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RESIDUAL A-POSTERIORI ERROR ESTIMATES IN BEM: CONVERGENCE OF h-ADAPTIVE ALGORITHMS

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Abstract. Galerkin methods for FEM and BEM based on uniform mesh refinement have a guaranteed rate of convergence. Unfortunately, this rate may be suboptimal due to singularities present in the exact solution. In numerical experiments, the optimal rate of convergence is regained when algorithms based on a-posteriori error estimation and adaptive mesh-refinement are used. This observation was proved mathematically for the FEM in the last few years, cf. [5]. In constrast, the mathematical understanding of adaptive strategies is wide open in BEM. One reason for this is the non-locality of the boundary integral operators involved and the appearance of fractional-order or negative Sobolev norms.

In our prior works on adaptive BEM [1], we considered $h-h/2$ error estimators. Reliability of such estimators is, however, equivalent to the so-called saturation assumption. Although this is widely believed to hold in practice, it still remains mathematically open. For this reason, these convergence results are not fully satisfactory.

In our talk, we consider weighted-residual error estimators for some weakly-singular integral equations in 2D or 3D. These estimators are reliable, irrespective of the saturation assumption. We prove a certain (local) inverse-type estimate which allows us to conclude that the discrete solutions generated by the usual h-adaptive algorithm converge towards the exact solution of the integral equation. In a second step we prove quasi-optimality in a certain approximation class. From this, we infer that the rate of convergence of adaptive mesh-refinement is at least as good as for uniform approaches.

1. Boundary element method

1.1. Model problem. For a bounded Lipschitz domain $\Omega \subset \mathbb{R}^3$ with boundary Γ, we consider the Dirichlet problem

$$\begin{aligned} -\Delta u &= 0 \quad \text{in } \Omega, \\ u &= g \quad \text{on } \Gamma. \end{aligned} \tag{1}$$

The given boundary data satisifies $g \in H^{1/2}(\Gamma)$. By $\langle \cdot\,,\cdot\rangle$, we denote the extended $L_2(\Gamma)$-scalar product. Problem (1) is equivalently stated as boundary integral equation

$$\langle V\phi\,,\psi\rangle = \langle f\,,\psi\rangle \qquad \text{for all } \psi \in H^{-1/2}(\Gamma) \tag{2}$$

with $f = (K + 1/2)g$. Here, V and K denote the simple- and double-layer potential

$$\begin{aligned} (V\psi)(\mathbf{x}) &= \int_\Gamma G(\mathbf{x}-\mathbf{y})\psi(\mathbf{y})d\Gamma(\mathbf{y}), \\ (Kv)(\mathbf{x}) &= \int_\Gamma \partial_{n(\mathbf{y})}G(\mathbf{x}-\mathbf{y})v(\mathbf{y})d\Gamma(\mathbf{y}) \end{aligned}$$

with the fundamental solution $G(x) = \frac{1}{4\pi}\frac{1}{|x|}$ of the 3D-Laplacian. Equation (2) has a unique solution ϕ which depends continuously on the given data. The link between (1) and (2) is provided by $\phi = \partial_n u$. The operator V is symmetric and elliptic and thus induces an energy norm $|||\cdot|||$ via

$$|||\phi|||^2 := \langle V\phi\,, \phi\rangle.$$

1.2. **Galerkin boundary element method.** Replacing the infinite-dimensional space in (2) by some finite element space leads to the Galerkin formulation: Find $\Phi_\ell \in \mathcal{P}^0(\mathcal{E}_\ell)$ such that

$$\langle V\Phi_\ell\,, \Psi_\ell\rangle = \langle f\,, \Psi_\ell\rangle \quad \text{for all } \Psi_\ell \in \mathcal{P}^0(\mathcal{E}_\ell). \tag{3}$$

Here, $\mathcal{E}_\ell$ is a mesh on Γ, and $\mathcal{P}^0(\mathcal{E}_\ell)$ is the space of piecewise constant functions on $\mathcal{E}_\ell$. We introduce $h_\ell(E) := |E|^{1/2}$ to be the square root of the area of $E \in \mathcal{E}_\ell$. The local mesh width function $h_\ell \in L_\infty(\Gamma)$ is defined as $h_\ell|_E := h_\ell(E) = |E|^{1/2}$. The Galerkin equation (3) allows for a unique solution Φ_ℓ which is obtained solving a linear system of equations.

2. Convergent adaptive algorithm

For the effective solution of (3), we propose to use an adaptive algorithm of the type

`solve` $\longrightarrow$ `estimate` $\longrightarrow$ `mark` $\longrightarrow$ `refine`

Having computed the Galerkin solution Φ_ℓ in step `solve`, we use the weighted-residual a-posteriori error estimator μ_ℓ of [4] in step `estimate` to measure the local energy error. The estimator is defined elementwise as

$$\mu_\ell^2 := \sum_{E\in\mathcal{E}_\ell} \mu_\ell(E)^2 \quad \text{with } \mu_\ell(E)^2 := h_\ell(E)\|\nabla_\Gamma(V\Phi_\ell - f)\|_{L_2(E)}^2,$$

and it was shown in [4] that μ_ℓ is reliable, i.e.,

$$|||\phi - \Phi_\ell||| \lesssim \mu_\ell. \tag{4}$$

By $A \lesssim B$ we mean that $A \leq C * B$, where $C > 0$ is a constant that does not depend on crucial quantities such as the current step ℓ of the adaptive algorithm. If $A \lesssim B$ and $B \lesssim A$, we write $A \simeq B$.

With $\ell := 0$, an initial mesh $\mathcal{E}_0$, and the parameter $0 < \theta < 1$ as input, the adaptive algorithm now takes the following form:

(i) Compute the Galerkin solution Φ_ℓ on the mesh $\mathcal{E}_\ell$.

(ii) Compute refinement indicator $\mu_\ell(E)$ for all $E \in \mathcal{E}_\ell$

(iii) Determine the set $\mathcal{M}_\ell \subseteq \mathcal{E}_\ell$ such that

$$\sum_{E\in\mathcal{M}_\ell} \mu_\ell(E)^2 \geq \theta \sum_{E\in\mathcal{E}_\ell} \mu_\ell(E)^2, \tag{5}$$

under the condition that $\mathcal{M}_\ell$ has minimal cardinality.

(iv) Refine at least elements in $\mathcal{M}_\ell$ by newest-vertex-bisection to obtain $\mathcal{E}_{\ell+1}$.

(v) Update $\ell := \ell + 1$, goto (i).

The marking criterion (5) has been introduced in [6], and in [8] it was shown that it is in some sense even necessary for quasi-optimality of adaptive FEM.

A first important observation from [2] is that the computed discrete solutions converge to some limit Φ_∞ and thus fulfill

$$|||\Phi_\ell - \Phi_{\ell+1}||| \to 0 \quad \text{as } \ell \to \infty. \tag{6}$$

To prove $\Phi_\infty = \phi$, we show $\mu_\ell \to 0$ and use (4).
In [7], we prove that the adaptive algorithm ensures

$$\mu_{\ell+1}^2 \leq \widetilde{\kappa}\mu_\ell^2 + C\|h_{\ell+1}^{1/2}\nabla_\Gamma V\,(\Phi_{\ell+1} - \Phi_\ell)\,\|_{L_2(\Gamma)}^2 \tag{7}$$

with some $0 < \widetilde{\kappa} < 1$ and $\widetilde{\kappa} \simeq (1 - \theta(1-q))$. Here, q describes the shrinking factor of the mesh width h_ℓ on refined elements. Since marked elements are at least split into 2 sons with at most halve area, we have $q = 1/2$. To estimate the last term in (7), we provide the novel inverse estimate

$$\|h_k^{1/2}\nabla_\Gamma V\Psi_k\|_{L_2(\Gamma)} \lesssim |||\Psi_k||| \quad \text{for all } \Psi_k \in \mathcal{P}^0(\mathcal{E}_k). \tag{8}$$

We stress that on quasi-uniform meshes, (8) follows immediately from the theory of interpolation spaces. On locally refined meshes, the proof of (8) is much more technical due to the non-locality of V and the fact that $|||\cdot|||$ is equivalent to a negative Sobolev norm of fractional order. Choosing $k = \ell + 1$ and $\Psi_k = \Phi_{\ell+1} - \Phi_\ell$ in (8) yields the so-called *estimator reduction*

$$\mu_{\ell+1}^2 \leq \widetilde{\kappa}\mu_\ell^2 + C|||\Phi_{\ell+1} - \Phi_\ell|||^2. \tag{9}$$

Together with (6), the estimator reduction can be written in Landau-notation as

$$\mu_\ell^2 \leq \widetilde{\kappa}\mu_\ell^2 + \mathrm{o}(1),$$

and elementary calculus shows $\mu_\ell \to 0$, i.e. the discrete solutions computed from the adaptive algorithm converge to the exact solution of the boundary integral equation. From (9), we conclude in [7] that a weighted sum of energy error and error estimator, the so-called quasi-error, is indeed a contraction. This is a stronger result than pure convergence, because the quasi-error is reduced uniformly in each step of the adaptive loop.

Theorem 1. *There are constants $0 < \kappa, \lambda < 1$ such that the quasi-error is contractive, i.e.*

$$\Delta_{\ell+1} \leq \kappa\Delta_\ell \quad \text{with } \Delta_\ell := |||\phi - \Phi_\ell|||^2 + \lambda\mu_\ell^2. \tag{10}$$

The constants κ and λ do not depend on ϕ or ℓ. In particular, this implies convergence

$$|||\phi - \Phi_\ell||| \to 0 \quad \text{as } \ell \to \infty.$$

3. A Quasi-Optimality Result

Contemporary proofs of quasi-optimality of adaptive FEM mainly follow [5] and have three main ingredients:

(I) The contraction property for the quasi-error, cf. Theorem 10,
(II) the optimality of the marking strategy (5),
(III) and the definition of a proper approximation class.

The proof of (II), i.e. the optimality of the marking strategy, needs the *discrete local reliability* of the error estimator. It states that, for two successive meshes, the difference of the corresponding Galerkin solutions is bounded by the error estimator on the refined elements. In contrast to the FEM, the energy norm $|||\cdot|||$ is equivalent to the non-local norm $\|\cdot\|_{H^{-1/2}(\Gamma)}$, which introduces technical difficulties. Nevertheless, the discrete local reliability of μ_ℓ can be proven. We emphasize that the set $\mathcal{E}_\ell \setminus \mathcal{E}_\star$ in the next Lemma is actually the set of elements from $\mathcal{E}_\ell$ that are refined when $\mathcal{E}_\ell$ is refined to $\mathcal{E}_\star$. The set $\mathcal{R}_\ell$ is basically this last set plus one additional layer of elements.

Lemma 1. *Let $\mathcal{E}_\star$ be an arbitrary refinement of $\mathcal{E}_\ell$ with corresponding Galerkin solution $\Phi_\star \in \mathcal{P}^0(\mathcal{E}_\star)$. Then,*

$$|||\Phi_\star - \Phi_\ell|||^2 \lesssim \sum_{E \in \mathcal{R}_\ell} \mu_\ell(E)^2.$$

Here, $\mathcal{R}_\ell := \{E \in \mathcal{E}_\ell : \exists E' \in \mathcal{E}_\ell \setminus \mathcal{E}_\star : E \in \omega_\ell(E')\}$, and $\omega_\ell(E')$ is the patch of all elements touching E'.

So far, we have proven in Theorem 1 that the marking criterion (5) implies a contraction of Δ_ℓ. The following proposition states the converse implication, i.e. the ingredient (II) is available.

Proposition 1. *There are constants $0 < \kappa_\star, \theta_\star < 1$ such that for all $0 < \theta \le \theta_\star$ and all refinements $\mathcal{E}_\star$ of $\mathcal{E}_\ell$ holds*

$$\Delta_\star \le \kappa_\star \Delta_\ell \implies \theta \sum_{E \in \mathcal{E}_\ell} \mu_\ell(E)^2 \le \sum_{E \in \mathcal{R}_\ell} \mu_\ell(E)^2,$$

i.e. the set $\mathcal{R}_\ell$ defined in Lemma 1 satisfies the marking criterion (5). The constants $\kappa_\star$, $\theta_\star$ do not depend on ℓ.

In [5], the point (III) contains the definition of an *optimal* approximation class, i.e. a prescribed accuracy for the approximation error should be reached with the optimal order of complexity of the corresponding Galerkin method. Note that the adaptive algorithm only sees the estimator μ_ℓ, thus it is neccesary to link μ_ℓ to the approximation error introduced within the approximation class. This leads us to the definition of *oscillation*. In adaptive FEM, it can be proven that, on locally refined meshes,

$$\eta_\ell \simeq |||u - U_\ell||| + \mathrm{osc}_\ell \tag{11}$$

for the residual error estimator η_ℓ, the exact solution u and the FEM-solution U_ℓ. The term osc_ℓ is of higher order compared to $|||u - U_\ell|||$, and thus the right hand side of (11) is a prospective candidate for the definition of the approximation class.
Up to now, a result of the form (11) for BEM is available only on quasi-uniform meshes. Using an efficiency result from [3], we have in this special case the equivalence

$$\mu_\ell \simeq |||\phi - \Phi_\ell|||. \tag{12}$$

On locally refined meshes, we provide a weak efficiency result of the form

$$\mu_\ell \simeq |||\phi - \Phi_\ell||| + \mathrm{osc}_\ell,$$

with an oscillation term osc_ℓ that is generically not of higher order. We then use the right hand side as approximation error in an artificial approximation class.

Let $\mathbb{T}_N$ be the set of all meshes $\mathcal{E}$ with at most $\#\mathcal{E} = N$ elements, which can be generated from the initial mesh $\mathcal{E}_0$ by newest-vertex-bisection. We introduce the approximation class

$$\mathbb{A}_s := \left\{ \phi : |\phi|_s := \sup_{N>0} \left(N^s \inf_{\mathcal{E}_\star \in \mathbb{T}_N} \mu_\star \right) < \infty \right\}. \tag{13}$$

Here, $\mu_\star$ denotes the weighted-residual error estimator with respect to $\mathcal{E}_\star$. The interpretation of this definition is the following: If $\phi \in \mathbb{A}_s$, then the estimator μ_ℓ and in particular the error $|||\phi - \Phi_\ell|||$ can be as small as $\epsilon > 0$ with $\mathcal{O}(\epsilon^{-s})$ elements. In [7], we prove the following quasi-optimality result:

Theorem 2. *For sufficient small adaptivity parameter θ, the adaptive algorithm ensures*

$$\mu_\ell \lesssim \#\mathcal{E}_\ell^{-s}$$

provided that $\phi \in \mathbb{A}_s$ for some $s > 0$. With reliability (4), this yields

$$|||\phi - \Phi_\ell||| \lesssim \#\mathcal{E}_\ell^{-s}$$

The consequence is the following: If a uniform approach yields $|||\phi - \Phi_\ell||| = \mathcal{O}(N^{-t})$ for some $t > 0$, then it follows from (12) that $\phi \in \mathbb{A}_t$. But then Theorem 2 states that the adaptive algorithm also achieves an order of convergence $\mathcal{O}(N^{-t})$. This concludes that the adaptive algorithm achieves at least the same order of convergence as the uniform approach.

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