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COMPACT FAMILIES OF PIECEWISE CONSTANT FUNCTIONS IN $L^p(0,T;B)$

MICHAEL DREHER AND ANSGAR JÜNGEL

Abstract. A strong compactness result in the spirit of the Lions-Aubin-Simon lemma is proven for piecewise constant functions in time (u_τ) with values in a Banach space. The main feature of our result is that it is sufficient to verify one uniform estimate for the time shifts $u_\tau(t) - u_\tau(t-\tau)$ instead of all time shifts $u_\tau(t) - u_\tau(t-h)$ for $h > 0$, as required in Simon's compactness theorem. This simplifies significantly the application of the Rothe method in the existence analysis of parabolic problems.

1. Introduction

A useful technique to prove the existence of weak solutions to nonlinear evolution equations and their systems is to semi-discretize the equations in time by the implicit Euler method (also called Rothe method [5]):

$$\frac{1}{\tau}\big(u_\tau(t) - u_\tau(t-\tau)\big) + A(u_\tau(t)) = f_\tau(t), \quad 0 < t < T, \quad u(0) \text{ given,} \tag{1}$$

where $\tau > 0$ is the time step, A is an abstract (nonlinear) operator defined on a certain Banach space, and $f_\tau(t)$ is some (piecewise constant) function with values in a Banach space. In this way, nonlinear elliptic problems are obtained which are sometimes easier to solve. In order to pass to the limit of vanishing time steps, $\tau \to 0$, (relative) compactness for the sequence of piecewise constant approximate solutions (u_τ) is needed. Since the problem is nonlinear, we need strong convergence of (a subsequence of) (u_τ) to identify the limits. Having suitable *a priori* estimates at hand, strong compactness can be concluded from the Aubin (or Lions-Aubin-Simon) lemma [6] which is a consequence of a compactness criterium due to Kolmogorov. However, the results of [6] are not directly applicable. Indeed, typically one can derive the uniform estimate

$$\|u_\tau(t) - u_\tau(t-\tau)\|_{L^1(\tau,T;Y)} = \tau\,\|-A(u_\tau) + f_\tau\|_{L^1(\tau,T;Y)} \le C\tau, \tag{2}$$

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where $C > 0$ does not depend on τ, and Y is some Banach space. On the other hand, in order to apply the Aubin lemma, one needs [6, Theorem 3]

$$\|u_\tau(t) - u_\tau(t-h)\|_{L^1(h,T;Y)} \to 0 \quad \text{as } h \to 0, \text{ uniformly in } \tau > 0.$$

A possible way to avoid this problem is to construct linear interpolants of u_τ, say $\widetilde{u}_\tau$, for which a continuous time-derivative version of the Aubin lemma can be applied (e.g., Corollary 4 in [6]). Since we need strong convergence of (u_τ), one has to show that $u_\tau - \widetilde{u}_\tau \to 0$ a.e. as $\tau \to 0$, which might be difficult or even impossible to prove.

In this note, we show that estimate (2) is sufficient to infer strong compactness of (u_τ). The main feature of our result is that it is sufficient to study the time shifts $u_\tau(t) - u_\tau(t-\tau)$ instead of all time shifts $u_\tau(t) - u_\tau(t-h)$ for all $h > 0$. This simplifies the proof of the limit $\tau \to 0$ in (1) significantly.

For our main results, let $T > 0$, $N \in \mathbb{N}$, $\tau = T/N$, and set $t_k = k\tau$, $k = 0, \ldots, N$. Furthermore, let $(S_h u)(x,t) = u(x, t-h)$, $t \geq h > 0$, be the shift operator. We notice that non-uniform time steps may be considered too [3], but they are of minor interest in the existence analysis.

Theorem 1. *Let X, B, and Y be Banach spaces such that the embedding $X \hookrightarrow B$ is compact and the embedding $B \hookrightarrow Y$ is continuous. Furthermore, let either $1 \leq p < \infty$, $r = 1$ or $p = \infty$, $r > 1$, and let (u_τ) be a sequence of functions, which are constant on each subinterval (t_{k-1}, t_k), satisfying*

$$\tau^{-1}\|u_\tau - S_\tau u_\tau\|_{L^r(\tau,T;Y)} + \|u_\tau\|_{L^p(0,T;X)} \leq C \quad \textit{for all } \tau > 0, \tag{3}$$

where $C > 0$ is a constant which is independent of τ. Then (u_τ) is relatively compact in $L^p(0,T;B)$ if $p < \infty$ and in $C^0([0,T];B)$ if $p = \infty$.

A related result in finite-dimensional spaces was recently proven by Gallouët and Latché [4, Theorem 3.4]. The same setting for degenerate elliptic-parabolic equations in L^1 was considered by Andreianov [2].

Proposition 2. *The factor τ^{-1} in inequality (3) cannot be replaced by $\tau^{-\alpha}$ for $0 < \alpha < 1$.*

This note is organized as follows. In Section 2, Theorem 1 is shown; the proof of Proposition 2 is presented in Section 3. Finally, we comment these results in Section 4.

2. Proof of Theorem 1

The proof of Theorem 1 is based on a characterisation of the norm of fractional Sobolev spaces. Let $1 \leq q < \infty$, $0 < \sigma < 1$, and let Y be a Banach space. The fractional Sobolev space $W^{\sigma,q}(0,T;Y)$ is the space of (equivalence classes of) functions $u \in L^q(0,T;Y)$ with finite norm

$$\|u\|_{W^{\sigma,q}(0,T;Y)} = \left(\|u\|^q_{L^q(0,T;Y)} + |u|^q_{\dot{W}^{\sigma,q}(0,T;Y)}\right)^{1/q},$$

where

$$|u|_{\dot{W}^{\sigma,q}(0,T;Y)} = \left(\int_0^T \int_0^T \frac{\|u(t) - u(s)\|^q_Y}{|t-s|^{1+\sigma q}} \,\mathrm{d}s\,\mathrm{d}t\right)^{1/q}$$

is a semi-norm. Fractional Sobolev spaces in time have also been proven to be a useful tool in [3].

Lemma 3. *Let $1 \le q < \infty$, $0 < \sigma < 1$ with $\sigma q < 1$ and let $u \in L^q(0,T;Y)$ be a piecewise constant function with (a finite number of) jumps of height $[u]_k \in Y$ at points t_k, $k = 1, \ldots, N-1$. Then $u \in W^{\sigma,q}(0,T;Y)$ and*

$$\|u\|_{W^{\sigma,q}(0,T;Y)} \le \|u\|_{L^q(0,T;Y)} + C_{\sigma q,T}^{1/q} \sum_{k=1}^{N-1} \|[u]_k\|_Y \,,$$

where $C_{\sigma q,T} = 2(2^{\sigma q} - 1)T^{1-\sigma q}/(\sigma q(1-\sigma q))$.

Proof. We may assume that $0 = t_0 < t_1 < \cdots < t_{N-1} < t_N = T$ and that $u(\cdot,t) = u_k$ for $t_{k-1} < t < t_k$ where $k = 1, \ldots, N$. Then $[u]_k = u_{k+1} - u_k$, $k = 1, \ldots, N-1$, and

$$u(\cdot,t) = u_k = u_1 + \sum_{j=1}^{k-1}(u_{j+1} - u_j) = u_1 + \sum_{j=1}^{N-1}[u]_j H_{t_j}(t)$$

for $t_{k-1} < t < t_k$, where H_{t_j} is the shifted Heaviside function

$$H_{t_j}(t) = \begin{cases} 0 & : 0 < t < t_j, \\ 1 & : t_j < t < T. \end{cases}$$

By definition of the $W^{\sigma,q}(0,T;Y)$ norm and the semi-norm property of $|\cdot|_{\dot W^{\sigma,q}(0,T;Y)}$, we find that

$$\begin{aligned} \|u\|_{W^{\sigma,q}(0,T;Y)} &= \Big(\|u\|^q_{L^q(0,T;Y)} + |u|^q_{\dot W^{\sigma,q}(0,T;Y)}\Big)^{1/q} \\ &\le \|u\|_{L^q(0,T;Y)} + |u|_{\dot W^{\sigma,q}(0,T;Y)} \\ &\le \|u\|_{L^q(0,T;Y)} + |u_1|_{\dot W^{\sigma,q}(0,T;Y)} + \sum_{j=1}^{N-1} \|[u]_j\|_Y \; |H_{t_j}|_{\dot W^{\sigma,q}(0,T)} \\ &= \|u\|_{L^q(0,T;Y)} + \sum_{j=1}^{N-1} \|[u]_j\|_Y \; |H_{t_j}|_{\dot W^{\sigma,q}(0,T)}. \end{aligned} \tag{4}$$

It remains to compute the seminorm of H_{t_j}:

$$\begin{aligned} |H_{t_j}|^q_{\dot W^{\sigma,q}} &= \int_0^T \int_0^T \frac{|H_{t_j}(t) - H_{t_j}(s)|^q}{|t-s|^{1+\sigma q}}\,\mathrm{d}s\,\mathrm{d}t = 2\int_0^{t_j}\int_{t_j}^T \frac{|H_{t_j}(t) - H_{t_j}(s)|^q}{|t-s|^{1+\sigma q}}\,\mathrm{d}s\,\mathrm{d}t \\ &= 2\int_0^{t_j}\int_{t_j}^T |t-s|^{-1-\sigma q}\,\mathrm{d}s\,\mathrm{d}t = \frac{2}{\sigma q(1-\sigma q)}\big((T-t_j)^{1-\sigma q} + t_j^{1-\sigma q} - T^{1-\sigma q}\big). \end{aligned}$$

The right-hand side can be interpreted as a function of $\vartheta = t_j \in [0,T]$ whose maximum is achieved at $\vartheta = T/2$. Therefore,

$$|H_{t_j}|^q_{\dot W^{\sigma,q}} \le \frac{2}{\sigma q(1-\sigma q)}\left(2\Big(\frac{T}{2}\Big)^{1-\sigma q} - T^{1-\sigma q}\right) = \frac{2}{\sigma q(1-\sigma q)}(2^{\sigma q} - 1)T^{1-\sigma q} = C_{\sigma q,T}.$$

Inserting this estimate in (4), the result follows. ☐

For later use, we remark that the calculations in (4) and below show that

$$|u|_{\dot W^{\sigma,1}(0,T;Y)} \le C_{\sigma q,T}^{1/q} \sum_{k=1}^{N-1} \|[u]_k\|_Y . \tag{5}$$

Proof of Theorem 1. The idea of the proof is to apply Corollary 5 in [6]: If (u_τ) is bounded in $L^p(0,T;X)\cap W^{\sigma,r}(0,T;Y)$, where $\sigma > \max\{0, 1/r-1/p\}$, then (u_τ) is relatively compact in $L^p(0,T;B)$ if $p<\infty$, $r=1$ and in $C^0([0,T];B)$ if $p=\infty$, $r>1$.

Let $\sigma \in (0,1)$ satisfy $\sigma > \max\{0,1/r-1/p\}$ and $u_\tau(\cdot,t) = u_k$ for $t_{k-1} < t < t_k$, $k=1,\ldots,N$. Then

$$\begin{aligned}\sum_{k=1}^{N-1}\|[u_\tau]_k\|_Y &= \sum_{k=1}^{N-1}\|u_{k+1}-u_k\|_Y = \tau^{-1}\sum_{k=1}^{N-1}\int_{t_k}^{t_{k+1}}\|u_{k+1}-u_k\|_Y \,\mathrm{d}t \\ &= \tau^{-1}\|u_\tau - S_\tau u_\tau\|_{L^1(\tau,T;Y)} \le \tau^{-1}C\|u_\tau - S_\tau u_\tau\|_{L^r(\tau,T;Y)} \le C. \end{aligned}\tag{6}$$

Since $L^p(0,T;X) \hookrightarrow L^1(0,T;Y)$ if $p<\infty$ and $L^p(0,T;X)\hookrightarrow L^r(0,T;Y)$ if $p=\infty$, Lemma 3 shows that (u_τ) is bounded in $W^{\sigma,r}(0,T;Y)$, and the corollary applies. ☐

3. Proof of Proposition 2

We construct a sequence (u_τ) satisfying the assumptions of Theorem 1 with $\tau^{-\alpha}$ $(0<\alpha<1)$ in (3) instead of τ^{-1}, but not possessing a convergent subsequence in $L^p(0,T;B)$, where $p<\infty$.

Take $X=Y=B=\mathbb{C}$ and $(0,T)=(0,1)$. For $\beta\ge 1$, define the function

$$f_\beta(t) := (\beta p+1)^{1/p}t^\beta, \quad 0\le t\le 1.$$

Then we have $\|f_\beta\|_{L^p(0,T)} = 1$. For later use, we remark that

$$\lim_{\beta\to\infty} f_\beta(t) = 0, \tag{7}$$

for each fixed $t\in[0,1)$, uniformly on compact sub-intervals $[0,t_*]\subset[0,1)$.

Since $\alpha<1$, we may choose a real number $0<\gamma\le\min\{1,p(1-\alpha)\}$. We set $\beta(\tau)=\tau^{-\gamma}$ and

$$u_\tau(t) := \begin{cases} f_{\beta(\tau)}(k\tau) & \text{for } k\tau\le t<(k+1)\tau,\ k\in\{0,1,\ldots,N-1\}, \\ f_{\beta(\tau)}((N-1)\tau) & \text{for } t=1. \end{cases}$$

The function u_τ has jumps of height $[u_\tau]_k$ at the values $t=k\tau$ for $1\le k\le N-1$, and all jumps have the same sign. In particular,

$$\begin{aligned} &\sum_{k=1}^{N-1}\|[u_\tau]_k\|_Y = \sum_{k=1}^{N-1}[u_\tau]_k = f_{\beta(\tau)}(1-\tau) = (\tau^{-\gamma}p+1)^{1/p}(1-\tau)^{\tau^{-\gamma}}, \\ &1\ge(1-\tau)^{\tau^{-\gamma}} \ge (1-\tau)^{1/\tau} \ge \frac{1}{2e}. \end{aligned}$$

Therefore, it follows that

$$\tau^{-\alpha} \|u_\tau - S_\tau u_\tau\|_{L^1(\tau,T,Y)} = \tau^{1-\alpha} \sum_{k=1}^{N-1} \|[u_\tau]_k\|_Y = \tau^{1-\alpha}(\tau^{-\gamma}p+1)^{1/p}(1-\tau)^{\tau^{-\gamma}}$$
$$\leq \tau^{1-\alpha}(\tau^{-\gamma}p+1)^{1/p} \leq K,$$

for all $\tau \in (0,1/2)$, since $1-\alpha-\gamma/p \geq 0$. Hence, (3) holds. But the sequence $(u_\tau) \subset L^p(0,T;B)$ does not possess a converging subsequence, which can be seen as follows. Fix $t \in [0,1)$. Then $0 \leq u_\tau(t) \leq f_{\beta(\tau)}(t)$, and (7) implies the pointwise convergence $\lim_{\tau\to 0} u_\tau(t) = 0$, uniform on compact sub-intervals $[0,t_*] \subset [0,1)$. Thus, the pointwise limit of the subsequence must be the zero function. However, this is impossible, because of the following uniform lower bound:

$$\int_0^1 |u_\tau(t)|^p \,\mathrm{d}t \geq \int_0^{1-\tau} |f_{\beta(\tau)}(t)|^p \,\mathrm{d}t = (1-\tau)^{\tau^{-\gamma}p+1} \geq \frac{1}{2}\left((1-\tau)^{\tau^{-\gamma}}\right)^p \geq \frac{1}{2}(2e)^{-p},$$

showing the claim.

4. Comments

Proposition 2 shows that the exponent of the factor τ in (3) cannot be raised. However, when allowing for arbitrary time shifts S_h, the factor can be replaced by $h^{-\alpha}$, where $0<\alpha<1$, under some conditions. An example, adapted to our situation, can be found in [1, Theorem 1.1]:

Theorem 4 (Amann)**.** *Let X, B, and Y be Banach spaces such that the embedding $X \hookrightarrow Y$ is compact and there exist $0<\theta<1$, $C_1>0$ such that for all $u \in X$, $\|u\|_B \leq C_1 \|u\|_X^\theta \|u\|_Y^{1-\theta}$. Furthermore, let $0<s<1$, $1 \leq p < \infty$, and $F \subset L^p(0,T;Y)$. Assume that there exists $C_2>0$ such that each $u \in F$ satisfies the following infinite collection of inequalities:*

$$h^{-s}\|u - S_h u\|_{L^1(\tau,T;Y)} + \|u\|_{L^p(0,T;X)} \leq C_2 \quad \textit{for all } h>0.$$

Then F is relatively compact in $L^q(0,T;B)$ for all $q < p/((1-\theta)(1-s)p+\theta)$.

Notice that $q=p$ is admissible if $(1-\theta)(1-s)p+\theta<1$ which is equivalent to $s>1-1/p$. Thus, if we wish to allow for arbitrary large $p \geq 1$, we have to require the condition $s=1$, which corresponds to the result of Theorem 1. On the other hand, in applications, often $p=2$, and compactness follows even for $s<1$, namely for any $s>1/2$.

In the special situation when we have the triple $X \hookrightarrow B \hookrightarrow X'$, where $Y = X'$ is the dual space of X and B is a Hilbert space, the assumptions of Amann's theorem hold with $\theta = 1/2$. Then $q < 2p/((1-s)p+1)$, and we see that $2p$ is an upper bound for q. This corresponds to the result of Walkington [7, Theorem 3.1 (1)].

Estimates (5) and (6) imply that, for all piecewise constant functions $u \in L^1(0,T;Y)$ with jumps at $t_k = k\tau$,

$$|u|_{\dot W^{\sigma,1}(0,T;Y)} \leq C_{\sigma q,T}^{1/q} \sum_{k=1}^{N-1} \|[u]_k\|_Y \leq \tau^{-1} C_{\sigma q,T}^{1/q} \|u - S_\tau u\|_{L^1(\tau,T;Y)}\,.$$

By Lemma 5 of [6], there exists an inverse inequality for all $u \in W^{\sigma,1}(0,T;Y)$ and all $\sigma \in (0,1)$:

$$\|u - S_\tau u\|_{L^1(\tau,T;Y)} \leq C_3 \tau^\sigma |u|_{\dot{W}^{\sigma,1}(0,T;Y)},$$

where $C_3 > 0$ depends on σ and T. In this sense, the chain of inequalities

$$\tau |u|_{\dot{W}^{\sigma,1}(0,T;Y)} \leq \tau^\sigma C_{\sigma q,T}^{1/q} C_3 |u|_{\dot{W}^{\sigma,1}(0,T;Y)}$$

is almost sharp since we can choose σ as close to one as we wish.

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