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Adaptive coupling of FEM and BEM: Simple error estimators and convergence

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A posteriori error estimators and adaptive mesh-refinement have themselves proven to be important tools for scientific computing. For error control in finite element methods (FEM), there is a broad variety of a posteriori error estimators available, and convergence as well as optimality of adaptive FEM is well-studied in the literature. This is, however, in sharp contrast to the boundary element method (BEM) and to the coupling of FEM and BEM. In our contribution, we present an easy-to-implement error estimator for some FEM-BEM coupling which, to the best of our knowledge, has not been proposed in the literature before. The derived mesh-refining algorithm provides the first adaptive coupling procedure which is mathematically proven to converge.

1 Symmetric FEM-BEM Coupling

For a bounded Lipschitz domain $\Omega\subset\mathbb{R}^2$ with $\Gamma=\partial\Omega$, we consider the nonlinear interface problem

$$\left\{\begin{array}{rcll} -\mathrm{div}(A\nabla u^{\rm int}) &=& f & \text{in } \Omega^{\rm int} := \Omega,\\ -\Delta u^{\rm ext} &=& 0 & \text{in } \Omega^{\rm ext} := \mathbb{R}^2\backslash\overline{\Omega},\\ u^{\rm int}-u^{\rm ext} &=& u_0 & \text{on } \Gamma,\\ (A\nabla u^{\rm int}-\nabla u^{\rm ext})\cdot n &=& \phi_0 & \text{on } \Gamma,\\ u^{\rm ext}(x) &=& a\,\log|x| + \mathcal{O}(1/|x|) & \text{as } |x|\to\infty, \end{array}\right. \tag{1}$$

where n denotes the outer unit normal vector. The given data satisfy $f\in L^2(\Omega)$, $u_0\in H^{1/2}(\Gamma)$, and $\phi_0\in H^{-1/2}(\Gamma)$, and the (possibly nonlinear) operator $A: L^2(\Omega)^2\to L^2(\Omega)^2$ is assumed to be strongly monotone and Lipschitz continuous.

Problem (1) is equivalently stated via the well-known symmetric FEM-BEM coupling: Find $(u,\phi)\in\mathcal{H} := H^1(\Omega)\times H^{-1/2}(\Gamma)$ such that, for all $(v,\psi)\in\mathcal{H}$,

$$\left\{\begin{array}{rcl} \langle A\nabla u\,,\,\nabla v\rangle_\Omega + \langle Wu + (K'-\tfrac12)\phi\,,\,v\rangle_\Gamma &=& \langle f\,,\,v\rangle_\Omega + \langle \phi_0 + Wu_0\,,\,v\rangle_\Gamma,\\ \langle\psi\,,\,V\phi - (K-\tfrac12)u\rangle_\Gamma &=& -\langle\psi\,,\,(K-\tfrac12)u_0\rangle_\Gamma. \end{array}\right. \tag{2}$$

Here, V denotes the simple-layer potential, K denotes the double-layer potential with adjoint K', and W denotes the hyper-singular integral operator. Then, (2) has a unique solution (u,ϕ) which depends continuously on the given data, see e.g. [4]. Moreover, (1) and (2) are linked through $(u,\phi) = (u^{\rm int},\partial_n u^{\rm ext})$ and $u^{\rm ext} = K(u-u_0) - V\phi$.

2 Galerkin Discretization

For the Galerkin discretization, let $\mathcal{T}_\ell$ be a regular triangulation of Ω into triangles $T_j\in\mathcal{T}_\ell$ and $\mathcal{E}_\ell = \mathcal{T}_\ell|_\Gamma$ be the induced partition of the coupling boundary Γ into piecewise affine line segments $E_j\in\mathcal{E}_\ell$. We then use P1-finite elements $u_\ell\in\mathcal{S}^1(\mathcal{T}_\ell)$ to approximate u and piecewise constants $\phi_\ell\in\mathcal{P}^0(\mathcal{E}_\ell)$ to approximate ϕ, i.e. the discrete space is defined by $\mathcal{X}_\ell := \mathcal{S}^1(\mathcal{T}_\ell)\times\mathcal{P}^0(\mathcal{E}_\ell)\subset\mathcal{H}$. Now, the Galerkin formulation reads: Find $(u_\ell,\phi_\ell)\in\mathcal{X}_\ell$ such that, for all $(v_\ell,\psi_\ell)\in\mathcal{X}_\ell$,

$$\left\{\begin{array}{rcl} \langle A\nabla u_\ell\,,\,\nabla v_\ell\rangle_\Omega + \langle Wu_\ell + (K'-\tfrac12)\phi_\ell\,,\,v_\ell\rangle_\Gamma &=& \langle f\,,\,v_\ell\rangle_\Omega + \langle \phi_0 + Wu_0\,,\,v_\ell\rangle_\Gamma,\\ \langle\psi_\ell\,,\,V\phi_\ell - (K-\tfrac12)u_\ell\rangle_\Gamma &=& -\langle\psi_\ell\,,\,(K-\tfrac12)u_0\rangle_\Gamma. \end{array}\right. \tag{3}$$

Again, we refer to [4] for the fact that the discretization (3) has a unique solution $(u_\ell,\phi_\ell)\in\mathcal{X}_\ell$.

3 A posteriori Error Control

For a posteriori error estimation, we employ the general concept of $h-h/2$ error estimation: We solve the discrete system (3) twice to obtain Galerkin solutions $(u_\ell,\phi_\ell)\in\mathcal{X}_\ell$ and $(\widehat{u}_\ell,\widehat{\phi}_\ell)\in\widehat{\mathcal{X}}_\ell$, where the enriched space $\widehat{\mathcal{X}}_\ell$ is induced by the uniform

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refinement $\widehat{\mathcal{T}}_\ell$ of $\mathcal{T}_\ell$ and $\widehat{\mathcal{E}}_\ell = \widehat{\mathcal{T}}_\ell|_\Gamma$. With

$$\eta_\ell = \|(\widehat{u}_\ell, \widehat{\phi}_\ell) - (u_\ell, \phi_\ell)\|_{H^1(\Omega)\times H^{-1/2}(\Gamma)}, \tag{4}$$

we observe that, up to some multiplicative constant, we always obtain a lower bound for the error

$$\eta_\ell \lesssim \|(u, \phi) - (u_\ell, \phi_\ell)\|_{H^1(\Omega)\times H^{-1/2}(\Gamma)}. \tag{5}$$

Moreover, the converse inequality $\gtrsim$ holds under a saturation assumption

$$\|(u, \phi) - (\widehat{u}_\ell, \widehat{\phi}_\ell)\|_{H^1(\Omega)\times H^{-1/2}(\Gamma)} \le q\,\|(u, \phi) - (u_\ell, \phi_\ell)\|_{H^1(\Omega)\times H^{-1/2}(\Gamma)} \tag{6}$$

with some uniform constant $0 < q < 1$. Note that (6) essentially states that the Galerkin scheme has reached some asymptotic regime, i.e. $\|(u, \phi) - (u_\ell, \phi_\ell)\|_{H^1(\Omega)\times H^{-1/2}(\Gamma)} = \mathcal{O}(h^\alpha)$.

Having defined η_ℓ in (4), we stress that, first, the $H^{-1/2}$-norm can hardly be computed and, second, (u_ℓ, ϕ_ℓ) is hardly ever used in practice since $(\widehat{u}_\ell, \widehat{\phi}_\ell)$ is a better approximation. The remedy for both objectives is given by the $(h - h/2)$-type error estimator

$$\mu_\ell^2 = \|\nabla(\widehat{u}_\ell - I_\ell \widehat{u}_\ell)\|^2_{L^2(\Omega)} + \|h_\ell^{1/2}(\widehat{\phi}_\ell - \Pi_\ell \widehat{\phi}_\ell)\|^2_{L^2(\Gamma)}, \tag{7}$$

which, up to multiplicative constants, coincides with η_ℓ. Here, $h_\ell|_E = \mathrm{diam}(E)$ is the local mesh-width of $\mathcal{E}_\ell$. Moreover, $I_\ell \widehat{\phi}_\ell \in \mathcal{S}^1(\mathcal{T}_\ell)$ is the nodal interpolant, and $\Pi_\ell \widehat{\phi}_\ell \in \mathcal{P}^0(\mathcal{T}_\ell)$ is the piecewise integral mean, i.e. having computing the improved Galerkin solution $(\widehat{u}_\ell, \widehat{\phi}_\ell)$ it is an elementary and easy-to-implement postprocessing step to compute μ_ℓ.

4 Convergent Adaptive Coupling

For triangles $T \in \mathcal{T}_\ell$ and boundary edges $E \in \mathcal{E}_\ell$, we define

$$\mu_\ell(T) = \|\nabla(\widehat{u}_\ell - I_\ell \widehat{u}_\ell)\|_{L^2(T)} \quad \text{and} \quad \mu_\ell(E) = \mathrm{diam}(E)^{1/2}\|\widehat{\phi}_\ell - \Pi_\ell \widehat{\phi}_\ell\|_{L^2(E)}. \tag{8}$$

Based on these local contributions of μ_ℓ and given some fixed parameter $0 < \theta < 1$ as well as an initial mesh $\mathcal{T}_0$, the usual adaptive algorithm reads as follows:

(i) Refine $\mathcal{T}_\ell$ and $\mathcal{E}_\ell = \mathcal{T}_\ell|_\Gamma$ uniformly to obtain $\widehat{\mathcal{T}}_\ell$ and $\widehat{\mathcal{E}}_\ell = \widehat{\mathcal{T}}_\ell|_\Gamma$.

(ii) Compute Galerkin solution $(\widehat{u}_\ell, \widehat{\phi}_\ell) \in \widehat{\mathcal{X}}_\ell = \mathcal{S}^1(\widehat{\mathcal{T}}_\ell) \times \mathcal{P}^0(\widehat{\mathcal{E}}_\ell)$.

(iii) Find minimal set $\mathcal{M}_\ell \subseteq \mathcal{T}_\ell \cup \mathcal{E}_\ell$ such that $\theta \sum_{\tau \in \mathcal{T}_\ell \cup \mathcal{E}_\ell} \mu_\ell(\tau)^2 \le \sum_{\tau \in \mathcal{M}_\ell} \mu_\ell(\tau)^2$.

(iv) Refine at least marked elements $T \in \mathcal{T}_\ell \cap \mathcal{M}_\ell$ and edges $E \in \mathcal{E}_\ell \cap \mathcal{M}_\ell$ to obtain $\mathcal{T}_{\ell+1}$.

(v) Increase counter $\ell \mapsto \ell + 1$ and iterate.

In the context of FEM, convergence of such an algorithm has first been proven by [6]. Even optimality is nowadays understood for linear problems [5]. For BEM, convergence of this algorithm has recently been shown by [8]. For the adaptive coupling, the following result from our work [2] is the first convergence result available: One can prove that the adaptive algorithm guarantees $\lim_{\ell\to\infty} \mu_\ell = 0$, whence $\lim_{\ell\to\infty}(\widehat{u}_\ell, \widehat{\phi}_\ell) = (u, \phi) = \lim_{\ell\to\infty}(u_\ell, \phi_\ell)$, where only the second convergence hings on the saturation assumption (6). Our proof follows the concept of estimator reduction proposed in [3].

Numerical experiments in [2] show that our adaptive algorithm empirically leads to optimal order of convergence with respect to the degrees of freedom. Moreover, if an error accuracy is prescribed, the introduced strategy is more effective than uniform mesh-refinement with respect to computational time and storage requirements.

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