilde{S}_h + \mathcal{O}(h^s)\big)y = f + \mathcal{O}(h^s). \tag{38}$$

On the other hand, for

$$\pi = \begin{pmatrix} \pi_0 \\ \pi_1 \\ \vdots \\ \pi_N \end{pmatrix}$$

we have

$$\tilde{S}_h\, \pi = f. \tag{39}$$

Taking the difference of (39) and (38) we conclude that the error $\pi_\nu - y_\nu$ at the end of the subintervals satisfies the uniform estimate

$$\|\pi_\nu - y_\nu\| = \mathcal{O}(h^s), \quad \nu = 0 \ldots N. \tag{40}$$

Now we consider the error function $e_\nu(t) = p_\nu(t) - y_\nu(t)$ for arbitrary t within an arbitrary subinterval $[t_{\nu-1}, t_\nu]$. From (26) we conclude

$$\begin{aligned} e_\nu(t) &= p_\nu(t) - y(t) \\ &= P_\nu(t)\, \pi_{\nu-1} - Y_\nu(t)\, y_{\nu-1} && \text{(41a)} \\ &\quad + h_\nu \sum_{j=1}^{s} \omega_{\nu,j}(t) \big(P_\nu(t)P_\nu^{-1}(\tau_{\nu,j}) - Y_\nu(t)Y_\nu^{-1}(\tau_{\nu,j})\big)\, g(\tau_{\nu,j}) + \mathcal{O}(h^{s+1}) && \text{(41b)} \\ &= \mathcal{O}(h^s) && \text{(41c)} \end{aligned}$$

uniformly in t, because

$$\begin{aligned} \text{(41a)} &= P_\nu(t)\, \pi_{\nu-1} - Y_\nu(t)\, y_{\nu-1} \\ &= P_\nu(t)\, \pi_{\nu-1} - Y_\nu(t)\, \pi_{\nu-1} + Y_\nu(t)\, \pi_{\nu-1} - Y_\nu(t)\, y_{\nu-1} \\ &= \underbrace{\big(P_\nu(t) - Y_\nu(t)\big)\, \pi_{\nu-1}}_{=\,\mathcal{O}(h^{s+1})} + \underbrace{Y_\nu(t)\, \big(\pi_{\nu-1} - y_{\nu-1}\big)}_{=\,\mathcal{O}(h^s)} \\ &= \mathcal{O}(h^s), \end{aligned} \tag{42}$$

and

$$
\begin{aligned}
(41\text{b}) &= h_\nu \sum_{j=1}^{s} \omega_{\nu,j}(t) \left(P_\nu(t) P_\nu^{-1}(\tau_{\nu,j}) - Y_\nu(t) Y_\nu^{-1}(\tau_{\nu,j}) \right) g(\tau_{\nu,j}) + \mathcal{O}(h^{s+1}) \\
&= \mathcal{O}(h^{s+1})
\end{aligned} \tag{43}
$$

due to (22c). This proves (7a).

Furthermore, for a uniform estimation of the first derivative of the error $e'(t) = p'(t) - y'(t)$ we first consider the error equation at the collocation nodes $\tau_{\nu,i}$,

$$
e'(\tau_{\nu,i}) = A(\tau_{\nu,i})\, e(\tau_{\nu,i}) = \mathcal{O}(1) \cdot \mathcal{O}(h^s) = \mathcal{O}(h^s). \tag{44}
$$

Next, Lagrange interpolation with degree $s-1$ of the $e'(\tau_{\nu,j})$ leads to

$$
e'(t) = p'(t) - y'(t) \approx \sum_{j=1}^{s} e'(\tau_{\nu,j}) L_{\nu,j}(t) =: \varepsilon(t), \tag{45}
$$

with $L_{\nu,j}(t)$ from (11), implying the uniform estimate

$$
\begin{aligned}
e'(t) &= \varepsilon(t) - \text{interpolation error} \\
&= \sum_{j=1}^{s} e'(\tau_{\nu,j}) L_{\nu,j}(t) + (t-\tau_{\nu,1}) \cdots (t-\tau_{\nu,s})\, e'[\tau_{\nu,1} \ldots \tau_{\nu,s}, t] \qquad (46\text{a}) \\
&= \sum_{j=1}^{s} \underbrace{e'(\tau_{\nu,j})}_{=\mathcal{O}(h^s)} L_{\nu,j}(t) - \underbrace{(t-\tau_{\nu,1}) \cdots (t-\tau_{\nu,s})}_{=\mathcal{O}(h^s)}\, y'[\tau_{\nu,1} \ldots \tau_{\nu,s}, t] \qquad (46\text{b}) \\
&= \mathcal{O}(h^s).
\end{aligned}
$$

This proves (7b).

Remarks:

- In the transition from (46a) to (46b), the divided difference $e'[\tau_{\nu,1} \ldots \tau_{\nu,s}, t]$ can be replaced by $y'[\tau_{\nu,1}, \ldots, \tau_{\nu,s}, t]$ because $p'(t)$ is a polynomial of degree $s-1$ and thus, $p'[\tau_{\nu,1}, \ldots, \tau_{\nu,s}, t]$ vanishes.

- Note that $y'[\tau_{\nu,1}, \ldots, \tau_{\nu,s}, t]$ can be estimated in terms of $y^{(s+1)}$.

3.4 Superconvergence

Consider the polynomial quadrature formula over $[0,1]$ of degree $s-1$ based on the nodes c_i, $i = 1 \ldots s$. The degree of exactness is at least $s-1$, but for a special distribution of nodes it may be higher. Assume that that the degree of exactness is[5] $m-1 \geq s-1$. Then, for a sufficiently smooth function $F(\tau)$ we have, with the corresponding quadrature coefficients b_j from (12b) for the interval $[t_{\nu-1}, t_\nu]$, it is well-known that

$$
\int_{t_{\nu-1}}^{t_\nu} F(\tau)\, d\tau = h_\nu \sum_{j=1}^{s} b_j\, F(\tau_{\nu,j}) + \mathcal{O}(h^{m+1}). \tag{47}
$$

[5]For a criterion, special cases (e.g., Gauss quadrature) and error estimates, see e.g. [2].

Now we define the defect as the residual obtained by inserting the collocation solution into the differential equation (1a),

$$\delta_\nu(t) := p_\nu'(t) - A(t)\, p_\nu(t) - g(t), \quad t \in [t_{\nu-1}, t_\nu].$$

Due to the collocation conditions (6) the defect vanishes at the collocation nodes $\tau_{\nu,i}$: $d(\tau_{\nu,i}) = 0$. The error $e(t) = p(t) - y(t)$ is the unique continuous solution of[6]

$$e'(t) = A(t)\, e(t) + \delta_\nu(t), \quad t \in (t_{\nu-1}, t_\nu),\ \nu = 1 \dots N, \tag{48}$$

$$R_a\, e(a) + R_b\, e(b) = 0, \tag{49}$$

and it can be represented as

$$e(t) = \sum_{\mu=1}^{N} \int_{t_{\mu-1}}^{t_\mu} G(t,\tau)\, \delta_\mu(\tau)\, d\tau, \tag{50}$$

where Green's matrix $G(t,\tau)$ is a smooth function (except for $t = \tau$).

Now we consider any particular endpoint $t = t_\nu$ of a subinterval and substitute the integrals in (50) by the quadrature formula with the weights b_j from (12b). With (47) this gives rise to

$$\int_{t_{\mu-1}}^{t_\mu} G(t_\nu,\tau)\, \delta_\mu(\tau)\, d\tau = h_\mu \sum_{j=1}^{s} b_j\, G(t_\nu, \tau_{\mu,j})\, \underbrace{\delta(\tau_{\mu,j})}_{=0} + \mathcal{O}(h^{m+1}) = \mathcal{O}(h^{m+1}). \tag{51}$$

After summation[7] (with $N = \mathcal{O}(h^{-1})$) we obtain an error bound uniformly valid at the endpoints of the subintervals,

$$e(t_\nu) = \mathcal{O}(h^m). \tag{52}$$

We stress that the $\mathcal{O}(h^m)$-term in (52) depends, via estimate for the quadrature error in (51), on a higher derivative of the defect, which has to be uniformly bounded for the estimate to be valid. This can be shown to hold true via an extended error analysis for $e''(t), e'''(t), \dots$ in a similar way as in the above analysis for $e'(t)$.

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[6] We note that the defect is uniformly $\mathcal{O}(h^s)$ due to (7).

[7] Implicitly, this requires that the variation in the length of the subintervals is not too extreme, otherwise relation $N = \mathcal{O}(h^{-1})$ would involve a large constant.