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SPECTRAL ANALYSIS AND LONG-TIME BEHAVIOUR OF A FOKKER-PLANCK EQUATION WITH A NON-LOCAL PERTURBATION

D. STÜRZER, A. ARNOLD

ABSTRACT. In this article we consider a Fokker-Planck equation with a non-local, mass preserving perturbation. We show that the perturbed Fokker-Planck operator generates a C_0 -semigroup on an exponentially weighted L^2 -space. Surprisingly, the spectrum of the Fokker-Planck operator is not affected by the perturbation. In particular there still exists a unique (normalized) stationary solution of the perturbed equation. And we have convergence towards the stationary state with exponential rate -1 , the same rate as for the unperturbed Fokker-Planck equation. Moreover, for any $k \in \mathbb{N}$ there exists an invariant subspace with finite codimension in which the exponential decay rate equals $-k$.

As a byproduct of our analysis we characterize the spectrum of the Fokker-Planck operator in L^2 -spaces with exponential weights.

1. Introduction

This work deals with the analysis of the following class of perturbed Fokker-Planck equations:

$$\partial_t f = \nabla \cdot (\nabla f + \mathbf{x}f) + \Theta f =: Lf + \Theta f \tag{1.1a}$$

$$f|_{t=0} = \varphi(\mathbf{x}), \tag{1.1b}$$

where $t \geq 0$, $\mathbf{x} \in \mathbb{R}^d$ with $d \in \mathbb{N}$, and $f = f(t, \mathbf{x})$. Here, $\partial_t f$ denotes the time derivative. The linear, non-local operator Θ is given by a convolution $\Theta f = \vartheta * f$ with respect to \mathbf{x} , where its kernel ϑ is assumed to be time-independent and massless, i.e. $\int_{\mathbb{R}^d} \vartheta(\mathbf{x}) \, d\mathbf{x} = 0$. Also, it is assumed to satisfy certain regularity conditions, which will be specified in the Sections 3 and 4.

The above equation is mainly motivated by the kinetic Wigner-Fokker-Planck equation, describing so-called open quantum systems, see [3, 4]. It is of the form

$$\begin{aligned} \partial_t u &= \nabla_{\mathbf{x}, \mathbf{v}} \cdot (\nabla_{\mathbf{x}, \mathbf{v}} u + (\nabla_{\mathbf{x}, \mathbf{v}} A + \mathbf{F})u) + \Xi[V]u \\ u|_{t=0} &= u_0, \end{aligned}$$

where $u = u(t, \mathbf{x}, \mathbf{v})$, with $\mathbf{x}, \mathbf{v} \in \mathbb{R}^d$. The coefficient function $\nabla_{\mathbf{x}, \mathbf{v}} A + \mathbf{F}$ is affine in (\mathbf{x}, \mathbf{v}) , and $\Xi[V]$ is a non-local operator (convolution in \mathbf{v}) determined by an external potential $V(\mathbf{x})$. One question of interest in this problem is to show the existence of a unique normalized stationary state, and to prove uniform exponential convergence of the solution to the stationary state. In the case of a quadratic confinement potential with a small perturbation these questions have been answered positively in [3], see also [2] for an operator-theoretic approach. However, from the physical point of view, the restriction to nearly quadratic potentials seems quite artificial. This raises the question if the results can be extended to a more general family of (confining) potentials. In order to gain insight into what can be expected and what mechanisms are responsible for the actual behaviour, we shall consider here (1.1) as a similar, yet simplified model, which still preserves the essential structure. The non-local operator $\Xi[V]$ is replaced by a convolution with kernel ϑ .

Other examples of non-local (and partly non-linear) perturbations in Fokker-Planck equations appear e.g. in the vorticity formulation of the 2D Navier-Stokes equations (cf. (12)-(14) in [14]) or in electronic transport models (cf. equations (1), (6), (7) in [21]).

For the unperturbed equation (1.1), i.e. the case $\vartheta = 0$, the natural functional setting is the space $L^2(\mu^{-1})$, where $\mu(\mathbf{x}) = \exp(-|\mathbf{x}|^2/2)$. Here, $\mu/(2\pi)^{d/2}$ is the unique steady state

with normalized mass, i.e. $\int_{\mathbb{R}^d} \mu / (2\pi)^{d/2} d\mathbf{x} = 1$, and all solutions to initial conditions with normalized mass decay towards this state with exponential rate of at least -1 , see e.g. [5]. However, if Θ is added, the situation often becomes more complicated. One reason is that many non-local (convolution) operators are unbounded in the space $L^2(\mu^{-1})$. This can be illustrated for the simple example with the convolution kernel $\vartheta = \delta_{-\alpha} - \delta_{\alpha}$, $\alpha \in \mathbb{R}$, in one dimension. It corresponds to the operator $(\Theta f)(x) = f(x + \alpha) - f(x - \alpha)$, $x \in \mathbb{R}$, which is unbounded in $L^2(\mu^{-1})$. In this case one can show that every (non-trivial) stationary state of (1.1) is *not* even an element of $L^2(\mu^{-1})$. Thus, this space is not suitable for our intended large-time analysis, since it is “too small”. This motivates to consider (1.1) in some larger space $L^2(\omega)$, with a weight ω growing slower than μ^{-1} . Due to the previous discussion we shall choose ω such that a large class of non-local operators becomes bounded. But the new space should not be “too large” either, since we would risk to lose many convenient (spectral) properties of the unperturbed Fokker-Planck operator. It will turn out that $\omega(\mathbf{x}) := \cosh \beta |\mathbf{x}|$, $\beta > 0$, is a convenient choice. Our spectral analysis of the Fokker-Planck operator in such weighted spaces is based on a technique developed in [15] and [23], which enables us to carry over certain spectral properties of L in $L^2(\mu^{-1})$ to larger spaces $L^2(\omega)$. Here we focus on exponentially weighted spaces, even though our results are applicable to polynomial weights as well. For the latter case the spectrum of L has been studied in more detail in [13]. Furthermore, our results complement the analysis of Metafuno [22], where a larger class of Ornstein-Uhlenbeck operators is investigated in unweighted L^p -spaces with $p \geq 1$.

This paper is organized as follows. Since the analysis in the d -dimensional case is very similar to the one-dimensional case, we first discuss (in Sections 2 and 3) the one-dimensional problem in great detail, to keep the notation and arguments more concise. In Section 4, we generalize the proofs to higher dimensions.

In Section 2 we investigate the one-dimensional Fokker-Planck operator in $L^2(\omega)$ (denoted by \mathcal{L}), and show that its spectrum is $-\mathbb{N}_0$, and consists entirely of eigenvalues. All eigenspaces are one-dimensional, in particular the stationary state is unique up to normalization. Moreover, the operator \mathcal{L} generates a C_0 -semigroup of uniformly bounded operators on $L^2(\omega)$, and any solution of (1.1) for $\Theta = 0$ converges towards the (appropriately scaled) stationary solution with exponential rate of at least -1 . More generally, for any $k \in \mathbb{N}_0$ there exists an \mathcal{L} -invariant subspace of $L^2(\omega)$ with codimension k in which the associated semigroup has an exponential decay rate of $-k$. Section 3 is dedicated to the perturbed Fokker-Planck operator $\mathcal{L} + \Theta$ in one dimension. Using the compactness of the resolvent of \mathcal{L} and ladder operators we show that the spectrum of \mathcal{L} is not affected by the addition of Θ , i.e. $\sigma(\mathcal{L} + \Theta) = \sigma(\mathcal{L}) = -\mathbb{N}_0$. It still consists only of eigenvalues with one-dimensional eigenspaces, which ensures the existence of a unique normalized steady state of (1.1) in $L^2(\omega)$. Finally we show that the semigroup generated by $\mathcal{L} + \Theta$ still has the same decay properties as the one generated by \mathcal{L} , in particular the solutions of (1.1) with normalized mass decay to the stationary state with exponential rate of at least -1 . In Section 5 we present simulation results, which illustrate the decay rates obtained before.

2. The Fokker-Planck Operator in Weighted L^2 -Spaces

Here and in Section 3 we shall consider the one-dimensional Fokker-Planck equation, i.e. $d = 1$. For the Fourier transform we use the convention

$$\mathcal{F}_{x \rightarrow \xi} f \equiv \hat{f}(\xi) := \int_{\mathbb{R}} f(x) e^{-ix\xi} dx.$$

With this scaling we may identify $\hat{f}(0)$ with the *mass* of f .

On a domain $\Omega \subseteq \mathbb{R}$ we call a real-valued function $w \in L^1_{\text{loc}}(\Omega)$ a *weight function* if it is bounded from below by a positive constant a.e. on every compact subset of \mathbb{R} . We denote the corresponding weighted L^p -space by $L^p(\Omega; w) \equiv L^p(\Omega; w(x) dx)$, where $1 \leq p \leq \infty$. The space $L^2(\Omega; w)$ is equipped with the inner product

$$\langle f, g \rangle_{\Omega, w} = \int_{\Omega} f \bar{g} w dx,$$

and the norm $\|\cdot\|_{\Omega, w}$.

Also, we introduce weighted Sobolev spaces. For two weight functions w_0 and w_1 and $1 \leq p \leq \infty$, the space $W^{1,p}(\Omega; w_0, w_1)$ consists of all functions $f \in L^p(\Omega; w_0)$, whose distributional derivative satisfies $f' \in L^p(\Omega; w_1)$. We equip the space $W^{1,2}(\Omega; w_0, w_1)$ with the norm

$$\|f\|_{\Omega, w_0, w_1} := (\|f\|_{\Omega, w_0}^2 + \|f'\|_{\Omega, w_1}^2)^{\frac{1}{2}},$$

see [20]. If $\Omega = \mathbb{R}$ we shall omit the symbol Ω in these notations.

Furthermore, we present some definitions and properties concerning unbounded operators and their spectrum. Let X, \mathcal{X} be Hilbert spaces. If X is continuously and densely embedded in \mathcal{X} , we write $X \hookrightarrow \mathcal{X}$, and $X \hookrightarrow\hookrightarrow \mathcal{X}$ indicates that the embedding is compact. $\mathcal{C}(X)$ denotes the set of all closed operators A in X with dense domain $D(A)$. The set of all bounded operators $A : X \rightarrow \mathcal{X}$ is $\mathcal{B}(X, \mathcal{X})$; if $X = \mathcal{X}$ we just write $\mathcal{B}(X)$. For $A \in \mathcal{C}(X)$ and $B \in \mathcal{B}(X)$, we define $A+B := A|_{D(A)} + B|_{D(A)}$, which is closed with the domain $D(A+B) := D(A)$. A closed, linear subspace $Y \subset X$ is said to be *invariant* under $A \in \mathcal{C}(X)$ (or *A-invariant*) iff $D(A) \cap Y$ is dense in Y and $\text{ran } A|_Y \subset Y$, see e.g. [1]. For an operator $A \in \mathcal{C}(X)$ its range is $\text{ran } A$, its null space is $\ker A$, and its algebraic null space is $M(A) := \bigcup_{k \geq 0} \ker A^k$. For any $\zeta \in \mathbb{C}$ lying in the resolvent set $\rho(A)$, we denote the resolvent by $R_A(\zeta) := (\zeta - A)^{-1}$. The complement of $\rho(A)$ is the spectrum $\sigma(A)$, and $\sigma_p(A)$ is the point spectrum. For an isolated point $\lambda \in \sigma(A)$ the corresponding *spectral projection* $P_{A,\lambda}$ is defined via the line integral

$$P_{A,\lambda} := \frac{1}{2\pi i} \int_{\Gamma} R_A(\zeta) d\zeta, \quad (2.1)$$

where Γ is a closed, sufficiently small Jordan curve around λ with counter-clockwise orientation, not enclosing or crossing any other point of $\sigma(A)$. Therefore, $P_{A,\lambda}$ is the residue of $R_A(\zeta)$ at its singularity at $\zeta = \lambda$. The spectral projection is a bounded projection operator, decomposing X into two A -invariant subspaces, namely $\text{ran } P_{A,\lambda}$ and $\ker P_{A,\lambda}$, cf. [29, Section V.9]. This property is referred to as the *reduction of A by $P_{A,\lambda}$* . A remarkable property of this decomposition is the fact that $\sigma(A|_{\text{ran } P_{A,\lambda}}) = \{\lambda\}$ and $\sigma(A|_{\ker P_{A,\lambda}}) = \sigma(A) \setminus \{\lambda\}$, see [19, Section III.6.4] and Appendix A.

It is convenient to define the open half plane $\Delta_a := \{z \in \mathbb{C} : \text{Re } z > a\}$ for $a \in \mathbb{R}$. When we introduce, for fixed $N \in \mathbb{N}_0$, constants $\zeta_0, \dots, \zeta_{N-1} \in \mathbb{C}$, we use the convention $\{\zeta_0, \dots, \zeta_{N-1}\} := \emptyset$ whenever $N = 0$. For a closed operator A we make the following definitions:

Definition 2.1 (Localization of the spectrum). Let $a \in \mathbb{R}$ and $\zeta_0, \dots, \zeta_{N-1} \in \Delta_a$ for $N \in \mathbb{N}_0$ given. Then we say that $A \in \mathcal{C}(X)$ satisfies the

- *weak localization of the spectrum (L1)* iff $\sigma(A) \cap \Delta_a = \{\zeta_0, \dots, \zeta_{N-1}\}$.
- *strong localization of the spectrum (L1')* iff A satisfies (L1), and furthermore the ranges of the spectral projections associated to the ζ_k are finite-dimensional.

Remark 2.2. The condition (L1') implies $\{\zeta_0, \dots, \zeta_{N-1}\} \subset \sigma_p(A)$, see Proposition A.2 (iv).

Definition 2.3 (Decomposition of A). Let two Hilbert spaces $X \hookrightarrow \mathcal{X}$ and scalars $a \in \mathbb{R}$ and $\zeta_0, \dots, \zeta_{N-1} \in \Delta_a$ be given for $N \in \mathbb{N}_0$. Then we say that an operator $\mathcal{A} \in \mathcal{C}(\mathcal{X})$ satisfies the *decomposition property (L2)* if there exist $\mathcal{B} \in \mathcal{B}(\mathcal{X}, X)$ and $\mathcal{S} \in \mathcal{C}(\mathcal{X})$ with $D(\mathcal{A}) = D(\mathcal{S})$, such that

- (i) $\mathcal{A} = \mathcal{B} + \mathcal{S}$,
- (ii) $\sigma(\mathcal{S}) \cap \Delta_a \subseteq \{\zeta_0, \dots, \zeta_{N-1}\}$,
- (iii) there is some $\lambda \in \Delta_a$ such that $\lambda - \mathcal{A}$ is injective in \mathcal{X} .

We begin our analysis by investigating the unperturbed one-dimensional Fokker-Planck operator $Lf := f'' + xf' + f$ in various weighted spaces. The natural space to consider L in is $E := L^2(1/\mu)$ with $\mu(x) := \exp(-x^2/2)$. We use the notation $\|\cdot\|_E$ for the norm and $\langle \cdot, \cdot \rangle_E$ for

the inner product. Writing the operator in the form

$$Lf = \left(\left(\frac{f}{\mu} \right)' \mu \right)'$$

shows that $L|_{C_0^\infty}$ is symmetric and dissipative in E . Then, the proper definition of L is obtained by the closure of $L|_{C_0^\infty}$, and this procedure yields its domain $D(L) \subset E$. In the subsequent theorem we summarize some important properties of L in E , see [22, 5, 18]. Since L in E is isometrically equivalent to the (dimensionless) quantum harmonic oscillator Hamiltonian $H = -\Delta - 1/2 + x^2/4$ in $L^2(\mathbb{R})$, we transfer many results of H (see [25] and [27, Theorem XIII.67]) to L . For the properties of the spectral projections, see also Proposition A.3.

Theorem 2.4. *The Fokker-Planck operator L in E has the following properties:*

- (i) L with $D(L) = \{f \in E : Lf \in E\}$ is self-adjoint and has a compact resolvent.
- (ii) The spectrum is $\sigma(L) = -\mathbb{N}_0$, and it consists only of eigenvalues.
- (iii) For each eigenvalue $-k \in \sigma(L)$ the corresponding eigenspace is one-dimensional, spanned by $\mu_k := \frac{1}{\sqrt{2\pi}} H_k \mu$, where

$$H_k(x) = \mu(x)^{-1} \frac{d^k}{dx^k} \mu(x)$$

is the k -th Hermite polynomial.

- (iv) The eigenvectors $(\mu_k)_{k \in \mathbb{N}_0}$ form an orthogonal basis of E .
- (v) There holds the spectral representation

$$L = \sum_{k \in \mathbb{N}_0} -k \Pi_{L,k}, \quad \text{where } \Pi_{L,k} := \frac{\sqrt{2\pi}}{k!} \mu_k \langle \cdot, \mu_k \rangle_E$$

is the spectral projection onto the k -th eigenspace.

- (vi) The operator L generates a C_0 -semigroup of contractions on E_k for all $k \in \mathbb{N}_0$, where $E_k := \ker(\Pi_{L,0} + \dots + \Pi_{L,k-1})$, $k \geq 1$, and $E_0 := E$. The semigroup satisfies the estimate

$$\|e^{tL}|_{E_k}\|_{\mathcal{B}(E_k)} \leq e^{-kt}, \quad \forall k \in \mathbb{N}_0.$$

Hence, the Fokker-Planck equation $\partial_t f = Lf$ has a unique stationary solution with normalized mass, given by μ_0 . Its orthogonal complement E_1 consists of all elements of E with zero mass. And according to Result (vi) for $k = 1$, any solution of $\partial_t f = Lf$ with unit mass converges towards μ_0 with exponential rate of at least -1 in the E -norm.

In order to analyze the perturbed equation (1.1), it is convenient that the non-local operator Θ is bounded in the considered L^2 -space. But, as mentioned in Section 1, even a simple shift operator is unbounded in E . This motivates to consider some larger space $L^2(\omega)$ instead of E , with a weight function ω growing more slowly than μ^{-1} such that Θ becomes a bounded operator in $L^2(\omega)$. At the same time, ω should be such that L still has a spectral gap in $L^2(\omega)$, i.e. there exists some $a < 0$ such that $\Delta_a \cap \sigma(L) = \{0\}$. The following theorem is the key ingredient to finding an appropriate weight ω :

Theorem 2.5. *Let $X \hookrightarrow \mathcal{X}$ be two Hilbert spaces, and $a \in \mathbb{R}$ and $\zeta_0, \dots, \zeta_{N-1} \in \Delta_a$ for some $N \in \mathbb{N}_0$. Consider an operator $A \in \mathcal{C}(X)$ with the following properties in X :*

- (i) A satisfies **(L1)** for the ζ_k and a , and $\dim M(\zeta_k - A) < \infty$ for all $0 \leq k \leq N-1$.
- (ii) The space $X_N := [M(\zeta_0 - A) \oplus \dots \oplus M(\zeta_{N-1} - A)]^\perp$ is invariant under A .
- (iii) $A - a$ is dissipative in X_N .
- (iv) $\text{ran}(A - b)|_{X_N} = X_N$ for some $b > a$.

Furthermore, assume in \mathcal{X} :

- (v) There exist $\mathcal{B} \in \mathcal{B}(\mathcal{X}, X)$ and an operator $\mathcal{S} \in \mathcal{C}(\mathcal{X})$ with the property that $\mathcal{S} - a$ is dissipative, and there holds the decomposition $A \subset \mathcal{B} + \mathcal{S}$.

Then we have:

1. A satisfies **(L1')** for the constants a and ζ_k , and $X_N = \ker(\Pi_{A,0} + \dots + \Pi_{A,N-1})$, where $\Pi_{A,k}$ denotes the spectral projection of A corresponding to ζ_k .
2. A generates a C_0 -semigroup on X and X_N satisfying $\|e^{tA}|_{X_N}\|_{\mathcal{B}(X_N)} \leq e^{at}$.
3. A is closable in \mathcal{X} , its closure $\mathcal{A} := \text{cl}_{\mathcal{X}} A$ also satisfies **(L1')** for a and the ζ_k . The corresponding algebraic eigenspaces $M(\zeta_k - \mathcal{A})$ coincide with the $M(\zeta_k - A) \subset X$.
4. For $0 \leq k \leq N-1$, the spectral projection $\Pi_{\mathcal{A},k}$ of \mathcal{A} corresponding to $\zeta_k \in \sigma(\mathcal{A})$ equals the closure of $\Pi_{A,k}$. In particular, $\text{ran } \Pi_{\mathcal{A},k} = \text{ran } \Pi_{A,k} = M(\zeta_k - A)$, and

$$\ker \Pi_{\mathcal{A},k} = \mathcal{X}_N \oplus \left(\bigoplus_{j \neq k} M(\zeta_j - A) \right),$$

where $\mathcal{X}_N := \text{cl}_{\mathcal{X}} X_N$.

5. \mathcal{A} generates a C_0 -semigroup on \mathcal{X} and \mathcal{X}_N , and for any $\tilde{a} > a$ there exists some $C_{\tilde{a}} \geq 1$ such that

$$\|e^{t\mathcal{A}}|_{\mathcal{X}_N}\|_{\mathcal{B}(\mathcal{X}_N)} \leq C_{\tilde{a}} e^{\tilde{a}t}, \quad \forall t \geq 0.$$

This result is based upon [15, Corollary 4.2]. Also, in [3] a simpler version of that theory was applied to the Wigner-Fokker-Planck operator. The proof of the above theorem requires several quite technical steps, so it is deferred to Appendix A.

Next we apply this theorem to the Fokker-Planck operator L . Our aim is to find a weight function ω such that the closure \mathcal{L} of L in $L^2(\omega)$ still satisfies $\sigma(\mathcal{L}) = \sigma(L)$ and that the semigroup $(e^{t\mathcal{L}})_{t \geq 0}$ still fulfills the same decay rates on the subspaces \mathcal{X}_k as $(e^{tL})_{t \geq 0}$ on X_k . In order to do so we are looking for operators \mathcal{B}, \mathcal{S} , such that the requirements of Theorem 2.5 are fulfilled for any $a \in \mathbb{R}$. The following lemma yields the appropriate condition on the weight function.

Lemma 2.6. *For any $a < 0$ there exists a weight function $\omega(x) \in C^2(\mathbb{R})$ that satisfies the conditions*

$$\begin{aligned} \exists R > 0 : \forall |x| > R : \quad \omega''(x) - x\omega'(x) + (1 - 2a)\omega(x) &\leq 0, \\ \exists \delta_2 > \delta_1 > 0 : \forall x \in \mathbb{R} : \quad \delta_1 \leq \omega(x) \leq \delta_2/\mu(x). \end{aligned} \quad (2.2)$$

With a weight $\omega(x)$ which satisfies this condition for some fixed $a < 0$, the Fokker-Planck operator L is closable in $\mathcal{E} := L^2(\omega)$, and its closure \mathcal{L} has the following properties (where $N := -\lfloor a \rfloor \in \mathbb{N}$, the floor function of a):

- (i) The spectrum satisfies $\sigma(\mathcal{L}) \cap \Delta_a = \{0, -1, \dots, -N+1\}$, and $M(\mathcal{L} + k) = \ker(\mathcal{L} + k) = \text{span}\{\mu_k\}$ for $0 \leq k \leq N-1$.
- (ii) For any $0 \leq k \leq N$ the closed subspace $\mathcal{E}_k := \text{cl}_{\mathcal{E}} \text{span}\{\mu_k, \mu_{k+1}, \dots\}$ is an \mathcal{L} -invariant subspace of \mathcal{E} , and $\text{span}\{\mu_0, \dots, \mu_{k-1}\}$ is a complement.
- (iii) The spectral projection $\Pi_{\mathcal{L},k}$ corresponding to the eigenvalue $-k \in -\mathbb{N}_0 \cap \Delta_a$ fulfills $\text{ran } \Pi_{\mathcal{L},k} = \text{span}\{\mu_k\}$ and $\ker \Pi_{\mathcal{L},k} = \mathcal{E}_{k+1} \oplus \text{span}\{\mu_{k-1}, \dots, \mu_0\}$.
- (iv) For any $0 \leq k \leq N-1$, the operator \mathcal{L} generates a C_0 -semigroup on \mathcal{E}_k , and there exists a constant $C_k \geq 1$ such that we have the estimate

$$\|e^{t\mathcal{L}}|_{\mathcal{E}_k}\|_{\mathcal{B}(\mathcal{E}_k)} \leq C_k e^{-kt}, \quad \forall t \geq 0.$$

Proof. First we note that for any $a < 0$ the function $\omega(x) = 1 + x^{2(-\lfloor a \rfloor + 1)}$ is a weight function that fulfills the condition (2.2).

The proof of the main results (i)-(iv) is achieved by application of Theorem 2.5 for $X = E$, $\mathcal{X} = \mathcal{E}$, $A = L$ and $\zeta_k = -k$. Due to the results of Theorem 2.4 the assumptions (i)-(iv) of Theorem 2.5 are obviously fulfilled. It still remains to show the requirement (v). In order to find an appropriate decomposition $L \subset \mathcal{B} + \mathcal{S}$ we begin by evaluating $\text{Re}\langle Lf, f \rangle_{\omega}$ for $f \in C_0^{\infty}(\mathbb{R})$:

$$\begin{aligned} \text{Re}\langle Lf, f \rangle_{\omega} &= \text{Re} \int_{\mathbb{R}} \left(\left(\frac{f}{\mu} \right)' \mu \right)' \bar{f} \omega \, dx \\ &= - \text{Re} \int_{\mathbb{R}} (xf + f')(\bar{f} \omega' + \omega \bar{f}') \, dx \end{aligned}$$

$$= - \int_{\mathbb{R}} \omega |f'|^2 dx + \frac{1}{2} \int_{\mathbb{R}} (\omega'' - x\omega' + \omega) |f|^2 dx.$$

Now we want to find $\mathcal{B} \in \mathcal{B}(\mathcal{E}, E)$ such that $\mathcal{S}|_{C_0^\infty} - a \equiv L|_{C_0^\infty} - \mathcal{B} - a$ is dissipative in \mathcal{E} for some fixed $a < 0$. According to the previous calculation this is the case iff there holds for all $f \in C_0^\infty(\mathbb{R})$:

$$\operatorname{Re}\langle Lf, f \rangle_\omega = - \int_{\mathbb{R}} \omega |f'|^2 dx + \frac{1}{2} \int_{\mathbb{R}} (\omega'' - x\omega' + \omega) |f|^2 dx \leq a \int_{\mathbb{R}} |f|^2 \omega dx + \operatorname{Re} \int_{\mathbb{R}} (\mathcal{B}f) \bar{f} \omega dx. \quad (2.3)$$

Let now $\omega \in C^2(\mathbb{R})$ be a weight function satisfying (2.2) for a and some $R > 0$ sufficiently large. It remains to find an appropriate bounded operator \mathcal{B} . For functions f with $\operatorname{supp} f \subset \mathbb{R} \setminus [-R-1, R+1]$ we allow \mathcal{B} to be zero, since (2.3) is already fulfilled due to (2.2). Hence, we choose \mathcal{B} as the multiplication $\mathcal{B}f := \delta_2 \eta_R f$, where $\eta_R \in C^2(\mathbb{R})$ with $\eta_R \geq 0$, $\operatorname{supp} \eta_R \subset [-R-1, R+1]$ and $\eta_R(x) \equiv 1$, $\forall |x| \leq R$. Due to the cutoff there holds $\mathcal{B} \in \mathcal{B}(\mathcal{E}, E)$. In order to choose the constant $\delta_2 > 0$ we observe that due to the assumption $\omega \in C^2(\mathbb{R})$ there exists some $M \in \mathbb{R}$ such that

$$\sup_{x \in \mathbb{R}} \frac{1}{2} (\omega'' - x\omega' + (1-2a)\omega) =: M_\omega < \infty.$$

Since also $\omega \geq \delta_1$, we may choose $\delta_2 := \max\{0, M_\omega/\delta_1\}$, and (2.3) holds for all $f \in C_0^\infty$. With this choice for \mathcal{B} the operator $\mathcal{S}|_{C_0^\infty} - a$ is dissipative, thus closable in \mathcal{E} . So $L|_{C_0^\infty} = \mathcal{B} + \mathcal{S}|_{C_0^\infty}$ is closable as well, and we define $\mathcal{S} := \operatorname{cl}_{\mathcal{E}} \mathcal{S}|_{C_0^\infty}$ and $\mathcal{L} := \operatorname{cl}_{\mathcal{E}} L|_{C_0^\infty} = \mathcal{B} + \mathcal{S} \supset L$. In particular, we have determined \mathcal{B}, \mathcal{S} such that condition (v) of Theorem 2.5 is fulfilled, so that we may now apply this theorem to L :

The property (i) follows directly from Result 3 of Theorem 2.5 and the spectral properties of L in E given in Theorem 2.4 (ii)-(iii). Next we prove the Results (ii) and (iii) simultaneously. For this we apply Theorem 2.5, Result 4, where X_N corresponds to $E_N = \operatorname{cl}_E \operatorname{span}\{\mu_N, \mu_{N+1}, \dots\}$, see Theorem 2.4. We obtain that for $0 \leq k \leq N-1$ the null space of the spectral projection $\Pi_{\mathcal{L},k}$ is given by $\mathcal{E}_N \oplus \operatorname{span}\{\mu_j : 0 \leq j \leq N-1, j \neq k\}$, where $\mathcal{E}_N := \operatorname{cl}_{\mathcal{E}} E_N$. Then we define $\mathcal{E}_k := \mathcal{E}_N \oplus \{\mu_{N-1}, \dots, \mu_k\} = \operatorname{cl}_{\mathcal{E}} \operatorname{span}\{\mu_k, \mu_{k+1}, \dots\}$ for all $0 \leq k \leq N$, which is \mathcal{L} -invariant because \mathcal{E}_N is. Using the \mathcal{E}_k , the null space of $\Pi_{\mathcal{L},k}$ can be rewritten as $\ker \Pi_{\mathcal{L},k} = \mathcal{E}_{k+1} \oplus \{\mu_{k-1}, \dots, \mu_0\}$, for $0 \leq k \leq N-1$, which yields the desired result for the spectral projection.

Finally, Result (iv) follows from Theorem 2.5, Result 5. First, we obtain that for any $\bar{a} \in (a, -N+1]$ there exists a constant $C_{\bar{a}} \geq 1$ such that we have

$$\|e^{t\mathcal{L}}|_{\mathcal{E}_N}\|_{\mathcal{B}(\mathcal{E}_N)} \leq C_{\bar{a}} e^{\bar{a}t}, \quad \forall t \geq 0. \quad (2.4)$$

If we consider the semigroup on $\mathcal{E}_k = \mathcal{E}_N \oplus \{\mu_{N-1}, \dots, \mu_k\}$ instead, we notice that \mathcal{L} generates a C_0 -semigroup with decay rate $-k$ on the finite dimensional, \mathcal{L} -invariant space $\{\mu_{N-1}, \dots, \mu_k\}$, for all $0 \leq k \leq N-1$. Combining this fact with the estimate (2.4) on \mathcal{E}_N completes the proof. \square

Remark 2.7. For $a < 0$ given, the polynomial weight function $\omega(x) = 1 + x^{2(-[a]+1)}$ fulfills (2.2), thus the conclusions of Lemma 2.6 apply to the Fokker-Planck operator \mathcal{L} in $L^2(\omega)$ for $N = -[a]$. Therefore, the spectrum of \mathcal{L} in the half plane Δ_a is given by $\{0, -1, \dots, -N+1\}$. These results are complemented in [13, Appendix A], where an exhaustive analysis of the spectral properties of \mathcal{L} in polynomially weighted spaces is given, and the complete spectrum is determined. For the one-dimensional case see also [12].

In the following we turn our attention to \mathcal{L} in exponentially weighted spaces. In this case we are able to completely characterize the spectrum, and deduce the following result:

Theorem 2.8. *Let $\omega(x) := \exp(\beta|x|^\gamma)$ for some $\gamma \in (0, 2]$ and $\beta > 0$. If $\gamma = 2$, we additionally require $\beta \leq \frac{1}{2}$. Then the closure $\mathcal{L} = \operatorname{cl}_{\mathcal{E}} L$ has the following properties in $\mathcal{E} = L^2(\omega)$:*

- (i) *The spectrum satisfies $\sigma(\mathcal{L}) = -\mathbb{N}_0$, and $M(\mathcal{L} + k) = \ker(\mathcal{L} + k) = \operatorname{span}\{\mu_k\}$ for any $k \in \mathbb{N}_0$. The eigenfunctions satisfy the relation $\mu_k = \mu_0^{(k)}$, the k -th derivative of μ_0 .*

- (ii) For any $k \in \mathbb{N}_0$ the closed subspace $\mathcal{E}_k := \text{cl}_{\mathcal{E}} \text{span}\{\mu_k, \mu_{k+1}, \dots\}$ is an \mathcal{L} -invariant subspace of \mathcal{E} , and $\text{span}\{\mu_0, \dots, \mu_{k-1}\}$ is a complement. In particular $\mathcal{E}_0 = \mathcal{E}$.
- (iii) The spectral projection $\Pi_{\mathcal{L},k}$ to the eigenvalue $-k \in -\mathbb{N}_0$ fulfills $\text{ran } \Pi_{\mathcal{L},k} = \text{span}\{\mu_k\}$ and $\ker \Pi_{\mathcal{L},k} = \mathcal{E}_{k+1} \oplus \text{span}\{\mu_{k-1}, \dots, \mu_0\}$.
- (iv) For any $k \in \mathbb{N}_0$ the operator \mathcal{L} generates a C_0 -semigroup on \mathcal{E}_k , and there exists a constant $C_k \geq 1$ such that we have the estimate

$$\|e^{t\mathcal{L}}|_{\mathcal{E}_k}\|_{\mathcal{B}(\mathcal{E}_k)} \leq C_k e^{-kt}, \quad \forall t \geq 0.$$

Proof. This is a direct application of Lemma 2.6. A short calculation shows that $\omega(x) = \exp(\beta|x|^\gamma)$ fulfills (2.2) for any $a < 0$, if the assumptions on β and γ hold. So we may take N arbitrarily large in Lemma 2.6. \square

Remark 2.9. The admissible weight functions from Theorem 2.8 coincide with those considered in Proposition 5.4 (i) of [15] (for the special case of a quadratic confinement potential). In comparison to Theorem 5.5 of [15], Theorem 2.8 even shows the independence of $\sigma(\mathcal{L})$ for a certain class of weight functions.

Remark 2.10. For $\gamma = 0$ in Theorem 2.8 (i.e. $\mathcal{E} = L^2(\mathbb{R})$) the situation changes completely. From Theorem 4.4 in [22] we know: In $L^2(\mathbb{R})$ there holds $\sigma(\mathcal{L}) = \{z \in \mathbb{C} : \text{Re } z \leq \frac{1}{2}\}$ and every $z \in \mathbb{C}$ with $\text{Re } z < \frac{1}{2}$ is an eigenvalue.

Next we characterize the subspaces \mathcal{E}_k . For this purpose we need the following technical lemma:

Lemma 2.11. *Let $X \hookrightarrow \mathcal{X}$ be Hilbert spaces, and $\psi_0, \dots, \psi_{k-1} \in \mathcal{B}(\mathcal{X}, \mathbb{C})$, $k \in \mathbb{N}$, be linearly independent functionals. Then $\tilde{\psi}_j := \psi_j|_X \in \mathcal{B}(X, \mathbb{C})$ for all $0 \leq j \leq k-1$, and*

$$\bigcap_{j=0}^{k-1} \ker \psi_j = \text{cl}_{\mathcal{X}} \bigcap_{j=0}^{k-1} \ker \tilde{\psi}_j.$$

Proof. The boundedness of the $\tilde{\psi}_j$ is an immediate consequence of $X \hookrightarrow \mathcal{X}$. In order to show the second statement, we notice that according to the Riesz representation theorem there exists a unique $x_j \in X$ such that $\tilde{\psi}_j(\cdot) = \langle \cdot, x_j \rangle_X$ for every $0 \leq j \leq k-1$, where $\langle \cdot, \cdot \rangle_X$ denotes the inner product in X . The set $\{x_0, \dots, x_{k-1}\}$ is linearly independent, because the corresponding functionals are. We now apply the Gram-Schmidt process to $\{x_0, \dots, x_{k-1}\}$ to obtain the orthonormal family $\{\hat{x}_0, \dots, \hat{x}_{k-1}\}$ with same linear hull. As a consequence, there exists a regular matrix $\Lambda := (\lambda_\ell^j)_{\ell,j} \in \mathbb{C}^{k \times k}$ such that $\hat{x}_\ell = \sum_{j=0}^{k-1} \lambda_\ell^j x_j$. With this we get

$$\hat{x}_\ell \langle \cdot, \hat{x}_\ell \rangle_X = \sum_{i,j=0}^{k-1} \lambda_\ell^i \bar{\lambda}_\ell^j x_i \langle \cdot, x_j \rangle_X = \sum_{i,j=0}^{k-1} \lambda_\ell^i \bar{\lambda}_\ell^j x_i \tilde{\psi}_j(\cdot), \quad 0 \leq \ell \leq k-1.$$

We may now define the orthogonal projection

$$P_X := \sum_{\ell=0}^{k-1} \hat{x}_\ell \langle \cdot, \hat{x}_\ell \rangle_X = \sum_{i,j,\ell=0}^{k-1} \lambda_\ell^i \bar{\lambda}_\ell^j x_i \tilde{\psi}_j(\cdot). \quad (2.5)$$

It can naturally be extended to a projection $P_{\mathcal{X}}$ in \mathcal{X} by replacing the $\tilde{\psi}_j$ by ψ_j . Since $\psi_j \in \mathcal{B}(\mathcal{X}, \mathbb{C})$ for all $0 \leq j \leq k-1$, there follows $P_{\mathcal{X}} \in \mathcal{B}(\mathcal{X})$ from (2.5). Now we apply Lemma A.6 to $P_X \subset P_{\mathcal{X}}$ to obtain $\ker P_{\mathcal{X}} = \text{cl}_{\mathcal{X}} \ker P_X$.

Now it remains to characterize the kernels of the projections. Due to (2.5) we have $P_X f = 0$ in X iff

$$\sum_{j=0}^{k-1} \tilde{\psi}_j(f) \sum_{\ell=0}^{k-1} \lambda_\ell^i \bar{\lambda}_\ell^j = 0, \quad 0 \leq i \leq k-1, \quad (2.6)$$

since the vectors x_i are linearly independent. We note that the sums $\sum_{\ell=0}^{k-1} \lambda_\ell^i \bar{\lambda}_\ell^j$ for $0 \leq i, j \leq k-1$ are the elements of the matrix $\Lambda_2 := \Lambda \Lambda^*$, where Λ^* is the Hermitian conjugate of Λ .

Since Λ_2 is regular, it follows that (2.6) holds iff $\tilde{\psi}_j(f) = 0$ for all $0 \leq j \leq k-1$. The proof of $P_{\mathcal{X}}f = 0$ iff $\psi_j(f) = 0$ for all $0 \leq j \leq k-1$ is analogous. \square

Proposition 2.12. *Under the assumptions of Theorem 2.8 and for any $k \in \mathbb{N}$, the subspace \mathcal{E}_k is explicitly given by*

$$\mathcal{E}_k = \left\{ f \in \mathcal{E} : \int_{\mathbb{R}} f(x)x^j dx = 0, 0 \leq j \leq k-1 \right\}. \quad (2.7)$$

Furthermore, there holds

$$\mathcal{E}_k = \left\{ f \in \mathcal{E} : \hat{f}^{(j)}(0) = 0, 0 \leq j \leq k-1 \right\}, \quad (2.8)$$

where $\hat{f}^{(j)}$ denotes the j -th derivative of the Fourier transform of f .

Proof. The functionals $\psi_j : f \mapsto \int_{\mathbb{R}} f(x)x^j dx$, $j \in \mathbb{N}_0$, are continuous in \mathcal{E} . We define $\tilde{\psi}_j := \psi_j|_E$. Let $f \in E_k = \{\mu_0, \dots, \mu_{k-1}\}^{\perp_E}$. The orthogonality condition then reads

$$0 = \langle f, \mu_j \rangle_E = \int_{\mathbb{R}} f(x)\mu_j(x)\mu(x)^{-1} dx = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(x)H_j(x) dx, \quad \forall 0 \leq j \leq k-1,$$

which is equivalent to $\tilde{\psi}_0(f) = \dots = \tilde{\psi}_{k-1}(f) = 0$. Applying Lemma 2.11 with $\mathcal{X} = \mathcal{E}$ and $X = E$ yields $\text{cl}_{\mathcal{E}} E_k = \{f \in \mathcal{E} : \psi_j(f) = 0, 0 \leq j \leq k-1\}$, which is equal to \mathcal{E}_k by definition. This proves (2.7).

The second equality (2.8) immediately follows from

$$\int_{\mathbb{R}} f(x)x^j dx = \mathcal{F}_{x \rightarrow \xi}[f(x)x^j](0) = i^j \hat{f}^{(j)}(0), \quad \forall j \in \mathbb{N}_0.$$

\square

Remark 2.13. The representation (2.7) of the \mathcal{E}_k also holds in polynomially weighted spaces, which is shown in [13, Appendix A].

As stated in the introduction, our goal is to choose a weight function ω such that \mathcal{L} still has a spectral gap and a “large class” of perturbation operators Θ is bounded in $L^2(\omega)$. E.g., one can easily verify that $\Theta f(x) := f(x+\alpha) - f(x-\alpha)$ is bounded in $L^2(\exp(\beta|x|^\gamma))$ iff $\gamma \in [0, 1]$ (for $\beta > 0$). It is thus convenient to make the choice $\gamma = 1$. So, from now on, we shall fix the weight $\omega(x) := \cosh \beta x$, for some $\beta > 0$. The corresponding space is $\mathcal{E} = L^2(\omega)$, and we adopt the rest of the notation from Theorem 2.8.

With this weight function, the elements of \mathcal{E} have a Fourier transform that extends analytically to a strip around the real axis:

Lemma 2.14. *For $f \in \mathcal{E}$ we have the following properties:*

- (i) *There holds $f \in \mathcal{E}$ iff its Fourier transform \hat{f} possesses an analytic continuation (still denoted by \hat{f}) to the open strip $\Omega_{\beta/2} := \{z \in \mathbb{C} : |\text{Im } z| < \beta/2\}$, which satisfies*

$$\sup_{\substack{|b| < \beta/2 \\ b \in \mathbb{R}}} \|\hat{f}(\cdot + ib)\|_{L^2} < \infty.$$

- (ii) *For $\xi \in \mathbb{R}$ and $|b| < \beta/2$, \hat{f} is explicitly given by $\hat{f}(\xi + ib) = \mathcal{F}_{x \rightarrow \xi}(e^{bx} f(x))$.*
 (iii) *The following function lies in $L^2(\mathbb{R})$:*

$$\xi \mapsto \hat{f}\left(\xi \pm i\frac{\beta}{2}\right) := \mathcal{F}_{x \rightarrow \xi}(e^{\pm \frac{\beta}{2}x} f(x)), \quad \text{for a.e. } \xi \in \mathbb{R}. \quad (2.9)$$

Then $b \mapsto \hat{f}(\cdot + ib)$ lies in $C([-\beta/2, \beta/2]; L^2(\mathbb{R}))$

Proof. According to [26, Theorem IX.13], \hat{f} is analytic in $\Omega_{\beta/2}$, and due to part (b) of the proof of this theorem, Result (ii) follows. Next we show (iii). For $f \in \mathcal{E}$ there holds $f(x)e^{\pm \frac{\beta}{2}x} \in L^2(\mathbb{R})$, and therefore $\hat{f}(\xi \pm i\beta/2)$ as defined in (2.9) is again an element of $L^2(\mathbb{R})$. Since the map $b \mapsto \|f(x)e^{bx}\|_{L^2(\mathbb{R})}$ is continuous on $[-\beta/2, \beta/2]$, Result (iii) follows from Plancherel’s formula

together with Result (ii) and (2.9). Finally, (i) follows from [26, Theorem IX.13] together with (iii). \square

In the following, \hat{f} always denotes the extension of the Fourier transform of $f \in \mathcal{E}$ according to Lemma 2.14 (ii)-(iii). Using this convention, we introduce an alternative norm on the space \mathcal{E} :

$$\|f\|_{\omega}^2 := \|\hat{f}(\cdot + i\beta/2)\|_{L^2}^2 + \|\hat{f}(\cdot - i\beta/2)\|_{L^2}^2, \quad (2.10)$$

which is equal to $4\pi\|f\|_{\omega}^2$.

Finally we notice that there holds a Poincaré-type inequality in \mathcal{E} :

Lemma 2.15 (Poincaré inequality). *The inequality*

$$\|f\|_{\omega} \leq C_{\beta}\|f'\|_{\omega} \quad (2.11)$$

holds for all $f \in W^{1,2}(\omega, \omega)$, where $C_{\beta} > 0$ is a constant only depending on β .

Proof. Use $|\widehat{f'}(\xi)| = |\xi\hat{f}(\xi)|$, and $|\xi| \geq \beta/2$ on $|\operatorname{Im} \xi| = \beta/2$. Then apply the norm $\|\cdot\|_{\omega}$. \square

Next we prove the compactness of the resolvent $R_{\mathcal{L}}(\zeta)$. To this end we shall use the following simplified version of [24, Theorem 2.4]:

Lemma 2.16. *Let w, w_0, w_1 be weight functions, and $(\Omega_n)_{n \in \mathbb{N}}$ a monotonically increasing sequence of subsets of \mathbb{R} that converges to \mathbb{R} . Assume that for all $n \in \mathbb{N}$ there holds the compact embedding $W^{1,2}(\Omega_n; w_0, w_1) \hookrightarrow L^2(\Omega_n; w)$. Then*

$$W^{1,2}(w_0, w_1) \hookrightarrow L^2(w) \iff \lim_{n \rightarrow \infty} \sup_{\|f\|_{w_0, w_1} \leq 1} \|f\|_{\mathbb{R} \setminus \Omega_n; w} = 0.$$

From this we deduce immediately the following lemma:

Lemma 2.17. *Let w, w_0, w_1 be weight functions. If $\lim_{|x| \rightarrow \infty} w(x)/w_0(x) = 0$, then the compact embedding holds:*

$$W^{1,2}(w_0, w_1) \hookrightarrow L^2(w).$$

This compact embedding allows to prove that $R_{\mathcal{L}}(\zeta)$ is compact:

Theorem 2.18. *For any $\zeta \in \rho(\mathcal{L})$ the resolvent operator $R_{\mathcal{L}}(\zeta) : \mathcal{E} \rightarrow \mathcal{E}$ is compact.*

Proof. Let us fix $\zeta \in \rho(\mathcal{L})$ with $\operatorname{Re} \zeta \geq 1 + \beta^2/2$. Now we consider the resolvent equation $(\zeta - \mathcal{L})f = g$ for $f, g \in \mathcal{E}$. Applying $\langle \cdot, f \rangle_{\omega}$ to both sides yields:

$$\begin{aligned} \int_{\mathbb{R}} \bar{f}g \, dx &= \int_{\mathbb{R}} \zeta |f|^2 \omega - (f' + xf)' \bar{f} \omega \, dx \\ &= \int_{\mathbb{R}} |f'|^2 \omega + |f|^2 (x\omega' + \zeta\omega) + f' \bar{f} \omega' + f \bar{f}' x \omega \, dx. \end{aligned}$$

Next we take the real part:

$$\begin{aligned} \operatorname{Re} \int_{\mathbb{R}} \bar{f}g \, dx &= \int_{\mathbb{R}} |f'|^2 \omega + |f|^2 (x\omega' + \operatorname{Re}(\zeta)\omega) + \frac{1}{2}|f^2|'(\omega' + x\omega) \, dx \\ &= \int_{\mathbb{R}} |f'|^2 \omega + \frac{1}{2}|f|^2 \varpi \, dx, \end{aligned} \quad (2.12)$$

with $\varpi := -\omega'' + x\omega' + (2\operatorname{Re} \zeta - 1)\omega$. For our choice $\omega(x) = \cosh \beta x$ we obtain $\varpi(x) = (2\operatorname{Re} \zeta - 1 - \beta^2)\omega(x) + x\beta \sinh \beta x$. For $\operatorname{Re} \zeta \geq 1 + \beta^2/2$, ϖ is strictly positive. Thus, ϖ is a weight function, and it has the asymptotic behaviour $\varpi(x) \sim \beta|x|\omega(x)$ as $x \rightarrow \pm\infty$. Applying the Cauchy-Schwarz inequality to the left hand side of (2.12) yields

$$\frac{1}{2}\|f\|_{\varpi}^2 + \|f'\|_{\omega}^2 \leq \|f\|_{\omega}\|g\|_{\omega}.$$

For the left hand side we use $\omega(x) \leq \varpi(x)$ and the Poincaré inequality (2.11) to obtain

$$\frac{1}{2}\|f\|_{\varpi} + \frac{1}{C_{\beta}}\|f'\|_{\omega} \leq \|g\|_{\omega}.$$

Hence $R_{\mathcal{L}}(\zeta) \in \mathcal{B}(\mathcal{E}, W^{1,2}(\varpi, \omega))$. Now we have the asymptotic behaviour $\omega(x)/\varpi(x) \sim 1/(\beta|x|) \rightarrow 0$ as $x \rightarrow \pm\infty$. Therefore we may apply Lemma 2.17 for $w = w_1 = \omega$ and $w_0 = \varpi$, which yields the compact embedding $W^{1,2}(\varpi, \omega) \hookrightarrow \mathcal{E}$. Thus, the resolvent $R_{\mathcal{L}}(\zeta) \in \mathcal{B}(\mathcal{E})$ is compact for $\operatorname{Re} \zeta \geq 1 + \beta^2/2$. But this already implies the compactness of $R_{\mathcal{L}}(\zeta)$ for all $\zeta \in \rho(\mathcal{L})$, cf. [19, Theorem III.6.29]. \square

3. Analysis of the Perturbed Operator

So far we have discussed the one-dimensional Fokker-Planck operator \mathcal{L} in $\mathcal{E} = L^2(\omega)$, with $\omega(x) = \cosh \beta x$. In this section we investigate the properties of the perturbed (one-dimensional) operator $\mathcal{L} + \Theta$ in \mathcal{E} , and we shall summarize the results in Theorem 3.17. We begin by specifying the assumptions we make on the perturbation Θ .

(C) Conditions on Θ : We assume that $\Theta f = \vartheta * f$, for $f \in \mathcal{E}$, where ϑ is a tempered distribution that fulfills the following properties in $\Omega_{\beta/2}$ for some $\beta > 0$:

- (i) The Fourier transform $\hat{\vartheta}$ can be extended to an analytic function in $\Omega_{\beta/2}$ (also denoted by $\hat{\vartheta}$), and $\hat{\vartheta} \in L^\infty(\Omega_{\beta/2})$.
- (ii) It holds $\hat{\vartheta}(0) = 0$, i.e. ϑ has zero mass.
- (iii) The mapping $\xi \mapsto \operatorname{Re} \int_0^1 \hat{\vartheta}(\xi s)/s \, ds$ is essentially bounded in $\Omega_{\beta/2}$.

Remark 3.1. If the conditions **(C)(i)-(ii)** hold for ϑ , then the mapping $\xi \mapsto \int_0^1 \hat{\vartheta}(\xi s)/s \, ds$ is analytic in $\Omega_{\beta/2}$. This becomes clear when writing $\hat{\vartheta}(\xi s)/s = \xi \hat{\vartheta}(\xi s)/(\xi s)$, which is analytic for all $s \in (0, 1]$ and can be continuously extended to $\hat{\vartheta}'(0)\xi$ for $s = 0$. The analyticity of $\xi \mapsto \int_0^1 \hat{\vartheta}(\xi s)/s \, ds$ on $\Omega_{\beta/2}$ then follows from [10, Theorem 4.9.1].

Lemma 3.2. *There holds $\Theta f \in \mathcal{E}$ for all $f \in \mathcal{E}$ iff the condition **(C)(i)** holds.*

Proof. According to Lemma 2.14 there holds $\Theta f \in \mathcal{E}$ iff

$$\sup_{|b| < \beta/2} \|(\hat{\vartheta} \hat{f})(\cdot + ib)\|_{L^2(\mathbb{R})} < \infty, \quad (3.1)$$

where we use $\widehat{\Theta f} = \hat{\vartheta} \hat{f}$. Now we apply Hölder's inequality and find that (3.1) holds for all $f \in \mathcal{E}$ iff ϑ satisfies **(C)(i)**. \square

As a consequence of the above lemma we may define $(\hat{\vartheta} \hat{f})(\cdot \pm i\beta/2) \in L^2(\mathbb{R})$ for $f \in \mathcal{E}$ according to (2.9) whenever ϑ satisfies **(C)(i)**. Thereby, the functions $\hat{\vartheta}(\cdot \pm i\beta/2)$ obtained by this procedure are the unique elements of $L^\infty(\mathbb{R})$ with the property

$$\lim_{b \nearrow \beta/2} \|\hat{\vartheta}(\cdot \pm ib) - \hat{\vartheta}(\cdot \pm i\beta/2)\|_{L^\infty(\mathbb{R})} = 0. \quad (3.2)$$

In the following, we define $\hat{\vartheta}(\cdot \pm i\beta/2)$ according to (3.2) for every ϑ which satisfies **(C)(i)**.

Corollary 3.3. *The convolution Θ is bounded in \mathcal{E} iff the condition **(C)(i)** holds.*

Proof. We apply the norm (2.10) to Θf . The Fourier transform turns the convolution into a multiplication, so we get

$$\|\Theta f\|_{\omega}^2 = \int_{\mathbb{R}} |\hat{f}(\xi - i\beta/2)|^2 |\hat{\vartheta}(\xi - i\beta/2)|^2 \, d\xi + \int_{\mathbb{R}} |\hat{f}(\xi + i\beta/2)|^2 |\hat{\vartheta}(\xi + i\beta/2)|^2 \, d\xi.$$

The result follows by using the estimate $|\hat{\vartheta}(\xi \pm i\beta/2)| \leq \|\hat{\vartheta}(\cdot \pm i\beta/2)\|_{L^\infty} < \infty$ for $\xi \in \mathbb{R}$, cf. (3.2). \square

Lemma 3.4. *Under the assumption **(C)** there holds $\Theta : \mathcal{E}_k \rightarrow \mathcal{E}_{k+1} \subset \mathcal{E}_k$ for every $k \in \mathbb{N}$.*

Proof. According to Proposition 2.12, $f \in \mathcal{E}_k$ iff $\xi = 0$ is a zero of $\hat{f}(\xi)$ of order greater or equal to k . Because of the assumption $\hat{\vartheta}(0) = 0$ the Fourier transform $\widehat{\Theta f} = \hat{\vartheta} \hat{f}$ has a zero at least of order $k + 1$ for $f \in \mathcal{E}_k$, so $\Theta f \in \mathcal{E}_{k+1}$. \square

Corollary 3.5. *Let (C) hold, and $k \in \mathbb{N}_0$. Then the space \mathcal{E}_k is a $(\mathcal{L} + \Theta)$ -invariant subspace of \mathcal{E} .*

Since the conditions (C) are not very handy for direct applications, the following lemma gives some criteria that are simpler to verify and sufficient for (C).

Lemma 3.6. *Let $\beta > 0$ and $\omega(x) = \cosh \beta x$, and assume that $\vartheta \in \mathcal{S}'$ fulfills*

- (i) $\hat{\vartheta}(0) = 0$,
- (ii) $\vartheta = \vartheta_W + \vartheta_D$ with $\vartheta_W \in W^{1,1}(\omega^{\frac{1}{2}}, \omega^{\frac{1}{2}})$ and $\vartheta_D \in D := \{\sum_{j=1}^n a_j \delta_{x_j} : a_j \in \mathbb{C}, x_j \in \mathbb{R}, n \in \mathbb{N}\}$, where δ_{x_j} denotes the delta distribution located at x_j .

Then $\Theta f = \vartheta * f$ satisfies (C) for this $\beta > 0$.

Proof. In general $\hat{\vartheta}_W(0)$ and $\hat{\vartheta}_D(0)$ are not zero, so it is convenient to define $\vartheta_W^* := \vartheta_W - M\mu$ and $\vartheta_D^* := \vartheta_D + M\mu$, where $M := \hat{\vartheta}_D(0)/\sqrt{2\pi}$. Then $\hat{\vartheta}_W^*$ and ϑ_D^* have zero mass, and we still have $\hat{\vartheta}_W^* \in W^{1,1}(\omega^{\frac{1}{2}}, \omega^{\frac{1}{2}})$. Since $\mathcal{F}_{x \rightarrow \xi} \delta_{x_j} = e^{-i\xi x_j}$ and $\hat{\mu}(\xi) = \sqrt{2\pi}\mu(\xi)$, it is immediate that ϑ_D^* satisfies (C)(i) and (C)(iii).

Now we verify the same properties for ϑ_W^* . Since $\vartheta_W^* \in L^1(\omega^{\frac{1}{2}})$, we may extend $\hat{\vartheta}_W^*$ to an analytic function in $\Omega_{\beta/2}$, and there holds (2.9), cf. [9, Proposition XVI.1.3]. The Fourier transform is a continuous map from $L^1(\mathbb{R})$ to $B_0(\mathbb{R})$, i.e. the continuous functions decaying at infinity, equipped with the uniform norm. Therefore, $\vartheta_W^* \in L^1(\omega^{\frac{1}{2}})$ implies

$$\|\hat{\vartheta}_W^*\|_{L^\infty(\Omega_{\beta/2})} = \sup_{|b| < \frac{\beta}{2}} \sup_{\xi \in \mathbb{R}} |\hat{\vartheta}_W^*(\xi + ib)| \leq \sup_{|b| < \frac{\beta}{2}} \|\vartheta_W^*(x)e^{bx}\|_{L^1(\mathbb{R})} \leq \|\vartheta_W^*(x)e^{\frac{\beta}{2}|x|}\|_{L^1(\mathbb{R})} < \infty,$$

So (C)(i) is satisfied. For (C)(iii) it is sufficient to show that for some $c > 0$ and all $\xi \in \Omega_{\beta/2}$ with $|\xi| \geq 1$ there holds $|\hat{\vartheta}_W^*(\xi)| \leq c/|\xi|$, which is fulfilled if $\mathcal{F}(\vartheta_W^{*\prime}) \in L^\infty(\Omega_{\beta/2})$. Analogously to the previous part of the proof we obtain that this is satisfied if $\vartheta_W^{*\prime} \in L^1(\omega^{\frac{1}{2}})$. We conclude that ϑ_W^* fulfills (C)(i) and (C)(iii) if $\vartheta_W^* \in W^{1,1}(\omega^{\frac{1}{2}}, \omega^{\frac{1}{2}})$.

Finally, ϑ satisfies the condition (C)(ii) due to the assumption (i). \square

For the rest of the article, we shall always assume that Θ satisfies the condition (C) for some fixed $\beta > 0$, and we choose the weight function $\omega(x) = \cosh \beta x$ with this particular β . The first result about the perturbed Fokker-Planck operator is the following lemma:

Lemma 3.7. *The operator $\mathcal{L} + \Theta$ has compact resolvent in \mathcal{E} .*

Proof. A bounded perturbation of an infinitesimal generator with compact resolvent has compact resolvent again, see [11, Proposition III.1.12]. Then the result follows by combining the results of Theorems 2.18 and 2.8 for \mathcal{L} , and Corollary 3.3 for Θ . \square

As a consequence, the spectrum of $\mathcal{L} + \Theta$ in \mathcal{E} is non-empty and consists only of eigenvalues. In order to characterize the entire spectrum, we introduce the following ladder operators¹, namely the *annihilation operator*

$$\alpha^- : \mathcal{E}_1 \rightarrow \mathcal{E} : f \mapsto \int_{-\infty}^x f(y) dy,$$

and its formal inverse $\alpha^+ : f \mapsto f'$, the *creation operator*.

Lemma 3.8. *The annihilation operator α^- has the following properties:*

- (i) For any $k \in \mathbb{N}$ there holds $\alpha^- \in \mathcal{B}(\mathcal{E}_k, \mathcal{E}_{k-1})$.
- (ii) In \mathcal{E}_1 the operators Θ and α^- commute.
- (iii) Let $f \in \mathcal{E}_1$, $\zeta \in \mathbb{C}$ such that $(\mathcal{L} + \Theta)f = \zeta f$. Then

$$(\mathcal{L} + \Theta)(\alpha^- f) = (\zeta + 1)(\alpha^- f).$$

¹One of the best-known applications of ladder operators occurs in the spectral analysis of the quantum harmonic oscillator, see e.g. [17].

Proof. First we show (i). The property $\alpha^- : \mathcal{E}_k \rightarrow \mathcal{E}_{k-1}$ can be verified by using the explicit representation (2.7) of the \mathcal{E}_k , and integration by parts (first for $f \in C_0^\infty(\mathbb{R})$). The boundedness of α^- follows immediately from the Poincaré inequality (2.11). Property (ii) holds true since Θ is a convolution. For Result (iii) one applies α^- to the equation $(\mathcal{L} + \Theta)f = \zeta f$, and uses the identity $\alpha^-(\mathcal{L}f) = \mathcal{L}(\alpha^-f) - \alpha^-f$ and the property (ii). \square

By using the annihilation operator, we are able to prove:

Proposition 3.9. *We have the following spectral properties of $\mathcal{L} + \Theta$ in \mathcal{E} :*

- (i) $\sigma(\mathcal{L} + \Theta) = -\mathbb{N}_0$.
- (ii) For each $k \in \mathbb{N}_0$, the eigenspace $\ker(\mathcal{L} + \Theta + k)$ is one-dimensional.
- (iii) Under appropriate scaling, the eigenfunction f_k to the eigenvalue $-k \in \mathbb{N}_0$ is explicitly given by

$$f_k = (\alpha^+)^k f_0 = f_0^{(k)}, \quad \text{and} \quad \hat{f}_0(\xi) = \exp\left(-\frac{\xi^2}{2} + \int_0^1 \frac{\hat{\vartheta}(\xi s)}{s} ds\right), \quad \xi \in \Omega_{\beta/2}. \quad (3.3)$$

In particular, f_0 is the unique stationary solution with unit mass of the perturbed Fokker-Planck equation (1.1) in one dimension.

Proof. In order to show (i) we first prove that $\bigcap_{k \in \mathbb{N}} \mathcal{E}_k = \{0\}$. According to (2.8) there holds

$$\bigcap_{k \in \mathbb{N}} \mathcal{E}_k = \left\{ f \in \mathcal{E} : \hat{f}^{(k)}(0) = 0, k \in \mathbb{N}_0 \right\}.$$

But for $f \in \mathcal{E}$, \hat{f} is analytic, and the only analytic function with a zero of infinite order is the zero function, which proves the statement.

Thus, for any eigenfunction f , there exists a unique $k \in \mathbb{N}_0$ such that $f \in \mathcal{E}_k \setminus \mathcal{E}_{k+1}$, which is the minimal $k \in \mathbb{N}_0$ with the property $\Pi_{\mathcal{L},k} f \neq 0$. Applying this projection to the eigenvalue equation yields

$$\Pi_{\mathcal{L},k}(\mathcal{L} + \Theta)f = -k\Pi_{\mathcal{L},k}f = \zeta\Pi_{\mathcal{L},k}f,$$

where we used $\Theta f \in \mathcal{E}_{k+1}$ (cf. Lemma 3.4). Hence, the eigenvalue corresponding to f satisfies $\zeta = -k$. Thus $\sigma(\mathcal{L} + \Theta) \subseteq -\mathbb{N}_0$. If now f_k is an eigenfunction with eigenvalue $-k$, we can apply k times the continuous operator α^- to f_k , and create eigenfunctions to all eigenvalues $\{-k+1, \dots, 0\}$. So either $\sigma(\mathcal{L} + \Theta) = -\mathbb{N}_0$ or $\sigma(\mathcal{L} + \Theta) = \{-k_0, \dots, 0\}$, i.e. there exists some minimal eigenvalue $-k_0$. But the latter scenario is actually not possible, because then the operator $(\mathcal{L} + \Theta)|_{\mathcal{E}_{k_0+1}}$ would have empty spectrum in \mathcal{E}_{k_0+1} , which contradicts the fact that it still has a compact resolvent in \mathcal{E}_{k_0+1} .

In order to verify (ii) we recall from the first part of the proof that if f is an eigenfunction of $\mathcal{L} + \Theta$ to the eigenvalue $-k$, then $k = \operatorname{argmin}\{\Pi_{\mathcal{L},j} f \neq 0 : j \in \mathbb{N}_0\}$. Assume that $\dim \ker(\mathcal{L} + \Theta + k) > 1$ for some $k \in \mathbb{N}_0$. Since $\dim \operatorname{ran} \Pi_{\mathcal{L},k} = 1$, we could construct an eigenfunction f to the eigenvalue $-k$ such that $\Pi_{\mathcal{L},k} f = 0$, i.e. the eigenvalue of f would be strictly less than $-k$, which is a contradiction.

For the third result (iii) we consider the Fourier transform of the eigenvalue equation $(\mathcal{L} + \Theta)f_k = -kf_k$ for $k \in \mathbb{N}_0$. This yields the following differential equation for \hat{f}_k :

$$\xi \hat{f}_k'(\xi) = (\hat{\vartheta}(\xi) + k - \xi^2) \hat{f}_k(\xi).$$

Its general solution reads

$$\hat{f}_k(\xi) = c_k \xi^k q(\xi), \quad \text{with} \quad q(\xi) := \exp\left(-\frac{\xi^2}{2} + \int_0^1 \frac{\hat{\vartheta}(\xi s)}{s} ds\right),$$

for all $k \in \mathbb{N}_0$, with $c_k \in \mathbb{C}$. We may now fix $c_k := i^k$, which completes the proof. \square

Lemma 3.10. *The spectral projection \mathcal{P}_k of $\mathcal{L} + \Theta$ corresponding to the eigenvalue $-k \in -\mathbb{N}_0$ fulfills*

$$\operatorname{ran} \mathcal{P}_k = \operatorname{span}\{f_k\}, \quad \ker \mathcal{P}_k = \mathcal{E}_{k+1} \oplus \operatorname{span}\{f_{k-1}, \dots, f_0\},$$

with the eigenfunctions f_k, \dots, f_0 given in (3.3). Therefore, all singularities of the resolvent are of order one, and for all $k \in \mathbb{N}_0$ there holds $M(\mathcal{L} + \Theta + k) = \ker(\mathcal{L} + \Theta + k)$.

Proof. The set $\mathcal{K}_k := \mathcal{E}_{k+1} \oplus \text{span}\{f_{k-1}, \dots, f_0\}$ is invariant under $\mathcal{L} + \Theta$, cf. Corollary 3.5. Therefore the algebraic eigenspace satisfies $M(\mathcal{L} + \Theta + k) = \ker(\mathcal{L} + \Theta + k) = \text{span}\{f_k\}$. In particular we obtain the $(\mathcal{L} + \Theta)$ -invariant decomposition $\mathcal{E} = \mathcal{K}_k \oplus M(\mathcal{L} + \Theta + k)$, and $\sigma((\mathcal{L} + \Theta)|_{\mathcal{K}_k}) = -\mathbb{N}_0 \setminus \{-k\}$. So we can apply Lemma A.4 from the appendix, which yields the properties of the spectral projections.

Since $\dim \mathcal{P}_k = 1$ and $M(\mathcal{L} + \Theta + k) = \ker(\mathcal{L} + \Theta + k)$, the singularity of $R_{\mathcal{L} + \Theta}(\zeta)$ at $\zeta = -k$ is a pole of order one, see Proposition A.2 (iv)-(v). \square

Having explicitly determined the spectrum of the perturbed Fokker-Planck operator, we now turn to the generated semigroup and the corresponding decay rates. We start with the fact that $\mathcal{L} + \Theta$ generates a C_0 -semigroup:

Proposition 3.11. *For each $k \in \mathbb{N}_0$ the operator $(\mathcal{L} + \Theta)|_{\mathcal{E}_k}$ is the infinitesimal generator of a C_0 -semigroup on \mathcal{E}_k . The semigroup on \mathcal{E} preserves mass, i.e.*

$$\int_{\mathbb{R}} f(x) dx = \int_{\mathbb{R}} [e^{t(\mathcal{L} + \Theta)} f](x) dx, \quad \forall t \geq 0.$$

Proof. According to Theorem 2.8 the operator \mathcal{L} generates a C_0 -semigroup on \mathcal{E}_k for every $k \in \mathbb{N}_0$, and due to Lemma 3.4 and Corollary 3.3 we have $\Theta|_{\mathcal{E}_k} \in \mathcal{B}(\mathcal{E}_k)$. Now a bounded perturbation of the infinitesimal generator of a C_0 -semigroup is again infinitesimal generator, see [11, Theorem III.1.3], and so the first result follows.

To show the conservation of mass we use the reduction of $(e^{t(\mathcal{L} + \Theta)})_{t \geq 0}$ by \mathcal{P}_0 given by $\mathcal{E} = \mathcal{E}_1 \oplus \text{span}\{f_0\}$. The space \mathcal{E}_1 consists of all massless functions, so the part $\mathcal{P}_0 f$ alone determines the mass of any $f \in \mathcal{E}$. Since \mathcal{E}_1 and $\text{span}\{f_0\}$ are both invariant under the semigroup, \mathcal{P}_0 and $(e^{t(\mathcal{L} + \Theta)})_{t \geq 0}$ commute, i.e. $\mathcal{P}_0 e^{t(\mathcal{L} + \Theta)} f = e^{t(\mathcal{L} + \Theta)} \mathcal{P}_0 f$ for all $f \in \mathcal{E}$, $t \geq 0$. Furthermore we have $\mathcal{P}_0 f \in \ker(\mathcal{L} + \Theta)$, and so $e^{t(\mathcal{L} + \Theta)} \mathcal{P}_0 f = \mathcal{P}_0 f$ for all $t \geq 0$. Altogether we obtain $\mathcal{P}_0 e^{t(\mathcal{L} + \Theta)} f = \mathcal{P}_0 f$ for all $f \in \mathcal{E}$, $t \geq 0$, i.e. the semigroup preserves mass. \square

Next we investigate the decay rate of $(e^{t(\mathcal{L} + \Theta)})_{t \geq 0}$ on the subspaces \mathcal{E}_k . To this end we define:

$$\hat{\psi}(\xi) := \exp\left(\int_0^1 \frac{\hat{\vartheta}(\xi s)}{s} ds\right), \quad \xi \in \Omega_{\beta/2},$$

which is analytic in $\Omega_{\beta/2}$ according to Remark 3.1.

Lemma 3.12. *The map $\Psi : f \mapsto f * \psi$ has the properties:*

- (i) *For each $k \in \mathbb{N}_0$, $\Psi : \mathcal{E}_k \rightarrow \mathcal{E}_k$ is a bijection, with inverse $\Psi^{-1} : f \mapsto f * \mathcal{F}^{-1}[1/\hat{\psi}]$.*
- (ii) *$\Psi, \Psi^{-1} \in \mathcal{B}(\mathcal{E})$.*

Proof. We define $\bar{\Psi} : f \mapsto f * \mathcal{F}^{-1}[1/\hat{\psi}]$. Due to the condition **(C)(iii)** there holds $\Psi f, \bar{\Psi} f \in \mathcal{E}$ for all $f \in \mathcal{E}$, which is shown analogously to Lemma 3.2. Let now $f \in \mathcal{E}_k$ for some $k \in \mathbb{N}_0$. Then $\hat{f}(\xi)$ has a zero of order greater or equal to k at $\xi = 0$, cf. Proposition 2.12. Since $\hat{\psi}$ and $1/\hat{\psi}$ are analytic in $\Omega_{\beta/2}$, the zero at $\xi = 0$ of $\mathcal{F}_{x \rightarrow \xi} \Psi f = \hat{f}(\xi) \hat{\psi}(\xi)$ and of $\mathcal{F}_{x \rightarrow \xi} \bar{\Psi} f = \hat{f}(\xi) / \hat{\psi}(\xi)$ is of the same order as of \hat{f} . So $\Psi, \bar{\Psi} : \mathcal{E}_k \rightarrow \mathcal{E}_k$ for all $k \in \mathbb{N}_0$.

By applying the Fourier transform, we see that $\Psi \circ \bar{\Psi} f = \bar{\Psi} \circ \Psi f = f$ for all $f \in \mathcal{E}$, i.e. $\bar{\Psi} = \Psi^{-1}$, and $\Psi, \Psi^{-1} : \mathcal{E}_k \rightarrow \mathcal{E}_k$ are bijections for all $k \in \mathbb{N}_0$.

Finally, as in Corollary 3.3 one proves the boundedness of Ψ and Ψ^{-1} by using the assumption **(C)(iii)**. \square

The map Ψ plays a crucial role in the analysis of the perturbed Fokker-Planck operator $\mathcal{L} + \Theta$, because it relates the eigenspaces of \mathcal{L} with the eigenspaces of $\mathcal{L} + \Theta$: According to Proposition 3.9 we have:

$$f_k = \Psi \mu_k, \quad k \in \mathbb{N}_0. \quad (3.4)$$

By using this property of Ψ we obtain the following result:

Proposition 3.13. *Let $k \in \mathbb{N}_0$ and $\zeta \in \mathbb{C} \setminus \{-k, -k-1, \dots\}$. Then there holds*

$$R_{\mathcal{L}+\Theta}(\zeta)|_{\mathcal{E}_k} = \Psi \circ R_{\mathcal{L}}(\zeta) \circ \Psi^{-1}|_{\mathcal{E}_k}. \quad (3.5)$$

In particular there exists a constant $\tilde{C}_k > 0$ such that

$$\|(R_{\mathcal{L}+\Theta}(\zeta)|_{\mathcal{E}_k})^n\|_{\mathcal{B}(\mathcal{E}_k)} \leq \frac{\tilde{C}_k}{(\operatorname{Re} \zeta + k)^n}, \quad \operatorname{Re} \zeta > -k, n \in \mathbb{N}. \quad (3.6)$$

Proof. We fix $k \in \mathbb{N}_0$. Then for all $j \geq k$ and $\zeta \in \mathbb{C} \setminus \{-k, -k-1, \dots\}$ there holds due to (3.4):

$$R_{\mathcal{L}}(\zeta)\mu_j = \frac{\mu_j}{\zeta + j} = \Psi^{-1} \circ R_{\mathcal{L}+\Theta}(\zeta)f_j = \Psi^{-1} \circ R_{\mathcal{L}+\Theta}(\zeta) \circ \Psi\mu_j.$$

So we have $R_{\mathcal{L}}(\zeta) = \Psi^{-1} \circ R_{\mathcal{L}+\Theta}(\zeta) \circ \Psi$ in the space $\operatorname{span}\{\mu_j : j \geq k\} \subset E_k$, which is dense in \mathcal{E}_k . Then this identity extends to \mathcal{E}_k due to the continuity of the occurring operators.

In order to prove the resolvent estimate (3.6) we use

$$(R_{\mathcal{L}+\Theta}(\zeta)|_{\mathcal{E}_k})^n = R_{\mathcal{L}+\Theta}(\zeta)^n|_{\mathcal{E}_k} = \Psi \circ R_{\mathcal{L}}(\zeta)^n \circ \Psi^{-1}|_{\mathcal{E}_k},$$

which follows from (3.5) and Lemma 3.12 (i). Because of $\Psi, \Psi^{-1} \in \mathcal{B}(\mathcal{E}_k)$ we conclude

$$\|(R_{\mathcal{L}+\Theta}(\zeta)|_{\mathcal{E}_k})^n\|_{\mathcal{B}(\mathcal{E}_k)} \leq \|\Psi\|_{\mathcal{B}(\mathcal{E}_k)} \|(R_{\mathcal{L}}(\zeta)|_{\mathcal{E}_k})^n\|_{\mathcal{B}(\mathcal{E}_k)} \|\Psi^{-1}\|_{\mathcal{B}(\mathcal{E}_k)}. \quad (3.7)$$

Due to the semigroup estimate in Theorem 2.8 (iv) there holds

$$\|(R_{\mathcal{L}}(\zeta)|_{\mathcal{E}_k})^n\|_{\mathcal{B}(\mathcal{E}_k)} \leq \frac{C_k}{(\operatorname{Re} \zeta + k)^n}, \quad \operatorname{Re} \zeta > -k, n \in \mathbb{N},$$

according to the Hille-Yosida theorem. Inserting this estimate in (3.7) shows (3.6). \square

Corollary 3.14. *Let $k \in \mathbb{N}_0$. Then there exists a constant $\tilde{C}_k > 0$ such that*

$$\|e^{t(\mathcal{L}+\Theta)}|_{\mathcal{E}_k}\|_{\mathcal{B}(\mathcal{E}_k)} \leq \tilde{C}_k e^{-kt}, \quad t \geq 0. \quad (3.8)$$

Proof. The result immediately follows from (3.6) by application of the Hille-Yosida theorem. \square

Remark 3.15. The above result implies the exponential convergence of any solution of (1.1) towards the (appropriately scaled) stationary state: Choose any $f \in \mathcal{E}$. Then there exists a unique constant $m \in \mathbb{C}$ (the ‘‘mass’’ of f) such that $\mathcal{P}_0 f = m f_0$. So $f - m f_0 = (1 - \mathcal{P}_0)f \in \mathcal{E}_1$, cf. Lemma 3.10, which implies $e^{t(\mathcal{L}+\Theta)}f - m f_0 = e^{t(\mathcal{L}+\Theta)}(f - m f_0) \in \mathcal{E}_1$ for all $t \geq 0$, due to Proposition 3.11. With (3.8) and $k = 1$ this implies

$$\|e^{t(\mathcal{L}+\Theta)}f - m f_0\|_{\mathcal{E}} \leq \tilde{C}_1 \|f - m f_0\|_{\mathcal{E}} e^{-t}, \quad t \geq 0.$$

Remark 3.16. In the one dimensional case we can explicitly compute the Fourier transform of $R_{\mathcal{L}+\Theta}(\zeta)g$, see Proposition B.1: For any $k \in \mathbb{N}_0$, $\operatorname{Re} \zeta > -k$, and $g \in \mathcal{E}_k$, the unique solution $f \in \mathcal{E}_k$ of $(\zeta - \mathcal{L} - \Theta)f = g$ satisfies

$$\hat{f}(\xi) = \mathcal{F}_{x \rightarrow \xi}[R_{\mathcal{L}+\Theta}(\zeta)g] = \hat{f}_0(\xi) \int_0^1 \frac{\hat{g}(s\xi)}{\hat{f}_0(s\xi)} s^{\zeta-1} ds, \quad \xi \in \Omega_{\beta/2},$$

where $s^\zeta = e^{\zeta \log s}$ and \log is the natural logarithm on \mathbb{R}^+ . One can use this representation for an alternative proof of the resolvent estimate (3.6). However, this becomes less convenient in higher dimensions, since it is then not clear how to properly compute the explicit Fourier transform of $R_{\mathcal{L}+\Theta}(\zeta)$.

Now we summarize our results in the final theorem:

Theorem 3.17. *Let $\mathcal{E} = L^2(\omega)$, where $\omega(x) = \cosh \beta x$, for some $\beta > 0$, and let Θ fulfill the condition (C) for this $\beta > 0$. Then the perturbed operator $\mathcal{L} + \Theta$ has the following properties in \mathcal{E} :*

- (i) *It has compact resolvent, and $\sigma(\mathcal{L} + \Theta) = \sigma_p(\mathcal{L} + \Theta) = -\mathbb{N}_0$.*
- (ii) *There holds $M(\mathcal{L} + \Theta + k) = \ker(\mathcal{L} + \Theta + k) = \operatorname{span}\{f_k\}$, where f_k is the eigenfunction to the eigenvalue $-k$ given by (3.3). The eigenfunctions are related by $f_k = f_0^{(k)}$.*

(iii) The spectral projection \mathcal{P}_k corresponding to $-k \in \sigma(\mathcal{L} + \Theta)$ fulfills

$$\text{ran } \mathcal{P}_k = \text{span}\{f_k\}, \quad \ker \mathcal{P}_k = \mathcal{E}_{k+1} \oplus \text{span}\{f_{k-1}, \dots, f_0\},$$

where the $(\mathcal{L} + \Theta)$ -invariant spaces \mathcal{E}_k are explicitly given in (2.7).

(iv) For every $k \in \mathbb{N}_0$, the operator $(\mathcal{L} + \Theta)|_{\mathcal{E}_k}$ generates a C_0 -semigroup $(e^{t(\mathcal{L} + \Theta)}|_{\mathcal{E}_k})_{t \geq 0}$ in \mathcal{E}_k , which satisfies the estimate

$$\|e^{t(\mathcal{L} + \Theta)}|_{\mathcal{E}_k}\|_{\mathcal{B}(\mathcal{E}_k)} \leq \tilde{C}_k e^{-kt}, \quad t \geq 0,$$

where the constant $\tilde{C}_k > 0$ is independent of t .

Remark 3.18. Apparently, the particular choice of $\beta > 0$ has no influence on the above results, except possibly for the constants \tilde{C}_k . In practice, the constant β may therefore be chosen arbitrarily small, such that Θ satisfies (C) for this β .

Remark 3.19. The sequence of eigenfunctions $(\mu_k)_{k \in \mathbb{N}_0}$ is an orthogonal basis of E . In the larger space \mathcal{E} , the linear hull $\text{span}\{\mu_k : k \in \mathbb{N}_0\}$ is still dense, due to the continuous embedding $E \hookrightarrow \mathcal{E}$. Also, each $f \in \mathcal{E}$ can (formally) uniquely be decomposed according to the sequence $(\Pi_{\mathcal{L}, k})_{k \in \mathbb{N}_0}$, see the proof of the Proposition 3.9. But in some cases, the obtained series is divergent in \mathcal{E} . As an example we consider $f(x) := \exp(-|x|) \in L^2(\cosh x)$. Since f is symmetric, we have $\Pi_{\mathcal{L}, k} f = 0$ if k is odd. For $k = 2n$, $n \in \mathbb{N}_0$, we can show the asymptotic behaviour for $n \rightarrow \infty$:

$$\|\Pi_{\mathcal{L}, 2n} f\|_{\mathcal{E}} = \mathcal{O}\left(\frac{\sqrt{(2n)!}}{n^{1/4}}\right),$$

where we use the explicit representation for the Hermite polynomials H_{2n} from (5.5.4) in [28], and the asymptotic expansions for H_{2n} given in [28, Theorem 8.22.9]. Therefore, the formal series $\sum_{n \in \mathbb{N}_0} \Pi_{\mathcal{L}, 2n} f$ is divergent in \mathcal{E} . So the sequence $(\mu_k)_{k \in \mathbb{N}_0}$ is neither a Schauder basis nor a representation system of \mathcal{E} . However, the sequence $(\mu_k / \|\mu_k\|_E)_{k \in \mathbb{N}_0}$ is still a Bessel system, see [7, 6] for the definitions.

4. The Higher-Dimensional Case

As already mentioned in the introduction, the preceding results can be generalized to higher dimensions without much additional effort. Most proofs are analogous to the ones in the one-dimensional case. Therefore we give here only an outline of the steps leading to the extension of Theorem 3.17 to higher dimensions.

In this section we consider the perturbed Fokker-Planck equation (1.1) on \mathbb{R}^d , where $d \in \mathbb{N}$ is the spatial dimension. Elements of \mathbb{R}^d resp. \mathbb{C}^d are represented by bold letters, e.g. $\mathbf{x} \in \mathbb{R}^d$, $\boldsymbol{\xi} \in \mathbb{C}^d$, and we write $\mathbf{x} = (x_1, \dots, x_d)$. For a multi-index $\mathbf{k} \in \mathbb{N}_0^d$ we define $|\mathbf{k}| := k_1 + \dots + k_d$, $\mathbf{x}^{\mathbf{k}} := x_1^{k_1} \dots x_d^{k_d}$ and $\mathbf{k}! := k_1! \dots k_d!$. Furthermore

$$D^{\mathbf{k}} := \frac{\partial^{|\mathbf{k}|}}{\partial x_1^{k_1} \dots \partial x_d^{k_d}}.$$

We adopt the notation for weighted Sobolev spaces on \mathbb{R}^d from Section 2, as well as the normalization of the Fourier transform.

We consider the Fokker-Planck operator on \mathbb{R}^d given by

$$Lf := \nabla \cdot \left(\mu \nabla \left(\frac{f}{\mu} \right) \right) = \Delta f + \mathbf{x} \cdot \nabla f + df,$$

where $\mu(\mathbf{x}) := \exp(-|\mathbf{x}|^2/2)$. The natural space to consider L in is $E := L^2(1/\mu(\mathbf{x}))$. For spectral properties see [22, 5, 18]. Since it is isometrically equivalent to the harmonic oscillator $H := -\Delta - d/2 + |\mathbf{x}|^2/4$ in $L^2(\mathbb{R}^d)$, we transfer many results of H (see [25] and [27, Theorem XIII.67]) to L . In the following we summarize some properties of L in E :

Theorem 4.1. *The Fokker-Planck operator L in E has the following properties:*

- (i) L with $D(L) = \{f \in E : Lf \in E\}$ is self-adjoint and has a compact resolvent.

- (ii) The spectrum is $\sigma(L) = -\mathbb{N}_0$, and it consists only of eigenvalues.
- (iii) For each eigenvalue $-k \in \sigma(L)$ the corresponding eigenspace has the dimension $\binom{k+d-1}{k}$, and it is spanned by the eigenfunctions

$$\mu_{\mathbf{k}}(\mathbf{x}) := \prod_{\ell=1}^d \mu_{k_\ell}(x_\ell), \quad |\mathbf{k}| = k,$$

where the μ_j are defined in Theorem 2.4.

- (iv) The eigenfunctions $(\mu_{\mathbf{k}})_{\mathbf{k} \in \mathbb{N}_0^d}$ form an orthogonal basis of E .
- (v) The spectral projection $\Pi_{L,k}$ onto the k -th eigenspace is given by

$$\Pi_{L,k} = \sum_{|\mathbf{k}|=k} \Pi_{L,\mathbf{k}}, \quad \text{where } \Pi_{L,\mathbf{k}} := \frac{(2\pi)^{d/2}}{\mathbf{k}!} \mu_{\mathbf{k}} \langle \cdot, \mu_{\mathbf{k}} \rangle_E. \quad (4.1)$$

There holds the spectral representation $L = \sum_{k \in \mathbb{N}_0} -k \Pi_{L,k}$.

- (vi) The operator L generates a C_0 -semigroup of contractions on E_k for all $k \in \mathbb{N}_0$, where $E_k := \ker(\Pi_{L,0} + \cdots + \Pi_{L,k-1})$, $k \geq 1$, and $E_0 := E$. The semigroup satisfies the estimate

$$\|e^{tL}|_{E_k}\|_{\mathcal{B}(E_k)} \leq e^{-kt}, \quad \forall k \in \mathbb{N}_0.$$

In order to consider L in $L^2(\omega)$ with a weight ω growing more slowly than $1/\mu$, we may apply Theorem 2.5. This yields the following generalization of the condition (2.2) on the weight function:

$$\begin{aligned} \exists R > 0 : \forall |\mathbf{x}| > R : \quad \Delta\omega(\mathbf{x}) - \mathbf{x} \cdot \nabla\omega(\mathbf{x}) + (d-2a)\omega(\mathbf{x}) &\leq 0, \\ \exists \delta_2 > \delta_1 > 0 : \forall \mathbf{x} \in \mathbb{R}^d : \quad \delta_1 \leq \omega(\mathbf{x}) \leq \delta_2/\mu(\mathbf{x}). \end{aligned} \quad (4.2)$$

The weight function $\omega(\mathbf{x}) := \cosh \beta|\mathbf{x}|$, with $\beta > 0$, satisfies this condition for any $a < 0$. Analogously to Theorem 2.8 we conclude from Theorem 2.5:

Theorem 4.2. *In $\mathcal{E} := L^2(\omega)$, with $\omega(\mathbf{x}) = \cosh \beta|\mathbf{x}|$ and $\beta > 0$, the operator L is closable, and $\mathcal{L} := \text{cl}_{\mathcal{E}} L$ has the following properties:*

- (i) The spectrum satisfies $\sigma(\mathcal{L}) = -\mathbb{N}_0$, and $M(\mathcal{L} + k) = \ker(\mathcal{L} + k) = \text{span}\{\mu_{\mathbf{k}} : |\mathbf{k}| = k\}$ for any $k \in \mathbb{N}_0$.
- (ii) For any $k \in \mathbb{N}_0$ the closed subspace $\mathcal{E}_k := \text{cl}_{\mathcal{E}} \text{span}\{\mu_{\mathbf{k}} : |\mathbf{k}| \geq k\}$ is an \mathcal{L} -invariant subspace of \mathcal{E} , and $\text{span}\{\mu_{\mathbf{k}} : |\mathbf{k}| < k\}$ is a complement. In particular $\mathcal{E}_0 = \mathcal{E}$.
- (iii) The spectral projection $\Pi_{\mathcal{L},k}$ to the eigenvalue $-k \in -\mathbb{N}_0$ fulfills $\text{ran } \Pi_{\mathcal{L},k} = \text{span}\{\mu_{\mathbf{k}} : |\mathbf{k}| = k\}$ and $\ker \Pi_{\mathcal{L},k} = \mathcal{E}_{k+1} \oplus \text{span}\{\mu_{\mathbf{k}} : |\mathbf{k}| < k\}$.
- (iv) For any $k \in \mathbb{N}_0$ the operator \mathcal{L} generates a C_0 -semigroup on \mathcal{E}_k , and there exists a constant $C_k \geq 1$ such that we have the estimate

$$\|e^{t\mathcal{L}}|_{\mathcal{E}_k}\|_{\mathcal{B}(\mathcal{E}_k)} \leq C_k e^{-kt}, \quad \forall t \geq 0.$$

By applying Lemma 2.11 we get by induction

$$\mathcal{E}_k = \left\{ f \in \mathcal{E} : \int_{\mathbb{R}^d} f(\mathbf{x}) \mathbf{x}^{\mathbf{k}} d\mathbf{x} = 0, |\mathbf{k}| \leq k-1 \right\} = \left\{ f \in \mathcal{E} : D^{\mathbf{k}} \hat{f}(0) = 0, |\mathbf{k}| \leq k-1 \right\}. \quad (4.3)$$

In \mathcal{E} there holds a Poincaré-type inequality, namely there exists a constant $C_\beta > 0$ such that for all $f \in W^{1,2}(\omega, \omega)$:

$$\|f\|_\omega \leq C_\beta \|\nabla f\|_\omega. \quad (4.4)$$

This follows from [16, Theorem 14.5]. The resolvent $R_{\mathcal{L}}(\zeta)$ is compact in \mathcal{E} for all $\zeta \in \rho(\mathcal{L})$, which is obtained analogously to Theorem 2.18 by using (4.4), only that here $\varpi(\mathbf{x}) := -\Delta\omega(\mathbf{x}) + \mathbf{x} \cdot \nabla\omega(\mathbf{x}) - (d-2\text{Re}\zeta)\omega(\mathbf{x})$. And [24, Theorem 2.4] yields the required compact embedding $W^{1,2}(\varpi, \omega) \hookrightarrow \mathcal{E}$.

Due to (a small variant of) [26, Theorem IX.13] we have: There holds $f \in \mathcal{E}$ iff \hat{f} has an analytic extension (denoted by \hat{f} as well) to the set $\Omega_{\beta/2} := \{\mathbf{z} \in \mathbb{C}^d : |\operatorname{Im} \mathbf{z}| < \beta/2\}$ and

$$\sup_{\substack{|\mathbf{b}| < \beta/2 \\ \mathbf{b} \in \mathbb{R}^d}} \|\hat{f}(\cdot + i\mathbf{b})\|_{L^2(\mathbb{R}^d)} < \infty. \quad (4.5)$$

For any $\mathbf{b} \in \mathbb{R}^d$ with $|\mathbf{b}| < \beta/2$ we have $\hat{f}(\boldsymbol{\xi} + i\mathbf{b}) = \mathcal{F}_{\mathbf{x} \rightarrow \boldsymbol{\xi}}(e^{\mathbf{b} \cdot \mathbf{x}} f(\mathbf{x}))$. The right hand side still makes sense for $|\mathbf{b}| = \beta/2$ as an $L^2(\mathbb{R}^d)$ -function. And according to this identity and Plancherel's formula there holds $\mathbf{b} \mapsto \hat{f}(\cdot + i\mathbf{b}) \in C(\overline{B(\beta/2, 0)}; L^2(\mathbb{R}^d))$, where $B(\beta/2, 0) := \{\mathbf{b} \in \mathbb{R}^d : |\mathbf{b}| < \beta/2\}$. We can use this fact to define the norm

$$\|f\|_{\omega}^2 := \sum_{\ell=1}^d \left\| \hat{f}\left(\cdot + i\frac{\beta}{2}\boldsymbol{\delta}_{\ell}\right) \right\|_{L^2(\mathbb{R}^d)}^2 + \left\| \hat{f}\left(\cdot - i\frac{\beta}{2}\boldsymbol{\delta}_{\ell}\right) \right\|_{L^2(\mathbb{R}^d)}^2,$$

where $\boldsymbol{\delta}_{\ell} \in \mathbb{R}^d$ is the vector whose ℓ -th component is one, and all others are zero. The norm $\|\cdot\|_{\omega}$ is equivalent to $\|\cdot\|_{\omega}$.

Next we specify the conditions on the perturbation Θ .

(C_d) Conditions on Θ : We assume that $\Theta f = \vartheta * f$, for $f \in \mathcal{E}$, where ϑ is a tempered distribution that fulfills the following properties in $\Omega_{\beta/2}$ for some $\beta > 0$:

- (i) The Fourier transform $\hat{\vartheta}$ can be extended to an analytic function in $\Omega_{\beta/2}$ (also denoted by $\hat{\vartheta}$), and $\hat{\vartheta} \in L^{\infty}(\Omega_{\beta/2})$.
- (ii) It holds $\hat{\vartheta}(\mathbf{0}) = 0$, i.e. ϑ has zero mass.
- (iii) The mapping $\boldsymbol{\xi} \mapsto \operatorname{Re} \int_0^1 \hat{\vartheta}(\boldsymbol{\xi}s)/s \, ds$ is essentially bounded in $\Omega_{\beta/2}$.

Condition **(C_d)(i)** ensures that $\Theta \in \mathcal{B}(\mathcal{E})$, which is seen by using the norm $\|\cdot\|_{\omega}$, and due to **(C_d)(ii)** we have $\Theta : \mathcal{E}_k \rightarrow \mathcal{E}_{k+1}$ for all $k \in \mathbb{N}_0$. In the following we always assume that **(C_d)** holds.

Proposition 4.3. *We have the following spectral properties of $\mathcal{L} + \Theta$ in \mathcal{E} :*

- (i) $\sigma(\mathcal{L} + \Theta) = -\mathbb{N}_0$.
- (ii) For each $k \in \mathbb{N}_0$, the eigenspace $\ker(\mathcal{L} + \Theta + k)$ has the dimension $\binom{k+d-1}{k}$.
- (iii) Under appropriate scaling, the eigenfunctions $f_{\mathbf{k}}$ to the eigenvalue $-k \in \mathbb{N}_0$ are explicitly given by

$$f_{\mathbf{k}} = D^{\mathbf{k}} f_{\mathbf{0}}, \quad |\mathbf{k}| = k, \quad (4.6)$$

where

$$\hat{f}_{\mathbf{0}}(\boldsymbol{\xi}) := \exp\left(-\frac{\boldsymbol{\xi} \cdot \boldsymbol{\xi}}{2} + \int_0^1 \frac{\hat{\vartheta}(\boldsymbol{\xi}s)}{s} \, ds\right), \quad \boldsymbol{\xi} \in \Omega_{\beta/2} \subset \mathbb{C}^d. \quad (4.7)$$

Thereby $f_{\mathbf{0}}$ is the unique stationary solution of the perturbed Fokker-Planck equation (1.1) with unit mass.

Proof (Sketch). Since the resolvent is compact (see the discussion above), the spectrum consists only of eigenvalues. As in the one-dimensional case one shows $\sigma(\mathcal{L} + \Theta) \subset -\mathbb{N}_0$ by applying $\Pi_{\mathcal{L}, k}$ to the eigenvalue equation. This also implies $\dim \ker(k + \mathcal{L} + \Theta) \leq \dim \operatorname{ran} \Pi_{\mathcal{L}, k} = \binom{k+d-1}{k}$. Then one verifies that the functions $f_{\mathbf{k}}$ given in (4.6) are eigenfunctions, and lie in \mathcal{E} , according to the condition (4.5). Since $\dim \operatorname{span}\{f_{\mathbf{k}} : |\mathbf{k}| = k\} = \binom{k+d-1}{k}$, there are no further eigenfunctions, due to the previous estimate on the dimension of the eigenspaces. So $\ker(k + \mathcal{L} + \Theta) = \operatorname{span}\{f_{\mathbf{k}} : |\mathbf{k}| = k\}$ for all $k \in \mathbb{N}_0$. \square

Now we introduce

$$\hat{\psi}(\boldsymbol{\xi}) := \exp\left(\int_0^1 \frac{\hat{\vartheta}(\boldsymbol{\xi}s)}{s} \, ds\right), \quad \boldsymbol{\xi} \in \Omega_{\beta/2},$$

and the mapping $\Psi : f \mapsto f * \psi$. The results of Lemma 3.12 for Ψ still hold, and due to (4.7) we have for all $\mathbf{k} \in \mathbb{N}_0$:

$$f_{\mathbf{k}} = \Psi \mu_{\mathbf{k}}.$$

As in Proposition 3.13 we obtain $R_{\mathcal{L}+\Theta}(\zeta)|_{\mathcal{E}_k} = \Psi \circ R_{\mathcal{L}}(\zeta) \circ \Psi^{-1}|_{\mathcal{E}_k}$, for all $k \in \mathbb{N}_0$ and $\zeta \in \mathbb{C} \setminus \{-k, -k-1, \dots\}$. The estimates (3.6) and (3.8) immediately follow, and for the convergence to the stationary solution see Remark 3.15. As in Section 3 we finally have:

Theorem 4.4. *Let $\mathcal{E} = L^2(\omega(\mathbf{x}) \, d\mathbf{x})$, where $\omega(\mathbf{x}) = \cosh \beta|\mathbf{x}|$, for some $\beta > 0$ and $\mathbf{x} \in \mathbb{R}^d$, and let Θ fulfill the condition (\mathbf{C}_d) for this $\beta > 0$. Then the perturbed operator $\mathcal{L} + \Theta$ has the following properties in \mathcal{E} :*

- (i) *It has compact resolvent, and $\sigma(\mathcal{L} + \Theta) = \sigma_p(\mathcal{L} + \Theta) = -\mathbb{N}_0$.*
- (ii) *There holds $M(\mathcal{L} + \Theta + k) = \ker(\mathcal{L} + \Theta + k) = \text{span}\{f_{\mathbf{k}} : |\mathbf{k}| = k\}$, where the $f_{\mathbf{k}}$ are the eigenfunctions given by (4.6). They are related by $f_{\mathbf{k}} = D^{\mathbf{k}}f_0$.*
- (iii) *The spectral projection \mathcal{P}_k to the eigenvalue $-k \in -\mathbb{N}_0$ fulfills $\text{ran } \mathcal{P}_k = \text{span}\{f_{\mathbf{k}} : |\mathbf{k}| = k\}$ and $\ker \mathcal{P}_k = \mathcal{E}_{k+1} \oplus \text{span}\{f_{\mathbf{k}} : |\mathbf{k}| < k\}$, where the $(\mathcal{L} + \Theta)$ -invariant spaces \mathcal{E}_k are explicitly given in (4.3).*
- (iv) *For every $k \in \mathbb{N}_0$, the operator $(\mathcal{L} + \Theta)|_{\mathcal{E}_k}$ generates a C_0 -semigroup $(e^{t(\mathcal{L}+\Theta)}|_{\mathcal{E}_k})_{t \geq 0}$ in \mathcal{E}_k , which satisfies the estimate*

$$\|e^{t(\mathcal{L}+\Theta)}|_{\mathcal{E}_k}\|_{\mathcal{B}(\mathcal{E}_k)} \leq \tilde{C}_k e^{-kt}, \quad t \geq 0,$$

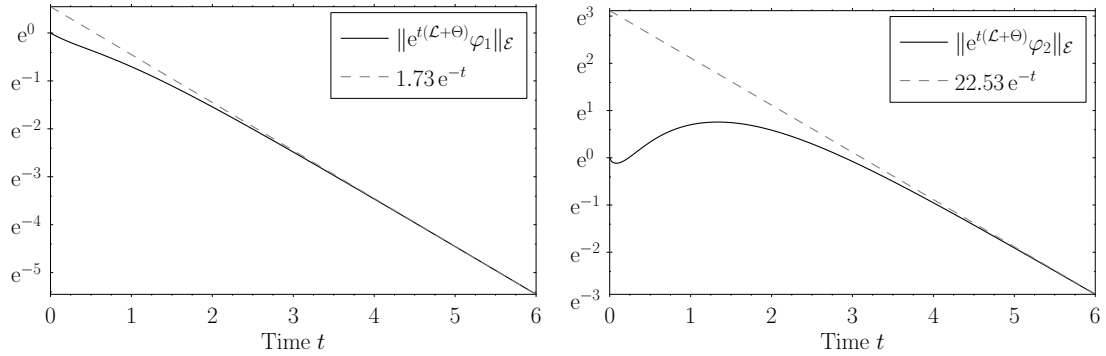
where the constant $\tilde{C}_k > 0$ is independent of t .

5. Simulation Results

In this section we shall illustrate numerically the exponential convergence for the one-dimensional perturbed Fokker-Planck equation (1.1), with $\vartheta := \varepsilon(\delta_{-\alpha} - \delta_{\alpha})$, i.e. $\Theta f(x) = \varepsilon(f(x + \alpha) - f(x - \alpha))$, for some $\varepsilon, \alpha \in \mathbb{R}$. The eigenfunctions f_k of the evolution operator $\mathcal{L} + \Theta$ can be obtained by an inverse Fourier transform, with \hat{f}_k explicitly given in (3.3). If the initial condition φ is a (finite) linear combination of the f_k , the solution to (1.1) reads explicitly

$$f(t, x) = e^{t(\mathcal{L}+\Theta)} \left[\sum_{j=1}^n a_j f_{k_j} \right] = \sum_{j=1}^n a_j e^{-k_j t} f_{k_j}, \quad \forall t \geq 0. \quad (5.1)$$

In the simulation we use a mass conserving Crank-Nicolson finite difference scheme for (1.1). It is employed on the spatial interval $[-25, 25]$ (with 1500 gridpoints) along with zero-flux boundary conditions. Moreover, we choose $\alpha = \varepsilon = 2$ and $\beta = 1$, i.e. $\mathcal{E} = L^2(\cosh x)$.



(a) Initial condition $\varphi_1 = (f_1 - 1.32f_2)/\|f_1 - 1.32f_2\|_{\mathcal{E}}$. (b) Initial condition $\varphi_2 = (\chi_{[-4,0]} - \chi_{[0,4]})/\|\chi_{[-4,0]} - \chi_{[0,4]}\|_{\mathcal{E}} \in \mathcal{E}_1$.

Figure 1. Evolution of the norm $\|\cdot\|_{\mathcal{E}}$ of solutions of the perturbed equation for different initial conditions φ .

The following numerical results verify the decaying behaviour of solutions to (1.1), and yield an estimate to the constants \tilde{C}_k from Theorem 3.17. First we consider the initial condition $\varphi_1 = (f_1 - 1.32f_2)/\|f_1 - 1.32f_2\|_{\mathcal{E}}$. For the corresponding solution we plot $\|f(t, \cdot)\|_{\mathcal{E}}$ in Figure

1(a). Since the sequence $(f_k)_{k \in \mathbb{N}}$ is not orthogonal in \mathcal{E} , the initial decay rate is here smaller than the individual decay rate of f_1 (i.e. -1). But after some time, the f_1 -term becomes dominant, and the decay rate approaches -1 . For large times, the norm behaves approximately like $1.73 e^{-t}$, so we have the lower bound $\tilde{C}_1 \geq 1.73$.

As a second example we choose the initial condition $\varphi_2 = (\chi_{[-4,0]} - \chi_{[0,4]}) / \|\chi_{[-4,0]} - \chi_{[0,4]}\|_{\mathcal{E}}$. It lies in \mathcal{E}_1 since it is massless. The evolution of the norm is displayed in Figure 1(b). Here, the norm even increases initially. Only after some time, the norm begins to decay with a rate tending to -1 . For large times t , the norm behaves approximately like $22.53 e^{-t}$, which shows $\tilde{C}_1 \geq 22.53$.

Appendix A. Space Enlargement

To begin with, we review some properties of spectral projections and resolvents, cf. [29, Chapters V.9-10], [30, Chapter VIII.8] and [19, Sections III.6.4-5].

In the sequel, X is a Hilbert space, $A \in \mathcal{C}(X)$, and we assume $\lambda \in \sigma(A)$ to be an isolated point of the spectrum. Then the corresponding spectral projection $P_{A,\lambda}$ is defined by (2.1), and λ is an isolated singularity of the resolvent $R_A(\zeta)$. Moreover, $R_A(\zeta)$ can be expanded in a Laurent series in a neighbourhood of λ :

$$R_A(\zeta) = \sum_{n=-\infty}^{\infty} (\zeta - \lambda)^n A_n, \quad \text{where } A_n := \frac{1}{2\pi i} \int_{\Gamma} \frac{R_A(z)}{(z - \lambda)^{n+1}} dz, \quad (\text{A.1})$$

and Γ is a sufficiently small, closed, counter-clockwise curve around λ , see also (2.1). For every $n \in \mathbb{Z}$ there holds $A_n \in \mathcal{B}(X)$, and A_{-1} is equal to $P_{A,\lambda}$.

Proposition A.1. *For every $n \in \mathbb{N}$ we have*

$$\begin{aligned} \text{ran}(\lambda - A)^n &\supseteq \ker P_{A,\lambda}, \\ \ker(\lambda - A)^n &\subseteq \text{ran} P_{A,\lambda}. \end{aligned}$$

There exists some $n \in \mathbb{N}$ such that both inclusion relations become equalities iff λ is a pole of $R_A(\zeta)$. In this case $\lambda \in \sigma_p(A)$, i.e. an eigenvalue.

Proposition A.2. *For the reduction of A by a fixed spectral projection $P_{A,\lambda}$ we have:*

- (i) *There holds $P_{A,\lambda} D(A) \subset D(A)$, and $\ker P_{A,\lambda}$ and $\text{ran} P_{A,\lambda}$ are A -invariant subspaces of X .*
- (ii) *$A|_{\text{ran} P_{A,\lambda}} \in \mathcal{C}(\text{ran} P_{A,\lambda})$ and $A|_{\ker P_{A,\lambda}} \in \mathcal{C}(\ker P_{A,\lambda})$.*
- (iii) *There holds $\sigma(A|_{\text{ran} P_{A,\lambda}}) = \{\lambda\}$ and $\sigma(A|_{\ker P_{A,\lambda}}) = \sigma(A) \setminus \{\lambda\}$. Furthermore $A|_{\text{ran} P_{A,\lambda}} \in \mathcal{B}(\text{ran} P_{A,\lambda})$.*
- (iv) *If $\dim \text{ran} P_{A,\lambda} < \infty$, then $\lambda - A|_{\text{ran} P_{A,\lambda}}$ is nilpotent, λ is a pole of $R_A(\zeta)$, and $\lambda \in \sigma_p(A)$.*
- (v) *If λ is a pole of $R_A(\zeta)$, then $M(\lambda - A) = \ker(\lambda - A)$ iff the pole has order one.*

In the self-adjoint case we have the following result, cf. [19, Section V.3.5]:

Proposition A.3. *If A is self-adjoint, then any isolated point λ of the spectrum is an eigenvalue, and the algebraic eigenspace coincides with the (geometric) eigenspace. Furthermore, λ is a pole of $R_A(\zeta)$ of order 1, and the corresponding spectral projection is orthogonal.*

For a finite number of isolated points of the spectrum we have:

Lemma A.4. *For $N \in \mathbb{N}_0$, let A have isolated points of the spectrum $\zeta_0, \dots, \zeta_{N-1}$, which are eigenvalues with $\dim M(\zeta_k - A) < \infty$ for all $0 \leq k \leq N-1$. Assume there exists a closed subspace $Y \subset X$, such that*

- (i) *Y is A -invariant, and $\sigma(A|_Y) \cap \{\zeta_0, \dots, \zeta_{N-1}\} = \emptyset$.*
- (ii) *X can be decomposed as $X = Y \oplus M(\zeta_0 - A) \oplus \dots \oplus M(\zeta_{N-1} - A)$.*

Then $Y = \ker \Pi_A$, where $\Pi_A := \Pi_{A,0} + \dots + \Pi_{A,N-1}$ is the sum of the spectral projections $\Pi_{A,k}$ corresponding to the ζ_k , and $M(\zeta_k - A) = \text{ran} \Pi_{A,k}$ for all $0 \leq k \leq N-1$.

Proof. According to the assumptions there holds $\sigma(A|_Y) = \sigma(A) \setminus \{\zeta_0, \dots, \zeta_{N-1}\}$, and therefore the map $\zeta \mapsto R_A(\zeta)|_Y$ is analytic in $\rho(A) \cup \{\zeta_0, \dots, \zeta_{N-1}\}$. Due to the definition (2.1) of spectral projections this implies that $\Pi_{A,k}Y \equiv 0$ for every $\Pi_{A,k}$, and therefore $Y \subseteq \ker \Pi_A$. On the other hand we have $M(\zeta_k - A) \subseteq \text{ran } \Pi_{A,k}$ for all $0 \leq k \leq N-1$, according to Proposition A.1. From (ii) we conclude that the inclusions have to be equalities, otherwise $\ker \Pi_A \cap \text{ran } \Pi_A \neq \{0\}$, which is impossible. \square

Next we give an explicit bound for the resolvent in case that A generates a C_0 -semigroup and $R_A(\zeta)$ has a finite number of poles in Δ_a .

Lemma A.5. *Assume that A satisfies (L1') for $a \in \mathbb{R}$ and $\zeta_0, \dots, \zeta_{N-1} \in \Delta_a$, for some $N \in \mathbb{N}_0$. Furthermore, let A generate a C_0 -semigroup $(e^{tA})_{t \geq 0}$ satisfying the growth bound*

$$\left\| e^{tA} \left(1 - \sum_{k=0}^{N-1} \Pi_{A,k} \right) \right\|_{\mathcal{B}(X)} \leq C_a e^{at}, \quad \forall t \geq 0, \quad (\text{A.2})$$

for some constant $C_a \geq 1$, where $\Pi_{A,k}$ denotes the spectral projection of A corresponding to ζ_k . Then there exists a constant $C > 0$ such that for all $\zeta \in \Delta_a \setminus \{\zeta_0, \dots, \zeta_{N-1}\}$:

$$\|R_A(\zeta)\|_{\mathcal{B}(X)} \leq C \max \left\{ \frac{C_a}{\text{Re } \zeta - a}, \frac{|\zeta - \zeta_0|^{d_0-1} + 1}{|\zeta - \zeta_0|^{d_0}}, \dots, \frac{|\zeta - \zeta_{N-1}|^{d_{N-1}-1} + 1}{|\zeta - \zeta_{N-1}|^{d_{N-1}}} \right\}, \quad (\text{A.3})$$

where d_k is the order of the pole of $R_A(\zeta)$ at ζ_k .

Proof. Let $\zeta \in \Delta_a \setminus \{\zeta_0, \dots, \zeta_{N-1}\}$. We have the decomposition of X according to $X = Y \oplus M(\zeta_0 - A) \oplus \dots \oplus M(\zeta_{N-1} - A)$ into a finite number of closed subspaces, where $Y := \ker(\Pi_{A,0} + \dots + \Pi_{A,N-1})$. First we consider $R_A(\zeta)$ on each subspace individually. On Y we have, due to (A.2) and the Hille-Yosida theorem, the estimate

$$\|R_A(\zeta)|_Y\|_{\mathcal{B}(Y)} \leq \frac{C_a}{\text{Re } \zeta - a}. \quad (\text{A.4})$$

Next we consider $R_A(\zeta)$ on $Y_k := M(\zeta_k - A)$ for some fixed $0 \leq k \leq N-1$. According to Proposition A.1 there holds $Y_k = \text{ran } \Pi_{A,k}$. Now the coefficients A_n in (A.1) (with $\lambda := \zeta_k$) satisfy the relation $A_n \Pi_{A,k} = 0$ for all $n \in \mathbb{N}_0$ (using $A_{-1} = \Pi_{A,k}$ in [8, Proposition VIII.1.3]). Therefore we have due to (A.1)

$$R_A(\zeta)|_{Y_k} = R_A(\zeta) \Pi_{A,k} = \sum_{n=1}^{d_k} \frac{A_{-n}}{(\zeta - \zeta_k)^n}. \quad (\text{A.5})$$

Since the operators A_n are all bounded, we deduce for every $0 \leq k \leq N-1$ the estimate

$$\|R_A(\zeta)|_{Y_k}\|_{\mathcal{B}(Y_k)} \leq \sum_{n=1}^{d_k} \frac{\|A_n\|_{\mathcal{B}(X)}}{|\zeta - \zeta_k|^n} \leq D_k \left(\frac{1}{|\zeta - \zeta_k|} + \frac{1}{|\zeta - \zeta_k|^{d_k}} \right) = D_k \frac{|\zeta - \zeta_k|^{d_k-1} + 1}{|\zeta - \zeta_k|^{d_k}},$$

where $D_k > 0$ is a constant. Using the unique decomposition of elements of X according to $X = Y \oplus Y_0 \oplus \dots \oplus Y_{N-1}$, we may estimate the norm $\|R_A(\zeta)\|_{\mathcal{B}(X)}$ by the sum of the norms of $R_A(\zeta)$ on each of the subspaces of the decomposition of X . Combining (A.4) and (A.5) yields the desired estimate. \square

We also need to introduce the following technical lemma:

Lemma A.6. *Consider two Hilbert spaces $X \hookrightarrow \mathcal{X}$, and a projection $P_{\mathcal{X}} \in \mathcal{B}(\mathcal{X})$, such that $P_X := P_{\mathcal{X}}|_X \in \mathcal{B}(X)$. Then $\text{ran } P_{\mathcal{X}} = \text{cl}_{\mathcal{X}} \text{ran } P_X$ and $\ker P_{\mathcal{X}} = \text{cl}_{\mathcal{X}} \ker P_X$.*

Proof. We give here the proof of the equality of the ranges, the other identity can be shown analogously, using the complementary projections instead. On the one hand we have $\text{ran } P_X \subseteq \text{ran } P_{\mathcal{X}}$, and so $\text{cl}_{\mathcal{X}} \text{ran } P_X \subseteq \text{ran } P_{\mathcal{X}}$, since $\text{ran } P_{\mathcal{X}}$ is closed in \mathcal{X} due to the boundedness of $P_{\mathcal{X}}$. On the other hand $P_{\mathcal{X}} = \text{cl}_{\mathcal{X}} P_X$, which implies $\text{ran } P_{\mathcal{X}} \subseteq \text{cl}_{\mathcal{X}} \text{ran } P_X$. \square

Now we turn to the actual theory of enlarging the space X . The following theorem is an extension of [15, Theorem 2.1]. Here we cover the more general case in which the isolated points $\zeta_0, \dots, \zeta_{N-1}$ of the spectrum are not necessarily eigenvalues of A .

Theorem A.7. *Let $X \hookrightarrow \mathcal{X}$ be two Hilbert spaces, and $A \in \mathcal{C}(X)$, $\mathcal{A} \in \mathcal{C}(\mathcal{X})$ such that $\mathcal{A} = \text{cl}_{\mathcal{E}} A$. Assume there are numbers $a \in \mathbb{R}$ and $\zeta_0, \dots, \zeta_{N-1} \in \Delta_a$ for some $N \in \mathbb{N}_0$, such that A satisfies **(L1)** and \mathcal{A} in \mathcal{X} satisfies **(L2)**. Then there holds:*

1. \mathcal{A} fulfills **(L1)** for the same $a \in \mathbb{R}$ and $\zeta_0, \dots, \zeta_{N-1} \in \Delta_a$.
2. For any $\zeta \in \Delta_a \setminus \{\zeta_0, \dots, \zeta_{N-1}\}$ we have the representation

$$R_{\mathcal{A}}(\zeta) = (\zeta - \mathcal{S})^{-1} + R_A(\zeta)\mathcal{B}(\zeta - \mathcal{S})^{-1},$$

which implies the estimate

$$\|R_{\mathcal{A}}(\zeta)\|_{\mathcal{B}(\mathcal{X})} \leq \|(\zeta - \mathcal{S})^{-1}\|_{\mathcal{B}(\mathcal{X})} + \|R_A(\zeta)\|_{\mathcal{B}(X)} \|\mathcal{B}(\zeta - \mathcal{S})^{-1}\|_{\mathcal{B}(X, \mathcal{X})}.$$

3. If we even require **(L1')** for A , then \mathcal{A} also satisfies **(L1')** for the same $a \in \mathbb{R}$ and $\zeta_0, \dots, \zeta_{N-1} \in \Delta_a$, and the corresponding algebraic eigenspaces coincide, i.e. $M(\zeta_k - A) = M(\zeta_k - \mathcal{A})$.

Proof. For the proof of the inclusion $\sigma(\mathcal{A}) \cap \Delta_a \subseteq \{\zeta_0, \dots, \zeta_{N-1}\}$ and Result 2 we refer to [15, Theorem 2.1]. For the reverse inclusion, the proof in [15] covers only the case $\zeta_k \in \sigma_p(A)$. Hence, we provide here an independent demonstration: Choose some $\zeta_k \in \sigma(A) \cap \Delta_a$. The corresponding spectral projections satisfy $\Pi_{A,k} \subset \Pi_{\mathcal{A},k} \in \mathcal{B}(\mathcal{X})$, since $R_A(\zeta) \subset R_{\mathcal{A}}(\zeta)$ for any $\zeta \in \Delta_a \setminus \{\zeta_0, \dots, \zeta_{N-1}\}$, cf. (2.1). If now $\zeta_k \notin \sigma(\mathcal{A})$, the function $\zeta \mapsto R_{\mathcal{A}}(\zeta)$ would be analytic around ζ_k , and therefore $\Pi_{\mathcal{A},k} \equiv 0$. But this would imply $\Pi_{A,k} \equiv 0$, which is a contradiction to the fact that $\zeta_k \in \sigma(A)$, and so we actually obtain $\sigma(\mathcal{A}) \cap \Delta_a = \{\zeta_0, \dots, \zeta_{N-1}\}$.

Let us now replace the assumption **(L1)** on A by the stronger requirement **(L1')**. We may apply Lemma A.6 to the spectral projections $\Pi_{A,k} \subset \Pi_{\mathcal{A},k}$, and get $\text{ran } \Pi_{\mathcal{A},k} = \text{cl}_{\mathcal{X}} \text{ran } \Pi_{A,k}$. Since $\dim \text{ran } \Pi_{A,k} < \infty$, $\text{ran } \Pi_{A,k}$ is closed in \mathcal{X} , and we conclude that $\text{ran } \Pi_{\mathcal{A},k} = \text{ran } \Pi_{A,k}$. Due to Proposition A.2 (iv) and Proposition A.1 the equality of the eigenspaces follows. \square

Now we finally turn to the proof of Theorem 2.5, parts of which are based on [15] and [3]:

Proof of Theorem 2.5. From the assumptions (ii)-(iv) the Result 2 is obtained by applying the Lumer-Phillips theorem. From this we conclude $\sigma(A|_{X_N}) \cap \Delta_a = \emptyset$, and together with the assumptions (i)-(ii) the Result 1 follows from Lemma A.4.

For Result 3 we adopt the proof of [15, Corollary 4.2]. We begin by showing that \mathcal{S} generates a C_0 -semigroup on \mathcal{X} . Since we assume $\mathcal{B}|_X \in \mathcal{B}(X)$, and since A generates a C_0 -semigroup on X due to the Result 2 just proven, the operator $\mathcal{S}|_{D(A)} = A - \mathcal{B}|_{D(A)}$ also generates a C_0 -semigroup on X . Due to the continuous embedding $X \hookrightarrow \mathcal{X}$, the mapping $t \mapsto e^{t\mathcal{S}|_{D(A)}} f$ lies in $C^1([0, \infty]; \mathcal{X})$ for any $f \in D(A) \subset X$, and we compute

$$\frac{d}{dt} \|e^{t\mathcal{S}|_{D(A)}} f\|_{\mathcal{X}}^2 = 2 \text{Re} \langle \mathcal{S} e^{t\mathcal{S}|_{D(A)}} f, e^{t\mathcal{S}|_{D(A)}} f \rangle_{\mathcal{X}} \leq 2a \|e^{t\mathcal{S}|_{D(A)}} f\|_{\mathcal{X}}^2,$$

since $\mathcal{S} - a$ is dissipative. Applying the Gronwall inequality this implies

$$\|e^{t\mathcal{S}|_{D(A)}} f\|_{\mathcal{X}} \leq e^{at} \|f\|_{\mathcal{X}}, \quad \forall f \in D(A), t \geq 0. \quad (\text{A.6})$$

Because of $X \hookrightarrow \mathcal{X}$ we have $D(A) \subset \mathcal{X}$ dense. So we can consider the closure $T(t) := \text{cl}_{\mathcal{X}} e^{t\mathcal{S}|_{D(A)}}$. We conclude that $(T(t))_{t \geq 0}$ is a C_0 -semigroup on \mathcal{X} with generator $\text{cl}_{\mathcal{X}} \mathcal{S}|_{D(A)}$ (for this see [11, Section II.2.3]). On the basis of this result we now properly redefine $\mathcal{S} := \text{cl}_{\mathcal{E}} \mathcal{S}|_{D(A)}$. Together with $\mathcal{B} \in \mathcal{B}(\mathcal{X})$ this implies that also $\mathcal{A} := \text{cl}_{\mathcal{X}} A \equiv \mathcal{B} + \mathcal{S}$ is the generator of a C_0 -semigroup, which we denote by $(e^{t\mathcal{A}})_{t \geq 0}$. So the operator \mathcal{A} satisfies **(L2)**: Condition **(L2)(i)** is fulfilled by the definition of $\mathcal{A} = \mathcal{B} + \mathcal{S}$, and **(L2)(ii)** holds true for \mathcal{S} (with $N = 0$) due to the semigroup estimate (A.6). Finally **(L2)(iii)** is satisfied, since \mathcal{A} generates a C_0 -semigroup as well. Thus, the requirements of Theorem A.7 being fulfilled, we conclude that $\sigma(\mathcal{A}) \cap \Delta_a = \{\zeta_0, \dots, \zeta_{N-1}\}$. By Result 1 of Theorem 2.5 we know that A satisfies **(L1')**. Hence,

Theorem A.7 further implies that \mathcal{A} also satisfies **(L1')**, the algebraic eigenspaces $M(\zeta_k - \mathcal{A})$ and $M(\zeta_k - A)$ coincide for $0 \leq k \leq N-1$, and we have the representation of the resolvent

$$R_{\mathcal{A}}(\zeta) = (\zeta - \mathcal{S})^{-1} + R_{\mathcal{A}}(\zeta)\mathcal{B}(\zeta - \mathcal{S})^{-1}, \quad \forall \zeta \in \Delta_a \setminus \{\zeta_0, \dots, \zeta_{N-1}\}. \quad (\text{A.7})$$

For Result 4 we first characterize $\Pi_{A,k}$ in X . Since A satisfies **(L1')**, $R_A(\zeta)$ has poles at $\zeta = \zeta_k$, cf. Proposition A.2 (iv). Hence, Proposition A.1 implies that $\text{ran } \Pi_{A,k} = M(\zeta_k - A)$ for $0 \leq k \leq N-1$. Moreover, from the Result 1 shown above we have

$$\ker \Pi_{A,k} = X_N \oplus \left(\bigoplus_{j \neq k} M(\zeta_j - A) \right).$$

Applying Lemma A.6 then yields the desired result for $\Pi_{A,k}$.

We now prove Result 5 by using the representation (A.7) of the resolvent. For $(\zeta - \mathcal{S})^{-1}$ we have (A.6), so we may apply the Hille-Yosida theorem to get the resolvent estimate

$$\|(\zeta - \mathcal{S})^{-1}\|_{\mathcal{B}(\mathcal{X})} \leq \frac{1}{\text{Re } \zeta - a}, \quad \text{Re } \zeta > a.$$

Due to the Results 1 and 2 already shown we may use the Lemma A.5 for $R_{\mathcal{A}}(\zeta)$, where $\text{ran}(1 - (\Pi_{A,0} + \dots + \Pi_{A,N-1})) = X_N$ and $C_a := 1$. So we obtain the estimate (A.3) for $R_{\mathcal{A}}(\zeta)$ in $\mathcal{B}(X)$. Altogether we obtain from (A.7) for all $\zeta \in \Delta_a \setminus \{\zeta_0, \dots, \zeta_{N-1}\}$:

$$\|R_{\mathcal{A}}(\zeta)\|_{\mathcal{B}(\mathcal{X})} \leq \frac{1}{\text{Re } \zeta - a} \left[1 + C \|\mathcal{B}\|_{\mathcal{B}(\mathcal{X}, X)} \max \left\{ \frac{1}{\text{Re } \zeta - a}, \frac{|\zeta - \zeta_0|^{d_0-1} + 1}{|\zeta - \zeta_0|^{d_0}}, \dots \right. \right. \quad (\text{A.8}) \\ \left. \left. \dots, \frac{|\zeta - \zeta_{N-1}|^{d_{N-1}-1} + 1}{|\zeta - \zeta_{N-1}|^{d_{N-1}}} \right\} \right].$$

Now we follow an idea from the proof of [3, Proposition 4.8]: Due to the Result 4 just shown above, $\zeta \mapsto R_{\mathcal{A}}(\zeta)|_{\mathcal{X}_N}$ is analytic in Δ_a , so it has no poles and is uniformly bounded on every compact subset of Δ_a . Now we fix a compact set $\Omega_c \subset \Delta_a$, such that $\{\zeta_0, \dots, \zeta_{N-1}\} \subset \Omega_c^c$, and $R_{\mathcal{A}}(\zeta)|_{\mathcal{X}_N}$ is uniformly bounded in Ω_c . In the ‘‘complement’’ $\Delta_a \setminus \Omega_c$ we may apply (A.8). Since we have excluded the poles of $R_{\mathcal{A}}(\zeta)$ in Δ_a , it implies that $R_{\mathcal{A}}(\zeta)$ remains uniformly bounded on $\Delta_{\tilde{a}} \setminus \Omega_c$ for any $\tilde{a} > a$. Combining these estimates we get

$$\sup_{\text{Re } \zeta > \tilde{a}} \|R_{\mathcal{A}}(\zeta)|_{\mathcal{X}_N}\|_{\mathcal{B}(\mathcal{X}_N)} < \infty, \quad \forall \tilde{a} > a.$$

Then the Gearhart-Prüss-Greiner theorem (cf. [11, Theorem V.1.11]) yields the Result 5. \square

Appendix B. Fourier Transform of the Resolvent

This section deals with the explicit computation of the Fourier transform of the resolvent $R_{\mathcal{L}+\Theta}(\zeta)$ of the (one-dimensional) perturbed Fokker-Planck operator $\mathcal{L}+\Theta$ in \mathcal{E} , where Θ fulfills the condition **(C)**. We begin by considering the resolvent equation

$$(\zeta - \mathcal{L} - \Theta)f = g \quad (\text{B.1})$$

on \mathbb{R} , where we assume $\zeta \in \Delta_{-k}$ and $f, g \in \mathcal{E}_k$ for some $k \in \mathbb{N}_0$. We apply the Fourier transform, which yields the following differential equation:

$$\xi \left[\hat{f}'(\xi) + \left(\xi + \frac{\zeta - \hat{\vartheta}(\xi)}{\xi} \right) \hat{f}(\xi) \right] = \hat{g}(\xi). \quad (\text{B.2})$$

By defining $\tilde{f} := \hat{f}/\hat{f}_0$ and $\tilde{g} := \hat{g}/\hat{f}_0$ we obtain the equivalent equation

$$\xi \tilde{f}'(\xi) + \zeta \tilde{f}(\xi) = \tilde{g}(\xi). \quad (\text{B.3})$$

The general solution for $\xi \in \mathbb{R}^\pm$ reads

$$\tilde{f}(\xi) = \int_0^1 \tilde{g}(\xi s) s^{\zeta-1} ds + C_\pm \xi^{-\zeta} =: I(\xi) + C_\pm \xi^{-\zeta}, \quad (\text{B.4})$$

where the $C_{\pm} \in \mathbb{C}$ are integration constants to be determined. For defining $\xi^{\zeta} := \exp(\zeta \ln \xi)$, we choose the following branch of the logarithm: $\ln \xi := \log |\xi| + i \arg \xi$, with $\arg \xi \in [-\frac{\pi}{2}, \frac{3\pi}{2})$, and \log is the natural logarithm on \mathbb{R}^+ .

First we shall show that the integral $I(\xi)$ is an analytic function on $\Omega_{\beta/2}$: If $g \in \mathcal{E}_k$, then \tilde{g} is analytic in $\Omega_{\beta/2}$ and has a zero at $\xi = 0$ of order not less than k , see (2.8). Therefore, for any fixed $\zeta \in \Delta_{-k}$,

$$\tilde{g}(\xi s) s^{\zeta-1} = \frac{\tilde{g}(\xi s)}{s^k} s^{\zeta+k-1}, \quad s \in (0, 1],$$

is locally integrable at $s = 0$, and $I(\xi)$ is well defined for all $\xi \in \Omega_{\beta/2}$. To see that it is actually analytic, we define $I_{\varepsilon}(\xi) := \int_{\varepsilon}^1 G_k(\xi, s) s^{\zeta+k-1} ds$ for $\varepsilon \in [0, 1)$, where

$$G_k(\xi, s) := \begin{cases} \frac{\tilde{g}(\xi s)}{s^k}, & s \in (0, 1], \\ \frac{\tilde{g}^{(k)}(0) \xi^k}{k!}, & s = 0, \end{cases}$$

for $\xi \in \Omega_{\beta/2}$. The function $G_k(\cdot, s)$ is analytic in $\Omega_{\beta/2}$ for all (fixed) $s \in [0, 1]$, and G_k is continuous in $\Omega_{\beta/2} \times [0, 1]$. According to [10, Theorem 4.9.1], the functions $I_{\varepsilon}(\xi)$ are analytic in $\Omega_{\beta/2}$ for all $\varepsilon \in (0, 1)$. Now we show that $(I_{\varepsilon})_{\varepsilon \in (0, 1)}$ converges normally to I in $\Omega_{\beta/2}$ as $\varepsilon \rightarrow 0$: Let $K \subset \Omega_{\beta/2}$ be compact. Then we have

$$\begin{aligned} \sup_{\substack{\xi \in K \\ s \in [0, 1]}} |G_k(\xi, s)| &\leq \sup_{\substack{\xi \in K_0 \\ s \in [0, 1]}} |G_k(\xi, s)| = \sup_{\substack{\xi \in K_0 \setminus \{0\} \\ s \in (0, 1]}} \left| \frac{\tilde{g}(\xi s)}{(\xi s)^k} \xi^k \right| \\ &\leq \sup_{\xi \in K_0 \setminus \{0\}} \left| \frac{\tilde{g}(\xi)}{\xi^k} \right| \cdot \sup_{\xi \in K_0} |\xi^k| =: C_K < \infty, \end{aligned} \quad (\text{B.5})$$

since $\tilde{g}(\xi)/\xi^k$ is analytic in $\Omega_{\beta/2}$ (the singularity at $\xi = 0$ is removable). Thereby, K_0 is an appropriate convex, compact set with $\{0\} \cup K \subseteq K_0 \subset \Omega_{\beta/2}$, and $C_K > 0$ is a constant. With (B.5) we obtain the following estimate for $\xi \in K$ and $0 < \varepsilon \leq 1$:

$$|I(\xi) - I_{\varepsilon}(\xi)| = \left| \int_0^{\varepsilon} G_k(\xi, s) s^{\zeta+k-1} ds \right| \leq C_K \frac{e^{\operatorname{Re} \zeta + k}}{\operatorname{Re} \zeta + k}.$$

Since $\operatorname{Re} \zeta + k > 0$, this shows the normal convergence of the analytic functions I_{ε} towards I . According to [10, Theorem 4.2.3] this implies that $I(\xi)$ is analytic in $\Omega_{\beta/2}$.

Now it remains to determine the constants C_{\pm} in (B.4). If we require $f \in \mathcal{E}_k$, it is necessary that \tilde{f} is analytic in $\Omega_{\beta/2}$ and has a zero of order not less than k at $\xi = 0$. As already shown, $I(\xi)$ is analytic in $\Omega_{\beta/2}$. Furthermore, for $g \in \mathcal{E}_k$ and all (fixed) $s \in [0, 1]$, $\xi \mapsto G_k(\xi, s)$ has a zero of order not less than k at $\xi = 0$. Therefore $I(\xi) = \int_0^1 G_k(\xi, s) s^{\zeta+k-1} ds$ has the same property, so $\mathcal{F}^{-1}I \in \mathcal{E}_k$. Thus, it is sufficient to consider the term $C_{\pm} \xi^{-\zeta}$. If $\zeta \notin -\mathbb{N}_0$, then $\xi^{-\zeta}$ is not analytic in $\Omega_{\beta/2}$ anyway, hence $C_+ = C_- = 0$. If $\zeta \in \{-k+1, \dots, -1\}$ for $g \in \mathcal{E}_k$, $\xi^{-\zeta}$ is analytic, and we obtain $C_+ = C_-$ because we require continuity of the solution. But the order of the zero of $\xi^{-\zeta}$ is at most $k-1$. Since we need a zero of at least order k , we again obtain $C_+ = C_- = 0$. The conclusion of the above analysis is summarized in the following proposition:

Proposition B.1. *Let $g \in \mathcal{E}_k$ for some $k \in \mathbb{N}_0$, and $\zeta \in \Delta_{-k}$. Then the unique $f \in \mathcal{E}_k$ with $f = R_{\mathcal{L}+\Theta}(\zeta)g$ satisfies*

$$\hat{f}(\xi) = \hat{f}_0(\xi) \int_0^1 \frac{\hat{g}(s\xi)}{\hat{f}_0(s\xi)} s^{\zeta-1} ds, \quad \xi \in \Omega_{\beta/2}. \quad (\text{B.6})$$

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