ASC Report No. 31/2011

Robust exponential convergence of hp-FEM for singularly perturbed reaction diffusion systems with multiple scales

JM. Melenk, C. Xenophontos, L. Oberbroeckling

Institute for Analysis and Scientific Computing
Vienna University of Technology — TU Wien
www.asc.tuwien.ac.at ISBN 978-3-902627-04-9

Most recent ASC Reports

30/2011 *JM. Melenk, C. Xenophontos, L. Oberbroeckling*
Analytic regularity for a singularly perturbed system of reaction-diffusion equations with multiple scales: a raod map

29/2011 *JM. Melenk, C. Xenophontos, L. Oberbroeckling*
Analytic regularity for a singularly perturbed system of reaction-diffusion equations with multiple scales: proofs

28/2011 *M. Feischl, M. Karkulik, JM. Melenk, D. Praetorius*
Quasi-optimal convergence rate for an adaptive boundary element method

27/2011 *N. Happenhofer, O. Koch, F. Kupka*
IMEX Methods for the ANTARES Code

26/2011 *Michael Dreher, Angar Jüngel*
Compact families of piecewise constant functions in Lp(0, T;B)

25/2011 *Jens Geier, Anton Arnold*
WKB-based schemes for two-band Schrödinger equations in the highly ascillatory regime

24/2011 *Markus Aurada, Michael Ebner, Michael Feischl, Samuel Ferraz-Leite, Petra Goldenits, Michael Karkulik, Markus Mayr, Dirk Praetorius*
HILBERT-A MATLAB Implementation of Adaptive 2D-BEM

23/2011 *JinMyong Kim, Anton Arnold, Xiaohua Yao*
Estimates for a class of oscillatory integrals and decay rates for wave-type equations

22/2011 *Markus Aurada, Michael Feischl, Michael Karkulik, Dirk Praetorius*
Adaptive coupling of FEM and BEM: Simple error estimators and convergence

21/2011 *Michael Feischl, Michael Karkulik, Jens Markus Melenk, Dirk Praetorius*
Residual a-posteriori error stimates in BEM: Convergence of h-adaptive algorithms

Institute for Analysis and Scientific Computing
Vienna University of Technology
Wiedner Hauptstraße 8–10
1040 Wien, Austria

E-Mail: admin@asc.tuwien.ac.at
WWW: http://www.asc.tuwien.ac.at
FAX: +43-1-58801-10196

ISBN 978-3-902627-04-9

Robust exponential convergence of hp-FEM for singularly perturbed reaction diffusion systems with multiple scales

J. M. Melenk
Institut für Analysis und Scientific Computing
Vienna University of Technology
Wiedner Hauptstrasse 8-10, A-1040 Wien
AUSTRIA

C. Xenophontos
Department of Mathematics and Statistics
University of Cyprus
P.O. BOX 20537
Nicosia 1678
CYPRUS

and

L. Oberbroeckling
Department of Mathematics and Statistics
Loyola University Maryland
4501 N. Charles Street
Baltimore, MD 21210
USA

August 16, 2011

Abstract

We consider a coupled system of two singularly perturbed reaction-diffusion equations in one dimension. Associated with the two singular perturbation parameters $0 < \varepsilon \leq \mu \leq 1$, are boundary layers of length scales $O(\varepsilon)$ and $O(\mu)$. We propose and analyze an hp finite element scheme which includes elements of size $O(\varepsilon p)$ and $O(\mu p)$ near the boundary, where p is the degree of the approximating polynomials. We show that under the assumption of analytic input data, the method yields *exponential* rates of convergence, independently of ε and μ and independently of the relative size of ε to μ. In particular, the full range $0 < \varepsilon \leq \mu \leq 1$ is covered by our analysis. Numerical computations supporting the theory are also presented.

1 Introduction

The numerical solution of singularly perturbed problems has been studied extensively over the last couple of decades (see, e.g., the books [12], [13], [14] and the references therein). Besides the question of stability of discretizations (e.g., in the treatment of convection-dominated problems), a main difficulty in these problems is the presence of *boundary layers* in the solution, whose accurate approximation, independently of the singular perturbation parameter(s), is of great importance for the overall quality of the approximate solution. In the context of the Finite Element Method (FEM), the robust approximation of boundary layers requires either the use of the h version on non-uniform meshes (such as the Shishkin [17] or Bakhvalov [1] mesh), or the use of the high order p and hp versions on specially designed (variable) meshes [16]. In both cases, the *a priori* knowledge of the position of the layers is taken into account, and mesh-degree combinations can be chosen for which uniform error estimates can be established [2], [16].

In this article we consider a *system* of two coupled singularly perturbed linear reaction-diffusion equations, which have two overlapping boundary layers. In contrast to equations with a single singular perturbation parameter, systems with multiple parameters (and correspondingly multiple layers) are much less studied and understood. The problem under consideration here was studied by Matthews et al. [7, 8], Madden and Stynes [6], and by Linß and Madden [3, 4] in the context of finite differences, and by Linß and Madden [2] in the context of the h version of the FEM with piecewise linear basis functions. We refer also to [5] for a survey on the numerical solution of systems of singularly perturbed differential equations. In [18] an hp FEM was presented for a coupled system of reaction-diffusion equations, and its robust exponential convergence was demonstrated via several numerical experiments. The recent regularity results of [11] allow us to provide the mathematical justification of what was reported in [18], which is the purpose of this article.

The rest of the paper is organized as follows: In Section 2 we present the model problem and discuss the typical phenomena, along with the regularity of the solution as determined in [11]. In Section 3 we prove our main result, which is the exponential convergence of the proposed hp FEM, and in Section 5 we give some closing remarks.

We will utilize the usual Sobolev space notation $H^k(I)$ to denote the space of functions on I with $0, 1, 2, ..., k$ generalized derivatives in $L^2(I)$, equipped with norm and seminorm $\|\cdot\|_{k,I}$ and $|\cdot|_{k,I}$, respectively. For vector functions $\mathbf{U} := (u_1, u_2)^T$, we will write

$$\|\mathbf{U}\|_{k,I}^2 = \|u_1\|_{k,I}^2 + \|u_2\|_{k,I}^2 .$$

We will also use the space

$$H_0^1(I) = \left\{ u \in H^1(I) : u|_{\partial I} = 0 \right\},$$

where ∂I denotes the boundary of I. The norm of the space $L^\infty(I)$ of essentially bounded functions is denoted $\|\cdot\|_{\infty,I}$. Finally, the letter C will be used to denote a generic positive constant, independent of any discretization or singular perturbation parameters and possibly having different values in each occurrence.

2 The Model Problem and its Regularity

We consider the following model problem: Find a pair of functions (u, v) such that

$$\left\{\begin{array}{l} -\varepsilon^2 u''(x) + a_{11}(x)u(x) + a_{12}(x)v(x) = f(x) \text{ in } I = (0,1), \\ -\mu^2 v''(x) + a_{21}(x)u(x) + a_{22}(x)v(x) = g(x) \text{ in } I = (0,1), \end{array}\right. \tag{1a}$$

along with the boundary conditions

$$u(0) = u(1) = 0\ ,\ v(0) = v(1) = 0. \tag{1b}$$

With the abbreviations

$$\mathbf{U} = \begin{pmatrix} u \\ v \end{pmatrix}, \qquad \mathbf{E}^{\varepsilon,\mu} := \begin{pmatrix} \varepsilon^2 & 0 \\ 0 & \mu^2 \end{pmatrix}, \qquad \mathbf{A}(x) := \begin{pmatrix} a_{11}(x) & a_{12}(x) \\ a_{21}(x) & a_{22}(x) \end{pmatrix}, \qquad \mathbf{F} = \begin{pmatrix} f \\ g \end{pmatrix},$$

equations (1a)–(1b) may also be written in the following, more compact form:

$$L_{\varepsilon,\mu}\mathbf{U} := -\mathbf{E}^{\varepsilon,\mu}\mathbf{U}''(x) + \mathbf{A}(x)\mathbf{U} = \mathbf{F}, \qquad \mathbf{U}(0) = \mathbf{U}(1) = 0. \tag{2}$$

The parameters $0 < \varepsilon \le \mu \le 1$ are given, as are the functions f, g, and a_{ij}, $i,j \in \{1,2\}$, which are assumed to be analytic on $\overline{I} = [0,1]$. Moreover we assume that there exist constants $C_f, \gamma_f, C_g, \gamma_g, C_a, \gamma_a > 0$ such that

$$\left\{\begin{array}{r} \left\|f^{(n)}\right\|_{\infty,I} \le C_f \gamma_f^n n! \quad \forall\, n \in \mathbb{N}_0, \\ \left\|g^{(n)}\right\|_{\infty,I} \le C_g \gamma_g^n n! \quad \forall\, n \in \mathbb{N}_0, \\ \left\|a_{ij}^{(n)}\right\|_{\infty,I} \le C_a \gamma_a^n n! \quad \forall\, n \in \mathbb{N}_0, i,j \in \{1,2\} \end{array}\right. . \tag{3}$$

The variational formulation of (1a)–(1b) reads: Find $\mathbf{U} := (u,v) \in \left[H_0^1(I)\right]^2$ such that

$$B(\mathbf{U},\mathbf{V}) = F(\mathbf{V}) \quad \forall\, \mathbf{V} := (\overline{u},\overline{v}) \in \left[H_0^1(I)\right]^2, \tag{4}$$

where, with $\langle\cdot,\cdot\rangle_I$ the usual $L^2(I)$ inner product,

$$B(\mathbf{U},\mathbf{V}) = \varepsilon^2 \left\langle u',\overline{u}'\right\rangle_I + \mu^2 \left\langle v',\overline{v}'\right\rangle_I + \left\langle a_{11}u + a_{12}v, \overline{u}\right\rangle_I + \left\langle a_{21}u + a_{22}v, \overline{v}\right\rangle_I, \tag{5}$$

$$F(\mathbf{V}) = \langle f,\overline{u}\rangle_I + \langle g,\overline{v}\rangle_I. \tag{6}$$

The matrix-valued function $\mathbf{A}$ is assumed to be pointwise positive definite, i.e., for some fixed $\alpha > 0$

$$\overrightarrow{\xi}^T \mathbf{A} \overrightarrow{\xi} \ge \alpha^2 \overrightarrow{\xi}^T \overrightarrow{\xi} \quad \forall\, \overrightarrow{\xi} \in \mathbb{R}^2 \qquad \forall x \in \overline{I}. \tag{7}$$

It follows that the bilinear form $B(\cdot,\cdot)$ given by (5) is coercive with respect to the *energy norm*

$$\|\mathbf{U}\|_{E,I}^2 \equiv \|(u,v)\|_{E,I}^2 := \varepsilon^2 |u|_{1,I}^2 + \mu^2 |v|_{1,I}^2 + \alpha^2 \left(\|u\|_{0,I}^2 + \|v\|_{0,I}^2\right), \tag{8}$$

i.e.,

$$B(\mathbf{V},\mathbf{V}) \ge \|\mathbf{V}\|_{E,I}^2 \quad \forall\, \mathbf{V} \in \left[H_0^1(I)\right]^2. \tag{9}$$

This, along with the continuity of $B(\cdot,\cdot)$ and $F(\cdot)$ imply the unique solvability of (4). We also have the following standard *a priori* estimate

$$\|\mathbf{U}\|_{E,I} \le \max\left\{1, \frac{\|\mathbf{A}\|_{\infty,I}}{\alpha}\right\} \sqrt{\|f\|_{0,I}^2 + \|g\|_{0,I}^2}. \tag{10}$$

The finite element approximation of (1a)–(1b) reads: Find $\mathbf{U}_N := (u_N, v_N)^T \in [S_N]^2 \subset \left[H_0^1(I)\right]^2$ such that

$$B(\mathbf{U}_N, \mathbf{V}) = F(\mathbf{V}) \quad \forall\, \mathbf{V} := (\overline{u}, \overline{v})^T \in [S_N]^2, \tag{11}$$

where $[S_N]^2$ is an appropriately chosen finite dimensional subspace of $\left[H_0^1(I)\right]^2$. The unique solvability of the discrete problem (11) follows from (7), and by the well-known orthogonality relation, we have

$$\|\mathbf{U} - \mathbf{U}_N\|_{E,I} \leq \max\left\{1, \frac{\|\mathbf{A}\|_{\infty,I}}{\alpha}\right\} \inf_{\mathbf{V}\in[S_N]^2} \|\mathbf{U} - \mathbf{V}\|_{E,I}. \tag{12}$$

As is well-known for singularly perturbed problems, the choice of the space S_N must be carefully made in dependence on the layer structure of the solution $\mathbf{U}$, in order for the approximation to be *robust*, i.e., convergence is independent of ε or μ. As we will formalize in Theorem 2 below, the solution $\mathbf{U}$ has features on up to three different length scales ($O(1)$, $O(\mu)$, and $O(\varepsilon)$ with the features on the $O(\varepsilon)$ and $O(\mu)$ scale being of boundary layer type). These three different length scales have to be incorporated into the approximation space, and we will do this with the *Spectral Bounday Layer mesh* below in Definition 3.

Our design of the *Spectral Bounday Layer mesh* hinges on the regularity theory of [11] that we will discuss in more detail in Theorem 2 below. Essentially, Theorem 2 derives from asymptotic expansions of the solution. Such expansions rely on scale separation assumptions. For the present context of length scales $O(1)$, $O(\mu)$, and $O(\varepsilon)$, the following cases may occur:

- (I) The "no scale separation case" which occurs when *neither* $\mu/1$ *nor* ε/μ is small.
- (II) The "3-scale case" in which all scales are separated and occurs when $\mu/1$ is small *and* ε/μ is small.
- (III) The first "2-scale case" which occurs when $\mu/1$ is *not* small *and* ε/μ is small.
- (IV) The second "2-scale case" which occurs when $\mu/1$ is small *and* ε/μ is *not* small.

The concept of "small " (or "not small") mentioned above, is tied in two ways to the regularity theory in terms of asymptotic expansions. First, on the level of constructing asymptotic expansions, the decision which parameters are deemed small determines the ansatz to be made and thus the form of the expansion. Second, on the level of using asymptotic expansions for approximation purposes or the design of approximation spaces, the decision which parameters are deemed small depends on the desired accuracy, i.e., whether the remainder resulting from the asymptotic expansion can be regarded as small.

In order to be able to describe the regularity assertions for the solution $\mathbf{U}$, we need to introduce some notation:

Definition 1. *1. We say that a function w is analytic with length scale ν (and analyticity parameters C_w, γ_w), abbreviated $w \in \mathcal{A}(\nu, C_w, \gamma_w)$, if*

$$\|w^{(n)}\|_{\infty,I} \leq C_w \gamma_w^n \max\{n, \nu^{-1}\}^n \qquad \forall n \in \mathbb{N}_0.$$

2. *We say that that an entire function w is of L^∞-boundary layer type with length scale ν (and analyticity parameters C_w, γ_w), abbreviated $w \in \mathcal{BL}^\infty(\nu, C_w, \gamma_w)$, if for all $x \in I$*

$$|w^{(n)}(x)| \leq C_w \gamma_w^n \nu^{-n} e^{-\operatorname{dist}(x,\partial I)/\nu} \qquad \forall n \in \mathbb{N}_0.$$

Both definitions extend naturally to vector-valued functions by requiring that the above bounds hold componentwise.

The four cases $(I) - (IV)$ (of scale separation) listed earlier correspond to the four cases in the following theorem from [11].

Theorem 2. *Assume* (3) *and* (7) *hold. Then there exist constants C, b, δ, q, $\gamma > 0$, independent of $0 < \varepsilon \leq \mu \leq 1$, such that the following assertions are true for the solution $\mathbf{U}$ of* (1)*:*

(I) $\mathbf{U} \in \mathcal{A}(\varepsilon, C\varepsilon^{-1/2}, \gamma)$.

(II) $\mathbf{U}$ *can be written as* $\mathbf{U} = \mathbf{W} + \widetilde{\mathbf{U}}_{BL} + \widehat{\mathbf{U}}_{BL} + \mathbf{R}$, *where* $\mathbf{W} \in \mathcal{A}(1, C, \gamma)$, $\widetilde{\mathbf{U}}_{BL} \in \mathcal{BL}^\infty(\delta\mu, C, \gamma)$ $\widehat{\mathbf{U}}_{BL} \in \mathcal{BL}^\infty(\delta\varepsilon, C, \gamma)$, *and* $\|\mathbf{R}\|_{L^\infty(\partial I)} + \|\mathbf{R}\|_{E,I} \leq C\left[e^{-b/\mu} + e^{-b\mu/\varepsilon}\right]$. *Furthermore, the second component $\widehat{v}$ of $\widehat{\mathbf{U}}_{BL}$ satisfies the stronger assertion $\widehat{v} \in \mathcal{BL}^\infty(\delta\varepsilon, C(\varepsilon/\mu)^2, \gamma)$.*

(III) If $\varepsilon/\mu \leq q$ then $\mathbf{U}$ can be written as $\mathbf{U} = \mathbf{W} + \widehat{\mathbf{U}}_{BL} + \mathbf{R}$, where $\mathbf{W} \in \mathcal{A}(\mu, C, \gamma)$, $\widehat{\mathbf{U}}_{BL} \in \mathcal{BL}^\infty(\delta\varepsilon, C, \gamma)$, and $\|\mathbf{R}\|_{L^\infty(\partial I)} + \|\mathbf{R}\|_{E,I} \leq Ce^{-b/\varepsilon}$. Furthermore, the second component $\widehat{v}$ of $\widehat{\mathbf{U}}_{BL}$ satisfies the stronger assertion $\widehat{v} \in \mathcal{BL}^\infty(\delta\varepsilon, C(\varepsilon/\mu)^2, \gamma)$

(IV) $\mathbf{U}$ *can be written as* $\mathbf{U} = \mathbf{W} + \widetilde{\mathbf{U}}_{BL} + \mathbf{R}$, *where* $\mathbf{W} \in \mathcal{A}(1, C, \gamma)$, $\widetilde{\mathbf{U}}_{BL} \in \mathcal{BL}^\infty(\delta\mu, C\sqrt{\mu/\varepsilon}, \gamma\mu/\varepsilon)$, *and* $\|\mathbf{R}\|_{L^\infty(\partial I)} + \|\mathbf{R}\|_{E,I} \leq C(\mu/\varepsilon)^2 e^{-b/\mu}$.

The above results arise after asymptotic expansions are derived for each case. In the subsections that follow we will briefly explain what the ansatz is in each Case *(II)–(IV)* listed in the above theorem.

2.1 The three scale case (Case *(II)*)

Anticipating that boundary layers of length scales $O(\mu)$ and $O(\varepsilon)$ will appear at the endpoints $x = 0$ and $x = 1$, we introduce the stretched variables $\widetilde{x} = x/\mu$, $\widehat{x} = x/\varepsilon$ for the expected layers at the left endpoint $x = 0$ and variables $\widetilde{x}^R = (1 - x)/\mu$, $\widehat{x}^R = (1 - x)/\varepsilon$ for the expected behavior at right endpoint $x = 1$. We make the following formal ansatz for the solution $\mathbf{U}$:

$$\mathbf{U} \sim \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \left(\frac{\mu}{1}\right)^i \left(\frac{\varepsilon}{\mu}\right)^j \left[\mathbf{U}_{ij}(x) + \widetilde{\mathbf{U}}_{ij}^L(\widetilde{x}) + \widehat{\mathbf{U}}_{ij}^L(\widehat{x}) + \widetilde{\mathbf{U}}_{ij}^R(\widetilde{x}^R) + \widehat{\mathbf{U}}_{ij}^R(\widehat{x}^R)\right], \tag{13}$$

where the functions $\mathbf{U}_{ij}$, $\widetilde{\mathbf{U}}_{ij}^L$, $\widehat{\mathbf{U}}_{ij}^L$ $\widetilde{\mathbf{U}}_{ij}^R$, $\widehat{\mathbf{U}}_{ij}^R$ are to be determined by inserting the ansatz (13) into the boundary value problem (1), and equating like powers of $\mu/1$ and ε/μ. The functions $\mathbf{U}_{ij}$, $\widetilde{\mathbf{U}}_{ij}^L$, $\widetilde{\mathbf{U}}_{ij}^R$, $\widehat{\mathbf{U}}_{ij}^L$, $\widehat{\mathbf{U}}_{ij}^R$ can then be determined recursively as solutions of suitable boundary value problems (see [11] for details). The decomposition of Theorem 2 is obtained by truncating the asymptotic expansion (13) after a finite number of terms:

$$\mathbf{U}(x) = \mathbf{W}_M(x) + \widehat{\mathbf{U}}_{BL}^M(\widehat{x}) + \widehat{\mathbf{V}}_{BL}^M(\widehat{x}^R) + \widetilde{\mathbf{U}}_{BL}^M(\widetilde{x}) + \widetilde{\mathbf{V}}_{BL}^M(\widetilde{x}^R) + \mathbf{R}_M(x), \tag{14}$$

where

$$\mathbf{W}_M(x) = \sum_{i=0}^{M_1}\sum_{j=0}^{M_2} \mu^i \left(\frac{\varepsilon}{\mu}\right)^j \mathbf{U}_{ij}(x) \tag{15}$$

denotes the outer (smooth) expansion,

$$\widehat{\mathbf{U}}_{BL}^M(\widehat{x}) = \sum_{i=0}^{M_1}\sum_{j=0}^{M_2} \mu^i \left(\frac{\varepsilon}{\mu}\right)^j \widehat{\mathbf{U}}_{ij}^L(\widehat{x}), \qquad \widehat{\mathbf{V}}_{BL}^M(\widehat{x}^R) = \sum_{i=0}^{M_1}\sum_{j=0}^{M_2} \mu^i \left(\frac{\varepsilon}{\mu}\right)^j \widehat{\mathbf{U}}_{ij}^R(\widehat{x}^R), \tag{16}$$

denote the left and right inner (boundary layer) expansions asscociated with the variables $\widehat{x}$, $\widehat{x}^R$, respectively,

$$\widetilde{\mathbf{U}}_{BL}^M(\widetilde{x}) = \sum_{i=0}^{M_1}\sum_{j=0}^{M_2} \mu^i \left(\frac{\varepsilon}{\mu}\right)^j \widetilde{\mathbf{U}}_{ij}^L(\widetilde{x}), \qquad \widetilde{\mathbf{V}}_{BL}^M(\widetilde{x}) = \sum_{i=0}^{M_1}\sum_{j=0}^{M_2} \mu^i \left(\frac{\varepsilon}{\mu}\right)^j \widetilde{\mathbf{U}}_{ij}^R(\widetilde{x}), \tag{17}$$

denote the left and right inner (boundary layer) expansions asscociated with the variables $\widetilde{x}$, $\widetilde{x}^R$ respectively, and the remainder $\mathbf{R}_M$ is defined such that (14) holds. In establishing Theorem 2 the choices $M_1 = O(1/\mu)$, $M_2 = O(\mu/\varepsilon)$ are made. (For full details see [11].)

2.2 The first two scale case (Case *(III)*)

We employ again the notation of the stretched variables $\widehat{x} = x/\varepsilon$ and $\widehat{x}^R = (1-x)/\varepsilon$. Since $\mu/1$ is not assumed to be small, only the scales $O(1)$ and $O(\varepsilon/\mu)$ are expected to be present in the problem. Inserting the formal ansatz

$$\mathbf{U}(x) \sim \sum_{i=0}^{\infty} \left(\frac{\varepsilon}{\mu}\right)^i \left[\mathbf{U}_i(x) + \widehat{\mathbf{U}}_i^L(\widehat{x}) + \widehat{\mathbf{U}}_i^R(\widehat{x}^R)\right], \tag{18}$$

into the boundary value problem (1) and equating like powers of ε/μ, yields recursions for the functions $\mathbf{U}$, $\widehat{\mathbf{U}}^L$ and $\widehat{\mathbf{U}}^R$. Truncating (18) after M terms leads to the representation

$$\mathbf{U}(x) = \mathbf{W}_M(x) + \widehat{\mathbf{U}}_{BL}^M(\widehat{x}) + \widehat{\mathbf{V}}_{BL}^M(\widehat{x}) + \mathbf{R}_M(x), \tag{19}$$

where

$$\mathbf{W}_M(x) = \sum_{i=0}^{M} \left(\frac{\varepsilon}{\mu}\right)^i \mathbf{U}_i(x) \tag{20}$$

denotes the outer (smooth) expansion,

$$\widehat{\mathbf{U}}_{BL}^M(\widehat{x}) = \sum_{i=0}^{M} \left(\frac{\varepsilon}{\mu}\right)^i \widehat{\mathbf{U}}_i^L(\widehat{x}), \qquad \widehat{\mathbf{V}}_{BL}^M(\widehat{x}) = \sum_{i=0}^{M} \left(\frac{\varepsilon}{\mu}\right)^i \widehat{\mathbf{U}}_i^R(\widehat{x}) \tag{21}$$

denote the left and right inner (boundary layer) expansions, respectively, and the remainder $\mathbf{R}_M$ is such that (19) is valid. In establishing Theorem 2 the choice $M = O(\mu/\varepsilon)$ is made. (See [11] for more details.)

2.3 The second two scale case (Case *(IV)*)

In this case, $\mu/1$ is assumed to be small and ε/μ is not deemed small, which leads us to the ansatz

$$\mathbf{U} \sim \sum_{i=0}^{\infty} \mu^i \left[\mathbf{U}_i(x) + \widetilde{\mathbf{U}}_i^L(\widetilde{x}) + \widetilde{\mathbf{U}}_i^R(\widetilde{x}^R) \right], \tag{22}$$

with the stretched variables $\widetilde{x} = x/\mu, \widetilde{x}^R = (1-x)/\mu$. Inserting this ansatz into the boundary value problem (1) and equating like powers of μ yields recursions for the functions $\mathbf{U}_i$, $\widetilde{\mathbf{U}}_i^L$, and $\widetilde{\mathbf{U}}_i^R$. The truncated series $\mathbf{W}_M(x) := \sum_{i=0}^{M} \mu^i \mathbf{U}_i(x)$, $\widetilde{\mathbf{U}}_{BL}^M := \sum_{i=0}^{M} \mu^i \mathbf{U}_i^L(\widetilde{x})$, $\widetilde{\mathbf{V}}_{BL}^M := \sum_{i=0}^{M} \mu^i \mathbf{U}_i^R(\widetilde{x}^R)$ yield the decomposition stated in Theorem 2 if $M = O(1/\mu)$. (The details are given in [11].)

3 Approximation results

3.1 Main results

In this section we will describe the finite dimensional subspace $[S_N]^2$ which appears in (11), in order to construct an *hp* scheme for the approximation of the solution to (4). To this end, let $\Delta = \{0 = x_0 < x_1 < ... < x_{\mathcal{M}} = 1\}$ be an arbitrary partition of $I = (0, 1)$ and set

$$I_j = (x_{j-1}, x_j), \ h_j = x_j - x_{j-1}, \ j = 1, ..., \mathcal{M}.$$

Also, define the master (or standard) element $I_{ST} = (-1, 1)$, and note that it can be mapped onto the j^{th} element I_j by the linear mapping

$$x = Q_j(t) = \frac{1}{2}(1-t)\,x_{j-1} + \frac{1}{2}(1+t)\,x_j.$$

With $\Pi_p(I_{ST})$ the space of polynomials of degree $\leq p$ on I_{ST}, we define our finite dimensional subspace as

$$S_N \equiv S^p(\Delta) = \left\{ \mathbf{V} \in [H_0^1(I)]^2 : \mathbf{V} \circ Q_j \in \left(\Pi_{p_j}(I_{ST})\right)^2, j = 1, ..., \mathcal{M} \right\}$$

and set

$$S_0^p(\Delta) := S^p(\Delta) \cap [H_0^1(I)]^2, \tag{23}$$

We restrict our attention here to constant polynomial degree p for all elements, but clearly, more general settings with variable polynomial degree are possible.

The following *Spectral Boundary Layer mesh* is essentially the minimal mesh that yields robust exponential convergence. Loosely speaking, one inserts nodes to resolve the boundary layers, i.e., upon setting $x_\varepsilon := \kappa p \varepsilon$ and $x_\mu = \kappa p \mu$, one inserts the nodes x_ε and $1 - x_\varepsilon$ if $\kappa p \varepsilon < 1/2$; the nodes x_μ and $1 - x_\mu$ are inserted if $\kappa p \mu < 1/2$; here, $\kappa > 0$ is a user-specified parameter.

Definition 3 (Spectral Boundary Layer mesh). *For $\kappa > 0$, $p \in \mathbb{N}$ and $0 < \varepsilon \leq \mu \leq 1$, define the spaces $S(\kappa, p)$ of piecewise polynomials by*

$$S(\kappa,p) := \begin{cases} S_0^p(\Delta); & \Delta = \{0, \kappa p\varepsilon, \kappa p\mu, 1-\kappa p\mu, 1-\kappa p\varepsilon, 1\} \quad \text{if } \kappa p\mu < \frac{1}{2} \\ S_0^p(\Delta); & \Delta = \{0, \kappa p\varepsilon, 1-\kappa p\varepsilon, 1\} \quad \text{if } \kappa p\varepsilon < \frac{1}{2} \leq \kappa p\mu \\ S_0^p(\Delta); & \Delta = \{0, 1\} \quad \text{if } \kappa p\varepsilon \geq \frac{1}{2} \end{cases}$$

We now present the main result of this paper:

Theorem 4. *Let f, g and $\mathbf{A}$ be analytic on $\overline{I}$ and satisfy the conditions in (3) and (7). Let $\mathbf{U} = (u,v)^T$ be the solution to (1). Then there exists $\widetilde{\mathcal{I}}_p\mathbf{U} = [\mathcal{I}_p u, \mathcal{I}_p v]^T \in S(\kappa,p)$ with $\widetilde{\mathcal{I}}_p\mathbf{U} = \mathbf{U}$ on ∂I and*

$$\left\| \mathbf{U} - \widetilde{\mathcal{I}}_p\mathbf{U} \right\|_{E,I} \leq Ce^{-\beta\kappa p},$$

for all $\kappa \in (0, \kappa_0]$, where the constants κ_0, C, $\beta > 0$ depend only on f, g and $\mathbf{A}$.

Proof. The proof is given at the end of Section 3.2. ∎

Using the above theorem and the quasioptimality result (12) we have the following:

Corollary 5. *Let $\mathbf{U}$ be the solution to (4) and let $\mathbf{U}_N \in S_0^p(\Delta)$ be the solution to (11) based on the Spectral Boundary Layer mesh of Definition 3. Then there exist constants κ_0, C, $\sigma > 0$ depending* only *on the input data f, g and $\mathbf{A}$, such that for any $0 < \kappa \leq \kappa_0$*

$$\|\mathbf{U} - \mathbf{U}_N\|_{E,I} \leq Ce^{-\sigma\kappa p}.$$

3.2 Proof of Theorem 4

3.2.1 An approximation operator

We start with two lemmas:

Lemma 6. *There exists a linear operator $\mathcal{I}_p : H^1(I_{ST}) \to \Pi_p(I_{ST})$ with the following property:*

$$u(\pm 1) = \mathcal{I}_p u(\pm 1). \tag{24}$$

Furthermore, if $u \in C^\infty\left(\overline{I}_{ST}\right)$, then

$$| \|u - \mathcal{I}_p u\|_{0,I_{ST}}^2 \leq \frac{1}{p^2}\frac{(p-s)!}{(p+s)!}\left\|u^{(s+1)}\right\|_{0,I_{ST}}^2, \ \forall\, s = 0,1,...,p, \tag{25}$$

$$\left\|(u - \mathcal{I}_p u)'\right\|_{0,I_{ST}}^2 \leq \frac{(p-s)!}{(p+s)!}\left\|u^{(s+1)}\right\|_{0,I_{ST}}^2, \ \forall\, s = 0,1,...,p. \tag{26}$$

Proof. The result is taken from [15, Cor. 3.15]. ∎

Lemma 7. *Let $p \in \mathbb{N}, \lambda \in (0,1]$ such that $\lambda p \in \mathbb{N}$. Then*

$$\frac{(p-\lambda p)!}{(p+\lambda p)!} \leq \left[\frac{(1-\lambda)^{(1-\lambda)}}{(1+\lambda)^{(1+\lambda)}}\right]^p p^{-2\lambda p} e^{2\lambda p+1}.$$

Proof. This follows from Stirling's formula—see [18, Lemma 3.1] for the details. ∎

On the reference element I_{ST}, we have the following additional stability results.

Lemma 8. *Let $\mathcal{I}_p : H^1(I_{ST}) \to \Pi_p(I_{ST})$ be as in Lemma 6. Then, on the reference element I_{ST}, we have*

$$|\mathcal{I}_p u|_{1,I_{ST}} \leq C|u|_{1,I_{ST}}, \qquad \|\mathcal{I}_p u\|_{0,I_{ST}} \leq \|u\|_{0,I_{ST}} + C\frac{1}{p}|u|_{1,I_{ST}},$$
$$\|\mathcal{I}_1 u\|_{L^\infty(I_{ST})} \leq \|u\|_{L^\infty(I_{ST})}, \qquad \|(\mathcal{I}_1 u)'\|_{L^\infty(I_{ST})} \leq C\|u\|_{L^\infty(I_{ST})}.$$

Proof. The estimates for the linear interpolant $\mathcal{I}_1$ are standard. The stability in the H^1-seminorm is a direct consequence of the definition of the norm and the L^2-stability follows from a triangle inequality and an approximation result:

$$\|\mathcal{I}_p u\|_{0,I_{ST}} \leq \|u\|_{0,I_{ST}} + \|u - \mathcal{I}_p u\|_{0,I_{ST}} \leq \|u\|_{0,I_{ST}} + Cp^{-1}|u|_{1,I_{ST}}.$$

∎

The approximation operator $\mathcal{I}_p$ on the reference element can be used to define an approximation operator in an elementwise fashion: For a mesh Δ with elements I_j, $j = 1,\ldots,\mathcal{M}$, element maps Q_j and given degree vector $\overrightarrow{p} = (p_1,\ldots,p_{\mathcal{M}})$, with $1 \leq p_j \leq p$, we define the operator $\mathcal{I}_{\overrightarrow{p},\Delta} : [H^1(I)]^2 \to S^p(\Delta)$ elementwise in the standard way with the operators $\mathcal{I}_{p_j}$, by requiring

$$(\mathcal{I}_{\overrightarrow{p},\Delta}\mathbf{V})|_{I_j} \circ Q_j = \mathcal{I}_{p_j}(\mathbf{V}|_{I_j} \circ Q_j), \qquad j = 1,\ldots,\mathcal{M}. \tag{27}$$

Since the operators $\mathcal{I}_{p_j}$ interpolate in the endpoints of I_{ST}, this operator is indeed well-defined. We will write $\mathcal{I}_{p,\Delta}$ if $p_j = p$ for $j = 1,\ldots,\mathcal{M}$. We point out that $p_j = 1$ corresponds to the linear interpolant. Finally, since the operators $\mathcal{I}_{\overrightarrow{p},\Delta}$ are defined elementwise, we will work with the abbreviation

$$\mathcal{I}_{p_j,I_j}\mathbf{V} := (\mathcal{I}_{\overrightarrow{p},\Delta}\mathbf{V})|_{I_j} = (\mathcal{I}_{p_j}(\mathbf{V}|_{I_j} \circ Q_j)) \circ Q_j^{-1}. \tag{28}$$

The following approximation result on the reference element will be one of our main tools for the proof of Theorem 4.

Lemma 9. *Let $u \in C^\infty(I_{ST})$ satisfy for some C_u, $\gamma_u > 0$, $K \geq 1$, $h \in (0,1]$,*

$$\|u^{(n)}\|_{0,I_{ST}} \leq C_u(\gamma_u h)^n \max\{n,K\}^n \qquad \forall n \in \mathbb{N}. \tag{29}$$

Then there exist η, β, $C > 0$ depending solely on γ_u, such that under the condition

$$\frac{hK}{p} \leq \eta, \tag{30}$$

the approximation $\mathcal{I}_p u \in \Pi_p(I_{ST})$ *given by Lemma 6 satisfies* $u(\pm 1) = (\mathcal{I}_p u)(\pm 1)$ *and*

$$\|u - \mathcal{I}_p u\|_{0,I_{ST}} + \|(u - \mathcal{I}_p u)'\|_{0,I_{ST}} \quad \leq \quad C C_u \frac{hK}{p} e^{-\beta p}.$$

Proof. First note that $\sup_{j\geq 1}((j+1)/j)^{2j} \leq e^2$. Then, in view of $hK/p \leq \eta$, $h \leq 1$ and $\lambda \leq 1$, we compute for $s = \lambda p$ the following:

$$\begin{aligned} h^{2(s+1)} \left(\max\{s+1, K\}\right)^{2(s+1)} &\leq e^2 (hK)^2 \left(\max\{hs, hK\}\right)^{2s} \leq e^2 (hK)^2 \left(\max\{h\lambda p, \eta p\}\right)^{2s} \\ &\leq e^2 (hK)^2 \left(p \max\{\lambda, \eta\}\right)^{2s}. \end{aligned}$$

This and Lemma 7 give

$$\begin{aligned} \frac{(p-s)!}{(p+s)!} \|u^{(s+1)}\|^2_{0,I_{ST}} &\leq e^2 (hK)^2 C_u^2 \gamma_u^{2\lambda p} p^{2\lambda p} \left(\max\{\lambda, \eta\}\right)^{2\lambda p} \left[\frac{(1-\lambda)^{1-\lambda}}{(1+\lambda)^{1+\lambda}}\right]^p p^{-2\lambda p} e^{2\lambda p + 1} \\ &\leq C_u^2 e^3 (hK)^2 \left[\frac{(1-\lambda)^{1-\lambda}}{(1+\lambda)^{1+\lambda}}\right]^p \left[\gamma_u e \max\{\lambda, \eta\}\right]^{2\lambda p}. \end{aligned}$$

Select now $\lambda \in (0,1)$ and $\eta > 0$ such that $\gamma_u e \max\{\lambda, \eta\} \leq 1$. Since for this choice of λ we have $(1-\lambda)^{1-\lambda}/(1+\lambda)^{1+\lambda} =: q < 1$, we conclude

$$\frac{(p-s)!}{(p+s)!} \|u^{(s+1)}\|^2_{0,I_{ST}} \quad \leq \quad e^3 (hK)^2 C_u^2 e q^p = e^3 \left(\frac{hK}{p}\right)^2 C_u^2 p^2 q^p,$$

which is the desired bound, since the algebraic factor p^2 may be absorbed in the exponentially decaying one by suitably adjusting the constants. ∎

We reformulate the approximation result of Lemma 9 in a form that will be convenient for the approximation of the smooth and the boundary layers parts of the expansion (the latter within the layer):

Corollary 10. *Let* I_j *be an interval of length* h_j*, and let* $\mathbf{V} \in C^\infty(I_j)$ *satisfy for some* C_u*,* $\gamma_u > 0$*,* $K \geq 1$*,*

$$\|\mathbf{V}^{(n)}\|_{L^\infty(I_j)} \leq C_u \gamma_u^n \max\{n, K\}^n \qquad \forall n \in \mathbb{N}.$$

Then there exist constants C*,* η*,* $\beta > 0$ *depending only on* γ_u*, such that under the scale resolution condition*

$$\frac{h_j K}{p_j} \leq \eta, \tag{31}$$

the polynomial approximation $\mathcal{I}_{p_j,I_j}\mathbf{V}$ *satisfies*

$$h_j^{-1} \|\mathbf{V} - \mathcal{I}_{p_j,I_j}\mathbf{V}\|_{0,I_j} + |\mathbf{V} - \mathcal{I}_{p_j,I_j}\mathbf{V}|_{1,I_j} \leq C C_u \frac{h_j^{1/2} K}{p_j} e^{-\beta p_j}. \tag{32}$$

Proof. Let $\widehat{\mathbf{V}} := \mathbf{V} \circ Q_j$, where $Q_j : I_{ST} \to I_j$ is the affine bijection. Then $\widehat{\mathbf{V}}$ satisfies

$$\|\widehat{\mathbf{V}}^{(n)}\|_{L^2(I_{ST})} \leq C C_u (\gamma_u h_j/2)^n \max\{n, K\}^n \qquad \forall n \in \mathbb{N}.$$

Therefore, Lemma 9 gives the existence of C, β, η such that under the assumption (31), we have

$$\|\widehat{\mathbf{V}} - \mathcal{I}_{p_j}\widehat{\mathbf{V}}\|_{0,I_{ST}} + \|\left(\widehat{\mathbf{V}} - \mathcal{I}_{p_j}\widehat{\mathbf{V}}\right)'\|_{0,I_{ST}} \leq CC_u \frac{h_j K}{p_j} e^{-\beta p_j}.$$

Transforming back to I_j gives the result. ∎

The following result will be useful for the approximation of the remainder $\mathbf{R}$ and the boundary layer contrinbutions.

Lemma 11. *Let I_j be an interval of length h_j and $p_j \in \mathbb{N}$. Then, for scalar functions (and analogously for vector-valued ones):*

$$\|u - \mathcal{I}_{p_j,I_j}u\|_{0,I_j} \leq \|u\|_{0,I_j} + C\frac{h_j}{p_j}|u|_{1,I_j}, \tag{33}$$

$$|u - \mathcal{I}_{p_j,I_j}u|_{1,I_j} \leq C|u|_{1,I_j}, \tag{34}$$

$$\|u - \mathcal{I}_{1,I_j}u\|_{0,I_j} \leq \|u\|_{0,I_j} + Ch_j^{1/2}\|u\|_{L^\infty(I_j)}, \tag{35}$$

$$|u - \mathcal{I}_{1,I_j}u|_{1,I_j} \leq |u|_{1,I_j} + Ch_j^{-1/2}\|u\|_{L^\infty(I_j)}. \tag{36}$$

Proof. The estimates follow from Lemma 8 and standard scaling arguments. ∎

Finally, we formulate an approximation result for the approximation of functions of boundary layer type outside the layer:

Lemma 12. *Let $\nu > 0$ and let u satisfy*

$$|u(x)| + \nu|u'(x)| \leq C_u e^{-\operatorname{dist}(x,\partial I)/\nu} \qquad \forall x \in I.$$

Let Δ be an arbitrary mesh on I with mesh points ξ and $1-\xi$, where $\xi \in (0, 1/2)$. Then the piecewise linear interpolant $\mathcal{I}_{1,\Delta}u$ satisfies on $(\xi, 1-\xi)$:

$$\nu|u - \mathcal{I}_{1,\Delta}u|_{1,(\xi,1-\xi)} + \|u - \mathcal{I}_{1,\Delta}u\|_{0,(\xi,1-\xi)} \leq CC_u e^{-\xi/\nu},$$

for some $C > 0$ independent of ν.

Proof. Let I_j be an element in the interval $(\xi, 1-\xi)$ of length h_j. We distinguish between the cases $h_j \leq \nu$ and $h_j > \nu$. In the case $h_j \leq \nu$, we note that (33), (34) yield

$$\|u - \mathcal{I}_{1,I_j}\|_{0,I_j} + \nu\|(u - \mathcal{I}_{1,I_j})'\|_{0,I_j} \leq \|u\|_{0,I_j} + C\nu|u|_{1,I_j} \leq Ch_j^{1/2}C_u e^{-\xi/\nu}.$$

In the converse case $h_j \geq \nu$, we use (35), (36) to get

$$\begin{aligned}\|u - \mathcal{I}_{1,I_j}\|_{0,I_j} + \nu\|(u - \mathcal{I}_{1,I_j})'\|_{0,I_j} &\leq C\left[h_j^{1/2}\|u\|_{L^\infty(I_j)} + \nu|u|_{1,I_j} + C\nu h_j^{-1/2}\|u\|_{L^\infty(I_j)}\right] \\ &\leq Ch_j^{1/2}C_u e^{-\xi/\nu}.\end{aligned}$$

Summation over all elements in $(\xi, 1-\xi)$ then gives the result. ∎

3.2.2 Proof of Theorem 4

Proof of Theorem 4: The approximation of the exact solution $\mathbf{U}$ is constructed element by element with the aid of the operator $\mathcal{I}_p$. The basic ingredient of the proof is that the four regularity assertions of Theorem 2 permit us to show that, on each element, the features on all length scales can either be resolved or are sufficiently small to be safely ignored. The parameter κ_0 appearing in the statement of Theorem 4 will be determined in the course of the proof.

We remark here that we make the simplifying assumption that

$$\frac{\varepsilon}{\mu} \leq q \tag{37}$$

with q given by Theorem 2. In the converse case, μ and ε are comparable, and the regularity assertion *(III)* follows from *(IV)* by suitably adjusting constants.

Case 1: *$\kappa p \varepsilon \geq 1/2$, which implies in particular $\kappa p \mu \geq 1/2$ and $\kappa p \varepsilon / \mu \geq 1/2$. This is the "asymptotic case" and $\Delta = \{0, 1\}$.*

We may employ the regularity assertion of Case *(I)* of Theorem 2, for the solution $\mathbf{U}$, i.e., $\mathbf{U} \in \mathcal{A}(\varepsilon, C\varepsilon^{-1/2}, \gamma)$. The mesh Δ consists of the single element I with length 1. Corollary 10 then implies the existence of $\eta > 0$ (depending solely on γ) such that the condition

$$\frac{1}{p\varepsilon} \leq \eta, \tag{38}$$

implies $\|\mathbf{U} - \mathcal{I}_{p,I}\mathbf{U}\|_{H^1(I)} \leq Ce^{-\beta p}$, which is even stronger than what is required. The crucial condition (38) is easily satisfied by making sure that $\kappa_0 < \eta/2$, since then the assumption $\kappa p \varepsilon > 1/2$ produces

$$\frac{1}{p\varepsilon} = \frac{\kappa}{\kappa p \varepsilon} \leq 2\kappa \leq 2\kappa_0.$$

Case 2: *$\kappa p \mu < \frac{1}{2}$ and $\kappa p \frac{\varepsilon}{\mu} < \frac{1}{2}$ (pre-asymptotic case), $\Delta = \{0, \kappa p \varepsilon, \kappa p \mu, 1 - \kappa p \mu, 1 - \kappa p \varepsilon, 1\}$.* The mesh has 5 elements $I_1, \ldots, I_5$. The second statement of Theorem 2 gives the decomposition

$$\mathbf{U} = \mathbf{W} + \widetilde{\mathbf{U}}_{BL} + \widehat{\mathbf{U}}_{BL} + \mathbf{R}, \tag{39}$$

where the smooth part satisfies $\mathbf{W} \in \mathcal{A}(1, C_w, \gamma_w)$, the boundary layers satisfy $\widetilde{\mathbf{U}}_{BL} \in \mathcal{BL}^\infty(\delta\mu, C_{\widetilde{u}}, \gamma_{\widetilde{u}})$, $\widehat{\mathbf{U}}_{BL} \in \mathcal{BL}^\infty(\delta\varepsilon, C_{\widehat{u}}, \gamma_{\widehat{u}})$, $\widehat{v}_{BL} \in \mathcal{BL}^\infty(\delta\varepsilon, C_{\widehat{v}}(\varepsilon/\mu)^2, \gamma_{\widehat{v}})$, and the remainder satisfies

$$\|\mathbf{R}\|_{L^\infty(\partial I)} + \|\mathbf{R}\|_{E,I} \leq C\left[e^{-b/\mu} + e^{-b\mu/\varepsilon}\right] \leq Ce^{-2b\kappa p}. \tag{40}$$

These give $\forall\, n \in \mathbb{N}_0$,

$$\left\|\mathbf{W}^{(n)}\right\|_{L^\infty(I)} \leq C_w \gamma_w^n n^n, \tag{41}$$

$$\left|\widetilde{\mathbf{U}}_{BL}^{(n)}(x)\right| \leq C_{\widetilde{u}} \gamma_{\widetilde{u}}^n (\delta\mu)^{-n} e^{-\mathrm{dist}(x,\partial I)/(\delta\mu)}, \tag{42}$$

$$\left|\widehat{\mathbf{U}}_{BL}^{(n)}(x)\right| \leq C_{\widehat{u}} \gamma_{\widehat{u}}^n (\delta\varepsilon)^{-n} e^{-\mathrm{dist}(x,\partial I)/(\delta\varepsilon)}, \tag{43}$$

$$|\widehat{v}_{BL}^{(n)}(x)| \leq C_{\widehat{v}} \left(\frac{\varepsilon}{\mu}\right)^2 \gamma_{\widehat{v}}^n (\delta\varepsilon)^{-n} e^{-\mathrm{dist}(x,\partial I)/(\delta\varepsilon)}. \tag{44}$$

We approximate $\mathbf{W}$ by $\mathcal{I}_{p,\Delta}\mathbf{W}$, the first boundary layer function $\widetilde{\mathbf{U}}_{BL}$ by $\mathcal{I}_{(p,p,1,p,p),\Delta}\widetilde{\mathbf{U}}_{BL}$, the second boundary layer function $\widehat{\mathbf{U}}_{BL}$ by $\mathcal{I}_{(p,1,1,1,p),\Delta}\widehat{\mathbf{U}}_{BL}$, and the remainder $\mathbf{R}$ by its *global* linear interpolant $\mathcal{I}_{p,\{0,1\}}\mathbf{R}$.

With the aid of Corollary 10, it is easy to see that $\|\mathbf{W}-\mathcal{I}_{p,\Delta}\mathbf{W}\|_{1,I} \leq Ce^{-\beta p}$ for some C, $\beta > 0$, independent of ε and μ. For the remainder $\mathbf{R}$, we get in view of (40), which gives control of $\mathbf{R}$ at the endpoint of I, that

$$\|\mathbf{R}-\mathcal{I}_{1,\{0,1\}}\mathbf{R}\|_{E,I} \leq \|\mathbf{R}\|_{E,I} + \|\mathcal{I}_{1,\{0,1\}}\mathbf{R}\|_{E,I} \leq Ce^{-2b\kappa p}.$$

We now turn to the approximation of the boundary layer contribution $\widetilde{\mathbf{U}}_{BL}$. Lemma 12 (with $\nu = \delta\mu$) immediately produces, for the element $I_3 = (\kappa p\mu, 1-\kappa p\mu)$, the estimate $\|\widetilde{\mathbf{U}}_{BL}-\mathcal{I}_{1,I_3}\widetilde{\mathbf{U}}_{BL}\|_{E,I_3} \leq Ce^{-\kappa p/\delta}$, which is exponentially small. For the small elements I_1, I_2, I_4, and I_5, we note that their length is smaller than $\kappa p\mu$. Corollary 10 is applicable with $K = 1/\mu$, which produces for these elements I_j, $j \in \{1,2,4,5\}$,

$$h_j^{-1}\|\widetilde{\mathbf{U}}_{BL}-\mathcal{I}_{p,I_j}\widetilde{\mathbf{U}}_{BL}\|_{0,I_j} + |\widetilde{\mathbf{U}}_{BL}-\mathcal{I}_{p,I_j}\widetilde{\mathbf{U}}_{BL}|_{1,I_j} \leq Ch_j^{1/2}\frac{1}{p\mu}e^{-\beta p}, \tag{45}$$

if the scale resolution condition

$$\frac{\kappa p\mu}{p\mu} \leq \eta,$$

is satisfied, where the parameter η depends only on $\gamma_{\widetilde{u}}$. Taking κ_0 sufficiently small, this condition is satisfied. Recalling that $h_j \leq \kappa p\mu$, we see that (45) is a much stronger result than required.

We next study the approximation of $\widehat{\mathbf{U}}_{BL}$. We first consider the approximation on the elements I_2, I_3, and I_4, which are all in the interval $(\kappa p\varepsilon, 1-\kappa p\varepsilon)$. For the $\widehat{u}$-component of $\widehat{\mathbf{U}}_{BL}$, Lemma 12 (with $\nu = \delta\varepsilon$, $C_u = C_{\widehat{u}}$) yields the desired exponential approximation result. For the $\widehat{v}$-component, we also apply Lemma 12 (with $\nu = \delta\varepsilon$, $C_u = C_{\widehat{v}}(\varepsilon/\mu)^2$) to get

$$\|\widehat{v}-\mathcal{I}_{1,\Delta}\widehat{v}\|_{0,(\kappa p\varepsilon,1-\kappa p\varepsilon)} + \varepsilon|\widehat{v}-\mathcal{I}_{1,\Delta}\widehat{v}|_{1,(\kappa p\varepsilon,1-\kappa p\varepsilon)} \leq CC_{\widehat{v}}\left(\frac{\varepsilon}{\mu}\right)^2 e^{-\kappa p/\delta}.$$

This implies

$$\|\widehat{v}-\mathcal{I}_{1,\Delta}\widehat{v}\|_{0,(\kappa p\varepsilon,1-\kappa p\varepsilon)} + \mu|\widehat{v}-\mathcal{I}_{1,\Delta}\widehat{v}|_{1,(\kappa p\varepsilon,1-\kappa p\varepsilon)} \leq CC_{\widehat{v}}\left(\frac{\varepsilon}{\mu}\right) e^{-\kappa p/\delta} \leq CC_{\widehat{v}}e^{-\kappa p/\delta},$$

which is the desired bound.

The approximation of $\widehat{\mathbf{U}}_{BL}$ on the remaining elements I_1 and I_5, is achieved with the aid of Corollary 10 in exactly the same way as $\widetilde{\mathbf{U}}_{BL}$ was approximated on I_1, I_2, I_4, and I_5. Again, for the $\widehat{v}$-component, we may exploit the fact that the bounds (44) feature an additional factor $(\varepsilon/\mu)^2$.

Case 3: *$\kappa p\varepsilon \leq \kappa p\mu < \frac{1}{2}$ and $\kappa p\frac{\varepsilon}{\mu} \geq \frac{1}{2}$ ("semi-asymptotic case"), $\Delta = \{0, \kappa p\varepsilon, \kappa p\mu, 1-\kappa p\mu, 1-\kappa p\varepsilon, 1\}$.*

The third and fourth statements of Theorem 2, give the decompositions

$$\begin{aligned}\mathbf{U} &= \mathbf{W}^{III} + \widehat{\mathbf{U}}_{BL}^{III} + \mathbf{R}^{III}, \\ \mathbf{U} &= \mathbf{W}^{IV} + \widetilde{\mathbf{U}}_{BL}^{IV} + \mathbf{R}^{IV},\end{aligned}$$

where $\mathbf{W}^{IV} \in \mathcal{A}(1, C, \gamma)$, $\widetilde{\mathbf{U}}_{BL}^{IV} \in \mathcal{BL}^\infty(\delta\mu, C\sqrt{\mu/\varepsilon}, \gamma\mu/\varepsilon)$ and $\|\mathbf{R}^{IV}\|_{L^\infty(\partial I)} + \|\mathbf{R}^{IV}\|_{E,I} \leq C(\mu/\varepsilon)^2 e^{-b/\mu}$; furthermore, $\mathbf{W}^{III} \in \mathcal{A}(\mu, C, \gamma)$, $\widehat{\mathbf{U}}_{BL}^{III} \in \mathcal{BL}^\infty(\delta\varepsilon, C, \gamma)$ and $\|\mathbf{R}^{III}\|_{L^\infty(\partial I)} + \|\mathbf{R}^{III}\|_{E,I} \leq Ce^{-b/\varepsilon}$ and additionally, the $\widehat{v}$-component of $\widehat{\mathbf{U}}_{BL}^{III}$ satisfies the stronger estimate $\widehat{v} \in \mathcal{BL}^\infty(\delta\varepsilon, C(\varepsilon/\mu)^2, \gamma)$. The mesh consists of 5 elements $I_1, \ldots, I_5$. On $I_1 = (0, \kappa p\varepsilon)$ (and analogously on $I_5 = (1 - \kappa p\varepsilon, 1)$), we approximate $\mathbf{U}$ by

$$\mathcal{I}_{p,I_1}\mathbf{W}^{IV} + \mathcal{I}_{p,I_1}\widetilde{\mathbf{U}}_{BL}^{IV} + \mathcal{I}_{1,I_1}\mathbf{R}^{IV}.$$

On $I_2 = (\kappa p\varepsilon, \kappa p\mu)$ (and analogously on $I_4 = (1 - \kappa p\mu, 1 - \kappa p\varepsilon)$), we approximate $\mathbf{U}$ by

$$\mathcal{I}_{p,I_2}\mathbf{W}^{III} + \mathcal{I}_{1,I_2}\widehat{\mathbf{U}}_{BL}^{III} + \mathcal{I}_{1,I_2}\mathbf{R}^{III}.$$

For the middle element I_3, we use

$$\mathcal{I}_{p,I_3}\mathbf{W}^{IV} + \mathcal{I}_{1,I_3}\widetilde{\mathbf{U}}_{BL}^{IV} + \mathcal{I}_{1,I_3}\mathbf{R}^{IV}.$$

The approximation of the functions $\mathbf{W}^{IV}$ and $\mathbf{W}^{III}$ is done with the aid of Corollary 10. The interesting case is the approximation of $\mathbf{W}^{III}$ on I_2 and I_4, for which we note that the element size satisfies $h_j = \kappa p(\mu - \varepsilon) \leq \kappa p\mu$.

Next, we study the approximation of $\widetilde{\mathbf{U}}_{BL}^{IV}$ on the elements I_1 and I_5. In order to be able to employ Corollary 10, we rewrite the regularity assertion for $\widetilde{\mathbf{U}}_{BL}^{IV} \in \mathcal{BL}^\infty(\delta\mu, C\sqrt{\mu/\varepsilon}, \gamma\mu/\varepsilon)$ as follows:

$$\| \left(\widetilde{\mathbf{U}}_{BL}^{IV}\right)^{(n)} \|_{L^\infty(I_1)} \leq C\sqrt{\mu/\varepsilon}\left(\gamma\frac{\mu}{\varepsilon}\right)^n (\delta\mu)^{-n} \leq C\sqrt{\mu/\varepsilon}\left(\frac{\gamma}{\delta}\right)^n \varepsilon^{-n} \qquad \forall n \in \mathbb{N}.$$

Hence, Corollary 10 implies

$$\|\widetilde{\mathbf{U}}_{BL}^{IV} - \mathcal{I}_{p,I_1}\widetilde{\mathbf{U}}_{BL}^{IV}\|_{1,I_1} \leq C\sqrt{\mu/\varepsilon}e^{-\beta p},$$

if the scale resolution condition

$$\frac{h_1}{p_1\varepsilon} \leq \eta,$$

is satisfied, where η depends solely on γ/δ. In view of $h_1 = \kappa p\varepsilon$ and $p_1 = p$, this condition is satified if κ_0 is sufficiently small. Finally, the factor $\sqrt{\mu/\varepsilon}$ can be controlled since $\frac{\mu}{\varepsilon} \leq 2\kappa p$, and this algebraic factor can be absorbed in the exponentially decaying one.

The approximation of $\widehat{\mathbf{U}}_{BL}^{III}$ on I_2 and I_5 is achieved with Lemma 12 (as in Case 2, the $\widehat{u}$-component and the $\widehat{v}$-component have to be studied separately). Finally, the approximation $\widetilde{\mathbf{U}}_{BL}^{IV}$ on the middle element I_3 is covered by Lemma 12.

We next turn to the remainders. The stability properties of the linear interpolant stated in (33), (34) yield for each element I_j and arbitrary $\mathbf{R} \in H^1(I_j)$

$$\|\mathbf{R} - \mathcal{I}_{1,I_j}\mathbf{R}\|_{1,I_j} \leq C\|\mathbf{R}\|_{1,I_j}.$$

Hence, we get for arbitrary $\mathbf{R} \in H^1(I)$ that $\|\mathbf{R} - \mathcal{I}_{1,\Delta}\mathbf{R}\|_{1,I} \leq C\|\mathbf{R}\|_{1,I} \leq C\varepsilon^{-1}\|\mathbf{R}\|_{E,I}$ and thus

$$\begin{aligned} \varepsilon^{-1}\|\mathbf{R}^{IV}\|_{E,I} &\leq C\varepsilon^{-1}(\mu/\varepsilon)^2 e^{-b/\mu} \leq C(\mu/\varepsilon)^3\mu^{-1}e^{-b/\mu} \leq C(2\kappa p)^3 e^{-b'/\mu} \\ \varepsilon^{-1}\|\mathbf{R}^{III}\|_{E,I} &\leq C\varepsilon^{-1}e^{-b/\varepsilon} \leq Ce^{-b'/\varepsilon}. \end{aligned}$$

Using $\varepsilon \leq \mu$ and $2\kappa p \geq 1/\mu$ shows that these terms are exponentially small in κp as desired.

Case 4: *$\kappa p\mu \geq 1/2$ and $\kappa p\varepsilon < 1/2$, ('semi'-asymptotic case),* $\Delta = \{0, \kappa p\varepsilon, 1-\kappa p\varepsilon, 1\}$. The third statement of Theorem 2 gives the decomposition

$$\mathbf{U} = \mathbf{W} + \widehat{\mathbf{U}}_{BL} + \mathbf{R}, \tag{46}$$

where $\mathbf{W} \in \mathcal{A}(\mu, C_w, \gamma_w)$, $\widehat{\mathbf{U}}_{BL} \in \mathcal{BL}^\infty(\delta\varepsilon, C_{\widehat{u}}, \gamma_{\widehat{u}})$, and $\|\mathbf{R}\|_{L^\infty(\partial I)} + \|\mathbf{R}\|_{E,I} \leq Ce^{-b/\varepsilon}$. Furthermore, the second component $\widehat{v}_{BL}$ of $\widehat{\mathbf{U}}_{BL}$ satisfies the stronger assertion $\widehat{v}_{BL} \in \mathcal{BL}^\infty(\delta\varepsilon, C_{\widehat{v}}(\varepsilon/\mu)^2, \gamma_{\widehat{v}})$. Recall that the mesh consists of three elements

$$I_1 = (0, \kappa p\varepsilon), I_2 = (\kappa p\varepsilon, 1-\kappa p\varepsilon), I_3 = (1-\kappa p\varepsilon, 1)\,,$$

and we approximate each component as follows: $\mathbf{W}$ is approximated by $\mathcal{I}_{p,\Delta}\mathbf{W}$, the boundary layer $\widehat{\mathbf{U}}_{BL}$ is approximated by $\mathcal{I}_{p,1,p}\widehat{\mathbf{U}}_{BL}$, and $\mathbf{R}$ is approximated by $\mathcal{I}_{1,\Delta}\mathbf{R}$. For the approximation of $\mathbf{W}$, we use Corollary 10, which yields exponential convergence, since the scale resolution condition

$$\eta \overset{!}{\geq} \frac{h_j}{p\mu} = \frac{h_j\kappa}{\kappa p\mu} \geq 2\kappa h_j,$$

can be satisfied for all elements by taking κ_0 sufficiently small.

The approximation of $\widehat{\mathbf{U}}_{BL}$ follows by the same arguments as in Case 2.

Finally, for the remainder $\mathbf{R}$, we use again stability of the piecewise linear approximation and the fact that the algebraic factor ε^{-1} can be absorbed in the exponentially small factor $e^{-b/\varepsilon}$ at the expense of slightly reducing b:

$$\begin{aligned} \|\mathbf{R} - \mathcal{I}_{1,\Delta}\mathbf{R}\|_{E,I} &\leq \|\mathbf{R} - \mathcal{I}_{1,\Delta}\mathbf{R}\|_{1,I} \leq C\|\mathbf{R}\|_{1,I} \leq C\varepsilon^{-1}\|\mathbf{R}\|_{E,I} \leq C\varepsilon^{-1}e^{-b/\varepsilon} \\ &\leq Ce^{-b'/\varepsilon} \leq Ce^{-2b'\kappa p}. \end{aligned}$$

This completes the proof of Theorem 4. □

4 Numerical Results

In this section we present the results of numerical computations for the model problem considered in [2], [6] and [18]. The data is chosen as follows:

$$\mathbf{A} = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}, \mathbf{F} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

An exact solution is available, hence the computations we report are reliable. We will be plotting the percentage relative error in the energy norm, given by

$$100 \times \frac{\|\mathbf{U}_{EXACT} - \mathbf{U}_{FEM}\|_{E,I}}{\|\mathbf{U}_{EXACT}\|_{E,I}}, \tag{47}$$

versus the number of degrees of freedom N. We will focus on Cases *(III)* and *(IV)* since in [18] the other two cases where adequately studied and the error estimates along with the robustness of the method were verified.

4.1 Case *(III)*

Recall that in this case μ is not small but ε/μ is. Figure 1, corresponds to $\mu = 0.1, \varepsilon = 10^{-j}, j = 2, ..., 6$. The robustness and exponential convergence is readily visible.

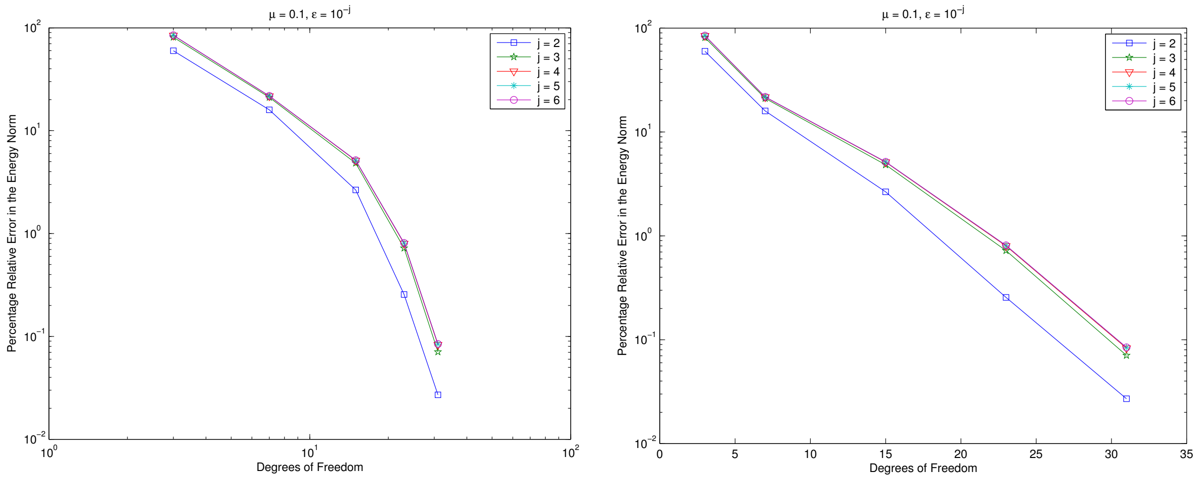

Figure 1: Energy norm convergence for the hp version on the Spectral Boundary Layer mesh $\Delta = \{0, p\varepsilon, 1 - p\varepsilon, 1\}$, with $\mu = 0.1$. (Left: log-log scale, Right: semi-log scale.)

Other values of μ produce almost identical results.

4.2 Case *(IV)*

In this case μ is small but ε/μ is not. Figure 2 corresponds to $\mu = 0.01, \varepsilon = 0.001$ with $\varepsilon/\mu = 0.1$.

Two things may be seen from Figure 2: First, the use of the five-element Spectral Boundary Layer mesh leads to robust exponential convergence. Second, the use of a three-element mesh also leads to exponential convergence in the regime of problem sizes studied here. This latter behavior can qualitatively be understood as follows: the three-element mesh employed is not truely capable to resolve the boundary layers on the $O(\varepsilon)$-scale but these layers are weak in the energy norm. A simple calculation shows that the boundary layers $\widehat{\mathbf{U}}_{BL}$ on the $O(\varepsilon)$-scale, which have the regularity

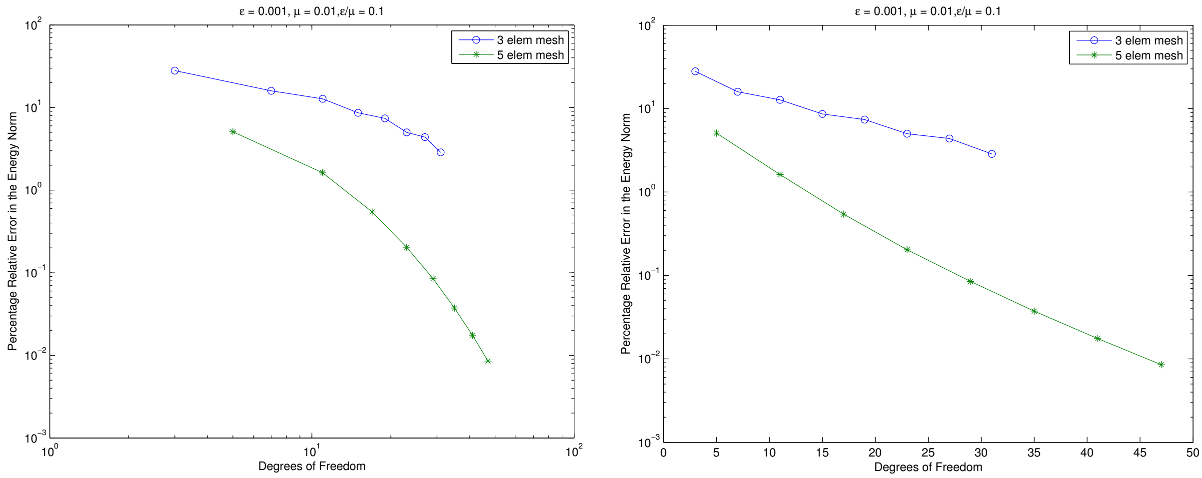

Figure 2: Comparison of the hp version on the Spectral Boundary Layer meshes $\Delta = \{0, p\varepsilon, 1-p\varepsilon, 1\}$ vs. $\Delta = \{0, p\varepsilon, p\mu, 1-p\mu, 1-p\varepsilon, 1\}$. (Left: log-log scale, Right: semi-log scale.)

described in Theorem 2, have energy norm $O(\varepsilon^{1/2}) + O((\varepsilon/\mu)^2(\mu\varepsilon^{-1/2} + \varepsilon^{1/2}))$. Hence, up this error level, one may expect preasymptotic exponential convergence even on meshes that are not able to resolve these solution features.

Figure 3 shows the same results for $\mu = 0.005, \varepsilon = 0.001$ with $\varepsilon/\mu = 0.2$.

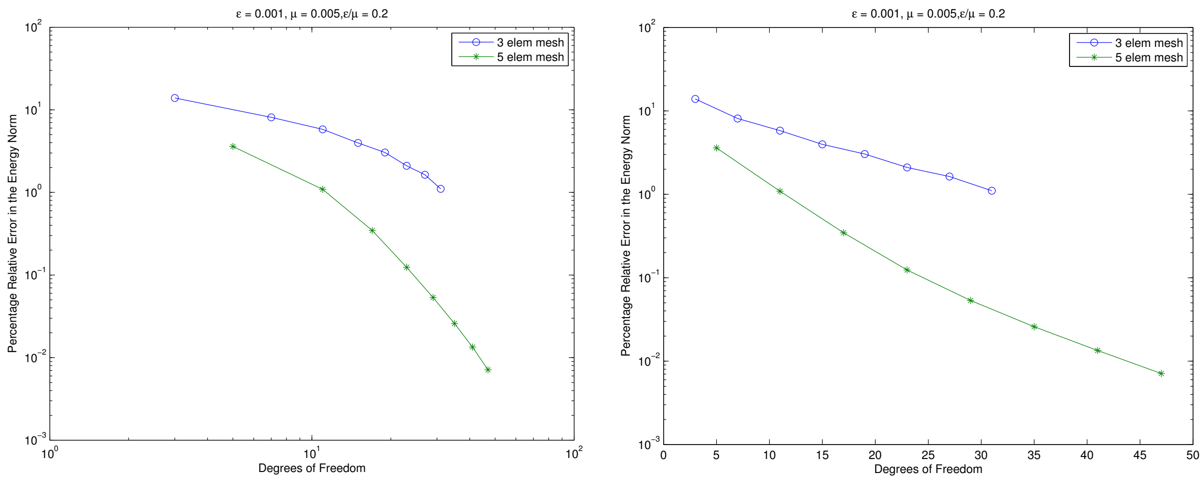

Figure 3: Comparison of the hp version on the Spectral Boundary Layer meshes $\Delta = \{0, p\varepsilon, 1-p\varepsilon, 1\}$ vs. $\Delta = \{0, p\varepsilon, p\mu, 1-p\mu, 1-p\varepsilon, 1\}$. (Left: log-log scale, Right: semi-log scale.)

5 Conclusions and Extensions

We considered a coupled system of two reaction diffusion equations with two singular perturbation parameters $0 < \varepsilon \leq \mu \leq 1$. We have proved that the hp FEM proposed in [18] for its approximation, indeed exhibits exponential rates of convergence, independently of the singular perturbation parameters ε and μ, as the degree p of the approximating polynomials is increased. The key ingredient in our proofs is the regularity theory of [11]. We believe the same ideas can be applied to systems of convection-diffusion problems in one dimension, as well as reaction-diffusion systems in two dimensions. This is the focus of our current research efforts.

Inspection of the proof of the approximation result Theorem 4 shows that some refinements are possible. We highlight three of them:

1. Our boundary layer approximation relies on Lemma 12, which provides estimates for the piecewise linear approximation of boundary layer functions outside the layer. This approximation can be improved using the technique employed in [16, Thm. 5.1] of adding an appropriate additional piecewise linear function. The end result is then that one can construct a piecewise polynomial approximation π to a function $u \in \mathcal{BL}^\infty(\nu, C, \gamma)$ on the three-element mesh $\Delta = \{0, \kappa\nu p, 1 - \kappa\nu p, 1\}$ that satisfies:
$$\|u - \pi\|_{0,I} + (\kappa p\nu)\|(u - \pi)'\|_{0,I} \leq C\sqrt{\kappa p\nu}e^{-\beta\kappa p}.$$
These boundary layer approximation results can, for example, lead to improved error bounds if the model problem (1) is considered with $f = g = 0$ and inhomogeneous Dirichlet boundary data. This parallels the case of a scalar singularly perturbed problem analyzed in [16] and visible in the numerics of [18].

2. The sublayer on the $O(\varepsilon)$-scale is particularly weak in the $\widehat{v}$-component. This could be exploited further. For example, in certain parameter ranges, one could remove the mesh points $\kappa p\varepsilon$ and $1 - \kappa p\varepsilon$ in the mesh for the second component without compromising, up to certain tolerances, the accuracy of the FEM.

3. The proof of Theorem 4 relies on the regularity assertions of Theorem 2. There, L^∞-based estimates are given for the solution $\mathbf{U}$. The proof of Theorem 4, however, mostly uses L^2-based regularity estimates.

References

[1] N. S. Bakhvalov, *Towards optimization of methods for solving boundary value problems in the presence of boundary layers* (in Russian), Zh. Vychisl. Mat. Mat. Fiz. **9** (1969) 841–859.

[2] T. Linß and N. Madden, *A finite element analysis of a coupled system of singularly perturbed reaction-diffusion equations*, Appl. Math. Comp. **148** (2004) 869–880.

[3] T. Linß and N. Madden, *An improved error estimate for a numerical method for a system of coupled singularly perturbed reaction-diffusion equations*, Comput. Meth. Appl. Math. **3** (2003) 417–423.

[4] T. Linß and N. Madden, *Accurate solution of a system of coupled singularly perturbed reaction-diffusion equations*, Computing **73** (2004) 121–1133.

[5] T. Linß and M. Stynes, *Numerical solution of systems of singularly perturbed differential equations*, Comput. Method. Appl. M., **9** (2009) 165–191.

[6] N. Madden and M. Stynes, *A uniformly convergent numerical method for a coupled system of two singularly perturbed linear reaction-diffusion problems*, IMA Journal of Numerical Analysis **23** (2003), 627-644.

[7] S. Matthews, E. O'Riordan and G. I. Shishkin, *A numerical method for a system of singularly perturbed reaction-diffusion equations*, J. Comput. Appl. Math. **145** (2002) 151–166.

[8] S. Matthews, J. J. H. Miller, E. O'Riordan and G. I. Shishkin, *A parameter robust numerical method for a system of singularly perturbed ordinary differential equations*, in: J. J. H. Miller, G. I. Shishkin, L. Vulkov (Eds.), Analytical and Numerical Methods for Convection-Dominated and Singularly Perturbed Problems, Nova Science Publishers, New York, 2000, pp. 219–224.

[9] J. M. Melenk, *On the robust exponential convergence of hp finite element methods for problems with boundary layers*, IMA Journal of Numerical Analysis **17** (1997), 577-601.

[10] J. M. Melenk, *hp-Finite Element Methods for Singular Perturbations*, Volume 1796 of Springer Lecture Notes in Mathematics, Springer Verlag, 2002.

[11] J. M. Melenk, C. Xenophontos and L. Oberbroeckling, *Analytic regularity for a singularly perturbed system of reaction-diffusion equations with multiple scales: proofs*, ASC-Report 29/2011, TU Vienna, arXiv:1108.2002, 2011.

[12] J. J. H. Miller, E. O'Riordan and G. I. Shishkin, *Fitted Numerical Methods Singular Perturbation Problems*, World Scientific, 1996.

[13] K. W. Morton, *Numerical Solution of Convection-Diffusion Problems*, Volume 12 of Applied Mathematics and Mathematical Computation, Chapman & Hall, 1996.

[14] Hans-Görg Roos, Martin Stynes, and Lutz Tobiska. *Robust numerical methods for singularly perturbed differential equations*, volume 24 of *Springer Series in Computational Mathematics*. Springer-Verlag, Berlin, second edition, 2008. Convection-diffusion-reaction and flow problems.

[15] C. Schwab, *p- and hp-Finite Element Methods*, Oxford Science Publications, 1998.

[16] C. Schwab and M. Suri, *The p and hp versions of the finite element method for problems with boundary layers*, Math. Comp. **65** (1996) 1403–1429.

[17] G. I. Shishkin, *Grid approximation of singularly perturbed boundary value problems with a regular boundary layer*, Sov. J. Numer. Anal. Math. Model. **4** (1989) 397–417.

[18] C. Xenophontos and L. Oberbroeckling, *A numerical study on the finite element solution of singularly perturbed systems of reaction-diffusion problems*, Appl. Math. Comp., Vol. 187, pp. 1351 – 1367 (2007).

[19] C. Xenophontos and L. Oberbroeckling, *On the finite element approximation of systems of reaction-diffusion equations by p/hp methods*, Journal of Computational Mathematics, Vol. 28, No. 3, pp. 386 – 400 (2010).