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# ADAPTIVE COUPLING OF FEM AND BEM: SIMPLE ERROR ESTIMATORS AND CONVERGENCE

MARKUS AURADA, MICHAEL FEISCHL, MICHAEL KARKULIK, AND DIRK PRAETORIUS

ABSTRACT. A posteriori error estimators and adaptive mesh-refinement have themselves proven to be important tools for scientific computing. For error control in finite element methods (FEM), there is a broad variety of a posteriori error estimators available, and convergence as well as quasi-optimality of adaptive FEM is well-studied in the literature, cf. e.g. [1] for error estimation and [2] and the references therein for convergence and quasi-optimality. This is, however, in sharp contrast to the boundary element method (BEM) and the coupling of FEM and BEM, cf. [3] for an overview on BEM error estimators and [4, 5] for first preliminary convergence results.

In our contribution, based on [6], we present an easy-to-implement  $(h - h/2)$ -type error estimator  $\mu_\ell$  for some FEM-BEM coupling which, to the best of our knowledge, has not been proposed in the literature before. The considered  $(h - h/2)$ -based approach is mathematically unifying in the sense that only stability of the FEM-BEM coupling as well as certain inverse estimates and approximation estimates for the energy norm are used. It is therefore applicable to symmetric as well as non-symmetric FEM-BEM formulations without any modification.

The lower bound  $\mu_\ell \leq C \cdot \text{error}_\ell$  does always hold, whereas the upper bound  $\text{error}_\ell \leq C \cdot \mu_\ell$  depends on a saturation assumption. In numerical experiments, this assumption, which is mathematically crucial, is empirically checked and verified.

The proposed mesh-refining algorithm provides the first adaptive coupling procedure which is mathematically proven to converge. More precisely, we show that the adaptive algorithm, based on Dörfler marking [7] and newest vertex bisection, drives the underlying error estimator to zero.

## 1. THE ABSTRACT IDEA OF $(h - h/2)$ -ERROR ESTIMATION

1.1. **Abstract setting.** We consider the following variational formulation: Find  $u \in \mathcal{H}$  such that

$$(1) \quad \langle\langle u, v \rangle\rangle = F(v) \quad \text{for all } v \in \mathcal{H}.$$

Here,  $\mathcal{H}$  is a Hilbert space with norm  $\|\cdot\|$ ,  $\langle\langle \cdot, \cdot \rangle\rangle$  is a given continuous bilinear form on  $\mathcal{H}$  (which may be even nonlinear in the first component), and  $F(\cdot)$  is a given linear and continuous functional.

We suppose that (1) admits a unique solution. For instance,  $\langle\langle \cdot, \cdot \rangle\rangle$  is elliptic so that the Lemma of Lax-Milgram can be applied, or, in general, the bilinear form  $\langle\langle \cdot, \cdot \rangle\rangle$  satisfies an inf-sup condition. If  $\langle\langle \cdot, \cdot \rangle\rangle$  depends nonlinearly on  $u$ , unique solvability of (1) is, for instance, satisfied if the induced operator  $\mathcal{A} : \mathcal{H} \rightarrow \mathcal{H}^*$ ,  $\mathcal{A}u = \langle\langle u, \cdot \rangle\rangle$  is strongly monotone and Lipschitz continuous.

For a Galerkin discretization of (1), let  $\mathcal{X}_\ell$  be a finite dimensional subspace, which in many applications is based on a certain mesh  $\mathcal{T}_\ell$  of some physical domain. Here and throughout, the index  $\ell$  denotes the  $\ell$ -th step of the adaptive loop below, and associated quantities are discrete and numerically computable. The discrete formulation then reads: Find  $U_\ell \in \mathcal{X}_\ell$  such that

$$(2) \quad \langle\langle U_\ell, V_\ell \rangle\rangle = F(V_\ell) \quad \text{for all } V_\ell \in \mathcal{X}_\ell.$$

Again, we assume existence and uniqueness of a discrete solution  $U_\ell \in \mathcal{X}_\ell$ .

**1.2. Adaptive mesh-refining algorithm.** We consider an adaptive loop of the usual type

$$(3) \quad \boxed{\text{solve}} \longrightarrow \boxed{\text{estimate}} \longrightarrow \boxed{\text{mark}} \longrightarrow \boxed{\text{refine}}$$

Formally, we start with some initial mesh  $\mathcal{T}_0$ . We fix an adaptivity parameter  $0 < \theta < 1$  and assume that we can compute certain refinement indicators  $\mu_\ell(T) \approx \|u - U_\ell\|_T$  for all  $T \in \mathcal{T}_\ell$  which measure, at least heuristically, the *local* error between  $u$  and  $U_\ell$  on some element  $T \in \mathcal{T}_\ell$ . Then, the adaptive loop, for  $\ell = 0, 1, 2, 3, \dots$ , reads as follows:

- (i) Compute discrete solution  $U_\ell \in \mathcal{X}_\ell$  for mesh  $\mathcal{T}_\ell$ .
- (ii) Compute refinement indicators  $\mu_\ell(T)$  for all  $T \in \mathcal{T}_\ell$ .
- (iii) Determine set  $\mathcal{M}_\ell \subseteq \mathcal{T}_\ell$  of (usually) minimal cardinality such that

$$(4) \quad \theta \sum_{T \in \mathcal{T}_\ell} \mu_\ell(T)^2 \leq \sum_{T \in \mathcal{M}_\ell} \mu_\ell(T)^2.$$

- (iv) Refine (at least) marked elements  $T \in \mathcal{M}_\ell$  to obtain  $\mathcal{T}_{\ell+1}$
- (v) Update counter  $\ell \mapsto \ell + 1$  and iterate.

The marking criterion (4) has been introduced in [7] to prove a first convergence result for adaptive FEM. In [2], it is proved that (4) is sufficient for convergence (with arbitrary  $\mathcal{M}_\ell \subseteq \mathcal{T}_\ell$ ) and in some sense even necessary for quasi-optimality of adaptive FEM (with minimal  $\mathcal{M}_\ell \subseteq \mathcal{T}_\ell$ ).

**1.3.  $(h - h/2)$ -error estimators.** The  $(h - h/2)$ -error estimation is a well-known technique for the a posteriori estimation of the error in the energy norm  $\|u - U_\ell\|$ , see [8] in the context of numerical integrators for ordinary differential equations, or the monograph [1, Chapter 5] in the context of FEM. For BEM,  $(h - h/2)$ -type estimators have first been considered in [9].

Let  $\widehat{\mathcal{T}}_\ell$  be the uniform refinement of  $\mathcal{T}_\ell$  and let  $\widehat{\mathcal{X}}_\ell$  be the associated discrete subspace of  $\mathcal{H}$  with corresponding Galerkin solution  $\widehat{U}_\ell \in \widehat{\mathcal{X}}_\ell$ . Then, the  $(h - h/2)$ -error estimator

$$(5) \quad \eta_\ell := \|\widehat{U}_\ell - U_\ell\|$$

is a computable quantity which can be used to estimate the error  $\|u - U_\ell\|$ :

First, provided that the discrete solution  $U_\ell$  has some best approximation property, i.e. it holds the Céa-type estimate

$$(6) \quad \|u - U_\ell\| \lesssim \min_{V_\ell \in \mathcal{X}_\ell} \|u - V_\ell\|,$$

it is an easy consequence of the triangle inequality and  $\mathcal{X}_\ell \subset \mathcal{X}_{\ell+1}$  that the  $(h - h/2)$ -error estimator satisfies the lower bound

$$(7) \quad \eta_\ell \lesssim \|u - U_\ell\|.$$

Here and throughout, the symbol  $\lesssim$  abbreviates  $\leq$  up to some multiplicative constant  $C > 0$  which may depend on the problem, but is independent of the discretization. We stress that for linear problems, the Céa-type estimate (6) is equivalent to stability. Moreover, it also holds in the context of strongly monotone operators.

Second, under the so-called saturation assumption

$$(8) \quad \|u - \widehat{U}_\ell\| \leq C_{\text{sat}} \|u - U_\ell\| \quad \text{with some } \ell\text{-independent constant } 0 < C_{\text{sat}} < 1,$$

the triangle inequality verifies that  $\eta_\ell$  satisfies also the upper bound

$$(9) \quad \|u - U_\ell\| \lesssim \eta_\ell.$$

For  $\langle \cdot, \cdot \rangle$  being symmetric and elliptic, it can easily be proved that the saturation assumption (8), stated in the energy norm, is not only sufficient, but even necessary to allow (9), cf. e.g. [9].

Although mathematical counter examples to (8) can be constructed, the saturation assumption essentially states that the numerical scheme has reached an asymptotic regime  $\|u - U_\ell\| = \mathcal{O}(h_\ell^{-\alpha})$  with  $h_\ell$  the mesh-width of  $\mathcal{T}_\ell$ . Then, even the quantitative value of  $C_{\text{sat}}$  can be predicted [9]. For FEM, it is rigorously proved in [10] that the saturation assumption (8) is satisfied if the given data are sufficiently resolved by the mesh  $\mathcal{T}_\ell$ .

In [4], an adaptive FE algorithm driven by  $(h - h/2)$ -type estimators is proposed and proven to converge. Therein, the resolution of the given data is part of the adaptive loop so that the saturation assumption (8) and hence reliability (9) is mathematically guaranteed.

So far, a result analogous to that of [10] is missing in the context of BEM and FEM-BEM coupling. The convergence result of [4] for an adaptive BE algorithm driven by  $(h - h/2)$ -type estimators therefore strongly depends on the saturation assumption (8).

## 2. SYMMETRIC FEM-BEM COUPLING

**2.1. Model problem.** For a bounded Lipschitz domain  $\Omega \subset \mathbb{R}^2$  with  $\Gamma = \partial\Omega$ , we consider the nonlinear interface problem

$$(10) \quad \begin{aligned} -\operatorname{div}(A\nabla u^{\text{int}}) &= f && \text{in } \Omega^{\text{int}} := \Omega, \\ -\Delta u^{\text{ext}} &= 0 && \text{in } \Omega^{\text{ext}} := \mathbb{R}^2 \setminus \overline{\Omega}, \\ u^{\text{int}} - u^{\text{ext}} &= u_0 && \text{on } \Gamma, \\ (A\nabla u^{\text{int}} - \nabla u^{\text{ext}}) \cdot \mathbf{n} &= \phi_0 && \text{on } \Gamma, \\ u^{\text{ext}}(x) &= a \log|x| + \mathcal{O}(1/|x|) && \text{as } |x| \rightarrow \infty, \end{aligned}$$

where  $\mathbf{n}$  denotes the outer unit normal vector. The given data satisfy  $f \in L^2(\Omega)$ ,  $u_0 \in H^{1/2}(\Gamma)$ , and  $\phi_0 \in H^{-1/2}(\Gamma)$ , and the (possibly nonlinear) operator  $A : L^2(\Omega)^2 \rightarrow L^2(\Omega)^2$  is assumed to be strongly monotone and Lipschitz continuous.

Problem (10) is equivalently stated via the well-known symmetric FEM-BEM coupling: Find  $(u, \phi) \in \mathcal{H} := H^1(\Omega) \times H^{-1/2}(\Gamma)$  such that, for all  $(v, \psi) \in \mathcal{H}$ ,

$$(11) \quad \begin{aligned} \langle A\nabla u, \nabla v \rangle_\Omega + \langle \mathcal{W}u + (\mathcal{K}' - \tfrac{1}{2})\phi, v \rangle_\Gamma &= \langle f, v \rangle_\Omega + \langle \phi_0 + \mathcal{W}u_0, v \rangle_\Gamma, \\ \langle \psi, \mathcal{V}\phi - (\mathcal{K} - \tfrac{1}{2})u \rangle_\Gamma &= -\langle \psi, (\mathcal{K} - \tfrac{1}{2})u_0 \rangle_\Gamma. \end{aligned}$$

Here,  $\mathcal{V}$  denotes the simple-layer potential,  $\mathcal{K}$  denotes the double-layer potential with adjoint  $\mathcal{K}'$ , and  $\mathcal{W}$  denotes the hypersingular integral operator. With the fundamental solution  $G(z) = -\frac{1}{2\pi} \log|z|$  of the 2D Laplacian, these boundary integral operators formally read, for  $x \in \Gamma$ ,

$$(12) \quad (\mathcal{V}\psi)(x) = \int_\Gamma G(x-y) \psi(y) d\Gamma(y),$$

$$(13) \quad (\mathcal{K}v)(x) = \int_\Gamma \partial_{\mathbf{n}(y)} G(x-y) v(y) d\Gamma(y),$$

$$(14) \quad (\mathcal{W}v)(x) = -\partial_{\mathbf{n}(x)} \int_\Gamma \partial_{\mathbf{n}(y)} G(x-y) v(y) d\Gamma(y).$$

By scaling of the domain, we may assume  $\operatorname{diam}(\Omega) < 1$ . Then, (11) has a unique solution  $(u, \phi)$  which depends continuously on the given data, see e.g. [11]. Moreover, (10) and (11) are linked through  $(u, \phi) = (u^{\text{int}}, \partial_{\mathbf{n}} u^{\text{ext}})$  and  $u^{\text{ext}} = \mathcal{K}(u - u_0) - \mathcal{V}\phi$ .

With  $\langle \cdot, \cdot \rangle$  being the sum of the left-hand side and  $F(\cdot)$  being the sum of the right-hand side of (11), the symmetric FEM-BEM coupling fits into the abstract setting of Section 1.1.

**2.2. Galerkin discretization.** For the Galerkin discretization, let  $\mathcal{T}_\ell$  be a regular triangulation of  $\Omega$  into triangles  $T_j \in \mathcal{T}_\ell$  and  $\mathcal{E}_\ell = \mathcal{T}_\ell|_\Gamma$  be the induced partition of the coupling boundary  $\Gamma$  into piecewise affine line segments  $E_j \in \mathcal{E}_\ell$ . We then use P1-finite elements  $U_\ell \in \mathcal{S}^1(\mathcal{T}_\ell)$  to approximate  $u$  and piecewise constants  $\Phi_\ell \in \mathcal{P}^0(\mathcal{E}_\ell)$  to approximate  $\phi$ , i.e. the discrete space is defined by  $\mathcal{X}_\ell := \mathcal{S}^1(\mathcal{T}_\ell) \times \mathcal{P}^0(\mathcal{E}_\ell) \subset \mathcal{H}$ . Now, the Galerkin formulation reads: Find  $(U_\ell, \Phi_\ell) \in \mathcal{X}_\ell$  such that, for all  $(v_\ell, \Psi_\ell) \in \mathcal{X}_\ell$ ,

$$(15) \quad \begin{aligned} \langle A\nabla U_\ell, \nabla V_\ell \rangle_\Omega + \langle \mathcal{W}U_\ell + (\mathcal{K}' - \frac{1}{2})\Phi_\ell, V_\ell \rangle_\Gamma &= \langle f, V_\ell \rangle_\Omega + \langle \phi_0 + \mathcal{W}u_0, V_\ell \rangle_\Gamma, \\ \langle \Psi_\ell, \mathcal{V}\Phi_\ell - (\mathcal{K} - \frac{1}{2})U_\ell \rangle_\Gamma &= -\langle \Psi_\ell, (\mathcal{K} - \frac{1}{2})u_0 \rangle_\Gamma. \end{aligned}$$

Again, we refer to [11] for the fact that the discretization (15) has a unique solution  $(U_\ell, \Phi_\ell) \in \mathcal{X}_\ell$  which satisfies Céa's lemma (6).

**2.3. Local  $(h - h/2)$ -type error estimator.** In this concrete setting with  $\|(v, \psi)\|^2 = \|v\|_{H^1(\Omega)}^2 + \|\psi\|_{H^{-1/2}(\Gamma)}^2$ , the  $(h - h/2)$ -error estimator from (5) takes the concrete form

$$(16) \quad \eta_\ell^2 = \|\widehat{U}_\ell - U_\ell\|_{H^1(\Omega)}^2 + \|\widehat{\Phi}_\ell - \Phi_\ell\|_{H^{-1/2}(\Gamma)}^2.$$

We obtain efficiency (7) in general and reliability (9) under the saturation assumption (8). Here,  $(\widehat{U}_\ell, \widehat{\Phi}_\ell) \in \widehat{\mathcal{X}}_\ell$  is the Galerkin solution for the uniform refinement  $\widehat{\mathcal{T}}_\ell$  of  $\mathcal{T}_\ell$  and  $\widehat{\mathcal{E}}_\ell = \widehat{\mathcal{T}}_\ell|_\Gamma$ .

However, we stress that, first, the  $H^{-1/2}$ -norm can hardly be computed and, second,  $(U_\ell, \Phi_\ell)$  is hardly ever used in practice since  $(\widehat{U}_\ell, \widehat{\Phi}_\ell)$  is a better approximation. The remedy for both objectives is given by the  $(h - h/2)$ -type error estimator

$$(17) \quad \mu_\ell^2 = \|\nabla(\widehat{U}_\ell - \mathcal{I}_\ell \widehat{U}_\ell)\|_{L^2(\Omega)}^2 + \|h_\ell^{1/2}(\widehat{\Phi}_\ell - \Pi_\ell \widehat{\Phi}_\ell)\|_{L^2(\Gamma)}^2,$$

which satisfies equivalence  $\mu_\ell \simeq \eta_\ell$ . Here,  $h_\ell|_E = \text{diam}(E)$  is the local mesh-width of  $\mathcal{E}_\ell$ . Moreover,  $\mathcal{I}_\ell \widehat{U}_\ell \in \mathcal{S}^1(\mathcal{T}_\ell)$  is the nodal interpolant, and  $\Pi_\ell \widehat{\Phi}_\ell \in \mathcal{P}^0(\mathcal{E}_\ell)$  is the piecewise integral mean, i.e. having computed the improved Galerkin solution  $(\widehat{U}_\ell, \widehat{\Phi}_\ell)$  it is a computationally cheap as well as elementary and easy-to-implement postprocessing step to compute  $\mu_\ell$ .

**2.4. Convergence of adaptive algorithm.** For triangles  $T \in \mathcal{T}_\ell$  and boundary edges  $E \in \mathcal{E}_\ell$ , we define

$$(18) \quad \mu_\ell(T) = \|\nabla(\widehat{U}_\ell - \mathcal{I}_\ell \widehat{U}_\ell)\|_{L^2(T)} \quad \text{and} \quad \mu_\ell(E) = \text{diam}(E)^{1/2} \|\widehat{\Phi}_\ell - \Pi_\ell \widehat{\Phi}_\ell\|_{L^2(E)}$$

and note that  $\eta_\ell^2 \simeq \mu_\ell^2 = \sum_{T \in \mathcal{T}_\ell} \mu_\ell(T)^2 + \sum_{E \in \mathcal{E}_\ell} \mu_\ell(E)^2$ .

Based on these local contributions of  $\mu_\ell$ , we consider the adaptive algorithm from Section 1.2, where the Dörfler marking (4) in step (iii) of the adaptive loop is now considered with  $\mathcal{T}_\ell$  replaced by  $\mathcal{T}_\ell \cup \mathcal{E}_\ell$ . In step (iv) of the adaptive loop, we use the newest vertex bisection algorithm to generate a new regular triangulation  $\mathcal{T}_{\ell+1}$ . To that end, we mark all edges  $E \in \mathcal{E}_\ell \cap \mathcal{M}_\ell$  and all edges of elements  $T \in \mathcal{T}_\ell \cap \mathcal{M}_\ell$  for bisection.

By use of this marking and refinement step, one can prove that the estimator  $\mu_\ell$  satisfies a perturbed contraction

$$(19) \quad \mu_{\ell+1} \leq q \mu_\ell + C \|(\widehat{U}_{\ell+1}, \widehat{\Phi}_{\ell+1}) - (\widehat{U}_\ell, \widehat{\Phi}_\ell)\|$$

with  $\ell$ -independent constants  $0 < q < 1$  and  $C > 0$ . Moreover, successive (local) mesh-refinement by newest vertex bisection guarantees nestedness  $\widehat{\mathcal{X}}_\ell \subset \widehat{\mathcal{X}}_{\ell+1}$  of the discrete spaces. Together with this observation, the Céa-type estimate (6) implies that the discrete solutions  $(\widehat{U}_\ell, \widehat{\Phi}_\ell)$  tend to some limit  $(\widehat{U}_\infty, \widehat{\Phi}_\infty)$  as  $\ell \rightarrow \infty$ . With this observation, the estimator reduction (19) may be written in Landau small- $\mathcal{O}$  notation

$$(20) \quad \mu_{\ell+1} \leq q \mu_\ell + \mathcal{O}(1),$$

and elementary calculus predicts  $\lim_\ell \mu_\ell = 0$ , i.e. the adaptive algorithm drives the underlying error estimator to zero. See [5] for this concept of *estimator reduction*.

Provided that the saturation assumption (8) holds—at least along the sequence of meshes  $\mathcal{T}_\ell$  generated by the adaptive loop—the adaptive algorithm thus reveals  $\lim_\ell \|(u, \phi) - (U_\ell, \Phi_\ell)\| = 0 = \|(u, \phi) - (\widehat{U}_\ell, \widehat{\Phi}_\ell)\|$ .

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